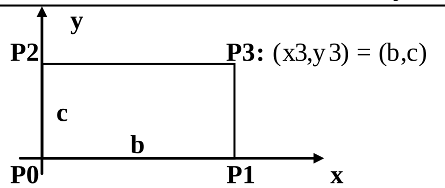
COUNT&ADD IN SPACE **S2**

Question	Answer
How to predict the position of points and lines?	By using a coordinate-system: If Po(x,y) = (3,4) and if $\Delta y/\Delta x = 2$, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14)
How to use the new calculation technology?	Computers can calculate set of numbers (vectors) and set of vectors (matrices)

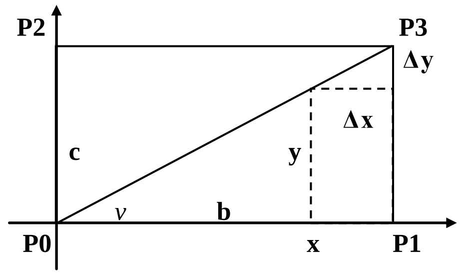
1 COUNTING POSITION

Question. How can we count the position of a point and a line?
Answer. By using a Cartesian coordinate-system assigning numbers to vertical and horizontal distances:
Example. A stack has 4 corners. The lower left corner is chosen as the centre form which steps are counted both in the horizontal direction (x-axis) and in the vertical direction (y-axis). The numbers are called the 1. and 2. coordinates coordinating points and numbers. The two axis are called a Cartesian coordinate system.

Points. In a c*b stack the 4 corner points have the coordinates
P0(x0,y0) = (0,0) is called the origin
P1(x1,y1) = (b,0) where b is the right/left-number
P2(x2,y2) = (0,c) where c is the up/down-number
P3(x3,y3) = (b,c)



Lines. The four side lines have the equations
P0P1: y=0, P2P3: y=c, P0P2: x=0, P1P3: x=b
The two diagonals have the equations:
P0P3: $\Delta y = \Delta y/\Delta x * \Delta x$
$y-0 = (c-0)/(b-0) * (x-0)$
$y = c/b * x$
or $y = \tan v * x$
where v is the altitude angle of the diagonal.
P2P1: $\Delta y = \Delta y/\Delta x*\Delta x = (0-c)/(b-0)*\Delta x = -c/b*\Delta x$
or $y-c = -c/b*(x-0)$, or $y = -c/b*x + c$



Distances. Horizontal and vertical distances are the differences between the coordinates: |P0P1|= |b-0| = b & |P0P2|= |0-c| = c where |b| means the numerical value of b: |±3| = 3.
The length of the diagonal is predicted by Pythagoras:
|P0P3|^2 = |P0P1|^2 + |P1P2|^2 = (x1-x0)^2 + (y3-y1)^2
= (b-0)^2 + (c-0)^2 = b^2 + c^2

The distance from P3 to the diagonal P2P1 is predicted by inserting the P3's coordinates in the distance-formula:
Distance = |a1x+b1y+c|/√(a1^2+b1^2)

The equation for a line y = m*x+b

Point&Point: (5,3) og (7,9)

y = ?	$\Delta y = \Delta y/\Delta x * \Delta x$
$\Delta y = y - 3$ $\Delta x = x - 5$ $\frac{\Delta y}{\Delta x} = \frac{9-3}{7-5} = \frac{6}{2} = 3$	$y - 3 = 3*(x - 5)$ $y = 3*x - 15 + 3$ **y = 3*x –12**

Point&Slope: (5,3) with the slope 2

y = ?	$\Delta y = \Delta y/\Delta x * \Delta x$
$\Delta y = y - 3$ $\Delta x = x - 5$ $\frac{\Delta y}{\Delta x} = 2$	$y - 3 = 2*(x - 5)$ $y = 2*x - 10 + 3$ **y = 2*x –7**

Exercise. Describe the corners, side lines and diagonals in a 3*5 stack.

