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Introduction

This is a selection of papers and other inputs produced for the 10th International Congress on Mathematical Education, ICME 10, in Denmark in 2004; and for ICME 11 in Mexico as well for ICME 12 in Souh Korea in 2012. The contributions are numbered 1xx, 2xx and 3xx respectively.

101 contains the paper 'One Digit Mathematics' written together with Saulius Zybartas, and presented at the topic study group 1, new development and trends in mathematics education at pre-school and primary level. The paper The paper suggests that to solve the relevance paradox in mathematics education, LIB-mathematics as 2+3 = 5 valid only in the library and not in the laboratory should be replaced by LAB-mathematics as 2*3 = 6, also valid in the laboratory. And that replacing modern authorized routines with postmodern authentic routines turns elementary mathematics upside down by bringing the authority back to the multiplicity-laboratory where mathematics may be learned from one digit numbers only. At the end of the chapter a German version of the paper is included.

102 contains the paper 'Adding PerNumbers' presented at the topic study group 2, new development and trends in mathematics education at secondary level. The paper suggests that to solve the relevance paradox in mathematics education postmodern sceptical Cinderella research could be used to look for new ways to teach mathematics at the secondary school. The paper introduces addition of per-numbers as a more user-friendly approach to the traditional subjects of proportionality, linear and exponential functions and calculus.

103 contains the paper 'Bundling & Stacking in a Count & Add Laboratory' written together with Saulius Zybartas, and presented at the topic study group 8, research and development in the teaching and learning of number and arithmetic. The paper suggests that to solve the relevance paradox in mathematics education, LIB-mathematics as 2+3 = 5 valid only in the library and not in the laboratory should be replaced with LAB-mathematics as 2*3 = 6, also valid in the laboratory. And that a reconstruction respecting Kronecker's and Russell's scepticism shows that multiplicity-based LAB-mathematics is rather different from set-based LIB-mathematics by allowing fundamental mathematics as per-numbers, re-counting and re-stacking to be introduced at the first year of school. Then differences between modern LIB- and postmodern LAB-mathematics are discussed, and finally a testing of postmodern LAB-Mathematics in the classroom of primary school and teacher education is described.

104 contains the paper 'Deconstructing Modern Top-Down Algebra into Postmodern Bottom-Up Algebra' written, first for the 24th conference of the International Group for the Psychology of Mathematics Education, PME, in Hiroshima in Japan in 2000 where it was rejected, then for the topic study group 9, research and development in the teaching and learning of algebra. The paper says it is a postmodern point that a phrasing constructs what it describes and that ruling phrasings and discourses clientifies humans. Inspired by this the paper asks: is it possible to re-describe and deconstruct mathematics? 'Geometry' means 'earth measuring' - but what does "Algebra" mean? The dictionary tells us that 'Algebra' means 'reunite'. Since 'low attainers' might be deconstructed into 'authenticity searchers' we could also ask: what will happen if we present authenticity searchers for authenticity by inviting them to join the social practice of reuniting that created Algebra? This paper then tells about what happened in such classes. The paper was accepted for distribution.

105 contains the paper 'Per-Number Calculus, A Postmodern Sceptical Fairy tale Study' written for the topic study group 12, research and development in the teaching and learning of calculus. To solve the relevance paradox in mathematics education this paper uses postmodern sceptical fairy tale research to look for new ways to teach calculus in the school. A renaming of 'calculus' to 'adding per-numbers' allows us to think differently about the reality 'sleeping' behind our words,

and all of a sudden we see a different calculus taking place both in elementary school, middle school and high school. Being a 'Cinderella-difference' by making a difference when tested, this postmodern calculus offers to the classroom an alternative to the thorns of traditional calculus. The paper was accepted for distribution.

106 contains the paper 'Applying Mathe-Matics, Mathe-Matism or Meta-Matics' written for the topic study group 20, Mathematical applications and modelling in the teaching and learning of mathematics. The paper suggests that to solve the relevance paradox in mathematics education a sceptical look should be taken at one of the taboos of mathematics education, the mathematical terminology. Two kinds of words are found, LAB-words abstracted from laboratory examples; and LIB-words exemplified from library abstractions, transforming mathe-matics to meta-matics. A third kind of mathematics is mathema-tism only valid in the library and not in the laboratory, and blending with meta-matics to meta-matism. This distinction suggests that the relevance paradox of mathematics education occurs when teaching and applying metamatism, and disappears when teaching and applying mathematics. The paper was accepted for presentation.

107 contains the paper 'Pastoral Power in Mathematics Education, A Postmodern Sceptical Fairy tale Study' written for the topic study group 25, language and communication in mathematics education. The paper suggests that to solve the irrelevance paradox in mathematics education we should look for help at institutional scepticism as it appeared in the Enlightenment and was implemented in its two democracies, the American and the French, in the form of pragmatism and post-structuralism. Inspired by Foucault's notion of 'pastoral power' the paper looks at the use of words in mathematics education, distinguishing between 'lib-words' coming from the library and 'lab-words' coming from the laboratory. From this distinction a hypothesis can be made saying that the irrelevance paradox is created by lib-words installing pastoral power, and that lab-words will make the irrelevance paradox disappear. Consequently mathematics education should be based upon verb-based 'ing'-words such as counting and adding and calculating etc. The paper was accepted for presentation.

108 is a short paper called 'FunctionFree PerNuber Calculus' made for my contribution to the Nordic presentation at ICME10. Observing that calculus did not call itself 'calculus' it is suggested that calculus could also be called something else, thus the non-action word 'calculus' could be reworded to the action-word 'adding per-numbers - taking place from K - 12. Then examples are given on adding per-numbers in primary school, in middle school and in high school. Also people were invited to the MATHeCADEMY.net stand at the conference to discuss details.

109 is a poster called 'A Kronecker-Russell Multiplicity-Based Mathematics'presented at the ICME 10. The poster is a short outline of the curriculum at the MATHeCADEMY.net. It was part of most of the papers at the conference, but the poster allowed for having additional time to discuss details with different people.

110 contains a proposal for a paper ' Multiplicity-Based Mathematics found by Postmodern Sceptical Fairy tale Research' written for the thematic afternoon E, perspectives on research in mathematics education. The focus of the paper is to solve the irrelevance paradox of the research industry. The paper uses as its theoretical framework institutional scepticism, as it appeared in the Enlightenment and was implemented in its two democracies, the American in the form of pragmatism and symbolic interactionism, and the French in the form of post-structuralism and post-modernism. On this basis the paper describes a methodology called 'sceptical fairy tale research' as a postmodern counter-seduction research based upon a post-structuralist 'pencil-paradox'. By its LIB-LAB-destinction between words and numbers, sceptical fairy tale research is inspired by the ancient Greek sophists always distinguishing between choice and necessity, between political and natural correctness, between nomos and logos. By transforming seduction back into interpretation scepticism transforms the library from a hall of fact to a hall of fiction to draw inspiration from,

especially from the tales that have been validated by surviving through countless generations, the fairy tales. Hence the preferred interpretation genre in counter-seduction (and to a certain extent grounded theory) is the fairy tale. Once a fairy tale interpretation has identified the 'evil' word, scepticism begins to look for hidden alternatives either by discovering forgotten or unnoticed alternatives at different times and places inspired by the genealogy and archaeology of Foucault; or by inventing alternatives using sociological imagination. The aim of sceptical fairy tale research is not to extend the existing seduction of the library, so no systematic reference to the existing 'research' literature will take place. The aim is to solve problems by searching for hidden Cinderella-alternatives in the laboratory, i.e. by 1) finding the word suspected to be the villain, 2) renaming the evil word through discovery and imagination, 3) testing the hidden alternative in the laboratory to see if it is a Cinderella-difference making a difference, and 4) publish the alternative so it can become an option. The proposal was rejected.

201 contains a paper called 'Avoiding Ten, a Cognitive Bomb'. The number ten is the basis of our number system. The traditional curriculum sees no problem in introducing ten as the follower of nine. However, being the only number with its own name but without its own icon, the number ten becomes a cognitive bomb if introduced to quickly. First 1digit mathematics should be learned through bundling & stacking reported by cup- and decimal-writing. The paper was written for the Topic Study Group 1: New developments and trends in mathematics education at preschool level. The paper was presented as a full paper.

202. contains a paper called 'A Fresh Start Presenting Mathematics as a Number-predicting Language'. It describes the website www.MATHeCADEMY.net that contains a CATS-approach to mathematics, Count&Add in Time&Space, offering to learn mathematics as a natural science investigating the nature of many. Also the website contains 7 papers from the ICME10 Congress describing the approach in details. The paper was written for the Topic Study Group 4: New developments and trends in mathematics education at upper secondary level. The paper war rejected.

203 contains a paper called 'Decimal-Counting, Disarming the Cognitive Bomb Ten'. The number ten is the base of our number system. The modern curriculum sees introducing 10 as the follower of 9 as nature. However, being the only number with its own name but without its own icon, ten becomes a cognitive bomb if introduced to quickly. Anti-pastoral sophist research, searching for alternatives to choices presented as nature, shows that ten is not 10 by nature but by choice, and that jumping directly from 1.order to 3.order counting means missing the learning opportunities of 2.order decimal-counting by bundling and stacking. The paper was written for the Topic Study Group10: Research and development in the teaching and learning of number systems and arithmetic. The paper war rejected.

204 contains a paper called 'Pastoral Algebra Deconstructed'. Presenting its choices as nature makes modern algebra pastoral, suppressing its natural alternatives. Seeing algebra as pattern seeking violates the original Arabic meaning, reuniting. Insisting that fractions can be added and equations solved in only one way violates the natural way of adding fractions and solving equations. Anti-pastoral grounded research identifying alternatives to choices presented as nature uncovers the natural alternatives by bringing algebra back to its roots, describing the nature of rearranging multiplicity through bundling & stacking. The paper was written for the Topic Study Group11: Research and Development in the Teaching and Learning of Algebra. The paper war accepted.

205 contains a paper called 'Pastoral Calculus Deconstructed'. Calculus becomes pastoral calculus killing the interest of the student by presenting limit- and function- based calculus as a choice suppressing its natural alternatives. Anti-pastoral sophist research searching for alternatives to choice presented as nature uncovers the natural alternatives by bringing calculus back to its roots,

adding and splitting stacks and per-numbers. The paper was written for the Topic Study Group16: Research and development in the teaching and learning of calculus. The paper war accepted.

206 contains a paper called ‘Applying Pastoral Metamatism or Re-Applying Grounded Mathematics’. When an application-based mathematics curriculum supposed to improve learning fails to do so, two questions may be raised: What prevents it from improving learning? And is ‘mathematics applications’ what it says, or something else? Skepticism towards wordings leads to postmodern thinking that, dating back to the ancient Greek sophists, warns against patronizing pastoral categories, theories and institutions. Anti-pastoral sophist research, identifying hidden alternatives to pastoral choices presented as nature, uncovers two kinds of mathematics: a grounded mathematics enlightening the physical world, and a pastoral self-referring mathematics wanting to ‘save’ humans through ‘metamatism’, a mixture of ‘metamatics’ presenting concepts as examples of abstractions instead of as abstractions from examples; and ‘mathematism’ true in the library, but seldom in the laboratory. Also ‘applying’ could be reworded to ‘re-applying’ to emphasize the physical roots of mathematics. Three preventing factors are identified: ‘ten=10’-centrism claiming that counting can only take place using ten-bundles; fraction-centrism claiming that proportionality can only be seen as applying fractions; and set-centrism claiming that modelling can only take place by applying set-based concepts as functions, limits etc. In contrast, an implying factor is grounded mathematics created through modelling the natural fact many by counting many in bundles &stacks; and by predicting many by a recount-formula T = (T/b)*b that can be re-applied at all school levels. The paper was written for the Topic Study Group21: Mathematical applications and modelling in the teaching and learning of mathematics. The paper war accepted.

207 contains a paper called ‘Mathematics: Grounded Enlightenment - or Pastoral Salvation; a Natural Science for All - or a Humboldt Mystification for the Elite’. Mathematics is taught differently in Anglo-American democratic enlightenment schools wanting as many as possible to learn as much as possible; and in European pastoral Humboldt counter-enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Humboldt Bildung schools pastoral ‘metamatism’ is exemplified from metaphysical mystifying concepts from above. To make mathematics a human right, pastoral Humboldt counter-enlightenment mathematics must be replaced with democratic grounded enlightenment mathematics. The paper was written for the Topic Study Group24: Research on classroom practice. The paper war rejected.

208 contains a paper called ‘Pastoral Humboldt Mathematics Deconstructed’. Having existed since antique Greece, pastoral and anti-pastoral curricula today exist at the Humboldt Bildung schools inside EU and Enlightenment schools outside. However, Humboldt anti-enlightenment seems to have influenced all mathematics curricula. Following the advice of the Greek sophists, this paper separates choice from nature in the mathematics curriculum to design an alternative natural enlightenment curriculum grounded in the roots of mathematics. The paper includes an appendix called ‘A General Enlightenment Curriculum’. The paper was written for the Topic Study Group25: The role of mathematics in the overall curriculum. The paper war rejected.

209 contains a paper called ‘CATS, Count&Add in Time&Space - a Natural Way to Become a Mathematics Teacher’. The CATS-approach, Count&Add in Time&Space, is a natural way to become a math teacher. It obeys the fundamental rule of good research, never to ask leading questions. To learn mathematics, students should not be taught mathematics; instead they should meet the roots of mathematics, multiplicity. Through guiding educational questions asking them to Count and Add in Time and Space, they learn mathematics without knowing it. The CATS-approach is rich on examples of recognition and new cognition to be observed, reflected and

reported by teachers and researchers. The paper was written for the Topic Study Group TSG 27: Mathematical knowledge for teaching. The paper war accepted.

210 contains a paper called ‘Pastoral Words in mathematics education’. Mathematical terminology is very fixed, almost dogmatic, which seems to indicate a metaphysical nature. The necessity of language shows the great advantages by having a fixed terminology. However, there is a fundamental difference between enlightening words labeling and pastoral words hiding differences. Following the advice of the ancient Greek sophists warning against mixing up nature and choice, this paper asks ‘what is nature and what is choice in mathematical terminology?’ The paper was written for the Topic Study Group31: Language and communication in mathematics education. The paper war rejected.

211 contains a paper called ‘Deconstructing the Mathematics Curriculum: Telling Choice from Nature’. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, seen from a skeptical sophist perspective wanting to tell nature from choice, three questions are raised: Are concepts grounded in nature or forcing choices upon nature? How can an ungrounded mathematics curriculum be deconstructed into a grounded curriculum? Does mathematics education mean pastoral patronization of humans, or anti-pastoral enlightenment of nature? The paper was written for the Topic Study Group TSG 35: Research on mathematics curriculum development. The paper war accepted.

212 contains a paper called ‘Mathematics Education: Pastoral Bildung - Or Anti-Pastoral Enlightenment’. Applying a postmodern philosophical perspective to mathematics education reveals different kinds of mathematics and different kinds of education and different kinds of philosophy. Based upon the ancient Greek controversy between the sophists and philosophers as to the nature of knowledge, two different forms of schooling has developed, an enlightenment school abstracting categories from physical examples; and a pastoral school exemplifying metaphysical categories; as well as two different kinds of mathematics, enlightenment mathematics seeing the world as the roots of mathematics, and pastoral mathematics seeing the world as applying mathematics. The paper was written for the Discussion Group 5: The role of philosophy in mathematics education. The paper war accepted.

213 contains a paper called ‘Concealing Choices to Teachers’. Teaching or preaching - this dilemma goes back to the ancient Greek controversy between the sophists advocating enlightenment, and the philosophers advocating patronization. Also today two kinds of schools exist, the North American enlightenment schools educating the people, and EU Humboldt Bildung Counter-enlightenment schools educating the elite for offices. Should teachers be told if they are trained for enlightenment or patronization? Or are they just expected unreflectively to follow the orders of the institution paying their wages? The paper was written for the Discussion Group 7: Dilemmas and controversies in the education of mathematics teachers. The paper war accepted.

214 contains a workshop in 1digit Mathematics, Cup-writing & Decimal-counting called ‘Avoiding 10, a Cognitive Bomb’. The workshop was accepted.

215. contains a poster called ‘The 12 Blunders of Pastoral Mathematics’. The poster was accepted.

216 contains a talk called ‘Mathematics as an anti-Pastoral Natural Science’ given to each visitor at the MATHeCADEMY.net booth in the exhibition area.

301 contains a paper called ‘Come Back with 1digit Mathematics’. Postmodern contingency research uncovers hidden alternatives to choices presented as nature e.g. by replacing choice with nature. Within traditional mathematics, numbers, operations, formulas, equations etc. turn out to be

choices hiding their natural alternatives. Presenting mathematics from its roots, the natural fact Many, help many dropouts to master mathematics as a natural science. The paper was written for the Topic Study Group 4: Activities and programs for students with special needs. The paper was rejected.

302 contains a paper called ‘Recounting as the Root of Grounded Mathematics‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize? The paper was written for the Topic Study Group 7: Teaching and learning of number systems and arithmetic - focusing especially on primary education. The paper was presented as a poster.

303 contains a paper called ‘Calculus Grounded in Adding Per-numbers‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? What are the roots of calculus? The paper was written for the Topic Study Group 13: Teaching and learning of calculus. The paper was presented as a poster.

304 contains a paper called ‘Saving Dropout Ryan With A Ti-82’. To lower the dropout rate in pre-calculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren’t even able to use a TI-30. A compendium called ‘Formula Predict’ replaced the textbook. A formula’s left and right hand side were put on the y-list as Y1 and Y2 and equations were solved by ‘solve Y1-Y2 = 0’. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects. The paper was written for the Topic Study Group 18: Analysis of uses of technology in the teaching of mathematics. The paper was presented as a full paper.

305 contains a paper called ‘Contingency Research Uncovers the Roots of Grounded Mathematics‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize? The paper was written for the Topic Study Group 21: Research on classroom practice. The paper was presented as a poster.

306 contains a paper called ‚Mathematics as Manyology‘. Mathematics education claims to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical perspective wanting to tell nature from choice, two questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? The paper was written for the Topic Study Group 23: Mathematical knowledge for teaching at primary level. The paper was rejected.

307 contains a paper called ‚Counting and Adding Roots Grounded Mathematics‘. Mathematics education claims to deliver well-proven knowledge about well-defined concepts applicable to the outside world. However, skepticism would ask: Are the concepts grounded in nature or forcing choices upon nature? Can ungrounded mathematics from above be replaced by grounded mathematics from below generalized in a natural way in secondary school? The paper was written for the Topic Study Group 24: Mathematical knowledge for teaching at secondary level. The paper was presented as a poster.

308 contains a paper called ‚Fractions Grounded as Decimals, or 3/5 as 0.3 5s‘. The tradition sees fractions as difficult to teach and learn. Skepticism asks: Are fractions difficult by nature or by choice? Are there hidden ways to understand and teach fractions? Contingency research seaching for hidden alternatives to traditions looks at the roots of factions, bundling the unbundled, described in a natural way by decimals. But why is the unnatural presented as natural? The paper was written for the Topic Study Group 25: In-services education, professional development of mathematics teachers. The paper was presented as a full paper.

309 contains a paper called ‚Counting and Adding - a Natural Way to Teach Mathematics‘. The CATS-approach, Count&Add in Time&Space, obeys the rule of good research, never to ask leading questions. To learn mathematics, students should not be taught mathematics; instead they should meet the roots of mathematics, Many. Through guiding educational questions asking them to Count and Add in Time and Space, they learn mathematics by doing it. The CATS-approach is rich on examples of recognition and new cognition to be observed, reflected and reported by teachers and researchers. The paper was written for the Topic Study Group 26: Preservice mathematical education of teachers. The paper was presented as a poster.

310 contains a paper called Hidden Understandings of Mathematics Education‘ To answer the question ‘are there hidden understandings of mathematics education’ this paper tries to reinvent mathematics as a natural science grounded in its natural roots, the study of the natural fact Many. It turns out that two different mathematics exist, metamatism from above and grounded mathematics from below: Also two different kinds of education exist: Line-organized Bildung forcing students to learn the same, and block-organized enlightenment allowing students to develop personal talents. The paper was written for the Topic Study Group 32: Mathematics curriculum development. The paper was rejected.

311 contains a paper called ‚Social Theory in Mathematics Education‘. Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize? The paper was written for the Topic Study Group 37 Theoretical issues in mathematics education. The paper was presented as a poster.

312 cointains a workshop in Recounting and Decimal-writing. To deal with the natural fact Many, we totalize. However, there are hidden ways to count and add. This workshop demonstrates the power of recounting made possible by counting in icons before counting in tens. Recounting shows that natural numbers are decimal numbers carrying units. And recounting allows both proportionality and integration to be introduced in grade one. The workshop was accepted.

313 contains the posters presented at the ICME12. First the poster for the general poster session. Then posters for the Topic Study Groups 7, 13, 21, 24, 26 and 37

314 contains a contribution to the blog of discussion group 11 on Mathematics Teacher Retention called ‘Three Teacher Taboos in Mathematics Education’. It describes ‘Three Teacher Taboos’ facing a mathematics teacher: 1) As to mathematics: shall I preach self-referring metamatism, or mathematics grounded in the outside world? 2) As to education: shall I choose a line-organized talent impeding school, or a block-organized talent developing school? 3) As to research: shall I seek guidance in self-referring discourse protection from monastery-like universities, or in grounded contingency research from Internet academies?

315 contains a contribution to the blog of discussion group 12 on Mathematics Teacher Educators’ Knowledge for Teaching called ‘To Math or to Totalize, That is the Question’. It asks the question:

How to educate math users, math teachers, and math teacher educators? And proposes the answer: By learning, not how to math, since math is not an action word, but how to deal with the natural fact Many by totalizing; in short, by becoming competent in counting and adding in time and space. It the describes the MATHeCADEMY.net is free for users and for franchise takers. It offers Internet PYRAMIDeDUCATION to teachers and educators wanting to learn about mathematics as a natural science investigating the natural fact Many through the CATS approach, Count&Add in Time&Space, building upon five principles.

316 contains a contribution to the blog of Discussion Group 6 on Postmodern Mathematics called 'Theses 1-7' proposing an answer to the question "To be able to participate in this discussion group, could you give me a short introduction to your view on postmodern thinking in mathematics education?"

317 contains a contribution to the Discussion Group 6 on Postmodern Mathematics called 'Postmodern Skepticism toward Mathematics and Education and Research'. The cornerstones of modern society is research and education, especially in the two basic languages, the word-language assigning words and sentences to qualities, and the number-language assigning numbers and calculations to quantities. And as an important institution, mathematics education is equipped with its own research to make it successful. Still the problems in mathematics education seem to grow with the number of research articles. This irrelevance paradox makes postmodern skepticism ask: Are mathematics, education and research what they claim to be? Or are they choices that presented as nature install patronization to be unmasked by postmodern contingency research?

318 contains the manuscript to a YoyTube Video on Postmodern Math Education. In the video Paul Ernest and Allan Tarp discuss 8 questions: What is meant by postmodern? What is meant by modern? What is the root of postmodern thinking? Who is the most important postmodern thinker? What is mathematics? What is postmodern mathematics? What is postmodern research? Used as introduction in the Discussion Group 6 on Postmodern Mathematics.

319 contains the manuscript to a YoyTube Video Manuscript to a YoyTube Video on A Postmodern Deconstruction of World History. This YouTube video on postmodern deconstruction describes world history as the history of trade. First eastern lowland pepper and silk was traded with western highland silver, then eastern cotton was moved west and traded with northern industrial products, and finally electrons replaced the silver and cotton economy with an information economy. Published after the ICME12.

320 contains the manuscript to a YoyTube Video on Deconstruction of Fractions. This YouTube video on deconstruction in mathematics education connects fractions to its root, the leftovers when performing icon-counting. To deal with Many, we total by bundling in icon-numbers less than ten, or in tens needing no icon as the standard bundle. When bundling in 5s, 3 leftovers becomes 0.3 5s or 3/5 5s, thus leftovers root both decimal fractions and ordinary fractions. Presented in the topic study group 25: In-services education, professional development of mathematics teachers

321 contains the manuscript to a YoyTube Video on a Postmodern Deconstruction of of PreCalculus. This YouTube video on deconstruction in mathematics education connects preCalculus to its root, the natural fact Many. To deal with Many, we total. Some totals are constant, some change in space or time. The change might be predictable, or not. Pre-calculus describes predictable constant change. Calculus describes predictable changing change. Presented in the topic study group 18 : Analysis of uses of technology in the teaching of mathematics.

322 contains the manuscript to a YoyTube Video ona Postmodern Deconstruction of Mathematics Education. This YouTube video on deconstruction in mathematics education describes how natural mathematics is made difficult by removing eight links to its roots, the natural fact Many. The

missing links make mathematics a privilege to a mandarin class wanting to monopolize public offices. Reopening the eight missing links will make mathematics easy and accessible to all. Published after the ICME12.

323 contains pictures from The MATHeCADEMY.net booth in the exhibition area. To share information about free web-based teacher education on mathematics as a natural science about Many, the MATHeCADEMY.net had a booth.

324 is the folder handed out in the MATHeCADEMY.net booth in the exhibition area.

101. One Digit Mathematics

*To solve the relevance paradox in mathematics education this paper suggests that LIB-mathematics as 2+3 = 5, valid only in the library and not in the laboratory, should be replaced by LAB-mathematics as 2*3 = 6, also valid in the laboratory. Replacing modern authorized routines with postmodern authentic routines turns elementary mathematics upside down by bringing the authority back to the multiplicity-laboratory where mathematics may be learned from one digit numbers only.*

Lib-Mathematics And Lab-Mathematics

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998) and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371). From the first day at school mathematics teaches that 2+3 = 5 and 2*3 = 6. That 2*3 = 6 is easily validated in the laboratory by re-counting 2 3s as 6 1s: *** *** -> * * * * * *. 2+3 = 5 is true in the library, but not in the laboratory where countless counter-examples exist: 2 weeks+3 days = 17 days, 2m+3cm = 203 cm etc. This '2&3-paradox' makes it possible to distinguish between LIB-mathematics from the library and LAB-mathematics from the laboratory where only the latter can be validated in the laboratory. And we may assume that the relevance paradox disappears when LIB-mathematics is replaced with LAB-mathematics. This hypothesis can be tested with a proper sceptical method.

Scepticism Towards Words

Modern natural science has established research as a number-based 'LAB-LIB research' where the LIB-statements of the library are induced from and validated by reliable LAB-data from the laboratory as illustrated by e.g. Brahe, Kepler and Newton, where Brahe by studying the motion of the planets provided LAB-data, from which Kepler induced LIB-equations that later were deduced from Newton's LIB-theory about gravity. So research is based upon numbers, not upon words.

The scepticism towards words is validated by a simple 'number&word-observation': Placed between a ruler and a dictionary a thing can point to a number but not to a word, so a thing can falsify a number-statement in the laboratory but not a word-statement in the library; thus numbers carry research, while words carry interpretations, which presented as research become seduction - to be met by sceptical counter-research replacing LIB-authority with LAB-authenticity (Tarp 2003).

From Authorized Routines To Authentic Routines

Postmodernism means scepticism, especially towards authorized routines creating problems to modern society (Bauman 1989: 21). To replace authorized routines with authentic routines the authority must be moved from the library back to the laboratory to allow 12 educational meetings with the root of mathematics, multiplicity, teaching us through educational questions and activities.

1 Repetition Becomes Multiplicity

The first educational question is: How can we represent temporal repetition in space? One answer is iconisation: Put a finger to the throat and add a match or a stroke for each beat of the heart:

. . . . -> I I I I I I I I

2 Multiplicity Becomes Bundles

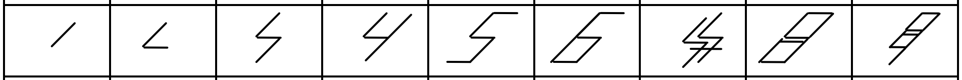
The next educational question is: How can we organise multiplicity? One answer is bundling: line up the total and divide it into bundles:

I I I I I I I I I -> II II II II I or I I I I I I I I I -> IIIII IIII or I I I I I I I I I -> III III III or …

3 Bundles Become Icons

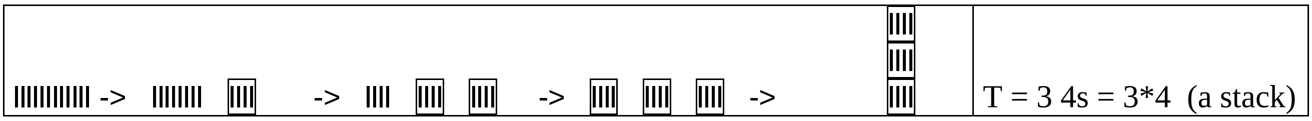
The next educational question is: How can we represent the different degrees of multiplicity? Again iconisation is an answer: the strokes of the different degrees of multiplicity are rearranged as icons, realising that there would be four strokes in the number-icon 4 etc., if written in a less sloppy way.

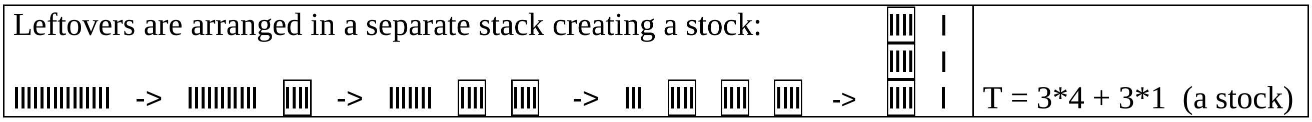
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
1	2	3	4	5	6	7	8	9

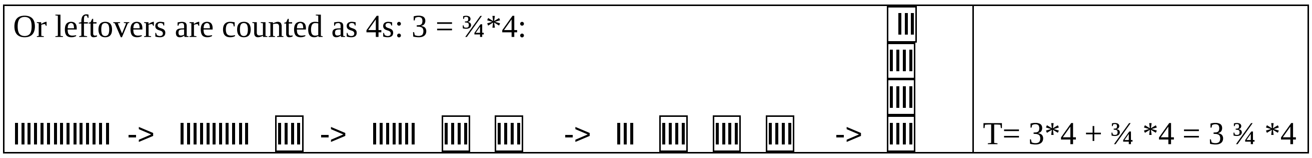


4 Multiplicity Is Counted As A Stack

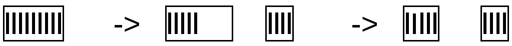
The next educational question is: How can we account for the different degrees of multiplicity? One answer is counting by bundling and stacking: First the total is lined up, then it is bundled and equal bundles are stacked and finally the height is counted as e.g. T = 3 4s = 3*4.







We count in 4s by taking away 4s. The process ‘from T take away 4’ may be iconized as ‘T-4’ and worded as ‘T minus 4’. The 4 taken away does not disappear, they are just put aside so the original total T is divided into two totals, one containing T-4 and the other containing 4:

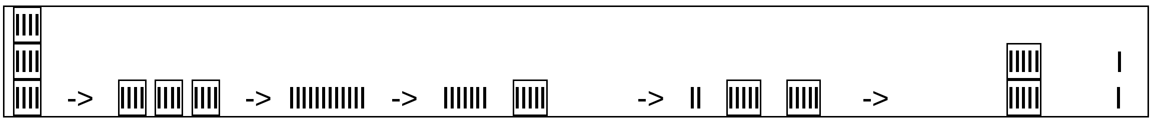


9 = (9-4) + 4 = 5 + 4 as predicted by the ‘restack-equation’ T = (T-b)+b.

The repeated process ‘from T take away 4s’ may be iconized as ‘T/4’ and worded as ‘T counted in 4s’. So the ‘recount-equation’ T = (T/4)*4 predicts the result of recounting the total T in 4-bundles: T =(T/4)*4 =3*4+3*1 =3 ¾*4. T/4 is called a per-number, T a stack-number or a total, and 4 a unit.

5 Stacks Are Recounted

The next educational question is: How can we change the bundle-size in a stack (T = 3 4s = ? 5s). One answer is de-stacking, de-bundling, re-bundling and re-stacking: First the stack is de-stacked into separate bundles, then the bundles are de-bundled into a total, then the total is bundled and equal bundles are stacked and finally the heights are counted.



3*4 = 4 +4 +4 = T = (T-5) +5 = (T/5)*5 = 2*5 + 2*1

Again the result can be predicted by the recount-equation T = (T/5)*5 = (3*4/5)*5 = 2*5 + 2*1 and displayed on a calculator able to do integer division as e.g. the Texas Instruments’ Math Explorer.

6 Stacks Are Symbolised

The next educational question is: How can we symbolise a stack? One answer is to use symbols as C and S to symbolise different bundle sizes, e.g. C = 4 and S = 5. Then the recount-question ‘T = 3*4 = ?*5’ is reformulated to ‘T = 3*C = ?*S ‘.

C												
C											S	I
C	->	C C C	->	IIIIIIIIIIII	->	IIIIIII	S	->	II S S	->	S	I
3*C	=	C+C+C	=	T	=	(T-S)+	S	=	(T/S)*S	=	2*S + 2*1	

Also the recounting can be done as in trade by skipping the 1s: Thus if 1*C = 2*S we have

CCC	-> SS SS SS	-> SSSSSS
T = 3*C	= 3*2*S	= 6*S

SSSSSS	-> SS SS SS	-> CCC	
T = 6*S	= 3*2*S	= 3*C	predicted by 6*S = ((6/2)*2)*S = 3*2*S = 3*C

SSSSSSS	-> SS SS SS S	-> CCC S	
T = 7*S	= 3*2*S + S	= 3*C + 1*S	predicted by 7*S = ((7/2)*2)*S = 3*2*S+1*S = 3*C+1*S

With units a stack can by counted in Cs to practise tables. Thus C C C C can be counted as ‘2 4 6 8’ or ‘ 3 6 9 twelve’. Here ‘twelve’ regains its original meaning ‘two left’ making it possible to count from ten to ‘two-ten’ as: ten and: one left, two left, three left, …, nine left, two ten etc.

7 Stocks Are Symbolised

The next educational question is: How can we symbolise stocks? One answer is to use short writing by leaving out the stack symbols: T = 2*C+3*1 -> 2)3) = 23. Here we need a symbol ‘0’ for ‘none’ to distinguish T = 2*C = 20 from T = 2*1 = 2. Thus stocks slowly become many-digit numbers.

With this short writing we can still practise recounting stocks by reformulating the recount-question ‘T = 3*C + 1 = ?*S’ to ‘T = 31 = ?*S’ , e.g. with C = 4 and S = 5:

													T	= 3*C + 1
C												I		= 31
C											S	I	T	= 2*S + 3*1
C I	->	C C C I	->	IIIIIIIIIIIII	->	IIIIIIII	S	->	III S S	->	S	I		= 23
3 1	=			T	=				(T/S)*S	=	2	3		

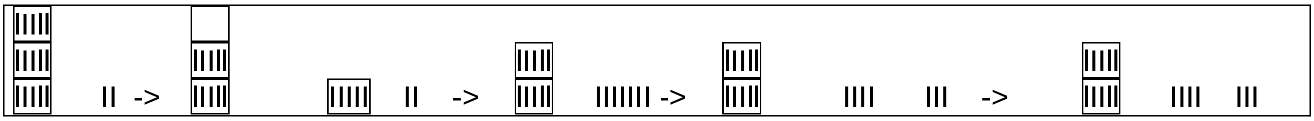
When the icons stop we can use symbols. Thus the Romans used the symbol X for the number ten.

The recount-question ‘T = 3*8 = ?*X’ recounts 3 8s as 2 tens and 4 1s: T = 3*8 = 2*X+4*1 = 24. This leads to traditional multiplication. However 3*8 is 3 8s; only if recounted in tens it is 24.

The recount-question ‘T = 3*X = ?*8’ recounts 3 tens as 3 8s and 6 1s: T= 3*X= 30/8*8= 3*8+6*1. This connects traditional division with recounting, and not with sharing. Thus sharing is applying and not creating division.

8 Stacks Are Sold

The next educational question is: How can we sell from an empty stack? From the stock T = 3 5s + 2 1s we want to sell 3 1s, but we only have 2 1s in stock. The answer is to restack 1 5s to 5 1s:



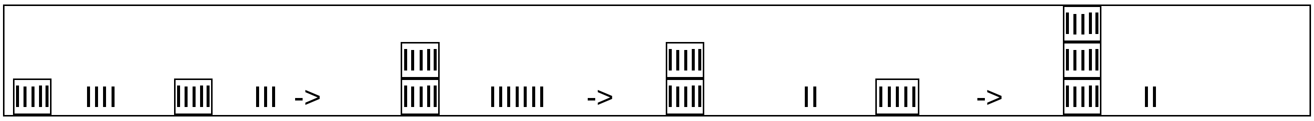
3*5 + 2 = (3*5-1*5) + 1*5 + 2 = 2*5 + 7 = 2*5 + (7-3) + 3 = 2*5 + 4 + 3

Working with symbols we use ‘internal trade’: T = 32 -3 = 3)2) -3 = 3-1)5+2) -3 = 27 -3 = 24.

In the case of tens we have T = 32 -3 = 3)2) -3 = 3-1)10+2) -3 = 2)12) -3 = 29

9 Stacks Are Bought

The next educational question is: How can different stocks be added? To the stock T = 1 5s + 4 1s we add the stock T' = 1 5s + 3 1s. Adding the 1s we are able to recount 7 1s to 1 5s + 2 1s as predicted by the restack-equation: T = 7 = (7-5) + 5 = 2 + 1*5



1*5+ 4 + 1*5 + 3 = 2*5 + 7 = 2*5 + (7-5) + 5 = 3*5 + 2

Working with symbols we have T = 14 + 13 = 1)4) + 1)3) = 2)7) = 2+1)-5+7) = 3)2) = 32.

In the case of tens we have T = 14 + 17 = 1)4) + 1)7) = 2)11) = 2+1)-10+11) = 3)1) = 31

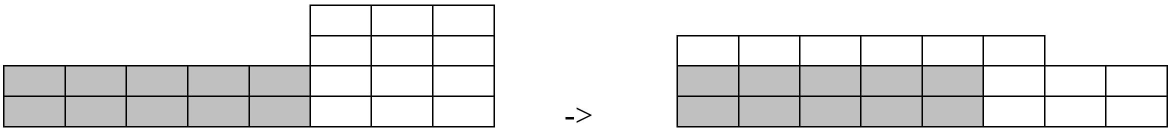
10 One Digit Calculus

The next educational question is: How can stocks be added differently? The stacks 2 5s and 4 3s can be 'added in time' as 3s or as 5s, or 'added in space' as 8s, which is called integration or calculus.

Added as 3s: T = 2 5s + 4 3s = 2*5 + 4*3 = (2*5)/3*3 + 4*3 = 3 1/3 * 3 + 4*3 = 7 1/3 * 3

Added as 5s: T = 2 5s + 4 3s = 2*5 + 4*3 = 2*5 + (4*3/5)*5 = 2*5 + 2 2/5 * 5 = 4 2/5 * 5

Added as 8s: T = 2 5s + 4 3s = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8

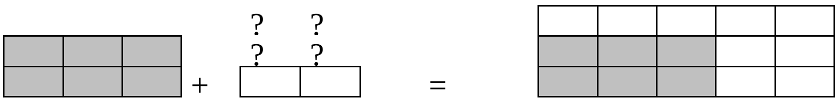


T = 2*5 + 4*3 = 2 6/8 *8 = (2*5+4*3)/8*8

Thus integration adds the per-numbers 2 and 4 as heights in stacks: 2 + 4 = 2 6/8. So 2 + 4 can give many different results, unless the units are the same:

T= 2*3 + 4*3 = 6*3 if added in time; and T= 2*3 + 4*3 = (2*3 + 4*3)/6*6 = 3*6 if added in space.

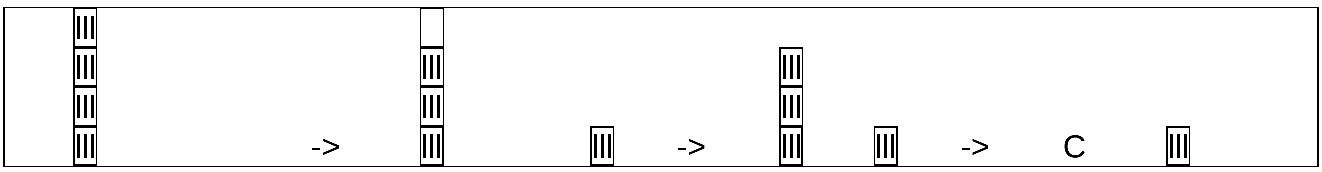
The addition process can be reversed by asking 2 3s + ? 2s = 3 5s:



The answer can be obtained by removing the 2 3s from the 3 5s and then recounting the remaining 9 in 2s as (9/2)*2 = 4 1/2 * 2. Thus ? = 4 ½. This process is later called differentiation.

11 Stacks Are Overloaded

The next educational question is: What do we do about overloads? If a stack is higher than its unit, the stack is overloaded. The overload then can be restacked to a new stack leaving a full stack C.



T = 4*3 = overload = (4*3 – 1*3) + 1*3 = 3*3 + 1*3 = 1*C + 1*3

12 One Digit Equations

The last educational question is: How can we reverse addition? One answer is reversed calculations, also called solving equations. The recount- and the restack-equation show that equations are solved when moving a number to the other side of the equation sign reversing its calculation sign:

Recounting:	T	= (T/4) * 4	Restacking:	T	= (T-4) + 4
Equation Solution	T T/4	= x * 4 = x	Equation Solution	T T-4	= x + 4 = x

The equation 2*x+1 = 9 is solved by combining restacking and recounting, not by neutralising.

2*x+1	= 9 = (9-1) +1	= 8 + 1	restacking 9 into 8 + 1	or by moving:	2*x + 1	= 9
2*x	= 8 = (8/2)*2	= 4*2	recounting 8 into 4*2		2*x	= 9 - 1 = 8
x	= 4		the solution		x	= 8/2 = 4

Educated By Multiplicity

Through these 12 educational meetings with multiplicity we are introduced to a radical different mathematics, a postmodern LAB-mathematics turning the modern LIB-mathematics upside down:

- To LIB-mathematics number-symbols are cultural artefacts to be learned by heart. To LAB-mathematics number-symbols are natural icons representing the number of strokes they contain.

- LIB-mathematics has many number-types: Whole numbers, integers, rational and irrational numbers etc. LAB-mathematics only has two number-types, stack-numbers and per-numbers.

- LIB-mathematics has addition as its fundamental operation to be followed by subtraction and multiplication, and finally the hard part, division. LAB-mathematics has division as its fundamental operation to be followed by multiplication, subtraction, and finally the hard part, addition.

- LIB-mathematics introduces many-digit numbers before mathematics. LAB-mathematics introduces mathematics before many-digit numbers.

- LIB-mathematics introduces many-digit numbers as the abstract idea of a positions system. LAB-mathematics introduces many-digit numbers as a physical fact, a stock i.e. a many-stack.

- LIB-mathematics introduces addition of many-digit numbers as the abstract idea of carrying. LAB-mathematics introduces addition of many-digit numbers as an internal trade inside a stock.

- LIB-mathematics postpones calculus to tertiary education. LAB-mathematics includes calculus in both primary and secondary education.

- LIB-mathematics postpones algebra and equations to secondary education. LAB-mathematics includes algebra and equations in primary education.

Summing up, LIB-mathematics is defined as examples of abstractions, derived from the concept set, part of a metaphysical axiom system; thus becoming a purely abstract mental activity, that cannot always by validated in the laboratory, as shown by the 2&3-paradox. LAB-mathematics is defined as abstractions from examples, derived from multiplicity, part of a physical reality; thus becoming a natural science induced from and validated in a multiplicity-laboratory.

Tested in a pre-calculus classroom LAB-mathematics solved the relevance paradox (Tarp 2003).

Tested in the classroom of teacher education student teachers overwhelmingly voted for including postmodern LAB-mathematics in teacher education (Zybartas et al 2001). This positive response has lead to the development of teaching and learning material in postmodern ‘LAB-mathematics from below’ for free teacher PYRAMIDeDUCATION at www.MATHeCADEMY.net (Tarp 2003).

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Jensen J H, Niss M & Wedege T (1998) *Justification and Enrolment Problems in Education Involving Mathematics or Physics*, Roskilde: Roskilde University Press.

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Einstellige Mathematik

*Der Mathematikunterricht hat eine Relevanz Problem. Die Lösung ist eine Substitution von BIB-Mathematik wie 2+3=5 nur gültig in der Bibliothek, aber nicht in dem Laboratorium, mit LAB-Mathematik wie 2*3=6 auch gültig in dem Laboratorium. Eine Substitution moderne autorisierte Routinen mit postmodernen authentische Routinen macht die elementar Mathematik kopf-stehen und gebt die Autorität zurück von die Menge-Bibliotek zu das Vielheit-Laboratorium, wo man Mathematik von einstelligen Zahlen lernen kann.*

Bibliotheks-Mathematik und Laboratorium-Mathematik

Der Hintergrund dieser Artikel ist das erdumfassende Rekrutierungsproblem in der Mathematik basierte Ausbildungen (Jensen et al 1998) und ‚der Relevanzparadox hervorruft von der simultane objektive Relevanz und subjektive Irrelevanz der Mathematik' (Niss in Biehler et al, 1994: 371).

Von der erste Schultag erzählt die Schule das 2+3 = 5 und 2*3 = 6. Das 2*3 = 6 wahr ist kann man einfach in das Laboratorium berichtigen, denn 2 3s kann immer als 6 1s umgezahlt werden: *** *** -> * * * * * *. 2+3 = 5 ist wahr in der Bibliothek, in das Laboratorium gibt es aber viele gegen Beispiele: 2 Wochen+3 Tagen = 17 Tagen, 2m+3cm = 203 cm usw. Mit dieser ‚2&3-paradox' kann man zwischen BIB-Mathematik und LAB-Mathematik unterscheiden, wo man nur das letzte in dem Laboratorium berichtigen kann. Und wir können die Hypothese machen das das Relevanzparadox verschwinden wird wenn wir BIB-Mathematik mit LAB-Mathematik umtaucht. Dieser Hypothese kann geprobt werden mit einer skeptische Methode.

Skeptizismus gegen Wörter

Die moderne Naturwissenschaft hat Forschung als zahl-basiert 'BIB-LAB Forschung etabliert, wo de BIB-Aussagen der Bibliothek von glaubwürdige daten von der Laboratorium induziert und berichtigt werden. In der Fall von Brahe, Kepler und Newton, Brahe hat die LAB-data geschafft bei die Bewegung der Sternen studieren. Von dieser daten hat Kepler ein BIB-gleichung induziert, die später von Newtons Theorie über die universelle Gravitation deduziert wurde. Forschung ist auf zahlen basiert, nicht auf Wörter. Dieser Skeptizismus gegen Wörter ist bei einer einfache ‚Zahl&Wort-Paradox' berichtigt: Zwischen ein Lineal und ein Wortbuch kann ein Ding zu Zahlen zeigen aber nicht zu Wörter. D.h. ein Ding kann falsche Aussagen falschmachen in der Laboratorium, aber ein Ding kann nicht ein falsche aussage falschmachen in der Bibliothek. In dieser Weg kann zahlen Forschung tragen, wo Wörter nur Interpretation tragen können, das präsentiert als Forschung Verführung bekommt – wozu skeptisch Gegen-Forschung ist nötig, das BIB-Autorität mit LAB-Authentizität ersetzen kann. (Tarp 2003).

Auch drei Fransoziesche Philosophen haben gegen die Problemen der Wörter gewarnt. Mit das Wort ‚logocentrismus' warnt Derrida gegen repräsentierende Wörter, die statt installieren was sie beschreiben. Mit dem Wort ‚postmodernismus' warnt Lyotard gegen wissenschaftliche Setze, die statt Ideologie installieren. Mit dem Wort ‚Pastorale Macht' warnt Foucault gegen Hilfinstitutionen, die statt Menschen zum Klienten machen.

Von autorisierten Routinen zum authentischen Routinen

Postmodernismus bedeutet Skeptizismus, in besonders gegen autorisierte Routinen die Problemen für das modern Gesellschaft machen (Bauman 1989: 21). Soll autorisierte Routinen mit authentische Routinen umtauscht werden, muss die autorität von der Bibliothek zu dem Laboratorium zurück gebracht werden. In dieser Weise kann man 12 Bildungsbegegnungen mit der Wurzel der Mathematik, die Vielheit, arrangieren, wo wir durch Bildungsfragen und Aktivitäten gebildet werden.

1 Repetition wird Vielheit

Die erste Bildungsfrage ist: Wie kann man seitliche Repetition in Raum repräsentieren? Eine Antwort ist: Mach die Fingern am Hals and mach ein Strich für jeden Gehschlag des Herzen.

. . . . -> I I I I I I I I

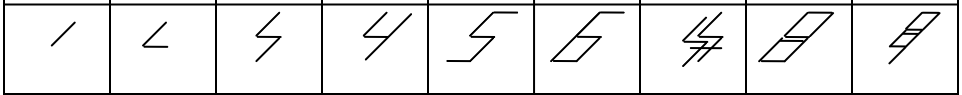
2 Vielheit werden Bündel

Die nächste Bildungsfrage ist: Wie kann man Vielheit organisieren? Eine Antwort ist bei bündeln: Man leint der totalen auf und teilt es in Bündel:

I I I I I I I I I -> II II II II I oder I I I I I I I I I -> IIIII IIII oder I I I I I I I I I -> III III III oder …

3 Bündel werden Ikonen

Die nächste Bildungsfrage ist: wie kann man verschiedene Formen von Vielheit repräsentieren? Wieder Ikonen sind eine Antwort: Die streiche die verscheiden Formen von Vielheit ist weiderarrangiert als Ikonen, und wir sehen das in der 4-ikon gibt es 4 Streiche usw., wenn geschrieben in einer nicht so schlechte weg.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

4 Vielheit ist wie ein Schober gezahlt

Die nächste Bildungsfrage ist: wie kann man verschiedene Formen von Vielheit geschreieben? Eine Antwort ist in Bunde und Schober zählen. Zum erst ist die total im Linie gemacht, dann in Bunde arrangiert, die in Schober schobern werden. Zuletzt ist die Höhe der Schober gezahlt als z.B. T = 3 4e = 3*4

IIIIIIIIIIII -> IIIIIIII [IIII] -> IIII [IIII] [IIII] -> [IIII] [IIII] [IIII] ->	[IIII] [IIII] [IIII]	T = 3 4e = 3*4 (Schober)

Zurückgebliebene sind in seine eigene Schober arrangiert (Lager): IIIIIIIIIIIIIII -> IIIIIIIIIII [IIII] -> IIIIIII [IIII] [IIII] -> III [IIII] [IIII] [IIII] ->	[IIII] I [IIII] I [IIII] I	T = 3*4 + 3*1 (Lager)

Oder sie sind als 4e gezahlt: 3 = ¾*4: IIIIIIIIIIIIIII -> IIIIIIIIIII [IIII] -> IIIIIII [IIII] [IIII] -> III [IIII] [IIII] [IIII] ->	[III] [IIII] [IIII] [IIII]	T= 3*4 + ¾ *4 = 3 ¾ *4

Wir zählen in 4e bei 4e wegtragen. Die Prozess ‚von T 4 wegtragen' kann man als ‚T-4' ikonmachen, und als ‚T minus 4' beschreiben. Die 4 weggenommene verschwinden nicht, sie sind nur zu Seite gelegen. In dieser weg ist der originale Total T in zwei teile geteilt, der eine erhalt T-4, und der andere erhalt 4, wie vorgesagt bei der ‚Umschobern-gleichung' T = (T-b)+b

[IIIIIIIII] -> [IIIII] [IIII] -> [IIIII] [IIII]
9 = (9-4) + 4 = 5 + 4

Die weiderholte Prozess 'von T 4e wegtragen' kann als 'T/4' ikonmachen und als 'T zählt in 4e' ernennen. In dieser weg kann die ‚Umzählungs-gleichung' T = (T/4)*4 vorsagen das Resultat von zählen T in 4-Bünde: T =(T/4)*4 =3*4+3*1 =3 ¾*4. T/4 ist eine per-zahl genant, T ist eine Schober-zahl oder eine total genant, und 4 ist ein Bündel oder eine Einheit genant.

5 Schober sind umgezählt

Die nächste Bildungsfrage ist: wie kann man die Bündel-größe einem Schober verändern (T = 3 4e = ? 5e)? Eine Antwort ist bei abschobern, abbündeln, wiederbündeln und wiederschobern: Erst ist der Schober abgeschobert in viel Bündel, dann sind die Bündel abgebündelt zu eine Total. Nächst ist die Total gezählt in die neue Bündel-größe , und zuletzt sind die Bündel wider geschobert.

IIII
IIII   IIIII I
IIII -> IIII IIII IIII -> IIIIIIIIIIII -> IIIIIII IIIII -> II IIIII IIIII -> IIIII I

3*4 = 4 +4 +4 = T = (T-5) +5 = (T/5)*5 = 2*5 + 2*1

Das Resultat kann wieder bei der Umzählungsgleichung voraussagt werden: T = (T/5)*5 = (3*4/5)*5 = 2*5 + 2*1. Der Texas Instruments' Math Explorer kann das Resultat zeigen wenn man Division mit Ganze zahlen wählt.

6 Schober sind symbolisiert

Die nächste Bildungsfrage ist: wie kann man eine Schober symbolisieren? Eine Antwort ist das man C's und S's braucht um die verschiedene Bündel-größe zu symbolisieren, z.B. C = 4 und S = 5. In dieser fall kann die Umzählungsgleichung T = 3*4 = ?*5' zum 'T = 3*C = ?*S ' umformuliert werden.

C
C   S I
C -> C C C -> IIIIIIIIIIII -> IIIIIII S -> II S S -> S I

3*C = C+C+C = T = (T-S)+ S = (T/S)*S = 2*S + 2*1

Man kann auch direkt von C bis S gehen und die 1e zu zählen. Wen z.B. 1*C = 2*S geltet:

CCC	-> SS SS SS	-> SSSSSS	
T = 3*C	= 3*2*S	= 6*S	

SSSSSS	-> SS SS SS	-> CCC	
T = 6*S	= 3*2*S	= 3*C	voraussagt bei 6*S = ((6/2)*2)*S= 3*2*S= 3*C

SSSSSSS	-> SS SS SS S	-> CCC S	
T = 7*S	= 3*2*S + S	= 3*C + 1*S	voraussagt bei 7*S = ((7/2)*2)*S = 3*2*S+1*S = 3*C+1*S

Ein Schober kann in Cs gezählt werden um tafeln zu üben: C C C C kann als '2 4 6 8' gezahlt worden, oder als ' 3 6 9 zwölf'. Hier zwölf kriegt seine Originalbedeutung: ‚zwo liefert' so man von zehn zum zwei-zehn zählen kann als: zehn & ein liefert, zwei liefert, drei liefert, ..., neun liefert, zwei zehn.

7 Lager sind symbolisiert

Die nächste Bildungsfrage ist: wie kann man ein Lager symbolisieren? Eine Antwort ist das man

‚Kurzschreiben' oder ‚Tasseschreiben' brauchen kann so das man die Schobersymbolen auslasst:

T = 2*C+3*1 -> 2)3) = 23. Hier brauchen wir die Symbol ‚0' für keine um nicht T = 2*C = 20 und T = 2*1 = 2 zu verwechseln. In dieser weg werden Lager langsam mehrstellige zahlen.

Mit Kurzschreiben oder Tasseschreiben können wir immer Lager zählen mit der Umzählungs-gleichung: 'T = 3*C + 1 = ?*S' bleibt 'T = 31 = ?*S' , z.B. mit C = 4 und S = 5:

C -> C C C I -> IIIIIIIIIIIII -> IIIIIIII S -> III S S ->	I	T	= 3*C + 1

C C I	S I S I	T	= 31 = 2*S + 3*1 = 23

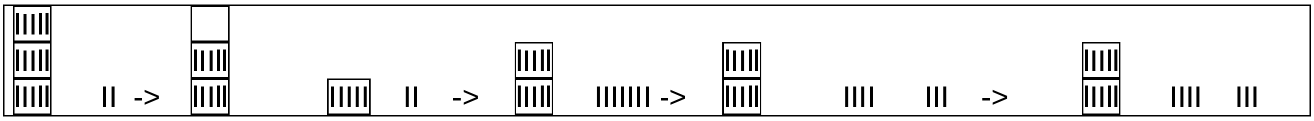
3 1 = T = (T/S)*S = 2 3

Wenn die Ikonen stoppen, kann man Symbolen benützen. Die Romanen brauchten das Symbol X für das zahl zehn. Die Umzählungs-gleichung' T = 3*8 = ?*X' zählt 3 8e als 2 zehn und 4 eins: T = 3*8 = 2*X+4*1 = 24. In dieser weg ist die traditionelle Multiplikation eingeführt. Aber 3*8 ist 3 8e, nur wenn man als zehne zahlt bekommt es 24.

Die Umzählungs-gleichung' 'T = 3*X = ?*8' zählt 3 zehne als 3 8e und 6 1e: T= 3*X= 30/8*8= 3*8+6*1. Dieser verbindet traditionelle Division mit zählen, nicht mit teilen. In dieser weg teilen ist eine Anwendung von Division, nicht eine Erschaffer von Division.

8 Schober sind verkauft

Die nächste Bildungsfrage ist: wie kann man von eine leere Schober etwas verkaufen? Von dem Lager T = 3 5e + 2 1e will man 3 1e verkaufen, aber man hat nur 2 1e im Lager. Eine Antwort ist das man 1 5e als 5 1s zählt.



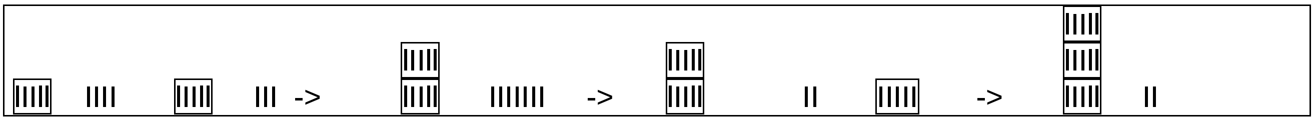
3*5 + 2 = (3*5-1*5) + 1*5 + 2 = 2*5 + 7 = 2*5 + (7-3) + 3 = 2*5 + 4 + 3

Mit Symbolen kann man 'inner-Handel' machen: T = 32 -3 = 3)2) -3 = 3-1)5+2) -3 = 27 -3 = 24.

In dem Fall Zehns haben wir T = 32 -3 = 3)2) -3 = 3-1)10+2) -3 = 2)12) -3 = 29

9 Schober sind kauft

Die nächste Bildungsfrage ist: wie kann man mehre Lager sammeln? Wir sollen die zwei Lager T = 1 5s + 4 1s und T' = 1 5s + 3 1s sammeln. Nach wir die 1s gesammelt hat können wir die 7 1e als 1 5e und 2 1e zählen als voraussagt bei der Umschobern-gleichung': T = 7 = (7-5) + 5 = 2 + 1*5



1*5+ 4 + 1*5 + 3 = 2*5 + 7 = 2*5 + (7-5) + 5 = 3*5 + 2

Mit Symbolen haben wir T = 14 + 13 = 1)4) + 1)3) = 2)7) = 2+1)-5+7) = 3)2) = 32.

In dem Fall Zehns haben wir T = 14 + 17 = 1)4) + 1)7) = 2)11) = 2+1)-10+11) = 3)1) = 31

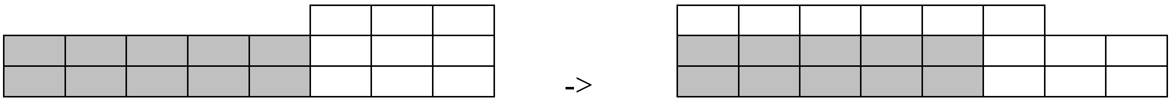
10 Einstellige Differential- und Integralrechnung

Die nächste Bildungsfrage ist: kann man Lager sammeln in verschiedene Weise? Die Lager 2 5e and 4 3e können in zeit gesammelt werden als 3e oder als 5e. Oder man kann sie in Raum als 8e sammeln. Das nennt man dann Differential- und Integralrechnung.

Sammelt vie 3e: T = 2 5e + 4 3e = 2*5 + 4*3 = (2*5)/3*3 + 4*3 = 3 1/3 * 3 + 4*3 = 7 1/3 * 3

Sammelt vie 5e: T = 2 5e + 4 3e = 2*5 + 4*3 = 2*5 + (4*3/5)*5 = 2*5 + 2 2/5 * 5 = 4 2/5 * 5

Sammelt vie 8e: T = 2 5e + 4 3e = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8



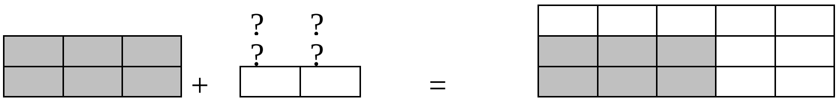
T = 2*5 + 4*3 = 2 6/8 *8 = (2*5+4*3)/8*8

In dieser weg kann man mit Integration die Perzahlen 2 und 4 sammeln als die Höhen der Schober: 2 + 4 = 2 6/8

2 + 4 kann viel verscheiden Resultaten geben, wenn die Einheiten nicht dieselbe sind:

T= 2*3 + 4*3 = 6*3 wenn sammelt in Zeit; and T= 2*3 + 4*3 = (2*3 + 4*3)/6*6 = 3*6 wenn sammelt in Raum.

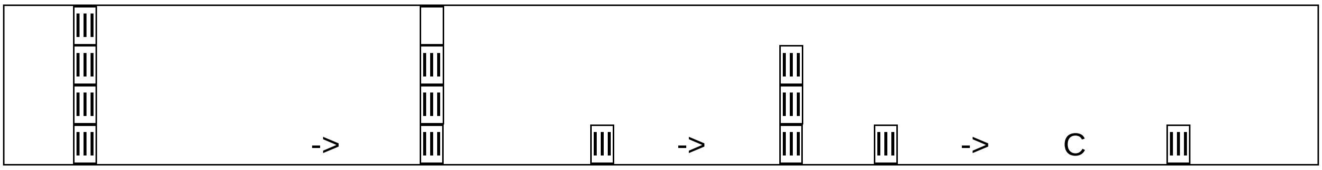
Der Prozess ‘sammeln’ kann man umkehren: 2 3e + ? 2e = 3 5e:



Die antwort kommt wenn man von die 3 5e die 2 3e entfernt, und danach die zurückgebliebene 9 als 2e zählt: 9 = (9/2)*2 = 4 1/2 * 2. Später nennt man dieser Prozess differenzieren.

11 Schober sind überladet

Die nächste Bildungsfrage ist: Was kann wir mit Überladen machen? Ein Schober ist übergeladen wenn es höher ist als seiner Bündel. Die Überladung kann man umschobern zu ein neuen Schober, und zurück ist ein voller Schober C.



T = 4*3 = überladen = (4*3 – 1*3) + 1*3 = 3*3 + 1*3 = 1*C + 1*3

12 Einstellige Gleichungen

Die letzte Bildungsfrage ist: Wie kann man eine Sammlung umkehren? Eine Antwort ist Zurück-rechnen, das man auch die Lösung eine Gleichung nennen kann. Bei umzählen und umschobern sieht man dass Gleichungen gelöst werden bei die zahle zu dem andre Seite zu rücken, und die Zeichen wechseln.

Umzählen: T = (T/4) * 4	Umschobern: T = (T-4) + 4
Gleichung T = x * 4 Lösen T/4 = x	Gleichung T = x + 4 Lösen T-4 = x

Die Gleichung 2*x+1 = 9 ist gelöst bei eine Kombination von umzählen und um schobern, nicht bei Neutralisierung.

2*x+1 = 9 = (9-1) +1 = 8 + 1 umschobern 9 zu 8 + 1 2*x = 8 = (8/2)*2 = 4*2 umzählen 8 zu 4*2 x = 4 die lösung	Oder bei rücken: 2*x + 1 = 9 2*x = 9 - 1 = 8 x = 8/2 = 4

Gebildet von Vielheit

Durch diese 12 Bildungsbegegnungen mit Vielheit ist eine neue radikale verschiedene Mathematik eingeführt, eine postmoderne LAB-Mathematik das die moderne BIB-Mathematik kopf-stehen macht.

• BIB-MATHEMATIK sieht Zahlsymbolen als kulturelle Artefakten, das man auswendig lernen muss. LAB-MATHEMATIK sieht Zahlsymbolen als naturelle Ikonen die seiner Anzahl von streichen repräsentiert.

• BIB-MATHEMATIK hat verschiedene Zahltypen: Natürliche zahlen, Ganze Zahlen, Rationale Zahlen, Irrationale Zahlen usw. LAB-MATHEMATIK hat nur zwei Zahltypen, Schober-zahle und Per-zahle.

• BIB-MATHEMATIK hat Addition als seine primäre Operation, dann folgt Subtraktion, und Multiplikation; zuletzt kommt die schwierige Division. LAB-MATHEMATIK has Division als seine primäre Operation, dann folget Multiplikation, Subtraktion, und zuletzt die schwierige Addition.

• BIB-MATHEMATIK einführt mehrstellige Zahle bevor Mathematik. LAB-MATHEMATIK einführt Mathematik bevor mehrstellige Zahle.

• BIB-MATHEMATIK einführt mehrstellige Zahle als ein abstraktes Idee von einer Positionssystem. LAB-MATHEMATIK einführt mehrstellige Zahle als einer physische Faktum, als ein Lager , ein multi-schober.

• BIB-MATHEMATIK einführt die Addition von mehrstellige Zahle als einer abstrakte Idee von Übertrag. LAB-MATHEMATIK einführt die Addition von mehrstellige Zahle als einer Innerhandel in ein Lager.

• BIB-MATHEMATIK aussetzt Differential- und Integralrechnung zu der tertiären Bildung. LAB-MATHEMATIK inkludiert Differential- und Integralrechnung in sowohl primäre als auch sekundäre Bildung.

• BIB-MATHEMATIK aussetzt Algebra und Gleichungen zum sekundäre Bildung. LAB-MATHEMATIK inkludiert Algebra und Gleichungen in primäre Bildung.

Konklusion

BIB-MATHEMATIK ist als beispiele von Abstraktionen definiert, von die Begriff Menge, die zu eine metaphysische Axiomsystem gehört. In dieser weg bleibt BIB-Mathematik ein reine abstraktes mentale Aktivität, das man nicht notwendig immer in das Laboratorium berichtigen kann, als das 2&3-paradox zeigt.

LAB-mathematik ist als Abstraktionen von Beispiele definiert, von Vielheit aufbaut, die zu einer physische Realität gehört. In dieser weg bleibt LAB-MATHEMATIK eine natürliche Wissenschaft das in das Laboratorium induziert und berichtigt ist.

Geprüft in das Klassezimmer der sekundäre Schule ist LAB-MATHEMATIK eine Lösung zu dem relevanz-paradox. (Tarp 2003).

Geprüft in das Lehrerbildung die Mehrheit der Studenten hatten anbefohlen das LAB-MATHEMATIK in der Lehrerbildung inkludiert werden. (Zybartas et al 2001).

Dieser positive Reaktion hat zu dem Konstruktion von Bildungs- und Lehrungsmateriale in einer postmoderne , LAB-MATHEMATIK von unten' geführt, das für freie ‚Teacher PYRAMIDeDUCATION at www.MATHeCADEMY.net' benutzt wird (Tarp 2003).

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102. Adding PerNumbers

To solve the relevance paradox in mathematics education this paper uses postmodern sceptical Cinderella research to look for new ways to teach mathematics at the secondary school. The paper introduces addition of per-numbers as a more user-friendly approach to the traditional subjects of proportionality, linear and exponential functions and calculus.

1 The Focus: Unnoticed Ways to Teach Mathematics

Mathematics education has a relevance paradox 'formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994). At a talk in the late 1970's Freudenthal described the didactical expert as the reflective practitioner. This inspired the author to use the mathematics classroom as a laboratory to develop different ways to teach secondary mathematics at the pre-calculus and the calculus level. Later as a research student the author used the inspiration from postmodern thinking to develop a new action focused methodology called sceptical Cinderella research. This postmodern methodology has proved to be a Cinderella-difference making a difference by solving the relevance paradox in the classrooms of the authors and cooperating teachers. So now the time has come to make its results more widely known.

2 The Theoretical Framework: Institutional Scepticism

The theoretical framework of the study is institutional scepticism, as it appeared in the Enlightenment and was implemented in its two democracies, the American in the form of pragmatism and symbolic interactionism, and the French in the form of post-structuralism and post-modernism.

3 Literature: Symbolic Interactionism & Postmodernism

The source of inspiration is American symbolic interactionism (Blumer etc.) and its methodology 'Grounded Theory' (Glaser & Strauss, etc.) as well as French poststructuralism (Derrida, Lyotard, Foucault) and postmodern thinking (Bauman). These sources have lead to a definition saying 'postmodernism means institutional scepticism towards and deconstruction of the pastoral power of non-democratic traditions'.

4 Methodology: Sceptical Cinderella-Research

The methodology 'sceptical Cinderella research' is a postmodern counter-seduction research based upon a post-structuralist 'pencil-paradox': Placed between a ruler and a dictionary a thing can point to numbers, but not to words - thus a thing can falsify a number-statement about its length, but not a word-statement about its name; i.e. a thing can defend itself against a number-accusation by making a statement of difference in a laboratory; but is forced to pay deference to any word-accusation from the library. (Tarp 2001, 2003).

A number is an ill written icon showing the degree of multiplicity (there are 4 strokes in the number sign 4, etc.); a word is a sound made by a person and recognised in some groups and not in others. Words can be questioned and put to a vote in a courtroom, numbers cannot. Numbers can carry valid conclusions based upon reliable data, i.e. research. Words can carry only interpretations, that if presented as research become seduction; words carry no truth, but hide differences to be uncovered by counter-seduction as e.g. sceptical research, having quality if it can produce a 'Cinderella-difference', i.e. a difference that makes a difference.

Thus the sceptical part of the methodology comes from the ancient Greek sophists always distinguishing between choice and necessity, between political and natural correctness. This part allows the researcher to see the difference between what could be different and what could not. Once an area with a hidden difference has been identified, the Cinderella part is used to identify hidden alternatives either by discovering forgotten or unnoticed alternatives at different times and places, inspired by the genealogy of Foucault (Dreyfus et al 1982); or by inventing alternatives using sociological imagination, inspired by Mills (Mills 1959).

Counter-seduction research is always sceptical towards word-based research in the library, especially if it has not been generated from and validated in the laboratory, thus unable to meet the 'LibLab' criteria of the research genre saying that a Lib-acceptance implies a Lab-validation of findings induced from Lab-observations or deduced from a Lib-hypothesis. Thus reference to the existing research literature is not possible within a counter-seduction methodology. The aim of sceptical Cinderella research is not to extend the existing seduction of the library. The aim is to search for hidden Cinderella-alternatives in the learning-laboratory, i.e. in the classroom, by 1) finding a non-democratic tradition hiding its alternatives, 2) finding hidden alternatives through discovery and imagination, 3) testing the alternative in the laboratory to see if it is a Cinderella-difference making a difference, and 4) publish the alternative so it can be an option in the classroom.

5 The Classroom Tradition: Fractions and Their Applications

Having developed the mathematics of natural numbers and decimal numbers and their applications in the primary school the time has come to introduce two new types of numbers in the secondary school. The rational numbers are treated in the pre-calculus classes, and the real numbers are introduced in the calculus classes.

Both of these numbers base their definition on the concept of 'sets of sets'. Thus the set Q of rational numbers is defined as a set of equivalence sets in a product set of two sets of sets of equivalence sets in a product set of two sets of sets of equivalence sets in a product set of two sets of Peano-numbers; such that the pair (a,b) is equivalent to the pair (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same rational number 1/2 (see e.g. Griffith et al 1970).

The set-concept however is controversial. Numbers differentiate between degrees of multiplicity; and sets differentiate between degrees of infinity.

Kronecker objected that infinity is not a quantity, but a quality; so potential infinity exists, but actual infinity does not. Instead Kronecker meant that mathematics should be build upon the natural numbers, our gift from God. The intuitionists later pursued this view.

Russell objected to sets of sets through his paradox: If $M=\{A \mid A\notin A\}$, then $M\in M \Leftrightarrow M\notin M$. Instead Russell made self-reference illegal by introducing his type-theory saying that a given type can only be described by a higher type. Computer science later pursued this view.

Historically fractions are left overs from the medieval times, when strokes were used to separate the two processes of using numbers as input and output: '3 | 4' meant that 3 is an input-number and that 4 is an output-number as used in book-keeping: 20 + (3 | 4) = 20 + 3 – 4 = 19. And 3/4 meant that 3 is a repeated input-number and that 4 is a repeated output-number: 15*3/4 = (15/4)*3. Modern mathematics instead talks about minus and reciprocal signs of numbers: '- 4' (take away 4) and '/4' (take away 4s).

Another problem is illustrated by the typical 'welcome ceremony' in the Danish upper secondary school:

The teacher:	The students:
Welcome! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No, 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
Inside this classroom 1/2 + 2/3 = 7/6!	

Apparently we have a fraction-paradox:

Inside the classroom	20/100 + 10/100 = = = 20% + 10% =	30/100 = 30%
Outside the classroom e.g. in the laboratory	20% + 10% = or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

Of course there are examples where 20% + 10% = 30% as in 20% of 300 + 10% of 300 = 30% of 300. But as the fraction-paradox shows, that there are also many counter-examples, so there is no general rule saying that 20% + 10% = 30%

If research is valid conclusions based on reliable data, then addition of fractions is not research since it cannot be validated outside the classroom. So we can distinguish between ‘mathematics’, which is a science that can be validated in the laboratory, and ‘mathematism’, which is a doctrine, an ideology, which cannot be validated in the laboratory.

This gives a possible solution to the relevance paradox: What we call ‘mathematics’ is really ‘mathematism’ teaching ‘killer-mathematics’ only existing inside classrooms, where it kills the relevance of mathematics. Presenting the students to mathematics instead of mathematism may validate this solution.

6 A Classroom Alternative: The History of Per-Numbers

Spices and silk from the East has always been popular in the West; but being a higher culture the East only accepted silver and gold as means of exchange. So history has witnessed a steady flow of silver from the West to the East creating wealth and culture along the road.

Thus silver mines outside Athens in Greece financed the antique culture. Later the silver mines in Spain financed the Roman Empire. After the Arabic conquest of Spain the dark Middle Ages came to Europe until silver was found again, this time in Germany. Italy became the middleman between Germany and the East and this position created the wealth that financed the Renaissance. The next trade centres were Portugal and Spain using the seaway around Africa to the East, and using America for new mines of silver and gold.

Later England wanted to sail to the East on open sea to avoid the hostile fortresses of Africa. This entailed a closer study of the motion of the moon. Opposing the traditional view that the moon is moving across the sky Newton instead suggested that the moon is falling towards the earth just as an apple. Newton thus discovered that the behaviour of physical objects is determined by a physical will to change, a natural force called gravity. Unlike the will of the metaphysical Lord this physical will was predictable through calculations, once the art of change-calculations was developed and given a name, differential and integral calculus.

Thus change-calculations, or calculus, became the basis of modern physics and other natural sciences. And thus calculus became the basis of the Enlightenment period in the 18th century rebelling against the will of the masters by claiming that man was his own master fully able to choose what was best. The Enlightenment installed two democracies, one in the US and one in France, as well as the modern industrial state based upon the applications of modern science. This changed the western economy from being based on silver to being based on production and knowledge.

So in the early modern time the West saw two golden periods, the renaissance and the Enlightenment. And these two periods both created and applied new mathematics: ‘Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights’ (Kline 1972: 399).

The two typical trade questions of the Renaissance were that of proportionality 'if 3 kg cost 5$, what is the price for 7 kg, and what will 12$ buy?'; and that of renting '20 days at 5$/day totals? $'

Because of Italy's immense wealth also money became a commodity that could be rented. But if the rent was not paid back also the rent had to be rented crating the compound interest '20 days at 0.5%/day totals ? $'

In England Newton described the movement of the moon as a fall towards the earth just like the fall of an apple. A falling object is described by its velocity, which is a per-number meters per seconds, m/s. If the velocity is constant then the total distance can be calculated by a simple multiplication: 5 seconds at 3 m/s totals 5*3 = 15 meters.

But in the case of a falling object the velocity is not constant, but constant accelerated, giving raise to a question that cannot be solved by simple multiplication: 5 seconds at 3 m/s changing to 4 m/s totals ? meters. So Newton was faced with the question: How to add variable per-numbers?

To sum up the two rich periods of the early modern world were connected with the mathematical development of per-numbers where the question 'how do we add per-numbers' were posed in different connections. In renaissance Italy the addition of per-numbers occurred within trade, both at prices as $/kg-numbers, at rent as $/day-numbers and at interest as %/day-numbers. And in the Enlightenment the addition of per-numbers occurred within physics at velocity as m/s-numbers.

Thus the Cinderella-principle has lead to the discovery of a forgotten hidden mathematics, the addition of per-numbers. Now it is time to use the imagination to set up a school curriculum on addition of per-numbers so that the school can tell the grand narratives about the growth and success of mathematics in the Renaissance and in the Enlightenment by making the addition of per-numbers the core of the curriculum in secondary school.

According to Kronecker and Russell we will try to build a set-free, fraction-free and function-free mathematics on the basis of a thing that exists with necessity, repetition in time.

7 Building Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process 'from T take away 4' can by iconised as 'T-4'. The repeated process 'from T take away 4s' can by iconised as 'T/4, a 'per-number'. So the count&stack calculation T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting.

5. A calculation T=3*4= 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T	= 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T	= 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T	= 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal's length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by re-counting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:
C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads:

T = 38 + 29 = 3ten 8 + 2ten 9 = 5ten 17 = 5ten 1ten 7 = (5+1)ten 7 = 6ten 7 = 67

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number a is multiplied with the day-number b before being added to the total $-number T: T2 = T1 + a*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated and reversed addition of per-numbers leads to integration and differentiation:

Repeated addition of per-numbers		Reversed addition of per-numbers	
T2	= T1 + a*b	T2	= T1 + a*b
T2 - T1	= + a*b	a	= (T2-T1)/b
ΔT	= $\sum$ a*b	a	= ΔT/Δb
ΔT	= $\int$ y*dx	a	= dy/dx

Only in the case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers:

4 years at 5 % /year = 21.6% , since 105%*105%*105%*105% = 105%^4 = 121,6%

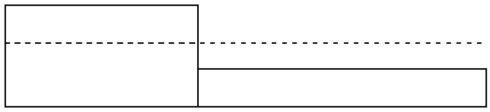
So a Kronecker-Russel multiplicity-based mathematics can be summarised as a 'count&add-laboratory' adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word 'algebra', reuniting.

	Constant	Variable
Totals *m, s, kg, $*	T = a*n T/n = a	T2 = T1 + a*n *T2-T1 = a*n*
Per-numbers *m/s, $/kg, $/100$ = %*	T = a^n $\sqrt[n]{T} = a$ $\log_a T = n$	T2 = T1 + $\int$a*dx *dT/dx = a*

The Count&Add-Laboratory

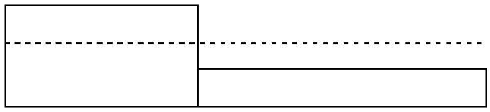
8 Bringing the Multiplicity-Based Mathematics to the Classroom

The elementary school introduces the practise of counting multiplicity by bundling and stacking to be predicted by per-numbers, thus making the students acquainted with the geometrical representation of per-numbers as the height of a stack. Also simple additions as T = 3 4s + 2 5s = ? 9s are carried out both by recounting and by prediction thus realising that mathematics is our language of prediction:



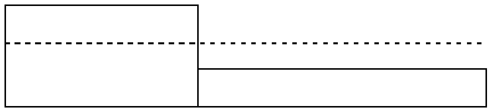
T = 3 4s+2 5s = 3*4+2*5 = (3*4+2*5)/9*9 = 2 4/9 *9

In secondary school addition of per-numbers follows the same pattern within trade:



T = 4kg at 3$/kg + 5kg at 2$/kg = 9kg at 2 4/9$/kg

And so does addition of per-numbers within physics:



T = 4s at 3m/s + 5s at 2m/s = 9s at 2 4/9m/s

This also applies if the m/s-number is locally constant and not just piecewise constant, i.e. if ε and δ changes places in the formal definition of constancy:

A variable y is globally constant c	∀ε>0:	|y-c|	< ε all over
A variable y is piecewise constant c	∃δ>0 ∀ε>0:	|y-c|	< ε in the interval δ
A variable y is locally constant c (continuous)	∀ε>0 ∃δ>0:	|y-c|	< ε in the interval δ

Thus a general pattern occurs: Per-numbers are added by the area under its curve. Since any smooth curve is locally constant it can be approximated by a series of stacks to be summed by integration: ∫ y*dx ≈ ∑ y*Δx. However if y*Δx can be written as the change of another variable z (y*Δx = Δz, or y = Δz/Δx) then the sum can be predicted since the sum of single changes= the total change= Terminal-number – Initial-number: ∑ y*Δx= ∑ Δz= Δz= z2 – z1.

This relation does not depend on the size of the change, so also ∫ y*dx = ∫ dz = z2 – z1.

Therefore the time has come to study the behaviour of changing variables by going back to the two addition tasks of the Renaissance:

5 days at 3 $/day = 5*3 = 15 $
5 days at 3 %/day is 5*3, i.e. 15 %, + CI = simple interest + compound interest

The compound interest CI can be calculated from the total interest TI:

TI = n*I + CI, where 1 + TI = (1+i)^n.

Thus the compound interest CI is what makes exponential change and linear change different. Or in other words, if we only consider the simple interest and neglect the compound interest then exponential change becomes linear. And since the compound interest first matters in the long run, there is no big harm done by neglecting it in the short run. This ‘neglecting the compound interest in the short run’ is called differential calculus. In differential calculus a smooth curve is considered locally linear having a locally constant per-number or slope:

The y-curve is locally linear if its per-number Δy/Δx is locally constant c = dy/dx	∀ε>0 ∃δ>0:	|(Δy/Δx) - c|	< ε in the interval δ

The change of a stack can be studied geometrically

Thus differential calculus means neglecting the upper right corner in a changing stack.

So in differential calculus (1+dx)^2 = 1+2*dx, and (1+dx)^3 = 1+3*dx as seen below:

dx	dx	2*dx^2
1	1	2*dx
	1	2*dx

In general (1+dx)^n = 1+n*dx.

If y = x^n, then a change in x, dx, produces a change in y, dy, and

y + dy = (x+dx)^n = (x*(1+dx/x))^n = x^n*(1+dx/x)^n = y*(1+n*dx/x) = y + n*y*dx/x

So dy = n*y*dx/x , or dy/dx = n*y/x = n*x^n/x = n*x^(n-1)

Now we are able to predict the result of adding variable per-numbers through a calculation:

5 sec. at 3m/sec increasing to 4 m/sec total $\int_0^5 (3+\frac{4-3}{5}x)\,dx = \int_0^5 (3+0.2x)\,dx = ?$ m

Since d/dx (3x+0.1x^2) = 3+0.2x we get that d (3x+0.1x^2) = (3+0.2x) dx, so

$$\int_0^5 (3+0.2x)\,dx = \int_0^5 d(3x+0.1x^2) = \Delta(3x+0.1x^2) = (3\cdot5+0.1\cdot5^2) - 0 = 17.5 \text{ m}$$

9 Multiplicity-Based Mathematics: A Difference in the Classroom

Kronecker-Russell mathematics, multiplicity-based stack-mathematics, Enlightenment-mathematics makes a difference in the Danish pre-calculus classroom (Tarp 2003) and in teacher education in Eastern Europe (Zybartas et al 2001) and in Africa (Tarp 2002). Thus by being a Cinderella-difference it brings a successful conclusion to the sceptical Cinderella study.

10 Conclusion

The purpose of this study was to apply sceptical Cinderella research to solve the relevance problem in the mathematics classroom. First scepticism was used to identify a non-democratic tradition hiding its alternatives. In this case fractions was identified bringing ‘killer-mathematics’ into the classroom killing the relevance of mathematics by being only valid inside the classroom and not outside, thus transforming mathematics to mathematism.

Then Cinderella thinking was used. First to travel in time and discover a forgotten alternative to fractions, per-numbers. Then to use sociological imagination to imagine alternative ways to bring the grand narratives of the Renaissance and the Enlightenment to the classroom in the form of a Kronecker-Russell multiplicity-based mathematics.

Finally this difference turned out to be a Cinderella-difference making a difference. Thus the sceptical Cinderella study has been successful by offering a hidden alternative to policy makers wanting to solve the relevance paradox of mathematics education.

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103. Bundling & Stacking in a Count & Add Laboratory

*To solve the relevance paradox in mathematics education this paper suggests that LIB-mathematics as 2+3 = 5, valid only in the library and not in the laboratory, should be replaced with LAB-mathematics as 2*3 = 6, also valid in the laboratory. A reconstruction respecting Kronecker's and Russell's scepticism shows that multiplicity-based LAB-mathematics is rather different from set-based LIB-mathematics by allowing fundamental mathematics as per-numbers, re-counting and re-stacking to be introduced at the first year of school and even with one digit numbers only.*

Lib-Mathematics And Lab-Mathematics

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998) and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371).

From the first day at school mathematics teaches that 2+3 = 5 and 2*3 = 6. That 2 3s can be re-counted as 6 1s is easily validated in the laboratory: ooo ooo -> oooooo. 2+3 = 5 might be true in the library, but not in the laboratory where countless counter-examples exist: 2 weeks + 3 days = 17 days, 2 cm + 2 m = 202 cm etc.

Thus we can distinguish between LIB-mathematics from the library and LAB-mathematics from the laboratory where only the latter can be validated in the laboratory. And we may assume that the relevance paradox disappears when LIB-mathematics is replaced with LAB-mathematics. Choosing a proper method can test this hypothesis.

Institutional Scepticism

Historically research is 'LAB-LIB research' where the LIB-statements of the library are induced from and validated by reliable LAB-data from the laboratory. Today 'LIB-LIB research' using LIB-words induced from and validated by other examples of LIB-research has lead to scepticism towards research. Teaching about democracy the ancient Greek sophists introduced scepticism to distinguish between necessity and choice by saying as e.g. Antiphon that 'the command of the law is only a decision without roots in nature, whereas the command of nature has grown from nature itself not depending on any decisions' (Antifon in Haastrup et al 1984: 82, my translation).

Later scepticism was revived as the institutional scepticism of the Enlightenment and implemented in its two democracies; the American in the form of pragmatism, symbolic interactionism and grounded theory; and the French in the form of post-structuralism and post-modernism.

Derrida calls the belief that words represent the world for 'logocentrism'. Lyotard defines modern as 'any science that legitimates itself with reference to a metadiscourse'; and postmodern as 'incredulity towards metanarratives' (Lyotard, 1984: xxiii-xxiv). Foucault describes pastoral power that after having over centuries 'been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers'(Foucault in Dreyfus et al, 1982: 215).