2 PREDICTING INTERSECTION POINTS

Question. How can we predict the intersection point between two lines?
Answer. By inserting one line equation in the other, or by reversing vector calculation as in 4.
Intersection points. The two diagonals have the intersection point S(x,y):

(x,y) = ?	y = y
y = c/b*x y = -c/b*x+c	c/b*x = -c/b*x + c 2*c/b*x = c **x = b/2** **y** = c/b*b/2 = **c/2**
Check:	*c/b*b/2 = -c/b*b/2 + c* *c/2 = -c/2 + c* *c/2 = c/2*

(x,y)=?	y = y
y = a1x+b1 y = a2x+b2 D = a1b2-a2b1 $= \begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$ D: Determinant a1 = c/b, b1 = 0 a2 = -c/b, b2 = c	a1x+b1 = a2x+b2 x(a1-a2) = b2-b1 x = (b2-b1)/(a1-a2) y = a1x+b1 = (a1*(b2-b1)+ b1*(a1-a2))/(a1-a2) = (a1*b2-a1*b1+ b1*a1-b1*a2)/(a1-a2) = (a1b2 - a2b1)/(a1-a2) = D/(a1-a2) x = (c-0)/(c/b+c/b) = c/(2*c/b) = c/2/c*b = b/2 y = (c/b*c-(-c/b)*0)/(c/b+c/b) = c/2

Exercise. Predict the intersection point of the diagonals in a 3*5 stack. And in a parallelogram.

3 VECTOR AND MATRIX

Question. How can we perform multiple simultaneous calculations?
Answer. By using number sets (vectors) and vector sets (matrices).
Number sets (vectors) are used to describe position in a coordinate system, and to describe goods and prices.
Example. Two goods weighing 15 and 35 kg are priced at 4$/kg and 6$/kg. The total value then is
T = 15*4 + 35*6. This can be described by two vectors, vertical or horizontal:

kg	$/kg	$
$\begin{pmatrix}15\\35\end{pmatrix}$ $\begin{pmatrix}15 & 35\end{pmatrix}$	$\begin{pmatrix}4\\6\end{pmatrix}$ $\begin{pmatrix}4 & 6\end{pmatrix}$	$\begin{pmatrix}15\\35\end{pmatrix} * \begin{pmatrix}4\\6\end{pmatrix} = 15*4 + 35*6 = 270$ $\begin{pmatrix}15 & 35\end{pmatrix} * \begin{pmatrix}4 & 6\end{pmatrix} = 15*4 + 35*6 = 270$

Vectors can be added	Vectors can be multiplied (a scalar product):
$\begin{pmatrix}15\\35\\10\\12\end{pmatrix} + \begin{pmatrix}40\\10\\5\\30\end{pmatrix} = \begin{pmatrix}15+40\\35+10\\10+5\\12+30\end{pmatrix} = \begin{pmatrix}55\\45\\15\\42\end{pmatrix}$	$\begin{pmatrix}15\\35\\10\\12\end{pmatrix} * \begin{pmatrix}40\\10\\5\\30\end{pmatrix} = 15*40 + 35*10 + 10*5 + 12*30 = 1360$

Exercise. Describe blending 5 sorts of tea by vectors.

4 REVERSING VECTOR CALCULATIONS

Question. How can we reverse vector calculations?
Answer. By using determinants.

Two vectors equations are united to	one matrix equation
$\begin{pmatrix}7 & 5\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = 7*x+5*y = 29$ & $\begin{pmatrix}8 & 3\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = 8*x+3*y = 25$	$\begin{pmatrix}7 & 5\\8 & 3\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}7*x+5*y\\8*x+3*y\end{pmatrix} = \begin{pmatrix}29\\25\end{pmatrix}$

A general matrix equation looks like this $\begin{pmatrix}a1 & a2\\b1 & b2\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a1*x+a2*y\\b1*x+b2*y\end{pmatrix} = \begin{pmatrix}c1\\c2\end{pmatrix}$. It can be solved by introducing the determinant of a matrix: Determinant $\begin{pmatrix}a1 & a2\\b1 & b2\end{pmatrix} = \begin{vmatrix}a1 & a2\\b1 & b2\end{vmatrix} = a1*b2 - a2*b1$

a1*x + a2*y = c1 (I)		b1*x + b2*y = c2 (II)
a1*x = c1 - a2*y a1*b1*x = c1*b1 - a2*b1*y →	a1*c2 – a1*b2*y = c1*b1 - a2*b1*y a1*c2 - b1*c1 = a1*b2*y - a2*b1*y a1*c2 - b1*c1 = (a1*b2 - a2*b1)*y $\frac{a1*c2 - b1*c1}{a1*b2 - a2*b1} = y$ $\frac{\begin{vmatrix}a1 & c1\\b1 & c2\end{vmatrix}}{\begin{vmatrix}a1 & a2\\b1 & b2\end{vmatrix}} = y$	b1*x = c2 - b2*y a1*b1*x = a1*c2 – a1*b2*y ←
a2*y = c1 - a1*x a2*b2*y = c1*b2 - a1*b2*x →	c1*b2 - a1*b2*x = a2*c2 - a2*b1*x c1*b2 - a2*c2 = a1*b2*x - a2*b1*x c1*b2 - a2*c2 = (a1*b2 - a2*b1)*x $\frac{c1*b2 - a2*c2}{a1*b2 - a2*b1} = x$ $\frac{\begin{vmatrix}c1 & a2\\c2 & b2\end{vmatrix}}{\begin{vmatrix}a1 & a2\\b1 & b2\end{vmatrix}} = x$	b2*y = c2 - b1*x a2*b2*y = a2*c2 – a2*b1*x ←

Example. Three different goods or spatial positioning involves 3x3 matrices leading to solving 3 equations with 3 unknowns by letting Excel calculate the determinants:

$\begin{pmatrix}3 & 5 & 6\\2 & 4 & 3\\6 & 4 & 2\end{pmatrix} * \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3*x+5*y+6*z\\2*x+4*y+3*z\\6*x+4*y+2*z\end{pmatrix} = \begin{pmatrix}52\\31\\42\end{pmatrix}$ or as 3 equations:	3*x + 5*y + 6*z = 52 2*x + 4*y + 3*z = 31 6*x + 4*y + 2*z = 42

$$x = \frac{\begin{vmatrix} 52 & 5 & 6 \\ 31 & 4 & 3 \\ 42 & 4 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 6 & 4 & 2 \end{vmatrix}} = \frac{-152}{-38} = 4 \qquad y = \frac{\begin{vmatrix} 3 & 52 & 6 \\ 2 & 31 & 3 \\ 6 & 42 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 6 & 4 & 2 \end{vmatrix}} = \frac{-76}{-38} = 2 \qquad z = \frac{\begin{vmatrix} 3 & 5 & 52 \\ 2 & 4 & 31 \\ 6 & 4 & 42 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 6 & 4 & 2 \end{vmatrix}} = \frac{-190}{-38} = 5$$

DETERMINANTS						
2x2 Matrix	7	29	3x3 Matrix	3	5	6
-57	8	25	**-38**	2	4	3
DoubleClick&Edit				6	4	2

Exercise. Solve the equation system

5 DISPLACEMENT AND ROTATION WITH VECTORS AND MATRICES

Question. How can we use vectors and matrices in geometry?
Answer. To predict displacements and rotations in the plane and in the space.

Example 1. A displacement vector $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ displaces the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ going from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

to a vector going from $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

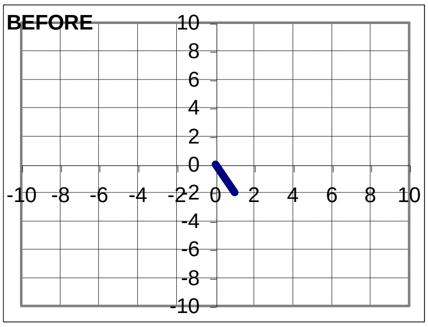
DISPLACEMENT

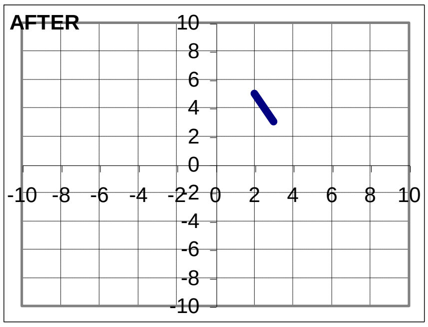
Δx	2
Δy	5

Enter Δx & Δy. Enter point2 before

		Point1	Point2
Before	x	***0***	***1***
	y	***0***	***-2***
After	x	***2***	***3***
	y	***5***	***3***

DoubleClick&Edit





Example 2. A rotation matrix $\begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}$ rotates a vector v degrees around its starting point.