This scepticism towards words is summarized in a 'number&word-paradox': Placed between a ruler and a dictionary a thing can point to a number but not to a word, so a thing can falsify a number-claim in the laboratory but not a word-claim in the library; thus numbers are reliable LAB-data able to carry research, whereas words carry interpretations that presented as research become seduction - to be lifted by sceptical counter-seduction research looking for a hidden 'Cinderella-difference' making a difference (Tarp 2003).

Kronecker And Russell Critizising Lib-Mathematics

The history of mathematics is a natural place to look for hidden differences. We begin at the golden period of mathematics, the Enlightenment century, where the mathematicians

> were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

However, in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. This thinking lead Cantor to introduce the controversial concept ‘set’ to distinguish between different degrees of infinity having the natural numbers as a unit, just as numbers were introduced to distinguish between different degrees of multiplicity having 1 as a unit.

> To Kronecker (..) Cantor’s work on transfinite numbers and set theory was not mathematics but mysticism. Kronecker was willing to accept the whole numbers because these are clear to the intuition. These ‘were the work of God.’ All else was the work of man and suspect. (1197).

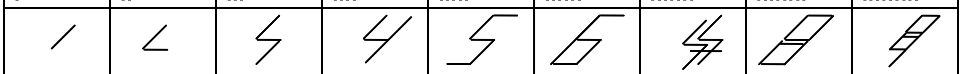
As to paradoxes in set-theory even Cantor saw problems asking Dedekind in 1899 whether the set of all cardinal numbers is itself a set; because if it is, it would have a cardinal number larger than any other cardinal (1003). Another paradox was Russell’s paradox showing that talking about sets of sets leads to self-reference and contradiction as in the classical liar-paradox ‘this sentence is false’: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$. Russell solves this paradox by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Thus if a fraction is defined as a set of numbers it cannot be a number itself making e.g. the addition ‘2+3/4’ meaningless. Not wanting a fraction-problem modern LIB-mathematics has chosen to neglect Russell’s type-theory and to base its definitions on the set-concept derived from the Zermelo-Fraenkel axiom system making self-reference legal by not distinguishing between an element of a set and the set itself in spite of this leading to syntax errors in ordinary languages.

Reconstructing A Lab-Based Mathematics

Modern mathematics is a LIB-based mathematics defining its concepts as examples of abstractions and validating its sentences from axioms. Postmodern sceptical mathematics is a LAB-based mathematics defining its concepts as abstractions from examples and validating its sentences in the laboratory, thus respecting two fundamental principles: A Kronecker-principle saying that only the natural numbers can be taken for granted, and a Russell-principle saying that we cannot use self-reference and talk about sets of sets. So the postmodern mathematics classroom is a sceptical laboratory where quantitative competences are developed through educational meetings with ‘many-ness’ occurring as repetition in time and multiplicity in space.

First a representation competence is developed through working with different examples where temporal repetition is represented as spatial multiplicity: Put a finger to the throat and add a match and a stroke for each beat of the heart. Add a stone and a stroke for each blink of the eye. Etc.

Then a rearrangement competence is developed rearranging different degrees of multiplicity into icons, realising that there would be four strokes in the number symbol 4 had not the number symbols been written in such a sloppy way.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

Next a count&stack competence is developed by piling a total T through bundling to a stack and some leftovers as e.g. T = 3 4s + 2 1s = 3*4 + 2*1

IIIIIIIIIIIIII -> IIIIIIIIII [IIII] -> IIIIII [IIII] [IIII] -> II [IIII] [IIII] [IIII] -> [IIII] [IIII] [IIII] II	T = 3*4 + 2*1

We count in 4s by taking away 4s. The process ‘from T take away 4’ may be iconized as ‘T-4’ and worded as ‘T minus 4’. The repeated process ‘from T take away 4s’ may be iconized as ‘T/4’ and worded as ‘T counted or divided in 4s’. So the ‘recount-equation’ T = (T/4)*4 predicts the result of recounting T in bundles of 4s: T= (T/4)*4= 3*4 + 2*1. T/4 is called a ‘per-number’ or a ‘recounter’.

Then a recounting competence is developed transforming e.g. 4s to 5s by changing the bundle-size: T = 3*4 = ?*5. This is done manually by de-stacking, de-bundling, re-bundling and re-stacking:

[IIII] [IIII] [IIII] -> [IIII] [IIII] [IIII] -> IIIIIIIIIIII -> IIIIIII [IIIII] -> II [IIIII] [IIIII] -> [IIIII] [IIIII] II	T = 3*4 = 2*5 + 1*2

Using per-numbers also the leftovers can be counted as 5s: IIIIIIIIIIII -> IIIIIII [IIIII] -> II [IIIII] [IIIII] -> [II] [IIIII] [IIIII]	T = 2*5 + 2/5*5 = 2 2/5*5

Now it is time to apply the recounting competence to develop a trading competence to deal with typical trade problems: If 2 S = 1 C, then 8 S = ?C. Recounting the 8 in 2s tells us how many times we have a 2 on the S-side, but also how many times we have a 1 on the C-side:

SSSSSSSS -> SS SS SS SS -> C C C C -> CCCC	T = 8*S = (8/2)*2*S = 4*2*S = 4*1*C = 4*C

Again the result is predicted by the recount-equation T = (T/b)*b.

Next a coding competence is developed to hide the bundle-size: T = 2*3 + 1 -> T = 2*S + 1.

And to guess the Total when coded: T = 2*S + 1 = ? if C = 1; if C = 2; if C = 3; if C = 4; etc.

Then a decoding competence is developed in three steps:

First the question S + 3 = 8 is answered by restacking: From the 8-stack we take away 3 to a new stack leaving 8-3 = 5 in the original stack as predicted by the ‘restack-equation’ T = (T-3) + 3.

S ooo = oooooooo -> ooooo ooo	S + 3 = 8 = (8-3) + 3 = 5 + 3 S = 5	or quicker	S + 3 = 8 S = 8-3 = 5

The restack question ?+3 = 8 is answered by restacking 8 to (8-3)+3 making ? = 8-3. This later becomes an equation x+b = T solved by x = T-b to be found by moving the number b across the equation sign and reversing its calculation sign from + to -.

Next the question 2*S = 6 is answered by recounting: The 6 is recounted in 2s and overturned as predicted by the recount-equation T = (T/2)*2.

oo S oo ooo S= oooooo -> oo oo oo -> oo -> ooo	2*S = 6 = (6/2)*2 = 3*2 S = 3	or quicker	2*S = 6 S = 6/2 = 3

The recount question 2*? = 6 is answered by rebundling 6 to (6/2)*2 making ? = 6/2. This later becomes an equation b*x = T solved by x = T/b to be found by moving the number b across the equation sign and reversing its calculation sign from * to /.

Finally the problem 2*S+1 = 7 is solved by first restacking 7 by taking away 1: 7 = (7-1)+1 = 6+1. Then the 6 is recounted in 2s and overturned.

oo	2*S+1= 7 = (7-1)+1 = 6+1	2*S+1 = 7
S oo ooo	2*S = 6 = (6/2)*2 = 3*2	2*S = 7-1 = 6
S o = ooooooo -> oooooo o -> oo oo oo o -> oo o -> ooo o	S = 3	S = 6/2 = 3

Here the result is predicted by applying both the restack-equation and the recount-equation.

Now the time has come to say farewell to one digit mathematics and deal with more digits by choosing a standard bundle-size. Using the number of fingers on our hands we will chose ten as a standard bundle-size where Donald Duck will choose eight as a standard bundle-size.

First we discuss with Donald how to count 2 dozens of stones.

Donald:	1, 2, 3, 4 ,5 ,6 ,7, bundle, bundle+1, … , bundle+7, 2*bundle+0, 2*bundle+1,…, 3*bundle+0, i.e. 30.
We:	1, 2, 3, 4 ,5 ,6 ,7, 8, 9, bundle, bundle+1, … , bundle+9, 2*bundle+0, 2*bundle+1,…, 2*bundle+4, i.e 24

So when Donald counts 30 = 3eight0, we count 24 = 2ten4 and M. Douze counts 20 = 2dozen0.

Thus if we count 2*bundle+6 = 26, Donald counts 3*bundle+2 = 32 since T = 26 = 2ten6 = (2*ten+6)/eight*eight = 3eight2 = 32; and M. Douze counts 2*bundle+2 = 22 since T = 26 = 2ten6 = (2*ten+6)/dozen*dozen = 2dozen2 = 22.

Likewise, if Donald counts 753 = 7eighteight5eight3 we count 491 = 4tenten9ten1 since

T = 7eighteight5eight3 = 7*8*8 + 5*8 + 3 = 491 which can be found directly on a calculator.

Thus multiplication is really division recounting a total in the standard-bundles 1, 10, 100 etc.

Now we can develop competences in decimals and percentages by rebundling in tens:

T = 3 6s = 3*6 = (3*6/10)*10 = (1 8/10)*10 = 1.8*10, where 1 8/10 = 1.8
T = 3 6s = 3*6 = 18 = (18/100)*100 = 18%*100, where 18/100 = 18%

The 3 percentage-problems can be solved by being reformulated as trade-problems and recounted:

If 100% = 60$, then 20% = ?	If 100% = 60$, then 15$ = ?	If 20% = 15$, then 100% = ?
20% = (20/100)*100% = (20/100)*60$ = 12$	15$ = (15/60)*60$ = (15/60)*100% = 25%	100% = (100/20)*20% = (100/20)*15$ = 75$

Next we develop an adding competence by dealing with the overloads created by uniting stacks as e.g. T = 38 + 24 = 3ten8 + 2ten4 = 5ten12 = ?

```
                                         O            O
                                         O            O
                                         O           |O|
                                         O           |O|
       O                                 O           |O|
       O                                 O           |O|
       O                                 O           |O|                      X
       O                             X   O       X   |O|       X              X
       O              O              X   O       X   |O|       X              X
   X   O              O              X   O       X   |O|       X   O          X
   X   O          X   O              X   O       X   |O|       X   O          X   O
   X   O          X   O              X   O       X   |O|       X   X          X   O
T = 3ten  8   +   2ten  4   =   5ten twelve  =  5ten+(1ten+2)  =  (5ten+1ten)+2  =  6ten  2  = 62
```

Thus the overload 12 is restacked and rebundled. First it is restacked: From 12 we take away 10 1s: T = 12 = (12-10)+10 = 2+10. Then the 10 1s is rebundled to 1 10s and added to the 10s as in book-keeping where an internal trade takes place between mr. Ten and mr. One:

Mr. Ten

Total	IN	OUT
3	2	
5	1	
6		

Mr. One

Total	IN	OUT
8	4	
12		10
2		

Repeating adding stacks is multiplication leading to a big overload:

T = 4*18 = 4*(1ten8) = 4ten32 = 4ten3ten2 = 7ten2 = 72

Finally we can develop a per-number competence. Per-numbers can be added after being transformed to stacks. Thus the $/day-number ‘b’ is multiplied with the day-number ‘n’ before being added to the total $-number T giving T2 = T1 + n*b:

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + a*b	T2	= T1 + a*b
T2 - T1	= + a*b	(T2-T1)/b	= a
ΔT	= ∑ a*b	ΔT/Δb	= a
ΔT	= ∫ a*db	dT/db	= a

Only in the case of adding constant per-numbers as e.g. a constant interest of 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%

Differences Between Modern LIB- And Postmodern LAB-Mathematics

There are many differences between modern LIB-mathematics and postmodern LAB-mathematics:

Modern set-based LIB-Mathematics:	Postmodern multiplicity-based LAB-Mathematics:
Is set-based defining its concepts from the library above as examples of abstractions	Is multiplicity-based defining its concepts from the laboratory below as abstractions from examples
Accepts natural, whole, rational and real numbers thus regarding fractions and roots as numbers	Accepts stack-numbers, unit-numbers and per-numbers thus regarding fractions and roots as calculations
Has addition as its fundamental operation, presenting multiplication as repeated addition	Has multiplication as its fundamental operation. Only allows addition inside parenthesis where units are the same
Sees division as reversed multiplication finding the unit	Sees division as repeated subtraction finding the per-number
Postpones algebra until middle school	Introduces algebra in grade one

Within concept education as e.g. mathematics the two fundamental questions are: ‘How does concepts come into the world - from above or from below?’ and ‘How does concepts come into the head - from the outside or from the inside?’

Modern LIB-based mathematics answers ‘from above, and from outside’ presenting an abstract concept as an example of a more abstract concept: ‘A function is an example of a relation between two sets, that …’. The students hear this as ‘bublibub *is an example of* bablibab’ giving no meaning. The lack of meaning forces students to construct their own meaning e.g. by using a metaphor to ‘carry over’ meaning from the same abstraction level: ‘A function *is like* a machine processing numbers'. Constructivist mathematics thus answers 'from above, and from inside' seeing concepts as being constructed by the individual student through activity and communication.

Postmodern LAB-based mathematics answers 'from below, and from outside' presenting an abstract concept as a name for a less abstract concept: 'A function *is a name for* a calculation with a variable quantity'. The students hear this as 'bublibub is a name for a calculation' thus obtaining meaning from below, so they do not have to construct their own meaning. Practise-learning, e.g. through apprenticeship, answers 'from below, and from inside'. The apprentice constructs meaning from observing and participating in the practise: 'A function *is for example* 2+x or 2·s, but not 2+3'.

Testing Postmodern LAB-Mathematics In The Classroom

At an in-service course in special education a teacher volunteered to try out multiplicity based mathematics with her first graders. So for 3 hours the classroom was transformed into a laboratory for recounting and restacking, first performed manually with stones, matches and plastic pearls on a plastic board, then reported on paper. At the end a test was given showing a stack T = 1*3 + 1*2 + 2 and its restacking in 2s (T = ?*2) and 3s (T = ?*3). Then the students were given three stacks to write as totals; and to restack in 2s and 3s both as drawings and as total-stories.

The stacks were T = 2 3s + 1 2s + 1 1s, T = 2 3s + 2 1s and T = 1 4s + 1 3s + 1 1s giving a total of 9 problems to solve. The 15 students scored 3, 4, 4.5, 5, 5, 5.5, 6.5, 7, 7, 8, 8.5, 9, 9, 9, 9 giving the mean 6.7, the median 7 and the mode 9. Thus almost 2/3 of the students mastered at least 2/3 of the restacking problems after a 3 hours introduction, showing that restacking is a familiar activity.

At teacher education postmodern LAB-mathematics was tested in 5 2-hour lectures over a week. Afterwards the students were given a questionnaire asking them to express their attitude on a scale from very negative to very positive, 0: very negative, 1: negative, 2: a little negative, 3: neutral, 4: a little positive, 5: positive, 6: very positive. The numbers report the answers as to the mean, the median and the mode and its frequency in the parenthesis (Zybartas et al 2001).

What should be taught	in primary school class 1-4			in basic school class 5-10			in secondary school cl. 11-12			in teacher education		
Only modern math	3,6	4	5 (8)	3,9	4	3 (11)	3,9	4	4 (11)	3,5	4	4 (8)
Only postmodern math	2,3	2	1 (9)	2,7	3	1,3 (8)	3,0	3	3 (13)	3,8	4	3,4,5 (7)
Both	3,1	3	5 (8)	3,6	4	3,4 (8)	4,2	4	4 (13)	5,4	6	6 (17)

Bundling leads to Total stories as T = 3*4 & 2*1	3,8	4	4 (15)
Describing parts of bundles leads to fractions as T = 2/3*3	4,0	4	4 (13)
Describing parts of bundles leads to decimals as T = 2/10*10 = 0.2*10	3,9	4	4 (12)
Rebundling leads to the rebundle-rule T = T/b*b	4,3	5	5 (14)
Coding and decoding can lead to solving an equation as 300 = 2*a	4,5	5	5 (17)
Restacking leads to the restacking-rule T = T – b + b	4,4	5	5 (14)
Writing bills leads to coding and decoding and solving the equation T = b + a*n	4,4	4	4 (16)

Teaching both modern and postmodern mathematics got a partial positive reaction as to secondary school and a very positive reaction as to teacher education. Concerning the specific topics rebundling, restacking, coding & decoding equations got a positive reaction.

The positive responses lead to the design of teaching and learning material in postmodern sceptical LAB-mathematics for free teacher PYRAMID**e**DUCATION at www.MATH**e**CADEMY.net (Tarp 2003).

Conclusion

To solve the relevance paradox in mathematics education this paper used institutional scepticism to find a hidden alternative to modern set-based LIB-mathematics valid only in the library and not in the laboratory. Inspired by Kronecker's and Russell's scepticism towards the set-concept a set-free fraction-free postmodern sceptical LAB-mathematics was uncovered being based upon quantity as it exists in repetition in time and multiplicity in space. Being quite different from set-based mathematics it allows fundamental mathematics as per-numbers, rebundling and restacking to be introduced the first year of school and even with one digit numbers only. Furthermore when tested in the classroom it shows a promising potential for becoming a Cinderella-difference making a difference by solving the relevance problem of mathematics education.

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104. Deconstructing Modern Top-Down Algebra into Postmodern Bottom-Up Algebra

It is a postmodern point, that a phrasing constructs what it describes and that ruling phrasings and discourses clientifies humans.[1] *Inspired by this we could ask: Is it possible to redescribe*[2] *and deconstruct mathematics? "Geometry" means "earth measuring" - but what does "Algebra" mean? The dictionary tells us that "Algebra" means "reunite". Since "low attainers" might be deconstructed into "authenticity searchers" we could also ask: what will happen if we present authenticity searchers for authenticity by inviting them to join the social practice of reuniting that created Algebra? This paper is a story about what happened in such classes.*

Background

After nine years of compulsory education some Danish students continue directly into three years of high school. Others leave school but might return later for a two years Higher Preparation exam (HP). Such HP classes normally have many "low attainers" in mathematics. In both cases a scanning of math competencies finds lacking knowledge of fractions and algebra, thus creating a tradition for a 20 hours introductory course on fractions and algebra. Especially in the HP classes this course has little positive effect just presenting students eager for a new start with old defeats.

The aim of the present study was to create an alternative micro curriculum for a 20 hours introductory course in a HP class covering both the fundamentals and the core of the curriculum, i.e. calculations and linear and exponential change. The micro curriculum was described in the teacher's journal asking for volunteers to try it out in their own classes. Two teachers offered themselves and they both tried it out in different classes over a three-year period.

Methodology - Modern and Postmodern Research

Among several other things[3] postmodernity could also mean accepting contingency, accepting that things could also be otherwise. Out of the breakdown of premodern order, modernity saw the emergence of contingency. Scared by the idea of a contingent world modernity desperately began to reinstate order.[4] Where modernity means hiding contingency, postmodernity means accepting contingency. Modern and postmodern research are both working in the borderland between nature and culture, between necessity and contingency, between what is given and what could also be otherwise trying respectively to present contingency as hidden necessity and necessity as hidden contingency.

Postmodernity realises that although the world may be able to speak in numbers through measuring instruments it is unable to speak in words since no word-measuring instrument exists: words are contingent human constructs making all phrasings contingent except this one; we are not grasping but being grasped by what we grasp, we are being conceited by our concepts. Postmodernity is always looking for alternative phrasings, redescriptions and deconstructions to become de-clientified from ruling phrasings and discourses. A postmodern study thus will try to uncover hidden contingency in traditions, use inspiration and imagination to design an alternative, and report the effects of practising this alternative.

Uncovering Hidden Alternatives in a Contingent Tradition

Within mathematics the prevailing tradition is an "example-of" tradition using a Platonic Top-Down understanding defining a concept as an *example of* a more abstract concept: Linear and exponential change are *examples of* functions, that are *examples of* relations, that are *examples of*

[1] See e.g. Foucault 1972
[2] See Rorty 1989
[3] See Bertens 1995
[4] See Bauman 1992

products, that are *examples of* sets. In a Top-Down tradition it seems self-evident that functions has to be taught (and learned, hopefully) before linear and exponential change can be introduced. This tradition creates learning and teaching problems for many students and teachers.[5]

A hidden alternative to the Platonic Top-Down understanding is a nominalistic Bottom-Up understanding where a new concept is defined as a *name for* a common property of less abstract examples: "A function is a *name for* a calculation with variable quantities", or in Euler's own words:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.[6]

In Arabic the word "Algebra" means reunite. If we buy five items in a store we don't have to pay all the single prices, we can ask for them to be united into a total. If the total is 17 $ we are allowed to pay e.g. 20 $. This new total is then split in the price and the change. So living in a money based culture means being engaged in a "social practice of totalling" consisting of uniting and splitting totals. The operations "+" and "·" unite variable and constant unit-numbers; "∫" and "^" unite variable and constant per-numbers. The inverse operations "–" and "/" split a total into variable and constant unit-numbers; "d/dx" and "√ and log" split a total into variable and constant per-numbers:

Totals unite/split into	variable	constant
unit-numbers $, m, s, ...	$T = a+n$ $T–n = a$	$T = a\cdot n$ $T/n = a$
per-numbers $/m, m/100m=%, ...	$\Delta T = \int f dx$ $dT/dx = f$	$T = a$^n $\sqrt[n]{T} = a$ $\log_a T = n$

From this an alternative can be designed rephrasing and deconstructing linear and exponential functions into constant-change-stories emerging from questions as: "100 $ plus n days at 5 $/day total ? $" and "100 $ plus n days at 5 %/day total ? $"

Likewise differential and integral calculus could emerge as variable-change-stories from questions as: "100 $ plus n times at (10 %/n)/time total ? $" and "100 m plus 5 seconds at 3 m/sec increasing to 4 m/sec total ? m".[7]

An Alternative Micro-Curriculum

Inspired by the above Bottom-Up understanding of linear- and exponential change as constant-change-stories and by several student comments over the years a micro curriculum was designed and tested:

• A change is linear if it is constant: +5$, +5$, +5$. Linear change is also called "+change"

• A change is exponential if it is constant in percent: +5%, +5%, +5%, i.e. if the change-multiplier is constant: ·1.05, ·1.05, ·1.05. Exponential change is also called "·change" or interest change.

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[5] See Tarp 1998
[6] Euler 1988 p. 3
[7] See Tarp 2000

Linear change *Exponential change*

b+a·n = **T** **= b·a^n** *a = 1+r*

	+a ·a		T: final value
	etc		+a: change per time
+a$ n times	+a ·a	*+r% n times*	·a: multiplier per time
	+a ·a		+r: %-change per time
	b		n: number of changes
			b: initial value

Top-Down Calculations

The Top-Down tradition writes a calculation as e.g. “3+5=8”. An equation is written as e.g. “3+x=8”. “3, 5 and 8” are considered examples of a number set e.g. the set of rational numbers Q. “x” is an example of a variable. ”=“ is an example of an equivalence relation on Q, i.e. a subset of the set product QxQ with certain properties. “+” is an example of a function from the set product QxQ to Q, i.e. a uni-valued relation on Q. “x+3=8” is considered an example of a propositional form. To solve an equation means to determine the truth set of the equation, i.e. the set of numbers that makes the propositional form a true proposition. Solving takes place through identical operations on both sides of the equal sign keeping the truth set unchanged.

Since Q is a group under addition, it has a neutral element “0“ and all non-zero elements a have an inverse element (–a).

3+x = 8	a propositional form
(3+x)+(-3) = 8+(-3)	(-3), the inverse element of 3, is added to both sides
(x+3)+(-3) = 8-3	addition is commutative
x+(3+(-3)) = 5	addition is associative
x+0 = 5	definition of an inverse element
x = 5	definition of a neutral element

The statement x=5 has the truth set {5}. Since the two statements “x+3=8” and “x=5” are logically equivalent they have the same truth set. Hence the solution of the equation 3+x=8 is
{xεQ| 3+x = 8} = {5}.

Bottom-Up Calculations

In a Bottom-UP tradition “3+5=8” is strictly speaking a syntax error violating Russell’s type theory. “3+5” is a calculation and “8” is a number i.e. a different type, hence they cannot be identical. To be meaningful the statement “3+5=8” should be interpreted as “3+5->8” or “(3+5)=8”. “3+5->8” means ”the calculation 3+5 gives 8”. “(3+5)=8” means ”the result of the calculation 3+5 is 8”.

An equation “3+x=8” is considered an example of a question “3+?=8” to be answered by guessing or by reversing the calculation introducing “–” as the inverse calculation to “+” defined as: “8–3 is the number that added to 3 gives 8”. In this way the answer can be *calculated* as x = 8–3 = 5. Solving equations with other operations leads to the introduction of the other reverse operations all obeying the “move&reverse” method: You can move a number to the other side by reversing its calculation sign.

3 + x -> 8	3 + x = 8	3 · x = 15	x^3 = 125	3^ x = 125
x <- 8–3	x = 8–3	x = 15/3	x = $\sqrt[3]{125}$	x = $\log_3 125$

Alternatively an equation can be considered a “calculation story” telling both what is calculated and how. T is calculated from a by first multiplying a by n, and then adding b. If the calculation is reversed a can be calculated from T. We see that using reversed calculations is the same as using the move&reverse method.

Calculation direction:

Forward		*Reverse*
b+(a·n)	=	T
	+b ↑ ↓ –b	
a·n	=	T–b
	·n ↑ ↓ /n	
a	=	(T–b)/n

"Hidden parentheses" reduces multiple calculations to a single calculation: 2+3·4 = 2+(3·4). This is useful e.g. if we want to find "a" in the equation T = b+a·n.

T = b+(a·n)	The multiple calculation "b+a·n" is reduced to a single calculation by means of a hidden parenthesis
T–b = a·n	b is moved from the forward calculation side to the reverse calculation side by reversing its calculation sign
(T–b)/n = a	n is moved from the forward calculation side to the reverse calculation side by reversing its calculation sign. Since the left side already contains one calculation this has to be performed before the arrival of n. Thus this calculation needs a parenthesis.

Calculation Tables

A calculation table used to report calculations has two columns, a number column containing number equations, and a calculation column containing calculation equations. The two columns are divided into two parts splitting the calculation table into four sections. Top-left shows the quantity to be calculated and top-right shows the equation to do the job. Bottom-left shows the values to be used in the calculation, and bottom-right shows the processing or solving of the equation. The two calculations below answer the two questions:

"80$ plus 4 days at ? $/day total 100$" "80$ plus 4 days at ? %/day total 100$"

a = ?	**T = b + (a·n)**
T = 100	T–b = a·n
b = 80	(T–b)/n = a
n = 4	(100–80)/4 = a
	5 = a

r = ?	**T = b · (a^n)**
T = 100	T/b = a^n
b = 80	n√(T/b) = a
n = 4	4√(100/80) = a
a = 1+r	1.057 = a = 1+r
	0.057 = **r = 5.7%**

Letter calculation before inserting numbers

In the calculation tables the students were asked to do letter calculation before inserting numbers to prevent to many miscalculations and miswritings of long numbers. The students were reluctant to this in the beginning but most students accepted it at the end.

Routine Building in the Classroom

Having introduced a new field in class traditional teaching asks the students to do some problems at home before proceeding to the next field. In this way the whole curriculum is covered with equal thoroughness according to the 50-50 principle: 50% of the fields takes 50% of the time.

An alternative is a 30-70 principle in which the core 30% is given 70% of the time to allow the students to build up routines in the classroom. The 30-70 principle accepts that the students have three brains to be nourished, a human brain for understandings, a mammal brain for feelings and a reptile brain for routines.

Emphasising routines might make the students feel they learn by hart without getting any understanding. So a question regarding this concern was included in the final questionnaire together with nine other questions.

Bringing the Class to the Same Level

The classes were interviewed in groups of 2-3 asked to tell their opinion about the introductory course. Many students were happy about the effort to bring students with different backgrounds to the same initial level. Hence a question about this was included in the questionnaire together with a question about the format of the learning material not being a traditional textbook but rather a concentrated compendium not exceeding 10 pages.

The Evaluation

The students were asked to grade their answers on a scale from -3 to 3 covering seven categories: really bad, bad, somewhat bad, neutral, somewhat good, good and really good. The table shows the class averages from five classes totalling 83 students.

1.	Calculation table	1.9
2.	Solving by moving instead of "doing the same on both sides"	2.0
3.	Letter calculations before inserting numbers	2.0
4.	Routine building through many like problems	1.4
5.	Class time used for routine building	2.0
6.	Did you obtain routine?	1.8
7.	Understanding instead of rote learning?	1.6
8.	Short compendium instead of a traditional textbook	1.5
9.	Bringing the class to the same initial level	2.4
10.	Total evaluation of the course	2.1

From the relative success of the Bottom-Up understanding of linear and exponential change two questions arise: Why is a Bottom-Up understanding user-friendlier? Why is a Bottom-Up understanding unrecognised?

Why Is Bottom-Up Algebra User-Friendlier?

According to the British sociologist Anthony Giddens routinisation is the cornerstone of a traditional society.[8] By echoing routines actors build up tacit practical consciousness enabling them to become participants in the practice and providing them with a basic ontological security and an identity. Actors also possess a discursive consciousness by which they can verbalise and reflect upon the world.

Today's society however is a globalised post-traditional society.[9] It is especially the development of the information and communication technology that has made global communication possible e.g. through the multichannel, satellite and Internet connected television. This has made visible alternatives to existing traditions making these appear contingent and ambiguous. With the loss of an external identity to echo, identity becomes self-identity, a reflexive project, where the individual actor has to create his own biographical narrative or self-story looking for authenticity and shunning meaninglessness.[10]

Modern traditional students might opt for gaining access to the social practise by becoming "echo-learners" and later "echo-teachers". But with postmodern post-traditional students it is different. They are engaged in building their own biographical narratives looking for meaning. By referring

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[8] See Giddens 1984
[9] See Giddens in Beck et al. 1994
[10] Giddens 1991 p. 5

upwards a Top-Down sentence ("a function is an example of (or child of) a relation") can give only one answer thus creating "echo teaching". And by referring upwards Top-Down sentences become "unknown-unknown" relations that cannot be anchored to the students' existing learning narrative. They become meaningless producing "echo-resistance" and become only accessible through "echo-learning". Since a Bottom-Up sentence ("a function is a name for a calculation with variable quantities") is an "unknown-known" relation it can be anchored to the students' existing narrative thus extending this. The students have learned in the sense that they are "able to tell something new about something they already knew". This definition of learning could be called postmodern Bottom-Up learning.

So it is through their ability/inability to extend the students existing narrative Bottom-Up/ Top-Down algebra becomes user friendly/hostile in a post-traditional society.

Why Is Bottom-Up Algebra Unrecognised?

It is a postmodern point, that a phrasing constructs what it describes and that ruling phrasings and discourses clientifies humans.[11] Inspired by this we could ask: Are the actors (students and teachers) and the system clientified, caught and frozen in a "mathematics" discourse forcing them to subscribe to the "mathematics before mathematics application" conviction?

A rephrasing of this ruling discourse could reveal an alternative silenced "word/ number language" discourse in which grammar/mathematics becomes the meta-language of the word/number language.[12] This deconstruction turns the Top-Down "Mathematics before application" conviction upside-down to the opposite Bottom-Up "Language before grammar" conviction. Through its captivity modern mathematics implements a "grammar before language" practice, which would create a revolution if practised within the word language.

Conclusion

This postmodern study has tried to uncover hidden contingency within the modern algebra tradition and reported the consequences of implementing an alternative postmodern algebra. Being postmodern the aim of the study is not to convince about a new and better way to teach algebra but to inspire others to look for hidden contingency in their own traditions and to maybe try to change them. However the study has a morale: What we phrase as "low attaining students" might be fellow value searchers in disguise. If we kiss them with postmodern algebra they might transform into beautiful princesses and princes wanting to inherit our beautiful subject, algebra, instead of evading math-based educations as science and technology thus creating public concern for future productivity and welfare.[13]

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[11] See e.g. Foucault 1972
[12] See Tarp 1998
[13] See Jensen et al. 1998

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105. Per-Number Calculus

A Postmodern Sceptical Fairy tale Study

To solve the relevance paradox in mathematics education this paper uses postmodern sceptical fairy tale research to look for new ways to teach calculus in the school. A renaming of 'calculus' to 'adding per-numbers' allows us to think differently about the reality 'sleeping' behind our words, and all of a sudden we see a different calculus taking place both in elementary school, middle school and high school. Being a 'Cinderella-difference' by making a difference when tested this postmodern calculus offers to the classroom an alternative to the thorns of traditional calculus.

How To Find A Hidden User-Friendly Calculus

The background of this study is the worldwide enrolment problem in mathematical based educations and teacher education (Jensen et al 1998). And 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371). In mathematics education 'calculus is difficult' is a widespread observation, and many different efforts have been made to penetrate the hawthorn of calculus with limited success (see e.g. Steen 1986).

This paper reports on a radical different approach using a postmodern 'sceptical fairy tale research' seeing humans as bewitched by words. This seduction can be counteracted by counter-seduction renaming 'echo-words' by asking 'Sleeping Beauty questions' like:

Calculus did not call itself calculus, so calculus can also be called something else? Mathematics did not name itself, so mathematics can also be named differently?

Counter-seduction is based upon a simple observation, the 'number&word-paradox': Placed between a ruler and a dictionary a thing can point to a number, but not to a word; thus a thing can falsify a number-statement about its length in the laboratory, but not a word-statement about its name in the library; i.e. numbers carry reliable information inducing and validating research; words carry debate i.e. interpretations, which if presented as research become seduction, to be met with counter-seduction searching for hidden 'Cinderella-differences' making a difference (Tarp 2003).

A Different Name For Calculus

According to the number&word-paradox calculus did not call itself 'calculus', so calculus could also be called something else; and this different name might be an eye-opener to a different approach. So let us free ourselves from the seduction of the library and go to the laboratory to see what kind of questions calculus is dealing with and arose from.

The historical library said 'the moon moves among the stars'. Newton freed himself from this seduction with a counter-formulation saying that the moon falls to the earth just as an apple, both drawn by the same will to change, the same force called gravity.

Unlike the unpredictable will to rule of the metaphysical Lord this physical will to change was predictable through calculations, once the art of change-calculations was developed and given a name, differential and integral calculus.

Newton used multiplication to solve the velocity-problem '5 sec *at* 2 m/s totals ? m': T = 5*2 = 10 m. But gravity changes the velocity thus changing the problem to e.g. '5 sec *at* 2 m/s increasing to 4 m/s totals ? m', which cannot be solved by a simple multiplication. Thus Newton was faced with the following problem: We know how to add constant per-numbers, how do we add variable per-numbers?

So the lesson from the laboratory is that calculus arose from and is dealing with addition of variable per-numbers. Hence ‘calculus’ can be renamed to ‘addition of per-numbers’. This renaming allows us to think differently by translating the question ‘how to teach calculus’ to ‘how to teach addition of per-numbers’.

Mathematics And Mathematism

If we say that research is library-statements induced from and validated by reliable laboratory data then we can distinguish between ‘mathematics’ being validated in the laboratory and ’mathematism’ being validated only in the library.

An example of mathematics is the statement that T = 2*3 = 6 claiming that 2 3s can be recounted as 6 1s. This claim is easily verified in the lab: *** *** -> * * * * * *.

An example of mathematism is the statement that T = 2+3 = 5, a claim that has countless counter-examples in the lab:

T = 2 m + 3 cm = 203 cm, T = 2 weeks + 3 days = 17 days, T = 2 tens + 3 ones = 23

That seduction by mathematism is costly is witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. The units must be alike and put outside a parenthesis to give a valid addition inside. Thus 20% + 10% = 30% only if taken of the same total. Else the ‘fraction-paradox’ applies:

Inside the classroom	20/100 + = 20% +	10/10 = = 10% =	30/100 = 30%
Outside the classroom	20% +	10% = or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

Brought to the classroom mathematism introduces ‘killer-mathematics’ only valid inside the classroom where it is applied to kill the relevance of mathematics.

Another example of mathematism and killer-mathematics is the modern function concept: If ‘f(x) = x+2’ means ‘let f(x) be a place-holder for the calculation ‘x+2’ having x as a variable number’ then f(3) = 5 means that 5 is a calculation with 3 as a variable number, which is a double syntax error (Tarp 2002).

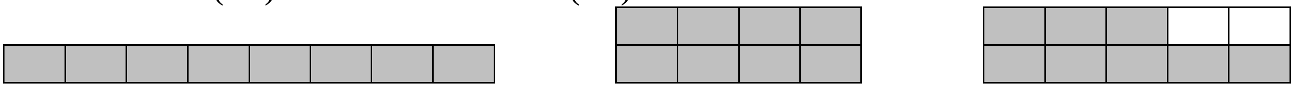
So to fulfil the social and ethical obligation to deliver lab-mathematics and not lib-mathematism, a different seduction-free mathematics has to be created, not from the library, but from the laboratory. One example of such a ‘deconstructed’ mathematics is a Kronecker-Russell multiplicity-based mathematics as is seen in the appendix.

Lab-Based Mathematics

In the laboratory multiplicity is organised through counting taking away bundles of e.g. 4s to produce a stack of e.g. T = 2 4s = 2*4. The process ‘from T take away 4’ can be iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4’, which is a called a ‘per-number’ counting how many times we have a 4. So numbers are never abstract entities but parts of a stack, as a height, 2, or as a base, 4.

Thus the ‘recount-equation’ T = (T/4)*4 is a prediction of the result of recounting T in 4s to be tested by performing the recounting T = 1*8 = ?*4, or T = ?*5.

T = 1*8 = (8/4)*4 = 2*4 = (8/5)*5 = 1 3/5 * 5



Counting in tens per-numbers or ‘recounters’ become decimals and percentages:

T = 32*1 = (32/10)*10 = 3 2/10*10 = 3.2*10 = (32/100)*100 = 0.32*100 = 32%*100

Per-numbers are like numerators, not of the abstract fractions of the fraction-paradox, but of ‘fractions-of’ always being multiplied with their base before they are added.

Also with units per-numbers are recounters recounting one unit by another unit:

40 $ per 2 kg = 40 per 2 $ per kg = 40/2 $/kg = 20 $/kg = rate.

Thus the question ‘5 kg = ? $’ is answered by recounting the number (5 kg = (5/2)*2 kg = (5/2)*40 $ = 100 $); or by recounting the unit ($ = ($/kg)*kg = 20*5 = 100).

Thus the question ‘how to add per-numbers’ occurs in three different forms:

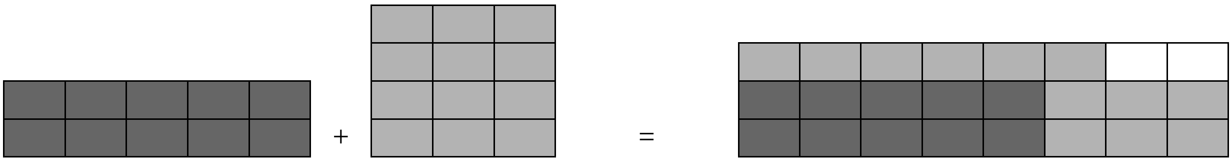
Primary school: T = 2 5s + 4 3s = ? 8s

Middle school: T = 5 kg *at* 2 $/kg + 3 kg *at* 4 $/kg = 8 kg *at* ? $/kg

Secondary school: T = 5 sec *at* 2 m/s increasing to 4 m/s = 5 sec *at* ? m/s

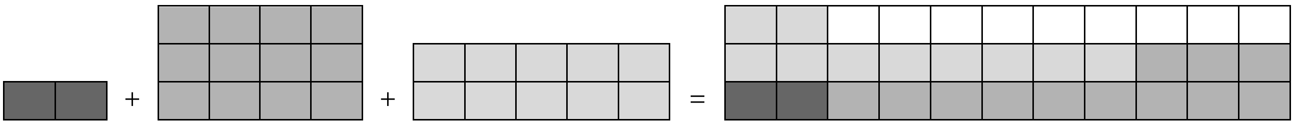
Calculus In Primary School

Adding two stacks as e.g. 2 5s and 4 3s is done by a recounting predicted by a recount-equation T = 2 5s + 4 3s = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8



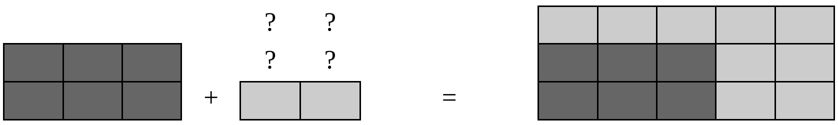
Repeated addition of stacks is later called integration:

T = 1 2s + 3 4s + 2 5s = 1*2 + 3*4 + 2*5 = (1*2 + 3*4 + 2*5)/11*11 = 2 2/11*11

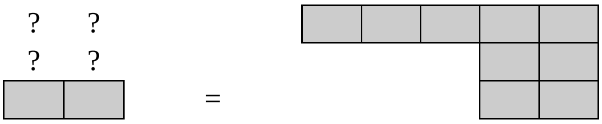


So the lesson from the primary school count&add-laboratory is that per-numbers can be added by adding the stacks in which they are the heights.

The addition process can be reversed by asking 2 3s+? 2s = 3 5s:



The answer can be obtained by removing the 2 3s from the 3 5s and then count the remaining 9 in 2s as (9/2)*2 = 4 1/2 * 2. Thus ? = 4 1/2



Or the answer can be predicated by a reversed calculation, later called differentiation. In this way solving equations becomes another name for a reversed calculations.

2 3s	+ ? 2s	=	3 5s	The question
2*3	+ x*2	=	3*5	The equation
	x*2	=	3*5-2*3 = 9	The 2 3s are removed from the 3 5s leaving 9
	x*2	=	(9/2)*2 = 4 1/2 * 2	The 9 is recounted as 2s
	x	=	4 1/2	The answer

Per-Numbers In Middle School

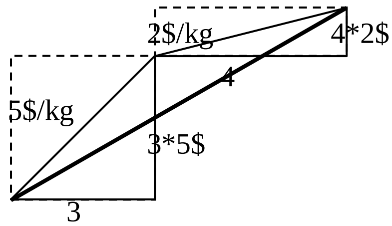
In middle school per-numbers occur in Renaissance trade as price-numbers 4 $/kg or rent-numbers 4 $/day, i.e. as rates counting the number of $ per kg or day. Again the per-number is the height of a stack, or the slope of the diagonal in a change-triangle.

And again per-numbers are added as stacks: The $/kg-number b is multiplied with the day-number n before being added to the total $-number T = T1 + n*b.

T = T1 + T2 = 3 kg at 5 $/ kg + 4 kg at 2 $/ kg = (3+4) kg at (∑n*b)/(3+4) $/ kg

Adding per-numbers can take place in tables, or by adding slopes in triangles:

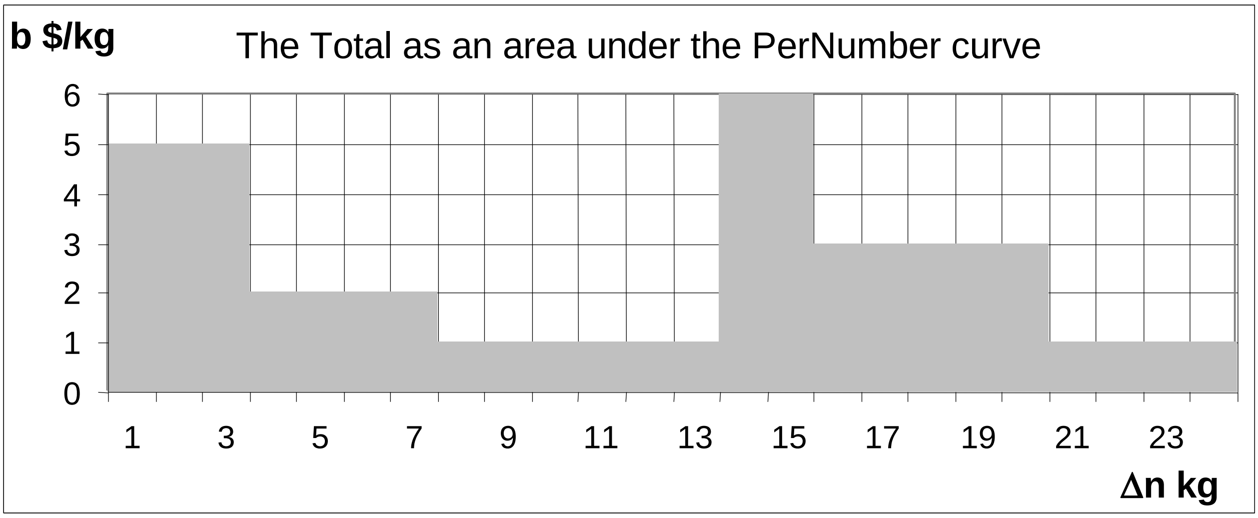
n kg at	b $/kg =	n*b $ = T
3 kg at 4 kg at 7 kg at	5 $/kg = 2 $/kg = x $/kg =	3 * 5 = 15 $ 4 * 2 = 8 $ 7 * x = ∑a*b= 23= (23/7)*7 $ x = 23/7= 3 2/7 $



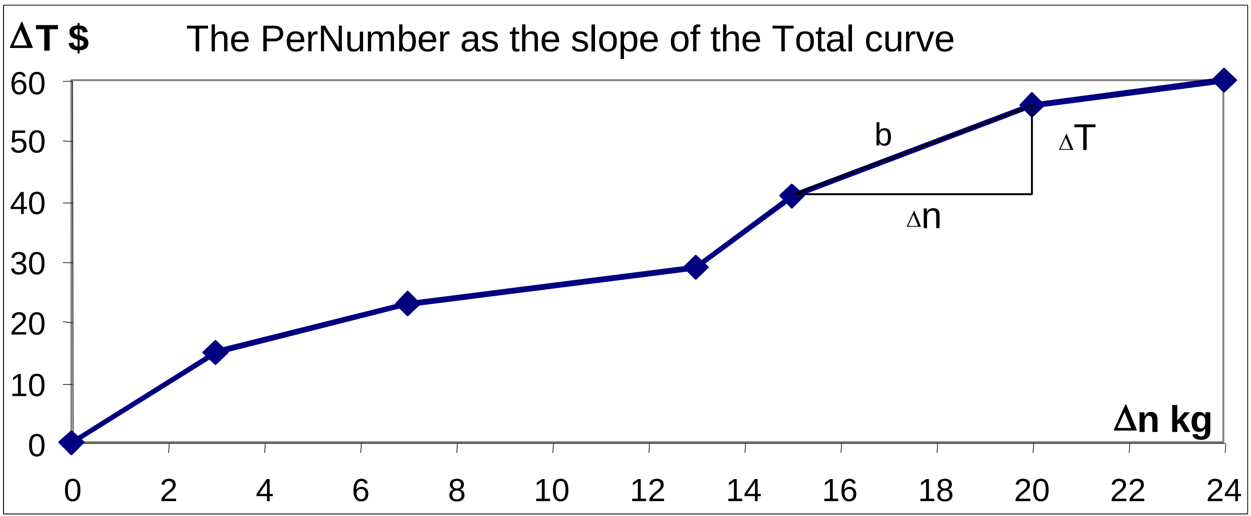
The table can be supplemented with two columns showing the added values of the kg-number Δn, of the $-number ΔT, and of the per-number Σ b occurring when re-counting or differentiating ΔT in Δns: ΔT = (ΔT/Δn) * Δn = b * Δn; as in this example where a teashop is adding different amounts with different prices to create a blending.

Δn kg	b $/kg	Δn*b = ΔT		Σ Δn = Δn	Σ ΔT = ΔT	Σ b $/kg = ΔT/Δn
3 kg at	5$/kg =	3 * 5 = 15		3	15$	15/3 = 5.0
4 kg at	2$/kg =	4 * 2 = 8		7	23$	23/7 = 3.3
6 kg at	1$/kg =	6 * 1 = 6		13	29$	29/13 = 2.2
2 kg at	6$/kg =	2 * 6 = 12		15	41$	41/15 = 2.7
5 kg at	3$/kg =	5 * 3 = 15		20	56$	56/20 = 2.8
4 kg at	1$/kg =	4 * 1 = 4		24	60$	60/24 = 2.5

When plotting the per-number b $/kg against Δn kg in a coordinate system the total $-number is the sum of the stacks, i.e. the area under the per-number curve.



When plotting ΔT against Δn in a coordinate system the curve shows both the added kg-number Δn, the added total ΔT, and the single per-numbers b = ΔT/Δn as the slopes.



Thus from blending tea in a shop we learn that:

The Total is the area under the per-number curve predicted by an integration formula: ΔT = ∑ $/kg*kg = ∑ b*Δn adding the per-number stacks.

The per-number is the slope of the Total curve predicted by a differentiation formula: b = Δ$/Δkg = ΔT/Δn recounting the ΔT in Δns.

Adding stacks implies that the initial stack is changed by 3 new stacks:

Δ(b*a) = Δb*a+b*Δa+Δb*Δa ≈ Δb*a + b*Δa neglecting the small upper right stack.

Δb	Δb*a	Δb*Δa
b	b*a	b*Δa
	a	Δa

To get the percent-change we divide by the initial number b*a:

$$\frac{\Delta(b*a)}{b*a} = \frac{\Delta b*a}{b*a} + \frac{b*\Delta a}{b*a} + \frac{\Delta b*\Delta a}{b*a}, \text{ or } \frac{\Delta(b*a)}{b*a} = \frac{\Delta b}{b} + \frac{\Delta a}{a} + \frac{\Delta b}{b} * \frac{\Delta a}{a} \approx \frac{\Delta b}{b} + \frac{\Delta a}{a}$$

Thus Δ%(b*a) ≈ Δ%b + Δ%a.

This change-formula is widely used in social science: If the total production increases by 2.5% and the population increases by 3.2% then the income per capita decreases by 3.2% - 2.5% = 0.7%.

That repeated addition and reversed addition of per-numbers leads to integration and differentiation follows from the formula for adding per-numbers as stacks:

T2	= T1 + a*b	T2	= T1 + a*b
T2 - T1	= + a*b	T2 - T1	= + a*b
ΔT	= + a*b	ΔT/a	= b
ΔT	= ∑ a*b	ΔT/Δa	= b
ΔT	= ∫ y*dx	dy/dx	= b

Until now the variation of the per-numbers has been arbitrary. It can however also be predictable:

In a trade the following discount is offered: The prise is 10$/kg for the first kg. Then the price is reduced with 0.5$ for each extra kg until 11 kg, after which the price stays constant. The total cost for 11 kg then can be calculated by the sum

S = 10 + 9.5 + 9 + 8.5 + ... + 5 = ?

This sum can be calculated by a spreadsheet or it can be predicted through a calculation leading to the study of arithmetic growth giving the result

S = n*(an + a1)/2 = 11*(5+10)/2 = 82.5

In another trade the following discount is offered: The prise is 10$/kg for the first kg. Then the price is reduced with 10% for each extra kg until 11 kg, after which the price stays constant. The total cost for 11 kg then can be calculated by the sum

S = 10 + 10*0.9 + 10*0.9^2 + 10*0.9^3 + ... + 10*0.9^10 = ?

This sum can be calculated by a spreadsheet or it can be predicted through a calculation leading to the study of geometric growth giving the result

S = 10*(1-a^n)/(1-a) = 10*(1-0.9^11)/(1-0.9) = 68.6

As in elementary school reversing addition of per-numbers leads to equations that now are sat up in a more formal way in calculation tables

4 kg at 3 $/kg+ 5 kg ? $/kg = 18 $, or 4*3 + 5*x = 18

x = ?	4*3 + 5*x	= 18	The problem
	(4*3) + (5*x)	= 18	Add the invisible parenthesis
	12 + (5*x)	= 18 = (18-12) + 12	Calculate and restack 18
	5*x	= 6 = (6/5)*5	Calculate and recount 6
	x	= 6/5	The solution as a calculation
	x	= 1.2	The solution as a calculated number
Test:	*4*3 + 5*1.2*	*= 18*	Insert the solution
	18	*= 18*	Calculate and check

Per-Numbers In High School

In high school per-numbers occur in Enlightenment physics as e.g. a velocity of 2 meter/second. If the per-number is constant again it can be represented as a stack with the per-number as the height, or as the slope of the Total curve.

However in many cases the per-number is not constant. Thus in the case of falling apples and planets the per-number is increasing continuously. If it had been constant then we could have used our knowledge about per-numbers from middle school. So the question arises: Can a continuous change be regarded as constant?

To answer this question we can set up a formal definition of constancy by saying that a variable y is equal to a constant c if the numerical difference between y and c is less than a number ε for all positive numbers ε.

This definition of global constancy is recycled in the definition of piecewise constancy; and in the definition of local constancy, which is called continuity:

A variable y is globally constant c if	$\forall\varepsilon>0$:	$\lvert y-c\rvert < \varepsilon$ in all intervals
A variable y is piecewise constant c if	$\exists\delta>0\ \forall\varepsilon>0$:	$\lvert y-c\rvert < \varepsilon$ in the interval δ
A variable y is locally constant c if	$\forall\varepsilon>0\ \exists\delta>0$:	$\lvert y-c\rvert < \varepsilon$ in the interval δ

So our knowledge about constant per-numbers still applies: Per-numbers are added by the area under the per-number curve.

Since a smooth curve is locally constant its area ΔA can be approximated by a sum of many stacks: $\Delta A \approx \sum b*\Delta n$. b is a per-number $b = \Delta T/\Delta n$, so $b*\Delta n = \Delta T$, which gives

$$\Delta A \approx \sum b*\Delta n = \sum \Delta T = \Delta T = T2 - T1$$

since a sum of single changes = the total change = terminal-number – initial-number

This relation does not depend on the size of the change, so it is also valid for small local changes where we use the symbols '∫' and 'd' for '∑' and 'Δ':

$$\Delta A = \sum b*\Delta n = \sum \Delta T = \Delta T = T2 - T1 \quad \text{->} \quad \Delta A = \int b*dn = \int dT = \Delta T = T2 - T1.$$

Thus we can substitute the change-symbol in our middle school change-formula:

$$\frac{\Delta(f*g)}{f*g} \approx \frac{\Delta f}{f} + \frac{\Delta g}{g} \quad \text{->} \quad \frac{d(f*g)}{f*g} = \frac{df}{f} + \frac{dg}{g}$$

This formula applies to powers of x:

$T = f*g = x*x = x\text{^}2$ $\frac{dT}{T} = \frac{dx}{x} + \frac{dx}{x} = 2*\frac{dx}{x}$ $\frac{dT}{dx} = 2*\frac{T}{x} = 2*\frac{x\text{^}2}{x} = 2*x$	$T = x*x*x*x*x = x\text{^}5$ $\frac{dT}{T} = \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} = 5*\frac{dx}{x}$ $\frac{dT}{dx} = 5*\frac{T}{x} = 5*\frac{x\text{^}5}{x} = 5*x\text{^}(5-1)$

Substituting f *g = h we get f = h/g:

$$\frac{dh}{h} = \frac{d(h/g)}{h/g} + \frac{dg}{g} \quad \text{giving} \quad \frac{d(h/g)}{h/g} = \frac{dh}{h} - \frac{dg}{g}$$

Now we are able to predict the result of adding variable per-numbers through a calculation as in the classical case of a falling body:

$$\text{5 sec. at 2m/sec increasing to 4 m/sec total} = \int_0^5 (2 + \frac{4-2}{5} x)\, dx = \int_0^5 (2 + 0.4x)\, dx = ?\ \text{m}$$

Since d/dx (2x + 0.2x^2) = 2 + 0.4x we get that d(2x + 0.2x^2) = (2 + 0.4x) dx, so

$$\int_0^5 (2 + 0.4x)\, dx = \int_0^5 d(2x + 0.2x\text{^}2) = \Delta_0^5\ (2x + 0.2x\text{^}2) = (2\cdot 5 + 0.2\cdot 5\text{^}2) – 0 = 15\ \text{m}$$

Thus in the case of a constant acceleration g the velocity v and position s after t seconds can be predicted by the calculations

v = vo + t seconds at g (m/s)/s = vo + g*t

$$s = so + \text{t seconds at v m/s} = so + \int_0^t v\, dt = so + \int_0^t (vo+g*t)\, dt = so + vo*t+½*g*t\text{^}2$$

At the surface of the earth the acceleration from gravitation is locally constant, but further out in space it decreases. So let us return to Newton's laboratory to see how he was able to perform the historical calculation that predicted how the moon and the planets fall under the will of gravity. The social impact of this calculation cannot be overestimated. By moving the authority from the scriptures in the library to the rulers in the laboratory this calculation created a standard for a modern lab-based research induced from and validated in the laboratory; which in turn created the Enlightenment scepticism that changed of our world from the autocracy of two superior Lords to a democracy building upon self-government and science.

First Brahe used a lifetime to collect data about the motion of the planets. Then Kepler used these data to conclude that the planets are moving in ellipses with the sun in the focus; and in such a way that the ratio between the cube of the average radius r and the square of the period T is a constant, T^2= c*r^3. This makes the acceleration proportional to 1/r^2 since v= (2*π*r)/T:

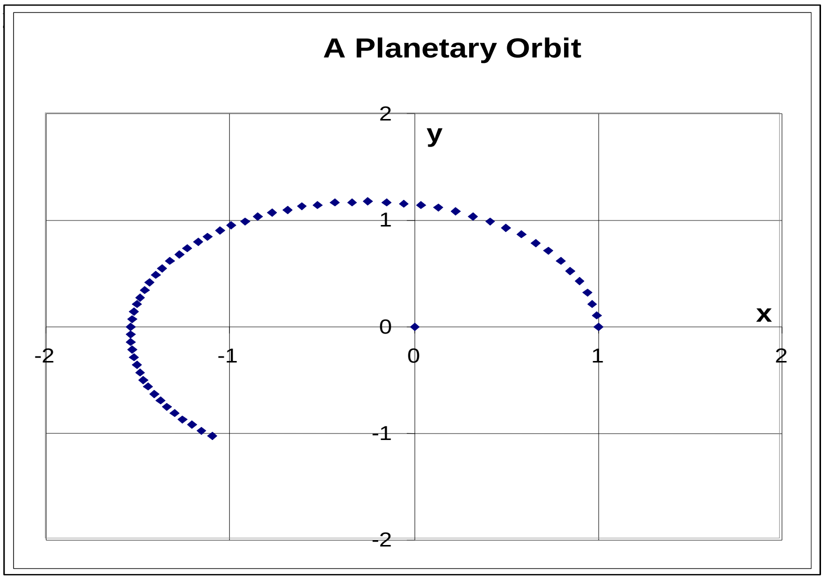
a = (2*π*v)/T = (2*π/T)*(2*π/T)*r = (2*π)^2*(1/T)^2*r = (4*π^2/c)*(1/r^2)

The library said that the force is proportional to the velocity. Newton neglected this and chose instead to test the counter-hypothesis that the force is proportional to the *change* of velocity, i.e. to the acceleration, i.e. to 1/r^2. To test the predictions from this hypothesis we set up a system of 8 quantities with their corresponding change-equations:

	Initial numbers		Change-numbers		Terminal numbers	
Acceleration	a =	1			a =	1/r^2
Horizontal acceleration	ax =	1			ax =	-x/r^3
Vertical acceleration	ay =	0			ay =	-y/r^3
Horizontal velocity	vx =	0	Δ vx =	ax*Dt	vx =	vx + Δ vx
Vertical velocity	vy =	1.1	Δ vy =	ay*Dt	vy =	vy + Δ vy
Horizontal position	x =	1	Δ x =	vx*Dt	x =	x + Δ x
Vertical position	y =	0	Δ y =	vy*Dt	y =	y + Δ y
Radius	r =	1			r =	√(x^2+y^2)

Today Newton could make a spreadsheet do the calculations and the plotting. The result validates his hypothesis since the orbit is an ellipse obeying Kepler's laws:

Sun:	0	0		Δt:	0,1					
ax	ay	Δvx	vx	Δvy	vy	Δx	x	Δy	y	r
			0		1,10		1		0	1,00
-1,00	0,00	-0,10	-0,10	0,00	1,10	-0,01	0,99	0,11	0,11	1,00
-1,00	-0,11	-0,10	-0,20	-0,01	1,09	-0,02	0,97	0,11	0,22	0,99
-0,99	-0,22	-0,10	-0,30	-0,02	1,07	-0,03	0,94	0,11	0,33	0,99
-0,95	-0,33	-0,10	-0,39	-0,03	1,03	-0,04	0,90	0,10	0,43	1,00
-0,91	-0,43	-0,09	-0,49	-0,04	0,99	-0,05	0,85	0,10	0,53	1,00
-0,85	-0,52	-0,08	-0,57	-0,05	0,94	-0,06	0,80	0,09	0,62	1,01
-0,77	-0,60	-0,08	-0,65	-0,06	0,88	-0,06	0,73	0,09	0,71	1,02
-0,69	-0,67	-0,07	-0,72	-0,07	0,81	-0,07	0,66	0,08	0,79	1,03
-0,60	-0,73	-0,06	-0,78	-0,07	0,74	-0,08	0,58	0,07	0,86	1,04
-0,51	-0,76	-0,05	-0,83	-0,08	0,66	-0,08	0,50	0,07	0,93	1,06
-0,42	-0,79	-0,04	-0,87	-0,08	0,58	-0,09	0,41	0,06	0,99	1,07
-0,34	-0,81	-0,03	-0,90	-0,08	0,50	-0,09	0,32	0,05	1,04	1,09
-0,25	-0,81	-0,02	-0,93	-0,08	0,42	-0,09	0,23	0,04	1,08	1,10
-0,17	-0,80	-0,02	-0,95	-0,08	0,34	-0,09	0,13	0,03	1,11	1,12



Does The Difference Make A Difference?

At an African teacher college the instructors had problems with the traditional ε–δ definition of a limit. All textbooks taught it in the same way so the instructors thought it could not be taught otherwise until we asked the 'Sleeping Beauty question': Limits and continuity did not name themselves so they can also be called something else?

Searching for hidden differences we realised that by interchanging the ε and δ the definition define a different kind of constancy as described above. So we tested the three constancy definitions in the laboratory, i.e. in the classrooms. After the double-lesson the student teachers were asked how they found the ε–δ definition of limits before and after this new introduction on a scale from –2 to 2 covering very bad, bad, neutral, good and very good. Among the 76 answers the average scores were –0.9 before and 0.7 after, giving an average difference of 1.5 in favour of the new one. 3 students preferred the old definition to the new and 73 students preferred the new definition to the old.

Also the differential calculus approach above was tested in a double lesson. Afterwards the student teachers were asked to express their opinion as above as to whether this approach should be introduced at the teacher college, and whether both the traditional and the alternative approach should be introduced at the secondary schools. Among the 117 answers the average scores were 1.2 and 0.9. Also the integral calculus approach above was tested with some success (Tarp 2005).

Adding per-numbers in primary school has been introduced at an East European teacher College with a positive result (Zybartas et al 2001).

So by being a Cinderella-difference making a difference in the mathematics classroom the adding per-numbers approach constitutes a user-friendly alternative to traditional calculus, which was the aim of this postmodern sceptical fairy tale study.

Where Is The Function, The Limit And The Computer?

Addition of per-numbers is a calculus that is set-free, fraction-free, function-free, limit-free, computer-free, etc. Thus this postmodern calculus lacks many ingredients of modern calculus, and seems to be more like a pre-modern ‘Enlightenment-calculus’ from when calculus was born in the absence of both functions and limits. However, to be considered part of the Enlightenment period is just an advantage since mathematics seems to blossom in the laboratory and wither with the winter breeze of rigor in the library:

> The enormous seventeenth-century advances in algebra, analytic geometry, and the calculus; the heavy involvement of mathematics in science, which provided deep and intriguing problems; the excitement generated by Newton's astonishing successes in celestial mechanics; and the improvement in communications provided by the academies and journals all pointed to additional major developments and served to create immense exuberance about the future of mathematics. (..) The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 398-99)

When teaching modern calculus computers make a positive difference (see e.g. Tall in Biehler et al). In postmodern calculus computers play a different role computing the ‘mega-sums’ that cannot be added manually: By renaming a ‘differential equation’ to a ‘change equation’ a computer elegantly solves any differential equation by recursion saying ‘y := y+dy’ as exemplified by the planetary orbits about. Thus the computer is not used to improve the teaching of a calculation form made obsolete by the computer, but to get access to the quantitative literature that improved our history by enabling us to predict and install changes in our conditions.

Conclusion

To solve the relevance paradox in mathematics education this paper used a postmodern sceptical fairy tale research to rename ‘calculus’ to ‘adding per-numbers’ allowing us to see that calculus takes place both in elementary school, middle school and high school. When tested in the laboratory of teacher education this hidden alternative proved to be a Cinderella-difference making a difference. This raises a political question: What is most important, to learn how to add per-numbers, or to learn to read calculus books? What is most essential, to be prepared for a future in the laboratory, or in the library? Once the policy makers have decided this we can answer the question: Shall we continue to use our institution to make half of the students calculus dropouts, or shall we allow all students to learn how to add per-numbers?

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Appendix. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ‘from T take away 4’ can by iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4, a ‘per-number’. So the ‘recount-equation’ T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking: T = 8 = (8/4)*4 = 2*4, T = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation T=3*4= 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal’s length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by recounting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference: C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads (predicted by the ‘restack-equation’ T = (T-b)+b):

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = (5+1)ten7 = 6ten7 = 67 (5ten17 = 5ten(17-10+10) = 6ten7)

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + n*b	T2	= T1 + n*b
T2 - T1	= + n*b	(T2-T1)/n	= b
ΔT	= ∑ n*b	ΔT/Δn	= b
ΔT	= ∫ b*dn	dT/dn	= b

Only in the case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell multiplicity-based mathematics can be summarised as a ‘count&add-laboratory’ adding to predict the result of counting stacks and per-numbers, in accordance with the original meaning of the Arabic word ‘algebra’, reuniting.

ADDING	Constant	Variable
Stacks m, s, kg, $	T = n*b T/n = b	T2 = T1 + n*b T2-T1 = n*b
Per-numbers m/s, $/kg, $/100$ = %	T = b^n $\sqrt[n]{T} = b$ $\log_b T = n$	T2 = T1 + ∫b*dn *dT/dn = b*

The Count&Add-Laboratory

106. Applying Mathe-Matics, Mathe-Matism or Meta-Matics

To solve the relevance paradox in mathematics education this paper takes a sceptical look at one of the taboos of mathematics education, the mathematical terminology. Two kinds of words are found, LAB-words abstracted from laboratory examples; and LIB-words exemplified from library abstractions, transforming mathe-matics to meta-matics. A third kind of mathematics is mathematism only valid in the library and not in the laboratory, and blending with meta-matics to metamatism. This distinction suggests that the relevance paradox of mathematics education occurs when teaching and applying metamatism and disappears when teaching and applying mathematics.

The Relevance Of Mathematics Applications

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998). And 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371). In order to make mathematics more relevant to the students it has been suggested that applications and modelling should play a more central role in mathematics education. However when tested in the classroom the result is not always positive: At the Danish Preparation High School pre-calculus was changed from an application-free curriculum to an application-based curriculum by replacing e.g. quadratic functions with exponential functions. Still the student performance kept deteriorating to such a degree that at the present reform the teacher union and the headmaster union have suggested that pre-calculus should no more be a compulsory subject. Thus the question can be raised: Why did this application-based curriculum not solve the relevance paradox? Is there an alternative unnoticed application-based curriculum that can make a difference? This paper argues that the relevance paradox is an effect of our terminology; it occurs when using LIB-words to teach LIB-mathematics and disappears when using LAB-words to teach LAB-mathematics.

Institutional Scepticism

Modern natural science has established research as a number-based 'LAB-LIB research' where the LIB-statements of the library are induced from and validated by reliable LAB-data from the laboratory as illustrated by e.g. Brahe, Kepler and Newton, where Brahe by studying the motion of the planets provided LAB-data, from which Kepler induced LIB-equations that later were deduced from Newton's LIB-theory about gravity.

Word-based research copies number-based research by operationalising its theories in order to validate them by reliable data. This however raises two questions: A quantitative theory can be operationalised through a calculation, how can a qualitative theory be operationalised? The reliability of quantitative data is checked through measuring, how is the reliability of qualitative data checked? These questions have led to scepticism towards modern word-based research.

This scepticism towards words is validated by a simple 'number&word-observation': placed between a ruler and a dictionary a thing can point to a number but not to a word, so a thing can falsify a number-statement in the laboratory but not a word-statement in the library; thus numbers are reliable, and words are unreliable; numbers represent natural correctness, and words represent political correctness; numbers carry research, words carry interpretations, which presented as research become seduction - to be counteracted by sceptical counter-seduction research using rewording to produce different words that are validated by being, not 'true', but 'Cinderella-differences' making a difference by solving problems and paradoxes (Tarp 2003).

Scepticism is part of democracy. The ancient Greek sophists used scepticism to distinguish between necessity and choice. Later it was revived as institutional scepticism in the Enlightenment and became part of its two democracies; the American in the form of pragmatism, symbolic interactionism and grounded theory; and the French in the form of post-structuralism and post-modernism.

In America Grounded Theory only allows the use of words grounded in data retrieved by never asking what you are looking for (Glaser 1992: 25). This creates two kinds of words, LAB-words abstracted from laboratory examples, and LIB-words exemplified from library abstractions. Thus a function is a LAB-word if presented as a name for certain calculations as in the Euler-definition:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities (Euler 1748: 3).

And a function is a LIB-word if presented as an example of a set-relation with the property that first-component identity implies second-component identity, as done in modern mathematics.

In France Derrida uses the word ‘logocentrism’ to warn against believing that words represent the world, and Lyotard uses the word ‘postmodern’ to warn against believing that ‘metanarratives’ describes the world (Cahoone 1996: 343, 482). Foucault describes how words are used to cancel democracy by installing ‘pastoral power’ ‘which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers’ (Foucault in Dreyfus et al, 1982: 213, 215).

In this way Foucault opens our eyes to the salvation promise of the modern generalised church: ‘You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction centre, the hospital, and do exactly what we tell you’.

Summing up, institutional scepticism suggests that we direct scepticism towards the taboo-part of mathematics education, the mathematical terminology, by asking ‘what kind of words are used in mathematics education? Would other words make a difference and solve the relevance paradox?

Lab-Mathematics And Lib-Mathematics

Euclidean geometry is the model for modern set-based mathematics defining its concepts as examples of the concept set, and deducing its statements from axioms. Thus modern mathematics is presented from above as LIB-mathematics using LIB-words in spite of the fact that it developed from below as LAB-mathematics using LAB-words as e.g. in the golden Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

The success was so overwhelming that mathematicians feared that mathematics (called geometry at that time) had come to a standstill at then end of the 18th century:

> Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century’s taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities’. (Lagrange in Kline: 623)

However, in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. This thinking lead Cantor to introduce the word ‘set’ to distinguish between different degrees of infinity having the natural numbers as a unit, just as numbers were introduced to distinguish between different degrees of multiplicity having 1 as its unit. However changing infinity from a quality to a quantity was controversial:

> To Kronecker (..) Cantor’s work on transfinite numbers and set theory was not mathematics but mysticism. Kronecker was willing to accept the whole numbers because these are clear to the intuition. These ‘were the work of God.’ All else was the work of man and suspect. (Kline 1972: 1197).

As to the paradoxes in set-theory even Cantor saw problems asking Dedekind in 1899 whether the set of all cardinal numbers is itself a set; because if it is, it would have a cardinal number larger than any other cardinal (Kline: 1003). Another paradox was Russell’s paradox showing that talking about sets of sets leads to self-reference and contradiction as in the classical liar-paradox ‘this sentence is false’: If $M = \{A \mid A\notin A)\}$ then $M\in M \Leftrightarrow M\notin M$. Russell solves this paradox by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Thus if a fraction is defined as a set of numbers it cannot be a number itself making e.g. the addition ‘2+3/4’ meaningless. And set-based mathematics defines a fraction as an equivalence set in a product set of two sets of numbers such that the pair (a,b) is equivalent to the pair (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same rational number ½.

Not wanting a fraction-problem, set-based mathematics has chosen to neglect Russell’s type-theory by accepting the Zermelo-Fraenkel axiom system making self-reference legal by not distinguishing between an element of a set and the set itself. But removing the distinction between examples and abstractions and between different abstraction levels is hiding that mathematics historically developed through layers of abstractions. And by changing from being a LAB-word to being a LIB-word, set also changed mathematics from being LAB-mathematics to being LIB-mathematics.

Mathematism

Mathematism is mathematics that is valid only in the library, and not in the laboratory (Tarp 2004). The ‘2&3-paradox’ illustrates the difference:

2*3 = 6 is a LAB-claim easily verified by recounting 2 3s as 6 1s: *** *** -> * * * * * *.

2+3 = 5 is a LIB-claim with countless counter-examples in the lab: T = 2 m + 3 cm = 203 cm, T = 2 weeks + 3 days = 17 days, T = 2 tens + 3 ones = 23 etc.

The US Mars-program crashing two probes by neglecting the units when adding cm to inches witnesses that seduction by mathematism is costly. The units must be alike and put outside a parenthesis; so addition can only take place inside a parenthesis to be valid. Thus 20%+10% = 30% only if taken of the same total; else the ‘fraction-paradox’ applies containing ‘killer-mathematics’ only valid inside the classroom where it is ‘applied’ to kill the relevance of mathematics:

Inside the classroom	20/100 + 10/100 = = = 20% + 10% =	30/100 = 30%
Outside the classroom	20% + 10% = or =	32% in the case of compound interest b% ($10<b<20$) in the case of the total average

Another example of mathematism and killer-mathematics is the modern function concept: If ‘f(x) = x+2’ means ‘let f(x) be a place-holder for the calculation ‘x+2’ containing x as a variable number’ then f(3) = 5 means that ‘5 is a calculation containing 3 as a variable number’. This is a multiple syntax error: 5 is a number, not a calculation; 5 does not contain 3; and 3 is a constant number, not a variable number (Tarp 2002).

Mathematics And Metamatism

Thus it seems we have three kinds of activities sharing the same name ‘mathematics’: LAB-mathematics abstracted from LAB-words below and validated in the laboratory, LIB-mathematics or ‘meta-matics’ exemplified from LIB-words above, and ‘mathematism’ containing statements that are valid in the library but not in the laboratory. By reducing the laboratory to an applier and not a creator of mathematics, LIB-mathematics cannot distinguish mathematics from mathematism as witnessed by the crashing Mars-probes, the 2&3-paradox, the fraction-paradox etc. Thus LIB-mathematics becomes a mixture of metamatics and mathematism that can be called ‘metamatism’.

This makes it possible to formulate some hypothesis: Maybe the relevance paradox is an effect of teaching, not mathematics, but metamatism? Maybe the lacking success of introducing application of mathematics is an effect of introducing application of, not mathematics, but metamatism? Maybe ‘education in mathematics’ is rather ‘indoctrination in metamatism’ teaching ‘killer-mathematics’ only existing and valid inside classrooms, where it kills the relevance of mathematics? And maybe replacing application of metamatism with application of mathematics will turn out to be a ‘Cinderella-difference’ making a difference by solving the relevance paradox?

To test these hypothesis we need a clear profile of LAB-mathematics: what words does it use, and what content does it have?

Constructing A Lab-Mathematics

A LAB-mathematics should respect two fundamental principles: A Kronecker-principle saying that only the natural numbers can be taken for granted. And a Russell-principle saying that we cannot use self-reference and talk about sets of sets. The appendix shows one example of a Kronecker-Russell mathematics based on the LAB-words ‘repetition in time’ and ‘multiplicity in space’ creating a set-free, fraction-free and function-free ‘Count&Add-laboratory’ where addition predicts counting-results making mathematics our language of prediction.

Set-based LIB-mathematics has different number sets as integer, rational numbers etc. Multiplicity-based LAB-mathematics only has stack-numbers and per-numbers. This has a consequence when talking about applications as illustrated by the trade problem ‘if 3kg = 2$, then 8$ = ?kg’. Writing ‘3kg = 2$’ seems problematic since kilos and dollars cannot be equal in a traditional sense. But in LAB-mathematics it expresses that a physical quantity can be counted in both kg and dollar. This use of the equation sign is more rational that the traditional use, making a syntax error when identifying a calculation with a number by writing ‘2*3 = 6’ instead of ‘2*3->6’ or ‘(2*3) = 6’.

Trade: Applying Per-Numbers, Fractions Or Functions

In primary school the trade problem ‘if 3kg = 2$, then 8$ = ?kg’ is solved by applying the ‘recount-equation’ T = (T/b)*b: by recounting the 8$ in 2s we know how many times we have, not only 2$, but also 3kg since 2$ can be replaced with 3kg: T = 8$ = (8/2)*2$ = (8/2)*3kg = 12kg.

In secondary school this ‘recount&replace-method’ is supplemented with a method expressing corresponding quantities as a per-number: 3kg per 2$ = ‘3 per 2’ ‘kg per $’ = 3/2 kg/$. Then we recount the units: kg = (kg/$)*$ = (3/2)*8 = 12; or write a bill: 8$ at 3/2 kg/$ = 8*3/2kg = 12kg.

In all the cases per-numbers are applied; the only difference is the order of the calculations.

In traditional mathematics the trade question applies fractions and equations to express that the price can be calculated in two different ways by dividing the $ with the kg: 2/3 = 8/x.

In modern mathematics the trade question applies functions and equations to set up a linear model f(x) = b+a*x that translates the trade question to the equation f(x) = 8. But first a linear function must be understood as an example of a homomorphism f(x+y) = f(x)+f(y). This definition is satisfied by the proportionality function f(x) = a*x; but not by the linear function f(x) = b+a*x, thus creating the bizarre mathematical theorem ‘the linear function is not linear’; indicating that the layer of rationality in modern mathematics is rather thin hiding a corpus of authorized routines so typical and dangerous for modern western society (Bauman 1989: 21).

So the trade question can be regarded as an application of per-numbers, of fractions&equations, or of functions&equations. By saying directly that ‘A is an application of B’ we indirectly say that B must be taught and learned before being applied to A. So in order to teach the trade problem we have to choose between three different curricula, one based on per-numbers, one based on fractions, and one based on functions. Since both fractions and functions are part of mathematism the choice is between a curriculum in mathematics and a curriculum in metamatism.

Testing A Per-Number Based Curriculum In The Classroom

The Danish Preparation High School has a function-based curriculum introducing functions before linear and exponential functions. As an alternative the author designed a 20-lesson per-number based micro-curriculum including the first two of the three fundamental change-questions:

T is the total of 200$ + 5 days at 4 $/day
T is the total of 200$ + 5 days at 4 %/day
T is the total of 200$ + 5 days at 4 $/day constantly increasing to 6 $/day

Also the genre concepts of ‘fact models’ and ‘fiction models’ were included saying that a fact model is a ‘since-hence’ model calculating the reliable consequences of a constant per-number; while a fiction model is an ‘if-then’ model calculating the consequences of an assumed-constant per-number supplemented with scenarios based upon alternative assumptions (Tarp 2001 a & b).

When tested in different classes with different teachers the micro-curriculum turned out to be a Cinderella-difference by solving the relevance paradox (Tarp 2003).

The per-number approach was also tested in a science-class having the same problems as the preparation classes when confronted with typical science-questions as ‘if 3kg = 2liters, then 8liters = ?kg’. Again it made a difference by excluding no one from the calculation laboratory.

The per-number approach has been developed into four full curricula in pre-calculus, calculus, science and economics (Tarp 2001 c). The first three curricula were sent to the Danish Education Ministry. The author had to ask for permission to test it in the classroom since Denmark, still deeply rooted in its continental feudal history, does not allow a High School to set up its own curriculum.

The Ministry however refused arguing that ‘it is a central part of these curriculum proposals that the students’ problems are caused by the terminology and can be changed by changing this. Modern educational research however points to a quite different direction, towards mathematical competences. The road forward does not go via a ‘softening’ of the terminology, but via a greater degree of teacher-insight into the students’ everyday thinking in order to lead them from this everyday thinking to scientific thinking’.

This Ministry reaction was to be expected from Foucault’s description of pastoral power. Only the pastoral terminology is accepted as ‘scientific’, and the relevance paradox can only be solved through introducing the new pastoral term ‘competence’ installing both students and teachers as ‘incompetent’. However the students can be cured if they ‘develop competence’ in ‘scientific thinking’. And the teachers can be cured if they ‘develop competences’ in diagnosing the students’ way of thinking, and in guiding the students from the ‘wrong path’ to the ‘right path’.

By replacing 'qualification' with 'competence' a verb-based LAB-word is replaced by a LIB-word. This increases the pastoral power since students and teachers, knowing when they are qualifying themselves, cannot know when they are 'competencing' themselves, only the pastors can.

Again we see how pastoral power is installed through words. At the same time a normality is worded, an abnormality is installed together with a normalizing institution meant to cure this abnormality. Failing its 'cure' it is 'cured' by the institution 'research' installing new 'scientific' words installing both a new normality and a new abnormality etc. In this way the research institution turns into an industry producing new pastoral LIB-words that are irrelevant to the LAB, but highly relevant to the survival and growth of the research industry (Tarp 2004).

Conclusion

Replacing authorized LIB-routines with the authentic LAB-routines can solve the relevance paradox in mathematics education. However the educational institution might not be interested in replacing pastoral power with enlightenment, and the research institution might not want its funding to decrease by solving the relevance paradox. So to publish an alternative to the modern set-based LIB-mathematics a website has been set up called MATHeCADEMY.net offering free teacher PYRAMIDeDUCATION in multiplicity-based LAB-mathematics from below (Tarp 2004).

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Appendix. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ‘ from T take away 4’ can by iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4, a ‘per-number’. So the ‘recount-equation’ T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking: T = 8 = (8/4)*4 = 2*4, T = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation T=3*4= 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg	T = 7 kg	T = 7 kg
= (7/2)*2kg	= (7/2)*2kg	= (7/2)*2kg
= (7/2)*6 litres	= (7/2)*100 %	= (7/2)*5 $
= 21 litres	= 350 %	= 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal’s length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by recounting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:
C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads (predicted by the ‘restack-equation’ T = (T-b)+b):

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = (5+1)ten7 = 6ten7 = 67 (5ten17 = 5ten(17-10+10) = 6ten7)

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + n*b	T2	= T1 + n*b
T2 - T1	= + n*b	(T2-T1)/n	= b
ΔT	= $\sum$ n*b	$\Delta T/\Delta n$	= b
ΔT	= $\int$ b*dn	dT/dn	= b

Only in the case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell multiplicity-based mathematics can be summarised as a ‘count&add-laboratory’ adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word ‘algebra’, reuniting.

ADDING	Constant	Variable
Stack-numbers or Totals m, s, kg, $	T = n*b T/n = b	T2 = T1 + n*b *T2-T1 = n*b*
Per-numbers m/s, $/kg, $/100$ = %	T = b^n $\sqrt[n]{T} = b$ $\log_b T = n$	T2 = T1 + $\int$b*dn $dT/dn = b$

The Count&Add-Laboratory

107. Pastoral Power in Mathematics Education

A Postmodern Sceptical Fairy tale Study

To solve the irrelevance paradox in mathematics education this paper looks for help at institutional scepticism as it appeared in the Enlightenment and was implemented in its two democracies, the American and the French, in the form of pragmatism and post-structuralism. Inspired by Foucault's notion of 'pastoral power' the paper looks at the use of words in mathematics education, distinguishing between 'lib-words' coming from the library and 'lab-words' coming from the laboratory. From this distinction a hypothesis can be made saying that the irrelevance paradox is created by lib-words installing pastoral power, and that lab-words will make the irrelevance paradox disappear. Consequently mathematics education should be based upon verb-based 'ing'-words such as counting and adding and calculating etc.

Lib-Words And Lab-Words

The background of this paper is the worldwide crisis in mathematics education created by enrolment problems in mathematical based educations and teacher education (Jensen et al 1998); by 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371); and by an 'irrelevance paradox' created by the fact that the volume of the mathematics education research increases together with the volume of problems it studies and aims to solve, thus being unable to be validated through solving the problems of the mathematics classroom (A Tarp 2004).

To solve the irrelevance paradox we must use a non-traditional method as e.g. institutional scepticism as it appeared in the Enlightenment and was implemented in its two democracies, the American in the form of pragmatism, and the French in the form of post-structuralism. A hypothesis can be made saying that the irrelevance paradox is a consequence of placing the authority in the library making mathematics 'mathematism' by using 'lib-words' defined through internal reference as examples of more abstract lib-words; and disappears when the authority is brought back to the laboratory building mathematics from 'lab-words' defined through external reference as abstractions from laboratory examples. To check this hypothesis a lab-based mathematics has to be designed and taken to the classroom to be tested.

Mathematics And Mathematism

Research is created through a 'LibLab-cooperation', where library statements are induced from and validated by reliable laboratory data. 'Mathematism' is mathematics that is validated only in the library, and not in the laboratory.

An example of mathematics is the statement that T = 2*3 = 6 claiming that 2 3s can be recounted as 6 1s. This claim is easily verified in the lab: *** *** -> * * * * * *.

An example of mathematism is the statement that T = 2+3 = 5, a claim that has countless counter-examples in the lab:

T = 2 m + 3 cm = 203 cm, T = 2 weeks + 3 days = 17 days, T = 2 tens + 3 ones = 23

That seduction by mathematism is costly is witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. The units must be alike and put outside a parenthesis to give a valid addition inside. Thus 20% + 10% = 30% only if taken of the same total; else the 'fraction-paradox' applies:

Inside the classroom	20/100 + = 20% +	10/100 = = 10% =	30/100 = 30%
Outside the classroom	20% +	10% = or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

Brought to the classroom mathematism introduces ‘killer-mathematics’ only valid inside the classroom where it is applied to kill the relevance of mathematics.

Another example of mathematism and killer-mathematics is the modern function concept: If ‘f(x) = x+2’ means ‘let f(x) be a place-holder for the calculation ‘x+2’ having x as a variable number’ then f(3) = 5 means that ‘5 is a calculation with 3 as a variable number’, which is a syntax error and pure gibberish (A Tarp 2002).

Mathematics And Metamatics

‘Lib-words’ are words defined through self-reference as examples of other library words; ‘lab-words’ are words defined through external reference as abstractions from the laboratory. As examples we can look at the two different definitions of a function, the historical and the modern. Historically Euler in 1748 defined a function as an abstraction, as a name for a calculation with a variable quantity:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1988: 3)

In contrast to this, the modern definition defines a function as an example of the ultimate mathematical abstraction, the set, by saying that ‘a function is a set of ordered pairs such that first-component identity implies second-component identity’.

Through external reference to the laboratory habit of calculating the Euler-function becomes a lab-word introduced to distinguish between calculating on quantities (T = 3*5^2 = ?) and calculating on calculations (T = 3*x^2, dT/dx = ?) as in calculus.

Through internal reference to the library concept ‘set’ the modern function is a lib-word introduced to give a ‘rigorous’ definition of a concept that is ‘applied’ in all the definitions of modern mathematics.

Having defined lib-words and lab-words we now turn to different theories on words.

Structuralism

Saussure has introduced the notions of the signified and the signifier to describe the difference between a concept and a sound-image:

> I propose to retain the word *sign* to designate the whole and to replace *concept* and *sound-image* respectively by *signified* and *signifier*; the last two terms have the advantage of indicating the opposition that separates them from each other and from the whole of which they are part (Saussure in Cahoone 1996: 179).

From these terms Saussure is able to formulate two principles; one about the arbitrary nature of the sign saying that the bond between the signifier and the signified is arbitrary; and one about the linear nature of the signifier saying that, being auditory, the signifier is unfolded solely in time. From these two principles Saussure can make his famous conclusion:

> Everything that has been said up to this point boils down to this: in language there are only differences (182).

In this description Saussure integrates the concept and the sound-image into a whole, a sign. However the origin of the concept is not discussed; so we do not know if the concept has emerged as an abstraction from examples from a lower abstraction level, i.e. from below; or as an example of an abstraction from a higher abstraction level, i.e. from above.

By avoiding this discussion it seems that a Saussurean sign approach to language resonates with a Platonic structural view seeing the world as examples of metaphysical structures being imprinted first upon the physical world through things and actions and then upon human minds as concepts giving raise to sound-images.

Historically this Platonic structuralism was developed in opposition to the sophist thinking of the ancient Greek democracy always emphasizing the importance of distinguishing between choice and necessity, between political correctness and natural correctness according to the three constituents of democracy: information, debate and decision. Thus Plato's half-brother, the sophist Antifon, writes:

> Correctness means not breaking any law in your own country. So the most advantageous way to be correct is to follow the correct laws in the presence of witnesses, and to follow nature's laws when alone. For the command of the law follows from arbitrariness, and the command of nature follows from necessity. The command of the law is only a decision without roots in nature, whereas the command of nature has grown from nature itself not depending on any decisions. (Antifon in Haastrup et al 1984: 82, my translation).

Plato saw democratic debate as ignorance and recommended that the power should be given to the philosophers who could make wise decisions based upon information coming from insight and knowledge, thus needing neither debate nor democracy. In this way Plato instituted the patronisation that Foucault calls 'pastoral power' to be continued first by the Christian church and later by modern universities.

This pastoral protection of human beings was unopposed until the emergence of the Enlightenment, which was a result Newton's discovery that the behaviour of physical objects are determined, not by a metaphysical Lord's incalculable will to rule, but by the physical mass' calculable will to change. So if people stopped to prey and began to calculate they would need patronising no more. Thus calculating was the sceptical basis of the Enlightenment and the two democracies it installed, the American and the French. However the two democracies developed two different forms of counter-structuralism. In America scepticism developed into pragmatism, and in France scepticism developed into post-structuralism.

One understanding of this difference can be based on the fact that, whereas America still has its first republic, France now has its fifth republic. Meaning that what Foucault calls pastoral power was not present in America, since the American settlers emigrated to avoid the pastoral power of Europe and to install 'freedom under God'. Whereas the pastoral power was very much present both inside France and around it, so several revolutions had to be fought forcing the French republic to organise the state as a military camp where French philosophers developed a special sensitivity towards any attempt to overthrow the democracy of 'la Republique'.

American Counter-Structuralism

The American counter-structuralism is called pragmatism created by Peirce arguing that the focus should be shifted away from laws to habits:

> (..) three elements are active in the world, first, chance; second, law; and third, habit-taking (..) conformity to laws exists only within a limited range of events and even there is not perfect (..) To develop its meaning, we have, therefore, simply to determine what habits it produces, for what a thing means is simply what habits it involves (Peirce in Menand 1997: 51, 49, 35)

Peirce added a third term, interpretation, to the Saussurean sign model. Thus to Peirce a sign is a triad of signified, signifiers and interpretation, to be included in a new sign as a new signified

> The word *pragmatism* was invented to express a certain (..) method for the analysis of concepts. A concept is (..) the rational part of the purport of a word. (..) This maxim is (..) a far-reaching theorem solidly grounded upon an elaborate study of the nature of signs. ‘Representation’ and ‘sign’ are synonymous. The whole purpose of a sign is that it shall be interpreted in another sign; and its whole purport lies in the special character which it imparts to that interpretation. (..) This maxim once accepted (..) speedily sweeps all metaphysical rubbish out of one’s house. Each abstraction is either pronounced to be gibberish or is provided with a plain, practical definition. (56-58)

Later American pragmatism received an input from German phenomenology through Schutz. This lead to ‘symbolic interactionism’ developing its own methodology called ‘grounded theory’.

As to ordinary sociology and symbolic interactionism Blumer says:

> Sociological thought rarely recognizes or treats human societies as composed of individuals who have selves. Instead they assume human beings to be merely organisms with some kind of organization, responding to forces which play upon them. (..) Respect the nature of the empirical world and organize a methodological stance to reflect that respect. This is what I think symbolic interactionism strives to do. (Blumer, 1998: 83, 60)

Thus grounded theory does not allow retrieving data through interrogation. Instead you listen to the agent’s own accounts and narratives from which you try to ‘find the hidden rationale behind the apparent irrationality’ (G Tarp 2003: 32).

Later American pragmatism received an input from the French post-structuralism giving it a more radical postmodern form as expressed e.g. by Rorty:

> When the notion of ‘description of the world’ is moved from the level of criterion-governed sentences within language games to language games as wholes, games which we do not choose between by reference to criteria, the idea that the world decides which descriptions are true can no longer be given a clear sense. It becomes hard to think that that vocabulary is somehow already out there in the world, waiting for us to discover it. (..) The world does not speak. Only we do. The world can, once we have programmed ourselves with a language, cause us to hold beliefs. But it cannot propose a language for us to speak. Only other human beings can do that. (Rorty 1989: 5-6)

Rorty thus shares the French scepticism towards ‘truth’ inspired by Nietzsche:

> It was Nietzsche who first explicitly suggested that we drop the whole idea of ‘knowing the truth.’ His definition of truth as a ‘mobile army of metaphors’ amounted to saying that the whole idea of ‘representing reality’ by means of language, and thus the idea of finding a single context for all human lives, should be abandoned. His perspectivism amounted to the claim that the universe had no lading-list to be known, no determinate

> length. He hoped that once we realized that Plato's 'true world' was just a fable, we would seek consolation, at the moment of death, not in having transcended the animal condition but in being that peculiar sort of dying animal who, by describing himself in his own terms, had created himself. (27)

Seeing language as just an army of metaphors brings Nietzsche in opposition to the first principle of Saussure regarding the arbitrariness of signs. To Nietzsche truth is

> a mobile army of metaphors, metonyms, and anthromorphisms – in short a sum of human relations, which have been enhanced, transposed, and embellished poetically and rhetorically and which after long use seem firm, canonical, and obligatory to people. (..) In some remote corner (..) of the universe there was once a star on which clever animals invented knowledge. It was the most arrogant and mendacious moment of 'universal history' (Nietzsche 1954: 46-47, 42)

French Counter-Structuralism

The French counter-structuralism is quite different from the American by turning the question of representation upside down and focusing upon, not how outside structure installs internal concepts, but how internal concepts install outside structure; and how words can be used as counter-enlightenment to patronise and 'clientify' people by installing pastoral power. Consequently to install democracy we need to identify the pastoral power and substitute the pastoral words with democratic words.

The French counter-structuralism, called post-structuralism, is inspired by Nietzsche and by German phenomenology developed in opposition to Platonic structuralism wanting philosophers to study metaphysical structures and not physical phenomena. Contrary to this, phenomenology finds it interesting to study, not the world in itself, but how the world presents itself to the human mind.

In Germany Husserl's phenomenology was carried on by Heidegger posing his famous question 'what is *is*?' A question that cannot be answered by a sentence containing *is*, so instead Heidegger enables recursive thinking by introducing an entity 'Dasein' that *has* 'Being':

> This entity which each of us is himself and which includes inquiring as one of the possibilities of its Being, we shall denote by the term 'Dasein'. (..) This guiding activity of taking a look at Being arises from the average understanding of Being in which we always operate and *which in the end belongs to the essential constitution of Dasein itself.* (..) Dasein accordingly takes priority over all other entities in several ways. The first priority is an *ontical* one: Dasein is an entity whose Being has the determinate character of existence. (Heidegger 1962: 27-28, 34)

Heidegger's thinking was so influential that Bauman calls it the second Copernican revolution:

> Of this demolition of false pretences the postmodern mind claims to be performing, the 'second Copernican revolution' of Heidegger is often seen as the archetype and trendsetter. (Bauman 1992: ix)

In France Heidegger inspired Derrida to inaugurate

> a project of deconstructing Western metaphysics or 'logocentrism' with its characteristic hierarchizing oppositions (..) Derrida's claim is that these conceptual orderings are not in the nature of things, but reflect strategies of exclusion and repression that philosophical systems have been able to maintain only at the cost of internal contradictions and suppressed paradoxes. The task of 'deconstruction' is to

> bring these contradictions and paradoxes to light, to undo, rather that to reverse, these hierarchies, and thereby to call into question the notions of Being as presence that give rise to them (Baynes 1987: 119)

Derrida talks about ‘deconstruction’ and ‘logocentrism’. Logocentrism is the belief that words represent what they name, and deconstruction is the inversion of representation: the named is destructed and reconstructed by the reader:

> But beyond theoretical mathematics, the development of practical methods of information retrieval extends the possibility of the ‘message’ vastly, to the point where it is no longer the ‘written’ translation of a language (..) The ‘rationality’ (..) which governs a writing thus enlarged, no longer issues from a logos. Further it inaugurates the destruction, not the demolition but the de-sedimentation, the de-construction, of all the significations that have their source in that of the logos. Particularly the signification of *truth*. … The notion of the sign always implies within itself the destruction between signifier and signified, even if, as Saussure argues, they are distinguished simply as the two faces of one and the same leaf. This notion remains therefore within the heritage of that logocentrism which is also a phonocentrism: absolute proximity of voice and being… (Derrida in Cahoone 1996: 341-343)

Lyotard coins the term ‘postmodern’ when describing ‘the crisis of narratives’:

> Science has always been in conflict with narratives. (..) I will use the term *modern* to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..) Simplifying to the extreme, I define *postmodern* as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

In his inaugural speech ‘Orders of discourse’ Foucault says:

> The fundamental notions now imposed upon us are no longer those of consciousness and continuity (with their correlative problems of liberty and causality), nor are they those of sign and structure. They are notions, rather, of event and of series, with the group of notions linked to these (Foucault 1970: 23).

As to pastoral power Foucault says

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word *salvation* takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1983: 213, 215)

In this way Foucault opens our eyes to the generalised salvation promise of the generalised church: ‘You are un-saved, un-healthy, un-social, un-educated. But do not fear! For we, the saved, healthy, social, educated, will cure you. All you have to do is to repent, and go to our institution, the church, hospital, correction centre, school, and do exactly what we tell you’.

So according to Foucault pastoral power comes from words installing an abnormality, and a normalizing institution to cure this abnormality through new words installing a new abnormality etc. (Foucault 1995). Thus the pastoral word 'educate' installs the 'un-educated' to be 'cured' by the institution 'education'; failing its 'cure' it is 'cured' by the institution 'research' installing new 'scientific' words as 'competence' installing the 'in-competent' to be 'cured' by 'competence development'; failing its 'cure' it is again being 'cured' by new 'research' installing new 'scientific' words etc.

Thus pastoral power is installed when the traditional bottom-up LibLab-cooperation is replaced by a top-down self-supporting LibLab-industry using self-created lab-problems to invent new 'scientific' lib-words that are exported to the lab through master educated inspectors creating new problems funding new research etc.

To increase its productivity the LibLab-industry lately has replaced verb-based words as 'educate' with words that are not verb-based such as 'competence'. So where the 'clients' themselves knew when they were 'educating' themselves or others, they do not know when they are 'competencing' themselves or others, only the pastors know.

Sceptical Fairy tale Research

A third alternative to American and French institutional scepticism is the ancient sophist distinction between choice and necessity. Today this distinction can be formulated as a LibLab-destinction or a 'number&word-paradox':

Placed between a ruler and a dictionary a thing can point to a number, but not to a word; thus a thing can falsify a number-statement about its length in the laboratory, but not a word-statement about its kind in the library; so numbers carry reliable information inducing and validating research, and words carry debate and interpretation that presented as information and research becomes seduction to be met with the counter-seduction of sceptical 'fairy tale' research searching for a hidden 'Cinderella-differences' making a difference (A Tarp 2003).

By transforming seduction back into interpretation scepticism transforms the word-library from a hall of facts to a hall of fiction to draw inspiration from, especially from the tales that have been validated by surviving through countless generations, the fairy tales. Hence the preferred interpretation genre in counter-seduction (and to a certain extent grounded theory) is the fairy tale. Once a fairy tale interpretation has identified the 'evil' word, scepticism begins to look for hidden alternatives either by discovering forgotten or unnoticed alternatives at different times and places inspired by the genealogy and archaeology of Foucault (Foucault 1980); or by inventing alternatives using sociological imagination, inspired by Mills (Mills 1959).

Looking For Pastoral Words In Mathematics

In modern mathematics a text consists of theorems, definitions, axioms and undefined terms as e.g. 'set'. An undefined term is a sign containing a 'transcendental' signified that becomes the foundation of definitions, as when a relation is defined as an example of a set-product, and a function is defined as an example of set-relation with certain properties.

The word 'is' can be used in 3 different connections relating different abstraction levels: as a many-1 relation defining one abstraction from many examples below; as a 1-many relation defining many examples from one abstraction above; or as a 1-1 relation defining a concept by another concept at the same abstraction level as in metaphors meaning 'carry-overs' in Greek.

Historically 'set' came from the laboratory below as a many-1 abstraction naming a collection of well-defined things. This simple set-concept however was controversial and criticized by e.g. Kronecker and Russell. So in the Zermelo-Fraenkel axiom system 'set' is an undefined term, thus changing from a lab-word to a lib-word.

Building its definitions as examples of this metaphysical set-concept modern mathematics can be given the name ‘meta-matics’ to distinguish it from the historical mathematics that arose from the laboratory as layers upon layers of abstractions. One example of a deconstruction of meta-matics to mathematics is the Kronecker-Russell multiplicity based mathematics in the appendix building upon ‘multiplicity’ instead of ‘set’ as the prime signified giving birth to other signifieds through abstraction.

Designing A Pastor-Free Lab-Based Mathematics

In the laboratory the signifier ‘multiplicity’ has as its signified temporal repetition as sensed by putting a finger to the throat; and spatial strokes made by a 1-1 iconisation of temporal repetition with 1 beat = 1 stroke. Strokes can be bundled and reorganised as icons with four strokes in the icon 4 etc.; and strokes can be counted by taking away bundles of e.g. 4s to produce a stack of e.g. T = 2 4s = 2*4. The process ‘from T take away 4’ can be iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4’, which is a called a ‘per-number’ or a ‘recounter’ counting how many times we have a 4 and leading to a ‘recount-equation’ T = (T/4)*4 predicting the result of recounting T in 4s to be tested by performing the recounting:

T= 8*1 = ?*4 = (8/4)*4 = 2*4 (thus the ‘equation’ x*4 = 8 has the solution x = 8/4)

Also with units per-numbers are recounters recounting one unit by another unit:

40 $ per 2 kg = 40 per 2 $ per kg = 40/2 $/kg = 20 $/kg = rate.

Thus the question ‘5 kg = ? $’ is answered by recounting the number (5 kg = (5/2)*2 kg = (5/2)*40 $ = 100 $); or by re-counting the unit ($ = ($/kg)*kg = 20*5 = 100).

Stacks can be added and overloads removed by the ‘restack-equation’ T = (T-b)+b:

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten(17-10+10) = 5ten1ten7 = 6ten7 = 67

Per-numbers can be added as stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b. Thus 1/2 + 2/3 might give what students say, 3/5, and certainly never the 7/6 of mathematism:

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

In this way a Kronecker-Russel multiplicity-based stack-mathematics can be summarised as a ‘count&add-laboratory’ adding to predict the result of counting stacks and per-numbers, where repeated addition of per-numbers leads to powers and integrations, and reversed addition of per-numbers leads to differentiation.

ADDING	Constant	Variable
Stacks *m, s, kg, $*	T = a*b T/a = b	T2 = T1 + a*b T2-T1 = a*b
Per-numbers *m/s, $/kg, $/100$=%*	T = b^n *n√T = b logbT = n*	T2 = T1 + ∫b*dx *dT/dx = b*

The Count&Add-Laboratory

Institutional Scepticism In Mathematics Education

In a paper Presmeg describes a high school research project based on the idea that

> mathematical elements in everyday activities of the students has an important role to play in making mathematics more meaningful to the students. The idea is that a cultural

> activity which is authentic to at least one student would be described in class by that student; then small group work and whole class sharing would be used to identify and symbolize patterns as a basis for developing mathematical concepts from the activity. (Presmeg 1998: 1-144)

In a chapter on semiotic analysis Presmeg uses

> Lacan's inversion of Saussure's dyadic model of semiosis (..) to analyse chaining of signifiers which involve mathematical symbolism, leading in two or more steps from an initial signified which is situated in a cultural practise, through various symbolic signifiers, to a mathematical structure which is isomorphic to the structure of the original practise and retains some of its properties. (1-146)

Presmeg includes several examples, one of which shows 'chaining of signifiers in a progression of generalisations from the Warlpiri kinship system to a dihedral group of order 8' (1-147). Thus Presmeg illustrates how semiotics can be used to show a relation between lab-activities and lib-concepts through a chain of abstractions. Presmeg suggests 'that teachers would be facilitating the development of curriculum by students' (1-145). However when a student describing the practise of barrel racing on horseback was asked if these activities were mathematical, she replied

> Naw! Well, maybe; if you count the barrels, one, two, three, four, five, six. (1-144)

So the actors don't see any mathematics in what they do. To study the actor's own understanding of their actions a phenomenological aspect could be added.

Interpreting hermeneutics and post-structuralism Brown suggests that mathematical understanding 'can be checked through the ability of the learner to tell convincing stories generated by himself or borrowed from the teacher' (Brown 1997:54). Also Brown raises the question: 'Can mathematics be seen existing outside the language that describes it?' (53).

Using the sophist LibLab-distinction the answer is yes. The Count&Add-Laboratory has as objective signifieds repetition and multiplicity having an educational effect by creating signifiers and tales of many in the learner through a 'grasping by grasping' principle (see e.g. www.MATHeCADEMY .net).

Conclusion

Multiplicity-based stack-mathematics has proven to be a Cinderella-difference making a difference in the Danish pre-calculus classroom (A Tarp 2003), and in teacher education in Eastern Europe (Zybartas et al 2001) and in Africa (A Tarp 2002). Thus institutional scepticism can help to solve the irrelevance paradox of mathematics education. With the American version we can direct scepticism towards the educational part, and with the French version we can direct scepticism towards the mathematical part by being able to see how historical mathematics has been transformed to meta-matics and mathematism through pastoral power replacing historical lab-words abstracted from and installing authority in the laboratory with lib-words exemplified from and installing authority in the library. So the irrelevance paradox is not here by necessity, it comes from choosing lib-words instead of lab-words. As expected the irrelevance paradox disappears when lib-words are replaced again with lab-words. However this replaces the teacher-metamatics with a student-mathematics (A Tarp 2003) free of sets, fractions, functions and other lib-words guarded by a 'research industry' being funded by an increase and not by a decrease of the irrelevance paradox (A Tarp 2004).

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Appendix. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ‘from T take away 4’ can by iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4, a ‘per-number’. So the ‘recount-equation’ T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking: T = 8 = (8/4)*4 = 2*4, T = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation T=3*4= 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by recounting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:
C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads (predicted by the ‘restack-equation’ T = (T-b)+b):

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = (5+1)ten7 = 6ten7 = 67 (5ten17 = 5ten(17-10+10) = 6ten7)

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + n*b	T2	= T1 + n*b
T2 - T1	= + n*b	(T2-T1)/n	= b
ΔT	= $\sum$ n*b	ΔT/Δn	= b
ΔT	= $\int$ b*dn	dT/dn	= b

Only in case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell multiplicity-based mathematics can be summarised as a ‘count&add-laboratory’ adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word ‘algebra’, reuniting.

ADDING	Constant	Variable
Stacks m, s, kg, $	T = n*b T/n = b	T2 = T1 + n*b *T2-T1 = n*b*
Per-numbers m/s, $/kg, $/100$ = %	T = b^n $\sqrt[n]{T} = b$ $\quad \log_b T = n$	T2 = T1 + $\int$b*dn *dT/dn = b*

The Count&Add-Laboratory

108. FunctionFree PerNumber Calculus

Calculus did not call itself 'calculus'. So calculus could also be called something else. Thus the non-action word 'calculus' could be reworded to the action-word 'adding per-numbers' - and then take place from K – 12.

Background

At a talk in the late 1970's Freudenthal described the didactical expert as a reflective practitioner. This inspired the author to use the mathematics classroom as a laboratory to develop different ways to teach secondary mathematics at the pre-calculus and the calculus level. Later as a research student the author used the inspiration from postmodern thinking to develop a new action focused methodology called sceptical Cinderella research.

The theoretical framework of Cinderella research is institutional scepticism as it appeared in the Enlightenment and was implemented in its two democracies; the American in the form of pragmatism, symbolic interactionism, and grounded theory; and the French in the form of post-structuralism and postmodernism.

Cinderella research sees the pragmatic and postmodern scepticism towards our most basic institution, words, validated by a simple observation 'the pencil-paradox': Placed between a ruler and a dictionary a thing can point to numbers, but not to words - thus a thing can falsify a number-statement, but not a word-statement. A number is an ill written icon showing the degree of multiplicity (there are 4 strokes in the number sign 4, etc.); a word is a sound made by a person and recognised in some groups and not in others. Numbers carry valid conclusions based upon reliable data, i.e. research. Words carry interpretations, that if presented as research become seduction (Tarp 2003).

Institutional Scepticism Directed Towards Calculus

Looking at calculus, a Cinderella study will use postmodern scepticism to note: Calculus did not call itself 'calculus', so calculus could also be called something else. And then use pragmatism to suggest a different word grounded on what calculus is doing. Thus the name 'adding variable per-numbers' respects calculus' grounding question: '5 seconds at 3m/s increasing to 4m/s totals ? m'. And pre-calculus' grounding questions: '5 days at 3$ (3%) totals ? $ (%).

Also the name 'adding variable per-numbers' respects that the original meaning of the Arabic word 'algebra' is 'reuniting':

Algebra: Reuniting	**Constant**	**Variable**
Totals m, s, kg, $	**T = a*n** T/n = a	**T2 = T1 + a*n** T2-T1 = a*n
Per-numbers m/s, $/kg, $/100$ = %	**T = a^n** $\sqrt[n]{T} = a$ $\log_a T = n$	**T2 = T1 + ∫a*dx** dT/dx = a

With 'adding variable per-numbers' as a parallel name for 'calculus' we can now take a sceptical view at the calculus traditions. The first thing we observe is the absence of the function-concept in the above grounding question, opposed to its presence in the modern tradition. This raises a Cinderella-question:

Is there a neglected different postmodern calculus that might be a Cinderella difference by making the prince dance (i.e. making the students learn)?

Postmodern thinking would use concept archaeology to look for an answer in history and study the social construction of calculus and the function concept. Doing this we find that calculus was constructed before 1700, and the function-concept after 1700, e.g.

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1748)

We see that Euler defines the function as an abstraction separating two kinds of expressions, calculations without and with a ‘variable quantity’. Thus instead of saying ‘calculus performs calculations on expressions with variable quantities’, he can just say ‘calculus performs calculations on functions’. However, today this function-concept has been turned upside down by defining a function as an example of an abstraction: ‘A function is an example of a relation between two sets’.

Thus we are faced with two different kinds of mathematics, a historical one defining a concept as an abstraction from examples, and a modern one defining a concept as an example from an abstraction. To distinguish we can introduce the name ‘meta-matics’ for the modern set-based mathematics.

Introducing new names allows you to introduce new actions. If we ask ‘will it be possible to introduce calculus in primary school?’ the answer right away would be: ‘Of course not since calculus builds upon functions, which belongs to upper secondary school!’ If instead we ask ‘will it be possible to introduce addition of per-numbers in primary school?’ the answer would be: ‘Yes, of course it will!’

Primary School

Adding Per-numbers: Integration

Multiplicity is counted in stacks, e.g. 3 4s. 3*4 is 3 4s, and only 12 1s if recounted in tens. 3*4 can be recounted in 3s as 4*3, i.e. 4 3s. Or 3*4 can be recounted in 5s as predicted by the ‘recount-equation’:

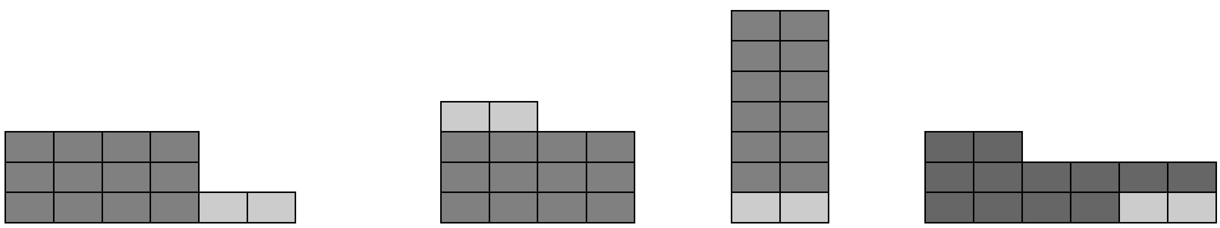
T=(T/5)*5: T = (3*4/5)*5 = 2*5+2*1 = 2*5 + 2/5*5 = 2 2/5*5 = 2.2*5.

Two stacks (a stock) can be added to 1 stack: T = 3 4s + 1 2s = 3*4 + 1*2 = ?

Added ‘in space’ as 4s: T = 3*4 + 1*2 = 3 2/4*4 = 3.2*4

Added ‘in space’ as 2s: T = 3*4 + 1*2 = 7*2

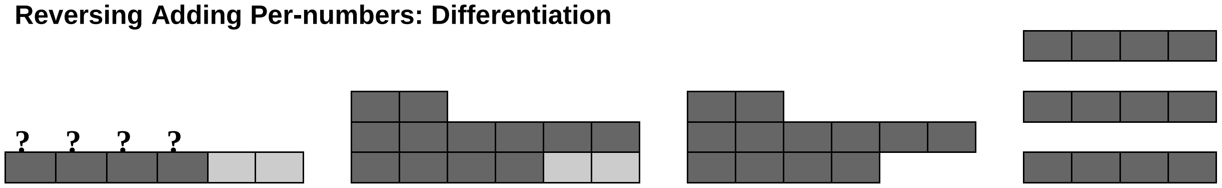
Added ‘in time’ as 6s: T = 3*4 + 1*2 = 2 2/6 * 6



T = 3*4 + 1*2 = 3 2/4*4 = 7*2 = 2 2/6 * 6

Recount-predictions: (1*2)/4*4=2/4*4, (3*4)/2*2=7*2, (3*4+1*2)/6*6=2 2/6*6

Reversing Adding Per-numbers: Differentiation



T = ?*4 + 1*2 = 2 2/6 * 6 T-1*2 (T-1*2)/4 *4

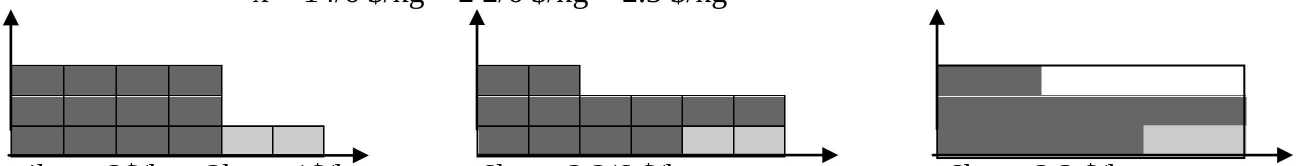
Recount-prediction:

?*n + T1 = T2, ? = (T2-T1)/n *Take away 4: T-4* *Take away 4s: T/4*

Middle School

Adding Per-numbers: Integration

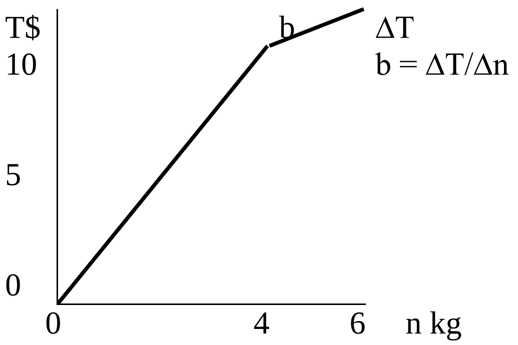
4kg at 3$/kg = 4*3 = 12$
<u>2kg at 1$/kg = 2*1 = 2$</u>
6kg at x$/kg = 6*x = 14$ = 14/6*6$
x = 14/6 $/kg = 2 2/6 $/kg = 2.3 $/kg



4kg at 3$/kg+ 2kg at 1$/kg = 6kg at 2 2/6 $/kg = 6kg at 2.3 $/kg

Reversing Adding Per-numbers: Differentiation

4kg at ?$/kg = 4*x = 4*x$
<u>2kg at 1$/kg = 2*1 = 2$</u>
6kg at 14/6$/kg = 4*x+2= 14$= 14 –2 +2 (the 'restack-equation' T= T–b+b)
x = (14-2)/4 = 3 = (T2-T1)/n = ΔT/Δn



Thus from adding per-numbers we learn that:
The Total is the area under the per-number curve predicted by an integration formula: ΔT = ∑ $/kg*kg = ∑ b*Δn adding the per-number stacks.
The per-number is the slope of the Total curve predicted by a differentiation formula: b = Δ$/Δkg = ΔT/Δn recounting the ΔT in Δns.

High School

The 3 Per-number Problems:

P1. The total of 5 days at 3$/day is ?$
P2. The total of 5 days at 3%/day is ?%
P3. The total of 5 days at 3$/day increasing to 4$/day is ?$

The 3 Per-number Solutions:

S1. The total of 5 days at 3$/day is 5*3$ = 15$
The total of n days at 3$/day is n*3$: T = n*3 *Linear change*

S2. The total of 5 days at 3%/day is 5*3%+0.9%=15.9% since (1+3%)^5=1.159
The total of n days at 3%/day is (1+3%)^n: T = (1+3%)^n *Exponential change*
Total interest I = simple interest n*i + compounded interest CI

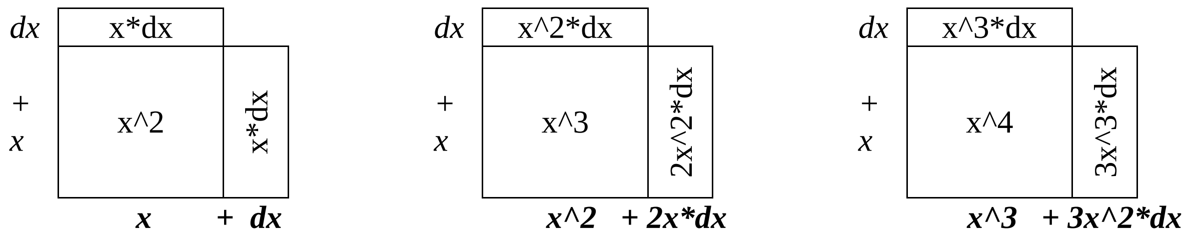
Compounded interest is what keeps exponential change from being linear.

So locally, exponential change is almost linear since the CI can be neglected at small interests:

The total of 5 days at 0.3%/day ≈ 5*0.3% = 1.5% since (1+0.3%)^5 = 1.01509 ≈ 1.015

Total interest I ≈ simple interest n*I; or (1+I) = (1+i)^n ≈ 1 + n*i

Neglecting compounded interest (or the upper right corner of the change stack) is called differential calculus. So in differential calculus the non-linear is considered locally linear.



In differential calculus d(x^2) = 2*x*dx, or d/dx(x^2) = 2x; d(x^3) = 3*x^2*dx,

or d/dx(x^3)= 3*x^2; d(x^4)= 4*x^3*dx, or d/dx(x^4) = 4*x^3 etc.

So d(x^n) = n*x^(n-1)*dx, or d/dx (x^n) = n*x^(n-1).

S3. Now we are able to predict the result of adding variable per-numbers through integration using that ∫dT = ΔT since the sum ∫ of single changes dT is the total change ΔT no matter the size of the changes:

5 sec. at 3m/sec increasing to 4 m/sec total $\int_0^5 (3+\frac{4-3}{5}x)\,dx = \int_0^5 (3+0.2x)\,dx = ?$ m

Since d/dx (3x+0.1x^2) = 3+0.2x we get that d (3x+0.1x^2) = (3+0.2x) dx, so

$$\int_0^5 (3+0.2x)\,dx = \int_0^5 d(3x+0.1x^2) = \Delta(3x+0.1x^2) = (3\cdot 5+0.1\cdot 5^2) - 0 = 17.5 \text{ m}$$

FunctionFree PerNuber PreCalculus makes a difference in the classroom (Tarp 2003). For details on FunctionFree PerNuber Calculus in primary, middle and high school please consult Zybartas et al. 2004, Tarp 2004 a, and Tarp 2004 b.

Conclusion

Modern calculus is turned upside down by being based upon functions that is based upon sets. This transformation of historical mathematics into unhistorical ‘meta-matics’ creates learning problems in the classroom, and a relevance paradox ‘formed by the simultaneous objective relevance and subjective irrelevance of mathematics’ (Niss in Biehler et al 1994). By renaming calculus to ‘adding variable per-numbers’, post-modern function-free calculus solves the relevance paradox by respecting the historical roots of pre-modern calculus.

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109. A Kronecker-Russell Multiplicity-Based Mathematics

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2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ‘from T take away 4’ can by iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4, a ‘per-number’. So the ‘recount-equation’ T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking: T = 8 = (8/4)*4 = 2*4, T = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation T=3*4= 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg	T = 7 kg	T = 7 kg
= (7/2)*2kg	= (7/2)*2kg	= (7/2)*2kg
= (7/2)*6 litres	= (7/2)*100 %	= (7/2)*5 $
= 21 litres	= 350 %	= 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by recounting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:
C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads (predicted by the ‘restack-equation’ T = (T-b)+b):

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = (5+1)ten7 = 6ten7 = 67 (5ten17 = 5ten(17-10+10) = 6ten7)

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + n*b	T2	= T1 + n*b
T2 - T1	= + n*b	(T2-T1)/n	= b
ΔT	= ∑ n*b	ΔT/Δn	= b
ΔT	= ∫ b*dn	dT/dn	= b

Only in case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell multiplicity-based mathematics can be summarised as a ‘count&add-laboratory’ adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word ‘algebra’, reuniting.

ADDING	Constant	Variable
Stacks m, s, kg, $	T = n*b T/n = b	T2 = T1 + n*b *T2-T1 = n*b*
Per-numbers m/s, $/kg, $/100$ = %	T = b^n $\sqrt[n]{T} = b$ $\quad log_b T = n$	T2 = T1 + ∫b*dn *dT/dn = b*

The Count&Add-Laboratory

110. Multiplicity-Based Mathematics Found by Postmodern Sceptical Fairy tale Research

To increase enrolment this paper looks for hidden differences within mathematics education using the theoretical framework of institutional scepticism as it appeared in the Enlightenment and was implemented in its two democracies, the American and the French in the form of symbolic interactionism and post-structuralism. As a hidden alternative to modern set-based mathematics postmodern sceptical Fairy tale research identifies an intuitionist mathematics build upon a Kronecker-principle wanting to base mathematics on the natural numbers; and upon a Russell-principle not allowing the syntax-errors of self-reference when talking about sets of sets. A website is set up for free teacher PYRAMIDeDUCATION with agendas for seduction-free educational meetings with the subject of mathematics, i.e. multiplicity, asking 'How can multiplicity be counted in bundles and stacks? How can stacks and per-numbers be added?'

The Focus of the Paper: To solve the Irrelevance Paradox of the Research Industry

The focus of the paper is to discuss under the perspective of philosophy in research in mathematics education the question 'how can postmodern thinking contribute to research in mathematics education'. The background is the worldwide crisis in mathematics education and in enrolment to mathematical based educations and teacher education (Jensen et al 1998). And the paradox that the output of the modern research 'industry' increases together with the amount of problems it studies and aims to solve. If research is valid conclusions based upon reliable data then solving and not creating problems in the classroom should validate mathematics education research. Being irrelevant to mathematics education the research in mathematics education has become an industry avoiding solving classroom problems in order to keep funding coming. So to solve this 'irrelevance paradox' the time has come to use postmodern thinking to develop a new kind of sceptical 'Cinderella research' searching for hidden differences that makes a difference in the educational laboratory, the classroom.

The Theoretical Framework of the Study: Institutional Scepticism

The theoretical framework of the study is institutional scepticism, as it appeared in the Enlightenment and was implemented in its two democracies, the American in the form of pragmatism and symbolic interactionism, and the French in the form of post-structuralism and post-modernism.

References to Related Literature: Symbolic Interactionism & Postmodernism

References to related literature will include quotes from American pragmatism (Pierce, Rorty), from symbolic interactionism (Blumer etc.) and from its methodology 'Grounded Theory' (Glaser & Strauss, etc.) as well as from French poststructuralism (Derrida, Lyotard, Foucault) and postmodern thinking (Nietzsche, Bauman) leading to a definition saying 'postmodernism means institutional scepticism towards the pastoral power of words'.

Methodology: a Postmodern Sceptical Fairy tale Research

The methodology 'sceptical fairy tale research' is a postmodern counter-seduction research based upon a post-structuralist 'pencil-paradox' or 'LibLab-paradox': Placed between a ruler and a dictionary a thing can point to numbers, but not to words - thus a thing can falsify a number-statement about its length, but not a word-statement about its name; i.e. a thing can defend itself against a number-accusation by making a statement of difference in a laboratory; but a thing is forced to pay deference to any word-accusation from the library (Tarp 2003).

A number is an ill written icon showing the degree of multiplicity (there are 4 strokes in the number sign 4 etc.). A word is a sound made by a person and recognised in some groups and not in others. Words can be questioned and put to a vote in a courtroom, numbers cannot. Numbers from the 'lab' carry valid conclusions based upon reliable data, i.e. research. Words from the 'lib' carry interpretations, that if presented as research become seduction; words carry no truth, words hide alternative perspectives to be uncovered by counter-seduction as e.g. sceptical research using a 'Sleeping Beauty' principle to look for hidden differences in a world frozen by words, by constantly renaming the words in order to find a 'Cinderella-difference', i.e. a difference that makes a difference.

By its LibLab-destinction between words and numbers, sceptical fairy tale research is inspired by the ancient Greek sophists always distinguishing between choice and necessity, between political and natural correctness, between nomos and logos. By transforming seduction back into interpretation scepticism transforms the library from a hall of fact to a hall of fiction to draw inspiration from, especially from the tales that have been validated by surviving through countless generations, the fairy tales. Hence the preferred interpretation genre in counter-seduction (and to a certain extent grounded theory) is the fairy tale. Once a fairy tale interpretation has identified the 'evil' word, scepticism begins to look for hidden alternatives either by discovering forgotten or unnoticed alternatives at different times and places inspired by the genealogy and archaeology of Foucault (Foucault 1980); or by inventing alternatives using sociological imagination, inspired by Mills (Mills 1959).

The aim of sceptical fairy tale research is not to extend the existing seduction of the library, so no systematic reference to the existing 'research' literature will take place. The aim is to solve problems by searching for hidden Cinderella-alternatives in the laboratory, i.e. by 1) finding the word suspected to be the villain, 2) renaming the evil word through discovery and imagination, 3) testing the hidden alternative in the laboratory to see if it is a Cinderella-difference making a difference, and 4) publish the alternative so it can become an option.

To compare with neighbouring research genres two cases are discussed, a paper using pragmatism by Presmeg called 'A semiotic framework for linking cultural practice and classroom mathematics'; and a paper using critical research by Skovsmose and Borba called 'Research Methodology and Critical Mathematics Education'.

Research Implications and Results: Locating a Hidden set-free Mathematics

In the contemporary crises in mathematics education and enrolment the blame has been put on the human actors: the students, the teachers, the politicians etc. Mathematics itself is never blamed. Hence sceptical research will be looking for a hidden Cinderella within mathematics itself.

Modern mathematics is basing its definitions on the concept of set. The different number-sets are defined as sets of sets, and the function is defined as a set-relation. However the concepts of sets and sets of sets are controversial being objected to by e.g. Kronecker and Russell. And in the history of mathematics it is possible to identify an alternative to modern set-based mathematics in the form of intuitionist mathematics build upon a Kronecker-principle wanting to base mathematics on the natural numbers, i.e. on multiplicity; and upon a Russell-principle not allowing the syntax-errors of self-reference when talking about sets of sets. The paper presents a possible design for a Kronecker-Russell mathematics, i.e. a set-free, fraction-free, function-free, seduction-free multiplicity-based mathematics as an alternative to the traditional modern 'meta-matics' of set-based mathematics.

Multiplicity-based mathematics is a Cinderella-difference making a difference in different classrooms. It has been successful in the Danish pre-calculus classroom (Tarp 2003) and in teacher education in Southern Africa; and in Eastern Europe (Zybartas et al 2001) as well as at a Danish e-learning college where it was made into web-based learning material for a self-instructing free teacher PYRAMIDeDUCATION, where 1 teacher coach 2 instructors chosen by turn from 2 teams of 4 students. By importing into the living room a count&add-laboratory, multiplicity moves the authority from the library to the laboratory. Each teacher continually produces 8 new teachers in multiplicity-based mathematics, who pay for their education by each coaching a new group of 8 students. Thus multiplicity-based mathematics will multiply as a self-reproducing virus on the Internet, until it can surface to the real world in ten years when today's mathematics teachers have retired unable to reproduce themselves by failing to make set-based meta-matics relevant to the mathematics students.

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Appendix. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ‘from T take away 4’ can by iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4, a ‘per-number’. So the ‘recount-equation’ T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking: T = 8 = (8/4)*4 = 2*4, T = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation T=3*4= 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by recounting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:
C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads (predicted by the ‘restack-equation’ T = (T-b)+b):

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = (5+1)ten7 = 6ten7 = 67 (5ten17 = 5ten(17-10+10) = 6ten7)

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + n*b	T2	= T1 + n*b
T2 - T1	= + n*b	(T2-T1)/n	= b
ΔT	= ∑ n*b	ΔT/Δn	= b
ΔT	= ∫ b*dn	dT/dn	= b

Only in case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell multiplicity-based mathematics can be summarised as a ‘count&add-laboratory’ adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word ‘algebra’, reuniting.

ADDING	Constant	Variable
Stacks m, s, kg, $	T = n*b T/n = b	T2 = T1 + n*b *T2-T1 = n*b*
Per-numbers m/s, $/kg, $/100$ = %	T = b^n $\sqrt[n]{T} = b$ $\log_b T = n$	T2 = T1 + ∫b*dn *dT/dn = b*

The Count&Add-Laboratory

201. Avoiding Ten, a Cognitive Bomb

The number ten is the basis of our number system. The traditional curriculum sees no problem in introducing ten as the follower of nine. However, being the only number with its own name but without its own icon, the number ten becomes a cognitive bomb if introduced to quickly. First 1digit mathematics should be learned through bundling & stacking reported by cup- and decimal-writing.

The Background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: 'Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$'.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots guided by a new kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

Ancient Greece saw a fierce controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people needed to be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers argued that patronization is the natural order since everything physical is an example of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become the natural patronising rulers.

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods; later to be was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing slow Spanish silver ships returning over the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it obeys its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight crated the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronization with democracy. Two democracies were installed, one in US, and one in France. US still has its first republic, France now has its fifth. The German autocracy tried to stop the French democracy by sending in an army. However the German army of mercenaries was no match to the French army of conscripts only to aware of the feudal alternative to stopping the German army. So the French stopped the Germans and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. Humboldt was asked to create an elite school and using Bildung as counter-enlightenment he created a school-system leading to the Humboldt University, which uses Luhmann System Theory to defend its chosen self-reference as nature (Luhmann 1995).

Inside the EU the sophist warning is kept alive only in France in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronising categories, discourses and institutions presenting their choices as nature (Tarp 2004b). Derrida recommends that pastoral categories be ‘deconstructed’. Lyotard recommends the use of postmodern ‘paralogy’ research to invent alternatives to pastoral discourses. And Foucault uses the term ‘pastoral power’ to warn against institutions legitimising their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

In descriptions, numbers and words are different as shown by the ‘number & word dilemma’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘pencil’, thus being able itself to falsify its number but not its word, which makes numbers nature and words choices, becoming pastoral if suppressing their alternatives; meaning that a thing behind a word only shows part of its nature through a word, needing deconstruction to show other parts.

Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what is pastoral choice presented as nature?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

The Nature of Numbers

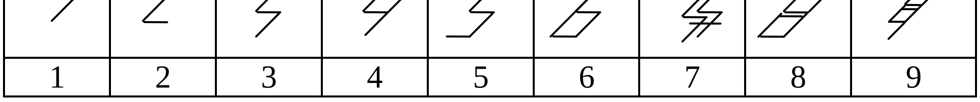
Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and repetition can be represented in space by one stick and one stroke per repetition. In this way multiplicity can be represented by a collection of matchsticks.

A collection or total of sticks can be treated in different ways.

A total can be lined up and counted by number-names invented for the different degrees of multiplicity: one, two, three, four, five, etc. written as T = lllll, a ‘stick-number’.

A total can have its sticks rearranged to form a number-icon having e.g. five sticks in the 5-icon if written less sloppy, T = 5, an ‘icon-number’.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
1	2	3	4	5	6	7	8	9



A total can be bundled in a bundle-size having both a name and an icon:

I I I I I -> ~~IIII~~ , written as T = 1*5, a ‘bundle-number’.

A total can be bundled in a bundle-size having a name but not an icon:

I I I I I I I I I I -> ǂǂǂǂǂǂǂǂǂǂ , written as T = 1*ten.

And a total can be counted by bundling and stacking: First it is bundled in e.g. 3s, then the 3-bundles are stacked in a 3-stack.

ǂǂǂ
I I I I I I -> ǂǂǂ ǂǂǂ -> ǂǂǂ written as T = 2*3, a ‘stack-number’.

Having icons for some numbers, I, V, X etc., the Roman number system mixes icon-numbers with stick-numbers as in XVIII.

The Arabic number system combines stack-numbers to a ‘stock-number’ or a polynomia: The number 23 means 2*ten + 3*1, i.e. a stock consisting of two stacks: a ten-stack containing 2 ten-bundles; and a 1-stack of unbundled singles consisting of 3 1s.

Today Arabic numbers are used almost everywhere. This makes them seem the natural way to describe numbers. However, historically it took a long time before they were established as the dominant number system. Therefore it seems natural to ask if young brains should be confronted with Arabic numbers right away, or if they should begin experimenting with other ways of counting and writing first. Why is it e.g. that the number that is the base of the Arabic numbers is the only number having its own name but not its own icon? Of course ten is the follower to nine, but why don’t we write X as the Romans did? And why are we caught in self-reference when after having explained the 2digit number 23 as 2 tens and 3 ones we try to explain the 2digit number ten a 1 ten and no ones? Isn’t there another way to count that lets us avoid this meaningless self-reference?

The Nature of Writing Numbers

To get an answer, let us try to change the base in the Arabic numbers from ten to five since there is no nature in choosing ten as the base. With six fingers on each hand, twelve had been the base number. And with four fingers, the base number would have been eight. We could also have chosen twenty as in some languages since we also have ten ‘fingers’ on our feet.

So to investigate a possible alternative, let us try investigating stock-numbers using a base-number that has both a name and an icon. Let us imagine, that five is chosen instead of ten And let us look closer at the activity of bundling & stacking sticks in stocks using 5-bundling.

With 5 as the bundle-size, a total of sticks might be bundled and stacked as a stock of e.g. 3 5-bundles and 2 singles; i.e. bundling & stacking produces a stock of two stacks: a stack of bundled sticks and a stack of unbundled single sticks.

The bundling & stacking process is performed manually and reported mentally.

Manually, in phase 1, the sticks are bundled in 5-bundles held together by a rubber band and stacked in a left bundle-cup with the unbundled single sticks stacked in a right single-cup. In phase 2, a trade exchanges 1 5-bundle of small sticks with 1 thick stick, e.g. a pencil, representing a 5-bundle of sticks glued together. In phase 3 a trade exchanges 1 pencil with 1 stick, now knowing that 1 stick in the left cup represents 1 5-bundle, and 1 stick in the right cup represents 1 single.

I I I I I I I -> ǂǂǂǂǂ) I I) -> █) II) -> I) II)

Mentally the counting is reported in words and in symbols.

The words reflect the counting sequence: 1, 2, 3, 4, bundle, bundle1, bundle2, bundle3, bundle4, 2bundle, 2bundle1, 2bundle2, etc. Using the fingers spread out, or clenched in a fist representing a

5-bundle, the counting sequence becomes: 1, 2, 3, 4, fist, fist1, fist2, etc. Including the toes where a full foot is called a fist, seventeen is counted as ‘3 5-bundles and 2’ or ‘3fist2’.

The symbols gradually emerge, first by introducing a bundle-symbol for the bundle and then by replacing it with cup-symbols illustrating the manual cup-procedure, and finally left out.

The a-row shows the oral counting sequence. In the b-row the word ‘bundle’ is symbolised with the letter B. The c-row uses ‘cup-writing’ referring directly to the two cups, the left bundle-cup and the right single-cup. The d-row is the c-row using 0 as a symbol for an empty cup. The e-row is ‘decimal-counting’ using a point to separate the left cup from the right. In the f-row the decimal point is left out choosing that the numbers now have place values where the left number counts the bundles in the left bundle-cup, and that the right number counts the singles in the right single-cup.

a	four	bundle	bundle 1	bundle 2	bundle 3	bundle 4	2 bundles	2 bundles 1	2 bundles 2
b	4	B	B1	B2	B3	B4	2B	2B1	2B2
c	4)	1))	1)1)	1)2)	1)3)	1)4)	2))	2)1)	2)2)
d	4)	1)0)	1)1)	1)2)	1)3)	1)4)	2)0)	2)1)	2)2)
e	0.4	1.0	1.1	1.2	1.3	1.4	2.0	2.1	2.2
f	04	10	11	12	13	14	20	21	22

Thus decimal-counting by bundling and stacking means putting all sticks in the right single-cup and then placing 1 stick in the left bundle-cup for each bundle taken from the single-cup.

) I I I I I I I I I I I I I I I I I) ->) ~~IIII~~ ~~IIII~~ ~~IIII~~ I I) -> I I I) I I) -> 3)2) -> 3.2 5s

A decimal-counting result can be written T = 3B2 = 3)2) = 3.2 Bs = 3.2 * B = 3.2 * 5 = 3.2 5s

We notice that if 5 is chosen as bundle-number, the 5-icon is not needed anymore since 5 now no more is a collection of five single sticks arranged in a 5-icon; instead five now is 1 bundle written

1 5s = 1 * B = 1)0) = 1.0 bundles.

Now we understand why the Arabic numbers don’t need a special icon for the number ten. No matter what number is chosen as the bundle-size, its icon is not needed since the chosen standard bundle-size always will be written as 1*B or 1)0) or 1.0 bundles, both in the case of 5 and ten.

In other words, it would be much easier if we used the word bundle instead of ten and counted

…, 8, 9, bundle, bundle1, bundle2, …, bundle9, 2bundles, 2bundles1 etc.

The Nature of Operations

Operations are icons describing the process of counting by bundling & stacking.

The division-icon ‘/2’ means ‘take away 2s’, i.e. a written report of the physical activity of taking away 2s when counting in 2s, e.g. 8/2 = 4 since 2s is taken away 4 times.

The multiplication-icon ‘4*’ means ‘stacked 4 times’, i.e. a written report of the physical activity of stacking 2-bundles 4 times, T = 4*2.

Subtraction ‘- 2’ means ‘take away 2’, i.e. a written report of the physical activity of taking away the bundles to see what rests as unbundled singles, e.g. R = 9 – 4*2 if 9 has 4 2s taken away.

And addition '+2' means 'plus 2', i.e. a written report of the physical activity of adding 2 singles to the stack of bundles either as singles or as a new stack of 1s making the original stack a stock of e.g. T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus the full process of 're-counting' or 're-bundling' 8 1s in 5s can be described by a 'recount or rebundle formula' containing three operations, together with a 'rest formula' finding the rest:

T = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is R = 8 – 1*5 = 3.

All the recount formula T = (T/b)*b says is: the total T is first counted in bs, then stacked in bs.

This recount formula cannot be used with ten as the bundle-size since we cannot ask a calculator to calculate T = (8/ten)*ten. However, this is no problem since the moment ten is chosen as the standard bundle-size, the operations take on new meanings. Now recounting any stack in tens is not done anymore by the recounting formula but by simple multiplication. To re-bundle 3 8s in tens, instead of writing T = (3*8)/10*10 = 2.4 * 10, we simply write T = 3*8 = 24. But when writing 24 instead of 2.4 bundles we leave out the unit, which might lead to 'mathematism' (Tarp 2004a).

With ten as the bundle-size we now want to know how to recount other stacks to ten-stacks. This is precisely what the tables predict: 7 3s = 7*3 = 21 = 2 tens 1, 8 3s = 8*3 = 24 = 2 tens 4, etc.

The Nature of Formulas

Using the recount formula, the counting result can be partly predicted on a calculator where 9/4 = 2.something. This predicts that recounting 9 in 4s will result in 2 4-bundles and some singles. The number of singles can be predicted by the rest formula R = 9 – 2*4 = 1. So (9/4)*4 is 2.1 4s.

Thus the calculator becomes a number-predictor using calculation for predictions. This shows the strength of mathematics as a language for number-prediction able to predict mentally a number that later is verified physically in the 'laboratory'. Historically, this enabled mathematics to replace pastoral belief with prediction, and to become the language of the natural sciences.

The Nature of Equations

The statement 4 + 3 = 7 describes a bundling where 1 4-bundle and 3 singles are rebundled to 7 1s. The equation x + 3 = 7 describes the reversed bundling asking what is the bundle-size that together with 3 singles can be rebundled to 7 1s. Obviously, we must take the 3 singles away from the 7 1s to get the unknown bundle-size: x = 7 – 3. So technically, moving a number to the other side reversing its calculation sign solves the equation: If x + 3 = 7, then x = 7 – 3.

The statement 2.1* 3 = 7 describes a bundling where 2.1 3-bundles are rebundled to 7 1s. The equation x * 3 = 7 describes the reversed bundling asking how 7 1s can be rebundled to 3s. Using the recount formula, T = 7 = (7/3)*3, the answer is x = 7/3. Technically, moving a number to the other side reversing its calculation sign solves also this equation: If x * 3 = 7, then x = 7/3.

With ten as the standard bundling-size, operations still are prediction techniques. Thus 5 + 3 predicts the end of the counting sequence when counting-on 3 times from 5: 6, 7, 8. In the same way multiplication predicts repeated addition of the same number: 3*2 predicts 2+2+2. And power predicts repeated multiplication of the same number: 2^3 predicts 2*2*2.

Any calculation can be turned around and become a reversed calculation predicted by the reversed operations: the answer to the reversed calculation 7 = 3 + ? is predicted by the reversed operation to plus, minus, i.e. by the calculation 7-3. The answer to the reversed calculation 7 = 3 * ? is predicted by the reversed operation to multiplication, division, i.e. by the calculation 7/3. The answer to the reversed calculation 7 = ? ^ 3 is predicted by the reversed operation to power, root, i.e. by the calculation $\sqrt[3]{7}$. The answer to the reversed calculation 7 = 3 ^ ? is predicted by the reversed operation to base, log, i.e. by the calculation $\log_3(7)$ = log7/log3.

Thus the nature of equations is reversed calculations solved by one more reversion, i.e. by moving numbers across the equation sign, and at the same time reversing their calculation signs.

3 + x = 7	3 * x = 7	x^3 = 7	3^x = 7
x = 7 - 3	x = 7/3	x = $\sqrt[3]{7}$	x = $\log_3(7)$

1Digit Mathematics

Decimal-counting by bundling & stacking allows for many learning-activities where the core of mathematics can be learned in preschool with 1decimal numbers only (Zybartas et al 2005).

A. Re-counting, later called proportionality: A total of 2.3 5s is recounted in another bundle-size, e.g. 4s: T = 2.3 5s = ? 4s. Using cup-handling and cup-writing, the 2 sticks are moved from the left 5bundle-cup to the right single-cup as 2*5 sticks. From the 2*5+3 sticks 3 4-bundles are moved from right single-cup to the left 4bundle-cup as 3 sticks giving a total of T = 3.1 4s

2.3 5s = I I) I I I) ->) IIIII IIIII I I I) ->) I I I I I I I I I I I I I) ->) IIII IIII IIII I) -> III)I) = 3.1 4s

2.3 5s = 2)3) =)2*5+3) =)3*4+1) = 3)1) = 3.1 4s

Prediction: T = (2*5+3)/4 *4 = 3*4 + 1 = 3.1*4 = 3.1 4s, since R = 2*5+3 – 3*4 = 1

B. Selling, later called subtraction: From a total of 3.2 5s is sold 1.4 5s, what is left? Using cup-handling and cup-writing, 1 stick is moved from the left 5bundle-cup to the right single-cup as 1*5 sticks now containing 1*5+2 sticks. From the 2 5s and 7 1s now 1.4 5s can be sold leaving 1.3 5s.

3.2 5s = I I I) I I) -> I I) IIIII I I) -> I I) I I I I I I I) -> I) I I I I) + I) I I I) = 1.4 5s + 1.3 5s

3.2 5s = 3)2) = 3-1)1*5+2) = 2)7) = 1)4) + 1)3) = 1.4 5s + 1.3 5s

C. Buying, later called addition, the core of algebra, which means reuniting in Arabic: A total of 2.4 5s is united with another total of 1.3 5s. Using cup-handling and cup-writing, 5 sticks are moved from the right single-cup to 1 stick in the left bundle-cup:

2.4 5s + 1.3 5s = I I) I I I I) + I) I I I) -> I I I) I I I I I I I) -> I I I) IIIII I I) -> I I I I) I I) = 4.2 5s

2.4 5s + 1.3 5s = 2)4) + 1)3) = 2+1)4+3) = 3)7) = 3+1)7-5) = 4)2) = 4.2 5s

D. Reversed addition, later called solving equations: Splitting the stack of 4.3 5s in 2.1 5s and another stack x leads to an equation 4.3 5s = 2.1 5s + x solved by subtraction: x = 4.3 5s - 2.1 5s.

E. Recounting and addition are combined when adding two stocks with different-bundle sizes:

T = 2.4 5s + 1.3 4s = ? 5s or T = 2.4 5s + 1.3 4s =? 4s. Using cup-handling and cup-writing, two sets of cups are present, a set for 5-bundling, and a set for 4-bundling. First the 4-bundling set is emptied on the table as 1*4+3 sticks and removed. Then the sticks on the table are moved to the right single-cup. From here 2 5s are moved as 2 sticks to the left 5bundle-cup.

2.4 5s+1.3 4s = II) IIII) + I) III) -> II) IIII) + IIII III -> II) IIIIIIIIIII) -> II) IIIII IIIII I) -> IIII) I) = 4.1 5s

2.4 5s+1.3 4s = 2)4+1*4+3) = 2+2)1) = 4)1) = 4.1 5s

F. Integration means adding 5s and 4s as 9s asking e.g. 2.4 5s+1.3 4s = ? 9s. Using cup-handling and cup-writing, two sets of cups are present, a set for 5-bundling, and a set for 4-bundling. First both sets are emptied on the table and one set is used for 9-bundling. Then the sticks on the table are moved to the right single-cup. From here 2 9s are moved as 2 sticks to the left 9bundle-cup.

2.4 5s+1.3 4s = II) IIII)+I) III) -> ~~IIII~~ ~~IIII~~ IIII ~~III~~ III -> IIIIIIIIIIIIIIIIIIIII) -> ~~IIIIIIII~~ ~~IIIIIIII~~ III) -> II)III) = 2.3 9s

2.4 5s+1.3 4s = 2*5 + 4*1 + 1*4 + 3*1 = 2*5 + 4*1 + 1*4 + 3*1) = 2)3) = 2.3 9s

G. Differentiation means reversed integration when e.g. splitting a total T2 = 2.3 9s in a total T1 = 1.3 4s and a 5-stack. Using cup-handling and cup-writing, the 2.3 9s is placed in a single-cup now containing 2*9+3 sticks from which 1*4+3 sticks are removed and the rest bundled in 5-bundles:

2.3 9s = II)III) -> ~~IIIIIIII~~ ~~IIIIIIII~~ III) -> IIIIIIIIIIIIIIIIIIIII) -> ~~III~~ III +~~IIII~~ ~~IIII~~ IIII) -> 1.3 4s+II) IIII) =1.3 4s+2.4 5s

T = (3*9+2 - 2*4+1)/5*5 = (T2-T1)/5 = ΔT/5 that later becomes the differential-quotient ΔT/Δx.

H. Handling overloads as e.g. 7.3 5s by inserting an extra cup to the left. With 7 5-bundles, 5 5-bundles can be bundled as 1 bundle of 5 5s, or 1 bundle of 5*5s, i.e. as 1 5*5-bundle. This calls for three cups, the first for the bundles of bundles, the next for the bundles and the last for the singles.

7.3 5s = I I I I I I I) I I I) -> ~~IIII~~ I I) I I I) -> I) I I) I I I) -> 1)2)3) = 12.3 5s

7.3 5s = 7)3) = 1B2)3) = 1)2)3) = 12.3 5s

I. Adding cups to the right. With T = 3.4 5s, adding 1 cup to the right means that the single-cup now becomes the bundle-cup; and that the bundle-cup now becomes the ‘bundle of bundles’-cup. This means that 3 now is 3*5 and that 4 is 4*5. So adding a cup to the right means two things, the decimal-number is multiplied with the bundle-number, and the decimal point, of course, moves one place to the right transforming 3.4 5s to 34.0 5s. So 3.4 5s * 5 = 34.0 5s.

3.4 5s = 3)4) = I I I) I I I I) , a cup is added to the right: I I I) I I I I)) = 3)4)) = 34.0 5s

J. Removing cups to the right. With T = 34.0 5s, removing 1 cup to the right means that the bundle-cup now becomes the single-cup; and that the bundle of bundles-cup now becomes the bundle-cup. This means that 3*5*5 is now 3*5 and that 4*5 is 4*1. So removing a cup to the right means two things, the decimal-number is divided with the bundle-number, and the decimal point, of course, moves one place to the left transforming 34.0 5s to 3.4 5s. So (34.0 5s) / 5 = 3.4 5s.

34.0 5s = 3)4)) = I I I) I I I I)) , a cup is removed to the right: I I I) I I I I) = 3)4) = 3.4 5s

Now multiplication and division with the bundle-size becomes very easy just meaning adding or removing a cup to the right since moving a stick to the left means adding ‘*B’ to its unit, and moving a stick to the right means removing a ‘*B’ from its unit.

B* 3.2 Bs = B* 3)2) = <u>3*B</u>)<u>2*B</u>) = 3)2)0) = 32.0 Bs ; or reversed as division:

32.0 Bs = 3)2)0) = <u>3*B</u>)<u>2*B</u>) = B* 3)2) = B * 3.2 Bs

Or in the case of 5-bundles:

5* 3.2 5s = 5* 3)2) = <u>3*5</u>)<u>2*5</u>) = 3)2)0) = 32.0 5s , and 3.2 5s = (3.2/5)*5 5s = 0.32 (5*5)s

Here the learning activities are performed with sticks. After that the same activities can be repeated with pellets on a plastic board, and later with squares on a squared paper.

The CATS-approach, Count&Add in Time&Space, at the MATHeCADEMY.net allows learners and teachers to experience a grounded approach to mathematics as a natural science investigating the natural fact multiplicity when counting by bundling & stacking (Tarp 2005).

Ten is not 10 by Nature, But by a Pastoral Choice

The traditional curriculum presents numbers as followers. When introducing the number five, different examples of five items are shown enabling the learner to practice counting five items in 1s, never in 2s or 3s, and writing the symbol 5. Numbers adding up to five revise the previous numbers.

Ten is introduced as the follower of the 1digit number nine, written with the 2digit symbol 10. If not prepared slowly and carefully, the learner might be confused by 2digit-numbers. A confusion that is understandable by the meaningless self-reference when later defining 10 = 1*10 + 0*1. And by the fact that while ten is the follower of nine by nature, 10 is the follower of 9 by choice.

First of all, nine isn't 9 by nature, but by choice. Counted in 4s nine is 2.1 4s; counted in 5s nine is 1.4 5s, counted in 8s nine is 1.1 8, counted in 9s nine is 1.0 9s. Nine is only 9 when counted in tens. So the follower of nine, ten, is not 10 by nature, it can be both 2.2 4s, 2.0 5s, 1.2 8s and 1.1 9s. And if eleven is chosen as bundle-size, then the follower of 9 is D if D is the chosen icon for ten. So there is no natural bound between ten and 10: with seven as bundle-size, ten is 13 and 10 is 7.

Only if ten is chosen as the bundling-size, ten becomes 1.0 tens, only written as 10 if we leave out both the units and the decimal point. So even here the follower to 9 is bundle or 1.0 tens, not 10.

Leaving out units might lead to 'mathematism' (Tarp 2004a) as e.g. '2 + 3 = 5' being true in the library but having countless counter-examples in the laboratory where 2 weeks + 3 days = 17 days, 2 m + 3 cm = 203 cm; in contrast to '2*3 = 6' including the units and being true both in the library and in the laboratory, merely claiming that 2 3s can be recounted to 6 1s.

So claiming that '9+1 IS 10' is a pastoral choice suppressing its alternatives. In the end, there is no natural way to write the degree of multiplicity called nine; it all depends upon the choice of bundle-size. Once a number is chosen as the standard bundle-size, icons are needed for all numbers before that number, but not for the bundle-number itself or for its followers. So saying that 9+1 = 10, is just saying that the follower of nine is bundle, or that ten has been chosen as bundle-size.

The sophist point is that there is a fundamental difference between a thing and its description, i.e. between nine and 9, and between ten and 10. This insight might be a little too abstract for young learners. Instead they should be allowed to experience the thrills of meeting examples of multiplicity, and of counting it by using both the hands and the head, i.e. both their physical and mental graspers to grasp the given multiplicity and bring it to order through counting in bundles and stacks to be named and written as icons, cup-numbers and decimal-numbers.

And to experience time and again the predicting power of a calculator predicting in under 10 seconds that the end-result of recounting e.g. 6 7s in 9s, while it takes several minutes to count up and bundle 6 7s, to de-bundle them to 1s, to re-bundle them in 9s, and finally to stack them as a 9-stack just to see that the prediction was correct when the recount formula said that T = (6*7)/9 = 4.R, R = 6*7-4*9 = 6, thus predicting the recount result T = 6 7s = 4.6 9s.

These recounting experiences are not possible if ten is presented as the natural bundling size. And with ten as the pastoral bundle-size suppressing its alternatives, in many cases the learner will not experience that counting means bundling & stacking. 32 just becomes some pastoral symbol needed to be learned and often being confused with 23, depending on it being named ‘thirty two’ or ‘two and thirty’; and also causing problems when pressing phone-numbers: 32 is easy to enter as thirty two, but can be confusing as two & thirty if the natural alternative ‘3bundle2’ is not available.

Introducing ten as 10 too early keeps the learner from being enlightened about the difference between the physical nature ten and its chosen description 10. So insisting that ‘ten IS 10’ means disregarding the sophist warning and installing pastoral power from day one. This is exactly what Humboldt counter-enlightenment wants, but hardly in the interest of democratic enlightenment who maybe should study the findings of anti-pastoral sophist research a little closer.

Conclusion

Today the Arabic numbers are used almost everywhere making them seem the only natural way to write down numbers; and consequently children should be exposed to them as soon as possible. However, behind any description there is a reality that might be reduced and distorted by the chosen description. And the description might contain concepts easy to use but hard to understand. The Arabic numbers have chosen ten as its base. But, being the only number with a name but without an icon, and being impossible to explain without using self-reference, ten might become a cognitive bomb if introduced to early. Anti-pastoral sophist research searching for alternatives to pastoral choices presented as nature uncovers a natural alternative in the form of an Arabic-like number system counting by bundling & stacking and reporting using cup-writing and decimal-writing; and by uncovering that ‘ten is 10’ by choice and not by nature, it enlightens ten as an anti-pastoral base chosen among alternatives, thus making Arabic numbers suited for anti-pastoral enlightenment.

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202. A Fresh Start Presenting Mathematics as a Number-predicting Language

Differentiating Between Choice and Nature

Disputing the nature of knowledge, the ancient Greek sophists and philosophers held opposite positions. To defend democracy, the sophists emphasized the importance of differentiating between nature and choice to prevent being patronized by ‘better-knowing’ presenting its choices as nature. To remove democracy, the philosophers claimed that choice is an illusion since all physical phenomena are examples of metaphysical forms only visible to the philosophers, educated at Plato’s academy, and obliged to offer patronisation to the ignorant people.

In the 17th century the dispute arose again when discussing the nature of falling objects as stones, moons and apples. The church claimed that falling objects obey the unpredictable will of Good only accessible thru faith, prayers and church attendance. No, Newton said, a falling stone obeys its own predictable will accessible through knowledge, calculation and school attendance.

Things obeying their own will motivated people to de the same and to replace patronisation with democracy, which let to the Enlightenment century and its two democracies, the American and the French.

To ensure that knowledge is about nature, the American democracy developed grounded theory. Although successful in the beginning, in the end the French democracy was overrun by the European autocracy; and developed postmodern thinking to repeat the sophist warning against false patronisation presenting choice as nature.

Thus Derrida, Lyotard and Foucault warn against ‘true’ words, sentences, and normalisation. Derrida recommends deconstruction of words installing choice as nature. Lyotard recommends paralogy as research methodology that by uncovering hidden differences produces parallel knowledge in dissension to the ruling consensus. And Foucault recommends knowledge archaeology to uncover the history of knowledge used as ‘research’ to legitimise pastoral salvation institutions.

Mathematics Education, Enlightening Nature or Forcing Choice

As to ‘mathematics education’ a postmodern analyses would ask: Are mathematical concepts and statements describing nature or installing a choice as nature? Is education enlightening the learner about nature, or forcing a choice upon the learner?

Using Derrida deconstruction it is easy to observe that the word ‘mathematics’ has installed a choice as nature when introducing concepts. Thus pastoral ‘metamatics’ from above defines a concept as an example of an abstraction (a function is as an example of a set-relation), in opposition to grounded mathematics from below that defines a concept as an abstraction from examples (a function is a name for a calculations containing a variable quantity).

Lyotard paralogy research uncovers a distinction between ungrounded ‘mathematism’ true in the library and grounded mathematics also true in the laboratory. Thus the statement ‘2+3=5’ is mathematism having countless counter-examples: 2w+3d=17d, 2m+3cm=203cm etc. Whereas the statement ‘2*3=6’ is true also in the laboratory stating that 2 3s can be recounted as 6 1s.

Using Foucault concept archaeology uncovers that the word ‘education’ covers over two different meanings. Wanting to prepare them for democracy, the sophists enlightened people about the difference between nature and choice. Wanting to protect them from democracy, the philosophers offered to the people pastoral guidance, later taken over by the Christian church just replacing guidance with salvation.

The Enlightenment democracies introduced enlightenment schools wit primary, secondary and tertiary schools allowing 95% of a youth population to enter university, and 50% to graduate. To stop democracy form spreading from France, the German autocracy asked Humboldt to create an elite school. Humboldt replaced enlightenment with Bildung and created the Humboldt University refusing to receive the students without an entrance exam from a Humboldt Gymnasium, which only allows the best half to enter and the best half to go on to the Humboldt university, where courses are so demanding that half drop out only letting an elite 13% get a university degree.

Identifying Alternative Micro Curricula

The roots of mathematics is the physical reality many. To represent different degrees of many icons are chosen containing the different degrees of many: 5 strokes in the 5-icon etc. Then counting by bundling and stacking produces numbers as groups of icons counting the unbundled, the bundles, the bundles of bundles etc. In primary school the bundle-number ten becomes a cognitive bomb if introduced too quickly since it violates the Piaget principle of learning, grasp before grasping. So as much mathematics as possible should be introduced with 1digit numbers alone using decimal numbers to describe the result of a counting process. Thus T = 3.2 5s means that the total has been bundled in 5s and stacked as 3 5s and 2 unbundled, predicted by the recount equation T = (T/5)*5. This grounded approach introduces division before multiplication in recounting questions as T = 3.2 5s = ? 6s. Later subtraction is introduced when selling part of a stack: T = 3.2 5s = 1.4 5s & ? 5s. And finally addition is introduced when uniting two stacks in three different ways: 3.2 5s + 2.3 4s = ? 5s = ? 4s = ? 9s, the last question leading directly to calculus.

In lower secondary school proportionality and percentages often give raise to many leaning problems. In a grounded approach proportionality is introduced in grade 1 in the recount equation, now used to recount a given quantity in two different units connected in a guide equation. Thus if 4kg = 5$, then 10kg = (10/4)*4kg = (10/4)* 5$ = 12.5$; and 18$ = (18/5)*5$ =(18/5)*4 kg = 14.4kg. Likewise percentages are introduced as special per-number expressing 3m per 100 m instead of 3 m per 100 sec.

Identifying an Alternative Upper Secondary New Start Curricula

An alternative upper secondary mathematics curriculum could aim at giving the students a new fresh start. Many students have turned their backs to mathematics finding it boring, uninteresting, irrelevant etc. So a fresh start should be given a try.

First the original meanings of the words ‘mathematics’, ‘algebra’ and geometry’ should be discussed.

Mathematics means knowledge in Greek, i.e. what can be used to predict with. So mathematics is our language for number-prediction. Thus the statement ‘2+3 = 5’ predicts that if 2 strokes and 3 strokes is counted not individually, but together, then the result is 5. In this way addition predicts the final result of a recounting process: 2 + 3 predicts 1, 2, 3, 4, 5. In the same way multiplication predicts repeated addition of the same number: 3*2 predicts 2+2+2. And power predicts repeated multiplication of the same number: 2^3 predicts 2*2*2.

Any calculation can be turned around and become a reversed calculation predicted by the reversed operations: the answer to the reversed calculation 6 = 3 + ? is predicted by the reversed operation to plus, minus, i.e. by the calculation 6-3. The answer to the reversed calculation 6 = 3 * ? is predicted by the reversed operation to multiplication, division, i.e. by the calculation 6/3. The answer to the reversed calculation 7 = ? ^ 3 is predicted by the reversed operation to exponent, root, i.e. by the calculation $\sqrt[3]{7}$. The answer to the reversed calculation 7 = 3 ^ ? is predicted by the reversed operation to base, log, i.e. by the calculation log3(7) = log7/log3.

Algebra means ‘reuniting’ in Arabic. With four different kinds of numbers in the world, like and unlike unit numbers and per-numbers, there are four different ways of uniting numbers. Addition unites unlike unit-numbers, multiplication unites like unit-numbers, power unites like per-numbers and integration unites unlike per-numbers. Thus algebra is generated by the four basic questions ‘The total is 3$ and 4$’ or ‘T = 3+4’; ‘The total is 3$ 4 times’ or ‘T=3*4’; ‘The total is 3% 4 times’ or ‘1+T=(103%)^4’; ‘The total is 5 seconds at 3 m/s increasing to 6m/s’ or $\Delta T = \int (3+(6-3)/5x)\, dx$. Here the reversed questions create the reverse operations subtraction, division, rot/logarithm and differentiation. ‘8$ is the total of 3$ and ?$’ or ‘8 = 3+x’; ‘12$ is the total of 3$? times’ or ‘T=3*x’; ‘15% is the total is ?% 4 times’ or ‘115%=(1+r)^4’; ‘15% is the total is 3% ? times’ or ‘115%=(103%)^x’; ‘20m is the total is 5 seconds at 3 m/s increasing to ?m/s’.

Geometry means ‘earth measuring’ in Greek. Since any form of earth can be divided into triangles, and since any triangle can be divided into right-angled triangles, right-angled triangles are the core of geometry. A right-angled triangle is half a rectangle, using the Arabic percentage equations sin and cos to express the length of the height and the width in percentage of the diagonal.

A formula ‘T = a+b*c’ expresses how a quantity T can be predicted by a calculation a+b*c. Containing one unknown, a formula becomes an equation, that can be solved manually by reversing the calculation, i.e. by moving numbers to the other side changing their calculation signs; and where the result can be checked by the ‘math solver’ on a graphical display calculator. Containing two unknowns, a formula becomes a function, that can be illustrated as a graph on a graphical display calculator; and where the two typical questions ‘given x find y’ and ‘given y find x’ reduces the function to an equation that can be solved manually; and where the result can be checked by the ‘math solver’, the ‘trace’ and the calc intersection’ on a graphical display calculator.

Now ‘regression mathematics’ can be introduces to show how mathematics is indeed a number-predicting language. Constant change situations as saving money at home or in the bank can be predicted by linear and exponential regression. Constantly changing change as playing golf can be predicted by quadratic regression. Driving a car is described by a per-number data in m/s, that can be predicted by cubic or quartic regression. In al cases the per-number graph needs to be integrated to give the total meter-number; and to be differentiated to give the acceleration.

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The website www.MATHeCADEMY.net contains a CATS-approach to mathematics, Count&Add in Time&Space, offering to learn mathematics as a natural science investigating the nature of many. Also the website contains 7 papers from the ICME10 Congress describing the approach in details.

203. Decimal-Counting, Disarming the Cognitive Bomb Ten

The number ten is the base of our number system. The modern curriculum sees introducing 10 as the follower of 9 as nature. However, being the only number with its own name but without its own icon, ten becomes a cognitive bomb if introduced to quickly. Anti-pastoral sophist research, searching for alternatives to choices presented as nature, shows that ten is not 10 by nature but by choice, and that jumping directly from 1.order to 3.order counting means missing the learning opportunities of 2.order decimal-counting by bundling and stacking.

1 The background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline 1972). Inspired by the invention of the set-concept, modern mathematics turns Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false:

Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment and justification problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots, multiplicity, guided by a new kind of research able at uncovering hidden alternatives to choices disguised as nature.

2 Anti-Pastoral Sophist Research

Ancient Greece saw a struggle between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers patronization was the natural order seeing everything physical as an example of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become the natural patronising rulers (Russell, 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spice. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it was following its own predictable physical will attainable by knowledge, calculations and school attendance.

This insight crated the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronization with democracy. Two democracies were installed, one in US, and one in France. US still has its first republic, France now has its fifth. The German autocracy tried to stop the French democracy by sending in an army. However the German army of mercenaries was no match to the French army of conscripts only too aware of the feudal alternative to stopping the German army. So the French stopped the Germans and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. Humboldt was asked to create an elite school and using Bildung as counter-enlightenment he created a school-system leading to the Humboldt University, which uses Luhmann System Theory to defend its chosen self-reference as nature (Luhmann, 1995).

Inside the EU the sophist warning is kept alive only in France in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronising categories, discourses and institutions presenting their choices as nature (Tarp, 2004). Derrida recommends that pastoral categories be ‘deconstructed’. Lyotard recommends the use of postmodern ‘paralogy’ research to invent alternatives to pastoral discourses. And Foucault uses the term ‘pastoral power’ to warn against institutions legitimising their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

In descriptions, numbers and words are different as shown by the ‘number & word dilemma’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘pencil’, thus being able itself to falsify its number but not its word, which makes numbers nature and words choices becoming pastoral if suppressing their alternatives; meaning that a thing behind a word only shows part of its nature through a word, needing deconstruction to show also other parts.

Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what is pastoral choice presented as nature?’

To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al, 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning (Piaget, 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

3 The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and repetition in time can be represented as multiplicity in space by one stick and one stroke per repetition. In this way multiplicity can be represented by a collection of matchsticks.

A collection or total of sticks can be seen as a ‘cardinal total’ of 1 united total, e.g. 1 5s; or it can be seen as an ‘ordinal total’ split up in many singles needed to be counted, e.g. 5 1s.

A cardinal total is described as a ‘1.order icon’ if the sticks are lined up, written T = I I I I I.

And is described as a ‘2.order icon’ if the sticks are rearranged to form a number-icon having five sticks in the five-icon if written in a less sloppy way than in the Arabic symbol 5.

2.order icons can be combined to a multi-digit number as e.g. T = 346.

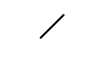
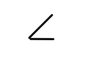
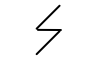
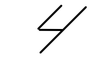
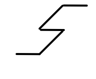
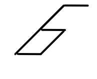
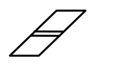
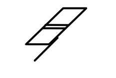
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

Figure 1: Numbers as strokes, icons and symbols

A cardinal total becomes an ordinal by counting, either in 1s, or in e.g. 5s, or in tens.

A total is ‘1.order counted’ if counted in 1s resulting in a ‘1.order stack’ as e.g. T= 5*1 = 5 1s.

A total is ‘2.order counted’ if counted in a bundle-size having both a name and an icon resulting in a ‘2.order stack’: I I I I I I I -> ~~III~~ ~~III~~ I, T = 2*3 + 1 = 2 3s + 1 1s.

A total is ‘3.order counted’ if counted in a bundle-size having a name but not an icon resulting in a ‘3.order stack’: I I I I I I I I I I I I -> ~~IIIIIIIIII~~ II, T = 1*ten + 2 = 1 tens + 2 1s = 12.

The Roman number system mixes 1.order and 2.order icons and symbols: In the number MXVIII, III is a 1.order icon; V and the X are 2.order icons showing one hand and two hands; and M is a symbol.

The Arabic number system uses 3.order stacks when counting in tens, the only number with a name but not an icon.

Apparently a number system is missing using 2.order counting and 2.order stacks. So the question arises: how might such an alternative system look like? Answering this question is a task for anti-pastoral sophist research searching for alternatives to choices disguised as nature.

4 Decimal-Counting

A number system using 2.order counting and 2.order stacks comes from the manual activity of bundling & stacking where the bundle-size has an icon, e.g. 5, resulting in a stock of e.g. 3 5-bundles and 2 singles, T = 3*5 + 2*1, which can be re-counted as 2 7-bundles and 3 singles if 7 is the bundle-size, T = 2*7 + 3*1. In both cases bundling & stacking produces a stock of two stacks: a stack of bundled and a stack of unbundled singles.

Counting by bundling & stacking is performed manually and reported mentally.

Manually, in phase 1, the sticks are bundled in 5-bundles held together by a rubber band and stacked in a left bundle-cup, while the unbundled singles are stacked in a right single-cup. In phase 2, a trade exchanges 1 5-bundle of thin sticks with 1 thick stick, e.g. a pencil, representing a 5-bundle of sticks glued together. In phase 3 a trade exchanges 1 thick with 1 thin stick, now knowing that 1 stick in the left cup represents 1 5-bundle, and 1 stick in the right cup represents 1 single.

IIIIIII -> IIIII) II) -> █) II) -> I) II)

First the five sticks are an ordinal total of five single sticks. The rubber band transforms the ordinal total 5 1s to a cardinal total 1 5s. Trading a 5-bundle to a thick stick and later to a left-cup stick allows the learner to manually grasp three different ways of representing a cardinal total: as 1 total held together by a rubber-band, as 1 thick stick symbolising a physical ‘gluing together’ of the thin sticks; and finally as 1 stick with a place value that depends on its place: in the left bundle-cup it counts bundles; and in the right single-cup it counts singles.

Mentally the counting is reported in words and in symbols.

Words are used in the counting sequence: 1, 2, 3, 4, bundle, bundle1, bundle2, bundle3, bundle4, 2bundle, 2bundle1, 2bundle2, etc. Using the fingers spread out, or clenched into a fist as a 5-bundle, the counting sequence becomes: 1, 2, 3, 4, fist, fist1, fist2, etc. Including the toes where a full foot is called a fist, seventeen is counted as 3 5-bundles and 2 or 3fist2.

The symbols gradually emerge, first by introducing a bundle-symbol for the bundle and then replacing it with cup-symbols illustrating the manual cup-procedure, and finally left out.

The a-row shows the oral counting sequence. In the b-row the word 'bundle' is symbolised with the letter B. The c-row uses 'cup-writing' referring directly to the two cups, the left bundle-cup and the right single-cup. The d-row is the c-row using 0 as a symbol for an empty cup. The e-row is decimal-counting using a point to separate the left cup from the right. In the f-row the decimal is left out so that the numbers now have place values where the left number counts the bundles in the left bundle-cup, and that the right number counts the singles in the right single-cup. Writing 13 instead of 1.3 5s leaves out the units, which might lead to 'mathematism' as '2+3=5' true in the library, but having countless counter-examples in the laboratory where 2 weeks + 3 days = 17 days, 2 m + 3 cm = 203 cm, etc. (Tarp, 2004).

a	four	bundle	bundle 1	bundle 2	bundle 3	bundle 4	2 bundles	2 bundles 1	2 bundles 2
b	4	B	B1	B2	B3	B4	2B	2B1	2B2
c	4)	1))	1)1)	1)2)	1)3)	1)4)	2))	2)1)	2)2)
d	4)	1)0)	1)1)	1)2)	1)3)	1)4)	2)0)	2)1)	2)2)
e	0.4	1.0	1.1	1.2	1.3	1.4	2.0	2.1	2.2
f	4	10	11	12	13	14	20	21	22

Figure 2: The emergence of numbers from bundling over cup-writing and decimal-numbers

Thus decimal-counting by bundling & stacking means putting all sticks in the right single-cup and then placing 1 stick in the left bundle-cup for each bundle taken from the single-cup.

When a number is chosen as bundle-size its icon is not used: 1 5s = 1B = 1)0) = 1.0 B = 10. This explains that when choosing 3.order counting in tens, ten does not need an icon.

5 The Nature of Operations

A decimal-counting process can be described by icons where the icon '/5' means 'take away 5s'; the icon '-5' means 'take away 5'; and the icon '3*' means 'stacked 3 times'. Thus 're-counting' 8 1s in 5s is described as T = (8/5)*5 = 1.3*5 leading to a general 'recount-formula' T = (T/B)*B merely saying that when counted by bundling & stacking, a total T is first bundled in bundles and then stacked in bundles. The recount formula connects cardinal and ordinal totals: The cardinal total 1*8 can recounted to ordinal totals as 1.3*5, 2*4, 2.2*3, etc.

The number of 5s can be found on a calculator as the number in front of the decimal point in 8/5. The decimal-part counting unbundled is the rest when the bundle is taken away: R = T – 1*5 = 3. In this way a calculation becomes number-prediction reflecting that the Greek word 'mathematics' means 'knowledge' i.e. what can be used for predications. So the word 'mathematics' could be translated to 'number-prediction'. This again explains the importance of mathematics, since it is costly not to be able to predict if a bridge will collapse or not.

With the recount-formula and the icons / and *, counting can be given a precise meaning. 1.order counting means counting in 1s: T = (T/1)*1. 2.order counting means counting in e.g. 5s: T = (T/5)*5. And 3.order counting means counting in tens: T = (T/ten)*ten. Here the predicting power of the recount-formula is only visible in the second case. The first case just states the banality that 1*8 = 8*1. The third case cannot be calculated since we don't have an icon for ten. And if we use the double-icon 10 for ten, when recounting e.g. 3 8s in tens we will not use the recount-formula saying T = (3*8/10)*10 = 2.4*10; instead we will just say T = 3*8 = 24, thinking of this as a fact from a table to be memorized; and never as a statement predicting that a physical recounting of 3 8s will give 2.4 tens.