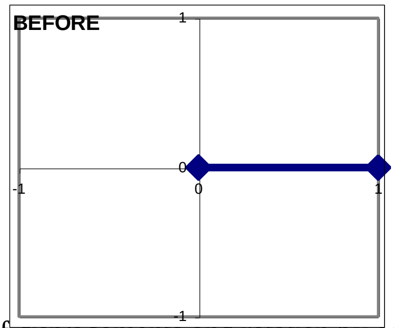
The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is rotated 63 degrees to the vector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 63 & -\sin 63 \\ \sin 63 & \cos 63 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 63 \\ \sin 63 \end{pmatrix} = \begin{pmatrix} 0.454 \\ 0.891 \end{pmatrix}$

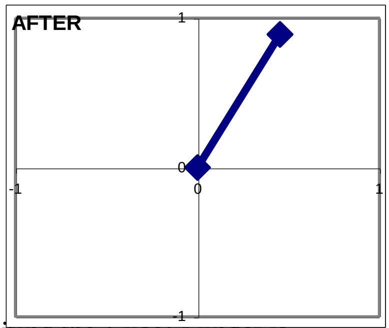
ROTATION *Enter Angle*

Angle	RotationMatrix	
63	**0,454**	**-0,891**
	0,891	**0,454**

DoubleClick&Edit

		Point1	Point2
Before	x	**0**	**1**
	y	**0**	**0**
After	x	**0**	**0,454**
	y	**0**	**0,891**





Exercise. Try other examples of displacements and rotation by editing the Excel-windows.

6 TRIANGLES WITH VECTORS AND MATRICES

Question. How can a triangle be predicted by vectors and matrices? **Answer**. By using the triangular vector formulas.
Example. A triangle has the vertices A(2,1), B(6,3) and C (4,6). Predict the sides, the angles and the area.
Method1. Frame&Cut. The triangle is framed in a rectangle, from which 3 right triangles are cut.
Method2. The triangle vectors are AB= (6-2, 3-1) = (4,2), BC= (4-6, 6-3) = (-2,3) and AC = (4-2, 6-1) = (2,5).
The side AB: IABI = √(AB*AB) = √(4,2)*(4,2) = √(4*4 + 2*2) = √20, IACI = √29 and IBCI = √13
The angle A: AB*AC=IABI*IACI*cosA, so cosA=AB*AC/IABI/IACI=(4*2+2*5)/√20/√29=0.747, A = 41.6
The area = ½* IÂB*ACI = ½*determinant(AB, AC) = ½*IABxACI = ½*I(-2,4)*(2,5)I = ½* I-4+20I = 8
Here ÂB is the AB's cross-vector, and x is the cross-product between the two vectors.
Exercise. A triangle has the vertices A(4,2), B(7,-1) and C (6,5). Predict the sides, the angles and the area.

7 PREDICTING CONIC SECTIONS

Question. How can we predict the shape of conic sections? **Answer**. By second degree equations.
Rotated around the y-axis a line with altitude angle v forms a vertical cone cutting out different conic sections from a plane depending on the angle of intersection u.
The shapes are predicted by the formula y^2 = 2*x – (1-e^2)*x^2, where e is called the eccentricity:
circle (u = 0, e = 0); ellipse (0<u<v, 0<e<1); parabola (u = v, e = 1); hyperbola (u = 90, e>1).

Parallel displacement. The coordinate system K is displaced x1 to the right and y1 up to K'. Thus the origin in K' has the coordinates (x,y) = (xo,yo) in K, and (x',y') = (0,0) in K'. Thus the coordinates are related by: x' = x-xo and y' = y-yo.

	Centre in (0,0)	Centre in (xo,yo)
Circle	A circle consists of the points P(x,y) having a constant distance r to a given centre C = P1(0,0): x^2 + y^2 = r^2 (= IPCI^2). The tangent passing through P(xo,yo) is predicted by x*xo + y*yo = r^2.	In K' a circle with centre in (x',y') = (0,0) and radius r is predicted by x'^2 + y'^2 = r^2. In K the circle is predicted by (x-x1)^2+ (y-y1)^2 = r^2.
Ellipse	An ellipse consists of the points P(x,y) having a constant distance-sum to two given centres B1 and B2 (the foci). An ellipse has a horizontal major axis b and a vertical minor axis c: (x/b)^2 + (y/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by x*xo/(b^2) + y*yo/(c^2) = 1.	In K' an ellipse with centre in (x',y') = (0,0) and major and minor axis b and c is predicted by (x'/b)^2 + (y'/c)^2 = 1. In K the ellipse is predicted by ((x-x1)/b)^2 + ((y-y1)/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by (x-x1)*xo/(b^2) + (y-y1)*yo/(c^2) = 1.
Hyperbola	A hyperbola consists of the points P(x,y) having a constant distance-difference to two given centres B1 and B2 (the foci). An hyperbola has a horizontal major axis b and a vertical minor axis c: (x/b)^2 - (y/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by x*xo/(b^2) - y*yo/(c^2) = 1. When a hyperbola is turned 45 degrees, its equation becomes y = k/x.	In K' an hyperbola with centre in (x',y') = (0,0) and major and minor axis b and c is predicted by (x'/b)^2 - (y'/c)^2 = 1. In K the hyperbola is predicted by ((x-x1)/b)^2 - ((y-y1)/c)^2 = 1. The tangent passing through P(xo,yo) is predicted by (x-x1)*xo/(b^2) - (y-y1)*yo/(c^2) = 1.
Parabola	A parabola consists of the points P(x,y) having the same distance to a given centre, the focus, and a given line, the directrix. The parameter p is twice the distance between the focus and the line. A vertical parabola is predicted by y = a*x^2 having the parameter p = 1/a. The tangent passing through P(xo,yo) is predicted by y+yo = 2*a*xo*x.	In K' an parabola with vertex in (x',y') = (0,0) and parameter p = 1/a is predicted by y'= a*x'^2. In K the ellipse is predicted by y-y1=a*(x-x1)^2. This can be transformed to: y = y+a*(x^2+x1^2 -2*x*x1) = a*x^2 + (-2*a*x1)*x + (y1+a*x1^2) = a*x^2 + b*x + c (#)