6 The Nature of Equations

The statement 4 + 3 = 7 describes a bundling where 1 4-bundle and 3 singles are rebundled to 7 1s. The equation x + 3 = 7 describes the reversed bundling asking what is the bundle-size that together with 3 singles can be rebundled to 7 1s. Obviously, to get the answer we must take away the 3 singles from the 7 1s: x = 7 – 3. So technically, moving a number to the other side reversing its calculation sign solves the equation: If x + 3 = 7 then x = 7 – 3.

The statement 2.1* 3 = 7 describes a bundling where 2.1 3-bundles are rebundled to 7 1s. The equation x * 3 = 7 describes the reversed bundling asking how 7 1s can be rebundled into 3s. Using the recount-formula T = 7 = (7/3)*3, the answer is x = 7/3. So moving a number to the other side reversing its calculation sign also solves this equation: If x * 3 = 7 then x = 7/3.

The statement 5^3 = T describes a bundling where the total consists of a 5-bundle of 5-bundles of 5-bundles. The equation x ^ 3 = 8 describes the reversed bundling asking what is the bundle-size when a bundle of bundles of bundles gives a total of 8 1s. To predict the answer a new operation is invented, x = 3√8. The equation 2 ^ x = 8 describes the reversed bundling asking how man times 2-bundles must be 2-bundled before the total is 8. To predict this answer a new operation is invented, x = log2(8).

Thus the nature of equations is reversed calculations solved by moving numbers across the equation sign, and at the same time reversing their calculation signs.

3 + x = 7 x = 7 - 3	3 * x = 7 x = 7/3	x^3 = 7 x = 3√7	3^x = 7 x = log3(7)

Figure 3: Equations solved by moving numbers across with reversed calculation sign

7 1Digit Mathematics

Decimal-counting by bundling & stacking allows many-activities where the mathematics core can be learned in primary school with 1decimal numbers only (Zybartas et al, 2005).

A. Re-counting, later called proportionality: A total of 2.3 5s is recounted in another bundle-size, e.g. 4s: T = 2.3 5s = ? 4s. Using cup-handling and cup-writing, the 2 sticks are moved from the left 5bundle-cup to the right single-cup as 2*5 sticks. Now 3 4-bundles are moved from right single-cup to the left 4bundle-cup as 3 sticks giving a total of T = 3.1 4s

2.3 5s = I I) I I I) ->) ~~IIII~~ ~~IIII~~ I I I) ->) I I I I I I I I I I I I I) ->) ~~III~~ ~~III~~ ~~III~~ I) -> III)I) = 3.1 4s

2.3 5s = 2)3) =)<u>2*5+3</u>) =)<u>3*4+1</u>) = 3)1) = 3.1 4s

Prediction: T = (2*5+3)/4 *4 = 3*4 + 1 = 3.1*4 = 3.1 4s, since R = 2*5+3 – 3*4 = 1

B. Selling, later called subtraction: From a total of 3.2 5s is sold 1.4 5s, what is left? Using cup-handling and cup-writing, 1 stick is moved from the left 5bundle-cup to the right single-cup as 1*5 sticks now containing 1*5+2 sticks. Now 1.4 5s can be sold leaving 1.3 5s.

3.2 5s = 3)2) = <u>3-1</u>)<u>1*5+2</u>) = 2)7) = 1)4) + 1)3) = 1.4 5s + 1.3 5s

C. Buying, later called addition, the core of algebra meaning reuniting in Arabic: A total of 2.4 5s is united with another total of 1.3 5s. Using cup-handling and cup-writing, 5 sticks are moved from the right single-cup to 1 stick in the left bundle-cup:

2.4 5s + 1.3 5s = 2)4) + 1)3) = 2+1)4+3) = 3)7) = 3+1)7-5) = 4)2) = 4.2 5s

D. Reversed addition, later called solving equations: Splitting the stack 4.3 5s in 2.1 5s and a stack x leads to an equation 4.3 5s = 2.1 5s + x solved by subtraction: x = 4.3 5s - 2.1 5s.

E. Re-counting and addition are combined if adding two stocks with different bundle-sizes:

T = 2.4 5s + 1.3 4s = ? 5s or T = 2.4 5s + 1.3 4s =? 4s. Using cup-handling and cup-writing, two sets of cups are present, a set for 5-bundling, and a set for 4-bundling. First the 4-bundling set is emptied on the table as 1*4+3 sticks and removed. Then the sticks on the table are moved to the right single-cup. Now 2 5s are moved as 2 sticks to the left 5bundle-cup.

2.4 5s + 1.3 4s = 2)4+1*4+3) = 2+2)1) = 4)1) = 4.1 5s

F. Integration means adding 5s and 4s as 9s asking e.g. 2.4 5s+1.3 4s = ? 9s. Using cup-handling and cup-writing, two sets of cups are present, a set for 5-bundling, and a set for 4-bundling. First both sets are emptied on the table and one set is used for 9-bundling. Then the sticks on the table are moved to the right single-cup. From here 2 9s are moved as 2 sticks to the left 9bundle-cup.

2.4 5s + 1.3 4s = 2*5 + 4*1 + 1*4 + 3*1) = 2)3) = 2.3 9s

G. Differentiation means reversed integration when e.g. splitting a total T2 = 3.7 9s in a total T1 = 2.3 4s and some 5s. Using cup-handling and cup-writing, the 3.7 9s is placed in a single-cup now containing 3*9+7 sticks from which 2*4 + 3*1 sticks are removed and the rest bundled in 5-bundles:

T = (3*9+7 – 2*4+3)/5*5 = (T2-T1)/5 = ΔT/5 later to become the gradient ΔT/Δx.

H. Overloads as e.g. 7.3 5s is handled by inserting an extra cup to the left. With 7 5-bundles, 5 5-bundles can be bundled as 1 bundle of 5 5s, or 1 bundle of 5*5s, i.e. as 1 5*5-bundle. This calls for three cups, the first for the bundles of bundles, the next for the bundles and the third for the singles.

7.3 5s = 7)3) = 1B2)3) = 1)2)3) = 12.3 5s

I. Adding cups to the right means transforming the single-cup into a bundle-cup, and the bundle-cup into a 'bundle of bundles'-cup. With T = 3.4 5s, 3 becomes 3*5*5, and 4 becomes 4*5. So adding a cup to the right means two things: that the decimal-number is multiplied with the bundle-number; and that the decimal point, of course, moves one place to the right transforming 3.4 5s to 34.0 5s. So 3.4 5s * 5 = 34.0 5s; and 4.2 7s * 7 = 42.0 7s etc.

3.4 5s = 3)4) = I I I) I I I I) , a cup is added to the right: I I I) I I I I)) = 3)4)) = 34.0 5s

J. Removing cups from the right means transforming the bundle-cup into a single-cup and the bundle of bundles-cup into a bundle-cup. With T = 34.0 5s, 3*5*5 becomes 3*5, and 4*5 becomes 4*1. So removing a cup from the right means two things: that the decimal-number is divided with the bundle-number, and that the decimal point, of course, moves one place to the left transforming 34.0 5s to 3.4 5s. So (34.0 5s) / 5 = 3.4 5s; and 42.0 7s / 7 = 4.2 7s etc.

34.0 5s = 3)4)) = I I I) I I I I)) , a cup is removed to the right: I I I) I I I I) = 3)4) = 3.4 5s

Now multiplication and division with the bundle-size becomes very easy just meaning adding or removing a cup to the right since moving a stick to the left means adding '*B' to its unit, and moving a stick to the right means removing a '*B' from its unit.

B* 3.2 Bs = B* 3)2) = <u>3*B</u>)<u>2*B</u>) = 3)2)0) = 32.0 Bs ; or reversed as division:

32.0 Bs = 3)2)0) = <u>3*B</u>)<u>2*B</u>) = B* 3)2) = B * 3.2 Bs; so 32.0 Bs / B = 3.2

Or in the case of 5-bundles, 5 * 32 = 320 and 320 / 5 = 32:

5* 3.2 5s = 5* 3)2) = <u>3*5</u>)<u>2*5</u>) = 3)2)0) = 32.0 5s , and 3.2 5s = (3.2/5)*5 5s = 0.32 (5*5)s

Here the learning activities are performed with sticks. After that the same activities can be repeated with pellets on a plastic board, and later with squares on a squared paper.

The CATS-approach, Count&Add in Time&Space, at the MATHeCADEMY.net allows learners and teachers to experience a grounded approach to mathematics as a natural science investigating the natural fact multiplicity when counting by bundling & stacking (Tarp, 2008).

8 Opportunities Lost by Overlooking 2.order Decimal Counting

Understanding the nature of mathematics as using formulas for number-prediction is possible when predicting recounting a stack of 4 5s in 6s by the recount-formula T=(4*5/6)*6 = 3.2*6; and impossible when using ten as the bundle-size, T = (4*5/ten)*ten.

Understanding that numbers always carry units is included in both stack-writing, cup-writing and decimal-writing where 3*5, 3)) and 3.0 all mean 3 5s; and impossible with ten as a unit since the natural way of writing two-ten-three as 2.3 tens showing the unit is changed to the short version 23 hiding its unit.

Experiencing the root of the most essential part of mathematics, proportionality or linearity, is possible when asking how e.g. 3 4s can be recounted in 5s, predicted by T = (3*4/5)*5, and impossible when ten as THE bundle-size makes recounting to other bundle-sizes meaningless.

Understanding that using decimals is the natural way to write down numbers is possible when recounting e.g. 4 5s in 6s as 3.2 6s shows that counting a total always results in a double-stack of bundles and unbundled singles; and impossible with ten as the bundle-size since decimals here have to wait for the introduction of fractions by being presented as a special fraction.

Understanding that the operations are icons describing the details of the counting process is possible when recounting 4 5s in 6s is predicted by T = (4*5/6)*6 = 3*6 + 2*1 where the singles can be predicted by the rest-formula R = T – 3*6 = 2. Here '/6' means 'take away 6s', '–6' means 'take away 6', '3*' means 'stacked 3 times' and '+2' means '2 juxtaposed'. And here the natural order of the operations turns out to be bundling, stacking, and finding the rest to juxtapose, i.e. /, *, –, +. This is not possible with ten as bundle-size, where the traditional order of operations, +, –, *, / leads directly to the number 10 and 2digit 10-based numbers.

Experiencing the root of calculus by 'adding by juxtaposing' is possible when asking 3 4s + 2 5s = ? 9s; and impossible with ten as bundle-size since all stacks are tens and added as tens.

Experiencing the gradual emergence of numbers through icons, bundles, double-stacks of bundles and unbundled singles, cup-handling, cup-writing and decimal-writing is possible when the numbers have both names and icons; and impossible when numbers are introduced as symbols where the follower-principle leads directly to ten having a name but no icon.

9 Conclusion

A total is assigned a number through counting, either 1.order counting in 1s, or 2.order counting in e.g. 5s, or 3.order counting in tens. The traditional curriculum overlooks 2.order counting and sees it as nature to jump directly form 1.order to 3.order counting by introducing 10 as the follower of nine. But meaning just ‘bundle’, 10 is not ten by nature: with 7 as bundle-size, 10 is the follower of six; and the follower of nine is 13. Furthermore being the only number with a name but without an icon, ten becomes a cognitive bomb if introduced to early. 2.order counting shows that the traditional curriculum contains many choices becoming pastoral by being disguised as nature. There is nothing natural about natural numbers; decimal-numbers are the natural numbers produced directly when counting by bundling & stacking. There is nothing natural in the operation sequence +, -, *, /; the natural order is the opposite. And there is nothing natural in introducing 2digit numbers in grade 1. Introducing the core of mathematics with 1digit numbers is the natural thing to do.

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204. Pastoral Algebra Deconstructed

Presenting its choices as nature makes modern algebra pastoral, suppressing its natural alternatives. Seeing algebra as pattern seeking violates the original Arabic meaning, reuniting. Insisting that fractions can be added and equations solved in only one way violates the natural way of adding fractions and solving equations. Anti-pastoral grounded research identifying alternatives to choices presented as nature uncovers the natural alternatives by bringing algebra back to its roots, describing the nature of rearranging multiplicity through bundling & stacking.

The Background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs the outside world no more and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: 'Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$'. Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots guided by a new kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

In ancient Greece a fierce debate took place between two different forms of knowledge represented by the sophists and the philosophers. The sophists argued that to protect democracy peopled needed to be enlightened to tell choice from nature in order to prevent the emergence of patronization presenting its choices as nature. The philosophers argued that democracy should be abolished since everything physical are examples of meta-physical forms only visible to philosophers educated at Plato's academy, and who then should become the patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment period: when an apple follows its own will, people could do the same and replace patronization with democracy. Two democracies were installed: one in the US, and one in France, now having its fifth republic.

In France, sophist skepticism is kept alive in the poststructuralist thinking of Derrida, Lyotard and Foucault warning against pastoral patronizing categories, discourses and institutions presenting their choices as nature (Tarp, 2004). Derrida recommends that pastoral categories be 'deconstructed'. Lyotard recommends the use of postmodern 'paralogy' research to invent alternatives to pastoral discourses. And Foucault uses the term 'pastoral power' to warn against institutions legitimizing their patronization with reference to 'scientific' categories and discourses.

On the basis of the ancient and the contemporary sophist skepticism, a research paradigm can be created called ‘anti-pastoral sophist research’ deconstructing pastoral choices presented as nature by uncovering hidden alternatives. Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘in this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?’

To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the American enlightenment democracy, resonating with Piaget’s principles of natural learning (Piaget, 1970).

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of e.g. eight sticks can be treated in different ways. They can be rearranged to an eight-icon containing the eight sticks, written as 8. They can be collected to one eight-bundle, written as 1 8s or 1*8. And they can be counted by bundling & stacking, bundling the sticks in e.g. 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled singles in a right single-cup. When writing down the counting-result, ‘cup-writing’ gradually leads to decimal-writing where the decimal separates the bundle-number from the single-number:

8 = 1 5s + 3 1s = 1)3) = 1.3 5s

So the nature of numbers is that any total can be decimal-counted by bundling & stacking and written as a decimal number including its unit, the chosen bundle-size.

The bundle-size icon, e.g. 5, is not used in decimal-counting as shown by the oral and written sequence: One, two, three, four, bundle, bundle1, bundle2, bundle3, bundle4, 2 bundle, 2bundle1 etc. written as 0.1, 0.2, 0.3, 0.4, 1.0, 1.1, 1.2, 1.3, 1.4, 2.0, 2.1.

Choosing ten as bundle-size, it needs no icon making ten a special number having its own name but not its own icon. The advantage of decimal-counting is seen when comparing decimal numbers with the Roman numbers, that has a special ten-icon X, but where multiplication as XXXIV times DXXVIX is almost impossible to do. Without its own icon, however, ten creates learning problems if introduced to early. So to avoid installing ten as a cognitive bomb in young brains, the core of mathematics can be introduced by using 1digit numbers alone (Zybartas et al, 2005).

Also, together with choosing ten as bundle-size, another choice is made, to leave out the unit of the stack thus transferring the stack-number 2.3 tens to what is called a natural number 23, but which is instead a choice becoming pastoral by suppressing its alternatives. Later leaving out units create ‘mathematism’ true in the library where 2+3 = 5 can be proven true, but not in the laboratory where countless counterexamples exist: 2weeks + 3days = 17days, 2m + 3cm = 203cm etc.

The Nature of Operations

Operations emerge as icons describing the process of counting by bundling & stacking. Saying ‘the total is the total bundled in bundles and stacked in bundles’ can be written in an abbreviated form as T = (T/b)*b, the recount-formula. In this way letters are introduced not as numbers, but as abbreviations of words. And here letters are replaced with numbers instead of numbers being replaced by letters, as is the tradition in algebra.

The division-icon ‘/2’ means ‘take away 2s’, i.e. a written report of the physical activity of taking away 2s when counting in 2s, e.g. 8/2 = 4. The multiplication-icon ‘4*’ means ‘(stacked) 4 times’, i.e. a written report of the physical activity of stacking a 2-bundle 4 times, e.g. T = 4*2

Subtraction '- 2' means 'take away 2', i.e. a written report of the physical activity of taking away the stack to see what rests as unbundled singles, e.g. R = 9 – 4*2, the rest-formula. And addition '+2' means 'plus 2', i.e. a written report of the physical activity of adding 2 singles to the stack of bundles as a new stack of 1s making the original stack a stock of e.g. T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus the result of re-counting 8 1s in 5s can be predicted by two formulas: the recount-formula finds the number to the left of the decimal point, and the rest-formula finds the number to the right:

T = 8 = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is R = T – 1*5 = 3.

With ten as bundle-size, recounting becomes multiplication: to recount 3 8s in tens, instead of writing T = (3*8)/10 * 10 = 2.4 * 10, we simply write T = 3*8 = 24. Now tables are practiced for rebundling 2s, 3s, 4s etc. in tens. Thus the root of multiplication is division, taking away tens.

The Nature of Equations

The root of equations is the fact that any operation and calculation can be reversed.

The statement 4 + 3 = 7 describes a bundling where 1 4-bundle and 3 singles are rebundled to 7 1s. The equation x + 3 = 7 describes the reversed bundling asking what is the bundle-size that together with 3 singles can be rebundled to 7 1s. Obviously, we must take the 3 singles away from the 7 1s to get the unknown bundle-size: x = 7 – 3. So moving a number to the other side of the equation sign, by changing its calculation sign solves this equation: If x + 3 = 7, then x = 7 – 3.

The statement 2.1* 3 = 7 describes a bundling where 2.1 3-bundles are rebundled to 7 1s. The equation x * 3 = 7 describes the reversed bundling asking how 7 1s can be rebundled to 3s. Using the rebundling procedure and formula, the answer is T = 7 = (7/3)*3, i.e. x = 7/3. Again, moving a number to the other side changing its calculation sign solves this equation: If x * 3 = 7, then x = 7/3.

The statement 2 * 3 + 1 = 7 describes a bundling where 2 3-bundles and 1 single are rebundled to 7 1s. The equation x * 3 + 1 = 7 describes the reversed bundling asking how 7 1s can be rebundled in 3s leaving 1 unbundled. Obviously, we first take the single unbundled away, 7 – 1, and then bundle the rest in 3s giving the result x = (7-1)/3. Again technically, moving a number to the other side changing its calculation sign solves this equation: If x * 3 + 1 = 7, then x = (7-1)/3.

A multiple calculation x * 3 + 2 can be reduced to a single calculation by inserting a 'hidden parenthesis': x*3+2 = (x*3)+2. This calculation consists of two steps: First x is multiplied by 3 to x*3. Then 2 is added to x*3+2, which is the total 14. Reversing the calculation consists of the two opposite steps: First 2 is subtracted from the total 14 to 12. Then 12 is divided by 3 to 4.

So a reversed calculation has two sides, a forward side, that cannot be performed because of the unknown quantity x; and a backward side that can be performed by moving the numbers from the forward side to the backward side reversing calculation signs. In the beginning an equation can be solved by walking forward and backward. Later just the numbers walks from one side to the other.

Calculation direction:	Forward		Backward
End	(x*3)+1	=	7
		+1 ↑ ↓ –1	
	x*3	=	7–1 = 6
		*3 ↑ ↓ /3	
Start	x	=	6/3 = 2

The 'Walk&Reverse' method

Forward		Backward
(x*3)+1	=	7
x*3	=	7–1 = 6
x	=	6/3 = 2

The 'Move&Reverse' method

With ten as bundle-size the addition 3 + 5 = 8 predicts the result of adding 3 to 5 by counting-on 6, 7, 8. In the same way multiplication predicts repeated addition of the same number: 3*2 predicts 2+2+2. And power predicts repeated multiplication with the same number: 2^3 predicts 2*2*2.

Any calculation can be turned around and become a reversed calculation predicted by the reversed operations: the answer to the reversed calculation 6 = 3 + ? is predicted by the reversed operation to plus, minus, i.e. by the calculation 6-3. The answer to the reversed calculation 6 = 3 * ? is predicted by the reversed operation to multiplication, division, i.e. by the calculation 6/3. The answer to the reversed calculation 7 = ? ^ 3 is predicted by the reversed operation to exponent, root, i.e. by the calculation $\sqrt[3]{7}$. The answer to the reversed calculation 7 = 3 ^ ? is predicted by the reversed operation to base, log, i.e. by the calculation $\log_3(7)$ = log7/log3.

So the natural way to solve equations is to move numbers over changing their calculation signs:

3 + x = 7	3 * x = 7	x^3 = 7	3^x = 7
x = 7 - 3	x = 7/3	x = $\sqrt[3]{7}$	x = $\log_3(7)$

The Nature of Formulas

The recount-formula and the rest-formula can be used to predict the result of a recounting-process.

Recounting 9 in 4s the calculator says 9/4 = 2.25. Here only the left part can be trusted since the decimals are valid only when bundling in tens, and here we bundle in 4s. So the decimal is found from the rest-formula: R = 9 – 2*4 = 1, predicting the recount-result to be 9 = 2 4s + 1 1s = 2.1 4s.

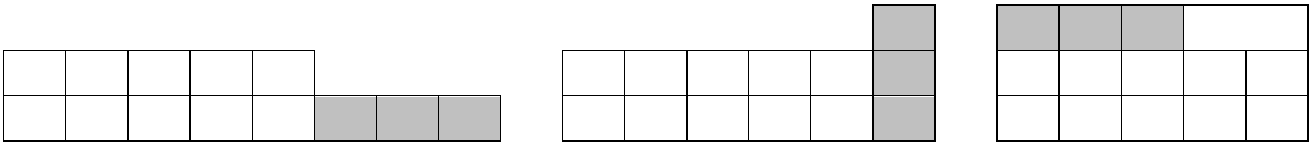
Formulas as number-predictor shows the strength of mathematics as a language for number-prediction able to predict mentally a number that later is verified physically in the 'laboratory'. Thus to avoid collapsing, the dimensions of physical constructions can be predicted by using formulas before being built. And historically religious belief was replaced with predictions with Newton's three contributions. The Kepler laws about the orbits of planets could not be tested by sending up new planets. Newton saw that the moon is falling towards the earth following its own will, enabling him to test his hypotheses on falling apples. Newton found the formula for this will, and saw that a will causes changes, so therefore he developed formulas for calculating changes.

A formula 'T = a+b*c' tells how a quantity T can be predicted by a calculation a b*c. With one unknown, a formula becomes an equation, that can be solved by reversing the calculation, i.e. by moving numbers to the other side changing their calculation signs; and the result can be tested by the 'math solver' on a Graphical Display Calculator, a GDC. Containing two unknowns, a formula becomes a function, that can be illustrated as a graph on a GDC; and where the two typical questions 'given x find y' and 'given y find x' reduces the function to an equation where the solution can be tested by the 'math solver', the 'trace' and the calc intersection' on a GDC.

A calculator, however, cannot be trusted when using multiplication. 3*7 means a stack of 3 7s, i.e. 3 7-bundles. 3*7 = 21 only when rebundling in tens, and even here the statement is not completely true since rebundling 3 7s in tens gives T = (3*7)/ten*ten = 2.1 tens, or 21 if we omit the unit 'tens' tacitly accepting that when nothing is specified, ten is the unit used for bundling. The knowledge that 3*7 = 21 is not needed for using the calculator to predict the result of rebundling 3 7s in e.g. 5s: T = 3 7s = 3*7 = (3*7)/5 *5 = 4*5 + R where R = 3*7 – 4*5 = 1, so T = 3 7s = 4.1 5s.

The Nature of Fractions

When counting by bundling & stacking, decimals and fractions are parallel ways of treating the unbundled singles. Thus counting in 5-bundles, 3 leftovers can be placed as a stack of 1-bundles next to the stack of 5-bundles and reported by a decimal number as 2.3; or they can be counted as 5s and put on top of the stack of 5-bundles giving a stack of T = 2*5 + (3/5)*5 = 2 3/5 *5 = 2 3/5 5s.



Introducing physical units creates 'double-counting' counting a quantity in both kg and $: If 4 kg cost 5 $, then the unit-price is 5$ per 4kg, i.e. 5$/4kg = 5/4 $/kg. And the per-equation '4kg per 5$' is used when recounting the actual kg-number in 4s, and recounting the actual $-number in 5s:

10kg = (10/4) * 4kg = (10/4) * 5$ = 12.5$, and 18$ = (18/5) * 5$ =(18/5) * 4kg = 14.4kg.

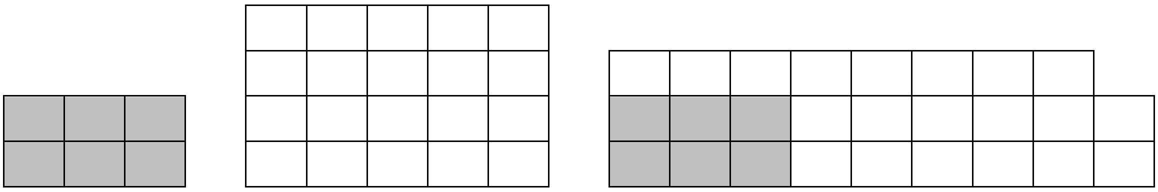
Fractions also occur when splitting a total in parts asking e.g. what is 3 8parts of 200$. Here double-counting produces the per-equation '200$ per 8parts' allowing to recount the parts in 8s:

3 parts = (3/8) * 8parts = (3/8) * 200$ = 75$

The Nature of Adding Fractions is Calculus

When counting by bundling & stacking, the statement 2 * 3 + 4 * 5 = 3.2 * 8 describes a bundling where 2 3-bundles and 4 5-bundles are rebundled in the united bundle-size 8s. This is 1digit integration. The equation 2 * 3 + x * 5 = 3.2 * 8 describes the reversed bundling asking how 3.2 8s can be rebundled to two stacks, 2 3s and some 5s. This is a 1digit differential equation solved by performing 1digit differentiation:

If 2 * 3 + x * 5 = 3.2 * 8, then x = (3.2 * 8 – 2*3)/5 = (T – T1)/5 = ΔT/5

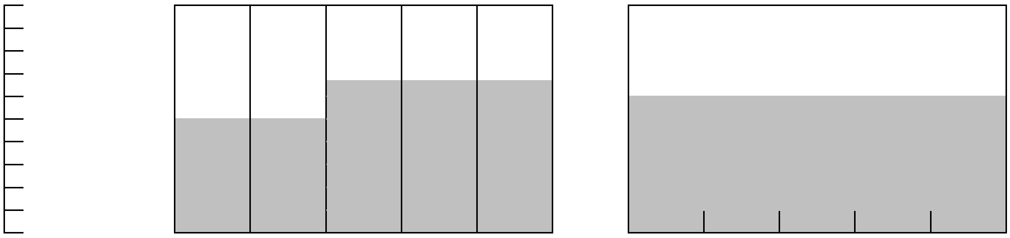


When adding fractions, it is important to include the units to avoid scaring the learners with mathematism as when performing the following 'fraction test' the first day in high school:

The teacher:	The students:
Welcome to high school! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

Seduction by mathematism can be costly as witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. So to add numbers the units must be included, also when adding fractions. And adding fractions f is the root of integration:

T = 1/2 *2 + 2/3 *3 = Σ (f*Δx) , later to become ∫ f dx



T = 1/2 *2 + 2/3 *3 = 3/5 *5

The Choices of Modern Pastoral Algebra

Modern algebra still believes in the existence of sets despite of Russell's paradox. It sees its task to present numbers and operations as examples of sets. Thus it defines the set Q of rational numbers as a set of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [Peano-numbers]]]; such that the number (a,b) is equivalent to the number (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same rational number ½. Real numbers are defined in a similar way (see any textbook in modern mathematics, e.g. Griffith et al, 1970).

Having defined the different number sets, modern algebra now defines operations as examples of relations between the product set of a number set and the number set itself. Thus addition in a number set N is an example of a subset of the set of (NxN)xN consisting of elements like (2,3,5) reflecting that 2+3=5. Since addition also contains the element (1,4,5) an equivalence set can be formed in NxN containing the elements (a,b) contained in (a,b,5).

An equation 2+3x = 14 is an open statement saying that the two numbers 2+3x and 14 belong to the same equivalence set thus in need of identical operations to stay in the same equivalence set. Using concepts and theorems from abstract algebra, the neutralizing method can now be introduced.

1+3*x	= 7	An open statement
(1+(3*x)) - 1	= 7 - 1	+1 has the inverse element -1
((3*x)+1) - 1	= 6	+ is commutative
(3*x) + (1-1)	= 6	+ is associative
(3*x) + 0	= 6	0 is the neutral element under +
3*x	= 6	by definition of the neutral element
(3*x) * 1/3	= 6 * 1/3	*3 has the inverse element 1/3
(x*3) * 1/3	= 2	* is commutative
x * (3*1/3)	= 2	* is associative
x*1	= 2	1 is the neutral element under *
x	= 2	by definition of the neutral element
L = {x∈R | 1+3*x = 7} = {2}		The solution set

The Arabic Meaning of Algebra

In Arabic, Algebra means ‘reuniting’. With constant and variable unit numbers and per-numbers providing four different kinds of numbers, there are four different ways of uniting numbers.

Addition unites variable unit-numbers, multiplication unites constant unit-numbers, power unites constant per-numbers and integration unites variable per-numbers.

Thus algebra is generated by the four basic questions:

What is the total of 3$ and 4$’ or ‘$T = 3+4$

What is the total of 3$ 4 times’ or ‘$T = 3*4$

What is the total of 3% 4 times’ or ‘$1+T = (103\%)^4$

What is the total of 5 seconds at 3 m/s increasing to 6m/s’ or $\Delta T = \int (3 + (6-3)/5 * x)\, dx$.

Algebra unites & splits	Variable	Constant
Unit-numbers m, s, kg, $, …	$T = a + x$ $T – a = x$	$T = a * x$ $T / a = x$
Per-numbers m/s, $/kg, m/100m = %	$T = \int a\, dx$ $dT/dx = a$	$T = a$ ^ x $x\sqrt{T} = a$ loga(T) = x

Grounded Algebra in Primary School

In primary school the tradition sees the purpose of algebra to be teaching the general rules for adding natural numbers, i.e. the commutative, the associative and the distributive laws.

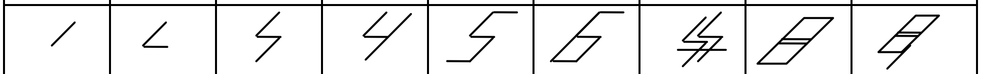
The hidden grounded alternative respects the original Arabic meaning of the word ‘algebra’, reuniting numbers, and the original Greek meaning of the word ‘mathematics’, knowledge that can predict. And it respects a grounded approach to the question ‘what are natural numbers?’

The tradition introduces the natural numbers using the follower-principle, leading directly to the ‘fact’ that 10 is the follower of nine. However, there is nothing natural about 10 as the follower of nine. Counting in 7-bundles, 10 is the follower of six, and the follower of nine is 13.

So a grounded approach to primary school algebra respects that the natural numbers are stacks of bundles as e.g. 3 4s, generated by a counting process predicted by the recount-law $T = (T/b)*b$ iconizing that a total T is counted by talking away bs T/b times.

Thus in primary school, mathematics is learned through counting and recounting by bundling and stacking, and through experiencing that operations predicts a counting result.

1.order counting counts in 1s by rearranging sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created until ten.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

To avoid introducing too many number-cons, 1.order counting is soon replaced with 2.order counting, counting by bundling&stacking using icon-bundles: First the sticks are bundled in e.g. 3-bundles, in 3s. Then the bundles are stacked in two stacks: a stack of 3s, and a stack of singles.

3.order counting counts in tens, the only number with its own name but without its own icon.

Introducing numbers by the follower-principle, the tradition skips both 1.order and 2.order counting and goes directly to 3.order counting by introducing 10 as the follower of 9; thus the tradition neglects the many leaning opportunities hidden in 1.oder and 2.order counting.

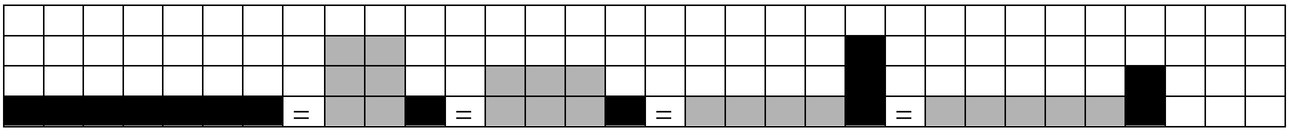
Using 1.oder counting, the learner experiences that the numerals are not just arbitrary symbols but icons written in a more or less sloppy way, in contrast to letters that are pure symbols.

Using 2.oder counting, the learner experiences that a given total can be counted in many different ways. Thus recounting in e.g. 3s, a total of seven sticks gives 2 3-bundles and 1 unbundled single. The stacks may then be placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents can be described by icons, first using 'cup-writing' 2)1), then using 'decimal-writing' to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

I I I I I I I -> III III I ->	III III I	-> ~~III~~ ~~III~~) I) -> ▮▮) I) -> II) I)
Or with icons:		-> 2 3s + 1 1s -> 2*3 + 1*1 -> 2)1) -> 2.1 3s

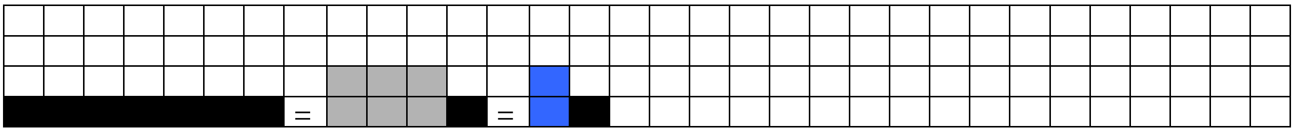
However, a total of 7 1s can also be counted and cup-written and decimal-written in other ways:

T = 7 1s = 3)1) or 3.1 2s = 2)1) or 2.1 3s = 1)3) or 1.3 4s = 1)2) or 1.2 5s = … =)7) or 0.7 8s etc.



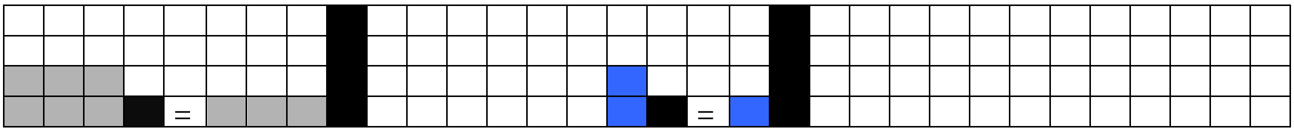
7 1s = 3.1 2s = 2.1 3s = 1.3 4s = 1.2 5s

Using squares, colors can represent the bundles, still maintaining cup-writing an decimal-writing:



7 1s = 2 3s + 1 = 2) 1) = 2.1 3s

Once decimal-numbers become the natural way to describe the result of a counting process, a new activity can take place where 'overloads' are created or removed by transforming a bundle into unbundled singles and vice versa. Thus 2.1 3s is the same as 1.4 3s and vice versa.



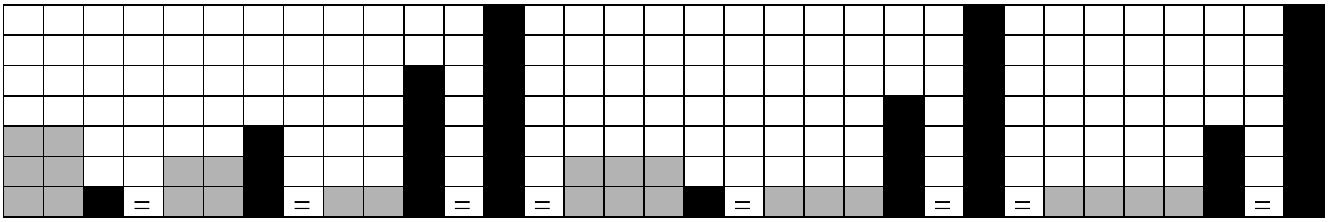
2 3s + 1 = 1 3s + 4 1s 2) 1) = 1) 4) or 2.1 3s = 1.4 3s

Using cup-writing, 1 3-bundle moves from the left bundle-cup to the right single-cup as 3 1s:

2) 1) = <u>2-1</u>) <u>1+3</u>) = 1) 4)

Using overloads allows a total of 7 1s to be decimal-written in many different ways:

T = 7 1s = 3.1 2s = 2.3 2s = 1.5 2s = 0.7 2s = 2.1 3s = 1.4 3s = 0.7 3s = 1.3 4s = 0.7 4s = 0.7 8s

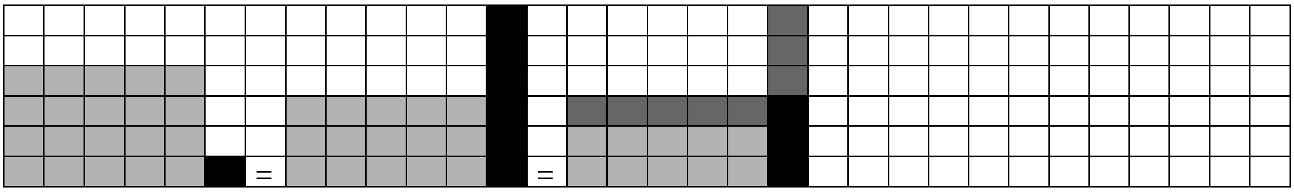


3.1 2s = 2.3 2s = 1.5 2s = 0.7 2s = 2.1 3s = 1.4 3s = 0.7 3s = 1.3 4s

Overloads might occur when selling or subtracting a stack from another stack, e.g. selling 2.3 5s from 4.1 5s, i.e. asking for the result of the subtraction 4.1 5s – 2.3 5s. Here the first stack is rewritten as an overload to 4.1 5s = 3.6 5s. Now

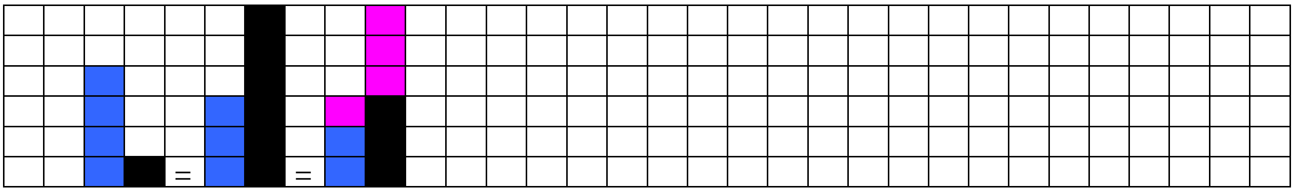
4.1 5s – 2.3 5s = 3.6 5s – 2.3 5s = 1.3 5s, or using cup-writing:

4) 1) – 2) 3) = 3) 6) – 2) 3) = 3-2) 6-3) = 1) 3)



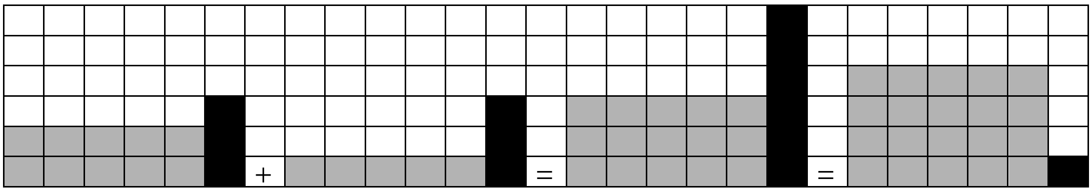
4.1 5s = 3.6 5s = 2.3 5s + 1.3 5s

Or with colors:



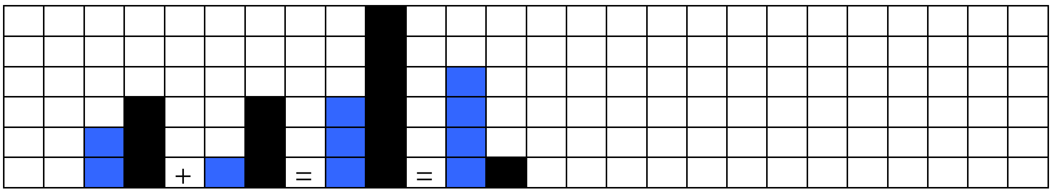
4.1 5s = 3.6 5s = 2.3 5s + 1.3 5s

Also adding one stack to another might lead to an overload that needs to be rebundled. Thus adding the two stacks 1.3 5s and 2.3 5s produces the stack 3.6 5s that can be rebundled to 4.1 5s.



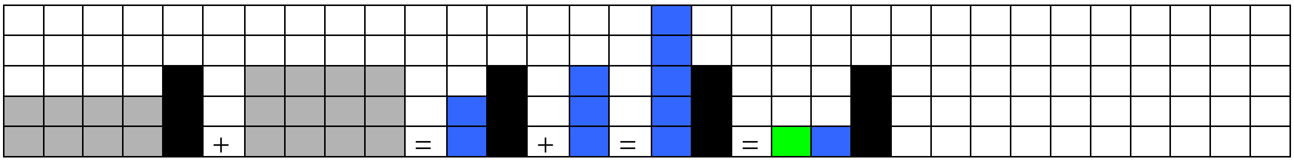
2.3 5s + 1.3 5s = 3.6 5s = 4.1 5s

Or with colors:



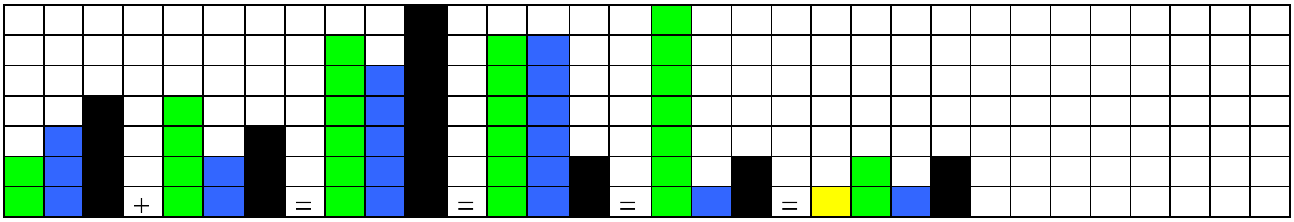
2.3 5s + 1.3 5s = 3.6 5s = 4.1 5s

Likewise, 2.3 4s + 3 4s gives 6 4s where the 4 4s can be rebundled into 1 4 4s, thus calling for an extra cup to the left for the bundles of bundles.



2.3 4s + 3 4s = 2) 3) + 3)) = 5) 3) = 1) 1) 3) = 11.3 4s

Since addition implies overloads, also bundles of bundles can be bundled, calling for extra cups. Thus 23.4 5s + 42.3 5s = 65.7 5s = 121.2 5s:



2) 3) 4) + 4) 2) 3) = 6) 5) 7) = 6) 6) 2) = 7) 1) 2) = 1) 2) 1) 2) = 121.2 5s

Using cup-writing, multiplication can be performed directly, followed by a re-bundling:

2* 34.2 6s = 2* 3)4)2) = 2*3) 2*4) 2*2) = 6)8)4) = 7)2)4) = 1)1)2)4) = 112.4 6s.

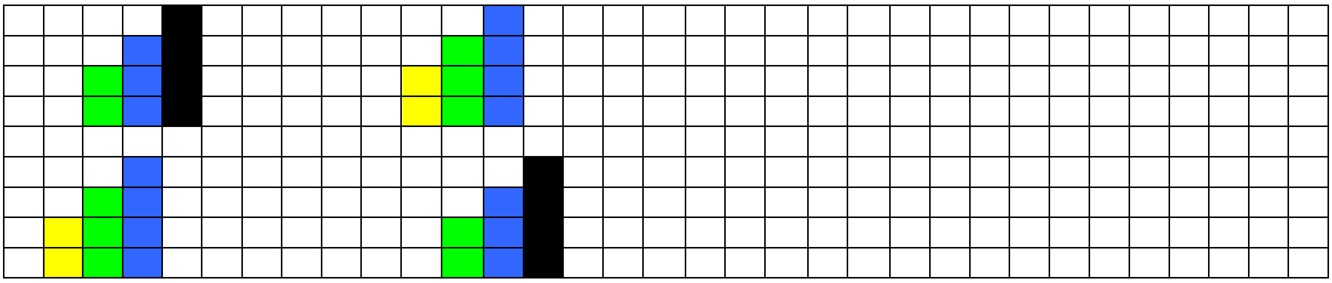
Likewise with division where 34.2 6s / 5 = 4.2 6s + 4 leftovers:

34.2 6s = ? 5S: 3)4)2) = 3*6+4) 2) = 4*5+2) 2) = 4*5) 2*6+2) = 4*5)2*5) + 4) = 5 * 4)2) + 4)

Multiplying a stack with the bundle-number, e.g. 5, just means moving all the stacks to the left, or the decimal to the right. Likewise, dividing a stack with the bundle-number just means moving all the stacks to the right, or the decimal to the left, as seen when using cup-writing:

5 * 2)3)4) = 5*2) 5*3) 5*4) = 2)3)4)0), and

2)3)4)0) = 5*2) 5*3) 5*4) = 5 * 2)3)4), so 2)3)4)0) / 5 = 2)3)4)



23.4 5s * 5 = 234 5s 234 5s /5 = 23.4 5s

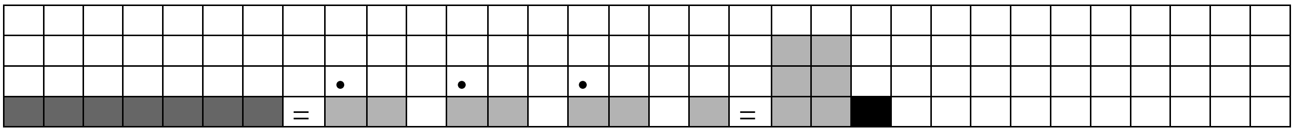
Using cup-writing for addition and subtraction we need not worry about overloads or ‘underloads’, we just rebundle. Thus in the case of 5s the results of the calculations 23.4+42.3 and 123.1–34.2 are

Using 5-bundles:	Using 5-bundles:	Using ten-bundles:	Using ten-bundles:
2)3)4)	1) 2) 3) 1)	7) 8) 9)	7) 3) 5)
+ 4)2)3)	– 3) 4) 2)	+ 8) 5) 6)	– 4) 5) 6)
6)5)7)	1)-1)-1)-1)	15)13)15)	3) -2) -1)
6)6)2)	0) 4)-1)-1)	15)14) 5)	2) 8) -1)
7)1)2)	3) 4)-1)	16) 4) 5)	2) 7) 9)
1)2)1)2)	3) 3) 4)	1) 6) 4) 5)	

Cup-writing can also be written down horizontally:

2)3)4) + 4)2)3) = 6)5)7) = 6)5+1)7–5) = 6)6)2) = 6+1)6–5)2) = 7)1)2) = 0+1)7–5)1)2) = 1)2)1)2).

The result of a bundling&stacking process can be predicted by the recount-formula T = (T/b)*b, iconizing that from the total T b-bundles can be taken away T/b times:



T = 7 = (7/2) 2s + 1 = 3*2 + 1 = 3 2s + 1 = 3.1 2s

The rest is found by using the Rest formula R = T – n*b iconizing that the rest is what is left when the stack is removed from the total.

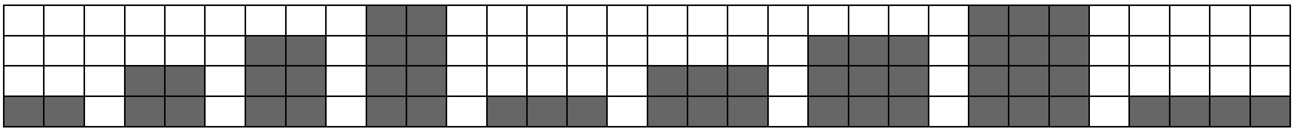
Thus to predict the result of recounting 7 1s in 2s can be predicted by the two formula:

T = 7 = (7/2)*2 = 3*2 + 1 = 3.1 2s since R = 7 – 3*2 = 1. Here the 3 is found on a calculator acting as number-predictor, as the number in front of the decimal point.

Now systematically 1 2s, 2 2s, 3 2s, 4 2s and 5 2s can by recounted in 3s and 4s and 5s etc.

Likewise, 1 3s, 2 3s, 3 3s, 4 3s and 5 3s etc. can by recounted in 4s and 5s and 6s etc.

Thus T = 4 2s = 2.2 3s = 2.0 4s = 1.3 5s etc.



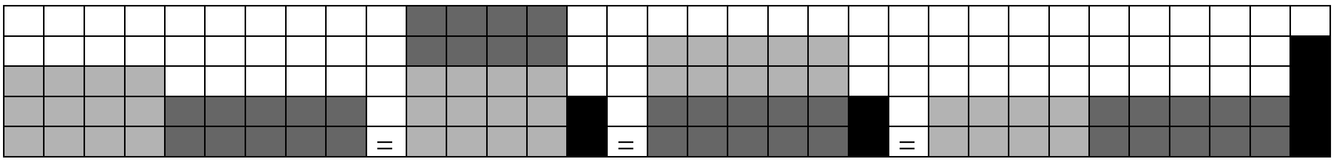
1 2s 2 2s 3 2s 4 2s 1 3s 2 3s 3 3s 4 3s 1 4s

As to adding, the stacks 3 4s and 2 5s can be added in three different ways, as 4s, as 5s or as 9s.

In the case of 4s the 2 5s must be recounted in 4s. In the case of 5s the 3 4s must be recounted in 5s.

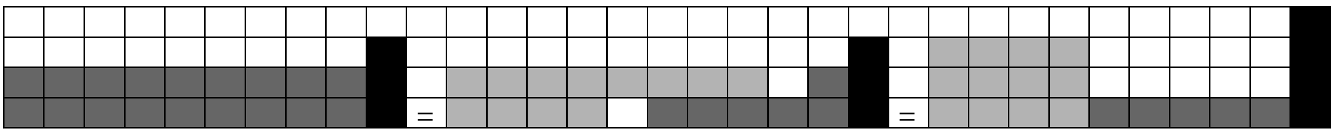
In the case of 9s the two stacks are juxtaposed next to each other, and the surplus is placed as 1s.

This might be called stack-integration by juxtaposition since it is a case of ordinary integration.



3 4s + 2 5s = 5.2 4s = 4.2 5s = 2.4 9s

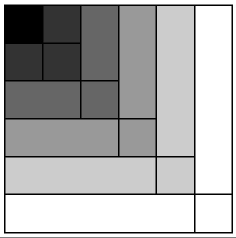
Reversing integration of stacks creates differentiation of stacks. When asking 3 4s + ? 5s = 2.3 9s, first the 3 4s is removed from the 2.3 9s, then the rest is counted in 5s: (2.3*9 – 3*4)/5 * 5 = 1.4 5s = (T – T1)/5 * 5 = □T/□x * 5, which later can be generalized to the rate of change.



T = 2.3 9s = T1 + (T – T1)/5 * 5 = 3 4s + 1.4 5s

When looking at full stacks as 1 1s, 2 2s, 3 3s etc. a pattern occurs when observing that adding two times the bundle-size plus one corner will produce the following full stack:

1 1s
+ 2.1 1s
= 2.0 2s
+ 2.1 2s
= 3.0 3s
+ 2.1 3s
= 4.0 4s



Discovering that n*n + 2.1*n = (n+1)*(n+1)

Grounded Algebra in Middle School

In middle school, the tradition sees the purpose of algebra to be teaching the general rules for adding fractions, i.e. laws for factorizing numbers and polynomials.

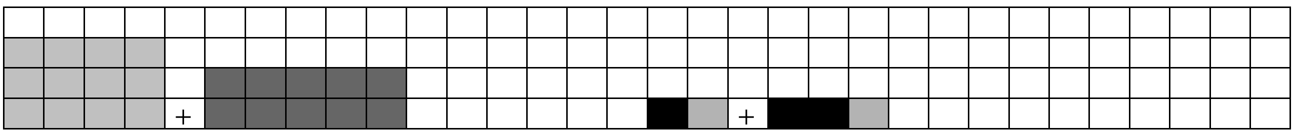
A grounded approach to fractions goes through double-counting, counting a quantity in two different units. A total of 3 can be counted in 1s as T = 3*1, or in 5s as T = (3/5)*5. Thus a grounded alternative does not see fractions as numbers but as operators needing a unit to be meaningful, 2/3 of 3, 2/3 of 7, etc. And a total of apples can be counted in kgs and in $s, resulting e.g. in 4$ per 3kg. The recount-formula from primary school now becomes a per-formula translating kgs to $s and vice versa. Thus to find the price for 12 kg, we just recount the 12kgs in 3s: T = 12 kg = (12/3)*3kg = (12/3)*4$ = 16$. Likewise if we want to find how many kgs 20$ will by, we just recount the 20$ in 4s: T = 20$ = (20/4)*4kg = (20/4)*3kg = 15kg.

Percent means per hundred. Thus we can use the recount-formula to translate per 8 to per 100 and vice versa. To ask ‘3u per 8 = ? per 100’, we just recount the 100 in 8s: T = (100/8)*8 = (100/8)*3u = 37.5u per 100 = 37.5%. Likewise, if we ask ‘20u per 100 = ? per 12’, we just recount the 12 in 100s: T = 12 = (12/100)*100 = (12/100)*20u = 2.4 u per 12.

As operators, fractions carry units when added: 1/2 of 2 + 2/3 of 3 = 3/5 of 5.

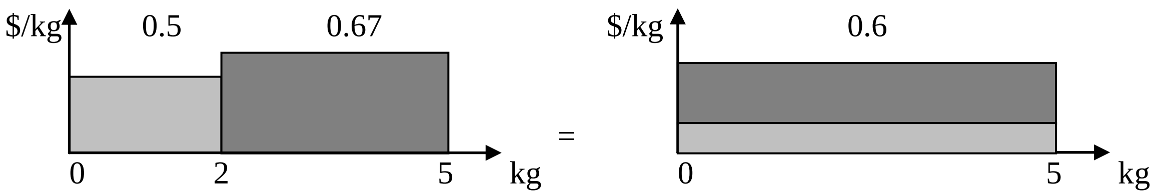
So geometrically, adding fractions is the same as adding stacks by combing their bundles:

3 4s + 2 5s = ? 9s, 3/4 of 4 + 2/5 of 5 = ? of 9.



3 4s + 2 5s = ? 9s 1/2 of 2 + 2/3 of 3 = ? 5s

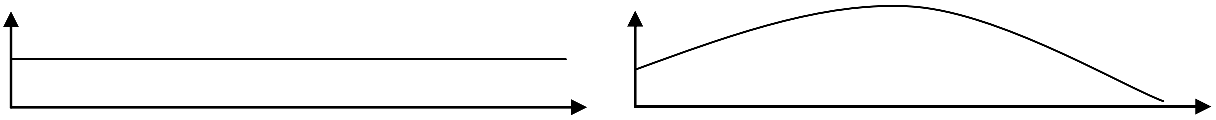
Using decimals instead of fractions, adding fractions becomes a mixture-problem adding quantities with different unit-prices: 1/2 of 2 + 2/3 of 3 becomes 2 kg at 1$/2kg + 3 kg at 2$/3kg, or 2 kg at 0.5 $/kg + 3 kg at 0.67 $/kg = 5 kg at 0.6 $/kg. Here adding fractions means adding the areas under the horizontal fraction-lines.



Grounded Algebra in High School

In high school, the tradition wants algebra to assist calculus in establishing rules for calculating the rate of change Δy/Δx, calling for extensive use of adding and subtracting fractions. The tradition sees calculus as operations on functions, seen as examples of many-to-one relations between sets.

A grounded approach to functions sees the graph as a line that is piecewise or locally constant, enabling the reuse of the mixture-situation where the total is the area under the per-number graph.



Piecewise constant: $\exists\delta>0\ \forall\varepsilon>0: |y-c| < \varepsilon$ in δ Locally constant: $\forall\varepsilon>0\ \exists\delta>0: |y-c| < \varepsilon$ in δ

A grounded alternative respects that algebra means reuniting numbers, in this case per-numbers.

First uniting constant interest rates r into a total rate R leads to a neglected formula 1+R = (1+r)^n, leading to another neglected formula, the saving-formula S/d = R/r predicting the total saving S from a periodical deposit d: If $ d/r is placed on account 1 and the interest r*d/r = d is transferred to account 2, then account 2 contains a saving with a periodical deposit d, but also the total interest of the d/r $ in account 1, so S = d/r*R, or S/d = R/r.

Then variable per-numbers are united when e.g. asking 'what is the total of 5 seconds at 3 m/s increasing to 6m/s' leads to summing up the m/s*s, written as $\Delta T = \int y\,dx = \int (3+\frac{6-3}{5}*x)\,dx$.

It is easy to observe that no matter the number of changes, adding many single changes results in a total change, which can be calculated as the difference between the terminal and the initial values.

y	Δy	ΣΔy
yo = 20		
32	12 = 32-20	12 = 32 – 20
27	-5 = 27-32	7 = 27 – 20
yn = 35	8 = 35-27	15 = yn – yo

In the case of small micro-changes, ΣΔy = yn – yo becomes ∫dy = yn – yo. The change dy can be recounted in x-changes dx as dy = (dy/dx)*dx = y'*dx. So ∫dy = ∫y'dx = yn – yo.

$$\text{Thus } \int_2^7 x\text{^}2\,dx = \int_2^7 \left(\frac{x\text{^}3}{3}\right)'\,dx = \frac{5\text{^}3}{3} - \frac{2\text{^}3}{3} = \frac{117}{3},\ \text{since } \left(\frac{x\text{^}3}{3}\right)' = x\text{^}2$$

The formulas for the rate of change, e.g. dy/dx (x^3), can be found by using 'Calc dy/dx' on a Graphical Display Calculator to set up the results in a table:

x	0	1	2	3	4
dy/dx	0	3	12	27	48

Using regression on a GDC, the formula dy/dx (x^3) = x^2 is generated as a hypothesis that then can deduce predications to be tested on a number of randomly chosen xs:

x	-8	-2	5	28	360	…
dy/dx predicted by the formula (x^3)' = 3*x^2	192	12	75	2352	388800	
dy/dx calculated on a GDC	192	12	75	2352	388800	

In this way we respect the nature of mathematics as a natural science investigating the natural fact many by using the method of the natural sciences to produce knowledge: observe, induce a hypothesis, and deduce predictions for testing the hypothesis.

Conclusion

To deconstruct pastoral choices presented as nature in contemporary algebra, anti-pastoral sophist research has uncovered several hidden alternatives. Historically, the roots of algebra is what its Arabic name says, the task of reuniting numbers. However two kinds of numbers exist, pastoral numbers claiming to be natural; and grounded numbers generated through counting by bundling & stacking as stacks reported by decimal-writing including the unit. Allowing 1.order and 2.order counting to take place before going on to the traditional choice, 3.order counting, means allowing the learners to profit from the many educational activities of 2.order counting, especially re-counting in different bundle-sizes and double-counting in different units. Fractions without units can be deconstructed to show the nature of fractions as operators always carrying units, and that when added with units becomes the root of calculus, which then can be deconstructed to adding variable per-numbers. Also equations seen as equivalence statements about number-names can be deconstructed to show the roots of equations, reversed calculations, which allows the natural method of solving equations, move over and reverse calculation sign, to be introduced as an alternative to the traditional neutralizing method. Finally, to improve the learning of algebra, an alternative algebra curriculum for primary school, middle school and high school is outlined.

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205. Pastoral Calculus Deconstructed

Calculus becomes pastoral calculus killing the interest of the student by presenting limit- and function- based calculus as a choice suppressing its natural alternatives. Anti-pastoral sophist research searching for alternatives to choice presented as nature uncovers the natural alternatives by bringing calculus back to its roots, adding and splitting stacks and per-numbers.

The background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become ’metamatics’ that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox ‘this statement is false’ being false when true and true when false: ’Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$’.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn’t be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and ‘the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics’ (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots guided by a new kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

Ancient Greece saw a fierce controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people needed to be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers argued that patronization is the natural order since everything physical is an example of meta-physical forms only visible to the philosophers educated at Plato’s academy, who then should become the natural patronising rulers.

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronisation with democracy. Two democracies were installed: one in US, and one in France, now having its fifth republic.

In France, sophist warning is kept alive in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronising categories, discourses and institutions presenting their choices as nature (Tarp 2004). Derrida recommends that pastoral categories be ‘deconstructed’. Lyotard recommends the use of postmodern ‘paralogy’ research to invent alternatives to pastoral discourses. And Foucault uses the term ‘pastoral power’ to warn against institutions legitimising their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

In descriptions, numbers and words are different as shown by the ‘number & word dilemma’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘pencil’, thus being able itself to falsify its number but not its word, which makes numbers nature and words choices, becoming pastoral if suppressing their alternatives; meaning that a thing behind a word only shows part of its nature through a word, needing deconstruction to show other parts.

Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what is pastoral choice presented as nature?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of e.g. eight sticks can be treated in different ways. The sticks can be rearranged to an eight-icon 8 containing the eight sticks, written as 8. The sticks can be collected to one eight-bundle, written as 1 8s. The sticks can be ‘decimal-counted’ in 5s by bundling & stacking, bundling the sticks in 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled singles in a right single-cup. When writing down the counting-result, cup-writing gradually leads to decimal-writing where the decimal separates the bundle-number from the single-number:

8 = 1 5s + 3 1s = 1)3) = 1.3 5s

So the nature of numbers is that any total can be decimal-counted by bundling & stacking and written as a decimal number including its unit, the chosen bundle-size.

Since ten is chosen as a standard bundle-size, no icon for ten exists making ten a very special number having its own name but not its own icon. This has big technical advantages as shown when comparing the Arabic numbers with the Roman numbers, that has a special ten-icon X, but where multiplication as XXXIV times DXXVIX is almost impossible to do. But without its own icon ten creates learning problems if introduced to early. So to avoid installing ten as a cognitive bomb in young brains, the core of mathematics should be introduced by using 1digit numbers alone (Zybartas et al 2005).

Also, together with choosing ten as the standard-bundle size, another choice is made, to leave out the unit of the stack thus transferring the stack-number 2.3 tens to what is called a natural number 23, but which is instead a choice becoming pastoral by suppressing its alternatives. Leaving out units might create ‘mathematism’ (Tarp 2004) true in the library where 2+3=is true, but not in the laboratory where countless counterexamples exist: 2weeks+3days = 17days, 2m+3cm = 203cm etc.

The Nature of Operations

Operations are icons describing the process of counting by bundling & stacking.

The division-icon ‘/2’ means ‘take away 2s’, i.e. a written report of the physical activity of taking away 2s when counting in 2s, e.g. 8/2 = 4. The multiplication-icon ‘4*’ means ‘(stacked) 4 times’, i.e. a written report of the physical activity of stacking 2-bundles 4 times, T = 4*2

Subtraction ‘- 2’ means ‘take away 2’, i.e. a written report of the physical activity of taking away the bundles to see what rests as unbundled singles, e.g. R = 9 – 4*2. And addition ‘+2’ means ‘plus 2’, i.e. a written report of the physical activity of adding 2 singles to the stack of bundles either as singles or as a new stack of 1s making the original stack a stock of e.g. T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus the full process of 're-counting' or 're-bundling' 8 1s in 5s can be described by a 'recount or rebundle formula' containing three operations, together with a 'rest formula' finding the rest:

T = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is R = 8 – 1*5 = 3.

All the recount formula T = (T/b)*b says is: the total T is first counted in bs, then stacked in bs.

This recount formula cannot be used with ten as the bundle-size since we cannot ask a calculator to calculate T = (8/ten)*ten. However, this is no problem since the moment ten is chosen as the standard bundle-size, the operations take on new meanings. Now recounting any stack in tens is not done anymore by the recounting formula but by simple multiplication. To re-bundle 3 8s in tens, instead of writing T = (3*8)/10*10 = 2.4 * 10, we simply write T = 3*8 = 24.

The Nature of Formulas

Using the recount formula, the counting result can be partly predicted on a calculator where 9/4 = 2.something. This predicts that recounting 9 in 4s will result in 2 4-bundles and some singles. The number of singles can be predicted by the rest formula R = 9 – 2*4 = 1. So (9/4)*4 is 2.1 4s.

Thus the calculator becomes a number-predictor using calculation for predictions. This shows the strength of mathematics as a language for number-prediction able to predict mentally a number that later is verified physically in the 'laboratory'. Historically, this enabled mathematics to replace pastoral belief with prediction, and to become the language of the natural sciences.

The Nature of Equations

The statement 4 + 3 = 7 describes a bundling where 1 4-bundle and 3 singles are re-bundled to 7 1s. The equation x + 3 = 7 describes the reversed bundling asking what is the bundle-size that together with 3 singles can be re-bundled to 7 1s. Obviously, we must take the 3 singles away from the 7 1s to get the unknown bundle-size: x = 7 – 3. So technically, moving a number to the other side changing its calculation sign solves this equation: If x + 3 = 7, then x = 7 – 3.

The statement 2.1* 3 = 7 describes a bundling where 2.1 3-bundles are re-bundled to 7 1s. The equation x * 3 = 7 describes the reversed bundling asking how 7 1s can be re-bundled to 3s. Using the re-bundling procedure and formula, the answer is T = 7 = (7/3)*3, i.e., x = 7/3. Again technically, moving a number to the other side changing its calculation sign solves this equation: If x * 3 = 7, then x = 7/3.

The statement 2 * 3 + 1 = 7 describes a bundling where 2 3-bundles and 1 single are re-bundled in 1s. The equation x * 3 + 1 = 7 describes the reversed bundling asking how 7 1s can be re-bundled in 3s leaving 1 unbundled. Obviously, we first take the single unbundled away, 7 – 1, and then bundle the rest in 3s, giving the result x = (7-1)/3. Again technically, moving a number to the other side changing its calculation sign solves this equation: If x * 3 + 1 = 7, then x = (7-1)/3.

The statement 2 * 3 + 4 * 5 = 4.2 * 6 describes a bundling where 2 3-bundles and 4 5-bundles are re-bundled in 6s. The equation 2 * 3 + x * 5 = 4.2 * 6 describes the reversed bundling asking how 4.2 6s can be re-bundled to two stacks, 2 3s and some 5s. Obviously, we first take the 2 3s away, and then bundle the rest in 5s. Again technically, moving a number to the other side changing its calculation sign solves this equation: If 2 * 3 + x * 5 = 4.2 * 6, then x = (4.2 * 6 – 2 * 3)/5

The Nature of Calculus

The statement 2 * 3 + 4 * 5 = 3.2 * 8 describes a bundling where 2 3-bundles and 4 5-bundles are re-bundled in the united bundle-size 8s. This is 1digit integration. The equation 2 * 3 + x * 5 = 3.2 * 8 describes the reversed bundling asking how 3.2 8s can be re-bundled to two stacks, 2 3s and some 5s. This is a 1digit differential equation solved by performing 1digit differentiation:

If 2 * 3 + x * 5 = 4.2 * 6, then x = (4.2 * 6 – 2*3)/5 = (T – T1)/5 = ΔT/5

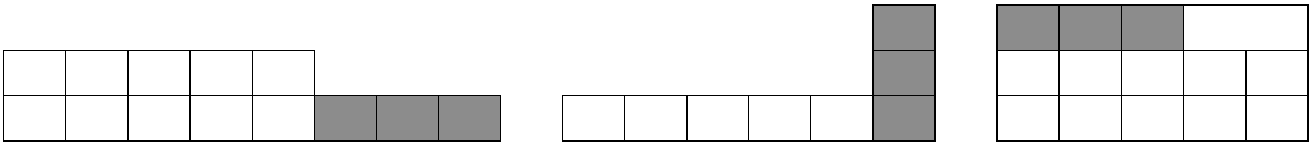
The Nature of Fractions

Once ten has been chosen as the standard bundle-size, the operations take on new meanings. Now recounting any stack in tens is not done anymore by the re-counting formula but by simple multiplication. To re-bundle 3 8s in tens, instead of writing T = (3*8)/10*10 = 2.4 * 10, we simply write T = 3*8 = 24. Now tables are practiced for re-bundling 2s, 3s, 4s etc. in tens.

With multiplication taking over there is no more need for division in re-bundling and recounting. So division takes on a new meaning in ‘per-numbers’: If 2 kg costs 8 $, then the unit-price is 8$ per 2kg, i.e. 8$/2kg = 8/2 $/kg. Thus if 4kg cost 5$, the guide-equation ‘4kg = 5$’ is used when re-counting the actual kg-number in 4s, and re-counting the actual $-number in 5s:

10kg = (10/4)*4kg = (10/4)* 5$ = 12.5$, and 18$ = (18/5)*5$ =(18/5)*4 kg = 14.4kg.

Division is also part of fractions, originally occurring if, instead of placing 3 singles besides the existing stack of 5-bundles, the 3 singles are bundled as a 5-bundle and put on top of the 5-stack giving a stack of T = 2*5 + (3/5)*5 = 2 3/5 *5 = 2 3/5 5s.



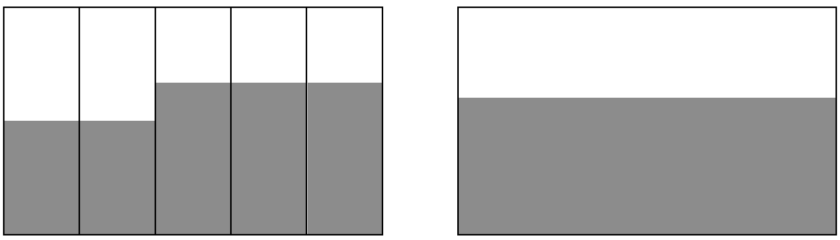
T = 2*5 + 1*3 = 2.3 *5 = 2 3/5 * 5

When adding fractions it is important to reintroduce the units to avoid scaring the learners with mathematism as when performing the following ‘fraction test’ the first day of secondary school:

The teacher:	The students:
Welcome to secondary School! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

That seduction by mathematism is costly is witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. So to add numbers the units must be included, also when adding fractions. And adding fractions f is basically integration:

T = 1/2 *2 + 2/3 *3 = Σ (f*Δx) later to become $\int$ f dx



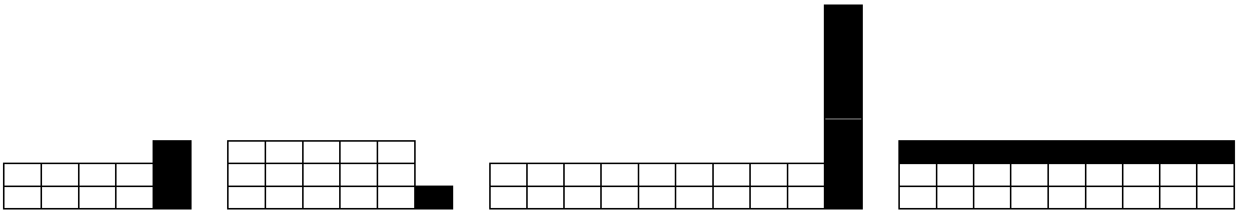
T = 1/2 *2 + 2/3 *3 = 3/5 *5

Primary School Calculus

In primary school integration means integrating two stacks into one where the bundle-size is the sum of the stacks' bundle-sizes. Thus a typical integration problem is 2.3 4s + 3.1 5s = ? 9s.

Manually, integration means placing the two bundle-stacks next to each other; then placing the two 1-stacks on top of each other; then moving any uncompleted bundle to the 1-stack; and finally move any bundles from the 1-stack to the bundle-stack. cup-writing reports this process:

2.3 4s + 3.1 5s = 2)3) + 3)1) = 2)3) + 2)1+1*5) = 2)3+6) = 2)9) = 2+1)0) = 3)0) = 3.0 9s

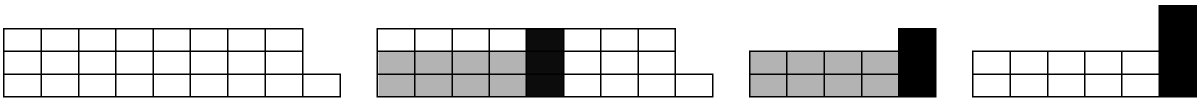


2.3 4s + 3.1 5s = 2.9 9s = 3.0 9s

Reversing the integration of two stacks becomes differentiation. Thus a typical differentiation question is 2.3 4s + ? 5s = 3.1 8s

Manually, differentiation means taking away a stack of bundles and a stack of 1s from a bigger stack; and then recount the rest in the given bundle size. Removing stacks reports this process:

? = (3.1 8s - 2.3 4s)/5 * 5 , later to be written as a differential quotient (T – T1)/5 = ΔT/5.



3.1 8s = 2.3 4s + ? 5s = 2.3 4s + 2.4 5s

Middle School Calculus

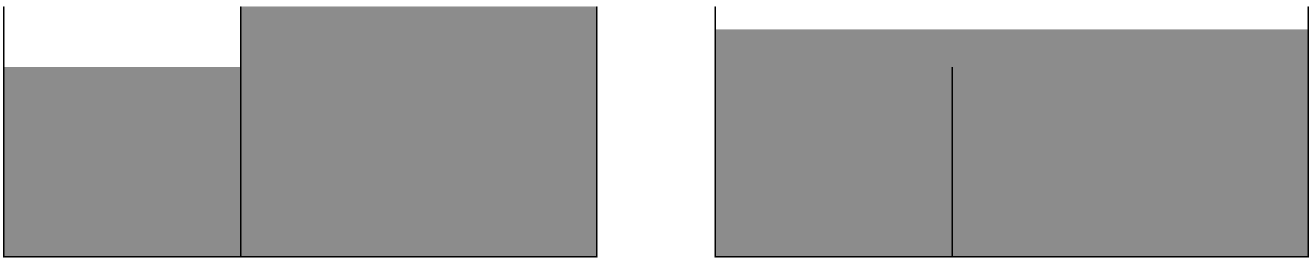
In middle school integration means integrating two fractions or per-numbers into one. Thus a typical integration problem is 1/2 of 2 + 2/3 of 3 = ? of 5; and 10% of 2 + 40% of 3 = ? of 5; and 2kg at 6$/kg + 3kg at 9$/kg = 5kg at ? $/kg.

Manually, integration means drawing next to each other two rectangular pools and then finding the average water-level if the separating wall is removed. Note-writing reports this process:

2 kg at 6 $/kg = 2*6 = 12 $

3 kg at 9 $/kg = 3*9 = 27 $

5 kg at ? x/kg = 5*x = 39 $, so x = 39/5 = 7.8 $/kg



1/2 of 2 + 2/3 of 3 = ? of 5

Reversing the integration of two pools becomes differentiation. Thus a typical differentiation question is 2kg at 5$/kg + 3kg at ?$/kg = 5kg at 6 $/kg.

Manually, differentiation means drawing two rectangular pools next to each other and then finding the resulting water-level of the right pool after water is pumped from the left pool.

Mentally, combined subtraction and division reports the manual process:

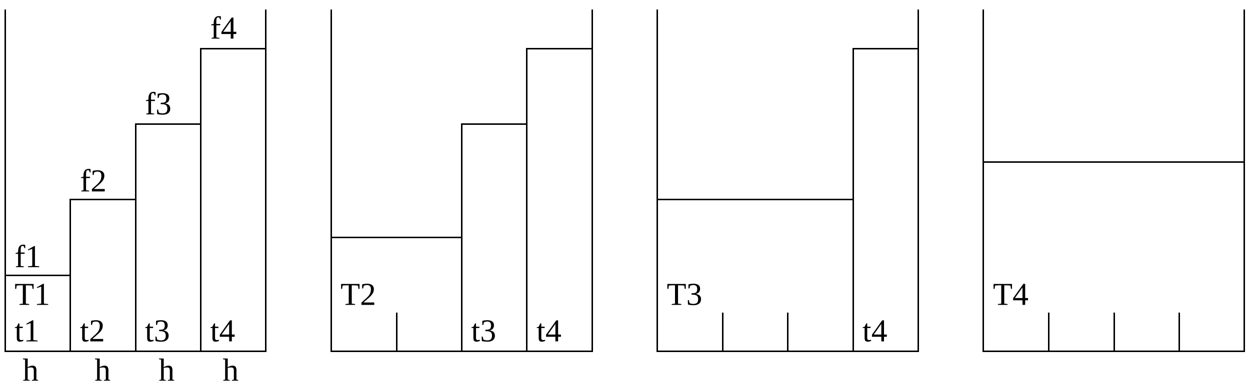
2*5 + 3*x = 5*6 , x = (5*6 – 2*5)/3 , later to be written as (T – T1)/5 = ΔT/5.

High School Calculus

In high school, integration means integrating many per-numbers into one. Thus a typical integration problem is: 7 seconds at 2 m/s increasing to 4 m/s totals 7 seconds at ? m/s in average.

Manually, integration means drawing next to each other many rectangular micro-pools with width h and water level described by a formula f; and then finding the formula describing the water level when the walls are removed one by one.

Mentally, cumulating describes the manual process: The volume of pool1 is t1 = f1*h. The total volume then is T4 = T3 + t4 = t1 + t2 + t3 + t4 = Σ ti = Σ fi * h

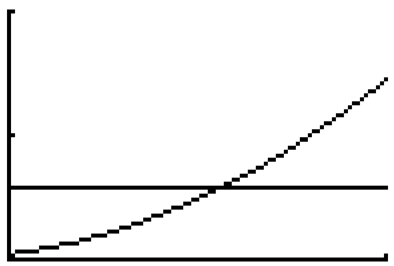


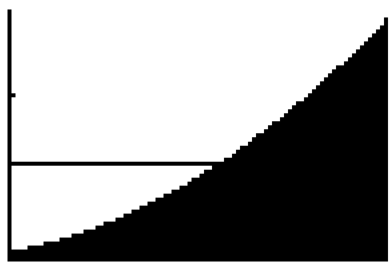
Reversing the integration of pools becomes differentiation. Thus a typical differentiation question is: 5 seconds at 1.82 m/s in average + 0.1 second at ? m/s totals 5.1 seconds at 1.84 m/s in average.

Manually differentiation means drawing two rectangular pools next to each other and then finding the resulting water-level of the right pool after water is pumped from the left pool.

Mentally combined subtraction and division reports the manual process:

T3 + f4*h = T4 , f4 = (T4 – T3)/h , now to be written as ΔT/h = dT/dx if h = dx is mikro-small.





In the case of many micro-pools a Graphical Display Calculator (GDC) can do the drawing.

The Choices of Pastoral Calculus

Modern calculus still believes in the existence of sets despite of Russell’s paradox. Instead of introducing calculus as integrating per-numbers in middle school it waits for algebra to define the real numbers. Then the concept of a limit can be given its ε–δ definition by calculus believing this gives a precise meaning to the term ‘mikro-small’; but forgetting that δ is the mikro-number giving the level of exactness described by ε.

So instead of working with mikro-numbers, modern calculus presents both the derivative and the integral as examples of the concept limit, which creates big problems to learners. Also the so-called fundamental theorem of calculus by it naming alone is presented as a deep insight where instead it is a mere banality as shown below. .

The Natural Alternatives of Anti-pastoral Calculus

To uncover natural alternatives to the choices of modern calculus, becoming pastoral by suppressing its alternatives, anti-pastoral sophist research sees calculus as a natural science exploring the natural fact multiplicity rearranged by bundling and stacking. In this approach the roots of calculus is found to be adding two stacks in a combined bundle-size, as shown above. Later calculus also applies when adding per-numbers as $/kg or m/s. Here integration means predicting the area under the per-number graph to predict the total $-number or m-number; and differentiation means predicting the gradient on the total $-graph or m-graph to predict the per-number.

As to finding gradient formulas given a total formula, and finding area-formulas given a per-number formula, the method of natural science can be used involving observation, induction and testing predictions.

First the GDC’s gradient calculator dy/dx is validated if its predictions are verified and never falsified on examples with known gradient. Thus no matter where the dy/dx-number is calculated on the graph y = 2.7 x + 4, the answer is the expected dy/dx = 2.7.

Now a table can be set up for the relation between x and the gradient-number dy/dx on the graph y = x^2. Using regression, it turns out that a gradient-formula dy/dx = 2x can be induced and used for deducing predictions that all are verified. In this way the experimental method of natural science can be used to find the different gradient-formulas.

Next the GDC’s area calculator ∫f(x)dx is validated if its predictions are verifying and never falsified on examples with known area. Thus no matter where the area-number is calculated on the graph y = 2, the answer agrees with the known formula A = 2*(x2-x1).

Now a table can be set up for the relation between x and the area-number under the graph y = x^2 from 0 to x. Using regression, it turns out that an area-formula A = x^3/3 can be induced and used for deducing predictions that are all verified. In this way the experimental method of natural science can be used to find the different area-formulas.

As to finding the area-number using the fundamental theorem of Calculus, as simple observation shows that the following statement cannot be falsified:

The total change = the cumulated step-change = y end – y start, or $\Delta y = \Sigma \Delta y = y2 - y1$.

This statement does not depend upon the size or number of changes, so it also applies for small mikro-changes:

$\Delta y = \int dy = y2 - y1$, or $\Delta y = \int y' \, dx = y2 - y1$ if dy is re-counted in dxs as dy = dy/dx * dx.

So when calculating the area under an f-graph, if f can be re-written as a change-formula f(x) = dy/dx, then the area ∫f(x) dx can be written as ∫dy and calculated as the difference y2 - y1.

Number y	Step-change Δy	Cumulated step-change ΣΔy	Total change Δy = y2 – y1
2			
5	3	3	3
4	-1	2	2
9	5	7	7

Conclusion

Modern mathematics finds it natural to postpone calculus until the end of high school or the beginning of university, wanting to present it as metamatics, i.e. as an example of the higher abstractions as sets, functions, real numbers and limits; in spite of the fact that historically calculus was developed before these abstractions. However, this choice turns out to be a pastoral choice suppressing its natural alternatives uncovered by anti-pastoral sophist research searching for alternatives to choice presented as nature. The natural alternative is to introduce calculus in primary school as adding stacks in united bundle-size, and to reintroduce calculus in middle school as adding per-number and fractions with units, and finally using the methods of natural science to make high school calculus limit-free.

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206. Applying Pastoral Metamatism or Re-Applying Grounded Mathematics

*When an application-based mathematics curriculum supposed to improve learning fails to do so, two questions may be raised: What prevents it from improving learning? And is 'mathematics applications' what it says, or something else? Skepticism towards wordings leads to postmodern thinking that, dating back to the ancient Greek sophists, warns against patronizing pastoral categories, theories and institutions. Anti-pastoral sophist research, identifying hidden alternatives to pastoral choices presented as nature, uncovers two kinds of mathematics: a grounded mathematics enlightening the physical world, and a pastoral self-referring mathematics wanting to 'save' humans through 'metamatism', a mixture of 'metamatics' presenting concepts as examples of abstractions instead of as abstractions from examples; and 'mathematism' true in the library, but seldom in the laboratory. Also 'applying' could be reworded to 're-applying' to emphasize the physical roots of mathematics. Three preventing factors are identified: 'ten=10'-centrism claiming that counting can only take place using ten-bundles; fraction-centrism claiming that proportionality can only be seen as applying fractions; and set-centrism claiming that modelling can only take place by applying set-based concepts as functions, limits etc. In contrast, an implying factor is grounded mathematics created through modelling the natural fact many by counting many in bundles &stacks; and by predicting many by a recount-formula T = (T/b)*b that can be re-applied at all school levels.*

Applying Mathematics Improves Learning – or Does it?

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998), and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994: 371). To improve learning it has been suggested that applications and modelling should play a more central role in mathematics education.

However, when tested in the classroom the result is not always positive: 30 years ago the pre-calculus course at the Danish second-chance high school changed from being application-free to being application-based by replacing e.g. quadratic functions with exponential functions. Still student performance deteriorated to such a degree that at the 2005 reform the teacher union and the headmasters suggested that pre-calculus should no more be a compulsory subject.

Thus two questions can be raised: Why did this application-based curriculum not improve learning? And is 'mathematics applications' what it says, or something else that might make a difference? Postmodern thinking, dating back to the ancient Greek sophists, has identified the hidden patronization in fixed wordings.

Anti-Pastoral Sophist Research

Ancient Greece saw a struggle between the sophists and the philosophers as to the nature of knowledge. The sophists warned that to protect democracy people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers, seeing everything physical as examples of meta-physical forms only visible to them, patronization was a natural order if left to the philosophers (Russell 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spices. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America.

Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea to avoid Portuguese forts was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people should do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. The US still has its first republic; France now has its fifth.

The German autocracy tried to stop the French democracy by sending in an army. However, the German mercenaries were no matches to the French conscripts only too aware of the feudal consequences of loosing. So the French stopped the Germans, and later occupied Germany.

Unable to use the army, the German autocracy used the school to stop the enlightenment spreading from France. Humboldt was asked to create an elite school, and used Bildung as counter-enlightenment to create the self-referring Humboldt University (Denzin et al 2000: 85).

Inside the EU the sophist warning is kept alive in the French postmodern or post-structural thinking of Derrida, Lyotard and Foucault warning against patronizing categories, discourses and institutions presenting their choices as nature (Tarp 2004).

Derrida recommends that patronizing categories, called logocentrism, be ‘deconstructed’:

> Derrida encourages us to be especially wary of the notion of the centre. We cannot get by without a concept of the centre, perhaps, but if one were looking for a single ‘central idea’ for Derrida’s work it might be that of decentring. It is in this very general context that we might situate the significance of ‘poststructuralism’ and ‘deconstruction’: in other words, in terms of a decentring, starting with a decentring of the human subject, a decentring of institutions, a decentring of the logos. (Logos is ancient Greek for ‘word’, with all its connotations of the authority of ‘truth’, ‘meaning’, etc.) (..) It is a question of the deconstruction of logocentrism, then, in other words of ‘the centrism of language in general’. (Royle 2003: 15-16)

As to discourses Lyotard coins the term ‘postmodern’ when describing ‘the crisis of narratives’:

> I will use the term modern to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..) Simplifying to the extreme, I define postmodern as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

Foucault calls institutional patronization for ‘pastoral power’:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1982: 213, 215)

In this way Foucault opens our eyes to the salvation promise of the generalized church: ‘you are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will save you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and accept becoming a docile lackey’.

To Foucault, institutions building on discourses building on categories build upon choice, so they all have a history, a ‘genealogy’ that can be uncovered by ‘knowledge archeology’.

The French skepticism towards words, our most fundamental institution, is validated by a ‘number&word observation’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘stick’; thus it can itself falsify its number but not its word, which makes numbers nature and words choices becoming pastoral if hiding their alternatives.

On this basis a research paradigm can be created called ‘anti-pastoral sophist research’ deconstructing pastoral choices presented as nature by discovering hidden alternatives. Anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘in this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the natural research method developed in the American enlightenment democracy and resonating with Piaget’s principles of natural learning (Piaget 1970).

A Historical Background

The natural fact many provoked the creation of mathematics as a natural science addressing the two fundamental human questions ‘how to divide the earth and what it produces?’

Distinguishing the different degrees of many leads to counting that leads to numbers.

1.order counting counts in 1s and creates number-icons by rearranging the sticks so that there are five sticks in the five-icon 5 if written in a less sloppy way.

2.order counting counts by bundling&stacking using numbers with a name and an icon, resulting in a double stack of bundled and unbundled, e.g. T = 3 5s + 2 1s = 3)2) = 3.2 5s = 3.2*5 if using cup-writing and decimal-writing separating the left bundle-cup from the right single-cup. The result can be predicted by the ‘recount-formula’ T = (T/b)*b iconizing that counting in bs means taking away bs T/b times, e.g. T = (4*5)/7*7 = 2*7 + 6*1 = 2)6) = 2.6*7.

3.order counting counts in tens, having a name but not an icon since the bundle-icon is never used: counting in 5s, T = 5 1s = 1 5s = 1.0 bundle = 10 if leaving out the decimal and the unit.

In Greek, mathematics means knowledge, i.e. what can be used to predict with, making mathematics a language for number-prediction: The calculation ‘2+3 = 5’ predicts that repeating counting 3 times from 2 will give 5. ‘2*3 = 6’ predicts that repeating adding 2 3 times will give 6. ‘2^3 = 8’ predicts that repeating multiplying with 2 3 times will give 8. Also, any calculation can be turned around and become a reversed calculation predicted by the reversed operation: In the question ‘3+x = 7’ the answer is predicted by the calculation x = 7–3, etc.

Thus the natural way to solve an equation is to move a number across the equation sign from the left forward- to the right backward-calculation side, reversing its calculation sign.

3+x = 7	3*x = 7	x^3 = 7	3^x = 7
x = 7–3	x = 7/3	x = 3√7	x = log3(7)

In Arabic, algebra means reuniting, i.e. splitting a total in parts and (re)uniting parts into a total. The operations + and * unite variable and constant unit-numbers; ∫ and ^ unite variable and constant per-numbers. The inverse operations – and / split a total into variable and constant unit-numbers; d/dx and √ & log split a total into variable and constant per-numbers:

Totals unite/split into	Variable	Constant
Unit-numbers $, m, s, ...	T = a + n $T - n = a$	T = a * n $T/b = a$
Per-numbers $/m, m/s, m/100m = %, ...	ΔT = ∫ f dx $dT/dx = f$	T = a ^ n $\sqrt[n]{T} = a$, $log_a T = n$

In Greek, geometry means earth measuring. Earth is measured by being divided into triangles, again being divided into right-angled triangles, each seen as a rectangle halved by a diagonal.

Recounting the height h and base b in the diagonal d produces three per-numbers:

sinA = height/diagonal = h/d, tanA = height/base = h/b, cosA = base/diagonal = b/d.

Also a circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: C = 2 * r * (n*sin(180/n)) = 2 * r * π for n sufficiently big.

However, having to do without the Arabic numbers, Greek geometry turned into Euclidean geometry, freezing the development of mathematics until the Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. The seventeen-century men had broken both of these bonds. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

The success was so overwhelming that mathematicians feared that mathematics (called geometry at that time) had come to a standstill at the end of the 18th century:

> Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities. (Lagrange in Kline 1972: 623)

But in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. So in the 1870s the concept 'set' reintroduced rigor into mathematics.

Mathematics Versus Metamatics

Using sets, a function is defined 'from above' as a set of ordered pairs where first-component identity implies second-component identity; or phrased differently, as a rule assigning exactly one number in a range-set to each number in a domain-set. The Enlightenment defined function 'from below' as an abstraction from calculations containing a variable quantity:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1748: 3)

So where the Enlightenment defined a concept as an abstraction from examples, the modern set-based definition does the opposite; it defines a concept as an example of an abstraction. To tell theses alternatives apart we can introduce the notions ‘grounded mathematics’ abstracting from examples versus ‘set-based metamatics’ exemplifying from abstractions, and that by proving its statements as deductions from meta-physical axioms becomes entirely self-referring needing no outside world. However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox ‘this statement is false’ being false when true and true when false:

Definition $M = \{A \mid A \notin A\}$, statement $M \in M \Leftrightarrow M \notin M$.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn’t be proven consistent since they will always contain statements that can neither be proved nor disproved.

Still, set-based mathematics soon found its way to the school even if it creates syntax errors:

A formula containing two variables becomes a function, e.g. y = 2*x+3 = f(x) where f(x) = 2*x+3 means that 2*x+3 is a formula containing x as the variable number. A function can be tabled and graphed, both describing if-then scenarios ‘if x = 6 then y = 15’. But writing f(6) = 15 means that 15 is a calculation containing 6 as the variable number. This is a syntax error since 15 is a number, not a calculation, and since 6 is a number, not a variable. Functions can be linear, quadratic, etc., but not numbers. So claiming that at function increases is a syntax error.

Set-based metamatics defines a fraction as an equivalence set in a product set of two sets of numbers such that the pair (a,b) is equivalent to the pair (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same fraction ½. However, this definition conflicts with Russell’s set paradox, solved by Russell by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Thus a fraction that is defined as a set of numbers is not a number itself, making additions as ‘2+3/4’ meaningless.

Wanting fractions to be ‘rational’ numbers, set-based mathematics has chosen to neglect Russell’s type-theory by accepting the Zermelo-Fraenkel axiom system making self-reference legal by not distinguishing between an element of a set and the set itself. But removing the distinction between examples and abstractions and between different abstraction levels means hiding that historically mathematics developed through layers of abstractions; and that mathematics can be defined through abstractions in a meaningful and uncontroversial way.

Mathematics Versus Mathematism

Traditionally, both 2+3 = 5 and 2*3 = 6 are considered universal true statements. The latter is grounded in the fact that 2 3s can be recounted as 6 1s. The first, however, is an example of ‘mathematism’ true in a library, but not in a laboratory where countless counter-examples exist: 2weeks + 3 days = 17 days, 2m + 3cm = 203cm etc. Thus addition only holds inside a bracket assuring that the units are the same: 2m + 3cm = 2*100cm + 3cm = (200 + 3)cm = 203cm.

Adding fractions without units is another example of mathematism:

Inside the classroom	20% (20/100) + 10% (10/100)	= 30% (30/100)
Outside the classroom e.g. in the laboratory	20% + 10%	= 32% in the case of compound interest = b% (10<b<20) in the case of a weighted average

Mathematics Modelling in Primary School

Having learned how to assign numbers to totals through counting by bundling&stacking, a real-world question as 'what is the total of 2 fours and 3 fives' can lead to two different models.

Model 1 says: This question is an application of addition. The mathematical problem is to find the total of 2 and 3. Applying simple addition, the mathematical solution is 2 + 3 = 5, leading to the real-world solution 'the total is 5'. An answer that is useless and incorrect because it has left out the unit.

Model 2 says: This question is a re-application of recounting. The mathematical problem is to find the total of 2 4s and 3 5s. Re-applying recounting to find a mathematical solution, the units must be the same before adding so we recount the 2 4s in 5s, predicted by the recount-formula T = (2*4)/5*5 = 1 5s + 3 1s, giving the total of T = 4 5s + 3 1s = 4.3 5s, which leads to the real-world solution 'the total is 4 fives and 3 ones'. A prediction that holds when tested:

IIII IIII + IIIII IIIII IIIII -> IIIII III + IIIII IIIII IIIII -> IIIII IIIII IIIII IIIII III = 4)3) = 4.3 5s

This example shows that applying mathematism may lead to incorrect solutions when modelling addition problems. Whereas applying grounded mathematics creates the categories 'stack' and 'recounting', and allows practicing recounting by asking e.g. 2 fours = ? fives.

Recounting a stack in a different bundle-size is a brilliant example of a modelling process:

	Problem		Solution
Mathematics	T = 2*4 = ?*5	→	T = (2*4)/5*5 = 1.3 5s
	↑		↓
Real-world	IIII IIII -> ? 5s		IIII IIII -> IIIII III

Rephrasing the problem to 'what is the total of nines in 2 fours and 3 fives' introduces integration already in primary school. As a matter of fact, the core of mathematics can be introduced as application of recounting, using 1digit numbers alone (Zybartas et al 2005).

However, this is impossible in a 'ten=10'-curriculum that by presenting 10 as the follower of nine introduces at once the number ten as the standard bundle-size, a pastoral choice hiding that also other numbers can be used as bundle-size. 10 simply means bundle, i.e. 1.0 bundle if not excluding the unit. Thus counting in 7s, 10 is the follower of 6, and the follower of nine is 13.

With ten as bundle-size, recounting-problems disappear, and all numbers loose their units, which creates the basis for teaching mathematism where 3 + 2 IS 5 without discussion.

Thus in primary school an application-based curriculum using recounting to learn the modelling process is prevented by a pastoral choice, 'ten=10'-centrism, hiding that also other numbers can be used when counting by bundling&stacking. And prevented by mathematism claiming that 3 + 2 IS 5.

Mathematical Modelling in Middle School

Middle school introduces fractions as rational numbers and allows them to be added without units in spite of the fact that fractions are multipliers carrying units: 1/3 of 6 = 1/3*6.

The real-world question 'what is the total of 1 coke among 2 bottles and 2 cokes among 3 bottles?' can lead to two different models.

Model 1 says: This question is an application of adding fractions. The mathematical problem is to find the total of 1/2 and 2/3. Applying simple addition of fractions, the mathematical solution is 1/2 + 2/3 = 3/6 + 4/6 = 7/6, leading to the real-world solution '7 out of the 6 bottles are cokes'. An answer that is meaningless and useless because it has left out the unit: we cannot have 7 cokes if we only have 6 bottles; and we do not have 6 bottles, we only have 5.

Model 2 says: This question is a re-application of adding stacks by integrating their bundles. The mathematical problem is to find the total of 1/2 of 2 and 2/3 of 3. Re-applying integration, the mathematical solution is T = 1/2 * 2 + 2/3 * 3 = 3 = 3/5 * 5, giving the real world solution 'the total is 3 cokes of 5 bottles'. A prediction that holds when tested on a lever carrying to the left 15 units in the distance 2 and 20 units in distance 3, and to the right 18 units in distance 5.

Sharing-problems asking 'the boys A, B and C paid $1, $2 and $3 to a pool buying a lottery ticket. How should they share a 300$ win?' can lead to two different models.

Model 1 says: This question is an application of fractions. The mathematical problem is to split a total of 300 in the proportions 1:2:3. Applying simple addition of fractions gives the answer: since boy A paid 1/(1+2+3) = 1/6 of the ticket he should receive 1/6 of the win, i.e. 1/6 of $300 = 1/6*300 = 50; likewise with the other boys: boy B will get 2/6 and boy C 3/6 of 300$. So the real-world solution is: boy A $50, boy B $100, and boy C $150. Of course such questions can only be answered after fractions and its algebra has been taught and learned.

Model 2 says: This question is simply a re-application of recounting. The mathematical problem is to recount the win in pools, i.e. in 6s, which then can be paid back to the boys a certain number of times. Since 300 = (300/6)*6 = 50*6, the boys are paid back 50 times. So the real world solution is A: $1*50 = $50, B: $2*50 = $100, and C: $3*50 = $150.

Trade-problems as 'if the cost is 2$ for 5kg, what then is the cost for 14kg, and how much can I buy for 6$?' can lead to three different models.

Model 1 says: This question is an application of proportionality, fractions and equations. The mathematical problem is to set up an equation relating the unknown to the 3 known numbers. Applying proportionality, fractions and equations we can set up a fraction-equation expressing that the cost c and the volume v is proportional, c/v = k. Hence c1/v1 = c2/v2, or 2/5 = x/14 and 2/5 = 6/x. Now the x can be found by solving the equations, or by cross-multiplication. So the real-world solution is 5.6$ and 15 kg. Of course such questions can only be answered after fractions and proportionality and equations has been taught and learned.

Model 2 says: This question is an application of linear functions. The mathematical problem is to set up a linear function expressing the price y as a function of the volume x, y = f(x) = m*x + c, given that the points (0,0) and (5,2) belongs to the graph of the function. The mathematical solution first finds c by inserting the point (0,0) in the formula: f(0) = m*0+c = 0, so c = 0; then we find m by inserting the point (5,2) in the formula: f(5) = m*5 = 2, so m = 2/5 = 0.4. Hence the linear formula is f(x) = 0.4*x. To answer the questions we insert the points (14,y) and (x,6) into the function: f(14) = 0.4*14 = y, and f(x) = 0.4*x = 6. Solving these equations give y = 5.6 and x = 15. So the real-world solution is 5.6$ and 15 kg. Of course such questions can only be answered when general functions, linear functions and equations has been taught and learned.

Model 3 says: This question is a re-application of recounting. The mathematical problem is to recount the 14kg in 5s and the 6$ in 2s since the cost is 2$ per 5kg. Thus the mathematical solution is 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$, and 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg.

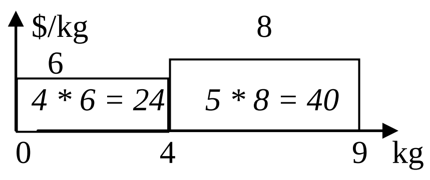
The examples show that many problems in middle school are re-applications of recounting from primary school; unless they are presented as being only solvable by applying fractions, proportionality and equations, in which case the modelling has to wait until these subjects has been taught and learned, which also excludes students unable to learn ungrounded mathematics.

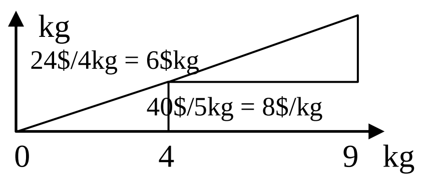
Again, unreflectively applying mathematism may lead to incorrect solutions. Whereas using grounded mathematics replaces fractions with the category per-numbers coming from double-counting in two different units, in 1s and 5s: 3 1s = (3/5)*5, or in $ and kg: 2$/5kg = 2/5 $/kg.

Adding numbers with units also occurs when modelling mixture situations, generalizing primary school's integrating stacks to middle school integral and differential calculus.

Thus asking 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg can be answered by using a table or a graph, realizing that integration means finding the area under the per-number graph; and vice versa, that the per-number is found as the gradient on the total-graph

4kg at 6$/kg	= 4 * 6 = 24
5kg at 8$/kg	= 5 * 8 = 40
9kg at ?$/kg	= 9 * ? = 64
	? = 64/9





Mathematical Modelling in High School

High school claims set-based functions to be its basis: a quantity growing by a constant number IS an example of a linear function; and a quantity growing by a constant percent IS an example of an exponential function; and both ARE examples of the set-based function concept.

Thus the real-world problem '200$ + ? days at 5$/day is 300$' leads to two different models: model 1 seeing the question as an application of linear functions; and model 2 seeing the question as a re-application of a formula stating that with constant change, the terminal number T is the initial number b added with the change m a certain number of times x: T = b + m*x. Inserting T = 300, b = 200 and m = 5 and using the Math Solver on a Graphical Display Calculator, the solution is found as x = 20. This prediction can be tested when graphing the function y = 200 + 5*x and observing that tracing x = 20 gives y = 300.

Cumulating a capital C by a yearly deposit p and interest rate r leads to two different models: model 1 seeing the question as an application of a geometric series; and model 2 setting up two accounts, one with the amount p/r from which the yearly interest p/r*r = p is transferred to the other, which after n years contains the cumulated interest p/r * R where by 1+R = (1+r)^n as well as the generated capital C. And C = p/r * R gives a beautiful a simple formula: C/p = R/r.

Two different models come out of the real-world problem 'Out driving, Peter observed the speed to be 6, 18, 11, 12 m/s after 5, 10, 15 and 20 seconds. What was the speed after 6 seconds? When was the speed 15m/s? When did he stop accelerating? When did he begin to accelerate again? What was the total distance traveled from 7 to 12 seconds?

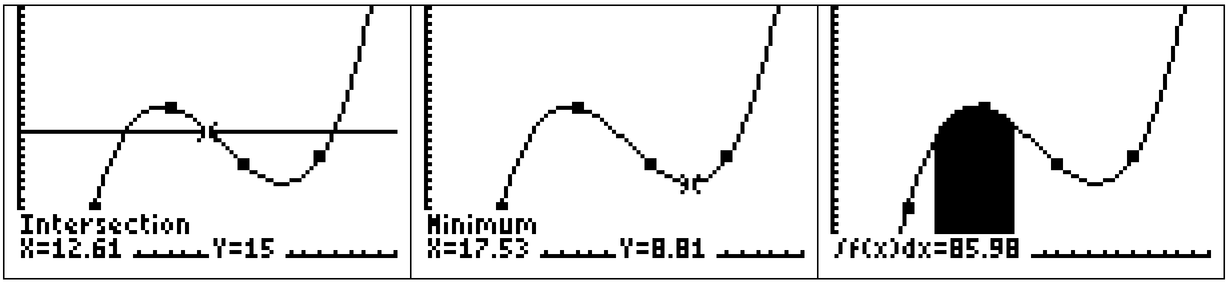
Model 1 says: This question is an application of matrices and differential and integral calculus. The mathematical problem is to set up a function expressing the distance y as a function of the time x, given that the function's graph contains the points (5,6), (10,18), (15,11), and (20,12). Applying matrices to solve 4 equations with 4 unknowns, the mathematical solution is y = 0.036 x^3 – 1.46 x^2 + 18 x – 52. Now the point (6,0) is inserted in the function to find y = 11.22. Inserting the point (x,15) in the function leads to a 3^{rd} degree equation.

To solve this equation we guess a solution in order to factorize the 3rd degree polynomial to $y = 0.036(x - 7.07)(x - 12.61)(x - 20.87)$. To find the turning points we must find the zeros of the derivative $y' = 0.108\,x^2 - 2.92\,x + 18$, i.e. $x = 9.51$ and $x = 17.53$, as well as the signs of the double-derivative $y'' = 0.216\,x - 2.92$ changing sign from minus to plus in $x = 13.52$. Finally the distance traveled from 7 seconds to 12 seconds comes from the integral:

$$\int_7^{12} (0.036\,x^3 - 1.46\,x^2 + 18\,x - 52)\,dx = \left[\,0.009\,x^4 - 0.49\,x^3 + 9\,x^2 - 52x\,\right]_7^{12} = 85.98.$$

Of course, this must wait till after matrices, polynomials and calculus are taught and learned.

Model 2 says: This question is a re-application of per-numbers. The mathematical problem is to find a per-number formula f(x) from a table of 4 data sets. On a Graphical Display Calculator Lists and CubicRegression do the job. Tracing x = 6 gives y = 11.22. Finding the intersection points with the line y = 15 using Calc Intersection gives x = 7.07, 12.61 and 20.87. Finding the turning points using Calc Minimum and Calc Maximum gives a local maximum at x = 9.51 and y = 18.10, and a local minimum at x = 17.53 and y = 8.81. The total meter-number from 7 to 12 seconds is found by summing up the m/s*s, i.e. by using Calc ∫f(x)*dx, which gives 85.98.



Change Equations
Solving any change-equation dy/dx = f(x,y) is easy when using technology. The change-equation calculates the change dy that added to the initial y-value gives the terminal y-value, becoming the initial y-value in the next period. Thus is dy = r*y, r = ro*(1 – y/M) is the change-equation if a population y grows with a rate r decreasing in a linear way with the population having M as its maximum. A spreadsheet can keep on calculating the formula y + dy -> y.

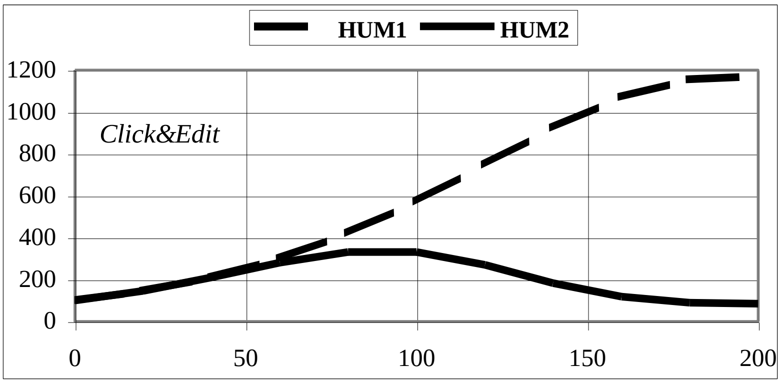
The Grand Narratives of the Quantitative Literature
Literature is narratives about real-world persons, actions or phenomena. Quantitative literature also has its grand narratives. That an infinity of numbers can be added by one difference if the numbers can be written as change-numbers is a grand narrative: If y-changes dy are recounted in x-changes dx: dy = (dy/dx)*dx = y'*dx, then ∫y'dx = ∫dy = Δy = y2 – y1:

Since x^2 = (x^3/3)', then ∫x^2 dx = ∫(x^3/3)' dx = 7^3/3 – 2^3/3 if summing from 2 to 7.

In physics, grand narratives can be found among those telling about the effect of forces, e.g. gravity, producing parabola orbits on earth, and circular and ellipse orbits in space. Jumping from a swing is a simple example of a complicated model. Physics' grand narratives enabled the rise of the Enlightenment and the modern democracy replacing religion with science.

In economics, an example of a grand narrative is Malthus' 'principle of population' comparing the linear growth of food production with the exponential growth of the population; and Keynes' model relating demand and employment creating the modern welfare society. As are the macroeconomic models predicting effects of taxation and reallocation policies. Also limit-to-growth models constitute grand narratives predicting the future population depending on different assumptions as to e.g. food and pollution: leaving out pollution only food will restrict human growth, but including pollution the population level might be different:

Time	HUM1	HUM2
0	100	100
20	145	145
40	210	210
60	301	280
80	421	331
100	572	330
120	747	269
140	925	182
160	1071	118
180	1156	89
200	1171	84



Using Applied Ethnography to Test a Set-free Pre-calculus Curriculum

Working together with Mogens Niss 30 years ago, we discussed ways to include applications and modelling in the Danish mathematics curriculum. Niss stayed at the university, I chose to become an applied ethnographer in the classroom. By replacing set-based functions with formulas at the pre-calculus level, I designed and tested an application and modelling based curriculum that made all students learn everything. However, Denmark is the only country in the world still practicing oral exams in mathematics, and the ministry of education's examiner didn't like a set-free curriculum. Likewise, countless articles to the mathematics teachers' journal on the advantages of a set-free formula-based curriculum were neglected. It took 30 years before the Ministry of Education finally followed my advice: to improve learning, replace functions with formulas and make the curriculum application and modelling based. However, care is needed since the phrasing 'apply mathematics' installs as self-evident that 'of course mathematics must be learned before it can be applied'. This contradicts the historical fact that mathematics was created as layers of abstractions coming from modelling real-world problems.

Factors Preventing an Application & Modelling Based Curriculum

Four factors preventing an application and modelling based curriculum have been identified.

1. Applying mathematism instead of mathematics, answers proven correct in a library may not hold in a laboratory. This makes mathematics totally self-referential and impossible to use as a prediction of real-world situations. Adding numbers without units in primary school and adding fractions without units in middle school are examples of mathematism.

Different examples of 'centrism' claiming to have monopoly in certain modelling situations prevent or postpone many fruitful modelling situations, and exclude many potential learners.

2. In primary school, 'ten=10'-centrism conceals that 'ten IS 10' is a pastoral choice hiding its alternatives, e.g. five=10 in the case of counting&bundling in 5s. This prevents modelling recounting-situations as e.g. 4 5s = ? 7s predicted by the recount-formula T = (T/b)*b.

3. In middle school, fraction-centrism presenting fraction without units is a pastoral choice hiding its alternative, per-numbers. This forces proportionality to become an application of fractions, and prevents it from being a re-application of recounting. Being defined as sets, fractions become an example of 'metamatism' merging metamatics with mathematism.

4. In high school, set-centrism demands all concepts be defined as examples of the concept set. Solving equations by the set-based neutralizing method prevents equations from being solved by

reversed calculation. Set-based functions and calculus prevent change situations from being modelled by re-applying per-numbers and using a Graphical Display Calculator.

Factors Implying an Application & Modelling Based Curriculum

Two factors implying an application and modelling based curriculum have been identified.

First, changing or deconstructing ‘applying mathematics’ to ‘re-applying mathematics’ will signal that historically mathematics was created as an application modelling real-world problems, i.e. as a language describing and predicting the natural fact many. This distinction is useful when answering the question ‘does ‘applying mathematics’ mean applying pastoral metamatics and mathematism, or re-applying grounded mathematics?’

Second, banning from school set-based metamatics and mathematism will bring back a new Enlightenment period with grounded mathematics. Thus in primary school ‘ten=10’-centrism is banned by practicing counting by bundling&stacking in 5-bundels, 7-bundles etc. before finally choosing ten as the standard bundle-size. In middle school fractions should be seen as per-numbers, and should always carry units; and equations should be introduced as reversed calculations. In high school functions and equations should be seen as formulas with two or one variables to be treated on a graphical display calculator using regression to produce formulas, describing per-numbers to be integrated to totals, or totals to be differentiated to per-numbers.

Conclusion

Now an answer can be given to the two initial questions. To improve learning, an application-based mathematics curriculum should re-apply grounded mathematics rooted in real-world problems; and not apply pastoral metamatism, a mixture of metamatics presenting concepts as examples of abstractions instead of as abstractions from examples, and mathematism true in a library but not in a laboratory and therefore unable to predict real-world situations.

Applying metamatism forces three cases of centrism upon mathematics as pastoral choices hiding their alternatives. The use of ‘ten=10’-centrism hides that also other numbers than ten can be used as bundle-size when counting by bundling&stacking. Fraction-centrism hides that proportionality and many other applications of fractions can also be solved by instead re-applying recounting. And set-centrism hides that modelling change can take place without the use of set-based concepts as functions and limits. Finally the wording ‘apply mathematics’ installs as self-evident that ‘of-course mathematics must be taught and learned before it can be applied’, thus hiding that historically mathematics is rooted in the real world as a model.

A grounded approach will respect the historical nature of mathematics as a natural science rooted in the physical fact many. Here mathematics is created through its real-world roots and then re-applied to similar situations. To avoid ‘ten=10’-centrism in primary school, before introducing 3.order counting installing ten as the only bundle-size, 2.order counting is used to emphasize that mathematics is a language for predicting real-world numbers, and to allow the learning of 1digit mathematics. To avoid fraction-centrism in middle school, proportionality is based upon recounting and per-numbers, and fractions always carry units when added. To avoid set-centrism in high school, the Graphical Display Calculator is used when modelling change, both the standard linear, exponential and polynomial models and the more complicated models.

Re-applying grounded mathematics invites the grand narratives of the quantitative literature into the mathematics curriculum, and allows the minor narratives to be introduced at an early stage. Applying pastoral metamatism means excluding the grand narratives and bringing the minor narratives to a halt until the metamatism applied is taught and learned. Does modelling want to generate grounded mathematics, or to be a docile lackey of pastoral metamatism?

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207 Mathematics: Grounded Enlightenment - or Pastoral Salvation

a Natural Science for All - or a Humboldt Mystification for the Elite

Mathematics is taught differently in Anglo-American democratic enlightenment schools wanting as many as possible to learn as much as possible; and in European pastoral Humboldt counter-enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Humboldt Bildung schools pastoral 'metamatism' is exemplified from metaphysical mystifying concepts from above. To make mathematics a human right, pastoral Humboldt counter-enlightenment mathematics must be replaced with democratic grounded enlightenment mathematics.

Introduction

This paper is reflecting upon the theme 'how to find new perspectives on mathematical knowledge', which translates into 'perspectives on knowledge knowledge' since in Greek 'mathematics' means knowledge. So to give it meaning, this paper interprets the theme as 'perspectives on the contemporary university discourse called mathematics.' This theme is an example of a more general theme called 'perspectives on the contemporary university discourse called knowledge production'. Thus a natural approach to such a theme is to identify perspectives in the general discourse and exemplify them in the mathematics discourse. At the general discourse level during the last three decades a fierce debate has taking place between modern and postmodern perspectives on knowledge. So it seems natural to import this discussion in to reflections about mathematical knowledge.

Postmodern Thinking, a Short Tour

As to defining the word 'postmodern', the literature often refers to Lyotard's scepticism towards modern science legitimising its truths as examples of a truth above, a meta-truth.

> I will use the term *modern* to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..) Simplifying to the extreme, I define *postmodern* as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

As to legitimising postmodern research, Lyotard says that postmodern research should produce paralogy in the sense of parallel knowledge that invents not truth, but differences and dissension. In other words, postmodern research means searching for hidden differences, contingency:

> Where, after the metanarratives, can legitimacy reside? The operativity criterion is technological; it has no relevance for judging what is true or just. Is legitimacy to be found in consensus obtained through discussion, as Jürgen Habermas thinks? Such consensus does violence to the heterogeneity of language games. And invention is always born of dissension. Postmodern knowledge is not simply a tool of the authorities; it refines our sensitivity to differences and reinforces our ability to tolerate the incommensurable. Its principle is not the expert's homology, but the inventor's paralogy. (xxiv-xxv)

Lyotard writes inside the French post-structural Enlightenment tradition, also including Derrida and Foucault. Inspired by Heidegger, Derrida has inaugurated

> a project of deconstructing Western metaphysics or 'logocentrism' with its characteristic hierarchizing oppositions (..) Derrida's claim is that these conceptual orderings are not in the nature of things, but reflect strategies of exclusion and repression that philosophical systems have been able to maintain only at the cost of

> internal contradictions and suppressed paradoxes. The task of 'deconstruction' is to bring these contradictions and paradoxes to light, to undo, rather that to reverse, these hierarchies, and thereby to call into question the notions of Being as presence that give rise to them (Baynes 1987: 119)

Later Derrida demystifies the term 'deconstruction' by saying in an interview '(..) in order to demystify or, if you prefer, to deconstruct (..) (Derrida in Royle 2003: 35). So Derrida expresses scepticism towards excessive trust in words, logocentrism. Some words might enlighten what they describe, others instead mystify; and thus needs to be demystified or deconstructed to be enlightening.

Inspired by Nietzsche, Foucault writes about knowledge-power, or pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1983: 213, 215)

Foucault thus sees modern institutions as generalised churches using pastoral discourses to offer salvation promises: 'You are un-saved, un-healthy, un-social, un-educated. But do not fear! For we, the saved, healthy, social, educated, will save you. All you have to do is to repent, and go to our salvation institution, the church, hospital, correction centre, school, and become a loyal lackey'.

Common to Derrida, Lyotard and Foucault is a revival of scepticism towards hidden patronisation. Together they describe the compulsion techniques of modern pastoral knowledge: compulsive pastoral mystifying words installing what they describe as nature instead of choice; compulsive pastoral statements installing their claims as nature instead of choice; and compulsive pastoral salvation institutions mediating discursive servility instead of enlightenment.

The first generation of sceptical thinkers were the ancient Greek sophists claiming that in order to practise democracy people must be enlightened to tell the difference between nature and choice; if not, patronisation in disguise would arise presenting its choice as nature. Thus Plato's half-brother, the sophist Antifon, writes:

> Correctness means not breaking any law in your own country. So the most advantageous way to be correct is to follow the correct laws in the presence of witnesses, and to follow nature's laws when alone. For the command of the law follows from arbitrariness, and the command of nature follows from necessity. The command of the law is only a decision without roots in nature, whereas the command of nature has grown from nature itself not depending on any decisions. (Antifon in Haastrup et al 1984: 82, my translation).

Plato claimed that choice is an illusion; all is nature since all physical phenomena are examples of metaphysical forms only visible to the philosophers who therefore are the only ones to name them. Hence people should abandon democracy and accept the pastoral patronisation of philosophers educated at Plato's academy.

In Greece democracy disappeared with the silver mines financing import of silk and spice from the Far East. The academy, however, survived and was later renamed to monasteries by the Christian church sympathising strongly with the academy's pastoral salvation techniques. Later some monasteries developed into universities, as visible in Cambridge and Oxford; and at universities in general still organised like a monastery with long corridors of cells where people sit and produce writings extending and referring to the ruling pastoral discourse.

Robbing Spanish silver on the Atlantic was no problem for the British. But sailing to the Far East only following the moon to avoid Portuguese fortification of Africa was. Newton rejected the official knowledge saying that the moon moves among the stars following the unpredictable will of a metaphysical Lord. Instead he claimed that the moon falls towards the earth as does the apple, both following an internal physical will that can be predicated through calculations and later tested.

Newton's scepticism led to the Enlightenment: when an apple only obeys its own will, why shouldn't people do the same and replace patronisation with democracy?

French Enlightenment and German Counter-Enlightenment

The Enlightenment established two democracies, in America and in France. America still has its first republic, France its fifth. In France the Germen autocracy send in the army to stop the French democracy. However, they send in an army of mercenaries that was no match to the French army of conscripts only to aware of the feudal alternative to democracy. So not only was the German armies rolled back to the border, the French occupied Germany itself.

Napoleon was shocked to see the many different measurers in the many principalities of Germany and Italy created to guard the silver on its journey from the Harz to Venice where it financed the import of spice and silk that financed the Italian Renaissance. So he cancelled the Second Reich, the Holy Roman Empire, having lasted almost 1000 years; and installed the meter system by force.

Being unable to use the army, the German autocracy turned to education to stop the spreading of democracy from France. So they asked the father of new-humanism, Humboldt, to develop counter-enlightenment and reinstall pastoral schools that could stop the democratic enlightenment schools.

Mixing Hegel philosophy with romanticism, Humboldt developed 'Bildung' to reinstall a metaphysical Spirit present all over nature, in minerals, plants, animals and humans, and expressing itself in art. To understand art, people need the Bildung of the Humboldt school system. However, Bildung is only accessible to the chosen few, so not everybody is allowed enter into the Humboldt schools. Thus today's Humboldt university refuses to receive the students directly from the democracy's secondary schools, first they must pass an entrance exam at a Humboldt-gymnasium. However, only the most gifted half of the students is allowed to enter the Humboldt gymnasium, and again only the best half is allowed to enter the Humboldt University, where a half is failed so that only 13% finally gets a university degree (OECD 2004: 6).

The elitism of the Humboldt schools was enthusiastically accepted by the other European autocracies. When later turning into democracies they kept the Humboldt Bildung system.

American Enlightenment and Grounded Action Theory

In America, Enlightenment developed into pragmatism showing scepticism towards traditional philosophy by developing 'symbolic interactionism' with its own methodology called 'grounded theory'. Grounded Theory respects agents as independent actors:

> Actors are seen as having, though not always utilizing, the means of controlling their destinies by their responses to conditions. They are able to make choices according to their perceptions, which are often accurate, about the options they encounter. Both

> Pragmatism and Symbolic Interactionism share this stance. Thus, grounded theory seeks not only to uncover relevant conditions, but also to determine how the actors respond to changing conditions and to the consequences of their actions. It is the researcher's responsibility to catch this interplay. (Corbin & Strauss 1990: 5)

As to the question about being guided by existing theory, Grounded Theory gives the advice to ignore the literature and theory on the area under study in order to assure that the emergence of categories will not be contaminated by concepts more suited to different areas:

> Although categories can be borrowed from existing theory, provided that the data are continually studied to make certain that the categories fit, generating theory does put a premium on emergent conceptualizations. (..) In short, our focus on the emergence of categories solves the problems of fit, relevance, forcing, and richness. An effective strategy is, at first, literally to ignore the literature of theory and fact on the area under study, in order to assure that the emergence of categories will not be contaminated by concepts more suited to different areas. Similarities and convergences with the literature can be established after the analytic core of categories has emerged. (Glaser et al 1967: 36-37)

So instead of going to the library, Grounded Theory listens to the agent's own accounts and narratives from which categories and relations are discover and constantly checked or accommodated through new data. In this way grounded research could be named 'systematic natural learning' reminding very much of the 'individual natural learning' described by Piaget:

> Is childhood capable of this activity, characteristic of the highest forms of adult behaviour: diligent and continuous research, springing from a spontaneous need? – that is the central problem of the new education. (..) But all these psychologists agree in accepting that intelligence begins by being practical, or sensorimotor, in nature before gradually interiorising itself to become thought in the strict sense, and in recognizing that its activity is a continuous process of construction. (..) In other words, intelligence is adaptation in its highest form, the balance between a continuous assimilation of things to activity proper and the accommodation of those assimilative schemata to things themselves. (Piaget 1969: 152, 158)

Piaget thus is the father of constructivist learning theories believing that learning takes place through a 'grasping before grasping' or 'greifen vor begreifen' process. With physical grasping always preceding mental grasping, the mental concepts will automatically enlighten the physically grasped. Contrary to this the Vygotsky social constructivism tries to adapt the learner to a pre-existing pastoral mystifying vocabulary calling itself 'scientific'. Likewise, the importance of physical grasping is absent in Luhmann's pragmatic constructivism seeing the individual embedded in two systems, a reflective and a communicational system, both being self-referential. Luhmann's theory of self-generating and self-referring systems seems to be created to support and legitimise the self-reference taking place at the pastoral Humboldt counter-enlightenment universities.

To avoid the self-reference of the Humboldt University and instead make research usable to the public, some American enlightenment universities recommend action research.

> Our universities have a monastic origin, and they have specialized in being centers of higher learning, functions originally given by the Church to monasteries. (..) The form of the university most familiar to us today is mainly a Prussian invention whose architect and champion was Wilhelm von Humboldt (..) The collegial system and its related peer review structures centered on an effort to gain intellectual freedom from the constraints of theological doctrine and political manipulation. Although addressing this

> problem was obviously important, the solution adopted has subsequently done much to weaken the social articulation of the university to all groups other than powerful elites. (..) Not surprisingly, society at large occasionally thinks it should be getting a more useful return for its investment and the freedom it gives to the professoriate. This situation is predictable because the autopoetic research process provides important supports for intellectual freedom but simultaneously opens the door to useless research and academic careerism divorced from attention to important public social issues. (..) While we advocate action research as a promising way of moving the academic social sciences to socially meaningful missions, we do not base our claims for action research only on its putative moral superiority. Central to our argument is the claim that action research creates the valid knowledge, theoretical development, and social improvements that the conventional social sciences have promised. Action research does better what academic social science claims to do. (Greenwood & Levin in Denzin & Lincoln. 2000: 85-89)

Deconstructing and Grounding Research

Lyotard's postmodern paralogy research creating dissension to the ruling consensus by searching for hidden differences, contingency, resonates with the ancient sophist advice: know the difference between nature and choice to avoid hidden patronisation presenting choice as nature. Also including the American enlightenment sociology advocating theory being grounded by assigning names to things that can be observed, it is now possible to design a postmodern research paradigm that could be called 'anti-pastoral enlightenment research': To avoid hidden patronisation, uncover pastoral choices presented as nature by replacing self-referring mystification with grounded enlightenment.

Thus linear and exponential functions are pastoral terms since they describe a Renaissance calculation formula using a word from around 1750. These terms can be demystified by terms grounded in and enlightening their nature as e.g. 'change by adding and by multiplying'.

Re-grounding mathematics in its historical roots, the nature of many, the names 'metamatics' and 'mathematism' can be given to ungrounded self-referring mathematics (Tarp 2004).

The roots of mathematics are revealed by its two sub-discourses, algebra and geometry. In Greek Geometry means 'earth measuring'; and in Arabic Algebra means 'reuniting'. Together they answer two fundamental questions 'How to divide the earth and its products?' Or simpler 'How to divide and unite many?' So mathematics is created as a grounded theory about many; and as such is was very successful in the Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 398-99)

Later with the set-concept, all concepts seemed to be examples of sets. This re-installed pastoral mathematics until Russell and Gödel showed that a self-referring mathematics can be neither well-defined nor well-proven. Russell's set-paradox 'if $M = \{ A \mid A \notin A) \}$ then $M \in M \Leftrightarrow M \notin M$' shows that concepts can't be self-referring. And Gödel proved that any axiom system would contain true statements that cannot be proven. So mathematics is still a natural grounded science.

Deconstructing and Grounding the Postmodern

To demystify and deconstruct the word ‘postmodern’ we can ask if the words ‘postmodernity’ and ‘postmodernism’ can be grounded in ‘laboratory’ observations.

As other biological animals also humans need a constant supply of matter, energy and information. Knowledge about supply techniques, technology, has developed through human history.

First matter technology using iron invented artificial hands, tools, enabling a transition from gather/hunter culture to agriculture. Then energy technology using electrons to carry energy invented artificial muscles, motors, combining with tools to machines, enabling a transition from agriculture to industrial culture. Then information technology using electrons to carry also information invented artificial brains, computers, combining with tools and motors to robots, enabling a transition from industrial culture to information culture, postmodernity. And since the robot is the end of the line, the term post-postmodernity has no meaning.

To control machines, the modern industrial culture needed the brain to be educated creating well-defined jobs. Modern thinking then means choosing between a set of well-defined identities.

In the postmodern information culture the human brain is not needed for routine jobs, making most traditional training redundant. Furthermore, by informing also about alternatives that were before hidden, information technology un-hides hidden contingency. Thus the individual now sees the world full of choices in areas where before was only nature to obey. Most identities now are liquid (Baumann 2000). To get an identity, the individual now has to build its own identity as a biographical narrative shunning meaninglessness and looking for authenticity (Giddens 1991).

Postmodernism means presenting choice as choice creating more personal and social choices. Post-postmodernism means presenting choice as nature resulting in a return of pastoral patronisation.

Deconstructing and Grounding Numbers

Names and icons enlighten the different degrees of many. Counting a given total T by bundling and stacking can be predicted by the recount-equations as T = (T/5)*5. Many different icons have been used. Today the most frequent icon systems are the Roman and the Arabic.

The Arabic system rearranges the given number of strokes into an icon so there are four strokes in the icon 4 etc. Ten is chosen as the standard bundle-size in which to bundle singles, bundles, bundles-of-bundles etc. Thus a total T can be iconised as e.g. T = 3BBB,5BB,7B,1, or leaving out the bundles, T = 3571; or T = 3501 if all bundles can be re-bundled into bundles-of-bundles.

The Roman system uses strokes for unbundled, and the letters V, X, L, D for certain bundle-sizes. However, bundling is not systematic: V means a 5-bundle, X means 2 V-bundles etc.

Arabic numbers are introduced from grade 1 in all pastoral mathematics curricula. In grounded mathematics 2digit numbers are banned from grade 1 since they refer to the number ten. As the only number with its own name but without its own icon the number ten becomes a cognitive bomb if presented too early: a 2digit number as 23 is explained as 2 10s and 3 1s, thus referring to ten. And the 2digit number 10 is explained as 1 10 and no 1s, i.e. through circular self-reference to ten. Instead 2digit numbers should be introduced slowly through bundling, stacking and cup-writing: A total of sixteen sticks can be counted in 5-bundles and stacked as 3 5-bundles and 1 unbundled: T = 3*5 + 1*1. Counted in 8-bundles produces 2 8-bundles that can be stacked as T = 2*8.

The bundles and the unbundled are put in a left and a right cup. Later a stone and later again a stick is used as a symbol of a full bundle, knowing that a stick in the left cup symbolises a full bundle.

The manual activity of cup-filling leads to the mental activity of 'cup-writing' T = 3)1) worded as 3 bundles and 1 unbundled in the case of 5-bundling; and T = 2)) worded as 2 bundles and no unbundled in the case of 8-bundling. Later the cups can be left out and a 0 introduced as an icon for an empty cup: T = 2)) = 20 worded as 2 bundles and no unbundled. Now 10 means 1)) thus being defined by a two-cup physical reality, which makes the circular self-reference disappear.

Likewise in a grounded approach, fractions and decimal numbers are introduced simultaneously in grade 1 as ways of dealing with the unbundled, where e.g. 2 can be counted in 5s as 2 = (2/5)*5 and put on top of the 5-stack and written as T = 3 2/5 *5; or the unbundled can be put next to the 5 stack as a separate stack of 1s written as T = 3.2 *5. In fact, all of mathematic can be introduced using 1digit numbers alone, including equations and calculus since equations is just another word for backward calculation (3 + ? = 8); and calculus is just another word for horizontal addition instead of vertical: 3 5s + 2 3s = ? 8 s instead of 3 5s + 2 3s = ? 5s, or 3 5s + 2 3s = ? 3s. (Zybartas 2005)

Deconstructing and Grounding Operations

In Greek, mathematics means knowledge, and knowledge can be used for prediction. Thus 'number-prediction' is one possible demystification or deconstruction of mathematics, which grounds operations as number-prediction techniques. Without addition, wanting to unite 32 and 64 becomes a very time-consuming task involving a high risk of making errors, since we have to count-on from 32 64 times: '33, 34,…, 96, 97, I think; or maybe it is 98?' To be sure, one has to make an accounting by writing down one stroke per count. It would be nice to be able to predict counting-results. Addition does this: T = 32 + 64 = 96. Likewise multiplication predicts adding many like numbers, and power predicts multiplying many like numbers.

To avoid trying out many numbers, it would be nice also to predict the answer to the questions 3+? = 8, 3*? = 15, 3^? = 81 and ?^5 = 32. This grounds inverse operations as the answers 8-3, 15/3, log3(81) and 5√(32); and offers an simple technique of solving equations: just move a number to the other side by changing its calculation sign:

3 + x = 8	3 * x = 15	3 ^ x = 81	x ^ 5 = 32
x = 8 - 3	$x = \frac{15}{3}$	x = log3(81)	$x = \sqrt[5]{32}$

Pastoral mathematics needs all the concepts of abstract algebra to solve the equation: neutral and inverse elements, commutative and associative laws:

2+3*x =14 , (2+3*x) + -2 =14 + -2 =12 , (3*x + 2) + -2 =12 , 3*x + (2 + -2) =12, 3*x+0 =12

3*x = 12 , (3*x)* 1/3 = 12* 1/3 = 4 , (x*3)* 1/3 = 4 , x*(3* 1/3) = 4 , x*1 = 4 , x = 4

However, this is impossible to bring to the classroom. Instead a lever is introduced to teach the method of doing the same to both sides, cheating students by reducing an understanding to a ritual.

Deconstructing and Grounding the Mathematics Curriculum

In a grounded mathematics curriculum mathematics is learned as a natural science exploring many. This means that both teachers and students re-discover mathematics through the CATS-approach: Count&Add in Time&Space as presented by the MATHeCADEMY.net. Thus in the lower primary school a grounded mathematics curriculum introduces the whole of mathematics working with 1digit cup-numbers alone (Zybartas 2005). Addition and subtraction of cup-numbers is learned through re-bundling and internal trade between neighbour cups: Thus, in the case of 5-bundles

T = 3)4) + 4)2) = 7)6) = 7+1)6-5) = 8)1) = 0+1)8-5)1) = 1)3)1)

In upper primary school this curriculum is repeated, now using multi-digit numbers. And per-numbers are introduced now using the recount-equation T = (T/b)*b to describe recounting in different units by recounting a given total in the given base unit, e.g. recounting 8 in 3s: If 3kg = 5$ then 8kg = (8/3)*3kg = (8/3)*5$ = 13.3 $. Geometry is introduced as trigonometry considering sin, cos and tan as percent-numbers and tan as an easy protractor.

Secondary school algebra deals with change equations: constant change, i.e. linear change (□y=a) and exponential change (□y=r%); variable predictable change (dy/dx = formula); and unpredictable change, i.e. statistics and probability. Geometry is extended to include non-linear forms, and later geometry becomes coordinate geometry and vector geometry.

A Grounded Perspective on Pastoral Mathematics

The pastoral approach to mathematics makes many learning-blunders (Tarp 2006) transforming it into metamatism only accessible to the elite. This is precisely what the Humboldt university wants: It witnessed how the Enlightenment was created by mathematics' ability to predict numbers, so a counter-enlightenment must reinstall mathematics as a pastoral knowledge descended from above. Thus in Germany teaching fractions as metamatism, e.g. 1/2 + 2/3 = 7/6 instead of 3/5 enables the Humboldt system to split the students into three groups: Realschule, Hauptschule und Gymnasium.

Still acting as a province governed from Holstein, Denmark has taken the Humboldt counter-enlightenment to an extreme. In school, most marks are oral being unreliable since they are based upon the personal subjective judgement of the person who has also given the education, and not on written performances. Being unable to prove the absent learning with written tests, the teachers are forced to give most students middle marks making it possible to sit off both school and teacher education since a teacher can function by just handing out middle marks. Sitting off of course means disaster at written exams. Thus the international standard of 60% correctness as passing limit is lowered to 40% in the Danish Gymnasium and to 20% in the secondary school. Likewise the Danish Humboldt university refuses to include other tertiary educations as e.g. teacher education.

The Humboldt Occupation of Europe

The Humboldt University's 200-years occupation of Europe created no problems in the industrial culture needing less that 10% to attend university. But in a postmodern information culture needing more that 50%, it presents an unmatched disaster since the Humboldt University will wipe out the population in 200 years by holding on to its youth in its Humboldt maze of uncoordinated non-modularized educations, that keep the youth from producing and keeps the reproduction rate at 1.5 child per couple. However, the European population is unaware of this since the counter-enlightenment of the Humboldt Bildung has kept the majority of the population including its politicians unenlightened while sorting out the elite for its own reproduction. Likewise as lackey-research supporting metamatism education, mathematics education research has turned into a research industry producing huge amounts of irrelevant research only useful for personal careerism.

Conclusion

A postmodern perspective on mathematical knowledge enlightens what is nature and what is choice within mathematical knowledge; and what is pastoral choice presented as nature. This again makes plain to Europe's democracies the choice they face: will they continue to support the occupation of Europe by the Humboldt counter-enlightenment Bildung system; that will wipe out the European population in 200 years by holding its youth caught in its pastoral salvation institutions in the crucial years where elsewhere they get their university degree, a job, and a family; that instead of teaching mathematics preach metamatism in order to sort out the elite; and that allows its universities to be self-referring and to produce useless research only usable for careerism. Or will they finally introduce democracy also into education; by changing the Humboldt counter-enlightenment system to the Anglo-American enlightenment system that has been adopted as international standard outside Europe; by changing pastoral metamatism salvation to mathematics enlightenment; and by only funding action research forcing research to ground its theories in society's needs and concerns. As a first step to this decision, the European democracies should privatise its Humboldt universities and Humboldt gymnasia in order to enable free competition with Anglo-American enlightenment education.

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208. Pastoral Humboldt Mathematics Deconstructed

Having existed since antique Greece, pastoral and anti-pastoral curricula today exist at the Humboldt Bildung schools inside EU and Enlightenment schools outside. However, Humboldt anti-enlightenment seems to have influenced all mathematics curricula. Following the advice of the Greek sophists, this paper separates choice from nature in the mathematics curriculum to design an alternative natural enlightenment curriculum grounded in the roots of mathematics.

The Background

An ethnographer only has to stay one day among first year students at a university in North America and in Denmark before realising that the word education has two different meanings.

At the first place the students are aged 19, at the second 23. At the first place the students have chosen their own combination of modules accessible for all; at the second place they are forced to follow one of several pre-designed educations only accessible to those with the highest marks. At the first place the students already took some university modules at the last year in high school; at the second place this is not possible. At the first place high school is attended by all and a high percentage goes on to university where around 50% gets a bachelor degree; at the second place only the best half of a year group is allowed to enter high school, and only the best half is allowed to go on to university where only the half graduates after having been forced to include a university directed master degree in their exam (OECD, 2005). At the first place some students are supplementing their bachelor with new modules in order to change career e.g. from teaching to engineering; at the second place they have to start all over. At the first place parents have different careers in their lifetime; at the second place parents are bound to the office they are educated for.

Looking for Explanations

Looking in the literature for explanations for this difference soon leads to Humboldt:

> Our universities have a monastic origin, and they have specialized in being centers of higher learning, functions originally given by the Church to monasteries. (..) The form of the university most familiar to us today is mainly a Prussian invention whose architect and champion was Wilhelm von Humboldt (..) The collegial system and its related peer review structures centered on an effort to gain intellectual freedom from the constraints of theological doctrine and political manipulation. Although addressing this problem was obviously important, the solution adopted has subsequently done much to weaken the social articulation of the university to all groups other than powerful elites. (..) Not surprisingly, society at large occasionally thinks it should be getting a more useful return for its investment and the freedom it gives to the professoriate. This situation is predictable because the autopoetic research process provides important supports for intellectual freedom but simultaneously opens the door to useless research and academic careerism divorced from attention to important public social issues. (..) While we advocate action research as a promising way of moving the academic social sciences to socially meaningful missions, we do not base our claims for action research only on its putative moral superiority. Central to our argument is the claim that action research creates the valid knowledge, theoretical development, and social improvements that the conventional social sciences have promised. Action research does better what academic social science claims to do. (Greenwood & Levin in Denzin & Lincoln, 2000, p. 85-89)

Should the university turn its back to the outside world and decide its foci itself; or should it create ‘valid knowledge, theoretical development, and social improvements’. This discussion goes back to the ancient Greece controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people needed to be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers argued that patronization is the natural order since everything physical is an example of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become the natural patronizing rulers (Russell, 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods; later it was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing slow Spanish silver ships returning over the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it obeys its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight crated the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. US still has its first republic, France now has its fifth. The German autocracy tried to stop the French democracy by sending in an army. However the German army of mercenaries was no match to the French army of conscripts only to aware of the feudal alternative to stopping the German army. So the French stopped the Germans and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment in spreading from France. Humboldt was asked to create an elite school and using Bildung as counter-enlightenment he created a school-system leading to the Humboldt University, which uses Luhmann System Theory to defend its chosen self-reference as nature (Luhmann, 1995); and which threatens to wipe out the EU population, sinking it to 10% over 200 years by holding on to its youth so only 1.5 child is born per family in the EU area, still protecting the autocratic Humboldt school system even after becoming democratic.

Inside the EU the sophist warning is kept alive in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronising categories, discourses and institutions presenting their choices as nature (Tarp, 2004). Derrida recommends that pastoral categories be 'deconstructed'. Lyotard recommends the use of postmodern 'paralogy research' to invent alternatives to pastoral discourses. And Foucault uses the term 'pastoral power' to warn against institutions legitimising their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

Enlightenment Schools and Anti-Enlightenment Bildung Schools

So in the US the Enlightenment created enlightenment schools wanting to enlighten as many as possible as much as possible in order to protect the democracy. In Europe the German counter-Enlightenment created anti-enlightenment Bildung schools wanting to identify the elite for offices in the central administration of then the autocracy now the elected government.

Enlightenment schools contain short serial modules finalized with written exams. The Bildung schools contain long parallel strings, in Denmark finalized with many oral and few written exams. Inspired by their roots in the ancient Greek knowledge controversy they have different curricula. Enlightenment schools have enlightenment curricula wanting to enlighten the physical world by abstracting categories and knowledge from it; Bildung schools have counter-enlightenment curricula presenting the physical world as exemplifying metaphysical categories and knowledge only visible to philosophers and transmitted through Bildung.

Does this imply two different mathematics curricula, enlightenment mathematics and pastoral mathematics; or has Humboldt counter-enlightenment occupied also the mathematics curriculum?

To get an answer, an ethnographer could follow the sophist advise and try to distinguish nature from choice in mathematics by considering mathematics a natural science grounding its categories and knowledge in its physical source, multiplicity, and using the method of natural research, American Grounded Theory (Glaser et al, 1967), resonating with Piaget's principles of natural learning (Piaget, 1970) and with the Enlightenment principles for research: Observe, abstract categories and relations to be accommodated through predicted deductions. But first, history again should be consulted.

Pre-modern and Modern Mathematics

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' (Tarp, 2004) that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: 'Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$'.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can be neither proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

These problems provide just one more reason for searching for the nature of mathematics.

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of e.g. eight sticks can be treated in different ways. The sticks can be rearranged to an eight-icon 8 containing the eight sticks, written as 8. The sticks can be collected to one eight-bundle, written as 1 8s. The sticks can be 'decimal-counted' in 5s by bundling & stacking, bundling the sticks in 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled singles in a right single-cup. When writing down the counting-result, cup-writing gradually leads to decimal-writing where the decimal separates the bundle-number from the single-number:

8 = 1 5s + 3 1s = 1)3) = 1.3 5s

So the nature of numbers is that any total can be decimal-counted by bundling & stacking and written as a decimal number including its unit, the chosen bundle-size. The bundle-size icon is not used in decimal-counting where using 5-bundles transforms 5 1s to 1.0 bundles. Since ten is chosen as a standard bundle-size, no icon for ten exists making ten a very special number having its own name but not its own icon. This installs ten as a cognitive bomb in young brains, if not the core of mathematics is introduced by using 1digit numbers alone (Zybartas et al, 2005).

Also, together with choosing ten as the standard-bundle size, another choice is made, to leave out the unit of the stack thus transferring the stack-number 2.3 tens to what is called a natural number 23, but which is instead a choice becoming pastoral by suppressing its alternatives. Leaving out units might create 'mathematism' (Tarp, 2004) true in the library where 2+3=is true, but not in the laboratory where countless counterexamples exist: 2weeks+3days = 17days, 2m+3cm = 203cm etc.

The Nature of Operations

Operations are icons describing the process of counting by bundling & stacking.

The division-icon '/2' means 'take away 2s', i.e. a written report of the physical activity of taking away 2s when counting in 2s, e.g. 8/2 = 4. The multiplication-icon '4*' means 'stacked 4 times', i.e. a written report of the physical activity of stacking 2-bundles 4 times, T = 4*2

Subtraction '- 2' means 'take away 2', i.e. a written report of the physical activity of taking away a stack to see what rests as unbundled singles, e.g. R = 9 – 4*2. And addition '+2' means 'plus 2', i.e. a written report of the physical activity of adding 2 singles to the stack of bundles as a new stack of 1s making the original stack a stock of e.g. T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus the full process of 're-counting' or 're-bundling' 8 1s in 5s can be described by a 'recount or rebundle formula' containing three operations, together with a 'rest formula' finding the rest:

T = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is R = 8 – 1*5 = 3.

Once ten has been chosen as the standard bundle-size, most operations take on new meanings. Plus now means 'adding on top of' instead of 'adding next to'. Division now means 'splitting in 4' instead of 'splitting in 4s'. And multiplication now means recounting in tens instead of stacking. Now recounting any stack in tens is not done anymore by the re-counting formula but by simple multiplication. To re-bundle 3 8s in tens, instead of writing T = (3*8)/10*10 = 2.4 * 10, we simply write T = 3*8 = 24. This allows tables to be practiced for re-bundling 2s, 3s, 4s etc. in tens.

The Nature of Formulas

Using the recount formula, the counting result can be partly predicted on a calculator where 9/4 = 2.something. This predicts that recounting 9 in 4s will result in 2 4-bundles and some singles.

With ten as the standard bundling-size, operations still are prediction techniques. Thus 5 + 3 predicts the end of the counting sequence when counting-on 3 times from 5: 6, 7, 8. In the same way multiplication predicts repeated addition of the same number: 3*2 predicts 2+2+2. And power predicts repeated multiplication of the same number: 2^3 predicts 2*2*2.

Any calculation can be turned around and become a reversed calculation predicted by the reversed operations: the answer to the reversed calculation 6 = 3 + ? is predicted by the reversed operation to plus, minus, i.e. by the calculation 6-3. The answer to the reversed calculation 6 = 3 * ? is predicted by the reversed operation to multiplication, division, i.e. by the calculation 6/3. The answer to the reversed calculation 6 = ? ^ 3 is predicted by the reversed operation to exponent, root, i.e. by the calculation $\sqrt[3]{6}$. The answer to the reversed calculation 6 = 3 ^ ? is predicted by the reversed operation to base, log, i.e. by the calculation $\log_3(6)$ = log6/log3.

Thus the calculator becomes a number-predictor using calculation for predictions. This shows the strength of mathematics as a language for number-prediction able to mentally predict a number that later is verified physically in the 'laboratory'. This enabled mathematics historically to replace pastoral belief with prediction, and presently to become the language of the natural sciences.

The Nature of Equations and of Calculus

A reversed calculation becomes an equation if the '?' is replaced with an x. Hence the natural way to solve equations is to move the number next to x to the other side reversing its calculation sign:

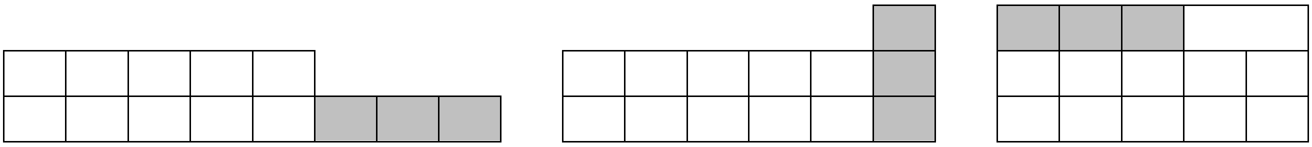
3 + x = 6	3 * x = 6	x^3 = 6	3^x = 6
x = 6 - 3	x = 6/3	$x = \sqrt[3]{6}$	$x = \log_3(6)$

The statement 2 * 3 + 4 * 5 = 3.2 * 8 describes a bundling where 2 3-bundles and 4 5-bundles are re-bundled in the united bundle-size 8s. This is 1digit integration. The equation 2 * 3 + x * 5 = 3.2 * 8 describes the reversed bundling asking how 3.2 8s can be re-bundled to two stacks, 2 3s and some 5s. This is a 1digit differential equation solved by performing 1digit differentiation:

If 2 * 3 + x * 5 = 4.2 * 6, then x = (4.2 * 6 – 2*3)/5 = (T – T1)/5 = ΔT/5

The Nature of Fractions

Fractions are created when, instead of placing 3 singles besides the stack of 5-bundles, they are bundled as a 5-bundle and put on top giving a stack of T = 2*5 + (3/5)*5 = 2 3/5 *5 = 2 3/5 5s.



T = 2*5 + 1*3 = 2.3 *5 = 2 3/5 * 5

With measuring comes 'double-counting' counting a quantity in two different units, creating 'per-numbers': If 2 kg costs 8 $, then the unit-price is 8$ per 2kg, i.e. 8$/2kg = 8/2 $/kg. Thus if 4kg cost 5$, the 'per-equation' 4kg = 5$ is used when recounting the actual kg-number in 4s, and recounting the actual $-number in 5s:

10kg = (10/4)*4kg = (10/4)*5$ = 12.5$, and 18$ = (18/5)*5$ =(18/5)*4 kg = 14.4kg.

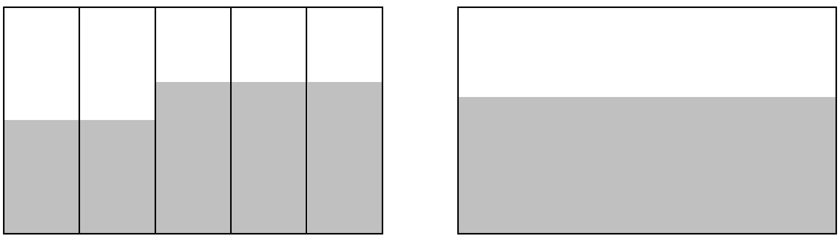
When adding fractions it is important to include the units to avoid mathematism as when performing the following 'fraction test' the first day of secondary school:

The teacher:	The students:
Welcome to secondary School! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

That seduction by mathematism is costly is witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. So to add numbers the units must be included, also when adding fractions. And adding fractions f is basically integration:

T = 1/2 *2 + 2/3 *3 = Σ (f*Δx) later to become ∫ f dx , or

T = 2 kg at 4 $kg + 3 kg at 9 $/kg = Σ ($/kg * kg) = Σ (f*Δx)



T = 1/2 *2 + 2/3 *3 = 3/5 *5

Nature and Choice in the Mathematics Curriculum

In primary school, an enlightenment curriculum would focus on the nature of numbers, operations and calculations to be learned through counting by bundling & stacking reported by cup-writing and decimals in accordance with the Piaget ‘from hand to head’ principle of natural learning (Piaget, 1970), thus postponing the introduction of ten and addition until the occurrence of several examples of the fact that for any bundle-size, its icon will not be used since a full bundle will always be counted as 1.0 bundles, or plain 10 if choosing to leave out both the decimal and the bundle-size.

The Humboldt curriculum introduces the ‘natural’ numbers one by one using the follower-principle. This leads to introducing ten as the follower of nine and in this way quickly introducing 2digit numbers and place values. Later comes multi-digit numbers. Addition is introduced to practise earlier numbers adding up to the actual number. Then subtraction is introduces as taking away and counting up to. Multiplication and the tables follow; and in the end follows division and simple fractions (see e.g. NCTM, 2000).

Introducing numbers using the follower-principle and addition, ten is introduced very quickly. Introducing 2digit numbers too early installs ten as a cognitive bomb being the only number with a name but without an icon. This leads to meaningless self-reference when after explaining 23 as 2 tens and 3 ones, ten has to be explained as 1 ten and no ones. What is called a ‘natural number’ is not a natural number but a pastoral choice suppressing its natural alternative coming from counting by bundling and stacking, i.e. decimal-numbers including the bundle-size as a unit. What is called the ‘natural order’ of operations, +, -, *, / is not natural but a pastoral choice suppressing its natural alternative coming from counting by bundling and stacking, i.e. first division for bundling, then multiplication for stacking, then subtraction for finding the unbundled and finally addition to place the stack of unbundled 1s next to the stack of bundled. Al four operations can be introduced without the need of 2digit numbers.

In middle school, an enlightenment curriculum would focus on the nature of per-numbers and triangles. Per-numbers occur when double-counting a quantity in two different units leads to fractions and percentages. Recounting now is called proportionality. When adding fractions and percentages the units are includes as in integration. Modelling leads to linear formulas as budget formulas $ax + by = c$ and constant change formulas $y = b + ax$. Formulas with two unknowns are graphed. Formulas with one unknown are equations solved using reversed calculations, first reducing a multiple calculation to a single by placing the hidden parentheses, and then moving numbers to the other side by reversing their calculation signs. Geometry is introduced via earth-splitting using a bisecting normal, angles, and polygons with diagonals, finally leading to the right-angled triangle seen as a rectangle halved by a diagonal, where the height and length can be recounted in diagonals, leading to sin, cos and tan as per-numbers and percentages.

The Humboldt curriculum enlarges the number domain with fractions, decimals and percentages where decimals and percentages are defined as examples of fractions. Again the order of operations is maintained starting with addition of fractions including factorisation in prime numbers. Equations are introduced and solved by the neutralising method. Proportionality is introduced as an example of equations and of a function that is graphed. Algebraic expressions are introduced to be factorised and simplified, and to be added as algebraic fractions. In geometry the focus is on 2- and 3-dimensional forms and translation groups.

Introducing decimals and percentages as examples of fractions makes them difficult to learn if fractions have not been learned. Presenting fractions as rational numbers is a choice, since fractions are not numbers, but operations always carrying a unit, 2/3 of 6. Adding fractions without units is mathematism true in the library but not in the laboratory. However, it legitimises the introduction of factorisation, and later of factorising algebraic expressions and adding algebraic fractions, tasks that are very time consuming and hard to learn. Insisting on using the neutralising method when solving equations creates problems with simple equations as 5 = 7/x, but it legitimises the introduction of the concepts of abstract algebra, if not at school then at teacher education. Likewise translation groups in geometry are included to legitimise the concepts of abstract algebra.

In high school, an enlightenment curriculum would focus on adding per-numbers, where adding constant per-numbers leads to power; and adding variable per-numbers leads to integration, generalising middle school's adding fractions with units and primary school adding two stacks in combined bundles to finding the area under the per-number graph. Reversed calculation then leads to roots and logarithms and to differentiation. This completes the project of algebra meaning 'reuniting' in Arabic where addition unites variable unit-numbers, multiplication unites constant unit-numbers, power unites constant per-numbers and integration unites variable per-numbers; and where the reversed operations splits a total into parts.

Algebra unites & splits	Variable	Constant
Unit-numbers m, s, kg, $, …	T = a + x T – a = x	T = a * x T / a = x
Per-numbers m/s, $/kg, m/100m = %	T = ∫ a dx dT/dx = a	T = a ^ x x√T = a loga(T) = x

Modelling now includes graphical display calculators, GDCs, able to test the solution of equations algebraically with 'math solver' and geometrically with 'trace' and 'calc intersection'. Change formulas are made for constant change: linear, exponential and power formulas; and for variable predictable change: polynomials and circular formulas. Unpredictable change leads to statistics using tables to create a 'post-diction' usable for interval-prediction. From data-tables, GDC regression can generate formulas. Steepness and area under graphs can be found as numbers on a GDC. And from tables, GDC regression can generate gradient- and area-formulas that can be tested by using the GDC to calculate gradient- and area-numbers.

As to models all three genres are introduced: since-then 'fact models' quantifying and predicting predictable quantities as e.g. areas and volumes, needing to be trusted; if-then 'fiction models' quantifying and predicting un predictable quantities as future interest rates, needing to be supplemented with scenarios based on different assumptions; and so-what 'fiddle models' quantifying and predicting unpredictable qualities as e.g. death and injuries, needing to be rejected and replaced by a democratic IDC Information-Debate-Choice process (Tarp, 2001).

The Humboldt curriculum enlarges the number domain with irrational and real numbers, and the number of operations is enlarged with power, root and log. The function concept is claimed to be the foundation of high school mathematics; and is defined as an example of a relation between two sets. Linear and exponential change is presented as examples of functions. The quadratic function is given an extended treatment. Its graph is studied using translations, and its formula is thoroughly factorised. Calculus is introduced as an example of the concept limit used to exemplify the concept continuity and differentiability and to define the gradient by the first principle and the integral as a Riemann sum. Geometry introduces coordinate geometry and vector geometry presenting a vector as an equivalence set of parallel arrows with the same length.

Introducing the function concept as an example of a set-relation instead of as a name for a calculation with a variable number as Euler did, means choosing an unknown-unknown relation instead of a unknown-known relation making the learners hear the definition as 'bublibub is an example of bablibab'. Choosing the function as the basic concept in both pre-calculus and calculus makes both hard to learn. Furthermore, choosing to base calculus on the limit concept makes calculus even harder and makes it impossible to make the point that while the operations +, -, *, /, √

and log operate on numbers and produces numbers, the calculus operations d/dx and ∫ operate on formulas and produce formulas.

How Successful is the Humboldt Curriculum

Contrary to the enlightenment curriculum wanting to enlighten as many as much as possible, the Humboldt counter-enlightenment curriculum wants to practise segregation to identify the elite.

It does so by systematically choosing the options making mathematics harder to learn. In primary school it introduces 2digit numbers and addition too quickly thus crating the first segregation between those having problems with multi-digit numbers and the elite. In middle school introducing the mathematism of adding fractions without units creates the next segregation, and later adding algebraic fractions also in need of being factorised provides the next segregation. In Germany itself adding fractions splits up the schools in three stream: Hauptschule, Realschule and Gymnasium. In Denmark the segregation is hidden by oral marks given by the same person who has done the teaching. In high school defining a function as an example of an abstraction instead of as an abstraction from examples introduces ‘metamatics’ enabling the next segregation. And finally presenting adding per-numbers with units as an example of the concepts function, limit and Riemann sum creates the final segregation. With segregation techniques, the Humboldt-gymnasium has no problem arguing that only the best half can enter, and only the best half of these can go on to the Humboldt-university happy to receive the elite candidates ready for further segregation.

Conclusion

The Humboldt Curriculum is very successful at using mathematics to segregate the elite for life-long office educations, as wanted in Berlin 200 years ago to protect the autocracy from the democracy. However, the Humboldt curriculum is not nature; its is a choice becoming pastoral by suppressing it democratic alternative, the North American enlightenment curriculum enlightening as many as possible as much as possible for a changing job market. Also within the mathematics curriculum a choice can be made between enlightenment mathematics presenting mathematics as a natural science grounded in the study of the natural fact multiplicity; and a pastoral mathematics presenting mathematics as deduced from metaphysical concepts and axioms. Nature shows that in order to survive you must adapt either by mutating your genes or by creating schemata that can be accommodated through learning. So if we want our children to survive we must seriously consider replacing pastoral mathematics and Humboldt elite Bildung with its natural alternative, mathematics as a natural science and democratic enlightenment schools.

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Appendix. A General Enlightenment Curriculum

The holes in the head provide humans with food for the body and knowledge for the brains: tacit knowledge for the reptile brain and discursive knowledge for the human brain. This curriculum sees a school as an institutionalised knowledge house providing humans with routines and stories by making them participants in social practices and narratives, and by respecting conceptual liberty.

The chaotic learning of tacit routine knowledge can be guided by attractors (Doll, 1993), in this case by social practices providing authenticity. In the case of mathematics the social practices will be those of bundling and totalling according to the Arabic meaning of the word Algebra: reuniting.

In today's post-traditional society (Giddens in Beck et al., 1994) humans can no longer obtain identity by echoing traditions, they have to create their self-identity by building biographical self-stories looking for meaning and authenticity (Giddens, 1991). Individual students have their own learning story, a network of concept-relations, sentences. Resembling a widespread organic carbon structure a learning story steadily grows by adding new sentences to existing words: Tell me something I don't know about something I know (Ausubel, 1968). Stories can tell about the metaphysical world above and about the physical world below. Pastoral stories from above connecting metaphysical concepts cannot be anchored to the existing learning story; they become encapsulated rote learning. Grounded stories from below can, i.e. stories about the social practices providing the daily bread. The three grounded mother stories are the stories about nature, culture and humans.

First the strong gravity force crunched the universe in a big bang, liberating the medium nuclear force trying to crunch the atoms of a star in small bangs liberating light. In the end the strong force crunches the star in a medium bang filling space with matter and planets; and liberating the weak electromagnetic force neutralising the strong force by distant electrons. Light makes motion flow through our planet's nature creating random micro-motion and cyclic macro-motion. Molecules transfer motion through collisions. The strong light, lightening, splits the strong nitrogen-nitrogen link in the air adding strength to the extended carbon-nitrogen structures from which life is build.

The three life forms are black, green and grey cells.

The black cells survive in oxygen free places in stomachs and on the bottom of lakes only able to take oxygen in small amounts from organic carbon-structures thus producing gas.

The weak light helps the green cells lifting the hydrogen from the water molecules to the carbon dioxide molecules thus producing both organic matter storing motion and the oxygen needed by the grey cells to release the motion again. Green cells form cell communities, plants, unable to move for the food and the light.

Grey cells form animals able to move for the food in form of green cells or other grey cells thus needing to collect and process information by senses and brains to decide which way to move. Animals come in three kinds. The reptiles have a reptile brain for routines. The mammals having live offspring in need of initial care have developed an additional mammal brain for feelings. Humans have developed human fingers to grasp the food, and a human brain to grasp the world in words and sentences. Thus humans can share and store not only food but also stories, e.g. stories about how to increase productivity by transforming nature to culture.

The agriculture transforms the human hand to an artificial hand, a tool, enabling humans to transform the forest into a field for growing crops. The industrial culture transforms the human muscle to an artificial muscle, a motor, integrating tools and motors to machines enabling humans to transform nature raw material to material goods. The information culture transforms the human reptile brain to an artificial brain, a computer, integrating the artificial hand, muscle and brain to an artificial human, a robot, freeing humans from routine work.

Human production and exchange of goods has developed a number language besides the word language to quantify the world and calculate totals. Agriculture totals crops and herds by adding. Trade totals stocks and costs by multiplying. Rich traders able to lend out money as bankers total interest percentages by raising to power. And finally industrial culture calculates the total change-effect of forces through integrating: by adding a certain amount of momentum per second and energy per meter a force changes the meter-per-second-number, which again changes the meter-number.

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209. CATS, Count&Add in Time&Space - a Natural Way to Become a Mathematics Teacher

The CATS-approach, Count&Add in Time&Space, is a natural way to become a math teacher. It obeys the fundamental rule of good research, never to ask leading questions. To learn mathematics, students should not be taught mathematics; instead they should meet the roots of mathematics, multiplicity. Through guiding educational questions asking them to Count and Add in Time and Space, they learn mathematics without knowing it. The CATS-approach is rich on examples of recognition and new cognition to be observed, reflected and reported by teachers and researchers.

The Background

Enlightenment mathematics treated mathematics as a natural science. Grounded in the natural fact many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline 1972: 398). Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics into a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: definition $M = \{A \mid A \notin A\}$, statement $M \in M \Leftrightarrow M \notin M$.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Without an alternative, modern mathematics creates big problems to mathematics education, as e.g. the worldwide enrolment and justification problems in mathematical based educations and teacher education (Jensen et al 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994: 371).

And teachers are forced to bring meaningless modern mathematics to the everyday classroom:

When insisting that general functions must be taught before linear and exponential functions that must be taught before teaching change by adding or multiplying a constant number. Students have no problems seeing that the change 200$ + 5$ x times can be generalized to y = b + a*x. But they refuse to learn that 'a functions is an example of a many-one set-relation', which they hear as 'bublibub is an example of bablibab'. A rational reaction since defining a 1700-concept as an example of a more abstract 1900-concept is just another example of 'metamatics'.

And when teaching that fractions can be added without units, thus transforming grounded mathematics into ungrounded ''mathematism' true in a library but not in a laboratory:

The teacher:	The students:
Welcome! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No, 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
Inside this classroom 1/2 + 2/3 = 7/6!	

So in reality, what is called ‘mathematics education’ is not education in mathematics, but instead education in ‘metamatism’ merging metamatics with mathematism. To design an alternative, mathematics should return to its roots, multiplicity, guided by a kind of research able at uncovering hidden alternatives to choices presented as nature, as recommended by the ancient Greek sophists.

Anti-pastoral Sophist Research

Ancient Greece saw a struggle between the sophists and the philosophers as to the nature of knowledge. The sophists warned that to protect democracy people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers, seeing everything physical as examples of meta-physical forms only visible to them, patronization was a natural order when left to the philosophers (Russell 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spices. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people should do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. The US still has its first republic; France now has its fifth.

The German autocracy tried to stop the French democracy by sending in an army. However, the German mercenaries were no matches to the French conscripts only too aware of the feudal consequences of loosing. So the French stopped the Germans, and later occupied Germany.

Unable to use the army, the German autocracy used the school to stop the enlightenment in spreading from France. Humboldt was asked to create an elite school, and used Bildung as counter-enlightenment to create the self-referring Humboldt University (Denzin et al 2000: 85).

Inside the EU the sophist warning is kept alive in the French postmodern or post-structural thinking of Derrida, Lyotard and Foucault warning against patronizing categories, discourses and institutions presenting their choices as nature (Tarp 2004).

Derrida recommends that patronizing categories, called logocentrism, be ‘deconstructed’:

> Derrida encourages us to be especially wary of the notion of the centre. We cannot get by without a concept of the centre, perhaps, but if one were looking for a single ‘central idea’ for Derrida’s work it might be that of decentring. It is in this very general context that we might situate the significance of ‘poststructuralism’ and ‘deconstruction’: in other words, in terms of a decentring, starting with a decentring of the human subject, a decentring of institutions, a decentring of the logos. (Logos is ancient Greek for ‘word’, with all its connotations of the authority of ‘truth’, ‘meaning’, etc.) (..) It is a question of the deconstruction of logocentrism, then, in other words of ‘the centrism of language in general’. (Royle 2003: 15-16)

As to discourses Lyotard coins the term ‘postmodern’ when describing ‘the crisis of narratives’:

> I will use the term modern to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..)

> Simplifying to the extreme, I define postmodern as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

Foucault calls institutional patronization for ‘pastoral power’:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1982: 213, 215)

In this way Foucault opens our eyes to the salvation promise of the generalized church: ‘you are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will save you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and accept becoming a docile lackey’.

To Foucault, institutions building on discourses building on categories build upon choice, so they all have a history, a ‘genealogy’, that can be uncovered by ‘knowledge archeology’.

The French skepticism towards words, our most fundamental institution, is validated by a ‘number&word observation’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘stick’; thus it can itself falsify its number but not its word, which makes numbers nature and words choices becoming pastoral if hiding their alternatives, and allows number-statements to be research, whereas word-statements will always be interpretations.

On the basis of the ancient and the contemporary sophist warning, a research paradigm can be created called ‘anti-pastoral sophist research’ deconstructing pastoral choices presented as nature by uncovering hidden alternatives. Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘in this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?’

Natural Learning and Natural Research

To make categories, discourses and institutions not pastoral but enlightening, they should be grounded, not in choice but in nature, by using Grounded Theory, the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning (Piaget 1970), and with the Enlightenment principles for research: Observe, abstract categories and relations to be accommodated through predicted deductions.

Grounded theory also shows skepticism towards existing research-based categories:

> Although categories can be borrowed from existing theory, provided that the data are continually studied to make certain that the categories fit, generating theory does put a premium on emergent conceptualizations. There are a number of reasons for this. Merely selecting data for a category that has been established by another theory tends to hinder the generation of new categories, because the major effort is not generation, but data selection. Also, emergent categories usually prove to be the most relevant and the best fitted to the data. As they are emerging, their fullest possible generality and meaning are continually being developed and checked for relevance. Also the adequacy of indicators for emergent categories is seldom a problem. (..) In short, our focus on the

> emergence of categories solves the problems of fit, relevance, forcing, and richness. An effective strategy is, at first, literally to ignore the literature of theory and fact on the area under study, in order to assure that the emergence of categories will not be contaminated by concepts more suited to different areas. Similarities and convergences with the literature can be established after the analytic core of categories has emerged. (Glaser et al 1967: 36-37)

A Historical Background

The natural fact many provoked the creation of mathematics as a natural science addressing the two fundamental human questions ‘how to divide the earth and what it produces?’

Distinguishing the different degrees of many leads to counting that leads to numbers.
1.order counting counts in 1s and creates number-icons by rearranging the sticks so that there are five sticks in the five-icon 5 etc. if written in a less sloppy way.

2.order counting counts by bundling&stacking in numbers with a name and an icon, resulting in a double stack of bundled and unbundled, e.g. T = 3 5s + 2 1s = 3)2) if using cup-writing, leading to decimal-writing separating the left bundle-cup from the right single-cup: T = 3)2) = 3.2 5s = 3.2*5. The result can be predicted by the ‘recount-formula’ T = (T/b)*b iconizing that counting in bs means taking away bs T/b times, e.g. T = (4*5)/7*7 = 2*7 + 6*1 = 2)6) = 2.6*7.

3.order counting counts in tens, having a name but not an icon since the bundle-icon is never used: counting in 5s, T = 5 1s = 1 5s = 1.0 bundle = 10 if leaving out the decimal and the unit.

In Greek, mathematics means knowledge, i.e. what can be used to predict with, making mathematics a language for number-prediction: The calculation ‘2+3 = 5’ predicts that counting-on 3 times from 2 will give 5. ‘2*3 = 6’ predicts that repeating adding 2 3 times will give 6. ‘2^3 = 8’ predicts that repeating multiplying with 2 3 times will give 8. Also, any calculation can be turned around and become a reversed calculation predicted by the reversed operation: In the question ‘3+x = 7’ the answer is predicted by the calculation x = 7–3, etc.

3+x = 7 x = 7–3	3*x = 7 x = 7/3	x^3 = 7 $x = \sqrt[3]{7}$	3^x = 7 $x = \log_3(7)$

Thus the natural way to solve an equation is to move a number across the equation sign from the left forward-calculation to the right backward-calculation side, reversing its calculation sign.

In Arabic, algebra means reuniting, i.e. splitting a total in parts and (re)uniting parts into a total. The operations + and * unite variable and constant unit-numbers; ∫ and ^ unite variable and constant per-numbers. The inverse operations – and / split a total into variable and constant unit-numbers; d/dx and √ & log split a total into variable and constant per-numbers:

Totals unite/split into	Variable	Constant
Unit-numbers $, m, s, ...	T = a + n *T – n = a*	T = a * n *T/b = a*
Per-numbers $/m, m/s, m/100m = %, ...	ΔT = ∫ f dx *dT/dx = f*	T = a ^ n $\sqrt[n]{T} = a$, $log_a T = n$

In Greek, geometry means earth measuring. Earth is measured by being divided into triangles, again being divided into right-angled triangles, each seen as a rectangle halved by a diagonal.

Recounting the height h and base b in the diagonal d produces three per-numbers:

sin A = height/diagonal = h/d, tan A = height/base = h/b, cos A = base/diagonal = b/d.

However, needing the Arabic numbers, Greek geometry turned into Euclidean geometry, freezing the development of mathematics until the Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. The seventeen-century men had broken both of these bonds. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

The success was so overwhelming that mathematicians feared that mathematics, called geometry at that time, had come to a standstill at then end of the 18th century:

> Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities. (Lagrange in Kline 1972: 623)

But in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. So in the 1870s the concept 'set' reintroduced rigor into mathematics, leading to modern set-based mathematics dreaming that it could become a collection of well-proven statements about well-defined concepts; a dream that broke with Russell and Gödel.

Where do concepts come from

The number one question in education is 'where do concepts come from - from above or from below - from the outside or from the inside?' The four different answers to this question lead to four different 'learning rooms'.

The traditional learning room is the 'transmitter room' saying 'from above&outside' and seeing knowledge as outside information to be transmitted into learners' brains with teachers and textbooks as channels. General communication theory is applied in this learning room.

Its traditional alternative is the 'constructivist room' saying and 'from above&inside' and seeing knowledge as library information that can be reached by learners through internal scaffolding guided by a teacher. Vygotsky and Bruner constructivism is applied in this learning room.

One hidden alternative is the 'story-telling room' saying 'from below&outside' and seeing knowledge as unknown stories about known subjects that the learner, biologically programmed to remember gossip, automatically learns if told as gossip by a story-teller, a teacher. Story-telling and fairy tales are applied in this learning room.

Another hidden alternative is the 'apprentice room' saying 'from below&inside' and seeing knowledge as both verbal and tacit that the learner, biologically programmed to adapt to the surroundings through assimilation and accommodation, automatically learns if placed in exemplary situations by a teacher. Piaget constructivism and Situated Learning is applied in this learning room.

The traditional learning rooms take mathematics for granted and see the world as applying mathematics. The hidden learning rooms have the opposite view taking the multiplicity of the outside world for granted and as a creator of mathematics through the principle ‘grip to grasp’, ‘through the hand to the head’ or ‘greifen vor begreifen’.

The transmission room versus the story-telling room arranges sentence-loaded educational meetings with sentences with abstract versus concrete subjects. The constructivist room versus the apprentice room arranges sentence-free educational meetings with abstract versus concrete subjects.

Thus the two ‘from above’ rooms, seeing mathematics as book-based knowledge, want learning to take place in a library; and the two ‘from below’ rooms, seeing mathematics as a natural science investigating the natural fact many, want learning to take place in a laboratory.

Humans develop from one biological species to another. Before puberty human children learn about the outside world by adapting, i.e. through feeling and experimenting, as mammal offspring do. Of course, human language makes human learning far more effective than other mammal learning. After puberty mammals stop learning and focus on feeding and guarding the offspring while adapting. Humans, however, have a language that can be used to communicate information between brains, most effectively in the form of gossip about known subjects.

Word-language and Number-language

A ruler and a dictionary help us to assign numbers and words to things using our number-language and our word-language. Thus we have word-sentences containing a subject, a verb, and an object; and we have number-sentences, equations, containing a quantity, an equation sign, and numbers and calculations. Both sentences are describing the world, and both are being described by a meta-language. The meta-language of the word-language is called grammar. The meta-language of the number-language is called mathematics.