*(#) Since -2*a*xo= b, xo= -b/(2*a). Since yo+a*xo^2= c, yo= c-a*xo^2= c-a*b^2/(4*a^2)= -(b^2-4*a*c)/4*a = -D/4*a.*
In K the parabola vertex has the coordinates (xo,yo) = (-b/(2*a), -D/(4*a)), where the discriminant D = b^2-4*a*c.
Thus the intersection points between a parabola and the x-axis is xo±Δx, where a*Δx^2=D/(4*a) giving x = (-b±√D)/(2*a).
Example. What is the intersection points between x^2-14*x+y^2+6*y+33=0 and x-2*y-8=0?
x^2-14*x+y^2+6*y+33=0 gives x^2-2*7x+7^2+y^2+2*3y+3^2=-33+7^2+3^2 or (x-7)^2+(y+3)^2 = 25 = 5^2 i.e. a circle with centre (7,-3) and radius 5. x-2*y-8=0 gives x-8=2*y or ½*x-4=y i.e. a line with slope ½ and y-intercept –4.
Inserting x=2*y+8 is in x^2-14*x+y^2+6*y+33=0 gives (2y+8)^2-14(2y+8)+y^2+6y+33=0 or 5y^2+10y-15=0 having the solutions y=(-10±√(10^2-4*5*(-15))/(2*5) = 1 & -3 giving x = 10 & 2. Intersection points: (x,y) = (10,1) & (2,-3).
Exercise. Predict & construct the intersections points between a conic section and a conic section or a line.

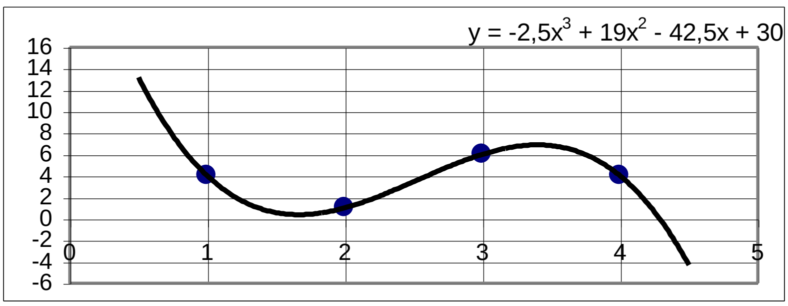
8 FITTNG CURVES

Question. How can we fit curves to points? **Answer**. By Excel trend lines or solving equations.
2 points. Through 2 points pass infinitely many 2nd degree polynomials (parabolas) but only 1 line.
3 points. Through 3 points pass infinitely many 3rd degree polynomials but only 1 2nd degree polynomial.
4 points. Through 4 points pass infinitely many 4th degree but only 1 3rd degree polynomial y = a*x^3+b*x^2 +c*x+d. d is the initial level, c is the initial slope, b is the initial curvature and a is the counter-curvature.
a, b, c and d can be predicted by solving 4 equations with med 4 unknown or by the Excel trend line.

4 POINTS

	x	y
Point1	1	4
Point2	2	1
Point3	3	6
Point4	4	4

DoubleClick&Edit



Exercise. Try other examples of curve fitting by editing the Excel-window.