Our two languages and their meta-languages constitute a language-house with two floors. In the lower floor the language is used to describe the world, and in the upper floor the meta-language is used to describe the language. Syntax errors occur if the meta-language is used to describe the world: ‘the verb got drunk’. So mathematics does not describe the world, mathematics describes the number-language, and the number-language describes the world.

		THE LANGUAGE HOUSE		
META-LANGUAGE	Grammar	Subject	Constants and variables	Mathematics
LANGUAGE	Word-language	The pencil is red	Area = length*height	Number-language
WORLD		THINGS IN TIME AND SPACE		

Constructing an Enlightening Mathematics From Below

Enlightening mathematics sees mathematics as a natural science investigating the natural fact many. Thus it respects two fundamental principles:

- a Kronecker-principle saying that only the natural numbers can be taken for granted; and

- a Russell-principle saying that we cannot use self-reference and talk about sets of sets.

The enlightening mathematics at the website MATHeCADEMY.net has been constructed as such an example of a Kronecker-Russell mathematics based on studying multiplicity in a set-free

'Count&Add laboratory' where addition predicts counting-results, thus making mathematics a language for number-prediction.
The website contains 2*4 study units in 'mathematics from below, the LIB-free LAB-approach', organized as laboratory activities where the learner learns 'CATS', i.e. to Count & Add in Time & Space guided by educational questions Q and answers A.

The study units CATS1 are for primary school and the study units CATS2 are for secondary school. The units were developed for a web-based distance-learning course in mathematics at a Danish teacher college.

Counting C1

Q: How to count multiplicity? **A:** By bundling and stacking the total T predicted by T = (T/b)*b

Q: How to recount 8 in 3s: T = 8 = ? 3s = ?*3. **A:** 8 = (8/3)*3 = 2*3+2*1= 2)2) = 2.2*3 = 2 2/3*3

Q: How to recount 6kg in $: T = 6kg = ?$. **A:** If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$

Q: How to count in standard bundles? **A:** Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2 + 2*B + 3

Section C1 looks at ways to count multiplicity. Spatial multiplicity is representing temporal repetition through sticks and strokes. A multiplicity of sticks can be rearranged in icons so that there are four sticks in the icon 4 etc. Then a given total T can be counted in e.g. 4s by repeating the process 'from T take away 4', which can be iconized as 'T–4 '; where the repeated process 'from T take away 4s' can be iconized as 'T/4'. This makes it possible to predict the counting-result through a calculation using the 'recount-equation' T = (T/b)*b. Leftovers are stacked as 1s creating a stock T = 2*3 + 2*1. The stacks can be placed in two cups, a left bundle-cup and a right single-cup, and described by cup-writing T = 2)2), or decimal-writing including the unit T = 2)2) = 2.2 3s = 2.2*3; or the leftovers can be counted in 3s and added on top of the 3-stack: T = 2 2/3 * 3 = 2 2/3 3s.

Changing units is another example of a recounting where a given total is double-counted in two different units e.g. T = 4$ = 5kg producing a per-number 4$/5kg = 4/5 $/kg. Thus to answer the question ' 7kg = ?$' we just have to recount the 7 in 5s: T = 7kg = (7/5)*5kg = (7/5)*4$ = 5 3/5$.

The number ten has a name but no icon, since the bundle-size is not used: Counting in 5s, 5 1s = 1 5s = 1 bundle. Before introducing ten as the standard-bundle and leaving out the units, 2.4 tens = 24, the core of mathematics can be leaned by using 1digit numbers alone. (Zybartas et al 2005).

Adding A1

Q: How to add stacks concretely? T = 27 + 16 = 2 ten 7 + 1ten6 = 3 ten 13 = ?.

A: By restacking overloads predicted by the 'restack-equation' T = (T–b) + b:

T = 27 + 16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43.

Q: How to add stacks abstractly? **A:** Vertical addition uses carrying. Horizontal addition uses FOIL.

Section A1 looks at how stacks can be added by removing overloads that often appears when one stack is placed on top of another stack. The overload leads to 'internal trade' between two stacks where a stack of 10 1s is rebundled and restacked as 1 10-bundle. The result can be predicted by a calculation on paper using either a vertical way of writing the stacks using carrying to symbolize the internal trade; or using a horizontal way of writing the stacks using the FOIL-principle (First, Outside, Inside, Last). In both cases the overload can be restacked predicted by the restack-equation (T–b) + b, and recounted predicted by the recount-equation T = (T/b)*b.

Time T1

Q: How can counting & adding be reversed? **A:** By calculating backwards, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign.

Q: Counting ? 3s and adding 2 gave 14. **A:** x * 3 + 2 = 14 is reversed to x = (14 – 2)/3.

Q: Can all calculations be reversed? **A:** Yes. x + a = b is reversed to x = b – a, x * a = b is reversed to x = b/a, x ^ a = b is reversed to x = $\sqrt[a]{b}$, a ^ x = b is reversed to x = logb/loga.

Section T1 looks at formulas, the sentences of the number-prediction language. Containing two unknown variables, a formula becomes a function to be tabled and graphed. Containing one unknown variable, a formula becomes an equation to be solved by reversing the calculations, moving numbers from the forward-calculation side to the backward-calculation side reversing their signs: x*3+2 = 14 is reversed to x = (14-2)/3. This forward and backward calculation method gives a new perspective on the classical quantitative literature consisting of word-problems.

Space S1

Q: How to count the plane and spatial properties of stacks, boxes and round objects?

A: By using a ruler, a protractor and a triangular shape; by the 3 Greek Pythagoras’, mini, midi & maxi; and by the 3 Arabic recount-equations: sin A = a/c, cos A = b/c, tan A = a/b.

Section S1 looks at how to describe plane properties of stacks as areas and diagonals by the 3 Greek Pythagoras’, mini, midi & maxi; and by the 3 Arabic recount-equations: sin A = a/c, cos A = b/c and tan A = a/b. A circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: C = 2 * r * (n*sin(180/n)) = 2 * r * π for n sufficiently big.

Finally we look at how to describe spatial properties of solids such as surfaces and volumes by formulas and by a 2-dimensional representation of 3-dimensional shapes.

Counting C2

Q: How to count possibilities? **A:** By using the numbers in Pascal’s triangle.

Q: How to predict unpredictable numbers? **A:** If a ‘post-diction’ gives the average 8.2 with deviation 2.3, the ‘pre-diction’ gives the confidence interval 8.2 ± 2*2.3 with 95% probability.

Section C2 looks at numbers that change unpredictably as e.g. in surveys. Through counting we can set up a frequency-table accounting for the previous behavior of the numbers. From this table their average level and their average change can be calculated. From this we can predict that with a 95% probability, future numbers will occur within an interval determined by the average level and double the average change. Counting the numbers of wins when repeating a game with winning probability p is another example of an unpredictable number, also called a stochastic variable.

Adding A2

Q: What is a per-number? **A:** Per-numbers occur when counting, when pricing and when splitting.

Q: How to add per-numbers? **A:** The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b

Section A2 looks at how to add per-numbers by transforming them to totals. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b. 2days at

6$/day + 3days at 8$/day = 5days at 7.2$/day. And 1/2 of 2 cans + 2/3 of 3 cans = 3 of 5 cans = 3/5 of 5 cans. Repeated and reversed addition of per-numbers leads to integration and differentiation:

$T2 = T1 + a*b$; $T2 - T1 = a*b$; $\Delta T = \sum a*b = \int y*dx$; and

$T2 = T1 + a*b$; $T2 - T1 = a*b$; $a = (T2 - T1)/b = \Delta T/\Delta b = dy/dx$

Time T2

Q: How to predict the terminal number when the change is constant?

A: By constant change-equations: If Ko = 30 and $\Delta K/n = a = 2$, then $K7 = Ko+a*n = 30+2*7 = 44$.

If Ko = 30 and $\Delta K/K = r = 2\%$, then $K7 = Ko*(1+r)^n = 30*1.02^7 = 34.46$

Q: How to predict the terminal number when the change is variable, but predictable?

A: By a variable change-equation: If Ko = 30 and dK/dx = K', then $K - 30 = \Delta K = \int dK = \int K'dx$

Section T2 looks at how a stack changes in time by adding a constant number, or by a constant percent where adding 5% means changing 100% to 105%, i.e. multiplying with 105% = 1.05.

If related by a formula y = f(x), a x-change Δx will effect a y-change Δy that can be recounted in the x-change as $\Delta y = (\Delta y/\Delta x)*\Delta x$, or $dy = (dy/dx)*dx = y'*dx$ in the case of micro-changes.

If a stack y changes by adding variable predictable numbers dy, summing up the single y-changes gives the total y-change, i.e. the terminal y2 minus the initial y1: $\int dy = \int y'*dx = y2 - y1$.

Space S2

Q: How to predict the position of points and lines? **A:** By using a coordinate-system: If Po(x,y) = (3,4) and $\Delta y/\Delta x = 2$, then $P1(8,y) = P1(x+\Delta x, y+\Delta y) = P1(3+(8-3),4+2*(8-3)) = (8,14)$

Section S2 looks at how to predict the position of points and lines and geometrical figures and graphs using a coordinate system. Then we look at how to use the new calculation technology such as computers and calculators to calculate a set of numbers, vectors, and a set of vectors, matrices.

Quantitative Literature

Q: What is quantitative literature? **A:** Quantitative literature is about multiplicity in time and space.

Q: Does quantitative literature share the 3 different genres: fact, fiction and fiddle? **A:** Yes.

In formulas as T = c*p we need to know what quantities are described to determine the truth-value of the formula's prediction. It turns out that both word-statements and number-statements share the same genres: fact, fiction and fiddle. Fact-models predict predictable quantities. Fiction-models predict unpredictable quantities. Fiddle-models predict qualities.

Modelling With Regression-Mathematics

Modelling consists of four parts: a real-world problem, a model problem, a model solution, and a real-world solution. First a real-world problem leads to a model problem, often a table relating two variables x and y. Then in the model solution, regression is used to find a formula connecting the variables. Containing one unknown, the formula becomes an equation that can be solved manually or by the Math Solver on a Graphical Display Calculator. Containing two unknowns, the formula becomes a function to be illustrated by a graph; and where the two typical questions 'given x find y' and 'given y find x' reduces the function to an equation to be solved by the Math Solver, or graphically by Trace and by Calc Intersection. Finally the solution is evaluated as to its applicability as a real-world solution. Computers have enabled the creation of huge models using system dynamics to model the interaction of complex systems of variables in science and in economics.

Fact Models

Fact models quantify and predict predictable quantities: 'What is the area of the walls in this room?' In fact models the predicted answer is what is observed. Hence numbers calculated by a fact model can be trusted. Geometry and the basic formulas T = 2+3, T = 2*3 and T = 2^3 etc. are fact models, as well as many models from physics, trade and financing. A fact model may also be called a 'since-then model or a 'room-model'.

Fiction Models

Fiction models quantify and predict unpredictable quantities: 'My debt will soon be paid off at this rate!' Fiction models produce predictions based upon presumed assumptions that should be supplemented with alternative assumptions, i.e. with parallel scenarios. Typical examples are average-models, simplifying complex economical or technical models by assuming some variables to be constants staying at their average level. Other examples are linear demand and supply curves in economical theory. A fiction model may also be called an 'if-then' or a 'rate-model'.

Fiddle Models

Fiddle models quantify qualities: 'Are the risk and casualty numbers of this road high enough to cost a bridge?' This question will install crosswalks instead of bridges on motorways since it is cheaper to be in a cemetery than at a hospital. Fiddle models should be rejected asking for a word description instead. Many risk-models are fiddle models. The basic risk model says: Risk = Consequence * Probability. A fiddle model may also be called a 'so-what' model or a 'risk-model'.

The Grand Narratives of the Quantitative Literature

Literature is narratives about real-world persons, actions and phenomena. Quantitative literature also has its grand narratives. That overwhelmingly many numbers can be added by one simple difference, providing the numbers can be written as change-numbers, is a grand narrative.

Another example is the geometry of a tennis ball, forcing much traditional geometry to be adapted: parallel lines intersect, the angles in a triangle add up to more than 180, two-angles pop up, etc.

In physics, grand narratives can be found among those telling about the effect of forces, e.g. gravity, producing parabola orbits on earth, and circular and ellipse orbits in space. Jumping from a swing is a simple example of a complicated model. The grand narratives of physics enabled the rise of the Enlightenment period, and of the modern democratic society replacing religion with science.

In economics, an example of a grand narrative is Malthus' 'principle of population' comparing the linear growth of food production with the exponential growth of the population; and the Keynes model relating demand and employment creating the modern welfare society. As are the macroeconomic models predicting the effects of different taxation and reallocation policies.

Also limit-to-growth models constitute grand narratives predicting the global economical and ecological future depending on different production, consumption, and pollution options.

Comparing Modern LIB-mathematics and Enlightening LAB-mathematics

Set-based LIB-mathematics has as number-sets integers, rational numbers etc. Multiplicity-based LAB-mathematics only has stack-numbers and per-numbers. In LIB-mathematics the order of introducing the operations are +, -, *, /. In LAB-mathematics the order is the opposite. In LIB-mathematics two digit numbers are introduced before mathematics. In LAB-mathematics it is opposite. LIB-mathematics considers multi-digit numbers as being easy and algorithms as being difficult. In LAB-mathematics it is opposite. LIB-mathematics introduces fractions and decimals and percentages late. LAB-mathematics introduces decimals and fractions in grade 1 as parallel ways of counting leftovers when bundling. LIB-mathematics adds fractions. LAB-mathematics adds per-numbers. LIB-mathematics solves equations by neutralizing. LAB-mathematics solves equations by backward calculations. LIB-mathematics talks about multiplicative structures and proportionality. LAB-mathematics just recounts. LIB-mathematics postpones calculus to late in secondary school. LAB-mathematics introduces 1digit calculus in primary school. LIB-mathematics postpones trigonometry to late in secondary school. LAB-mathematics introduces trigonometry in primary school. LIB-mathematics believes that sets, fractions and functions are fundamental mathematical concepts that are applied in many areas. LAB-mathematics is set-free, fraction-free and function-free. In LIB-mathematics 2+3 IS 5 and 2*3 IS 6. In LAB-mathematics 2+3 can be whatever depending on the units, while 2*3 IS 6 since the unit here is 3s: 2*3 = 2 3s = 6 1s = 6.

Learning Principles: Grip & Grasp and Gossip

The learning principles are 'grip&grasp', and 'learn from gossip'. In primary school children as mammal offspring learn by doing. This means that learning has to come through the hands, 'greifen vor begreifen', both as objects you can grip and as actions you can perform. Thus a LAB-approach means that the written learning material in primary school is brief since the learning takes place, not by reading, but gripping and moving. And in secondary school texts have the form of gossip.

One example of grip&grasp is the progression from 1.order over 2.order to 3.order counting. 1.order counting means counting in 1s rearranging sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for the numbers until ten that becomes a very special and strange number having its own name, but not its own icon.

2.order counting is counting by bundling&stacking using icon-bundles. First we bundle e.g. 7 sticks in e.g. 3-bundles, in 3s. Then we stack the total in two stacks: a stack of 3s, and a stack of unbundled singles. The stacks may then be placed in a left bundle-cup and in a right single-cup.

In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup.

Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit: T = 2.1 3s.

I I I I I I I -> III III I ->	III III I	-> ~~III~~ ~~III~~) I) -> █ █) I) -> II) I)
Or with icons:		-> 2 3s + 1 1s -> 2x3 + 1x1 -> 2)1) -> 2.1 3s

Later also bundles are bundled, calling for a new cup to the left. Thus 4 5s can be rebundled in 6 3s and 2 1s, i.e. as 6)2), where the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II).

Including ten as bundle-size means going on from 2.order counting, using bundles with both a name and an icon, to 3.order counting, using the bundle-size ten having a name, but not an icon.

Before introducing ten as 10, i.e. as the standard bundle-size, 5 is chosen as the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is never used when using place values: 1, 2, 3, 4, bundle.

In secondary school young people learn by listening, but the sentences need to be gossip with known subjects, so they tell you something new about something you already knew. This means that abstract concepts must be presented as abstractions from examples and not as examples of abstractions, i.e. from below and not from above.

Learning Steps

A student's learning-process has five steps: Do, name, write, reflect and communicate. On top of that a teacher should be able to design a learning experiment for the students, and be able to learn from observing it being carried out. The experiment is performed three times, first by the designing teacher, then by a student communicating with the teacher, finally by two students communicating with each other while the teacher is observing. During all three experiments the teacher looks for examples of cognition, both existing recognition and new cognition.

Afterwards the student teacher works out a learning report reporting the three experiments observed. The report finally formulates a hypothesis based on what has been learned from observing these three experiments. For each of the 2*4 final CATS-reports this hypothesis is validated by arranging a new learning experiment to be tested on one student and on two students; and by comparing the prediction from the hypothesis with the observations. Example:

Do: take 5 sticks from a matchbox and arrange them, first next to each other, then as the symbol 5. Say: five sticks can be rearranged to the number symbol or the number icon 5. Write: T = 5.

Reflect. That five sticks are called five is old cognition. That five sticks can be rearranged as the number symbol 5 is new cognition. That the number-symbols are icons containing the number of sticks they describe is new cognition. Also it is new cognition that this makes a fundamental difference between the ability of numbers and letters to represent the world: Numbers are icons representing what they describe; words are sounds often installing what they describe.

Communicate. Write a postcard: 'Dear Paul. I have just participated in an experiment where I was asked to take out five sticks from a matchbox and arrange them as the number symbol 5. All of a sudden I realized the difference between the symbols '5' and 'five', the first representing what it describes an the second representing a sound. See you next week. Best wishes'.

Design an experiment. 'Build the first twelve number-symbols by rearranging sticks'.

Hypothesis. This experiment will help Peter, who has problems understanding 2digit numbers. Once he tries to build a number symbol for ten, eleven and twelve, he will realize how smart it is to stop inventing new symbols and instead begin to double-count bundles and unbundled.

Test. After having finished reporting what Peter did and said, it is my impression that constructing the number icon for ten was what broke the ice for Peter. It seems as if this enabled him to separate number-names from number-icons, since it made him later ask, 'Why don't we say one-ten-seven instead of seventeen? It would make things much easier.' This resonates with what Piaget writes: 'Intellectual adaptation is thus a process of achieving a state of balance between the assimilation of experience into the deductive structures and the accommodation of those structures to the data of experience (Piaget 1970: 153-154)'.

PYRAMIDeDUCATION

In PYRAMIDeDUCATION, 8 student teachers are organized in 2 teams of 4 students choosing 3 pairs and 2 instructors by turn. The coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach helps the instructors when correcting the count&add problems. In each pair each student corrects the other student's routine-assignment. Each pair is the opponent on the essay of another pair. Having finished the course, each student teacher will 'pay' by coaching a new group of 8 student teachers.

Basic Teacher Training or Master Degree

The CATS-approach in teacher training can be used during the basic teacher training; or it can be used as a master degree for persons having already finished their basic training and wanting to become a math-coach offering part of the CATS approach as in-service training to other teachers. The training material will be available, partly at the MATHeCADEMY.net website, partly at a university licensed to offer the CATS-approach as part of a franchising agreement.

Conclusion

What is called 'mathematics education' is instead 'metamatism education', trying to fulfill the Platonic dream of presenting the world as examples of mathematics, itself being examples of the metaphysical form set; but instead making students turns their backs to the subject that through its number-predicting ability is the foundation of the modern industrialized well-fare society. To become what it says, mathematics education should respect the nature of mathematics as a natural science grounded in its physical root multiplicity, as was the case in the Enlightenment period.

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210. Pastoral Words in mathematics education

Mathematical terminology is very fixed, almost dogmatic, which seems to indicate a metaphysical nature. The necessity of language shows the great advantages by having a fixed terminology. However, there is a fundamental difference between enlightening words labeling and pastoral words hiding differences. Following the advice of the ancient Greek sophists warning against mixing up nature and choice, this paper asks 'what is nature and what is choice in mathematical terminology?'

The Background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: 'Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$'.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots guided by a new kind of research able at uncovering hidden alternatives to choices presented as nature, e.g. anti-pastoral sophist research, first identifying what is nature and then searching for pastoral choices hiding their alternatives.

Anti-Pastoral Sophist Research

Ancient Greece saw a fierce controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people needed to be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers argued that patronization is the natural order since everything physical is an example of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become the natural patronizing rulers.

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronization with democracy. Two democracies were installed: one in US, and one in France, now having its fifth republic. The German autocracy tried to stop the French democracy by sending in an army. However, the German army of mercenaries was no match to the French army of conscripts only to aware of the feudal alternative to stopping the German army. So the French stopped the Germans and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. Using Bildung as counter-enlightenment, Humboldt created an elite school system consisting of the Humboldt-gymnasium and the Humboldt University, still protected by the EU even after becoming democratic.

In France, sophist warning is kept alive in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronizing categories, discourses and institutions presenting their choices as nature (Tarp 2004). Derrida recommends that pastoral categories be ‘deconstructed’. Lyotard recommends the use of postmodern ‘paralogy’ research to invent alternatives to pastoral discourses. And Foucault uses the term ‘pastoral power’ to warn against institutions legitimizing their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

In descriptions, numbers and words are different as shown by the ‘number & word dilemma’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘pencil’, thus being able itself to falsify its number but not its word, which makes numbers nature, and words choices becoming pastoral if hiding their alternatives; and thus in need of deconstruction to unhide the alternatives.

Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what are pastoral choices presented as nature?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory, the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning, and with the Enlightenment principles for research: observe, abstract and test predictions.

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact. Adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of e.g. eight sticks can be treated in different ways. The sticks can be rearranged to an eight-icon 8 containing the eight sticks, written as 8.

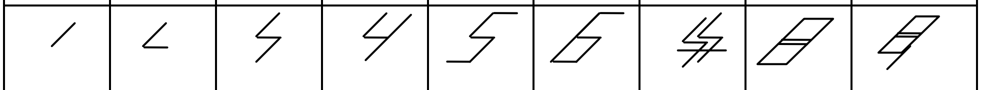
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

Figure 1: Numbers as strokes, icons and symbols.

The sticks can be collected to one eight-bundle, written as 1 8s. And the sticks can be ‘decimal-counted’ by bundling & stacking, bundling the sticks in e.g. 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled in a right single-cup. Writing down the counting-result, cup-writing gradually leads to decimal-writing where the decimal point separates the bundle-cup from the single-cup:

8 = 1 5s + 3 1s = 1)3) = 1.3 5s = 13 if leaving out both the decimal point and the unit.

With 5 as the bundle-size, icons are only needed for the numbers less than 5 since

1 5s = 1.0 fives = 10, and 1*6 = 1.1 fivess = 11.

Since ten has been chosen as the standard bundle-size, no icon for ten exists making ten a very special number having its own name but not its own icon. This has big technical advantages as shown when comparing the Arabic numbers with the Roman numbers, that has a special ten-icon X, but where multiplications as ‘XXXIV times DXXVIX’ is almost impossible to do. But without its own icon, ten might create learning problems if introduced to early.

Rebundling 3tens and 2 in 5s gives the result 6 5s + 2 1s = 6)2) = 6.2 5s. However, here also five 5-bundles can be rebundled to 1 5*5 bundle placed in an extra cup to the left of the two existing cups:

6 5s + 2 1s = 6)2) = <u>5+1</u>)2) = 1)1)2) = 11.2 5s

All cases of 5 1s can be rebundled into 1 5-bundle, 5)n)m) = 1)0)n)m), since 5 5*5-bundles = 1 5*5*5-bundle etc. as in

231.4 5s = 2)3)1)4) = 2 5*5*5s + 3 5*5s + 1 5s + 4 1s.

So, as indicated by number words as three hundred forty-six meaning three ten tens and four tens and six unbundled, the nature of numbers is that any total can be counted by bundling & stacking. And the natural way to write a stock consisting of stacks is to use a decimal number that includes its unit, the chosen bundle-size.

Pastoral Numbers

The tradition introduces numbers one by one using the follower-principle. On the textbook-page introducing the number five, different examples of five are shown enabling the learners to practice counting to five and writing the symbol 5. Numbers adding up to five revise the previous numbers. The numbers are called natural numbers, and the signs are called number-symbols or numerals (NCTM 2000).

In teacher education, first the set of Peano-numbers is defined as the set generated by a first number 0 and a successor-principle saying that each number has a successor. Then the natural numbers N are defined as equivalence sets in a product set of two sets of Peano-numbers created by the equivalence relation (a,b) ≈ (c,d) if a+b = c+d. Then the integers Z are defined as equivalence sets in a product set of two sets of natural numbers created by the equivalence relation (a,b) ≈ (c,d) if a+d = c+b. Then the rationals Q are defined as equivalence sets in a product set of two sets of integers created by the equivalence relation (a,b) ≈ (c,d) if a*d = c*b.

Naming numerals ‘symbols’ means choosing not to distinguish between number-symbols and letter-symbols, where the first are icons labeling the actual degree of many while the latter is a symbol installing one or more sounds to be different.

Saying ‘five IS 5’ hides its alternatives as ‘five is 10’ in the case of 5-bundling, or 11 in the case of 4-bundling. The follower of four is five by nature, but claiming that ‘the follower of 4 IS 5’ is hiding its alternative, 10 in the case of 5-bundling.

Calling 2digit numbers ‘natural’ presents them as nature instead of pastoral choices hiding their natural alternatives, decimal numbers with units. Leaving out units creates ‘mathematism’ (Tarp 2004) as 2+3=5, which is true in the library, but not in the laboratory where countless counterexamples exist, e.g. 2weeks+3days = 17days.

Using the follower-principle presenting 8+1 as 9 and 9+1 as 10 leads to 2digit numbers and also reduces counting to be counting in 1s not needing bundling & stacking. Again this is a pastoral choice hiding its alternative: to stop with nine, the last number having both its own name and its own icon; and to practice also counting in 2s and 3s etc. before introducing 2digit numbers. In fact, it is possible to introduce the core of mathematics with 1digit numbers alone (Zybartas et al 2005).

Also without bundling & stacking experiences, the follower-principle can't explain why the follower of 9 is 10, instead it is caught in meaningless self-reference when after explaining 23 as 2 10s and 3 1s, 10 has to be explained as 1 10s and 0 1s. Whereas with bundling & stacking experiences, 10 just means bundle; and experiences soon show that icons are not needed for the bundle-number and after. So with ten as the standard bundle-size no icon for ten is needed, ten just becomes 10.

So there is nothing natural about natural numbers; and there is nothing natural about counting in 1s. What is natural is counting by bundling & stacking using decimal-numbers including their units to describe the resulting stacks of bundles and unbundled with number icons.

The Nature of Operations and Formulas

The subtraction sign is an icon where '- 2' means 'take away 2'. The formula R = 5-2 = 3 predicts that 'if from 5 we take away 2 then we get 3'.

The division-icon '/2' means 'take away 2s'. The formula N = 8/2 = 4 predicts that 'if 8 is counted in 2s then we get 2s 4 times'.

The addition-icon '+ 2' means 'add 2'. The formula T = 5+2 has two meanings.

With the same units, addition means integrating: 5 1s + 2 1s = (5+2) 1s = 7 1s.

With different units, addition means juxtaposing: 1 5s + 2 1s = 1.2 5s

The multiplication-icon '4*' means 'stacked 4 times'. First T = 4*5 is not a formula but a specification of a stack containing 4 5-bundles, 4*5 = 4 5s, thus making 5 the unit of 4. With ten as the chosen bundle-size, T = 4*5 = 20 becomes a formula predicting that 4 5s can be recounted in tens as 2.0 10s or 2.0 * 10. So as a calculation multiplication becomes division recounting in tens.

Thus the full process of 're-counting' or 're-bundling' 8 1s in 5s can be described by a 'recount or rebundle formula' T = (T/b)*b saying that the total T is first counted in bs, then stacked in bs; together with a 'rest formula' R = T – n*b finding the rest:

T = (8/5)*5 = 1*5 + 3*1 = 1.3*5 , since the rest is R = 8 – 1*5 = 3.

With ten as standard bundle-size, division takes on an extra meaning in sharing situations where the calculation '8/2 = 4' predicts that 'if 8 is shared by 2, the shares are 4'. And with units, division occur in per-numbers where 'double-counting' in two different units produces 'guide-equations' used in proportionality and percentages: 3$ per 4kg = 3$/4kg = 3/4 $/kg leading to the guide equation '3kg = 4$' allowing the question '12 kg = ?$' to be answered by recounting the 12kg in 3s, and 20$ in 4s:

12 kg = (12/3)*3kg = (12/3)*4$ = 16$, and 20$ = (20/4)*4$ = (20/4)*3kg = 15 kg

In Greek, mathematics means knowledge, and knowledge can be used for prediction. Thus with ten as standard bundle-size, addition predicts a counting-on result: 4+3 = 7 predicting that counting-on 3 times from 4 result in 7. Likewise multiplication predicts the result of adding many like numbers, 3*5 = 5+5+5, and power predicts the result of multiplying many like numbers, 5^3 = 5*5*5. From this perspective -, /, and log becomes reversed operations giving the predictions 8-3, 15/3, log3(81) and 5√(32); a to the questions 3+? = 8, 3*? = 15, 3^? = 81 and ?^5 = 32.

So operations are icons describing the process of counting by bundling & stacking. And formulas combine numbers and operations in calculations that predict numbers.

Pastoral Operations and Formulas

The tradition introduces addition with the follower-principle defining 7+1 = 8 etc. and used for revising the previous numbers, as e.g. 6 + 2 = 8, 5 + 3 = 8 etc. Subtraction is introduced as the opposite operation to addition, either as taking away, 9-6 = 3; or as ‘counting up’ needing three counts to get from 6 to 9, i.e. as solving the equation 6+x = 9 without calling it an equation.

Multiplication is introduced as repeated addition with the same number: 3*5 = 5+5+5 = 15; and multiplication is practiced by memorizing tables. Division is introduced as reversed multiplication, i.e. sharing. Thus sharing 8 apples between 2 persons means calculating 8/2 or solving the equation 2*x = 8.

In teacher education, operations are defined as functions from a product set of a number set to itself. Thus in the set N of ‘natural’ numbers, addition is defined as a function f from NxN to N defined by f(a,b) = a+b. This function divides NxN into equivalence classes where (a,b) and (c,d) are equivalent if a+b = c+d.

In the classroom this perspective transforms calculations as ‘2+5’ and ‘3+4’ to being different ‘number-names’ for the same number. Other examples of different ‘number-names’ are 7-3 and 8-4; 2*3 and 6*1; 18/6 and 9/3 and 3/1.

Defining all operations from addition is a pastoral choice hiding its natural alternative defining the operations from subtraction: Division is repeated subtraction with the same number describing the counting process. Multiplication first is stacking the bundles, and later with ten as a standard bundle-size, multiplication becomes division recounting in tens. And addition is reversed subtraction.

Defining addition from the follower-principle by saying that 7+5 is 7 plus 1 5times is a pastoral choice hiding that addition has two different meanings, integrating and juxtaposing. Thus ‘+2’ means both ‘integrating 2’ when two stocks are integrated to one, e.g. 3.1 5s + 1.2 5s = 4.3 5s; and ‘juxtaposing 2’, when adding 2 singles transforms a stack of 3 5s to a stock of 3 5s and 2 1s: T = 3*5 + 2*1 = 3.2 5s.

Defining subtraction as the opposite of addition is a pastoral choice hiding that the nature of subtraction is taking away, not needing addition to be defined.

Defining multiplication as repeated addition with the same number is hiding that multiplication has two different meanings: specifying the height of a stack of bundles, and recounting stacks in tens. In stead a third meaning is introduced saying that 3*5 is the 3rd number in the 5-table to be memorized or found on a calculator. Also presenting the statement ‘3*5 IS 15’ as nature is hiding that 3*5 = 23 in the case of 6-bundling, and 3*5 = 21 in the case of 7-bundling etc.

Defining division as the opposite of multiplication is a pastoral choice hiding that division also has a different meaning, repeated subtraction. Furthermore this second meaning comes first as the icon for taking away e.g. 2s in the counting process.

Defining calculations as number-names is hiding that a calculation is a prediction predicting the number coming out of the described operation, thus hiding the fundamental role of mathematics as a language for number-prediction that helped the natural sciences to move the authority from the library back to the laboratory, which created the background for the Enlightenment period and the modern world.

The Nature of Equations and functions

Containing one unknown number, a formula is called an equation; containing two, a formula is called a function.

x+3=7 is a calculation that needs to be reversed to identify the starting-number x. The subtraction 7-3 predicts this result. So if x + 3 = 7, then x = 7 – 3.

x*3=7 is a calculation that needs to be reversed to identify the starting-number x. The division 7/3 predicts this result. So if x * 3 = 7, then x = 7/3.

x^3=7 is a calculation that needs to be reversed to identify the starting-number x. The root 3√7 predicts this result. So if x^3 = 7, then x = 3√7.

3^x=7 is a calculation that needs to be reversed to identify the starting-number x. The log log3(7) predicts this result. So if 3^x=7 , then x = 7/3. log3(7).

3 + x = 7 x = 7 - 3	3 * x = 7 x = 7/3	x^3 = 7 x = 3√7	3^x = 7 x = log3(7)

Figure 2: Equations solved by moving numbers across with reversed calculation sign.

A function can be solved if one of the unknowns is known, so a function is described by tables or graphs setting up scenarios of the form 'if x is this, then y is that'

Thus the nature of equations is reversed calculations to be solved by reversing the calculation. And the nature of functions is table- or graph-scenarios illustrating different outcome possibilities.

Pastoral Equations and Functions

The tradition presents an equation as an open equivalence-statement about number-names that can be changed by performing identical operations on both sides of the equal sign, finally finding the shortest version of the number-name as the solution. Teacher education uses equations to legitimize all the concepts of abstract algebra to solve the equation: neutral and inverse elements, commutative and associative laws:

2 +3 * x = 14	3 * x = 12	Equivalent number-names
(2 +3 * x) + -2 = 14 + -2 = 12	(3 * x) * 1/3 = 12* 1/3 = 4	Using the inverse element
(3 * x + 2) + -2 = 12	(x * 3) * 1/3 = 4	Using the commutative law
3 * x + (2 + -2) = 12	x * (3 * 1/3) = 4	Using the associative law
3 * x + 0 = 12	x * 1 = 4	Using the neutral element
3 * x = 12	x = 4	Equivalent number-names

Figure 3: Equations solved by moving numbers across with reversed calculation sign.

Seeing a calculation as a number-name is a pastoral choice hiding its alternative, where a calculation is a process that takes place and that can be reversed. This prevents experiencing solving equations by walking forward and backward.

Forward: x *(*3)*→ 3*x *(+2)*→ 2+3*x = 14. Backward: x = 4 ← *(/3)* 12 ← *(-2)* 14

The tradition presents a function, first as a machine with a built-in formula changing an input to an output, and later as an example of a set-relation with certain properties.

Presenting an abstraction as an example of a higher abstraction is using an unknown–unknown relation violating the historical Euler-definition using an unknown–known relation by defining a function as a calculation with a variable quantity. And introducing the function concept is violating the historical fact that the rest of school mathematics including calculus was created without a function-concept.

The Nature of fractions and Calculus

Fractions first occur in bundling & stacking when instead of placing 3 singles besides the existing stack of 5-bundles, the 3 singles are bundled as a 5-bundle and put on top of the 5-stack giving a stack of T = 2*5 + (3/5)*5 = 2 3/5 *5 = 2 3/5 5s.

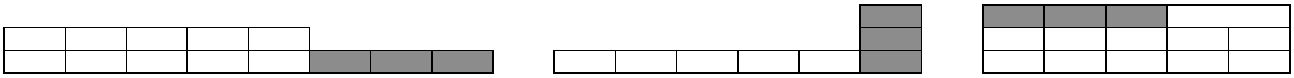


Figure 4: T = 2*5 + 1*3 = 2.3 *5 = 2 3/5 * 5.

With the introduction of units, fractions become ‘per-numbers’: If 4 kg costs 3 $, then the unit-price is 3$ per 4kg, i.e. 3$/4kg = 3/4 $/kg. In right-angled triangles fractions recounts the two small sides a and b in the diagonal c: sinA = a/c and cosA = b/c.

Fractions are not numbers but operators needing their units when added. Thus 1/2 of 2 bottles + 2/3 of 3 bottles adds up to 3/5 of 5 bottles, and not to 7/6 of 5 bottles. In this way adding fractions f is basically integration: T = 1/2 *2 + 2/3 *3 = Σ (f*Δx) , later to become $\int$ f dx

In primary school calculus takes place when integrating two stacks into one where the bundle-size is the sum of the stacks’ bundle-sizes, e.g. 2.3 4s + 3.1 5s = ? 9s. Reversing the integration of two stacks is differentiation as e.g. 2.3 4s + ? 5s = 3.1 9s.

? = (3.1 9s - 2.3 4s)/5 * 5 , later to be written as a quotient (T – T1)/5 = ΔT/5.

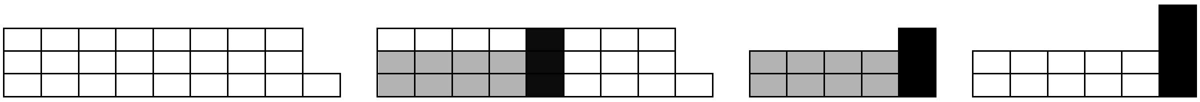


Figure 5: 3.1 8s = 2.3 4s + ? 5s = 2.3 4s + 2.4 5s.

In middle school calculus means integrating two fractions or per-numbers into one, e.g. 1/2 of 2 + 2/3 of 3 = ? of 5; or 10% of 2 + 40% of 3 = ? of 5; or 2kg at 6$/kg + 3kg at 9$/kg = 5kg at ? $/kg. Again reversed integration means differentiation.

In high school calculus means integrating many per-numbers into one, e.g. 7 sec. at 2 m/s increasing to 4 m/s totals 7 sec. at ? m/s in average: ΔT= 7*? = $\int$ (2+(4-2)/7*x) dx. Again differentiation means integration reversed.

Thus the nature of fractions is incomplete bundles and per-numbers. And the nature of calculus is adding stacks in united bundles, and adding fractions and per-numbers.

Pastoral Fractions

The tradition presents fractions as rational numbers that can be added if the denominators are the same, which legitimizes the introduction of factorizing and prime numbers, and later algebraic fractions. Later percentages and decimal-numbers are introduced as special fractions. And in calculus the gradient and the integral is introduced as examples of the limit-concept, so difficult to learners that often calculus is postponed to tertiary education.

Presenting fractions as numbers not needing units when added hides the fundamental role of fractions in the recount-equation and later as per-number in guide-equations. And mathematism comes from claiming that 1/2 + 2/3 IS 7/6 when it can also be 3/5. Claiming that fractions are the basis for percentages and decimal-numbers hides the parallelism of decimal-numbers and fractions when describing leftovers in bundling.

Presenting calculus as examples of the limit-concept hides its central role in primary school and in middle school providing a user-friendly approach to adding fractions and per-numbers.

Pastoral Words to be Deconstructed

Numerals could be called number-icons, not number-symbols to tell numbers representing multiplicity from letters installing special sounds. Multi-digit numbers could be called 'shortened numbers' reserving the wording 'natural numbers' to the natural numbers, i.e. numbers written as decimals including their units.

Fractions could be called per-numbers knowing that they are not numbers but operators needing a unit to make sense, and impossible to add without units.

Counting could be renamed to counting in tens to also allow for other bundle-sizes. And 356 could also be worded 3tenten5ten6 to show it comes from counting in tens.

With two different meanings, two different words for addition are needed. Thus integration-addition could be called 'add on top' to reflect that when adding 2.3 eights to 4.2 eights resulting in 6.5 eights, the 2 and the 3 are added on top of the stack. And juxtaposing-addition could be called 'add next to' reflecting that when adding 3 ones to 2 tens resulting in 2.3 tens, the stack of 3 ones is placed next to the stack of 2 tens.

With two different meanings, two different words for division are needed. Thus bundling-division /2, or repeated subtraction, could be called 'divided in 2s' to reflect that counting by bundling & stacking means taking away 2s several times. And sharing-division /2 could be called 'divided by 2' to reflect that a total is shared by 2.

With two different meanings, two different words for multiplication are needed. Thus stack-multiplication *5 could be called '5-bundles' to reflect that counting by bundling & stacking in 5s means producing a stack of e.g. 4*5 = 4 5s. And recount-multiplication *8 could be called '8s recounted in tens' to reflect that recounting from eights to tens, 3*8 = 24 predicts the result to be 2 tens and 4 unbundled singles.

Equations could be renamed to reversed calculations or formulas with one unknown to reflect that any calculation can be reversed to reproduce the input from the output. Functions could be renamed formulas with two unknowns to reflect that in this case a formula can only predict a collection of possibilities. Calculus could be called adding per-numbers with units to reflect that units should always be included when adding.

With three different meanings, three different words for mathematics are needed. With concepts as abstractions from examples it becomes 'grounded mathematics' to indicate its nature as a natural science. With concepts as examples of abstractions it becomes 'pastoral metamatics' to indicate its familiarity with other examples of patronizing truths. And if adding numbers without units it becomes 'mathematism' to indicate its familiarity with other beliefs true in theory but not in practice.

With two different meanings, two different words for education are needed. Thus presenting concepts as abstractions from examples could be called 'grounded enlightenment from below' to indicate its roots in the Enlightenment. And presenting concepts as examples of abstractions could be called 'pastoral counter-enlightenment from above' to indicate its roots in the German Humboldt Bildung.

Conclusion

To deconstruct means to change words from hiding to labeling differences. In mathematics education its whole basis needs to be deconstructed. Words as natural numbers, addition, multiplication, division, fractions, equations, functions and calculus are not labeling nature, but are pastoral choices hiding alternatives. A deconstruction exposes a hidden choice: Should mathematics education present mathematics as a pastoral subject wanting to patronize the learner; or as a natural science exploring its roots, multiplicity, wanting to enlighten the learner? Of course counter-enlightenment Bildung schools will choose pastoral mathematics. But maybe enlightenment schools should consider replacing pastoral mathematics with its natural alternative, enlightenment mathematics based upon decimal-counting and per-numbers, subtraction, reversed calculations; and always including units when adding?

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211. Deconstructing the Mathematics Curriculum: Telling Choice from Nature

Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, seen from a skeptical sophist perspective wanting to tell nature from choice, three questions are raised: Are concepts grounded in nature or forcing choices upon nature? How can an ungrounded mathematics curriculum be deconstructed into a grounded curriculum? Does mathematics education mean pastoral patronization of humans, or anti-pastoral enlightenment of nature?

The Background

Enlightenment mathematics treated mathematics as a natural science. Exploring the natural fact many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline, 1972, p. 398). Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics into a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needed no outside world and becomes entirely self-referring. However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: definition $M = \{A \mid A \notin A\}$, statement $M \in M \Leftrightarrow M \notin M$.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment and justification problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371). To design an alternative, mathematics should return to its roots, multiplicity, guided by a kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

Ancient Greece saw a struggle between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers, seeing everything physical as examples of meta-physical forms only visible to the philosophers educated at Plato's academy, patronization was a natural order when left to the philosophers (Russell, 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spices. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people should do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. The US still has its first republic; France now has its fifth. The German autocracy tried to stop the French democracy by sending in an army. However, the German army of mercenaries was no match to the French army of conscripts only too aware of the feudal consequences of loosing. So the French stopped the Germans, and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. Humboldt was asked to create an elite school, and using Bildung as counter-enlightenment he created the self-referring Humboldt University (Denzin et al, 2000, p. 85).

Inside the EU the sophist warning is kept alive only in the French postmodern or post-structural thinking of Derrida, Lyotard and Foucault warning against patronizing categories, discourses and institutions presenting their choices as nature (Tarp, 2004). Derrida recommends that patronizing categories be 'deconstructed'. Lyotard recommends the use of postmodern 'paralogy' research to invent alternatives to patronizing discourses. And Foucault uses the term 'pastoral power' to warn against institutions legitimizing their patronization with reference to categories and discourses presenting their choices as nature.

Anti-pastoral sophist research doesn't refer to but deconstruct existing research by asking 'In this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?' To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the American enlightenment democracy. As to the grounding of mathematics in the Enlightenment century, Morris Kline writes (Kline, 1972, p. 399)

The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. The seventeen-century men had broken both of these bonds. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights.

A Search Agenda

Confronted with the traditional mathematics curriculum wanting to provide the learners with well-proven knowledge about well-defined concepts applicable to the outside world (NCTM, 2000), anti-pastoral sophist research wanting to uncover hidden alternatives to pastoral choices presented as nature could raise three questions:

- Are concepts grounded in nature or forcing choices upon nature?

- How can an ungrounded mathematics curriculum be deconstructed into a grounded curriculum?

- Does mathematics education mean pastoral patronization or anti-pastoral enlightenment?

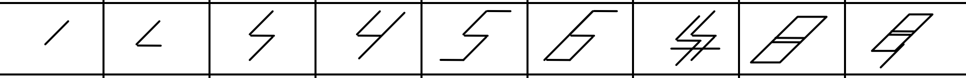
Thus the first question to ask is: are the roots of mathematics in nature or in human choices?

The Natural Roots of Primary School Mathematics

In primary school, the natural root of mathematics is double-counting the physical fact many.

Iconizing Many

Numbers coming from counting differentiates between degrees of many. 1.order counting means counting in 1s rearranging sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for the numbers until ten that becomes a very special and strange number having its own name, but not its own icon.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

2.order counting is counting by bundling&stacking using icon-bundles. First we bundle the sticks in e.g. 3-bundles, in 3s. Then we stack the total in two stacks: a stack of 3s, and a stack of unbundled singles. The stacks may then be placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

Thus a total of 7 1s is bundled&stacked as 2 3s + 1 1s, or as 2)1), or as 2.1 3s.

I I I I I I I -> III III I ->	III III I	-> ~~III~~ ~~III~~) I) -> █ █) I) -> II) I)
Or with icons:		-> 2 3s + 1 1s -> 2x3 + 1x1 -> 2)1) -> 2.1 3s

Later also bundles are bundled, calling for a new cup to the left. Thus 4 5s can be rebundled in 6 3s and 2 1s, i.e. as 6)2), where the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II).

And 4 8s can be rebundled in 1 3-bundle of 3-bundles of 3-bundles, and 1 3-bundle, and 2 1s, i.e. as 1))1)2) or 1)0)1)2), i.e. as the decimal number 101.2 3s:

IIIIIII IIIIIII IIIIIII IIIIIII -> III III III III III III III III III III II -> ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ ~~III~~ II

Iconizing Counting

Operations iconize the processes involved in counting by bundling&stacking. Taking away 4 is iconized as –4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 or 3*4 showing a 3 times lifting of the 4s. Juxtaposing a stack of 2 singles next to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. a 1 3^2-bundle.

Numbers and Operations Form Formulas for Prediction

Now numbers and operations can be combined to calculations predicting the counting results by two formulas. The 'recount-formula' T = (T/b)*b tells that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6/7) 7s. Using a calculator we get the result 2 7s and some leftovers. These can be found by the 'rest-formula' R = T – n*b telling that the rest is what is left when the full stack is taken away: R = 3*6 – 2*7. Using a calculator we get the result 4. So the combined prediction says T = 3*6 = 2*7 + 4*1. This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

Since formulas can be used to predict the result of counting, the scientific method using mathematical formulas for predictions to be tested can be introduced already in primary school.

Predicting and Practising Recounting

The recount-formula and the rest-formula now enables predicting and practising double-counting over and over, turning out to being the leitmotif of mathematics. First, only iconized numbers are used as bundle-size, recounting 3 5s in e.g. 7s, but not in tens that is postponed as long as possible. In this ten-free zone it becomes possible to introduce the core of mathematics using 1digit numbers only (Zybartas et al, 2005). The CATS-approach, Count&Add in Time&Space, is one example of a grounded approach to mathematics as a natural science investigating the natural fact many when counting by bundling&stacking, and when using double-counting at all school levels (Tarp, 2008).

Including ten as bundle-size means going on from 2.order counting, using bundles with both a name and an icon, to 3.order counting, using the bundle-size ten having a name, but not an icon.

Preparing for 10

Before introducing ten as 10, i.e. as the standard bundle-size, 5 is chosen as the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is never used when using place values, or in the counting sequence: 1, 2, 3, 4, bundle, 1B1, 1B2, 1B3, 1B4, 2B, 2B1, etc. So once chosen, ten needs no icon.

Likewise, in the beginning, counting in tens should use neither a ten-icon nor the ten-name, but count 8, 9, bundle, 1bundle1, 1B2, 1B3, …, 1B9, 2B, 2B1, etc. Then the name bundle can be replaced by the name ten counting 8, 9, ten, 1ten1, 1T2, …, 1T9, 2T, 2T1, etc. Finally the sloppy way eleven and twelve can be used meaning ‘1 left’ and ‘2 left’ in ‘Anglish’, i.e. in old English.

Finally Introducing 10

Introducing the number ten as the standard bundle-size changes almost everything.

Numbers are no more written as natural numbers, i.e. as decimals carrying units. Instead numbers are written using the sloppy place-value method hiding both the decimal and the unit.

Since ten has no icon, double-counting now becomes impossible to predict by formulas since asking 8 3s = ? tens leads to T = (8*3/ten)*ten that cannot be calculated. Now the answer is given by multiplication, 8*3 = 24 = 2 tens + 4 ones, thus transforming multiplication into division.

Almost all operations change meanings. *3 now means recounted in tens. /4 now means divided in 4, not divided into 4s. +3 now means adding 3 on top, not next to. Only -3 still means take away 3.

With 2.order counting in e.g. 5s the order of operations is: first /, then *, then –, and finally +. With 3.order counting in tens this order is turned around: first +, then –, then *, and finally /.

The Natural Roots of Middle School Mathematics

Also in middle school, the natural root of mathematics is double-counting the physical fact many, now occurring different places in the outside world, especially in time and space and in economy, thus reintroducing units, now as e.g. seconds, minutes, cm, m, m^2, m^3, liters, c, $, £ etc.

Double-Counting in Middle School

In middle school, double-counting means counting a given object in different units. Fractions emerge when counting 3 1s in 5s as 3 = (3/5)*5; and as ‘per-numbers’ when a quantity is counted e.g. both as 2$ and as 5kg thus containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg. Again the recount-formula predicts recounting results when asking ‘6$ = ?kg’, or ‘14kg = ?$:

T = 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg; and T = 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$, or

kg = kg/$ * $ = 5/2 * 6 = 15 , and $ = $/kg * kg = 2/5 * 14 = 5.6.

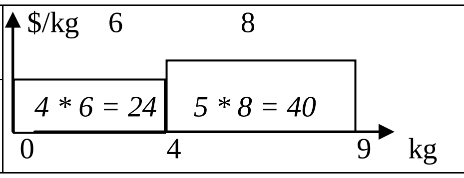
Percentages are per-numbers using the same unit. Thus the females might contribute with 3 females per 8 persons, or 3 females/8 persons, or 3/8 f/p, or 3/8 if we leave out the unit. Thus per hundred, the females will contribute with T = 100p = (100/8) * 8p = (100/8) * 3f = 37.5f, i.e. with 37.5 females per 100 persons, or 37.5 females /100persons, or 37.5/100 f/p, or 37.5/100, or 37.5%.

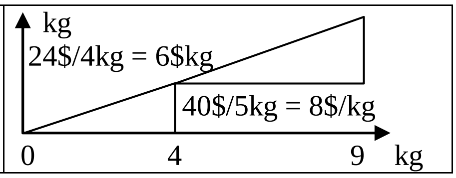
Thus percentages always carry a hidden unit. So when finding the number of females in a population of 75 having 20% females, the hidden unit is reintroduced: With 20% females we have 20 females per 100 persons, so the 75 persons must be recounted in 100s:

75 persons = (75/100) * 100 persons = (75/100) * 20 females = 15 females

Primary school integration adding stacks by adding bundles, e.g. 2 3s + 4 5s = ? 8s, now reoccurs as adding per-numbers when mixing two double-counted quantities, asking e.g. 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg. This question can be answered by using a table or a graph.

4kg at 6$/kg = 4 * 6 = 24 5kg at 8$/kg = 5 * 8 = 40
9kg at ?$/kg = 9 * ? = 64 ? = 64/9





Using a graph we see that integration leads to finding the area under a per-number graph; and opposite that the per-number is found as the gradient on the total-graph.

The Root of Equations: Reversed Calculations

In Greek, mathematics means knowledge, i.e. what can be used to predict with. So mathematics is our language for number-prediction: The calculation '2+3 = 5' predicts that counting on 3 times from 2 will give 5. '2*3 = 6' predicts that repeating adding 2 3 times will give 6. '2^3 = 8' predicts that repeating multiplying with 2 3 times will give 8. Also, any calculation can be turned around and become a reversed calculation predicted by the reversed operation:

- The answer to the reversed calculation 3 + x = 7 is predicted by the reversed operation x = 7 –3.
- The answer to the reversed calculation 3 * x = 7 is predicted by the reversed operation x = 7/3.
- The answer to the reversed calculation x ^ 3 = 7 is predicted by the reversed operation x = $\sqrt[3]{7}$.
- The answer to the reversed calculation 3 ^ x = 7 is predicted by the reversed operation x = $\log_3(7)$.

Thus the natural way to solve an equation is to move a number across the equation sign from the left forward-calculation to the right backward-calculation side, reversing its calculation sign:

3 + x = 7 x = 7 – 3	3 * x = 7 x = 7/3	x^3 = 7 x = $\sqrt[3]{7}$	3^x = 7 x = $\log_3(7)$

A calculation with more than one operation contains an invisible bracket placed around the calculation having priority, and disappearing again with the number outside the invisible bracket:

	2*x + 3 = 11	(2*x) + 3 = 11	2*x = 11-3	x = (11-3)/2
Or with letters:	a*x + b = c	(a*x) + b = c	a*x = c – b	x = (c – b)/a

The Root of Geometry: Triangulation

In Greek, geometry means earth-measuring. Earth is measured by being divided into triangles; which again can be divided into right-angled triangles, each seen as a rectangle halved by a diagonal. Recounting the height and base produces per-numbers: sinA = height/diagonal, tanA = height/base, cosA = base/diagonal. Additional formulas are A+B+C = 180, and a^2+b^2 = c^2.

Also a circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: C = 2 * r * (n * sin(180/n)) = 2 * r * π for n sufficiently big.

The Roots of Algebra: Reuniting

In Arabic, algebra means reuniting, i.e. splitting a total in parts and (re)uniting parts into a total. The operations + and * unite variable and constant unit-numbers; ∫ and ^ unite variable and constant per-numbers. The inverse operations – and / split a total into variable and constant unit-numbers; d/dx and √ and log split a total into variable and constant per-numbers:

Totals unite/split into	Variable	Constant
Unit-numbers $, m, s, ...	T = a + n $T - n = a$	T = a * n $\frac{T}{b} = a$
Per-numbers $/m, m/s, m/100m = %, ...	ΔT = ∫ f dx $\frac{dT}{dx} = f$	T = a ^ n $\sqrt[n]{T} = a$ $log_a T = n$

The Natural Roots of High School Mathematics

In high school, the natural root of mathematics is double-counting change: Being related by a formula y = f(x), how will a change in x, Δx, affect the change of y, Δy? Here the recount-equation gives the change-formula Δy = (Δy/Δx)*Δx, or dy = (dy/dx)*dx = y'dx for small micro-changes.

Constant Change

In trade, when the volume increases from 0 to x kg, the initial cost b increases to the final cost y = b + a*x if the cost increases a$/kg. This is called linear change or ++ change. Here Δy/Δx = a.

In a bank, when the years increase from 0 to x, the initial capital b increases to the final capital y = b*(1+r)^x if the capital increases with r %/year. This is called exponential change or +* change since we add 7% by multiplying with 107% = 1 + 7%. Here Δy/Δx = r*y.

In geometry, when the side-length is 3-doubled from 2 to 6, the area of a square y = x^2 is 3-doubled twice from 4 to 36. This is called potential change or ** change, in general given as y = b*x^a. Here Δy/Δx = a * y/x.

In a linear change formula y = b + a*x, the per-number a might itself change in a linear way as a = c + d*x. In this case we have y = b + a*x = b + (c + d*x)*x = b + c*x + d*x^2, a polynomial of degree 2. Here Δy'/Δx = d, where y' = Δy/Δx. If also d changes in a linear way as d = m + n*x then y becomes a polynomial of degree 3, y = b + c*x + d*x^2 = b + c*x + (m + n*x)*x^2 = b + c*x + m*x^2 + n*x^3. Here Δy''/Δx = n, with y'' = Δy'/Δx and y' = Δy/Δx.

Thus in y = a + b*x + c*x^2 + d*x^3, 'a' describes the initial height; 'b' the initial change or steepness; 'c' the acceleration or the curvature; and 'd' the counter-curvature. Thus the y-graph becomes a line bending first to one side, then to the other, i.e. a double-parabola.

Predictable Change

The change in x, Δx, might be a micro-change, dx. On a calculator we observe that, approximately, 1.001^5 = 1.005, 1.001^9 = 1.009, √1.001 = 1.0005 and 1.001^-3 = 0.997. From this a hypothesis can be made saying that, approximately, (1+dx)^n = 1 + ndx, allowing numerous predictions to be made as 1.0002^4 = 1.0008 etc., all being verified by the calculator. If y = x^n, then (x + dx)^n = (x(1+dx/x))^n = x^n(1+ndx/x) = x^n + ndx*x^n/n = y + ndx*x^(n-1) = y+dy, so dy/dx = n*x^(n-1).

If the changes in x, Δx, are micro-changes, dx, the area under a graph y = f(x) can be found by summing up the micro-strips under the graph, each having the area height*width = f*dx, written as

$\int_a^b y\,dx$. If y dx can be written as a micro-change df, then $\int_a^b y\,dx = \int_a^b df = f(b) – f(a)$ since summing up single changes always gives a total change = terminal number – initial number, no matter the size of the single changes. So with $x^3 = (\frac{x^4}{4})'$, $\int_2^5 x^3\,dx = \int_2^5 (\frac{x^4}{4})'dx = \frac{5^4}{4} - \frac{2^4}{4} = 152.25$.

Change Equations

Solving any change-equation dy/dx = f(x,y) is easy when using technology. In such equations, the change-equation calculates the change dy that added to the initial value gives the terminal y-value, becoming the initial value in the next period. Thus the change equation is dy = r*y, r = ro(1-y/M) if a population y grows with a rate r decreasing in a linear way with the population y having M as its maximum. This is easily solved using a spreadsheet to keep on calculating the formula y+dy -> y.

Unpredictable Change

From throwing a dice we know that some numbers change in a way that cannot be ‘pre-dicted’. However, such numbers x can be ‘post-dicted’ by a table describing their previous behaviour with relative frequencies f. From such a table we can calculate the average level m = Σ(x*f) and the average deviation d from the average level, d^2 = Σ((x – m)^2*f). This allows for predicting new numbers, not by a value, but by an interval, m ± 2d, containing the numbers with a 95% probability. Likewise, ordering the observed numbers and splitting them in four parts will produce a box-plot.

Relating Equations and Curves

Placed in a XY coordinate-system, the points of a curve are assigned coordinates (x,y). This allows a curve to be translated into an equation and vice versa. So now curve-problems can be translated into solving equations, and vice versa. Intersection points, turning points, tangents, parallelism, areas etc. now can be translated into solving simultaneous equations, differentiating and integrating formulas; and integrals is found as areas etc. The nature of formulas as means for prediction can be illustrated by using the geometry solution as a prediction of the algebra solution, and vice versa.

Using regression, a graphical calculator can translate any table into a formula and a curve.

Modelling With Regression-Mathematics

Modelling consists of four parts: a real-world problem, a model problem, a model solution, and a real-world solution. First a real-world problem leads to a model problem, often a table relating two variables x and y. Then in the model solution, regression is used to find a formula connecting the variables. Containing one unknown, the formula becomes an equation that can be solved manually or by the Math Solver. Containing two unknowns, the formula becomes a function, that can be illustrated by a graph; and where the two typical questions ‘given x find y’ and ‘given y find x’ reduces the function to an equation that can be solved manually or by the Math Solver, or by the Trace and the Calc Intersection. Using both provides the opportunity to use the first as a prediction of the second. Finally the solution can be evaluated as to its applicability as a real-world solution.

The Three Genres of Modelling

A formula can predict. However, in the formula T = c*p we need to know what quantities are described to determine the truth-value of the formula’s prediction. It turns out that both word-statements and number-statements share the same genres: fact, fiction and fiddle (Tarp, 2001).

Fact Models

Fact models quantify and predict predictable quantities: ‘What is the area of the walls in this room?’ In fact models the predicted answer is what is observed. Hence calculated values from a fact models can be trusted. The basic formulas T = 2*3 etc. are fact models, as well as many models from basic science and economy. A fact model may also be called a 'since-hence' model or a 'room-model'.

Fiction Models

Fiction models quantify and predict unpredictable quantities: 'My debt will soon be paid off at this rate!' Fiction models produce predictions based upon presumed assumptions that should be supplemented with alternative assumptions, i.e. with parallel scenarios. Typical examples of fiction models are average-models, simplifying complex economical or technical models by assuming some variables to stay as constants on their average level. Other examples are linear demand and supply curves in economical theory. A fiction model may also called an 'if-then' or a 'rate-model'.

Fiddle Models

Fiddle models quantify qualities: 'Are the risk and casualty numbers of this road high enough to cost a bridge?' This question will install crosswalks instead of bridges on motorways since it is cheaper to be in a cemetery than at a hospital. Fiddle models should be rejected asking for a word description instead. Many risk-models are fiddle models. The basic risk model says: Risk = Consequence * Probability. A fiddle model may also be called a 'so-what' model or a 'risk-model'.

The Grand Narratives of the Quantitative Literature

Literature is narratives about real-world persons, actions and phenomena. Quantitative literature also has its grand narratives. That overwhelmingly many numbers can be added by one simple difference, providing the numbers can be written as change-numbers, is a grand narrative.

In physics, grand narratives can be found among those telling about the effect of forces, e.g. gravity, producing parabola orbits on earth, and circular and ellipse orbits in space. Jumping from a swing is a simple example of a complicated model. These grand narratives of physics enabled the rise of the Enlightenment period and the modern democratic society replacing religion with science.

In economics, an example of a grand narrative is Malthus' 'principle of population' comparing the linear growth of food production with the exponential growth of the population; and the Keynes model relating demand and employment creating the modern welfare society. As are the macroeconomic models predicting the effects of different taxation and reallocation policies.

Also limit-to-growth models constitute grand narratives predicting the global economical and ecological future depending on different production, consumption, and pollution options.

Pastoral vs. Grounded Mathematics in Primary School

In primary school, 6 is introduced as a symbol for six being the follower of five, having the symbol 5. 10 is introduced as a symbol for ten, the follower of nine having the symbol 9. And the counting-numbers 1, 2, 3 etc. are presented as the natural numbers. The hidden grounded alternative to these pastoral choices introduces 5 and 6 as what they really are: icons rearranging the sticks they represent; and 10 as what it really is: a sloppy way of writing 1.0 bundle, thus being the follower to 4 in the case of 5-bundling making the follower of nine 20; and the natural numbers as what they really are: decimal-numbers with units, e.g. 0.4, 1.0, 1.1, …, 1.4, 2.0 when counting in 5-bundles.

The tradition introduces division as the last of the four operations, where /4 means to split in 4. The hidden grounded alternative introduces division as what it really is: an icon for taking away splitting in 4s, where 3*7/4 predicts how many times 4s can be taken away from 3 7s.

The tradition introduces multiplication as the third of the four basic operations, where the multiplication tables, 3*8 = 24 etc., forces all stacks to be recounted in tens. The hidden grounded alternative introduces multiplication as what it really is: 3*8 means a stack of 3 8s, not needing to be recounted into tens before after ten has been chosen as the standard bundle-size.

The tradition introduces addition as the first of the four operations, where e.g. 7 + 4 = 11 forces the immediate introduction of ten as the bundle-size; and forces the sloppy way of writing 2digit numbers without decimals and units. The hidden grounded alternative introduces addition as what it really is: 2 5s + 4 1s means juxtaposing a stack of 4 1s next to a stack of 2 5s.

The tradition introduces 'mathematism', true in the library but not in the laboratory, by teaching that '2 + 3 IS 5' in spite of the fact that 2weeks + 3days = 17days, 2m + 3c = 203cm etc. The hidden grounded alternative always includes the units when adding, e.g. 2 4s + 3 5s = 4.3 5s.

Pastoral vs. Grounded Mathematics in Middle School

Middle school introduces fractions as 'rational' numbers having decimals and percentages as examples; and allowed to be added without units. The hidden grounded alternative to this pastoral choice introduces decimals and fractions as what they really are: decimals occur as the natural numbers when counting in bundles; and fractions are per-numbers occurring when double-counting a quantity in two different units, as 1s and as 5s: 3*1 = (3/5)*5; and as $ and kg: 2$/5kg = 2/5 $/kg.

Factorization is introduced to create a common denominator when adding fractions, and for solving quadratic equations. The hidden grounded alternative includes units when adding fractions, and postpones quadratics to high school. Factorization unfolds folding numbers to prime numbers.

Equations are introduced as equating 2 number-names to be changed by identical operations aiming at neutralizing the numbers next to the unknown. The neutralizing method seems to have as a hidden agenda to legitimize the concepts of modern abstract algebra in teacher education solving a simple equation as 3 + x = 8 by using, not the definition of reversed operations, but both the commutative and associative laws, and the concepts of inverse and neutral elements of number sets:

3 + x = 8 , (3 + x) + (-3) = 8 + (-3) , (x + 3) + (-3) = 8 – 3 , x + (3 + (-3)) = 5 , x + 0 = 5 , x = 5.

The hidden grounded alternative to this pastoral choice introduces equations as what they really are: calculations being reversed since we know the result, but not the starting number.

Multiplying two digit brackets is introduced complaining about it is not possible to explain why (–1) * (–1) = +1. In a grounded approach this is no problem since the multiplication 9*9 = 81 implies that (10-1)(10-1) = 100 – 10 – 10 + (–1) * (–1), clearly showing that (–1) * (–1) = +1.

Geometry is introduced as forms and facts deduced from self-evident axioms. The hidden grounded alternative to this pastoral choice introduces geometry as what it really is: 'earth-measuring' realising that all forms can be split into right-angled triangles, where the relationship between the angle and the side can be expressed by the percentage numbers sinA, cosA and tanA.

Pastoral vs. Grounded Mathematics in High School

High school introduces formulas as set-relations: polynomial, exponential, and circular functions. The hidden grounded alternative introduces these formulas as what they really are: solutions to change-equations rooted in describing physical motion and population and economical growth.

Calculus is introduced as an example of a limit process, thus introducing limits and continuity before the derivative. The hidden grounded alternative generalizes primary school's adding stacks in combined bundle-sizes, and middle school's adding fractions with units into adding per-numbers with units; and introduces the terms continuous and differentiable as what they really are: foreign words for locally constant and locally linear in contrast to piecewise constant and piecewise linear.

Conclusion

Three questions have been answered using anti-pastoral sophist-research. In primary, middle and high school the core mathematical concepts are not grounded in nature. Replacing ungrounded concepts with grounded concepts constitutes a deconstruction of a self-referring mathematics curriculum into a grounded curriculum. This allows mathematics education to change from being pastoral patronization to democratic enlightenment. Now only the political decision remains. Is mathematics education meant to demonstrate how real-world phenomena are examples of metaphysical forms only visible to mathematicians training teachers to mediate this insight to the ordinary people? Or is mathematics education meant to demonstrate how mathematical concepts are rooted in and able to predict the behavior of real-world phenomena? In short, should mathematics education patronize the ignorant to become a docile lackey – or should mathematics education enlighten nature so that people can tell nature from choice and practice democracy?

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212. Mathematics Education: Pastoral Bildung - Or Anti-Pastoral Enlightenment

Applying a postmodern philosophical perspective to mathematics education reveals different kinds of mathematics and different kinds of education and different kinds of philosophy. Based upon the ancient Greek controversy between the sophists and philosophers as to the nature of knowledge, two different forms of schooling has developed, an enlightenment school abstracting categories from physical examples; and a pastoral school exemplifying metaphysical categories; as well as two different kinds of mathematics, enlightenment mathematics seeing the world as the roots of mathematics, and pastoral mathematics seeing the world as applying mathematics.

Pre-modern and Modern Mathematics

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact multiplicity, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false: 'Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$'.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel showed that theories can't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as e.g. the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998).

Pastoral and Anti-pastoral Philosophy

Ancient Greece saw a fierce controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people needed to be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers argued that patronization is the natural order since everything physical is an example of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become the natural patronising rulers.

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods; later the trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing slow Spanish silver ships returning over the Atlantic was no problem to the English; finding a route to India on open sea was. Until Newton found that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it obeys its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronization with democracy. Two were installed, one in US, and one in France. US still has its first republic, France now has its fifth. The German autocracy tried to stop the French democracy by sending in an army. However, the German army of mercenaries was no match to the French army of conscripts only to aware of the feudal alternative to stopping the German army. So the French army stopped the German army and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. Humboldt was asked to create an elite school. Using Bildung as counter-enlightenment, he created a school-system leading to the Humboldt University, which uses Luhmann System Theory to defend its chosen self-reference as nature (Luhmann 1995).

Inside the EU the sophist warning is kept alive only in France in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronising categories, discourses and institutions presenting their choices as nature (Tarp 2004). Derrida recommends that pastoral categories be ‘deconstructed’. Lyotard recommends the use of postmodern ‘paralogy’ research to invent alternatives to pastoral discourses. And Foucault uses the term ‘pastoral power’ to warn against institutions legitimising their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

In descriptions, numbers and words are different as shown by the ‘number & word dilemma’: Placed between a ruler and a dictionary, a’17 cm long stick’ can point to ‘15’, but not to ‘pencil’, thus being able itself to falsify its number but not its word, which makes numbers nature and words choices, becoming pastoral if suppressing their alternatives; meaning that a thing behind a word only shows part of its nature through a word, needing deconstruction to show other parts.

Thus anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what is pastoral choice presented as nature?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of e.g. eight sticks can be treated in different ways. The sticks can be rearranged to an eight-icon 8 containing the eight sticks, written as 8. The sticks can be collected to one eight-bundle, written as 1 8s. The sticks can be ‘decimal-counted’ in 5s by bundling & stacking, bundling the sticks in 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled singles in a right single-cup. When writing down the counting-result, cup-writing gradually leads to decimal-writing where the decimal separates the bundle-number from the single-number: 8 = 1 5s + 3 1s = 1)3) = 1.3 5s = 13 if leaving out the decimal and the unit.

So the nature of numbers is that any total can be decimal-counted by bundling & stacking and written as a decimal number including its unit, the chosen bundle-size. Choosing ten means that no icon for ten is needed since the bundle is 1.0 bundle. Choosing eight instead, ten becomes 12, and 10 becomes eight.

The Nature of Operations

Operations are icons describing the process of counting by bundling & stacking. The division-icon ‘/2’ means ‘take away 2s’ when counting in 2s, 8/2 = 4. The multiplication-icon ‘4*’ means ‘stacked 4 times’ when stacking 2-bundles, T = 4*2. Subtraction ‘- 2’ means ‘take away 2’ when taking away the bundles to see what rests as unbundled singles, R = 9 – 4*2. And addition ‘+2’ means ‘plus 2’ when adding 2 singles to the stack of bundles as a new stack of 1s making the original stack a stock of e.g. T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus the full process of ‘re-counting’ or ‘re-bundling’ 8 1s in 5s can be described by a ‘recount or rebundle formula’ T = (T/b)*b saying the total T is first counted in bs, then stacked in bs, together with a ‘rest formula’ finding the rest:

T = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is R = 8 – 1*5 = 3.

The Nature of Formulas

Using these formulas, the counting result can be predicted on a calculator thus becoming a number-predictor. This shows the strength of mathematics as a language for number-prediction able to predict mentally a number that later is verified physically in the ‘laboratory’. Historically, this enabled mathematics to replace pastoral belief with prediction, and to become the language of the natural sciences.

The Nature of Equations

The statement 2*4 + 1 = 9 describes a bundling where 2 4-bundle and 1 single is re-bundled to 9 1s. The equation x*4 + 1 = 9 describes the reversed bundling asking how many 4-bundles that together with 1 single can be re-bundled to 9 1s. Obviously, we must take the 1 single away from the 9 1s and count the rest in 4s. So technically, moving numbers to the other side changing their calculation sign solves an equation: If x*4 + 1 = 9, then x*4 = 9-1 = 8, and x = 8/2 = 2.

Enlightenment Mathematics and Pastoral Mathematics

In primary school, an enlightenment curriculum will focus on the nature of numbers, operations and calculations to be learned through counting by bundling & stacking reported by cup-writing and decimals, 8 = 1 5s + 3 1s = 1)3) = 1.3 5s, in accordance with the Piaget ‘from hand to head’ principle of natural learning; and postponing the introduction of ten and addition until after several examples of the fact that for any bundle-size, its icon will not be used since a full bundle will always be counted as 1.0 bundles or plain 10 if leaving out both the decimal and the bundle-size.

The pastoral curriculum introduces the ‘natural’ numbers one by one using the follower-principle. This leads to introducing 10 as the follower of 9 and in this way quickly introducing 2digit numbers and place values. Later comes multi-digit numbers. Likewise addition is introduced first to practise earlier numbers adding up to the actual number. Then subtraction is introduces as taking away and counting up to. Multiplication and the tables follow; and in the end division and simple fractions.

In middle school, an enlightenment curriculum will focus on the nature of per-numbers and triangles. Per-numbers occur when double-counting a quantity in two different units leads to fractions and percentages, 2$ per 3m = 2$/3m = 2/3 $/m. Recounting now is called proportionality. When adding fractions and percentages the units are includes as in integration. Formulas with two unknowns are graphed. Formulas with one unknown are equations solved using reversed calculations first reducing a multiple calculation to a single by placing the hidden parentheses, and then moving numbers to the other side reversing their calculation signs. Geometry is earth-splitting using triangles inside or outside coordinate systems, seeing a right-angled triangle as a rectangle halved by a diagonal, where the height and length can be recounted in diagonals making sine, cosine and tangent percentages.

The pastoral curriculum enlarges the number domain with fractions, and with decimals and percentages both defined as examples of fractions. Again the order of operations is maintained starting with addition of fractions including factorisation of ‘natural’ numbers in prime numbers. Equations are introduced and solved by the neutralising method. Proportionality is introduced as an example of equations and of a function that is graphed. Algebraic expressions are introduced in equations to be factorised and simplified, and to be added as algebraic letter fractions. In geometry the focus is on 2- and 3-dimensional forms and translation groups.

In high school, an enlightenment curriculum will focus on adding per-numbers, where adding constant per-numbers leads to power; and adding variable per-numbers leads to integration where primary school's adding stacks in combined bundle-sizes, and middle school's adding fractions with units are generalising to finding the total as the area under a per-number graph. Reversed addition then leads to roots and logarithms, and to differentiation finding the per-number as the gradient on a total-graph: 2s at 3m/s + 5s at ?m/s = 7m at 4m/s, ? = (7*4-2*3)/5 = ΔT/Δx.

The pastoral curriculum enlarges the number domain and operations with irrational and real numbers, and with power, root and log. The function concept is claimed to be the foundation of high school mathematics; and is defined as an example of a relation between two sets. Linear and exponential change is presented as examples of functions. The quadratic function is given an extended treatment. Its graph is studied using translations, and its formula is thoroughly factorised. Calculus is introduced as an example of the concept limit used to exemplify the concept continuity and differentiability and to define the gradient by the first principle and the integral as a Riemann sum. Geometry introduces coordinate geometry and vector geometry presenting a vector as an equivalence set of parallel arrows with the same length.

Enlightenment Schools and Pastoral Schools

After only a week, an ethnographer will see a fundamental difference between universities in North America and in the EU. At the first place the students are aged 19, at the second 23. At the first place the students have chosen their own combination of modules accessible for all; at the second place they are forced to follow one of several pre-designed educations only accessible to those with the highest marks. At the first place the students already took some university modules at the last year in high school; at the second place this is not possible. At the first place high school is attended by all and most go on to university where around the half gets a bachelor degree; at the second place only the best half of a year group is allowed to enter high school, and only best half is allowed to go on to university where only the half graduates after having been forced to include a university directed master degree in their exam. At the first place some students are supplementing their bachelor degree with new modules in order to change career e.g. from teaching to engineering; at the second place they have to start all over. At the first place parents have different careers in their lifetime; at the second place parents are bound to the office they are educated for. At the first place some students are studying education; at the second place education has to be studied outside the university. At the first the bachelor-degree takes four years and can be combined from different universities; at the second place the bachelor-degree takes three years and must be finished at one university, so the compulsory master-degree can't be taken outside the EU.

Looking in the literature for explanations for this difference soon leads to Humboldt:

> Our universities have a monastic origin, and they have specialized in being centers of higher learning, functions originally given by the Church to monasteries. (..) The form of the university most familiar to us today is mainly a Prussian invention whose architect and champion was Wilhelm von Humboldt (..) The collegial system and its related peer review structures centered on an effort to gain intellectual freedom from the constraints of theological doctrine and political manipulation. Although addressing this problem was obviously important, the solution adopted has subsequently done much to weaken the social articulation of the university to all groups other than powerful elites. (..) Not surprisingly, society at large occasionally thinks it should be getting a more useful return for its investment and the freedom it gives to the professoriate. This situation is predictable because the autopoetic research process provides important supports for intellectual freedom but simultaneously opens the door to useless research and academic careerism divorced from attention to important public social issues (Greenwood et al in Denzin et al. 2000: 85-89)

Facing a Choice: Democratic Enlightenment or Pastoral Patronisation

The ancient Greek controversy on the nature of knowledge between the sophists warning against patronisation and the philosophers recommending patronisation has been running up through human history. The Christian church gladly accepted the idea of metaphysical patronisation and transformed the Plato academy into a monastery. Brahe, Kepler and Newton rebelled against the library's monopoly on knowledge by pointing to laboratory observations as the knowledge source. This created the Enlightenment period believing that when enlightened through schooling, people could replace patronisation with democracy.

However, the two democracies installed, the American and the French, developed different forms of anti-pastoral thinking. The French post-structuralism is described above. America developed pragmatism created by Peirce and James arguing that the focus should be shifted away from laws to habits and ability to work. Later American pragmatism developed to Blumer's 'symbolic interactionism' developing its own methodology called 'Grounded Theory' grounding its categories and relations in data and being sceptical towards existing research categories (Tarp 2004).

To prevent enlightenment and democracy to spread from France, Germany invented Humboldt Bildung and Humboldt universities refusing to receive students without an entrance exam from a Humboldt-gymnasium only allowing the best half to enter, and the best half to go on, thus effectively identifying the elite. Today EU still has Humboldt universities while the rest of the world has enlightenment universities.

The invention of the controversial set-concept allowed mathematics to become self-referring defining its concepts as examples of sets. This 'metamatics' defining concepts as examples of abstractions instead of as abstractions from examples, came to mathematics education as modern mathematics, gladly accepted and guarded at Humboldt Bildung schools, and reluctantly adjusted at enlightenment schools.

However, a grounded approach to mathematics education reveals the existence of 'mathematism' (Tarp 2004) being true in the library, but not in the laboratory where e.g. 2+3 = 5 has countless counter-examples: 2weeks+3 days = 17 days, 2m+3cm = 203 cm etc.; in contrast to the statement that 2*3 = 6 stating that 2 3s can be recounted as 6 1s. Mixing metamatics and mathematism to 'metamatism' makes mathematics pastoral by suppressing its natural alternative, mathematics as a natural science studying multiplicity by counting and adding.

Questions

Should teachers be enlightened on the difference between enlightenment and Bildung schools? And on the different attitudes towards patronization, warned against by the sophists and recommended by the philosophers, as expressed today in modern and postmodern philosophy? And on the difference between mathematics, metamatics and mathematism? And on the existence of 1digit mathematics (Zybartas et al 2005)? And that mathematics becomes pastoral by suppressing its alternative, enlightenment mathematics? Or should teachers just unenlightened follow orders?

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213. Concealing Choices to Teachers

Teaching or preaching - this dilemma goes back to the ancient Greek controversy between the sophists advocating enlightenment, and the philosophers advocating patronization. Also today two kinds of schools exist, the North American enlightenment schools educating the people, and EU Humboldt Bildung Counter-enlightenment schools educating the elite for offices. Should teachers be told if they are trained for enlightenment or patronization? Or are they just expected unreflectively to follow the orders of the institution paying their wages?

Two Different Knowledge Institutions

After only one day, an ethnographer will see a fundamental difference between universities in North America and in the EU. At the first place the students begin at age 18, at the second at age 22. At the first place the students have chosen their own combination of modules accessible to all; at the second place they are forced to follow one of several pre-designed educations only accessible to those with the highest marks. At the first place the students already took some university modules at the last year in high school; at the second place this is not possible. At the first place high school is attended by all, and a high percentage goes on to university where around 50% gets a bachelor degree; at the second place only the best half of a year group is allowed to enter high school, and only the best half is allowed to go on to university where only the best half graduates after being forced to finish with a university directed master degree. At the first place some students are supplementing their bachelor with new modules in order to change career e.g. from teaching to engineering; at the second place they have to start all over. Wilhelm von Humboldt seems to hold the key to this difference:

> Our universities have a monastic origin, and they have specialized in being centers of higher learning, functions originally given by the Church to monasteries. (..) The form of the university most familiar to us today is mainly a Prussian invention whose architect and champion was Wilhelm von Humboldt. (..) The collegial system and its related peer review structures centered on an effort to gain intellectual freedom from the constraints of theological doctrine and political manipulation. Although addressing this problem was obviously important, the solution adopted has subsequently done much to weaken the social articulation of the university to all groups other than powerful elites. (..) This situation is predictable because the autopoetic research process provides important supports for intellectual freedom but simultaneously opens the door to useless research and academic careerism divorced from attention to important public social issues. (..) Central to our argument is the claim that action research creates the valid knowledge, theoretical development, and social improvements that the conventional social sciences have promised. (Greenwood & Levin in Denzin & Lincoln 2000: 85-89).

So today two different forms of knowledge institutions exist. A Humboldt institution for the elite practicing autopoetic self-reference producing knowledge of little relevance; and an enlightenment institution open to all and providing the public with relevant knowledge.

A Historical Background

To get a better understanding of the two different knowledge institutions we must go back in history. Apart form the gather/hunter economy, the world has witnessed three different economies, a silk&silver economy, a cotton&iron economy and today's knowledge-economy.

The Greek silver mines enabling trade with Far-East luxury goods as silk and spice lasted one hundred years, and financed the Greek democracy housing a controversy between two kinds of knowledge-men, the sophists and the philosophers. The sophists warned that to protect democracy, people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers seeing everything physical as examples of meta-physical forms only visible to them saw patronization as the natural order when conducted by the philosophers educated at Plato's academy. (Russell 1945).

Silver mines in Spain enabled the Romans to finance their empire that collapsed when the Arabs conquered the Spanish silver mines. After the dark Middle Ages, the silk&silver economy was reopened by German silver financing the Italian Renaissance, and by silver found in America. Robbing the slow Spanish silver ships was no problem to the English; finding a route to India on open sea was. Until Newton found that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

Bringing Indian cotton to North America created a cotton&iron economy, as well as the Enlightenment period: when an apple obeys its own will people could do the same and replace patronization with democracy. Two democracies were installed, one in US, and one in France. US still has its first republic, France now has its fifth. The German autocracy tried to stop the French democracy by sending in an army. However, the German army of mercenaries was no match to the French army of conscripts only too aware of the feudal alternative to stopping the German army. So the French stopped the Germans and later occupied Germany. Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. Humboldt was asked to create an elite school; and using Bildung as counter-enlightenment he created a school-system leading to the Humboldt University, today using Luhmann System Theory to defend its chosen self-reference as nature.

Inside the EU the sophist warning is kept alive in the French postmodern thinking of Derrida, Lyotard and Foucault warning against patronizing pastoral categories, discourses and institutions presenting their choices as nature. Derrida recommends that pastoral categories be 'deconstructed'. Lyotard recommends the use of postmodern 'paralogy' research to invent alternatives to pastoral discourses. And Foucault uses the term 'pastoral power' to warn against institutions legitimizing their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature (Tarp 2004).

Sophist and postmodern thinking form the base of 'anti-pastoral sophist research' searching for alternatives to pastoral choices concealed as nature. Looking at reproduction rates in North America and the EU, the US meets the 2.1 child/family that ensures stability; whereas the EU faces 1.5 child/family, in 200 years reducing EU population to 500 mio. * (1.5/2) 8 times, i.e. to 50 mio. or 10% . So created by the autocracy to identify the elite for state offices, the Humboldt system well suited a silver&silk economy only needing few to become university students. However, in a knowledge economy needing the majority of each year group to graduate, the Humboldt Bildung system is a disaster wiping out most of the EU-population in 200 years if not changed from a pastoral to an enlightenment system.

Enlightenment Mathematics and Pastoral Mathematics

In primary school, an enlightenment curriculum focuses on the nature of numbers, operations and calculations, first 'iconizing' five sticks in a 5-icon etc., then counting by bundling & stacking reported by cup-writing and decimals and predicted by the 'recount-formula' 8 = (8/5)*5 = 1*5 + 3*1 = 1)3) = 1.3*5 = 1.3 5s, in accordance with the Piaget 'from hand to head' principle of natural learning. Introducing 1digit mathematics allows postponing the 'cognitive bomb' 10 until several examples have shown that the icon of the bundle-size is never used since a full bundle will be counted as 1.0 bundles or 10 (Zybartas et al 2005).

The pastoral curriculum introduces the ‘natural numbers’ one by one using the follower-principle. This leads directly to 2digit numbers and place-values by introducing 10 as the follower of 9. Addition is introduced first to revise earlier numbers adding up to the actual number. Then subtraction is introduced as taking away and counting up to. Multiplication and tables follow; and finally division creating a new type of numbers, fractions. (NCTM 2000).

In middle school, an enlightenment curriculum focuses on the nature of per-numbers and triangles. Per-numbers occur when double-counting a quantity in two different units leads to fractions and percentages: 2$ per 3m = 2$/3m = 2/3 $/m. The recount-formula then enables changing units: 16$ = (16/2)*2$ = (16/2)*3m = 24m. When adding fractions and percentages, the units are included as in integration. Formulas with two unknowns are graphed. With only one unknown, a formula becomes an equation solved by using reversed calculations, first reducing a multiple calculation to a single by placing the hidden parentheses, and then moving numbers to the other side reversing their calculation signs. Geometry is introduced via earth-splitting leading to the right-angled triangle seen as a rectangle halved by a diagonal, where the height and length can be recounted in diagonals as the percentages sin and cos.

The pastoral curriculum enlarges the number domain with fractions, and with decimals and percentages as examples of fractions. Again the order of operations is maintained starting with addition of fractions including factorization of ‘natural’ numbers in prime numbers. Equations are statements about equivalent number-names to be transformed by identical operations aiming at neutralizing numbers. Recounting is called proportionality. Algebraic expressions are introduced to be factorized and simplified, and to be added as algebraic letter fractions. In geometry the focus is on 2- and 3-dimensional forms and translation groups.

In high school, an enlightenment curriculum focuses on adding per-numbers, where adding constant per-numbers leads to power, and adding variable per-numbers leads to integration where middle school’s adding fractions with units, 1/2 of 2 + 2/3 of 3 = 3/5 of 5, together with primary school’s adding stacks in combined bundles, 2 3s + 4 5s = 3.2 8s, is generalized to the area under the per-number m/s graph, A = ∫ m/s * s. Reversed calculation then leads to roots and logarithms, if x^3 = 10 then x = 3√10, if 3^x = 8 then x = log3(8); and to differentiation: 2s at 3m/s + 5s at ?m/s = 7m at 4m/s, ? = (7*4 – 2*3)/5 = Δm/Δs.

The pastoral curriculum enlarges the number domain with irrational and real numbers, and the number of operations is enlarged with power, root and log. The function concept is claimed to be the foundation of high school mathematics, and is defined as an example of a relation between two sets. Linear and exponential change is presented as examples of functions. The quadratic function is given an extended treatment. Its graph is studied using translations, and its formula is thoroughly factorized. Calculus is introduced as an example of the limit concept used to define continuity, and the gradient by the first principle, and the integral as a Riemann sum. Geometry introduces coordinate geometry and vector geometry presenting vectors as equivalence sets of parallel arrows with the same length.

Conceiled Choices in Mathematics Education

Through the glasses of anti-pastoral sophist research many concealed choices are revealed, rising the question ‘why conceal to teachers these choices between two opposite alternatives?’

Why conceal that ‘education’ is a choice? Education can mean enlightenment aiming at enlightening the outside world to enable students to practice democracy by being able to tell choice from nature; and testing using the real world things and actions it wants to enlighten. Or it can mean Bildung aiming at patronizing students by telling them how the physical world must be understood as examples of metaphysical forms only visible to university graduates who’s patronization therefore should be accepted with servility; and testing using concepts from a pastoral discourse called Didactics claiming to describe the content of Bildung.

Why conceal that ‘school’ is a choice? A school might be a democratic enlightenment-school aiming at enlightening as many as possible as much as possible. Or a pastoral Bildung-school wanting to identify the elite to be educated for offices.

Why conceal that ‘student’ is a choice? It can be a child trying as other mammal children to learn about the outside world through Piagetean adaptation. Or it can be a sexual mature adolescence creating a self-identity as a biographical narrative expanding through gossip, i.e. through statements with known subjects enlightened by the school.

Why conceal that ‘learning’ is a choice? It can be a bottom-up grounded pyramid growing through Piagetean induction and construction. Or it can be top-down metaphysical pyramid trying to be reached by Vygotskian deduction and scaffolding.

Why conceal that ‘teaching’ is a choice? It can be guidance arranging enlightening meetings with categories and facts grounded in the outside world. Or it can be preaching a pastoral discourse claiming to save humans from ignorance.

Why conceal that ‘teacher education’ is a choice? It can consist of a combination of coordinated modules that can be supplemented to facilitate a change in career, taken at a multi-faculty university with instructors carrying a PhD. Or it can be specially designed for a teaching job and impossible to integrate in a different bachelor degree in case of a change in career, taken at a mono-faculty academy where the instructors do not have PhDs.

Why conceal that ‘numerals’ is a choice? They can be icons showing the degree of multiplicity they represent: 5 strokes in the 5-icon etc. Or they can be arbitrary symbols.

Why conceal that ‘10’ is a choice? It can be a short version of writing 1.0 bundle, thus seven = 10 when counting in 7-bundles. Or it can be a pastoral choice ten = 10.

Why conceal that ‘natural numbers’ is a choice? Thus the follower of nine is 10 only when counting in ten-bundles. With seven-bundles we count 5, 6, 10, 11, 12, 13, …. So counting in seven-bundles, 10 is the follower of six, and the follower of nine is 13.

Why conceal that ‘addition’ is a choice? It can respect the laboratory observation that two stacks can be added side-by-side so that 3 5s and 2 1s become a double-stack of 3.2 5s; or be added on-top so that 2 5s and 1 6s becomes 3.1 5s or 2.4 6s. Or it can be pastoral addition where +5 means finding the fifth follower.

Why conceal that ‘multiplication’ is a choice? It can respect the laboratory observation that 6*7 means 6 7s. Or it can be pastoral multiplication where 6*7 means recounting 6 7s into tens as 4.2 tens, written in a sloppy way as 42 leaving out both the decimal and the unit.

Why conceal that ‘division’ is a choice? It can respect the laboratory observation that 8/2 means splitting 8 in 2s. Or be pastoral division where 8/2 means splitting 8 in 2.

Why conceal that ‘subtraction’ is the only operation with only one meaning saying that 8-2 means ‘from 8 take away 2’? And why conceal that subtraction is the most fundamental operation? Why conceal that ‘/’ and ‘–‘ are not just symbols, but icons, where ‘–4’ shows the dragging away of 4; and ‘/4’ shows the shoveling away of 4s?

Why conceal that ‘equation’ is a choice? It can be reversed calculations, reversing the calculation-signs when moved across the = sign. Or equivalence-statements relating number-names, to be transformed by identical operations aiming at neutralizing the known neighbors.

Why conceal that ‘calculus’ is a choice. It can be adding stacks side-by-side, and per-numbers or fractions carrying units. Or it can be introduced as an example of a limit process.

Why conceal that ‘definition’ is a choice? It can be grounded, defining its concept as an abstraction from examples. Or it can be metaphysical, defining its concept as an example from an abstraction, thus turning grounded mathematics upside down to ‘metamatics’.

Why conceal that ‘proof’ is a choice? It can be a laboratory-proof testing a prediction deduced from a hypothesis. Or it can be a library-proof showing how a statement can be deduced from others statements. Thus ‘2+3=5’ is ‘mathematism’ only true in a library, but having countless counter-examples in the laboratory: 2weeks+3days = 17days, 2m+3cm = 203cm, etc. Whereas ‘2*3=6’ is grounded mathematics true both in the library and in the laboratory by just stating the physical fact that 2 3s can always be recounted as 6 1s.

Why conceal that ‘function’ is a choice? It can follow Euler proposing the name function for calculations containing numbers and variables, becoming interesting after calculus enabled the calculation of changes in, not numbers, but calculations with variable numbers. Or it can present itself as metamatics as an example of a many-to-one set-relation.

Why conceal that ‘algebra’ is a choice? Respecting the Arabic meaning ‘re-uniting’, it can be grounded in the two opposite questions ‘how to unite parts into a total?’ and ‘how to split a total into parts?’ leading to the four different ways of adding and splitting, where + and * add variable and constant unit-numbers, and ∫ and ^ adds variable and constant per-numbers; and where a total is split by - and / into variable and constant unit-numbers, and by d/dx and √/log into variable and constant per-numbers. Or be pastoral algebra from above where numbers, operations and equations all are examples of the metaphysical concept ‘set’.

Why conceal that ‘geometry’ is a choice? Respecting the Greek meaning ‘earth-measuring’, it can be grounded geometry from below enlightening the problem of measuring a piece of earth in triangles. Or be pastoral geometry from above showing how physical forms and facts are examples of metaphysical undefined terms and axioms.

Why conceal that ‘applying mathematics’ is a choice? It can be ‘rooting mathematics’, a phrasing indicating that ‘of course the roots should be treated before the plant’, mathematics. Or by saying ‘applying mathematics’ it can seduce teachers and students to believe that ‘of course mathematics must be taught and learned before it can be applied’.

Why conceal that ‘mathematics’ in Greek means knowledge that can be used for prediction? And why conceal that the predicting ability of calculations can be observed all over mathematics? Thus the calculation 3+4 = 7 predicts the result of counting on 4 times from 3; and 3*4 = 12 predicts the result of adding 3 4 times; and 3^4 = 81 predicts the result of multiplying with 3 4 times. Likewise 6-3 predicts the answer to the question x+3 = 6; and 6/3 predicts the answer to the question x*3 = 6; and 3√6 predicts the answer to the question x^3 = 6; and log3(6) predicts the answer to the question 3^x = 6. And why conceal that the predicting ability of mathematics made physics replace patronization with enlightenment?

Why conceal that mathematics can’t be well-proven statements about well-defined concepts after Gödel proved that not all statements can be proved; and Russell proved that using sets leads to internal self-contradiction: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

Conclusion

A teacher is paid by an institution to influence students in a way prescribed by its curriculum. In his book 'Modernity and the Holocaust' Bauman points to three conditions, singly or together being able to produce both a welfare society and a Holocaust: authorization, routinization and dehumanization (Bauman 1989: 21). Also he points out that the first two 'have been spelled out repeatedly in those principles of rational action that have been given universal application by the most representative institutions of modern society'. Especially the education institution enforces authorization and routinization by using curricula and exams and special teacher education. Thus the question arises: How can we know if this authorization and routinization leads to cognitive welfare or to a cognitive Holocaust?

Being solidly based on the sophist idea of enlightening nature in order to tell choice from nature, enlightenment institutions have a strong defense: Nature is there and needs to be enlightened to protect democracy. Being created by Humboldt as counter-enlightenment, Bildung institutions have big problems legitimizing their authorization and routinization. They only refer to a Bildung-discourse, referring on to a Didactics-discourse for its content.

Additional problems occur when using oral marks, becoming unreliable when given by the person also giving the teaching. In Denmark the use of oral marks in mathematics has forced the authorities to lower the passing mark in written mathematics from the international level at 60% correctness to 40% in the gymnasium and to 20% in the secondary school. When students can pass an exam by solving correctly only 1 problem of 5, it seems to indicate that the teaching has made the students not cognitive enlightened, but cognitive stunted.

The Nuremberg process in 1946 tried the German leaders for war crimes. However, they all excused themselves for having just followed orders. Only Keitel changed his meaning when enlightened about the Holocaust. To prevent a new process against teachers at pastoral Bildung schools in 200 years it might be a good idea if the authorization and routinization in mathematics education come from nature, and not from choices becoming pastoral by concealing other possible choices. Thus in the case of mathematics, set-based top-down metamatics from above should be replaced by multiplicity-based bottom-up mathematics from below respecting mathematics as a natural science investigating the natural fact many.

The educational institution conceals choices at all levels. The choices concealed are revealed when comparing the international standard set by the North American enlightenment schools wanting as many as possible to be educated as much as possible, with the EU Humboldt Bildung schools wanting only the elite to be educated for offices. The teachers are the interface between the institution and the humans forced to attend it for several years. Are teachers expected just to follow the authorization and routinization of the institution paying their wages? Or should teachers also be enlightened about the many concealed choices made by this institution? The survival of the EU population depends upon this choice.

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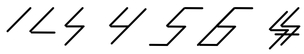
214. Workshop in 1digit Mathematics, Cup-writing & Decimal-counting

Avoiding 10, a Cognitive Bomb

A: Counting by bundling and stacking. Re-counting	**Expected answers**
A0. Place a total T of ten sticks on a table.	I I I I I I I I I I
A1. Rearrange the sticks in icons for 1, 2, etc. What about ten?	 Ten has no icon
A2. Count T in 2s. Write the result with units as T = __2s; and as a stack as T = __*2	T=5 2s , T = 5*2
A3. Recount T in 3s. Write the result with units as T = Write the result as a double stack T =	T = 3 3s + 1 T= 3*3 + 1
A4. Use the icon / to describe the recount process	T = T/3*3 The Recount Formula
A5. Can the result of recounting from 2s to 3s be predicted on a calculator?	T = (5*2) /3 *3 T = 3*3 + R R = 5*2-3*3 = 1
A6. Predict the result of recounting from 2s to 4s	T = (5*2) /4 *4 = 2*4 + 2
A7. Perform the recounting to 4s	
A8. Predict the result of and perform recounting from 2s to 5s	T = (5*2) /5 *5 = 2*5
A9. Predict the result of and perform recounting from 2s to 6s	T = (5*2) /6 *6 = 1*6 + 4
A10. Predict the result of and perform recounting from 2s to 7s	T = (5*2) /7 *7 = 1*7 + 3
B. Cup-writing using decimals	
B1. Count ten sticks in 6s and place the sticks in two cups, a left bundle-cup and a right single-cup. Write down the result using real 'cup-writing'.	T = IIIIII) I I I I)
B2. Change a bundle to a stick. Write down the result with symbolized 'cup-writing'.	T = I) I I I I)
B3. Change the sticks to icons.	T = 1) 4)
B4. Change to decimal-writing including the unit, using the dot to separate the left cup from the right.	T = 1.4 6s
C. Decimal Recounting	
C1. Recounting 1.3 6s to 5s, de-bundling Transform 1.3 6s to cup-writing Transform cup-writing to symbolized cup-writing Transform symbolized cup-writing to real cup-writing Transform real cup-writing to a total T of sticks	 T = 1.3 6s = 1) 3) T = I) I I I) T = IIIIII) I I I) T = I I I I I I I I I
C2. Recounting 1.3 6s to 5s, re-bundling Transform the total T of sticks to real cup-writing Transform the real cup-writing to symbolized cup-writing Transform the symbolized cup-writing to cup-writing Transform the cup-writing to 5 s	 T = IIIIII) I I I I) T = I) I I I I) T = 1) 4) T = 1.4 5s
C3. Predict the result when recounting 1.3 6s to 5s Use a calculator and the recount-formula to predict the result	 (1*6+3*1)/5 = 1.R R = 1*6+3*1-1*5 = 4
C4. Predict the result when recounting 1.3 6s to 4s	T = 1.3 6s = 1.4 5s (1*6+3*1)/4 = 2.1
C5. Perform the recounting of 1.3 6s to 4s	
C6. Predict the result of and perform recounting 1.3 6s to 3s	(1*6+3*1)/3 = 3.0

D. Selling from a stock I	
D1. From a stock of 3.2 5s is sold 1.4 5s. What is left?	
Transform 3.2 5s to cup-writing	T = 3.2 5s = 3) 2)
Transform cup-writing to symbolized cup-writing	T = I I I) I I)
Move a stick form the bundle-cup to the single cup as 5 1s	T = I I) IIIII I I)
Remove the 1.4 5s and count the rest in decimals	T = I) I I I I + I) I I I
Write down the subtraction result	3.2 5s – 1.4 5s = 1.3 5s
E. Selling from a stock II	
E1. From a stock of 4.2 5s is sold 1.3 5s. What is left?	
Transform 4.2 5s to cup-writing	T = 4.2 5s = 4) 2)
Move 1 5s from the bundle-cup to the single-cup as 5 1s	T = 4-1) 2+5) = 3) 7)
Remove the 1.3 5s and count the rest in decimals	T = 1) 3) + 2) 4)
Write down the subtraction result	4.2 5s – 1.3 5s = 2.4 5s
F. Adding stocks I	
F1. To a stock of 2.3 5s is bought 1.4 5s. What is the Total?	
Transform 2.3 5s and 1.4 5s to cup-writing	2.3 5s + 1.4 5s = 2)3) + 1)4)
Transform cup-writing to symbolized cup-writing	T = I I) I I I) + I) I I I I)
Move 1.4 5s to the 2.3 5s as 3.7 5s	T = I I I) I I I I I I I)
Move 5 1s from the single-cup to the bundle-cup as 1 5s	T = I I I) IIIII I I) -> I I I I) I I)
Write down the addition result	2.3 5s + 1.4 5s = 4.2 5s
G. Adding stocks II	
G1. Add the two stocks 2.3 5s and 3.2 4s as 4s.	
Recount the 2.3 5s in 4s	T = (2*5+3)/4 *4 = 3.1 *4
Add 3.1 4s and 3.2 4s	3.1 4s + 3.2 4s = 6.3 4s
Perform the addition	
H. Adding stocks III	
H1. Add the two stocks 2.3 5s and 3.2 4s as 5s.	
Recount the 3.2 4s in 5s	T = (3*4+2)/5 *5 = 2.4 *5
Add 2.3 5s and 2.4 5s	2.3 5s + 2.4 5s = 4.7 5s
Perform the addition	= 5.2 5s
I. Adding stocks as integration	
I1. Add the two stocks 2.3 5s and 3.2 4s as 9s (integration).	
Recount the 2.3 5s in 9s	T = (2*5+3)/9 *9 = 1.4 *9
Recount 3.2 4s in 9s	T = (3*4+2)/9 *9 = 1.5 *9
Perform the addition	1.4 9s + 1.5 9s = 2.9 9s
	= 3.0 9s
J. Handling overloads	
J1. In 7.3 5s introduce a new cup to the left meant for bundles of bundles	T =) 7) 3) = 1) 7-5) 3)
J2. Remove the overload in 9.5 8s, 7.3 4s and 45.2 3s	= 1) 2) 3) = 12.3 5s
K. Multiplying and dividing with the bundle-size	
K1. Multiply 3.2 5s with 5	T = 3.2 5s = 3)2)*5 = 3*5)2*5)
	= 3)2)0) = 32.0 5s
K2. Divide 14 5s with 5	T = 14 *5 = 1)4)0) = 1*5) 4*5)
	= 1)4) *5 = 1.4 5s *5
L. Solving equations	
L1. Solve the equation 2*x = 7 by rebundling	2*x = 7 = (7/2)*2, x = 7/2
L2. Solve the equation 2*x+1 = 7 by rebundling	2*x+1 = 7-1+1 = (7-1)/2*2 +1,
L3. Solve the equations by bundling and stacking	so x = (7-1)/2

1. Discuss the advantages and disadvantages of Cup-writing & Decimal-counting.

2. Discuss the advantages and disadvantages of 1digit mathematics.

3. Discuss if 10 is a cognitive bomb to be introduced as the last bundle-size.

215. The 12 Blunders of Pastoral Mathematics

Math-Blunder 1: Teaching Both Numbers and Letters as Symbols
A number is an icon showing a degree of multiplicity. A letter installs a sound to be distinct.

Math-Blunder 2: Teaching 2digit Numbers Before Decimal Numbers
Math grows from counting by bundling & stacking (T = 2 3s = 1 1/5 5s = 1.1 5s) using 1digit numbers only. Using two digit numbers directly means firing a cognitive bomb, 10.

Math-Blunder 3: Teaching Fractions Before Decimals
In a natural approach both fractions and decimals occur in grade 1 as different ways of accounting for leftovers: fractions on top of the stack, and decimals next to the stack.

Math-Blunder 4: Teaching Addition Before Division
Counting in 3-bundles can be predicted by the 'recount-equation' and division: T = (T/3)*3.

Adding without units leads to 'MatheMatism' true in the library but not in the laboratory.

MatheMatism: 2+3=5 since 2m+3cm=203cm etc.

Mathematics: 2*3=6 since 2 3s is 6 1s.

Math-Blunder 5: Forgetting the Units
Fraction paradox: 10/100 (10%) + 20/100 (20%) = 30/100 (32% or 12% or 18% or …).

Math-Blunder 6: Teaching Fractions Before Integration
Add = integrate: 6kg @ $\frac{5}{3}\frac{\$}{\text{kg}}$ + 8kg @ $\frac{9}{4}\frac{\$}{\text{kg}}$ = $\frac{5}{3}$*6 + $\frac{9}{4}$*8 = 28 = $\frac{28}{14}$*14 = 14kg @ $\frac{28}{14}\frac{\$}{\text{kg}}$

Math-Blunder 7: Teaching Proportionality Instead of DoubleCounting
'Per-numbers' occurs when double-counting in two different units using a 'guide-equation':

If 4kg =5$, then 12kg = (12/4)*4kg = (12/4)*4kg * 5$ = 3*5$ = 15$. (recounting 12 in 4s)

Math-Blunder 8: Teaching Balancing Instead of Backward Calculation
Forward calculation: 2+3*5 = ? ,

Backward calculation: 2+3*x = 14

Solution: x = (14-2)/3

Math-Blunder 9: Killer Equations Instead of Grounded Equations
2$ plus 3kg @ ? $/kg total 14$' leads to '2+3*x=14'.

Killer-equation: $2 + \frac{3 - 4x}{5x - 6} = 7x - \frac{8}{9x}$

Math-Blunder 10: Teaching Geometry Before Trigonometry
Geometry: earth-measuring.

Earth splits into triangles, triangles into right-angled triangles.

Math-Blunder 11: Postponing Calculus
Primary school integration: 2 3s + 4 5s = ? 8s.

Secondary school integration 6 kg @ 2 $\frac{\$}{\text{kg}}$ + 8 kg @ 5 $\frac{\$}{\text{kg}}$ = 14 kg @ ? $\frac{\$}{\text{kg}}$

Math-Blunder 12: The 5 Meta-Blunders of Math Education

1) Forgetting Prediction.

2) Interchanging Product & Process.

3) Interchanging Goal & Means.

4) Funding Library Research Instead of Laboratory Research.

5) Turning Enlightenment LAB-based Mathematics into Modern LIB-based MetaMatism.

Reality is the root, not an application of Mathematics

216. Mathematics as an anti-Pastoral Natural Science

Presentation given at the MATHeCADEMY.net booth at the ICME 11

0. Let us reinvent mathematics as a natural science grounded in the study of the natural fact many. And let us see if this grounded natural mathematics will be the same as the mathematics we know from the textbooks. If not, textbook-mathematics might be called pastoral mathematics, where pastoral means claiming that things can only be as in the Book, there are no alternatives, in which case the Book presents its choices as nature.

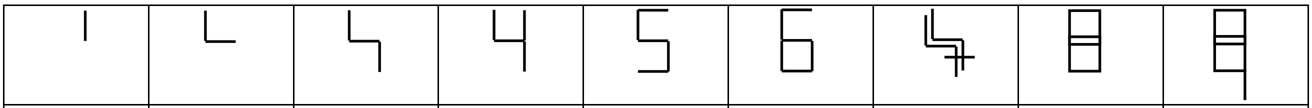
So an anti-pastoral question is: Can things be otherwise than in the Book? Does the Book hide alternatives?

Uncovering hidden alternatives to pastoral choices presented as nature might be called anti-pastoral research. Or sophist-research since the ancient Greek sophists were the first to warn, that to practice democracy people must know nature from choice to prevent being patronized by choices presented as nature.

1. So let us try to reinvent mathematics as a natural science dealing with the natural fact many. What do we do when we meet many? Two things, first we Count, then we Add, and we do that where we live, in Time and Space. So this approach can be called the CATS-approach to mathematics: Count&Add in Time&Space.

With a pile of sticks there are 3 ways of counting: 1.order-counting, 2.order-counting and 3.order-counting.

A 1.order-counting means rearranging the sticks in icons, so that there are five sticks in the five-icon 5 etc. So an icon contains the degree of many it describes. 1.order-counting stops after nine, thus ten has no icon.



1	2	3	4	5	6	7	8	9

A 2.order-counting counts by bundling and stacking in icon-bundles, i.e. counting in e.g. 5s, but not in tens.

A 3.order-counting counts in tens, a very special number: the only number with a name but without an icon.

2. As an example of 2.order-counting let us count 7 1s in 3s, 5s and 2s.

I I I I I I I -> III III) I) -> II) I) -> 2) 1) -> 2.1 3s

I I I I I I I -> IIIII) II) -> I) II) -> 1) 2) -> 1.2 5s

I I I I I I I -> II II II) I) -> III) I) -> II I) I) -> I) I) I) -> 1) 1) 1) -> 11.1 2s

Counting 7 1s in 3s, we take away a 3-bundle 2 times leaving 1 stick unbundled. The unbundled is placed in a right single-cup, and the 3-bundles are placed in a left bundle-cup, either as actual bundles, or as sticks counting bundles by being placed in the left bundle-cup. Thus the counting result is 2.1 3s, using a decimal point to separate the bundles to the left from then unbundled to the right; and including the unit 3s.

Likewise counting 7 1s in 5s gives 1.2 5s.

Counting 7 1s in 2s gives 3.1 2s. However, in the bundle-cup we also have a bundle of bundles that can be moved to a new cup to the left, counting the bundles of bundles. Thus counting 7 1s in 2s gives 11.1 2s.

Counting 3 8s in tens gives 2.4 tens, only this time we have no icon for ten: 3 8s = 2.4 tens.

In all cases, counting means bundling in a chosen bundle-size, and counting always produces decimal numbers carrying units. So natural numbers are decimal numbers carrying units.

3. Is this what the Book says? No. The book says: we only count in tens, and we do not write 2.4 tens. First we throw away the unit tens; then we misplace the decimal point one to the right. So instead of 2.4 tens we just write 24, which we call a natural number. Thus what the Book calls natural numbers are instead pastoral numbers hiding its natural alternative and creating problems to learners.

Counting in different bundle-sizes might also be called counting in different bases. However, base is a pastoral term hiding its alternative ‘counting in different bundle-sizes’. The term ‘bundle’ is grounded in experience, a bundle can be grasped. The term ‘base’ is not, it comes from the Book and it can’t be grasped.

4. Since pastoral numbers create learning problems by being unnatural, we ask: If ten is a cognitive bomb by having no icon but needing 2digits, how much mathematics can be learned from 1digit numbers alone?

Surprisingly the answer is that the core of mathematics can be learned as 1digit mathematics.

An example: My sister has 3.2 4s, and I have 2.3 5s. Now we would like to add them. However, to add, the units must be the same, so I must recount my 5s the 4s, or my sister must recount her 4s in 5s. Or we can add them as 9s by uniting the bundle-sizes.

Double-counting a given quantity in two different units, e.g. 4s and 5s, or kgs and £ is called proportionality, normally learned in middle school; and adding in the combined bundle-size is called integration, normally learned late in high school if ever. But using 1digit mathematics, both core concepts are learned in grade 1.

5. Furthermore, recounting 3.2 4s in 5s can be predicted by a calculator. We enter ‘(3*4+2*1)/5’ since counting in 5s means taking away 5s many times, which is iconised as division. The answer is 2.8 5s. To see if we can trust the .8 we take away the 2 5s by subtracting 2*5. Entering ‘(3*4+2*1)-2*5’ gives 4, so the recounting result can be predicted to be 2.4 5s. To test this prediction we perform the actual recounting by de-bundling the 3.2 4s in 1s and the re-bundling the 1s in 5s:

3.2 4s -> 3)2) -> IIII IIII IIII II -> IIII IIII IIII II -> IIIII IIIII IIII -> 2)4) -> 2.4 5s

So the prediction holds. So from now on we don’t have to perform the actual recounting by de-bundling and re-bundling since we can predict the result on a calculator thus becoming a number-predictor.

6. When the units are the same we can add the two ‘stocks’ using cup-writing:

3.2 4s + 2.3 5s = 2.4 5s + 2.3 5s = 5.7 5s = 5)7) = 5+1) 7-5) = 6)2) = 1) 6-5) 2) = 1)1)2) = 11.2 5s

Here the 7 1s can be recounted in 1.2 5s transferring 5 1s as 1 5s form the single-cup to the bundle-cup. Here the 6 5s can be recounted to 1.1 5*5s transferring the 5 5s as 1 5*5 from the bundle-cup to the bundles of bundles-cup, thus giving the total of 1 bundle of 5 5s and 1 bundle of 5s and 2 unbundled 1s.

7. With 2.3 5s, what happens if I add an extra cup to the right?

2.3 5s = 2)3)   <adding a cup to the right>   2)3)) = 23.0 5s.

Apparently adding an extra cup to the right means that the 3 1s becomes 3 5s, and that the 2 5s becomes 2 5*5s, i.e. means multiplying with the bundle-number and moving the decimal point 1 place to the right.

Likewise, removing 1 cup from the right means dividing with the bundle-number and moving the decimal point 1 place to the left:

23.0 5s = 2)3))   <removing a cup from the right>   2)3) = 2.3 5s.

8. Thus we see that 1digit mathematics respects the Piaget ‘through the hands to the head’-principle of natural learning: to grasp with the head, first grasp with the hand.

9. The CATS approach using 1digit mathematics conflicts with the traditional pastoral approach that introduces 2digit numbers in grade 1 by claiming that 10 is the follower of 9. Now ten is the follower of nine by nature, but to say that 10 IS the follower of 9 is a pastoral choice hiding the alternatives. With 8 as the bundle-number, 10 is the follower of 7, and the follower of nine is 12.

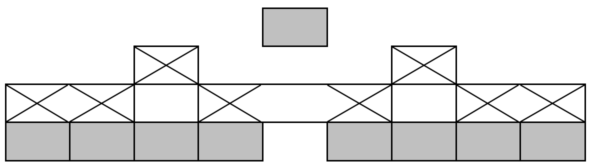
Thus the CATS approach treating mathematics as a natural science cannot be learned at traditional pastoral academies. Hence a web-based anti-pastoral academy www.MATHeCADEMY.net has been set up to teach the CATS approach to mathematics as an anti-pastoral natural science respecting the huge learning potential of 1digit mathematics. This is expressed in the rap presented at the booth:

Ten is a cognitive, cognitive bomb.
Save the worlds with 1digit math.
Count and Add in Time and Space.

10. The MATHeCADEMY.net offers free master degrees to teachers having learned pastoral mathematics at pastoral academies but wanting to learn mathematics as a natural science investigating the natural fact many.

The learners are organized in groups of 8 using PYRAMIDeDUCATION: the 8 learners are organized in 2 teams of 4 learners choosing 3 pairs and 2 instructors by turn. The teacher coaches the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In each pair each learner corrects the other learner’s routine-assignment. Each pair is the opponent on the essay of another pair. Each learner pays for the education by coaching a new group of 8 learners. It is not difficult to be a coach since the learners are educated, not by books but by counting and adding in time and space.

1 coach
2 instructors
3 pairs
2 teams



8 learners and 1 coach

The activities are divided into 2x4 parts, Count&Add in Time&Space 1 for primary school, C1, A1, T1 and S1; and Count&Add in Time&Space 2 for secondary school, C2, A2, T2 and S2. The material is activity-based and very short. It is accessible at the MATHeCADEMY.net website. The content is given in the summary below.

11. As research method the MATHeCADEMY.net uses anti-pastoral sophist research, following the ancient Greek warning 'in a democracy people must know nature from choice to prevent patronization by pastoral choices presented as nature' by uncovering hidden alternatives to pastoral choices presented as nature.

To demonstrate the power of this new research paradigm 12 papers was written to the ICME 11 conference, for Topic Study Groups (TSG) and Discussion Groups (DG) and most were accepted:

Avoiding Ten, a Cognitive Bomb	TSG 1: New developments and trends in mathematics education at preschool level. Acc.
A Fresh Start Presenting Mathematics as a Number-predicting Language	TSG 4: New developments and trends in mathematics education at upper secondary level. Rej.
Decimal-Counting, Disarming the Cognitive Bomb Ten	TSG 10: Research and development in the teaching and learning of number systems and arithmetic. Rej.
Pastoral Algebra Deconstructed	TSG 11: Research and Development in the Teaching and Learning of Algebra. Acc.
Pastoral Calculus Deconstructed	TSG 16: Research and development in the teaching and learning of calculus. Acc.
Applying Pastoral Metamatism or Re-Applying Grounded Mathematics	TSG 21: Mathematical applications and modelling in the teaching and learning of mathematics. Acc.
Pastoral Humboldt Mathematics Deconstructed	TSG 25: The role of mathematics in the overall curriculum. Rej.
CATS, Count&Add in Time&Space - a Natural Way to Become a Mathematics Teacher	TSG 27: Mathematical knowledge for teaching. Acc.
Pastoral Words in mathematics education	TSG 31: Language and communication in mathematics education. Rej.
Deconstructing the Mathematics Curriculum: Telling Choice from Nature	TSG 35: Research on mathematics curriculum development. Acc.
Mathematics Education: Pastoral Bildung - Or Anti-Pastoral Enlightenment	DG 5: The role of philosophy in mathematics education. Acc.
Concealing Choices to Teachers	DG 7: Dilemmas and controversies in the education of mathematics teachers. Acc.
Workshop in 1digit Mathematics, Cup-writing & Decimal-counting	Workshop. Acc.
The 12 Blunders of Pastoral Mathematics	Poster. Acc.

12. By reinventing mathematics as a natural science dealing with the natural fact many we have uncovered a hidden alternative to the traditional mathematics thus becoming a pastoral mathematics hiding its alternative. The main differences between natural mathematics and pastoral mathematics can be shown in two posters:

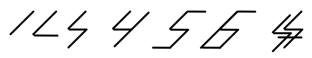
Mathe matics	A natural science investigating the natural fact MANY, grounding categories and relations in observations	Natural numbers: 5, 6, 7, 1.0, 1.1 Allows both 1.order and 2.order and 3.order counting, in icons, icon-bundles and tens. Natural operations: 8/2: From 8 take away 2s, 3*4: 3 times stack 4s 8–2: From 8 take away 2, 3+4: Add 3 next to 4 Natural order of operations: /, *, –, + Core Question: Double-counting Elementary school: 3.2 5s = ? 7s Middle school: 3kg = ? $ given that 4kg = 5$ High school: 4sec = ? meter, given that dy = (2x+3)dx Respects the original Greek and Arabic meaning of Geometry and Algebra: Geometry: Measuring earth through triangulation Algebra: Re-uniting constant & variable unit- & per-numbers Calculus: Adding variable per-numbers
Mathe matism	A pastoral science neglecting the units, presenting statements true in the library but not in the laboratory	Disregards Gödel's theorem: Not all true statements can be proven Allows only 3.order counting in tens. Claims that the natural numbers are : …, 8, 9, 10, 11, 12, … Neglects both the units and the decimal point by reeducating 2.3 tens to 23. Claims that '10 IS the follower of 9', in spite of the fact that counting in 8s, 10 is the follower of 7, and the follower of 9 is 12. Claims that the natural order of operations is: +, –, *, / Claims that '2+3 IS 5', in spite of 2w+3d=17d, 2m+3cm=203cm. Claims that '1/2+2/3 IS 7/6', in spite of 1/2 of 2 + 2/3 of 3 = 3/5 of 5.
Meta matics	A pastoral science claiming that all concepts can be defined as examples of the ultimate abstraction SET	Disregards Russell's paradox: If $M=\{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. Defines a Peano-set of followers. Defines numbers as equivalence sets in a product-set of Peano sets, i.e. as equivalent number-names: -3 = (0,3) # (5,8) # (6,9) etc, 2/3 = (4,6) # (6,8) # (8,12) etc. Defines a function as a many-one relation in a set-product. Defines operations as functions between a set-product and a set. Defines an equation as an equivalence statement to be transformed by identical operations on both number-names. Defines calculus as examples of a convergence concept. Defines geometry as the set of theorems deduced from a specific set of axioms. Defines algebra as a study of abstract patterns.

Summary of the MATHeCADEMY.net Study Units

	QUESTIONS	ANSWERS
COUNT 1	How to count multiplicity? How to recount 8 in 3s: T= 8 = ? 3s How to recount 6kg in $: T=6kg=?$ How to count in standard bundles?	By bundling and stacking the total T predicted by T = (T/b)*b T= 8= ?*3= ?3s, T= 8=(8/3)*3 = 2*3 +2 = 2*3 +2/3*3 = 2 2/3*3 If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$ Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle+2Bundle+3 = 4tenten2ten3 = 4*B^2+2*B+3
COUNT 2	How can we count possibilities? How can we predict unpredictable numbers?	By using the numbers in Pascal's triangle We 'post-dict' that the average number is 8.2 with the deviation 2.3. We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation)
ADD 1	How to add stacks concretely? T=27+16= 2ten7+1ten6= 3ten13=? How to add stacks abstractly?	By restacking overloads predicted by the restack-equation T= (T-b)+b T = 27+16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43 Vertical calculation uses carrying. Horizontal calculation uses FOIL
ADD 2	What is a prime number? What is a per-number? How to add per-numbers?	Fold-numbers can be folded: 10=2fold5. Prime-numbers cannot: 5=1fold5 Per-numbers occur when counting, when pricing and when splitting. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b
TIME 1	How can counting & adding be reversed ? Counting ? 3s and adding 2 gave 14. Can all calculations be reversed?	By calculating backward, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign. x*3+2=14 is reversed to x = (14-2)/3 Yes. x+a=b is reversed to x=b-a, x*a=b is reversed to x=b/a, x^a=b is reversed to x=a√b, a^x=b is reversed to x=logb/loga
TIME 2	How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable?	By using constant change-equations: If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44 If Ko = 30 and ΔK/K = r = 2%, then K7= Ko*(1+r)^n= 30*1.02^7= 34.46 By solving a variable change-equation: If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx
SPACE 1	How to count plane and spatial properties of stacks and boxes and round objects?	By using a ruler, a protractor and a triangular shape. By the 3 Greek Pythagoras', mini, midi & maxi By the 3 Arabic recount-equations: sinA=a/c, cosA=b/c, tanA=a/b
SPACE 2	How to predict the position of points and lines? How to use the new calculation technology?	By using a coordinate-system: If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14) Computers can calculate a set of numbers (vectors) and a set of vectors (matrices)
QL	What is quantitative literature? Does quantitative literature also have the 3 different genres: fact, fiction and fiddle?	Quantitative literature tells about multiplicity in time and space The word and the number language share genres: Fact is a since-so calculation or a room-calculation Fiction is if-then calculation or a rate-calculation Fiction is so-what calculation or a risk-calculation

301. Come Back with 1digit Mathematics

Postmodern contingency research uncovers hidden alternatives to choices presented as nature e.g. by replacing choice with nature. Within traditional mathematics, numbers, operations, formulas, equations etc. turn out to be choices hiding their natural alternatives. Presenting mathematics from its roots, the natural fact Many, help many dropouts to master mathematics as a natural science.

A: Counting by bundling and stacking. Re-counting	Expected answers
A0. Place a total T of ten sticks on a table.	I I I I I I I I I I
A1. Rearrange the sticks in icons for 1, 2, etc.	
What about ten?	Ten has no icon
A2. Count T in 2s. Write the result with units as T = __2s; and as a stack as T = __*2	T=5 2s , T = 5*2
A3. Recount T in 3s. Write the result with units as T = Write the result as a double stack T =	T = 3 3s + 1 T= 3*3 + 1
A4. Use the icon / to describe the recount process	T = T/3*3 **The Recount Formula**
A5. Can the result of recounting from 2s to 3s be predicted on a calculator?	T = (5*2) /3 *3 T = 3*3 + R R = 5*2-3*3 = 1
A6. Predict the result of recounting from 2s to 4s	T = (5*2) /4 *4 = 2*4 + 2
A7. Perform the recounting to 4s	
A8. Predict the result of and perform recounting from 2s to 5s	T = (5*2) /5 *5 = 2*5
A9. Predict the result of and perform recounting from 2s to 6s	T = (5*2) /6 *6 = 1*6 + 4
A10. Predict the result of and perform recounting from 2s to 7s	T = (5*2) /7 *7 = 1*7 + 3
B. Cup-writing using decimals	
B1. Count ten sticks in 6s and place the sticks in two cups, a left bundle-cup and a right single-cup. Write down the result using real 'cup-writing'.	T = IIIIII) I I I I)
B2. Change a bundle to a stick. Write down the result with symbolized 'cup-writing'.	T = I) I I I I)
B3. Change the sticks to icons.	T = 1) 4)
B4. Change to decimal-writing including the unit, using the dot to separate the left cup from the right.	T = 1.4 6s
C. Decimal Recounting	
C1. Recounting 1.3 6s to 5s, de-bundling Transform 1.3 6s to cup-writing Transform cup-writing to symbolized cup-writing Transform symbolized cup-writing to real cup-writing Transform real cup-writing to a total T of sticks	 T = 1.3 6s = 1) 3) T = I) I I I) T = IIIII) I I I) T = I I I I I I I I I
C2. Recounting 1.3 6s to 5s, re-bundling Transform the total T of sticks to real cup-writing Transform the real cup-writing to symbolized cup-writing Transform the symbolized cup-writing to cup-writing Transform the cup-writing to 5 s	 T = IIIII) I I I I) T = I) I I I I) T = 1) 4) T = 1.4 5s
C3. Predict the result when recounting 1.3 6s to 5s Use a calculator and the recount-formula to predict the result	 (1*6+3*1)/5 = 1.R
C4. Predict the result when recounting 1.3 6s to 4s	R = 1*6+3*1-1*5 = 4 T = 1.3 6s = 1.4 5s
C5. Perform the recounting of 1.3 6s to 4s	(1*6+3*1)/4 = 2.1
C6. Predict the result of and perform recounting 1.3 6s to 3s	(1*6+3*1)/3 = 3.0

D. Selling from a stock I D1. From a stock of 3.2 5s is sold 1.4 5s. What is left? Transform 3.2 5s to cup-writing Transform cup-writing to symbolized cup-writing Move a stick form the bundle-cup to the single cup as 5 1s Remove the 1.4 5s and count the rest in decimals Write down the subtraction result	T = 3.2 5s = 3) 2) T = I I I) I I) T = I I) IIIII I I) T = I) I I I I + I) I I I 3.2 5s – 1.4 5s = 1.3 5s
E. Selling from a stock II E1. From a stock of 4.2 5s is sold 1.3 5s. What is left? Transform 4.2 5s to cup-writing Move 1 5s from the bundle-cup to the single-cup as 5 1s Remove the 1.3 5s and count the rest in decimals Write down the subtraction result	T = 4.2 5s = 4) 2) T = 4-1) 2+5) = 3) 7) T = 1) 3) + 2) 4) 4.2 5s – 1.3 5s = 2.4 5s
F. Adding stocks I F1. To a stock of 2.3 5s is bought 1.4 5s. What is the Total? Transform 2.3 5s and 1.4 5s to cup-writing Transform cup-writing to symbolized cup-writing Move 1.4 5s to the 2.3 5s as 3.7 5s Move 5 1s from the single-cup to the bundle-cup as 1 5s Write down the addition result	2.3 5s + 1.4 5s = 2)3) + 1)4) T = I I) I I I) + I) I I I I) T = I I I) I I I I I I I) T = I I I) IIIII I I) -> I I I I) I I) 2.3 5s + 1.4 5s = 4.2 5s
G. Adding stocks II G1. Add the two stocks 2.3 5s and 3.2 4s as 4s. Recount the 2.3 5s in 4s Add 3.1 4s and 3.2 4s Perform the addition	T = (2*5+3)/4 *4 = 3.1 *4 3.1 4s + 3.2 4s = 6.3 4s
H. Adding stocks III H1. Add the two stocks 2.3 5s and 3.2 4s as 5s. Recount the 3.2 4s in 5s Add 2.3 5s and 2.4 5s Perform the addition	T = (3*4+2)/5 *5 = 2.4 *5 2.3 5s + 2.4 5s = 4.7 5s = 5.2 5s
I. Adding stocks as integration I1. Add the two stocks 2.3 5s and 3.2 4s as 9s (integration). Recount the 2.3 5s in 9s Recount 3.2 4s in 9s Perform the addition	T = (2*5+3)/9 *9 = 1.4 *9 T = (3*4+2)/9 *9 = 1.5 *9 1.4 9s + 1.5 9s = 2.9 9s = 3.0 9s
J. Handling overloads J1. In 7.3 5s introduce a new cup to the left meant for bundles of bundles J2. Remove the overload in 9.5 8s, 7.3 4s and 45.2 3s	T =) 7) 3) = 1) 7-5) 3) = 1) 2) 3) = 12.3 5s
K. Multiplying and dividing with the bundle-size K1. Multiply 3.2 5s with 5 K2. Divide 14 5s with 5	T = 3.2 5s = 3)2)*5 = 3*5)2*5) = 3)2)0) = 32.0 5s T = 14 *5 = 1)4)0) = 1*5) 4*5) = 1)4) *5 = 1.4 5s *5
L. Solving equations L1. Solve the equation 2*x = 7 by rebundling L2. Solve the equation 2*x+1 = 7 by rebundling L3. Solve the equations by bundling and stacking	2*x = 7 = (7/2)*2, x = 7/2 2*x+1 = 7-1+1 = (7-1)/2*2 +1, so x = (7-1)/2

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302. Recounting as the Root of Grounded Mathematics

Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize?

The Background

The Enlightenment period treated mathematics as a natural science. Grounded in the natural fact Many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline, 1972). Using the concept set, modern mathematics turned Enlightenment mathematics upside down to a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox 'this statement is false' inspired Russell to formulate a paradox about the set of sets not belonging to itself: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise, without using self-reference it is impossible to prove that a proof is a proof, as shown by Gödel.

To avoid becoming metamatics, mathematics must return to its roots, the natural fact Many, guided by a contingency research looking for hidden alternatives to choices presented as nature.

Postmodern Contingency Research

Skepticism towards hidden patronization is at the root of postmodern thinking as formulated in Lyotard's definition 'Simplifying to the extreme, I define postmodern as incredulity toward metanarratives (Lyotard, 1984: xxiv).'

The Enlightenment century created two republics, an American and a French. The US still has its first republic using pragmatism and grounded theory to exert skepticism towards philosophical claims. The French now has its fifth republic being turned over repeatedly by the German neighbors, which has caused France to develop a skeptical thinking warning against hidden patronization (Tarp, 2004).

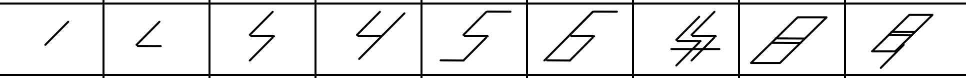
Derrida thus uses the terms 'logocentrism' and 'deconstruction' to warn against patronizing words installing instead of labeling what they describe. Lyotard uses the term 'postmodern' to warn against patronizing sentences stating political instead of natural correctness. Foucault uses the term 'pastoral power' to warn against patronizing institutions promising humans salvation from abnormalities installed by 'scientific' discourses. And Bourdieu uses the term 'symbolic violence' to label patronizing education that allows only the mandarin-class to acquire knowledge capital.

Based upon the French warning against hidden patronization, a research paradigm can be created called postmodern 'contingency research' deconstructing patronizing choices presented as nature by uncovering hidden alternatives.

To keep categories and discourses non-patronizing they should be grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the US enlighten-ment democracy; and resonating with Piaget's principles of natural learning (Piaget, 1970).

Recounting as the Root of Primary School Mathematics

To deal with the natural fact Many we iconize and bundle. 1.order counting rearranges sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for numbers until ten, the only number with a name, but without an icon.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

2.order counting bundles in icon-bundles. Thus a total of 7 can be bundled in 3s as T = 2 3s and 1; and placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III I -> III III) I) -> ▌▌) I) -> II) I) -> 2)1) -> 2.1 3s

Also bundles can be bundled and placed in a new cup to the left. Thus in 6 3s and 2 1s or 6)2) or 6.2 3s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II), or 6)2) = 2)0)2, or 6.2 3s = 20.2 3s.

Adding an extra cup to the right shows that multiplying or dividing with the bundle-size just moves the decimal point: T = 2.1 3s = 2)1) -> 2)1)) = 21.0 3s and vice versa.

Operations iconize the bundling and stacking processes. Taking away 4 is iconized as – 4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 showing a 3 times lifting of the 4s. Placing a stack of 2 singles next to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. a 1 3^2-bundle.

Now numbers and operations can be combined to calculations predicting the counting results. The ‘recount-formula’ T = (T/b)*b tells that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6)/7 7s. Using a calculator we get the result 2 7s and some leftovers that can be found by the ‘rest-formula’ T = (T–b) + b telling that T–b is what is left when b is taken away: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. So the combined prediction says T = 3*6 = 2*7 + 4*1 = 2.4 7s.

This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

So already in primary school, 2.order re-counting enable learners to predict and practice changing units, the leitmotif of mathematics reappearing later as proportionality and per-numbers. This also allows practicing the scientific method using formulas for predictions to be tested. In this ten-free zone it becomes possible to introduce the core of mathematics using 1digit numbers only (Zybartas et al, 2005). The CATS-approach, Count&Add in Time&Space, is one example of a grounded approach to mathematics as a natural science investigating the natural fact Many when counting by bundling&stacking, and when using double-counting at all school levels (Tarp, 2008).

3.order Counting in Tens Prevents Recounting

3.order counting in tens should be postponed as long as possible. Before introducing ten as 10, i.e. as the standard bundle-size, 5 should be the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is not used when using place values, or in the counting sequence: 1, 2, 3, 4, bundle, 1B1, 1B2, 1B3, 1B4, 2B, 2B1, etc.; or 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, etc.

Thus counting in tens can begin using neither a ten-icon nor the ten-name: 8, 9, bundle, 1bundle1, 1B2, 1B3, …, 1B9, 2B, 2B1, etc. Then the name bundle can be replaced by the name ten counting 8, 9, ten, 1ten1, 1T2, …, 1T9, 2T, 2T1, etc. Finally the sloppy words eleven and twelve can be used meaning ‘1 left’ and ‘2 left’ in ‘Anglish’, i.e. in old English.

A premature introduction of ten as THE standard bundle makes ten a ‘cognitive bomb’:

Numbers are no more written as natural numbers, i.e. as decimals carrying units. Instead numbers are written using the sloppy place-value method hiding the unit and misplacing the decimal point.

Almost all operations change meanings. Soft addition next-to as in T = 2.3 4s = 2*4 + 3*1 is changed to hard addition on-top as in 23 + 48 = 71. Soft multiplication where 3*8 means 3 8s is changed to hard multiplication, i.e. to division recounting the 3 8s in tens: 3*8 IS 24.

Now /4 now means divided in 4, not counted in 4s. Only –3 still means take away 3.

As shown by the recount- and rest-formulas, with 2.order counting the order of operations is: first /, then *, then –, and finally +. With 3.order counting in tens this order is turned around: first +, then –, then *, and finally /.

With ten as the only bundle-size, recounting is impossible to do and to predict by formulas since asking ‘3 8s = ? tens’ leads to T = (3*8/ten)*ten that cannot be calculated. Now the answer is given by multiplication, 3*8 = 24 = 2 tens + 4 ones, thus transforming multiplication into division.

Patronizing Versus Grounded Mathematics in Primary School

In primary school the tradition skips 1.order and 2.order counting and goes directly to 3.order counting claiming that 10 IS the follower of 9 in spite of the fact that 10 is the follower of 4 when counting in 5s. A grounded alternative postpones 3.order counting until after 2.order counting has introduced recounting in different units as an introduction to proportionality; and until after soft addition next-to has introduced integration.

The tradition presents one digit numbers as symbols and two digit numbers as natural numbers. A grounded alternative introduces one digit numbers as what they really are: icons rearranging the sticks they represent; and introduces the natural numbers as what they really are: decimal-numbers with units, e.g. 2.3 4s, and 2.3 tens instead of just 23.

The tradition presents hard addition on-top as the first of the four operations, where e.g. 7 + 4 = 11 forces the immediate introduction of ten as the bundle-size, and forces the sloppy way of writing 2digit numbers without decimals or units. A grounded alternative first introduces soft addition next-to so that 2 5s + 4 1s means placing a stack of 4 1s next to a stack of 2 5s, i.e. as 2.4 5s.

The tradition presents hard multiplication as the third operation with tables to be learned by heart, forcing all stacks to be recounted in tens, 3*8 IS 24 etc. A grounded alternative first introduces soft multiplication so that 3*8 means a stack of 3 8s, not needing to be recounted into tens before later.

The tradition presents division as the last of the four operations, where /4 means to split in 4. A grounded alternative introduces division as the first operation, where /4 means counting in 4s.

The tradition introduces 'mathematism' true in the library but not in the laboratory, by teaching that '2 + 3 IS 5' in spite of the fact that 2weeks + 3days = 17days, 2m + 3c = 203cm etc. A grounded alternative always includes the units when adding, e.g. 2 4s + 3 5s = 4.3 5s.

Conclusion

Postmodern contingency research has answered two questions: No, many mathematical concepts in primary school are not grounded in nature. And yes, ungrounded concepts can be replaced by concepts grounded in the root of mathematics, the natural fact Many. The question remaining is a choice to be made by the politicians: Is mathematics education meant to demonstrate how mathematical concepts are rooted in and able to predict the behavior of the natural fact Many?

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303. Calculus Grounded in Adding Per-numbers

Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? What are the roots of calculus?

The Background

The Enlightenment period treated mathematics as a natural science. Grounded in the natural fact Many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline, 1972). Using the concept set, modern mathematics turned Enlightenment mathematics upside down to a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox 'this statement is false' inspired Russell to formulate a paradox about the set of sets not belonging to itself: If $M = \{A \mid A \notin A)\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise, without using self-reference it is impossible to prove that a proof is a proof, as shown by Gödel.

To avoid becoming metamatics, mathematics must return to its roots, the natural fact Many, guided by a contingency research looking for hidden alternatives to choices presented as nature.

Postmodern Contingency Research

Skepticism towards hidden patronization is at the root of postmodern thinking as formulated in Lyotard's definition 'Simplifying to the extreme, I define postmodern as incredulity toward metanarratives (Lyotard, 1984: xxiv).'

In ancient Greece the need for patronization was debated between philosophers and sophists arguing that in a democracy the people must be enlightened to tell nature from choice to prevent patronization by choices presented as nature. To the philosophers choice didn't exist since the physical is examples of metaphysical forms, only accessible to philosophers educated at Plato's academy, thus obliged to patronize ordinary people (Russell, 1945). Later the academy was transformed into Christian monasteries, again changed into academies after the Reformation.

Brahe, Kepler and Newton rebelled against the patronization of clerical knowledge by bringing the authority from the library back to the laboratory. Based upon Brahe's observations, Kepler formulated a heliocentric hypothesis that could not be tested before Newton showed that universal gravitation makes both moons and apples fall to the earth following their own calculable will. When apples follow their own will instead of that of the patronizor, humans could do the same and replace autocratic patronization with enlightenment-based democracy.

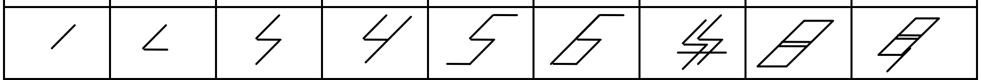
Thus the Enlightenment century created two republics, an American and a French. The US still has its first republic using pragmatism and grounded theory to exert skepticism towards philosophical claims. The French now has its fifth republic being turned over repeatedly by the German neighbors, which has caused France to develop a skeptical thinking warning against hidden patronization (Tarp, 2004).

Derrida thus uses the terms 'logocentrism' and 'deconstruction' to warn against patronizing words installing instead of labeling what they describe. Lyotard uses the term 'postmodern' to warn against patronizing sentences stating political instead of natural correctness. Foucault uses the term 'pastoral power' to warn against patronizing institutions promising humans salvation from abnormalities installed by 'scientific' discourses. And Bourdieu uses the term 'symbolic violence' to label patronizing education that allows only the mandarin-class to acquire knowledge capital. In short, postmodern thinking means skepticism towards hidden patronization of ungrounded words, sentences and institutions. In the case of mathematics this means skepticism towards mathematical concepts, statements and education.

Based upon the sophist and French warning against hidden patronization, a research paradigm can be created called postmodern 'contingency research' deconstructing patronizing choices presented as nature by uncovering hidden alternatives. To keep categories, discourses and institutions non-patronizing they should be grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the US enlightenment democracy; and resonating with Piaget's principles of natural learning (Piaget, 1970). Confronted with a tradition wanting mathematics to provide learners with well-proven knowledge about well-defined concepts applicable to the outside world (e.g. NCTM, 2000), contingency research instead looks for grounded concepts and verifiable statements that will enlighten the learner about the roots of mathematics, the natural fact Many.

Recounting as the Root of Primary School Mathematics

To deal with the natural fact Many we iconize and bundle. 1.order counting rearranges sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for numbers until ten, the only number with a name, but without an icon.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

2.order counting bundles in icon-bundles and 3.order counting bundles in tens, the only number with its own name but without an icon.

With 2.order counting a total of 7 can be bundled in 3s as T = 2 3s and 1, and placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III I -> III III) I) -> ▌▌) I) -> II) I) -> 2)1) -> 2.1 3s

The counting result can be predicted by a 'recount-formula' T = (T/b)*b telling that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6/7) 7s. Using a calculator we get the result 2 7s and some leftovers that can be found by the 'rest-formula' T = (T–b) + b telling that T–b is what is left when b is taken away: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. So the combined prediction says T = 3*6 = 2*7 + 4*1 = 2.4 7s.

This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

Once counted, totals can be added. However, two kinds of addition exist: Soft addition next-to and hard addition on-top.

Soft addition next-to adds T1 = 2 3s and T2 = 3 4s into T = 2.4 7s. Thus the totals are added by juxtaposing their stacks, i.e. by adding the areas of the stacks. In this way soft addition nex-to is the root of integration also using areas when adding

To perform hard addition on-top, the units must be the same, so the 3s must be recounted in 4s or vice versa. With T1 = 2 3s = 1.2 4s, T1 + T2 = 4.2 4s. And with T2 = 3 4s = 4 3s, T1 + T2 = 6 3s. In this way recounting and changing units becomes the root of proportionality.

With 3.order counting in tens only hard addition on-top is possible. Thus skipping 2.order counting means also skipping the roots of calculus and proportionality

Per-numbers as the Natural Roots of Middle School Mathematics

Also in middle school the root of mathematics is recounting the natural fact Many, now occurring different places in the outside world, e.g. in time and space and in economy with units as seconds, minutes, cm, m, m^2, m^3, liters, $, £ etc.

Recounting creates fractions when e.g. counting 3 1s in 5s as T = 3 = 0.3 5s = (3/5)*5, which shows that 3/b = 0.3 for any bundle-size b; and 'per-numbers' when a quantity is double-counted e.g. both as 2$ and as 5kg thus containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg.

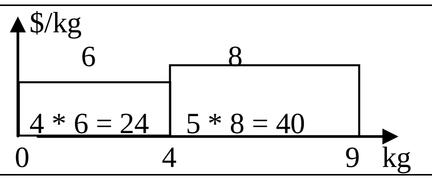
Again the recount-formula predicts recounting results when asking '6$ = ?kg', or '14kg = ?$:

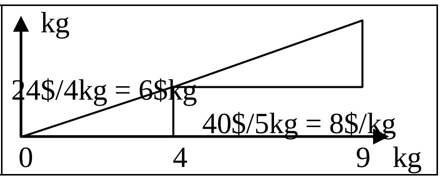
T = 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg; and T = 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$,

Or kg = kg/$ * $ = 5/2 * 6 = 15, and $ = $/kg * kg = 2/5 * 14 = 5.6.

In primary school soft addition next-to adds stacks by integrating bundles, e.g. 2 3s + 4 5s = ? 8s. Now integration adds per-numbers when adding two double-counted quantities, asking e.g. 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg. This question can be answered by using a table or a graph.

4kg at 6$/kg	= 4 * 6 = 24
5kg at 8$/kg	= 5 * 8 = 40
9kg at ?$/kg	= 9 * ? = 64
	? = 64/9





Using a graph we see that integration means finding the area under a per-number graph; and opposite that the per-number is found as the gradient on the total-graph.

Change as the Natural Roots of High School Mathematics

In high school, the natural root of mathematics is recounting change: Being related by a formula y = f(x), how will a change in x, Δx, affect the change of y, Δy? Here the recount-equation gives the change-formula Δy = (Δy/Δx)*Δx, or dy = (dy/dx)*dx = y'dx for small micro-changes.

In trade, when the volume increases from 0 to x kg, the initial cost b increases to the final cost y = b + a*x if the cost increases a$/kg. This is called linear change or ++ change. Here Δy/Δx = a.

In a bank, when the years increase from 0 to x, the initial capital b increases to the final capital y = b* (1+r)^x if the capital increases with r %/year. This is called exponential change or +* change since we add 7% by multiplying with 107% = 1 + 7%. Here Δy/Δx = r*y.

In geometry, when the side-length is 3-doubled from 2 to 6, the area of a square y = x^2 is 3-doubled twice from 4 to 36. This is called potential change or ** change, in general given as y = b*x^a. Here Δy/y = a*Δx/x., where a is called the elasticity.

The x-change □x, might be a micro-change, dx. On a calculator we observe that, approximately, 1.001^5 = 1.005, 1.001^9 = 1.009, √1.001 = 1.0005 and 1.001^-3 = 0.997. From this a hypothesis can be made saying that, approximately, (1+dx)^n = 1 + ndx, allowing numerous predictions to be tested as 1.0002^4 = 1.0008 etc., all being verified by the calculator. If y = x^n, then (x + dx)^n = (x(1+dx/x))^n = x^n(1+ndx/x) = x^n + ndx*x^n/n = y + ndx*x^(n-1) = y+dy, so dy/dx = n*x^(n-1). With micro-changes dx, the area under a graph f(x) can be split into micro-strips with area height *width = f*dx. The total area from a to b is then found by summing up the micro-strips, written as

$$A = \int_a^b f\,dx = \int_a^b dF = F(b) - F(a)$$

if f dx can be written as a micro-change dF, since summing up single changes gives a total change = terminal number – initial number, no matter their sizes.

Technology helps solve any change-equation dF = f dx by calculating the change dF that added to the initial F-value gives the terminal F-value, becoming the initial F-value in the next period.

Conclusion

The tradition presents calculus as an example of a limit process, thus introducing limits and continuity before the derivative. A grounded alternative generalizes primary school's soft addition of stacks in combined bundle-sizes, and middle school's adding fractions with units, to adding per-numbers with units; and introduces the terms continuous and differentiable as what they really are: foreign words for locally constant and locally linear in contrast to piecewise constant and piecewise linear.

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304. Saving Dropout Ryan With A Ti-82

To lower the dropout rate in pre-calculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' replaced the textbook. A formula's left and right hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects.

THE TASK: Reduce The Droput Rate!

The headmaster asked the mathematics teachers: "We have too many pre-calculus dropouts. What can we do?" I proposed buying the cheap TI-82 graphical calculator, but the other teachers rejected this proposal arguing that students weren't even able to use a simple TI-30. Still I was allowed to buy this calculator for my class allowing me to replace the textbook with a compendium emphasizing modeling with TI-82.

Concepts: Examples Of Abstractions Or Vice Versa

Enlightenment mathematics was as a natural science exploring the natural fact Many (Kline, 1972) by grounding its abstract concepts in examples, and by using the lack of falsifying examples to validate its theory. But after abstracting the set-concept, mathematics was turned upside down to modern mathematics or 'metamatism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathematism' true in the library, but not in the laboratory, as e.g. 2+3 = 5, which has countless counterexamples: 2m+3cm = 203 cm, 2weeks+3days = 17 days etc. Being self-referring, this modern mathematics did not need an outside world. However, a self-referring mathematics turned out to be a self-contradiction. With his paradox on the set of sets not belonging to itself, Russell proved that sets implies self-reference and self-contradiction as known from the classical liar-paradox 'this statement is false' being false when true and true when false: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise Gödel proved that a well-proven theory is a dream since it will always contain statements that can be neither proved nor disproved.

In spite of being neither well-defined nor well-proven, mathematics still teaches metamatism. This creates big problems to mathematics education as shown e.g. by 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

Contingency Research Unmasks Choices Presented As Nature

Alternatively, mathematics could return to its roots, Many, guided by contingency research uncovering hidden patronization by discovering alternatives to choices presented as nature.

Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists warned that to practice democracy, the people must be enlightened to tell choice from nature in order to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since to them all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should be given the role as natural patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment: when an apple obeys its own will, people could do the same and replace patronization with democracy.

Two democracies were installed: one in the US still having its first republic; and one in France, now having its fifth republic. German autocracy tried to stop the French democracy by sending in an army. However, a German mercenary was no match to a French conscript aware of the feudal consequence of defeat. So the French stopped the Germans and later occupied Germany. Unable to use the army, the German autocracy instead used the school to stop enlightenment in spreading from France. As counter-enlightenment, Humboldt used Hegel philosophy to create a patronizing line-organized Bildung school system based upon three principles: To avoid democracy, the people must not be enlightened; instead romanticism should install nationalism so the people sees itself as a ‘nation’ willing to fight other ‘nations’, especially the democratic ones; and the population elite should be extracted and receive ‘Bildung’ to become a knowledge-nobility for a new strong central administration replacing the former blood-nobility unable to stop the French democracy.

As democracies, EU still holds on to line-organized education instead of changing to block-organized education as in the North American republics allowing young students to uncover and develop their personal talent through individually chosen half-year knowledge blocks.

In France, the sophist warning against hidden patronization is kept alive in the post-structural thinking of Derrida, Lyotard, Foucault and Bourdieu. Derrida warns against ungrounded words installing what they label, such word should be ‘deconstructed’ into labels. Lyotard warns against ungrounded sentences installing political instead of natural correctness. Foucault warns against institutionalized disciplines claiming to express knowledge about humans; instead they install order by disciplining both themselves and their subject. And Bourdieu warns against using education as symbolic violence to monopolize the knowledge capital for a knowledge-nobility (Tarp, 2004).

Thus contingency research does not refer to, but questions existing research by asking ‘Is this nature or choice presented as nature?’ To prevent patronization, categories should be grounded in nature using Grounded Theory (Glaser et al, 1967), the method of natural research developed in the first Enlightenment democracy, the American, and resonating with Piaget’s principles of natural learning (Piaget, 1970).

The Case Of Teaching Math Dropouts

Being our language about quantities, mathematics is a core part of education in both primary and secondary education. Most parents accept the importance of learning mathematics, but many students fail to see the meaning in doing so. Consequently special core math courses for dropouts are developed.

Traditions Of Core Precalculus Courses For Dropouts

A typical core course for math dropouts is halving the content and doubling the text volume. So in a slow pace the students work their way through a textbook once more presenting mathematics as a subject about numbers, operations, equations and functions applied to space, time, mass and money. To prevent spending time on basic arithmetic, a TI-30 calculator is handed out without instruction.

As to numbers, the tradition focuses on fractions and how to add fractions.

Then solving equations is introduced using the traditional balancing method isolating the unknown by performing identical operations to both sides of the equation. Typically, the unknown occur on both sides of the equation as $2x + 3 = 4x – 5$; or in fractions as $5 = 40/x$.

Then relations between variables are introduced using tables, graphs and functions with emphasis on the linear function $y = f(x) = b+a*x$. In a traditional curriculum, a linear function is followed by the quadratic function. But a core course might instead go on to the exponential function $y = b*a^x$. To avoid solving its equations the solutions are given as formulas.

Problems In Traditional Core Courses

The intention of a traditional core course is to give a second chance to learners having dropped out op the traditional math course. However, from a skeptical viewpoint trying to avoid hidden patronization by presenting choice as nature, several questions can be raised.

As to numbers, are fractions numbers or calculations that can be expressed with as many decimals as we want, typical asking for three significant figures? Is it meaningful to add fractions without units as shown by the fraction-paradox above?

As to equations, is the balancing method nature or choice presented as nature? The number x = 8-3 is defined as the number x that added with 3 gives 8, x + 3 = 8. This can be restated as saying that the equation x + 3 = 8 has the solution x = 8-3, suggesting that the natural way to solve equations is the ‘move to opposite side with opposite sign’ method. This method can be applied to all cases of reversed calculation:

x + 3 = 15	x*3 = 15	x^3 = 125	3^x = 243
x = 15 – 3	x = 15/3	x = 3√125	x = log3(243)

Figure 1. Equations solved by the ‘opposite side & sign’ method of inverse calculation

Defining a function as an example of a set-relation, is that nature or choice presented as nature? In a formula as y = a+b all numbers might be known. If one number is unknown we have an equation to be solved, e.g. 5 = 3+x, if not already solved, x = 3+5. With two unknown numbers we have a function as in y = 3+x, or a relation as in 3 = x+y that can be changed into the function y = 3–x. So a function is just a label for a formula with two unknown variables.

Giving solution formulas to the exponential equations x^3 = 125 and 3^x = 243, is that nature or choice presented as nature? Since equation is just another name for inverse calculation, using the inverse operations root and log is the natural way to solve exponential equations.

The prime goal of education is that learners adapt to the outside world by proper actions. An action as ‘Peter eats apples’ is a three-term sentence with a subject, a verb and an object. Thus mathematics education should be described in this way. The learner is the subject, the object is the natural fact Many, and the verb is how we deal with Many: we totalize expressing the total as a formula, e.g. T = 345 = 3*10^2 + 4*10 + 5*1 showing that totalizing means counting and adding bundles, that all numbers carry units, and that there are four ways to add: +, *, ^ and integration adding unlike and like unit-numbers and like and unlike per-numbers.

Totalizing can also be called algebra if using the Arabic word for reuniting. Not being a verb, mathematics could be renamed to ‘totalizing’, ‘counting and adding’ or reckoning.

Desiging A Grounded Math Core Course

Real word problems translate to formulas by modeling, or triangulation in the case of forms. So, to adapt to the outside world, mathematics education has as its prime goal that persons learn how to totalize, i.e. how to count and add, how to model and how to triangulate.

A traditional core course seems to be filled with examples of choices presented as nature. This leads to the question: is it possible to design an alternative core course based upon nature instead of choices presented as nature? In other words, what would be the content of a core course in pre-calculus if grounded in the roots of mathematics, the natural fact Many?

Mathematics As A Number-Language Using Predicting Formulas

As to the nature of the subject itself, mathematics is a number-language that together with the word-language allows users to describe quantities and qualities in everyday life. Thus a calculator is a typewriter using numbers instead of letters. A typewriter combines letters to words and sentences. A calculator combines digits to numbers that combined with operations become formulas. Thus formulas are the sentences of the number-language.

A difference between the word-language and the number-language is that sentences describe whereas formulas predict the four different ways of uniting numbers:

Addition predicts the result of uniting unlike unit-numbers: uniting 4$ and 5$ gives a total that is predicted by the formula T = a+b = 4+5 = 9

Multiplication predicts the result of uniting like unit-numbers: uniting 4$ 5 times gives a total that is predicted by the formula T = a*b = 4*5 = 20.

Power predicts the result of uniting like per-numbers: uniting 4% 5 times gives a total that is predicted by the formula 1+T = a^b = 1.04^5 = 1.217, i.e. T = 0.217 = 21.7%.

Integration predicts the result of uniting unlike per-numbers: uniting 2kg at 7$/kg and 3kg at 8$/kg gives 5 kg at T$/5kg where T is the area under the $/kg per-number graph, T = Σ p*Δx.

Solving Equations With A Solver

As shown above, inverse operations solve equations, as do the TI-82 using a solver. Equations as 2+x = 6 always has a left hand side and a right hand side that can be entered on the calculators Y-list as Y1 and Y2. So any equation has the form Y1 = Y2, or Y1 – Y2 = 0 that only has to be entered to the solver once. After that, solving equations just means entering its two sides as Y1 and Y2. Using graphs, Y1 and Y2 becomes two curves having the same values at intersection points.

If one of the numbers in a calculation is unknown, then so is the result. A table describes a formula with two unknowns by answering the question ‘if x is this, then what is y?’ Graphing a table allows the inverse question to be addressed by reading from the y-axis.

Producing Formulas With Regression

Once a formula is known, it produces answers by being solved or graphed. Real world data often come as tables, so to model real world problems we need to be able to set up formulas from tables. Simple formulas describe levels as e.g. cost = price*volume. Calculus formulas describe predictable change where pre-calculus describes constant change.

If a variable y begins with the value b and x times changes by a number a, then y = b+a*x. This is called linear change and occurs in everyday trade and in interest-free saving.

If y begins with the value b and x times changes by a percentage r, then y = b*(1+r)^x = b*a^x since adding 5% means multiplying with 105% = 100%+5%= 1.05. This is called exponential change and occurs when saving money in a bank and when populations grow or decay.

In the case of linear and exponential change, a two-line table allows regression on a TI-82 to find the two constants b and a.

Multi-line tables can be modeled with polynomials, the formula used to describe numbers. Thus a 3-line table might be modeled by a degree 2 polynomial y = b + a*x + c*x^2 including also a bending-number c; and a 4-line table by a degree 3 polynomial y = b + a*x + c*x^2 + d*x^3 including also a counter-bending number d, etc.

Graphically, a degree 2 polynomial is a bending line, a parabola; and a degree 3 polynomial is a double parabola. The top and the bottom of a bending line as well as its zeros can be found directly by graphical methods of the TI-82.

Fractions as Per-numbers

Fractions are rooted in per-numbers: 3$ per 5 kg = 3$/5kg = 3/5 $/kg. To add per-numbers they first must be changed to unit numbers by being multiplied with their units:

3 kg at 4 $/kg + 5 kg at 6 $/kg = 8 kg at (3*4 + 5*6)/8 $/kg

Geometrically this means that the areas under their graph add per-numbers.

Her again TI-82 comes in handy when calculating areas under graphs. Areas can be calculated also in the case where the graph is not horizontal but a bending line, representing the case when the per-number is changing continuously as e.g. in a falling body: 3 seconds at 4 m/s increasing to 6 m/s totals 15 m because of a constant acceleration.

Models as Quantitative Literature

Using TI-82 as a quantitative typewriter able to set up formulas from tables and to answer both x- and y-questions, it becomes possible to include models as quantitative literature.

All models share the same structure: A real world problem is translated into a mathematical problem that is solved and translated back into a real world solution to be evaluated.

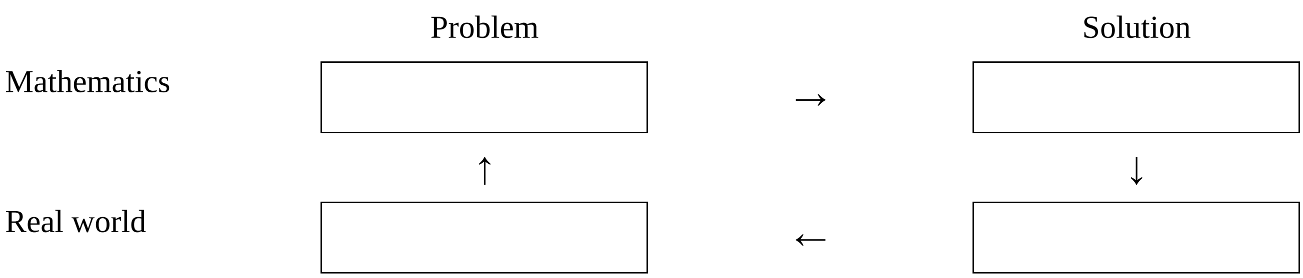


Figure 2. The four phases of mathematical modeling

The project ‘Population versus food’ looks at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2-line table regression can produce two formulas: with x counting years after 1850, the population is modeled by Y1= 815*1.013^x and the food by Y2= 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

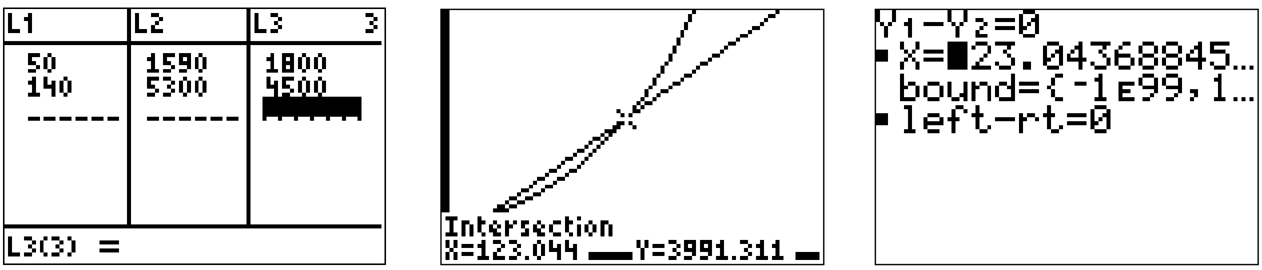


Figure 3. A Malthusian model of population and food levels

The project ‘Fundraising’ finds the revenue of a fundraising assuming all students will accept a free ticket, that 100 students will buy a 20$ ticket and that no one will buy a 40$ ticket. From this 3-line table the demand is modeled by a degree 2 polynomial Y1 = .375*x^2 – 27.5*x + 500. Thus the revenue formula is the product of the price and the demand, Y2 = x*Y1. Graphical methods shows that the maximum revenue will be 2688 $ at a ticket price of 12$.

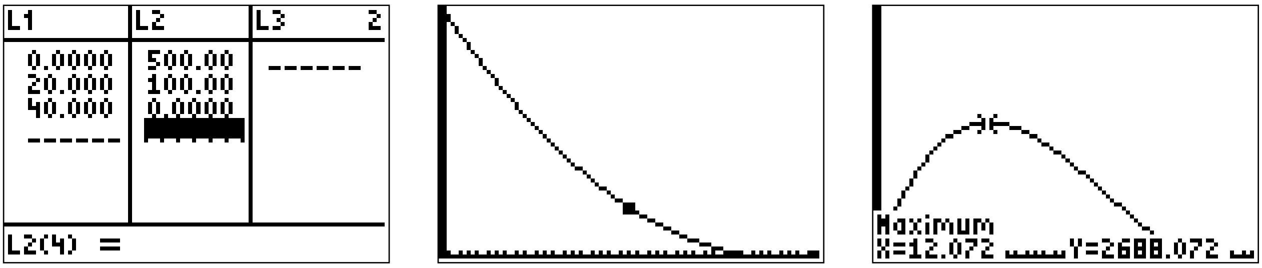


Figure 4. Modeling the optimal ticket price in a fundraising

In the project ‘Driving with Peter’ his velocity is measured five times. A model can answer many questions, e.g. when was Peter accelerating? And what distance did Peter travel in a given time interval? From a 5-line table the speed can be modeled by the degree 4 polynomial Y1 = -0.009x^4 + 0.53x^3 − 10.875x^2 + 91.25x − 235. Visually, the triple parabola fits the data points. Graphical methods shows that a minimum speed is attained after 14.2 seconds; and that Peter traveled 115 meters from the 10 th to the 15 th second.

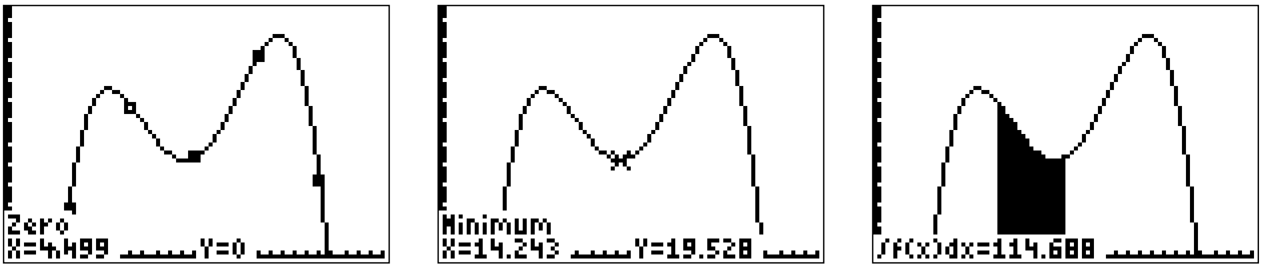


Figure 5. Modeling a car in motion

Six other projects were included in the course. The project ‘Forecasts’ modeled a capital growing constantly in three different ways: linear, exponential and potential. The project ‘Determining a Distance’ uses trigonometry to predict the distance to an inaccessible point on the other side of a river. The project ‘The Bridge’ uses trigonometry to predict the dimensions of a simple expansion bridge over a canyon. The project ‘Playing Golf’ predicts the formula for the orbit of a ball that has to pass three given points: a staring point, an ending point and the top of a hedge. The project ‘Saving and Pension’ predicts the size of a ten years monthly pension created by a thirty years monthly payment of 1000$ at an interest rate of 0.4% per month. And the project ‘The Takeover Attempt’ predicts how much company A has spent buying stocks in company B given an oscillating course described by a 4-line table.

Introducing The Three Genres Of Quantitative Litterature

Qualitative and quantitative literature has three genres: fact, fiction and fiddle (Tarp, 2001).

Fact models quantify and predict predictable quantities, as e.g. ‘What is the area of the walls in this room?’ In this case the calculated answer of the model is what is observed. Hence calculated values from a fact models can be trusted. A fact model can also be called a since- then model or a room-model. Most models from basic science and economy are fact models.

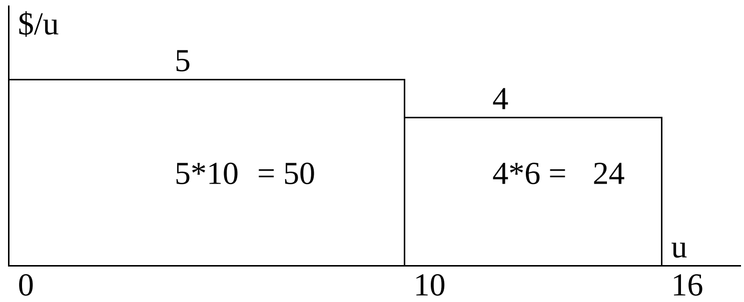
Fiction models quantify and predict unpredictable quantities, as e.g. ‘My debt will soon be paid off at this rate!’ Fiction models are based upon assumptions and its solutions should be supplemented with predictions based upon alternative assumptions or scenarios. A fiction model can also be called an if-then model or a rate-model. Models from basic economy assuming variables to be constant or predictable by a linear formula are fiction models.

Fiddle models quantify qualities: ‘Is the risk of this road high enough to cost a bridge?’ Many risk-models are fiddle models. The basic risk model says: Risk = Consequence * probability. Statistics can provide probabilities for casualties, but if casualties are quantified, it is much cheaper to stay in a cemetery than in a hospital, pointing to the solution ‘no bridge’. Fiddle models should be rejected asking for a word description instead of a number description.

Introducing The Core Of Calculus As Adding Per-Numbers

As an introduction to calculus the students looked at discounts: The price 5 $/u goes down to 4 $/u when buying more than 10 units, what is the price when buying 16 units?

10 units at 5 $/u gives 10*5 = 50 $
6 units at 4 $/u gives 6*4 = 24 $
16 units at 9 $/u gives 16*9 = 74 $
?? How do we add per-numbers??



The problem is that where unit-numbers are added directly, per-numbers are added as areas under the per-number graph, i.e. as Σ p*Δx, written by TI.82 as ∫ p dx.

So if a per-number p is constant, the total cost T for buying 5 units is T = p*5. And if the per-number p is a formula, the total cost T for buying 5 units is the area under the p-graph.

Completing The Algebra Project

Seeing the use of integration as adding per-numbers, the students enjoyed having completed the reuniting project of algebra since now they were able to add all four number types:

	Unlike	Like
Unit-numbers *m, s, kg, $*	**T = a + b** *T – b = a*	**T = a*b** *T/b = a*
Per-numbers *m/s, $/kg, $/(100$) = %*	**T = ∫ p dx** *dT/dx = p*	**T = a^b** *b√T = a* *loga(T) = b*

students asking for proofs

To stop fellow students mocking them by saying that the class was not on mathematics but on reckoning, the students asked for sophisticated proofs. Four were sufficient.

Depositing n times the interest r*(a/r) = a$ of a/r $ to an account makes this a saving account containing the total interest A = R*(a/r). Consequently A/a = R/r where 1+R = (1+r)^n.

In a triangle ABC with no angle above 90, the outside squares of the sides are divided in rectangles by the heights. Projections show that the two rectangles containing C have the same area, a*b*cosC. This gives c^2 = a^2 + b^2 – 2*a*b*cosC, or c^2 = a^2 + b^2 if C is 90.

Adding 100% interest in n parts n times gives the growth multiplier m = (1+1/n)^n. For n sufficiently big, m = 2.7182818 = e. Thus compounding 100% can add at most 71.8%.

Inscribing a symmetrical n-radius star in a unit circle gives n intersection points, from which tangents create a polygon with circumference c = 2*n*tan(180/n). For n big, c gets close to that of the unit-circle, 2*π. Hence π = n*tan(180/n) = 3.14159 for n sufficiently big.

Testing The Core Course

The students expressed surprise and content with the course. Their hand-ins were on time. And the course finished before time giving room for additional models from classical physics: vertical falling balls, projectile orbits, colliding balls, circular motion, pendulums, gravity points, drying wasted whisky with ice cubes and supplying bulbs with energy.

At the written and the oral exam, for the first time at the school, all the students passed. Some students wanted to move on to a calculus class, other were reluctant arguing that they had already learned the core of calculus so they didn't need an extra class to study engineering.

Reporting Back To The Headmaster

The headmaster expressed satisfaction, but the teachers didn't like setting aside the textbook and its traditional mathematics. To encourage the teachers, the headmaster ordered the TI-82 to be bought for all pre-calculus classes.

Discussing The Result

Discussing the result with teachers, an argument was that by not following the textbook, the students don't learn mathematics, only reckoning. My answer was that in the first place, the textbook does not teach mathematics but metamatism, so what dropouts reject is metamatism, not mathematics. Furthermore, teachers should respect the Nuremberg sentences from 1946: You can't just follow orders, you must evaluate the consequenses to your clients. Your order is to teach mathematics, but it must be useful and not harmful to the learners. So skepticism should reformulate the orders, e.g. to: the goal of mathematics education is to adapt learners to the natural fact Many by totalizing developing basic competences in how to count, add, model and triangulate; and how to cooperate with calculation technology able to perform both forward and backward calculation and to illustrate calculations with tables and graphs. Another argument was that by using technology, the students would not understand mathematics. My answer was that world problems can't be translated into formulas without understanding how the different operations are defined and used for forward and backward calculation. However, just as multiplication is useful to speed up addition, a graphical calculator should be allowed to speed up modeling so the human brain can be used to formulate questions and evaluate answers.

Discussing the result with researchers, a typical argument was that this work does not build on traditional theory in mathematics education research, e.g. theory about concept formation. My answer was that in most cases research is describing education in metamatism, not in mathematics. And in many cases research produces political instead of natural correctness as shown by the 'pencil paradox': Placed between a ruler and a dictionary, a '17 cm long pencil' can point to '15', but not to 'knife', so being itself able to falsify its number but not its word means that numbers and words produce natural and political correctness respectively. Only contingency research can produce natural correctness by uncovering hidden alternatives to choices presented as nature. Consequently, contingency research is very effective as action research assisting an actor in a field to implement change as in this case. But contingency research is banned from discourse protecting EU universities as predicted by Lyotard (1984).

Conclusion: How To Make Losers Users

In order to give dropout students in mathematics an extra chance it has good meaning to create a course boiling the mathematics content down to its core. However, to be successful, the core should be grounded in its roots, the natural fact Many. Numbers should be presented as polynomials to show the four operations uniting numbers according to Algebra's reuniting project. Also direct and inverse operations should be presented as means to predict the united total and its parts. In this way the core of basic algebra becomes solving equations with the move and change sign method, or with the solver of a graphical calculator. And the core of pre-calculus becomes regression, enabling tables to be translated to formulas that can be processed when entered into the y-list of the TI82. Thus grounding mathematics in its natural roots and including a graphical calculator provides ordinary students with a typewriter that can be used to model and predict the behavior of real world quantities (Tarp 2009). In this way a TI-82 develops competences in human-technology cooperation: Humans get the data and the questions, and technology provides the answers. Traditional metamatism makes losers of Ryan and other boys. Replacing metamatism with grounded mathematics and a TI-82 will not only save them, it will also install in them a confidence and a belief that they can become successful engineers cooperating with technology instead of being math dropouts. Thus learning technology based grounded mathematics in an educational system that is transformed from being line-organized to block-organized will transform boys from being dropouts to engineers, which again will help the EU solve its present economical crisis.

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305. Contingency Research Uncovers the Roots of Grounded Mathematics

Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize?

The Background

The Enlightenment period treated mathematics as a natural science. Grounded in the natural fact Many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline, 1972). Using the concept set, modern mathematics turned Enlightenment mathematics upside down to a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox 'this statement is false' inspired Russell to formulate a paradox about the set of sets not belonging to itself: Definition $M = \{A \mid A \notin A)\}$, statement $M \in M \Leftrightarrow M \notin M$. Likewise, without using self-reference it is impossible to prove that a proof is a proof, as shown by Gödel.

To avoid becoming metamatics, mathematics must return to its roots, the natural fact Many, guided by a contingency research looking for hidden alternatives to choices presented as nature.

Postmodern Contingency Research

Skepticism towards hidden patronization is at the root of postmodern thinking as formulated in Lyotard's definition 'Simplifying to the extreme, I define postmodern as incredulity toward metanarratives (Lyotard, 1984: xxiv).'

In ancient Greece the need for patronization was debated between philosophers and sophists arguing that in a democracy the people must be enlightened to tell nature from choice to prevent patronization by choices presented as nature. To the philosophers choice didn't exist since the physical is examples of metaphysical forms, only accessible to philosophers educated at Plato's academy, thus obliged to patronize ordinary people (Russell, 1945). Later the academy was transformed to Christian monasteries, again changed into academies after the Reformation.

Brahe, Kepler and Newton rebelled against the patronization of clerical knowledge by bringing the authority from the library back to the laboratory. Based upon Brahe's observations, Kepler formulated a heliocentric hypothesis that could not be tested before Newton showed that universal gravitation makes both moons and apples fall to the earth following their own calculable will. When apples follow their own will instead of that of the patronizors, humans could do the same and autocratic patronization with enlightenment-based democracy.

Thus the Enlightenment century created two republics, an American and a French. The US still has its first republic using pragmatism and grounded theory to exert skepticism towards philosophical claims. The French now has its fifth republic being turned over repeatedly by the German neighbors, which has caused France to develop a skeptical thinking warning against hidden patronization (Tarp, 2004).

Derrida thus uses the terms 'logocentrism' and 'deconstruction' to warn against patronizing words installing instead of labeling what they describe. Lyotard uses the term 'postmodern' to warn against patronizing sentences stating political instead of natural correctness. Foucault uses the term 'pastoral power' to warn against patronizing institutions promising humans salvation from abnormalities installed by ‘scientific’ discourses. And Bourdieu uses the term ‘symbolic violence’ to label patronizing education that allows only the mandarin-class to acquire knowledge capital.

The French skepticism towards words, our most fundamental institution, is validated by a ‘number&word observation’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘stick’; thus it can itself falsify its number but not its word, meaning that numbers and words express natural and political correctness respectively.

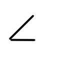
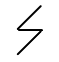
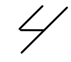
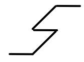
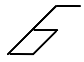
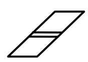
In short, postmodern thinking means skepticism towards hidden patronization of ungrounded words, sentences and institutions. In the case of mathematics this means skepticism towards mathematical concepts, statements and education.

Based upon the sophist and French warning against hidden patronization, a research paradigm can be created called postmodern ‘contingency research’ deconstructing patronizing choices presented as nature by uncovering hidden alternatives. Thus contingency research doesn’t refer to but deconstruct existing research by asking ‘is this nature or choice that presented as nature installs hidden patronization covering alternatives to be uncovered by contingency research?’

To keep categories, discourses and institutions non-patronizing they should be grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the US enlightenment democracy; and resonating with Piaget’s principles of natural learning (Piaget, 1970). Confronted with a tradition wanting mathematics to provide learners with well-proven knowledge about well-defined concepts applicable to the outside world (e.g. NCTM, 2000), contingency research instead looks for grounded concepts and verifiable statements that will enlighten the learner about the roots of mathematics, the natural fact Many.

Recounting as the Root of Primary School Mathematics

To deal with the natural fact Many we bundle. 1.order counting bundles in 1s by rearranging sticks to form an icon. Thus the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for numbers until ten, the only number with a name, but without an icon.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

2.order counting bundles in icon-bundles. First we bundle the total T in e.g. 3-bundles, in 3s. Then we stack T in two stacks: a stack of 3s, and a stack of unbundled singles. The stacks may then be placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

Also bundles can be bundled and placed in a new cup to the left. Thus in 6 3s and 2 1s or 6)2) or 6.2 3s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II), or 6)2) = 2)0)2, or 6.2 3s = 20.2 3s.

Adding an extra cup to the right shows that multiplying or dividing with the bundle-size just moves the decimal point: T = 2.1 3s = 2)1) -> 2)1)) = 21.0 3s and vice versa.

Operations iconize the bundling&stacking processes. Taking away 4 is iconized as – 4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 showing a 3 times lifting of the 4s. Placing a stack of 2 singles next to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. a 1 3^2-bundle.

Now numbers and operations can be combined to calculations predicting the counting results. The 'recount-formula' T = (T/b)*b tells that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6/7) 7s. Using a calculator we get the result 2 7s and some leftovers that can be found by the 'rest-formula' T = (T–b) + b telling that T–b is what is left when b is taken away: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. So the combined prediction says T = 3*6 = 2*7 + 4*1 = 2.4 7s.

This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

So already in primary school, 2.order re-counting enable learners to predict and practice changing units, the leitmotif of mathematics reappearing later as proportionality and per-numbers. This also allows practicing the scientific method using formulas for predictions to be tested. In this ten-free zone it becomes possible to introduce the core of mathematics using 1digit numbers only (Zybartas et al, 2005). The CATS-approach, Count&Add in Time&Space, is one example of a grounded approach to mathematics as a natural science investigating the natural fact Many when counting by bundling&stacking, and when using double-counting at all school levels (Tarp, 2008).

3.order Counting in Tens Prevents Recounting

3.order counting in tens should be postponed as long as possible. Before introducing ten as 10, i.e. as the standard bundle-size, 5 should be the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is not used when using place values, or in the counting sequence: 1, 2, 3, 4, bundle, 1B1, 1B2, 1B3, 1B4, 2B, 2B1, etc.; or 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, etc.

Thus counting in tens can begin using neither a ten-icon nor the ten-name: 8, 9, bundle, 1bundle1, 1B2, 1B3, …, 1B9, 2B, 2B1, etc. Then the name bundle can be replaced by the name ten counting 8, 9, ten, 1ten1, 1T2, …, 1T9, 2T, 2T1, etc. Finally the sloppy way eleven and twelve can be used meaning '1 left' and '2 left' in 'Anglish', i.e. in old English.

A premature introduction of ten as THE standard bundle makes ten a 'cognitive bomb':

Numbers are no more written as natural numbers, i.e. as decimals carrying units. Instead numbers are written using the sloppy place-value method hiding both the decimal and the unit.

Almost all operations change meanings. Soft addition next-to as in T = 2.3 4s = 2*4 + 3*1 is changed to hard addition on-top as in 23 + 48 = 71. Soft multiplication where 3*8 means 3 8s is changed to hard multiplication, i.e. to division recounting the 3 8s in tens: 3*8 IS 24.

Now /4 now means divided in 4, not counted in 4s. Only –3 still means take away 3.

As shown by the recount- and rest-formulas, with 2.order counting the order of operations is: first /, then *, then –, and finally +. With 3.order counting in tens this order is turned around: first +, then –, then *, and finally /.

With ten as the only bundle-size, recounting is impossible to do and to predict by formulas since asking ‘3 8s = ? tens’ leads to T = (3*8/ten)*ten that cannot be calculated. Now the answer is given by multiplication, 3*8 = 24 = 2 tens + 4 ones, thus transforming multiplication into division.

Per-numbers as the Natural Roots of Middle School Mathematics

Also in middle school the root of mathematics is recounting the natural fact Many, now occurring different places in the outside world, e.g. in time and space and in economy with units as seconds, minutes, cm, m, m^2, m^3, liters, c, $, £ etc.

Recounting creates fractions when e.g. counting 3 1s in 5s as T = 3 = 0.3 5s = (3/5)*5, which shows that 3/b = 0.3 for any bundle-size b; and ‘per-numbers’ when a quantity is double-counted e.g. both as 2$ and as 5kg thus containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg.

Again the recount-formula predicts recounting results when asking ‘6$ = ?kg’, or ‘14kg = ?$:

T = 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg; and T = 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$,

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306. Mathematics as Manyology

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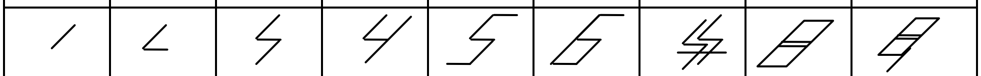								
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IIIIIII -> III III I -> III III) I) -> ▌▌) I) -> II) I) -> 2)1) -> 2.1 3s

Also bundles can be bundled and placed in a new cup to the left. Thus in 6 3s and 2 1s or 6)2) or 6.2 3s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II), or 6)2) = 2)0)2, or 6.2 3s = 20.2 3s.

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A premature introduction of ten as THE standard bundle makes ten a 'cognitive bomb':

Numbers are no more written as natural numbers, i.e. as decimals carrying units. Instead numbers are written using the sloppy place-value method hiding both the decimal and the unit.

Almost all operations change meanings. Soft addition next-to as in T = 2.3 4s = 2*4 + 3*1 is changed to hard addition on-top as in 23 + 48 = 71. Soft multiplication where 3*8 means 3 8s is changed to hard multiplication, i.e. to division recounting the 3 8s in tens: 3*8 IS 24.

Now /4 now means divided in 4, not counted in 4s. Only –3 still means take away 3.

As shown by the recount- and rest-formulas, with 2.order counting the order of operations is: first /, then *, then –, and finally +. With 3.order counting in tens this order is turned around: first +, then –, then *, and finally /.

With ten as the only bundle-size, recounting is impossible to do and to predict by formulas since asking '3 8s = ? tens' leads to T = (3*8/ten)*ten that cannot be calculated. Now the answer is given by multiplication, 3*8 = 24 = 2 tens + 4 ones, thus transforming multiplication into division.

Conclusion

Many mathematical concepts in primary school are not grounded in nature, but can be replaced by concepts grounded in the root of mathematics, the natural fact Many. So a question arises: Is mathematics education not meant to demonstrate how mathematical concepts are rooted in and able to predict the behavior of the natural fact Many?

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307. Counting and Adding Roots Grounded Mathematics

Mathematics education claims to deliver well-proven knowledge about well-defined concepts applicable to the outside world. However, skepticism would ask: Are the concepts grounded in nature or forcing choices upon nature? Can ungrounded mathematics from above be replaced by grounded mathematics from below generalized in a natural way in secondary school?

The Background

The Enlightenment period treated mathematics as a natural science. Grounded in the natural fact Many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline, 1972). Using the concept set, modern mathematics turned Enlightenment mathematics upside down to a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox 'this statement is false' inspired Russell to formulate a paradox about the set of sets not belonging to itself: If $M = \{A \mid A \notin A)\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise, without using self-reference it is impossible to prove that a proof is a proof, as shown by Gödel.

To avoid becoming metamatics, mathematics must return to its roots, the natural fact Many, guided by a contingency research looking for hidden alternatives to choices presented as nature.

Postmodern Contingency Research

Skepticism towards hidden patronization is at the root of postmodern thinking as expressed in the two Enlightenment republics, the American and the French. The US still has its first republic using pragmatism and grounded theory to exert skepticism towards philosophical claims. The French now has its fifth republic being turned over repeatedly by the German neighbors, which has caused France to develop a skeptical thinking warning against hidden patronization (Tarp, 2004).

Based upon skepticism towards hidden patronization, postmodern contingency research uncovers hidden alternatives to choices presented as nature.

To keep categories and discourses non-patronizing they should be grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the US enlightenment democracy; and resonating with Piaget's principles of natural learning (Piaget, 1970).

Recounting as the Root of Primary School Mathematics

To deal with the natural fact Many we count and add. 1.order counting rearranges sticks to form icons. 2.order counting bundles in icon-bundles and 3.order counting bundles in tens.

Using 2.order counting, first we bundle the total T in e.g. 3-bundles, in 3s. Then we place the total in a left bundle-cup and in a right single-cup. Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

Now numbers and operations can be combined to calculations predicting the counting results. The 'recount-formula' T = (T/b)*b tells that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6/7) 7s. Using a calculator we get the result 2 7s and some leftovers that can be found by the 'rest-formula' T = (T–b) + b telling that T–b is what is left when b is taken away: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. So the combined prediction says T = 3*6 = 2*7 + 4*1 = 2.4 7s.

This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

Once counted, totals can be added. However, two kinds of addition exist: Soft addition next-to and hard addition on-top.

Soft addition next-to adds T1 = 2 3s and T2 = 3 4s into T = 2.4 7s. Thus the totals are added by juxtaposing their stacks, i.e. by adding the areas of the stacks. In this way soft addition nex-to is the root of integration also using areas when adding

To perform hard addition on-top, the units must be the same, so the 3s must be recounted in 4s or vice versa. With T1 = 2 3s = 1.2 4s, T1 + T2 = 4.2 4s. And with T2 = 3 4s = 4 3s, T1 + T2 = 6 3s. In this way recounting and changing units becomes the root of proportionality.

With 3.order counting in tens only hard addition on-top is possible. Thus skipping 2.order counting means also skipping the roots of calculus and proportionality

So already in primary school, 2.order re-counting enable learners to predict and practice changing units, the leitmotif of mathematics reappearing later as proportionality and per-numbers. This also allows practicing the scientific method using formulas for predictions to be tested. In this ten-free zone it becomes possible to introduce the core of mathematics using 1digit numbers only (Zybartas et al, 2005). The CATS-approach, Count&Add in Time&Space, is one example of a grounded approach to mathematics as a natural science investigating the natural fact Many when counting by counting and adding at all school levels (Tarp, 2008).

Patronizing Versus Grounded Mathematics in Primary School

In primary school the tradition skips 1.order and 2.order counting and goes directly to 3.order counting claiming that 10 IS the follower of 9 in spite of the fact that 10 is the follower of 4 when counting in 5s. The grounded alternative postpones 3.order counting until after 2.order counting has introduced recounting in different units as an introduction to proportionality; and until after soft addition next-to has introduced integration.

The tradition presents one digit numbers as symbols and two digit numbers as natural numbers. The grounded alternative introduces one digit numbers as what they really are: icons rearranging the sticks they represent; and introduces the natural numbers as what they really are: decimal-numbers with units, e.g. 2.3 4s, and 2.3 tens instead of just 23.

Hard addition on-top is introduced as the first of the four operations, where e.g. 7 + 4 = 11 forces the immediate introduction of ten as the bundle-size, and forces the sloppy way of writing 2digit numbers without decimals or units. The grounded alternative first introduces soft addition next-to so that 2 5s + 4 1s means placing a stack of 4 1s next to a stack of 2 5s, i.e. as 2.4 5s.

Patronizing Versus Grounded Mathematics in Middle School

In middle school the tradition introduces fractions as ‘rational’ numbers, later having decimals and percentages as examples; and allows fractions to be added without units. The grounded alternative introduces decimals and fractions as what they really are: decimals occur as the natural numbers when counting in icon-bundles; and fractions are per-numbers occurring when double-counting a quantity in two different units, as 1s and as 5s: 3*1 = (3/5)*5; and as $ and kg: 2$/5kg = 2/5 $/kg.

Equations are introduced as equating 2 number-names to be changed by identical operations aiming at neutralizing the numbers next to the unknown. The neutralizing method seems to have as a hidden agenda to legitimize the concepts of modern abstract algebra in teacher education solving a simple equation as 3 + x = 8 by using, not the definition of reversed operations, but both the commutative and associative laws, and the concepts of inverse and neutral elements of number sets:

$3 + x = 8 , (3 + x) + (-3) = 8 + (-3) , (x + 3) + (-3) = 8 – 3 , x + (3 + (-3)) = 5 , x + 0 = 5 , x = 5.$

The grounded alternative introduces equations as what they really are, calculations being reversed since we know the end result, but not the initial number.

Geometry is introduced as forms and facts deduced from self-evident axioms. The grounded alternative introduces geometry as what it really is: 'earth-measuring' realizing that all forms can be split into right-angled triangles, where the relationship between the angle and the side can be expressed by recounting the sides thus creating the percentage numbers sinA, cosA and tanA.

Patronizing Versus Grounded Mathematics in High School

High school introduces formulas as set-relations: polynomial, exponential, circular functions etc. The grounded alternative introduces these formulas as what they really are: solutions to equations describing change in physical phenomena, in populations, in economical quantities etc.

Calculus is introduced as an example of a limit process, thus introducing limits and continuity before the derivative. The grounded alternative generalizes primary school's soft addition of stacks in combined bundle-sizes, and middle school's adding fractions with units, to adding per-numbers with units; and introduces the terms continuous and differentiable as what they really are: foreign words for locally constant and locally linear in contrast to piecewise constant and piecewise linear.

Conclusion

Many mathematical concepts in primary school are not grounded in nature, but can be replaced by concepts grounded in the root of mathematics, the natural fact Many. So a question arises: Is mathematics education not meant to demonstrate how mathematical concepts are rooted in and able to predict the behavior of the natural fact Many?

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308. Fractions Grounded as Decimals, or 3/5 as 0.3 5s

The tradition sees fractions as difficult to teach and learn. Skepticism asks: Are fractions difficult by nature or by choice? Are there hidden ways to understand and teach fractions? Contingency research seaching for hidden alternatives to traditions looks at the roots of factions, bundling the unbundled, described in a natural way by decimals. But why is the unnatural presented as natural?

The Background

Enlightenment mathematics was as a natural science exploring the natural fact Many (Kline, 1972) by grounding its abstract concepts in examples, and by using the lack of falsifying examples to validate its theory. But after abstracting the set-concept, mathematics was turned upside down to modern mathematics or 'metamatism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathema-tism' true in the library, but not in the laboratory, as e.g. 2+3 = 5, which has countless counterexamples: 2m+3cm = 203 cm, 2weeks+3days = 17 days etc. Being self-referring, this modern mathematics did not need an outside world.

However, a self-referring mathematics turned out to be a self-contradiction. With his paradox on the set of sets not belonging to itself, Russell proved that sets implies self-reference and self-contradiction as known from the classical liar-paradox 'this statement is false' being false when true and true when false: If $M = \{ A \mid A \notin A \}$, then $M \in M \Leftrightarrow M \notin M$.

Likewise Gödel proved that a well-proven theory is a dream since it will always contain statements that can be neither proved nor disproved.

In spite of being neither well-defined or well-proved, mathematics still teaches metamatism. This creates big problems to mathematics education as shown e.g. by 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest that when counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and not 7 cokes of 6 bottles.

Likewise modern metamatism forces natural way to write totals, T = 3.4 tens including the unit-bundle and a decimal point to separates the bundles from the unbundled, to be replaced with plain 34 hiding both the total and the unit, and misplacing the decimal point.

To design an alternative, mathematics can return to its roots, the natural fact Many, guided by contingency research that, inspired by contemporary and ancient skeptical thinking, uncovers hidden patronization by discovering alternatives to choices presented as nature.

Contingency Research Unmasks Choices Presented As Nature

Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists warned that to practice democracy, the people must be enlightened to tell choice from nature in order to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since to them all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should be given the role as natural patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment: when an apple obeys its own will, people could do the same and replace patronization with democracy.

Two democracies were installed: one in the US still having its first republic; and one in France, now having its fifth republic. German autocracy tried to stop the French democracy by sending in an army. However, a German mercenary was no match to a French conscript aware of the feudal consequence of defeat. So the French stopped the Germans and later occupied Germany. Unable to use the army, the German autocracy instead used the school to stop enlightenment in spreading from France. As counter-enlightenment, Humboldt used Hegel philosophy to create a patronizing line-organized Bildung school system based upon three principles: To avoid democracy, the people must not be enlightened; instead romanticism should install nationalism so the people sees itself as a 'nation' willing to fight other 'nations', especially the democratic ones; and the population elite should be extracted and receive 'Bildung' to become a knowledge-nobility for a new strong central administration replacing the former blood-nobility unable to stop the French democracy.

As democracies, EU still holds on to line-organized education instead of changing to block-organized education as in the North American republics allowing young students to uncover and develop their personal talent through individually chosen half-year knowledge blocks.

In France, the sophist warning against hidden patronization is kept alive in the post-structural thinking of Derrida, Lyotard, Foucault and Bourdieu. Derrida warns against ungrounded words installing what they label, such word should be 'deconstructed' into labels. Lyotard warns against ungrounded sentences installing political instead of natural correctness. Foucault warns against institutionalized disciplines claiming to express knowledge about humans; instead they install order by disciplining both themselves and their subject. And Bourdieu warns against using education and especially mathematics as symbolic violence to monopolize the knowledge capital for a knowledge-nobility (Tarp, 2004).

Bauman (1989) points out that by following authorized routines modernity can create both gas turbines and gas chambers. Arendt (1998) shows how in highly institutionalised societies patronization might become totalitarian, thus reintroducing evil actions this time rooted not in a devil but in the sheer banality of just following orders.

To prevent patronization, categories is grounded in nature using Grounded Theory (Glaser et al, 1967), the method of natural research developed in the first Enlightenment democracy, the American, and resonating with Piaget's principles of natural learning (Piaget, 1970).

The Case Of Fractions

The tradition says that fractions are difficult to teach and learn. Contingency research questions this by asking: Is fraction difficult by nature or by choice? And if so, whose choice? Can hidden alternatives be uncovered? Who has an interest in making fractions difficult?

Fractions In Textbooks

The fraction tradition can be observed in textbooks and in books on modern mathematics. Typically, the tradition postpones fractions to after all four basic operations have been introduced. Then unit fractions come in two versions. Geometric fractions are parts of pizzas or chocolate bars. And algebraic fractions are associated with simple division: 1/4 of the 12 apples is 12/4 apples. To find 4/5 of 20 apples, first 1/5 of 20 is found by dividing with 5 and then the result is multiplied by 4.

Then it is time for decimals as tenths, and for percentages as hundredths. Then adding or removing common factors in the numerator and in the denominator introduces the idea of similar fractions. Then, in late primary and early middle school, addition of fractions is introduced, first with like, then with unlike denominators.

Then decomposing a numbers into primes is introduced together with the lowest common multiple and the highest common factor to find the smallest common denominator when adding fractions with unlike denominators.

Then everything is repeated three times. Numerical fractions become algebraic fraction, first using monomials as (4abc)/(6ac) that are already factorized; then using polynomials as (4ab+8bc)/(3ab-6ac) that need to be factorized; and finally fractions enter equations.

Fractions In Modern Mathematics
Before modern mathematics, a fraction was a ratio between numbers, not allowing zero in the denominator. Some fractions could be reduced, e.g. 2/4 to 1/2. In modern mathematics they are made the same fraction, or more precisely, different representatives of the same fraction.

Wanting to define everything as examples of sets, modern mathematics defines a fraction as an equivalence set in the set product of ordered pairs of numbers created by an equivalence relation making (a,b) equivalent to (c,d) if cross multiplication holds: a*d = b*c. This definition creates a lot of activities aiming to show how operations defined on equivalence classes are independent upon the actual representatives.

Russell solved his set paradox by introducing type theory allowing only the natural numbers to be numbers, and therefore not accepting a pair of numbers to be a number, less an equivalence set in a set product. To Russell, a fraction is a calculation, not a number. The fractions of modern mathematics therefore violate Russell’s set paradox. To make fractions numbers, modern mathematics invented its own set theory, not distinguishing between elements and sets, thus being meaningless by mixing examples and abstractions: you can eat an example of an apple, but not the abstraction ‘apple’. Also Peano axioms were invented using a follower principle to prove that 1+1 = 2 is a natural correct statement. However, in a laboratory, 1 week + 1 day = 8 days, and 1 is not well defined since 1 threes = 3 ones, etc.

The Root Of Education: Adapting To The Ouside World
The prime goal of education is adapting to the outside world by proper actions. A typical action as ‘Peter eats apples’ is described by a three-term sentence with a subject, a verb and an object. Thus mathematics education should be described in this way. The learner is the subject, the object is the natural fact Many, and the verb is what we do to deal with Many, we totalize by counting and adding. Not being a verb, mathematics education should be renamed to ‘totalizing’, ‘counting and adding’ or reckoning. As to fractions, the fundamental question is: what outside world situations root fraction; and which occur in a natural way in primary, middle and high school?

The Root Of Mathematics: Counting Many By Bundling
To deal with Many, we totalize expressing the total as a formula, e.g. T = 345 = 3*10^2 + 4*10 + 5*1 showing that totalizing means counting and adding bundles, and that there are four ways of adding: +, *, ^ and integration. Totalizing can also be called algebra if using the Arabic word for reuniting. It turns out that there are three ways of counting:

1.order counting bundles sticks into icons, so that there are five sticks in the 5-icon etc. 2.order counting bundles the total in icon-bundles, and 3.order counting bundles in tens.

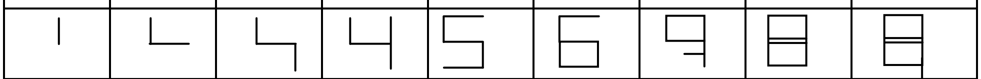
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

Figure 1. Rearranging sticks into icons transforms 5 1s into 1 5s, etc.

The calculator has no icon for ten since counting in tens, ten becomes 1 bundle, and eleven and twelve, meaning one left and two left in old Norse, becomes 1 bundle 1 and 1 bundle 2, or just 11 and 12, while ten becomes 10 symbolizing 1 bundle and no unbundled. However, ten is not 10 by nature, but by choice of bundle-size. Counting in fives, five becomes 10: 1, 2, 3, 4, 5 = 1 bundle = 10. Thus the icon for the bundle-size is never used.

Operations iconize the bundling and stacking processes. Taking away 4 is iconized as – 4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 showing a 3 times lifting of the 4s. Placing a stack of 2 singles next to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. as 1 3^2-bundle.

Now numbers and operations can be combined to calculations predicting the counting results. The 'recount-formula' T = T/b bs = (T/b)*b tells that from the total T, bs are taking away T/b times. Thus recounting a total T = 3 6s in 7s, the prediction says T = (3*6/7) 7s. A calculator gives the result 2 7s and some leftovers that can be found by the 'rest-formula' T = (T–b) + b telling that from the total T, b can be taken away and placed next-to: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. The combined prediction, T = 3*6 = 2*7 + 4*1 = 2.4 7s, holds when tested:

IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

Counting The Unbundled Roots Count-Fractions

With 2.order icon-counting a total may be bundled in 5s as e.g. T = 2 5s & 3. The sticks are placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIIIIIIIII -> IIIII IIIII III -> IIIII IIIII) III) -> ▌▌) III) -> II) III) -> 2)3) -> 2.3 5s

Also bundles can be bundled and placed in a new cup to the left. Thus in 6 3s & 2 1s or 6)2) or 6.2 3s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II), or 6)2) = 2)0)2 = 20.2 3s.

Adding an extra cup to the right shows that multiplying or dividing with the bundle-size just moves the decimal point: T = 2.1 3s = 2)1) -> 2)1)) = 2.1 3s * 3 = 21.0 3s and vice versa.

Using squares instead of cups, the 2 5-bundles are stacked and the 3 leftovers may be placed either next to the stack in a stack of 1s, which can be written as 2.3 5s; or on top of the stack of 5-bundles counted as 5s, i.e. as 3 = (3/5)*5 thus giving a total of T = 2 3/5 5s.

Thus 3/5 5s is just another way of writing 0.3 5s. And when bundling in tens, 3/ten tens is the same as 0.3 tens. So for any bundle-number, we can define 'count-fractions' by 3/b = 0.3. And we see that count-fractions are rooted in bundling the leftovers.

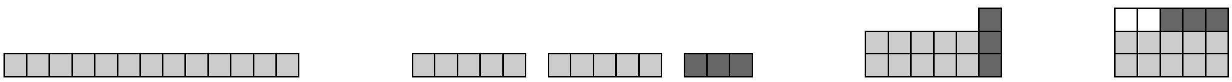


Figure 2. A Total has 3 unbundled added next-to or on-top: T = 2.3 * 5 = 2 3/5 * 5

So count-fractions are a geometrical way to describe the unbundled 1s when counting in bundles. Thus in the case of bundling in 5s or tens, 4 bundles and 3 unbundled can be written as respectively T = 4.3 5s = 4 3/5 5s or T = 4.3 tens = 4 3/10 tens. In this way both 3/5 of 5 and 3/10 of 10 is 3. So to give meaning to 3/5 of 20 we recount 20 in 5s: T = 20 = (20/5)*5 = 4*5 = 5*4 = 5 fours. Thus 3/5 of 20 = 3/5 of 5 fours = 3 4s = 3*4 = 12. Or the short version: 3/5 of 20 = 3 * (20/5).

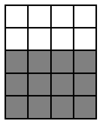


Figure 3. A Total of 20 recounted as 5 4s, thus 3/5 of 20 is 3 4s

Choosing ten as the standard-bundle means that all other bundles must be recounted in tens, which is what the tables does: 4 5s = 4*5 = 20 = 2.0 tens. However, the table can also be reversed recounting tens in e.g. 5s: 20 = (20/5)*5 = 4*5 = 5*4 = 5 fours. Thus taking fractions of numbers allow tables to be practiced both ways. In this way ‘fold-numbers’ can be folded or factorized partially or fully in non-foldable prime numbers: 12= 3*4= 3*2*2.

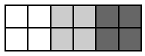


Figure 4. The fold-number 12 folded as 3 4s and as 3*(2 2s). So 12 = 3*2*2.

Sharing A Gain Roots Part-Fractions

Ann and Bo enters a bet with a 5$-stake where Ann pays 3$ and Bo 2$. A gain then should be shared in the same relation 3:2, meaning that the gain should be counted in 5s, each time giving back 3$ to Ann and 2$ to Bo. In this way Ann gets 3/5 of the gain or 3$ per 5$ of the gain. This fraction can be called a part-fraction or a per-number.

Adding Count-Fractions And Part-Fractions

Once counted, totals can be added. Counting unbundled, count-fractions add the unbundled if the units are the same: T = 1/5 5s + 2/5 5s = 3/5 5s. Different units can be made the same by recounting: T = T1+T2 = 2 1/3 3s +3 1/5 5s = 1 2/5 5s +3 1/5 5s = 4 3/5 5s.

As to part-fractions the totals have to be included when adding. Thus receiving 3/5 of 20$ and 2/3 of 30$ means receiving 3*(20/5)$ + 2*(30/3)$ = 12$ + 20$ = 32$ of 50 $ or 32/50 of 50$.

Recounting In Bundles And In Tens

Recounting can unbundle bundles or vice versa. Thus 13.2 5s is the same as 8.2 5s.

T = 13.2 5s = 1)3)2) = 1-1)+5+3)2) = 0)8)2) = 8.2 5s.

When counting in tens 4.5 tens is the same as 0.45 hundreds and 45.0 1s. This can be used when changing units when dealing with amounts, weight and distance:

T = 4.35 $ = 4.35 ten dimes = 43.5 dimes = 43.5 ten cents = 435 cents and vice versa.

T = 4.35 m = 4.35 ten dm = 43.5 dm = 43.5 ten cm = 435 cm and vice versa.

Recounting is useful in manual calculations:

Add	2.3 4s + 3.1 4s = 2)3) + 3)1) = 5)4) = 6)0) = 2)2)0) = 22.0 4s 28 + 45 = 2.8 tens + 4.5 tens = 2)8) + 4)5) = 6)13) = 7)3) = 7.3 tens = 73
Subtract	3.1 4s – 1.2 4s = 3)1) – 1)2) = 2)-1) = 2-1)+4-1) = 1)3) = 1.3 4s 52 – 18 = 5.2 tens – 1.8 tens = 5)2) – 1)8) = 4)-6) = 3)10-6) = 3.4 tens = 34
Multiply	3 * 2.3 4s = 3 * 2)3) = 6)9) = 6+2)-8+9) = 8)1) = 2)-8+8)1) = 2)0)1) = 20.1 4s 3* 58 = 3* 5.8tens = 3* 5)8) = 15)24) = 17)4) = 1)7)4) = 17.4 tens = 174

Divide	2.3 4s / 3 = 2)3) /3 = 2*4+3) /3 = 11) /3 = 3 rest 2, so 2.3 4s = 3* 3 + 2*1 48 /5 = 4.8 tens / 5 = 4)8) / 5 = 4*10+8) / 5 = 9 rest 3, so 48 = 9*5 + 3*1

Middle School

Middle school sees the introduction of physical quantities and units. Thus to find 3/5, a length or a weight first must be measured to produce a number. Again taking 3/5 of 46.8 cm means recounting in 5 cm, so T = 46.8 cm = (46.8/5)*5cm = 9.36 * 5cm = 5 (9.36 cm), and 3/5 of this is 3 (9.36 cm) = 28.08 cm. Again we can use the shortcut: 3/5 of 46.8 = 3*46.8/5 = 28.08.

Recounting physical quantities may give per-numbers double units. Thus 3kg sugar might give 5$ when recounted in dollars. So if 3kg cost 5 $, the unit price is 5$/3kg or 5/3 $/kg.

This per-number allows shifting units by recounting. Thus a weight of 12 kg can be recounted in 3s, and a price of 40$ can be recounted in 5s:

T = 12 kg = (12/3)*3kg = (12/3)*5$ = 20$, and

T = 40$ = (40/5)*5$ = (40/5)*3kg = 24 kg

Percent

The most frequently used per-number is percent. To transform a given per-number to percent, again we use recounting. To transform 3/5 to percent we recount 100 in 5s:

T = 100 = (100/5)*5 = 20 5s = 5 20s, so 3/5 is 3 20s = 60 meaning 3/5 = 60/100 = 60%.

Again we can use the shortcut: 3/5 of 100 = 3*100/5 = 60.

And to find 5% of a number means finding 5/100 of that number. Let us find 5% of 480:

T = 480 = (480/100)*100 = 4.8 hundreds, so 5% of 480 is 5*4.8 = 24

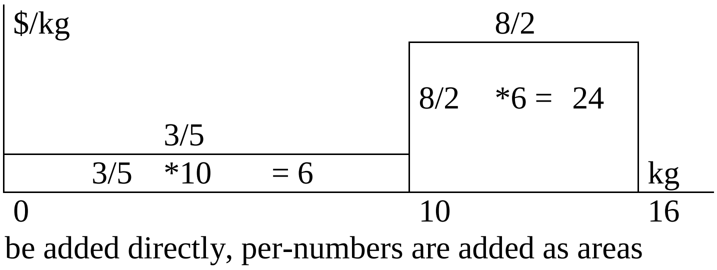
Simplifying Fractions

Recounting may be used to simplify fractions: In the fraction 4/6 both 4 and 6 can be recounted in 2s: 4 = (4/2)*2 = 2 twos; and 6 = 3 twos. So instead of writing 4/6 = (4 ones)/ (6 ones) we can write (2 twos)/(3 twos) = 2/3. So both above and below the fraction line common factors can be taken out as units and then cancelled out; or vice versa added as units.

Adding Per-Number Fractions

Adding per-numbers occurs when blending tea: If 10 kg at 3$/5kg is blended with 6 kg at 8$/2kg the result is 16 kg at a price that is found by calculating the weighted average:

10 kg at 3/5 $/kg gives 6 $
6 kg at 8/2 $/kg gives 24 $
16 kg at u $/kg gives 30 $
So u = 30$/16kg = 1.88 $/kg



The problem is that where unit-numbers can be added directly, per-numbers are added as areas under the per-number graph, i.e. as Σ p*Δx. This is the case no matter if the per-number is a number or a fraction. Adding per-numbers by their area thus roots addition by integration.

Statistics Roots Fractions

Reporting numbers from questionnaires includes fractions and percentages: Counting boys and girls in a group of two classes P and Q may result in the following 2x2 cross table:

	P	Q	Total
B (boys)	10	10	20
G (girls)	10	20	30
Total	20	30	50

The fraction describing the boys in class P depends of which total is used. Thus the P-boys make 10/50 or 20% of the group, and 10/20 or 50% of the class. Likewise, the Q-girls make 20/50 or 40% of the group, and 20/30 or 67% of the girls.

	P	Q	Total
B	10/50	10/50	20/50
G	10/50	20/50	30/50
Total	20/50	30/50	50/50

	P	Q	Total
B	50%	50%	100%
G	33%	67%	100%

	P	Q
B	50%	33%
G	50%	67%
Total	100%	100%

High School

In high school substituting numbers by letters allows letter fractions or algebraic fractions to occur. Again factoring and canceling out common units can simplify such fractions:

$$\frac{a*b*c}{c*d} = \frac{(a*b)\text{ cs}}{d\text{ cs}} = \frac{a*b}{d} \quad \text{and} \quad \frac{a*c + b*c}{c*d} = \frac{(a + b)*c}{c*d} = \frac{(a+b)\text{ cs}}{d\text{ cs}} = \frac{a+b}{d}$$

Middle school per-numbers are piecewise constant. In high school per-numbers are locally constant, continuous. Still, the area under its graph adds variable per-numbers. This area can be found by a graphical calculator, or by using integration, known from horizontal adding stacks next-to in primary school and from calculating weighted averages in middle school.

Middle school cross tables has as the task to go from data to fractions. In high school the task is to translate vertical fractions to horizontal or vice versa. Bayes' formula is introduced, but a more natural approach is to recreate the original data from which any fractions can be found.

In probability fractions describe probabilities leading to the binomial distribution.

In finance adding constant interest percentages r leads to exponential growth y = yo*(1+r)^n.

2/3 3s Becoming 2/3

In the laboratory both 2/3 3s and 0.2 3s exist as 2 leftovers when counting in 3s. Counting in tens, 2 leftovers become 2/10 10s or 0.2 tens. However, the tradition insists that when counting in tens, the unit is neglected and the decimal point is moved one place to the right, which transforms 2.3 tens to plain 23, and 0.2 tens to plain 2, i.e. 2 1s. In this way counting in tens is mixed up with counting in 1s, which has no meaning since 1 = 1^1 = 1^2 while 1 ≠ 10^1 ≠ 10^2. Thus counting in 1s makes it impossible to distinguish between 1s, bundles and bundles of bundles. The Romans never realized that 3 1s, III, could be transformed to 1 3s.

The tradition also insists that fractions as 2/3 3s should drop its unit to become 2/3 of 1, i.e. a calculation producing 0.0667 tens or 0.667. Also the tradition insists that 2/3 is a number instead of a calculation. Furthermore the tradition insists that fractions can be added without units as illustrated in the fraction paradox above.

So while 2/3 3s is labeling a physical fact, 2 leftovers when counting in 3s, 2/3 is not labeling, but installing what it mentions, thus installing hidden patronization that must be deconstructed if education shall serve as enlightenment. And deconstruction might lead to the discovery of 1digit mathematics (Zybartas et al, 2005).

The Two Goals Of Mathematics Education

In mathematics education, fractions can be used for two different purposes.

Mathematics may be seen as a natural science rooted in the natural fact Many. In this case fractions are introduced in grade one parallel with decimals to account for the unbundled when performing 2.order icon-counting. The core activity of icon-counting is recounting using the recount formula T = (T/b)*b showing directly why fractions occur as division. As a counting fraction, 3/5 only has meaning as 3/5 5s or 3/5 of 5. Simple sharing problems motivate seeing 3/5 as a per-number that can be taken of any number, and the ability to recount any number in 5s allows per-numbers to be introduced in primary school. Thus the core part of fractions is learned before entering middle and high school. Recounting also allows tables to be practiced both ways, both as multiplication tables and as factoring tables. Seeing mathematics as Manyology, a natural science about Many, makes it accessible to all.

However, mathematics may also be seen as metamatism, i.e. a self-referring body of knowledge that is meditated by what is called mathematics education. Metamatism allows only 3.order counting in ten-bundles to take place, thus banning both 1.order counting allowing the learners to build the digits themselves with sticks, and second order counting allowing recounting to introduce both fractions and decimals as well as changing units (proportionality) and horizontal addition next to (integration) in early primary school. Instead metamatism says that 2+3 IS 5 and that 1/2+2/3 IS 7/6, in spite of countless counterexamples in the laboratory. Changing mathematics to metamatism makes it accessible only to an elite.

Why Presenting The Unnatural As Natural

In a natural approach, fractions and decimals occur together when performing recounting, the most powerful operation in mathematics, producing natural numbers as decimals with units.

However, the tradition forces Many to be counted only in ten-bundles, and forces numbers to be presented without units and with displaced decimal points. This forces fractions to be postponed to after the introduction of all four operations when, all of a sudden, it is allowed to count in other units than ten, e.g. 7s; and forces decimals to be presented as examples of fractions. Finally, forcing fractions to be added without units creates mathematism.

Why all this force? One answer comes from Bourdieu and his modern version of the ancient Greek question: Should knowledge enlighten people to practice democracy; or should knowledge persuade people to accept patronization by the better knowing philo-sophers?

Bourdieu sees the social world divided into fields where people fight for the capital of the field. With the transition from industrial to information society, both economical and knowledge capital become important. Socialist parties have decentralized economical capital, but the knowledge capital is still centralized enabling also political power to be centralized to a knowledge-nobility whose offspring develops the relevant habitus to be successful in the knowledge field, much like the mandarin system of ancient China. And Bourdieu sees mathematics as specially suited to perform the symbolic violence that monopolizes knowledge to the nobility. Organizing knowledge in forced lines instead of in self-chosen blocks is another effective technique to guard the knowledge privilege (Bourdieu 1977).

To decentralize knowledge, the symbolic violence must be taken out of education by replacing line-organization with block-organization; and taken out of mathematics by replacing self-referring metamatism with mathematics as a natural science that explores the natural fact Many, and that by making 3.order ten-counting second to 2.order icon-counting also makes fractions second to decimals. With its ability to uncover hidden patronization presenting choice as nature, contingency research is an effective means to replace autocracy with democracy in the knowledge field.

Conclusion

This paper asked 'Are fractions difficult be nature or by choice?' Contingency research showed that fractions are difficult by choice. The nature of fractions is to account for the leftovers when counting in bundles. Introducing 2.order icon-counting before 3.order ten-counting shows that natural numbers always include units and a decimal point to separate the bundles form the unbundled. So fractions are natural parts of most counting or recounting results. However the tradition allows only 3.order ten-counting to be practiced and forces natural numbers to take on a unnatural identity as multi-digit numbers leaving out the unit and misplacing the decimal point one place to the right so that fractions are no longer needed to account for the unbundled. Instead 2.order counting is postponed to the end of primary school where all of a sudden is it allowed to count in icons to motivate the introduction of fractions. However, forcing fractions to be added without units transforms meaningful mathematics into meaningless metamatism repelling most learners. Except for the children of the knowledge nobility using mathematics and fractions as means to guard its knowledge- monopoly. To decentralize the knowledge capital, hidden patronization should be unmasked by research uncovering hidden contingency to choices presented as nature.

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309. Counting and Adding - a Natural Way to Teach Mathematics

The CATS-approach, Count&Add in Time&Space, obeys the rule of good research, never to ask leading questions. To learn mathematics, students should not be taught mathematics; instead they should meet the roots of mathematics, Many. Through guiding educational questions asking them to Count and Add in Time and Space, they learn mathematics by doing it. The CATS-approach is rich on examples of recognition and new cognition to be observed, reflected and reported by teachers and researchers.

The Background

The Enlightenment period treated mathematics as a natural science. Grounded in the natural fact Many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline 1972). Using the concept set, modern mathematics turned Enlightenment mathematics upside down to a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox 'this statement is false' inspired Russell to formulate a paradox about the set of sets not belonging to itself: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$.

To avoid becoming metamatics, mathematics must return to its roots, the natural fact Many, guided by a contingency research looking for hidden alternatives to choices presented as nature.

Contingency Research

Based upon the sophist and French warning against hidden patronization presenting choice as nature, a research paradigm can be created called postmodern 'contingency research' deconstructing patronizing choices presented as nature by uncovering hidden alternatives (Lyotard 1984, Tarp 2004). To keep categories, discourses and institutions non-patronizing they should be grounded in nature using Grounded Theory (Glaser et al 1967), the natural research method developed in the US enlightenment democracy; and resonating with Piaget's principles of natural learning (Piaget 1970).

Constructing a Grounded Mathematics From Below

Grounded in the natural fact Many, mathematics becomes a natural science. To deal with Many, first we count, then we add. So a grounded mathematics is about counting and adding in time and space, CATS.

The website MATHeCADEMY.net contains grounded mathematics organized in activities where the learner learns 'CATS', guided by educational questions Q and answers A. The study units CATS1 are for primary school and the study units CATS2 are for secondary school.

Counting C1

Q: How to count Many? A: By bundling and stacking the total T predicted by T = (T/b)*b

Q: How to recount 8 in 3s: T = 8 = ? 3s = ?*3. A: 8 = (8/3)*3 = 2*3+2*1= 2)2) = 2.2*3 = 2 2/3*3

Q: How to recount 6kg in $: T = 6kg = ?$. A: If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$

Q: How to count in standard bundles? A: Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2 + 2*B + 3

Section C1 looks at ways to count Many. Spatial multiplicity is representing temporal repetition through sticks and strokes. A multiplicity of sticks can be rearranged in icons so that there are four sticks in the icon 4 etc. Then a given total T can be counted in e.g. 4s by repeating the process 'from

T take away 4', which can be iconized as 'T–4 '; where the repeated process 'from T take away 4s' can be iconized as 'T/4'. This makes it possible to predict the counting-result through a calculation using the 'recount-equation' T = (T/b)*b. Leftovers are stacked as 1s creating a stock T = 2*3 + 2*1. The stacks can be placed in two cups, a left bundle-cup and a right single-cup, and described by cup-writing T = 2)2), or decimal-writing including the unit T = 2)2) = 2.2 3s = 2.2*3; or the leftovers can be counted in 3s and added on top of the 3-stack: T = 2 2/3 * 3 = 2 2/3 3s.

Changing units is another example of a recounting where a given total is double-counted in two different units e.g. T = 4$ = 5kg producing a per-number 4$/5kg = 4/5 $/kg. Thus to answer the question ' 7kg = ?$' we just have to recount the 7 in 5s: T = 7kg = (7/5)*5kg = (7/5)*4$ = 5 3/5$.

The number ten has a name but no icon, since the bundle-size is not used: Counting in 5s, 5 1s = 1 5s = 1 bundle. Before introducing ten as the standard-bundle and leaving out the units, 2.4 tens = 24, the core of mathematics can be leaned by using 1digit numbers alone. (Zybartas et al 2005).

Adding A1

Q: How to add stacks concretely? T = 27 + 16 = 2 ten 7 + 1ten6 = 3 ten 13 = ?.

A: By restacking overloads predicted by the 'restack-equation' T = (T–b) + b:

T = 27 + 16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43.

Q: How to add stacks abstractly? A: Vertical addition uses carrying. Horizontal addition uses FOIL.

Section A1 looks at how stacks can be added by removing overloads that often appears when one stack is placed on top of another stack. The overload leads to 'internal trade' between two stacks where a stack of 10 1s is rebundled and restacked as 1 10-bundle. The result can be predicted by a calculation on paper using either a vertical way of writing the stacks using carrying to symbolize the internal trade; or using a horizontal way of writing the stacks using the FOIL-principle (First, Outside, Inside, Last). In both cases the overload can be restacked predicted by the restack-equation (T–b) + b, and recounted predicted by the recount-equation T = (T/b)*b.

Time T1

Q: How can counting & adding be reversed? A: By calculating backwards, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign.

Q: Counting ? 3s and adding 2 gave 14. A: x * 3 + 2 = 14 is reversed to x = (14 – 2)/3.

Q: Can all calculations be reversed? A: Yes. x + a = b is reversed to x = b – a, x * a = b is reversed to x = b/a, x ^ a = b is reversed to x = $\sqrt[a]{b}$, a ^ x = b is reversed to x = logb/loga.

Section T1 looks at formulas, the sentences of the number-prediction language. Containing two unknown variables, a formula becomes a function to be tabled and graphed. Containing one unknown variable, a formula becomes an equation to be solved by reversing the calculations, moving numbers from the forward-calculation side to the backward-calculation side reversing their signs: x*3+2 = 14 is reversed to x = (14-2)/3. This forward and backward calculation method gives a new perspective on the classical quantitative literature consisting of word-problems.

Space S1

Q: How to count the plane and spatial properties of stacks, boxes and round objects?

A: By using a ruler, a protractor and a triangular shape; by the 3 Greek Pythagoras', mini, midi & maxi; and by the 3 Arabic recount-equations: sin A = a/c, cos A = b/c, tan A = a/b.

Section S1 looks at how to describe plane properties of stacks as areas and diagonals by the 3 Greek Pythagoras', mini, midi & maxi; and by the 3 Arabic recount-equations: sin A = a/c, cos A = b/c and tan A = a/b. A circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: $C = 2 * r * (n*\sin(180/n)) = 2 * r * \pi$ for n sufficiently big.

Finally we look at how to describe spatial properties of solids such as surfaces and volumes by formulas and by a 2-dimensional representation of 3-dimensional shapes.

Counting C2

Q: How to count possibilities? A: By using the numbers in Pascal's triangle.

Q: How to predict unpredictable numbers? A: If a 'post-diction' gives the average 8.2 with deviation 2.3, the 'pre-diction' gives the confidence interval $8.2 \pm 2*2.3$ with 95% probability.

Section C2 looks at numbers that change unpredictably as e.g. in surveys. Through counting we can set up a frequency-table accounting for the previous behavior of the numbers. From this table their average level and their average change can be calculated. From this we can predict that with a 95% probability, future numbers will occur within an interval determined by the average level and double the average change. Counting the numbers of wins when repeating a game with winning probability p is another example of an unpredictable number, also called a stochastic variable.

Adding A2

Q: What is a per-number? A: Per-numbers occur when counting, when pricing and when splitting.

Q: How to add per-numbers? A: The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b

Section A2 looks at how to add per-numbers by transforming them to totals. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b. 2days at 6$/day + 3days at 8$/day = 5days at 7.2$/day. And 1/2 of 2 cans + 2/3 of 3 cans = 3 of 5 cans = 3/5 of 5 cans. Repeated and reversed addition of per-numbers leads to integration and differentiation:

$T2 = T1 + a*b;\ T2 - T1 = a*b;\ \Delta T = \sum a*b = \int y*dx;$ and

$T2 = T1 + a*b;\ T2 - T1 = a*b;\ a = (T2 - T1)/b = \Delta T/\Delta b = dy/dx$

Time T2

Q: How to predict the terminal number when the change is constant?

A: By constant change-equations: If Ko = 30 and $\Delta K/n = a = 2$, then K7 = Ko+a*n = 30+2*7 = 44.

If Ko = 30 and $\Delta K/K = r = 2\%$, then K7 = Ko*(1+r)^n = 30*1.02^7 = 34.46

Q: How to predict the terminal number when the change is variable, but predictable?

A: By a variable change-equation: If Ko = 30 and dK/dx = K', then $K - 30 = \Delta K = \int dK = \int K'dx$

Section T2 looks at how a stack changes in time by adding a constant number, or by a constant percent where adding 5% means changing 100% to 105%, i.e. multiplying with 105% = 1.05.

If related by a formula y = f(x), a x-change Δx will effect a y-change Δy that can be recounted in the x-change as $\Delta y = (\Delta y/\Delta x)*\Delta x$, or $dy = (dy/dx)*dx = y'*dx$ in the case of micro-changes.

If a stack y changes by adding variable predictable numbers dy, summing up the single y-changes gives the total y-change, i.e. the terminal y2 minus the initial y1: $\int dy = \int y'*dx = y2 - y1$.

Space S2

Q: How to predict the position of points and lines? A: By using a coordinate-system: If Po(x,y) = (3,4) and $\Delta y/\Delta x = 2$, then P1(8,y) = P1(x+Δx, y+Δy) = P1(3+(8-3),4+2*(8-3)) = (8,14)

Section S2 looks at how to predict the position of points and lines and geometrical figures and graphs using a coordinate system. Then we look at how to use the new calculation technology such as computers and calculators to calculate a set of numbers, vectors, and a set of vectors, matrices.

Quantitative Literature

Q: What is quantitative literature? A: Quantitative literature is about multiplicity in time and space.

Q: Does quantitative literature share the 3 different genres: fact, fiction and fiddle? A: Yes.

In formulas as T = c*p we need to know what quantities are described to determine the truth-value of the formula's prediction. It turns out that both word-statements and number-statements share the same genres: fact, fiction and fiddle. Fact-models predict predictable quantities. Fiction-models predict unpredictable quantities. Fiddle-models predict qualities.

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310. Hidden Understandings of Mathematics Education

To answer the question 'are there hidden understandings of mathematics education' this paper tries to reinvent mathematics as a natural science grounded in its natural roots, the study of the natural fact Many. It turns out that two different mathematics exist, metamatism from above and grounded mathematics from below: Also two different kinds of education exist: Line-organized Bildung forcing students to learn the same, and block-organized enlightenment allowing students to develop personal talents.

Hidden Understandings of 'Mathematics'

Are the hidden understandings of the word 'mathematics', 'geometry' and 'algebra'? Simple observations say yes.

In Greek, the word 'mathematics' means knowledge, and since knowledge can be used for prediction, one understanding of mathematics is that mathematics is a language for number-prediction. This resonates with the fact that numbers are icons describing different degrees of many (there are five strokes in the 5-icon etc.) or combinations of icons describing stacks of bundles and unbundled (35 means 3 ten-bundles and 5 unbundled); and the fact that calculations predict counting-results: Thus 5+3 = 8 predicts that counting on 3 times from 5 gives 8, 5*3 = 15 predicts that adding 5 3 times gives 15, and 5^3 = 125 predicts that multiplying with 5 3 times gives 125. But number-prediction doesn't resonate with the traditional presentation of mathematics as a study of patterns, or as a study of the consequences of universal set-based definitions and axioms. Thus today's set-based mathematics is a choice becoming pastoral by hiding its alternative, mathematics developed as a natural science investigating the natural fact many.

In Greek, the word 'geometry' means 'earth-measuring'; and in Arabic the word 'algebra' means 'reunite'. Thus geometry and algebra seems to be rooted in the two fundamental human questions: how do we divide the earth and what it produces?

The Enlightenment treated mathematics as a natural science. Thus around 1750 Euler introduced the term 'function' as a name for a calculation with a variable quantity, thereby dividing calculations in two kinds, 3+5 to be calculated right away, and 3+x to be postponed until knowing the value of x. The latter calculation type he then called functions, becoming interesting after Newton introduced change-calculations by raising the question: what is the change of a calculation-result compared with a change in one of its numbers?

In this way Euler defined a concept as an abstraction from examples, or by generalizing examples to an abstraction, i.e. by categorizing calculations containing variables as functions.

Just before 1900 Cantor created the generalization set, as a collection of well-defined objects. Being the ultimate abstraction, all other concepts now could be presented as examples of this abstraction, which would make mathematics self-referring not needing an outside world for its concepts and thus presenting itself as pure reason unpolluted by the skepticism related to the reliability of outside observations. Thus a function could now be defined as an example of a many-to-one set-relation, i.e. defined as an example from an abstraction.

From these two different definitions of the function-concept we can distinguish between two different ways of defining a concept. A concept can be defined as an abstraction from examples, which we can call 'grounded mathematics from below'; or a concept can be defined as an example of an abstraction, which we can call 'ungrounded mathematics from above' or 'metamatics'.

Likewise with mathematical statements: defining the numbers using the follower principle, 10 becomes the follower of nine, and 2+3 = 5 becomes easy to prove: 2+3 = 2+1+1+1 = 5.

However, a grounded approach will use counting by bundling and stacking to assign numbers to a given total. That ten is the follower of nine is a natural fact, but counting in eight-bundles, 10 becomes the follower of seven, and the follower of nine becomes 12.

And contrary to the statement ‘2*3 = 6’, which is always true by stating that recounting 2 3s in 1s produces 6 1s, the statement ‘2+3 = 5’ is seldom true, having countless counter-examples: 2weeks + 3days = 17days, 2m+3cm = 203cm, etc. In 2*3 = 2 3s the ‘3s’ is the unit. In 2+3 the units are omitted. Neglecting that numbers always carry units may lead to statements that might be proved, but not validated.

Thus we can introduce the term ‘mathematism’ for statements that can be proved, but not validated, i.e. that are true in the library but not in the laboratory; and ‘grounded mathematics’ for statements that can be proved and validated, i.e. that are true both in the library and in the laboratory. The combination of metamatics and mathematism may be called ‘metamatism’.

Hidden Understandings of ‘Education’

Are the hidden understandings of the word ‘education’? Simple observations say yes.

In mathematics, the existence of both grounded mathematics and ungrounded metamatism shows that two different conceptions on education exist, agreeing that education is an institution aiming at transferring knowledge to the students, but disagreeing as to what knowledge is.

Metamatics sees knowledge as consisting of concepts and statements that are deduced from metaphysical forms and patterns that are discovered by mathematicians that are transmitting their knowledge to textbooks and to teachers that try to transmit textbook knowledge to students.

Grounded mathematics sees knowledge as categories and relations induced from real-world phenomena and actions; and accommodated to the real world by testing deduced predictions.

Thus metamatism sees the roots of mathematics to be in metaphysical forms only accessible to mathematicians educated at an academy, and therefore obliged to offer pastoral guidance to the people; and grounded mathematics sees the roots of mathematics to be the natural fact Many residing in the outside world needing to be enlightened in order to improve human adaptation to it.

This controversy between metamatism and grounded mathematics as to the nature of knowledge corresponds to the controversy between the ancient Greek philosophers and sophists, also disagreeing if the role of education should be pastoral salvation or enlightenment.

Newton’s discovery of a predictable physical will laid the foundation to the Enlightenment, using encyclopedias to enlighten the people and installing two democracies, in US and in France.

The US democracy sees education as enlightenment aiming at enlighten as many as possible as much as possible.

To stop democracy and enlightenment from spreading from France, Humboldt was asked by the Prussian autocracy to transform the classical university into a modern Bildung university, which should guard the Platonic pastoral attitude towards knowledge as metaphysical forms only visible to philosophers and transmitted through textbooks and teachers, and should wake up nationalism in the people, and should identify the elite for the central administration.

This Bildung educational system was gladly imported by the other European autocracies and guarded by the central administrations also after democracy was introduced.

Conclusion: Mathematics Education Must Make a Choice

From the identification of different understandings of mathematics and of education follows that the institution called mathematics education does not come from nature, instead it is based upon a choice between two options:

Mathematics education can choose to maintain its tradition as ‘metamatism Bildung’ preaching ungrounded mathematics to the elite. Or Mathematics education can choose to become ‘grounded mathematics enlightenment’ teaching grounded mathematics enlightening its outside roots.

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311. Social Theory in Mathematics Education

Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? Should mathematics education enlighten or patronize?

Social Theory

Social theory has human interaction as its main focus. As to communication, the most basic interaction, Berne has developed a transactional analysis describing three different ego-states called Parent, Adult and Child (Berne, 1964).

Berne's concepts reflect the social fact that interaction between human beings can be patronized and non-democratic, or it can be non-patronized and democratic. In a family the interaction between children and parents will typically be one of patronization. In a society adult interaction typically will be non-patronized, unless the society is a non-democratic autocracy where patronization is carried on into adulthood. In this way Berne describes the main problem in human interaction, the choice between patronization and self-determination or 'Mündigkeit'. The fact that the German word 'Mündigkeit' does not have en English equivalent indicates that social interaction is quite different outside the EU and inside where the presence of and resistance against patronization created the label 'Mündigkeit'; whereas an absence of patronization doesn't call for labeling resistance against patronization.

The debate on patronization runs all the way though the history of social theory (Russell, 1945). In ancient Greece the sophists warned against hidden patronization coming from choices presented as nature. Hence to protect democracy, people should be enlightened to tell choice from nature. To the philosophers choice was an illusion since according to their view everything physical is examples of meta-physical forms only visible to philosophers. Consequently patronization was a natural order with the philosophers as protectors.

In the middle age the patronization question reappeared in the controversy on universals between the realists and the nominalists. Here the realist took the Plato standpoint by renaming his metaphysical forms to universals claimed to have independent existence and to be exemplified in the physical world, and consequently waiting to be discovered by philosophers. In contrast to this the nominalist saw universals as invented names facilitating human interaction.

The Renaissance period saw a protestant uprising against the patronization of the Roman Catholic Church resulting in the bloody 30year war from 1618. To avoid the chaos of war, Hobbes in his book 'Leviathan' argues that to protect themselves against their natural egoistic state, humans would have a much better life if accepting the patronization of an autocratic monarch.

In natural science Newton discovered that the moon doesn't move among the stars, instead it falls towards the earth, as does the apple, both following their own physical will and not the will of a metaphysical patronizor. This discovery inspired Locke to argue against patronization: 'John Locke is the apostle of the Revolution of 1688, the most moderate and the most successful of all revolutions. Its aims were modest, but they were exactly achieved, and no subsequent revolution has hitherto been found necessary in England' (Russell, 1945). Locke's chief work, the Essay Concerning Human Understanding, was highly inspirational in the Enlightenment 1700-century, which resulted in two democracies being installed, one in the US and one in France.

American sociology describes human interaction based upon enlightenment and freed from patronization. Its 'it is true if it works'-pragmatism expressed by Peirce and James leads on to symbolic interactionism and to the natural empery-rooted research paradigm Grounded Theory resonating with the principles of natural learning expressed by Piaget. In harmony with this, the US enlightenment school, being organized in blocks and aiming at enlightening as many as possible as much as possible, has set the international standard followed worldwide outside Europe.

Inside Europe reaction against the Enlightenment came from Germany where Hegel reinstalled metaphysical patronization in the form of a Spirit expressing itself through the history of the people.

Marx develops Hegel thinking into Marxism claiming that until a socialist utopia has been established a socialist party serving the interest of the working people should patronize people. In contrast to this Nietzsche argued that only by freeing itself from meta-physical philosophical hegemony, western individuals would be able to realize their full potentials. Marxist thinking developed into critical theory in the Frankfurt school infiltrating the 1968 student revolt so that EU's Bildung universities could carry on protecting its Hegel-based patronization.

Wanting to protect its republic against patronization, France developed post-structuralism inspired by Nietzsche's opposition against Hegel, and by Heidegger's question 'what is IS?'

Derrida introduces 'logocentrism' to warn against patronizing words installing what they label and recommends that such categories be deconstructed. Lyotard uses the word 'postmodern' to warn against sentences taking the form of 'meta-narratives' claiming to be truths and recommends paralogy as research inventing dissensus to the ruling consensus. Foucault uses the word 'pastoral power' to warn against patronizing institutions promising to cure human abnormalities installed by discourses claiming to be disciplines producing truths about humans. He shows how disciplines discipline itself and its object, in contrast to a natural discipline disciplining itself by its objects. Foucault also describes doctrines and other techniques used for discourse protection.

Bourdieu sees the social world divided into fields where people fight for the capital of the field. With the transition from industrial to information culture two kinds of capital become important, economical and knowledge capital. Socialist parties have decentralized economical capital, but the knowledge capital is still centralized enabling also political power to be centralized to a knowledge-nobility whose offspring develops the relevant habitus to be successful in the knowledge field, much like the mandarin system of ancient China. And Bourdieu sees mathematics as specially suited to perform the symbolic violence that monopolizes knowledge to the nobility. Organizing knowledge in forced lines instead of in free blocks is another effective technique to guard the knowledge privilege.

Bauman points out that by following authorized routines modernity can create both gas turbines and gas chambers (Baumann, 1989). Analyzing the latter, Arendt shows how in industrialized societies patronization might become totalitarian, thus reintroducing evil actions this time rooted not in a devil but in the sheer banality of just following orders (Arendt, 1968).

Discourse Protection and Hegemony

Mathematics can be rooted in examples 'from-below' as well as in abstractions 'from-above', but only the latter exists in mathematics education. Can social theory explain this? If the question of patronization is the key issue in social theory, this question can be reformulated to 'does mathematics education contain elements of hidden patronization?'

From the perspective of the ancient Greek sophists, mathematics from-above is an example of hidden patronization installed by a choice presented as nature; a choice made by their opponents, the philosophers, seeing geometry as demonstrating how physical forms are examples of metaphysical structures only visible to philosophers educated at the Plato Academy, consequently needed for patronizing through education.

Ancient Greece thus created two different forms of schooling: an enlightening school wanting to inform the people about the difference between choice and nature to prevent hidden patronization by choices presented as nature; and a patronizing school wanting to demonstrate how philosophical knowledge is exemplified in everyday life, thus in the need of openly philosophical patronization.

The Enlightenment period installed two democracies, one in the US and one in France. The US democracy created an enlightening school organizing its secondary school in blocks to allow its students to uncover and develop their individual talents through daily lectures in self chosen half year blocks. Today block organization is the international school standard outside the EU.

In contrast to this, the EU Bildung schools are still line-organized forcing students to follow predetermined block combinations and forcing them to wait for years for an exam that cannot be retaken; in contrast to the block-organized schools having half-year exams that can always be retaken. At enlightening schools the outside world determines the curriculum and the exams. To determine the content of Bildung, EU needs to be patronized by strong central administrations and by a special educational discourse called didactics.

Historically, the Bildung schools were invented in Prussia just after 1800 using Hegel based romanticism to obtain three goals: to keep the people unenlightened so it will not ask for democracy as in France; to install nationalism into the people so that it could protect itself against the French and their democracy; and to sort out the population elite for central administration offices.

From the perspective of the contemporary sophists, the French poststructuralists, presentations can be seen as examples of discourses fighting each other to win the monopoly of representing truth and thus to establish what Foucault calls pastoral power and discourse protection.

At universities, the 'mathematics from-above' discourse took over power with the introduction of set-theory just before 1900. And it has managed to stay in power despite of its internal problems as demonstrated by Russell showing a set-based definition will never be well defined (If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$); and by Gödel showing that truths is not always provable.

At schools, 'the mathematics from-above' discourse took over power as 'modern mathematics'. The traditional 'Rechnung' discourse disappeared since it was no longer seen as the root of but only as a simple application of mathematics that 'of course' must be learned before it can be applied.

With a patronizing goal and wish to sort out the elite for the central administration, it seems natural that Bildung schools has chosen the mathematics from-above discourse as its curriculum But it seems odd that enlightening schools does the same since it keeps many students unenlightened by using its defining IS-statements to forces false identities upon the natural fact Many.

Thus 2 ten-bundles and 3 unbundled is sentenced to be an example of a position system description 23, instead of enjoying its nature as the double stack consisting of 2.3 tens. 3*6 is sentenced to be an example of the category 'number-name', instead of enjoying its nature as a calculation predicting that 3 6s can be recounted as 1.8 tens. 3*x = 18 is sentenced to be an example of an equation and is forced to be solved by performing identical operation to both sides of the equation sign, instead of enjoying its nature as a reversed calculation that that can be re-reversed by moving numbers to the other side and reversing its calculation sign. 1/2 and 2/3 are sentenced to be examples of rational numbers and are forced to be added without respect to their units, instead of enjoying their nature as per-numbers needing their units to be added. Shifting units as 2$ = ?£ is sentenced to be an example of proportionality, instead of enjoying its nature as a recounting problem. The question 2 3s + 4 5s is sentenced to deportation from the discourse, instead of enjoying its nature as two stacks being added either on-top or next-to thus constituting the root of proportionality and integration. A function is sentenced to be an example of a set-product where first component identity implies second component identity, instead of enjoying its nature as a formula containing two unknowns. The question ‘5 seconds at 4m/s increasing to 6m/s gives ?m’ is sentenced to be an example of integral calculus, again being sentenced to be an example of a limit process, instead of enjoying its natures as uniting per-numbers by the area under the per-number graph (Zybartas et al, 2005).

With false identities forced upon it by the ruling discourse, students are not allowed to meet the root of mathematics, Many, in its materiality but only as examples of false identities. Thus the ruling from-above discourse becomes a clear example of hidden patronization becoming pastoral by hiding its natural alternative, manyology, rooting mathematics from-below in the natural fact Many.

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312. Workshop in Recounting and Decimal-writing

To deal with the natural fact Many, we totalize. However, there are hidden ways to count and add. This workshop demonstrates the power of recounting made possible by counting in icons before counting in tens. Recounting shows that natural numbers are decimal numbers carrying units. And recounting allows both proportionality and integration to be introduced in grade one.

Introduction

Enlightenment mathematics was as a natural science exploring the natural fact Many (Kline, 1972) by grounding its abstract concepts in examples, and by using the lack of falsifying examples to validate its theory. But after abstracting the set-concept, mathematics was turned upside down to modern mathematics or 'metamatism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathema-tism' true in the library, but not in the laboratory, as e.g. 2+3 = 5, which has countless counterexamples: 2m+3cm = 203 cm, 2weeks+3days = 17 days etc. Being self-referring, this modern mathematics did not need an outside world. However, a self-referring mathematics turned out to be a self-contradiction. With his paradox on the set of sets not belonging to itself, Russell proved that sets implies self-reference and self-contradiction as known from the classical liar-paradox 'this statement is false' being false when true and true when false: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise Gödel proved that a well-proven theory is a dream since it will always contain statements that can be neither proved nor disproved. In spite of being neither well-defined nor well-proven, mathematics still teaches metamatism creating big problems to math education.

The primary school '10-IS-ten' claim prevents recounting a total in e.g. 5s or 7s, which removes shifting units and proportionality from grade one. Likewise the 'on-top' claim prevents next-to addition in introducing integration. And finally forcing the false identity 23 upon 2.3 tens prevents decimals and fractions to be introduced to describe the unbundled.

The middle school 'proportionality-IS-linearity' claim prevents recounting in different units to produce per-numbers as 3$/5kg. And the neutralization claim prevents equations to be solved in the natural way, by recounting and restacking.

And the high school 'calculus-IS-limits' claim prevents integration to remain adding per-numbers by finding the area under the per-numbers graph as known form primary school's next-to addition and middle school's adding per-numbers as weighted average.

Contingency Research Unmasks Choices Presented As Nature

Alternatively, mathematics could return to its roots, Many, guided by contingency research uncovering hidden patronization by discovering alternatives to choices presented as nature.

Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists warned that to practice democracy, the people must be enlightened to tell choice from nature in order to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since to them all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should be given the role as natural patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment: when an apple obeys its own will, people could do the same and replace patronization with democracy.

Two democracies were installed: one in the US still having its first republic; and one in France, now having its fifth republic. German autocracy tried to stop the French democracy by sending in an army. However, a German mercenary was no match to a French conscript aware of the feudal consequence of defeat. So the French stopped the Germans and later occupied Germany. Unable to use the army, the German autocracy instead used the school to stop enlightenment in spreading from France. As counter-enlightenment, Humboldt used Hegel philosophy to create a patronizing line-organized Bildung school system based upon three principles: To avoid democracy, the people must not be enlightened; instead romanticism should install nationalism so the people sees itself as a ‘nation’ willing to fight other ‘nations’, especially the democratic ones; and the population elite should be extracted and receive ‘Bildung’ to become a knowledge-nobility for a new strong central administration replacing the former blood-nobility unable to stop the French democracy.

As democracies, EU still holds on to line-organized education instead of changing to block-organized education as in the North American republics allowing young students to uncover and develop their personal talent through individually chosen half-year knowledge blocks.

In France, the sophist warning against hidden patronization is kept alive in the post-structural thinking of Derrida, Lyotard, Foucault and Bourdieu. Derrida warns against ungrounded words installing what they label, such word should be ‘deconstructed’ into labels. Lyotard warns against ungrounded sentences installing political instead of natural correctness. Foucault warns against institutionalized disciplines claiming to express knowledge about humans; instead they install order by disciplining both themselves and their subject. And Bourdieu warns against using education and especially mathematics as symbolic violence to monopolize the knowledge capital for a knowledge-nobility (Tarp, 2004).

Thus contingency research does not refer to, but questions existing research by asking ‘Is this nature or choice presented as nature?’ To prevent patronization, categories should be grounded in nature using Grounded Theory (Glaser et al, 1967), the method of natural research developed in the first Enlightenment democracy, the American, and resonating with Piaget’s principles of natural learning (Piaget, 1970).

Mathematics: A Natural Science About The Natural Fact Many

To deal with Many, we iconize, bundle and totalize. 1.order counting bundles sticks in icons with five sticks in the five-icon 5 thus making 5 1s to 1 5s. In this way icons are created for numbers until ten needing no icon since 3.order counting in tens has become the standard.

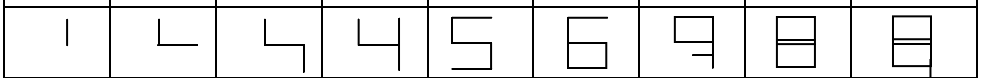
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

Figure 1. Rearranging sticks in icons transforms 5 1s into 1 5s etc.

2.order counting bundles in icon-bundles. Thus a total of 7 can be bundled in 3s as T = 2 3s & 1 to be placed in a right single-cup and in a left bundle-cup where a bundle can be traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III I -> III III) I) -> ▌▌) I) -> II) I) -> 2)1) -> 2.1 3s

Also bundles can be bundled and placed in a new cup to the left. Thus in 6 3s & 2 1s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II), or 6)2) = 2)0)2, thus 6.2 3s = 20.2 3s.

Adding an extra cup to the right shows that multiplying or dividing with the bundle-size just moves the decimal point: T = 2.1 3s = 2)1) -> 2)1)) = 2.1 3s * 3 = 21.0 3s and vice versa.

Operations iconize the bundling and stacking processes. Taking away 4 is iconized as – 4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 showing a 3 times lifting of the 4s. Placing a stack of 2 singles next-to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. as a 1 3^2-bundle.

Thus soft 'geometric addition' adds bundles next-to, e.g. T = 2 4s + 3 1s, while hard 'algebraic addition' adds on-top, e.g. T = 2 7s + 3 7s = 5 7s. Soft 'geometric subtraction' places part of a stack next to the stack, e.g. 3 4s = 2 4s + 1 4s, while hard 'algebraic subtraction' removes from the top, e.g. T 3 4s – 1 4s = 2 4s. Soft 'geometric multiplication' repeats adding on-top, e.g. T = 2* (3 7s) = 6 7s, while hard 'algebraic multiplication' recounts in tens, making 2* (3 7s) not 6 7s but 4 tens & 2 1s = 4.2 tens = 42. And soft 'geometric division' recounts a stack in a different bundle-size, e.g. 9 = (9/4)*4 = 2*4 + 1, while hard 'algebraic division' recounts in tens, e.g. 9/4 = 2.25 = 0.225 tens.

Numbers and operations can be combined to calculations predicting the counting results. The 'recount-formula' T = T/b bs = (T/b)*b tells that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6)/7 7s. Using a calculator we get the result 2 7s and some leftovers that can be found by the 'rest-formula'

T = (T–b) + b telling that T–b is the rest when b is taken away and placed next-to: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. So the combined prediction says T = 3*6 = 2*7 + 4*1 = 2.4 7s.

This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

A total as T = 2.3 5s can be described as a geometrical number by using squares. After stacking the two 5-bundles, the 3 leftovers may be placed either next to the stack as a stack of 1s, which can be written as 2.3 5s; or on top of the stack of 5-bundles counted as 5s, i.e. as 3 = (3/5)*5 thus giving a total of T = 2 3/5 5s.

Thus 3/5 5s is just another way of writing 0.3 5s. And when bundling in tens, 3/ten tens is the same as 0.3 tens. So for any bundle-number, we define 'count-fractions' by 3/b = 0.3. And we see that count-fractions are rooted in bundling the leftovers.

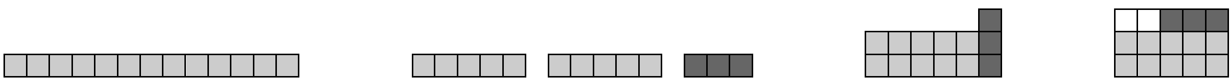


Figure 2. A Total has 3 unbundled added next-to or on-top: T = 2.3 * 5 = 2 3/5 * 5

Count-fractions describe the unbundled 1s when counting in bundles. In the case of bundling in 5s or tens, 4 bundles & 3 unbundled can be written as respectively T = 4.3 5s = 4 3/5 5s or T = 4.3 tens = 4 3/10 tens. In this way both 3/5 of 5 is 3 and 3/10 of 10 is 3. As to 3/5 of 20, 20 can be recounted to 5 fours, and since 3/5 of 5 units is 3 units, 3/5 of 5 4s = 3 4s = 3*4 = 12.

So already in primary school 2.order recounting in icon-bundles enables learners to predict and practice changing units, the leitmotif of mathematics, reappearing later as proportionality and per-numbers. This also allows practicing the scientific method using formulas for predictions to be tested. In this ten-free zone it becomes possible to introduce the core of mathematics using 1digit numbers only (Zybartas et al, 2005). The MATHeCADEMY.net has developed its CATS-approach, Count&Add in Time&Space, to introduce teachers to a grounded approach to mathematics as a natural science investigating the natural fact Many by counting by bundling and stacking, and using recounting at all school levels.

3.Order Counting In Tens Prevents Recounting

3.order counting in tens should be postponed as long as possible. Before introducing ten as 10, i.e. as the standard bundle-size, 5 should be the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is not used when using place values, nor in the counting sequence: 1, 2, 3, 4, Bundle, 1B1, 1B2, 1B3, 1B4, 2B, 2B1, etc.; or 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, etc.

Thus counting in tens can begin using neither a ten-icon nor the ten-name: 8, 9, Bundle, 1Bundle1, 1B2, 1B3, …, 1B9, 2B, 2B1, etc. Then the name bundle can be replaced by the name ten counting 8, 9, Ten, 1Ten1, 1T2, …, 1T9, 2T, 2T1, etc. Finally the sloppy words eleven and twelve can be used meaning '1 left' and '2 left' in 'Anglish', i.e. in old English.

A Premature Introduction of Ten as THE Standard Bundle May Make Ten a 'Cognitive Bomb'

When beginning to count in tens, numbers are no more written as natural numbers, i.e. as decimals carrying units. Instead numbers are written using the sloppy place-value method hiding both the total and the unit, and misplacing the decimal point: T = 2.3 tens -> plain 23.

Almost all operations change meanings. Soft addition next-to as in T = 2.3 4s = 2*4 + 3*1 is changed to hard addition on-top as in 23 + 48 = 71. Soft multiplication where 3*8 means 3 8s is changed to hard multiplication, i.e. to division recounting the 3 8s in tens: 3*8 IS 24.

Now /4 now means divided in 4, not counted in 4s. Only –3 still means take away 3.

As shown by the recount- and rest-formulas, with 2.order counting the order of operations is: first /, then *, then –, and finally +. With 3.order counting in tens this order is turned around: first +, then –, then *, and finally /.

With ten as the only bundle-size, recounting is impossible to do and to predict by formulas since asking '3 8s = ? tens' leads to T = (3*8/ten)*ten that cannot be calculated on a calculator. Now the answer is given by multiplication, 3*8 = 24 = 2 tens + 4 ones, thus transforming multiplication into division.

Patronizing Versus Grounded Mathematics in Primary School

In primary school the tradition skips 1.order and 2.order counting and goes directly to 3.order counting claiming that '10 IS ten', i.e. the follower of 9 in spite of the fact that 10 is the follower of 4 when counting in 5s. A grounded alternative postpones 3.order counting until after 2.order counting has introduced recounting in different units as an introduction to proportionality; and until after soft addition next-to has introduced integration.

The tradition presents 1digit numbers as symbols and 2digit numbers as natural numbers. A grounded alternative introduces 1digit numbers as what they really are: icons rearranging the sticks they represent; and introduces the natural numbers as what they really are: decimal-numbers with units, e.g. 2.3 4s, and 2.3 tens instead of just 23.

The tradition presents addition on-top as the first of the four operations, where e.g. 7 + 4 = 11 forces the introduction of ten as the bundle-size, and forces the sloppy way of writing 2digit numbers without decimals or units. A grounded alternative first introduces addition next-to so that 2 5s + 4 1s means placing a stack of 4 1s next to a stack of 2 5s, i.e. as 2.4 5s.

The tradition presents hard multiplication as the third operation with tables to be learned by heart, forcing all stacks to be recounted in tens, 3*8 IS 24 etc. A grounded alternative first introduces soft multiplication so that 3*8 means a stack of 3 8s, not needing to be recounted into tens before being recounted in other icon-bundles as 5s or 7s etc.

The tradition presents division as the last of the four operations, where /4 means to split in 4. A grounded alternative introduces division as the first operation, where /4 means counting in 4s. Likewise the recount formula T = (T/b)*b together with a calculator is introduced from grade one to allow formulas and calculators to predict results to be tested by counting.

The tradition introduces 'mathematism' true in the library but not in the laboratory, by teaching that '2 + 3 IS 5' in spite of the fact that 2weeks + 3days = 17days, 2m + 3c = 203cm etc. A grounded alternative always includes the units when adding, e.g. 2 4s + 3 5s = 4.3 5s.

The Workshop's Learning Principles

The workshop builds upon two learning principles. The first says that proper actions prepare individuals to meet the outside world. Wanting to prepare the learner for this meeting, education should be described in terms of actions. Humans deal with the natural fact Many by totalising. Consequently, education should teach learners how to count and how to add, but not how to math since math is not an action word, a verb. Thus counting and adding in time and space are the two core competences to acquire in mathematics education, resonating with the words algebra and geometry meaning to reunite in Arabic and to measure earth in Greek.

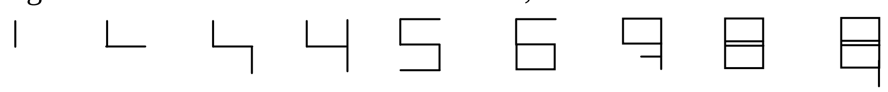
The second learning principle says 'greifen vor begreifen' or 'graps before grasping' or 'through the hands to the head'. Basically it says that categories are names associated with what is grasped by the hands or pointed to. This allows categories to be grounded in hand held objects and observations. This principle resonates with Piaget and with Grounded theory.

A. Counting By Bundling And Stacking, Recounting

In this section we deal with the natural fact Many by performing 1.order counting producing icons and 2.order counting bundling in icons. Thus 3.order counting in tens is postponed.

A0. We place a total of nine sticks on a table. I I I I I I I I I

A1. 1.order counting: We take one stick and form a 1-icon, two sticks to form a 2-icon etc.



A2. 3.order counting in tens: We count fingers in 5s to experience that 5 becomes 10 not needing and icon: 1, 2, 3, 4, B = 10, B1 = 11, B2 = 12, B3 = 13, B4 = 14, B5 = BB = 20, etc. Then we count fingers in 4s, first with whole fingers, then with finger parts. So choosing 3.order counting in ten makes ten to 10 not needing an icon.

A3. 2.order counting in icons: We add 3 sticks and recount the total in 2s, 3s and 4s:

T = 6 2s = 6*2, T = 4 3s = 4*3, T = 3 4s = 3*4.

A4. Cup-counting: We add two sticks and recount the total in 5s, using two cups, a left cup for the bundles and a right cup for the unbundled. We need not put all five sticks in the bundle cup, only 1 stick since 5 1s = 1 5s. Thus we get

T = I I I I I I I I I I I I I I -> IIIII IIIII I I I I -> I I) I I I I)

A5. Cup-writing: Writing down the result, we use parentheses for cups and a decimal point to separate the bundles from the unbundled.

T = I I) I I I I I) -> 2) 4) -> 2.4 so T = 2.4 5s

A6. We recount the total in 6s, 7s, 8s and 9s using cup-writing.

T = I I I I I I I I I I I I I I -> I I) I I) -> 2) 2) -> 2.2 so T = 2.2 6s

Likewise T = 2.0 7s, T = 1.6 8s and T = 1.5 9s.

A7. Overloads: We recount the total in 3s

T = I I I I I I I I I I I I I I -> I I I I) I I) -> 4) 2) -> 4.2 so T = 4.2 3s,

or since 3 threes = 1 three-threes

T = I I I I) I I) -> III I) I I) -> I) I) I I) -> 1)1)2) -> 11.2 so T = 11.2 3s

Consequently T = 4.2 3s = 11.2 3s = 1 three-threes & 1 threes & 2 ones.

A8. We recount the total in 2s

T = I I I I I I I I I I I I I I -> II II II II II II II-> 7)0) -> 1)5)0) -> 2)3)0) -> 3)1)0) -> 1)1)1)0)

Consequently T = 7 2s = 15 2s = 23 2s = 31 2s = 111 2s

A9. Recounting 8.7 4s, 4 1s in a right cup can be moved to 1 4s in its left cup and vice versa.

T = 8.7 4s = 8)7) = 8+1)-4+7) = 9)3) = 1)5)3) = 2)1)3) = 2)1-1)+4+3) = 2)0)7) = 1)4)7)

Consequently T = 8.7 4s = 9.3 4s = 15.3 4s = 21.3 4s = 20.7 4s = 14.7 4s

A10. We recount 2.3 5s in 6s, 7s, 8s, and 9s

T = 2.3 5s = 2)3) 5s -> 2)1) 6s -> 1)6) 7s -> 1)5) 8s -> 1)4) 9s

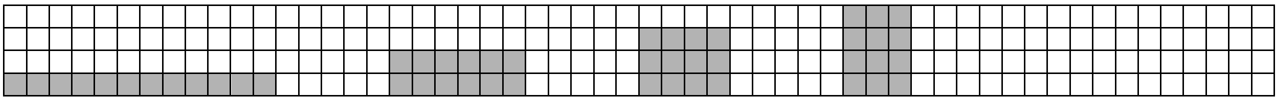
Consequently T = 2.3 5s = 2.1 6s = 1.6 7s = 1.5 8s = 1.4 9s

A11. We recount 2.3 5s in 4s, 3s, 2s, and 1s

T = 2.3 5s = 2)3) 5s -> 2)5) 4s -> 2)7) 3s -> 2)9) 2s -> ?? 1s

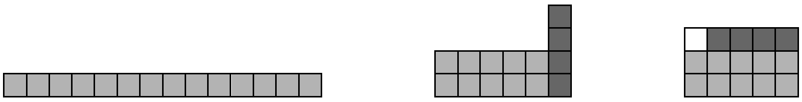
Consequently T = 2.3 5s = 2.5 4s = 2.7 3s = 2.9 2s. T cannot be bundled in 1s since 1^2 = 1.

A12. Grounding fractions. When counting, we use squares instead of sticks to get ‘geometric numbers’. We recount 9+3 squares in 6s, 4s and 3s.



T = 2 6s = 2*6 = 3*4 = 4*3

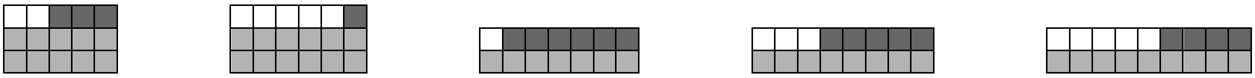
A13. We recount 9+5 squares in 5s



T = 2.4 5s = 2 4/5 5s

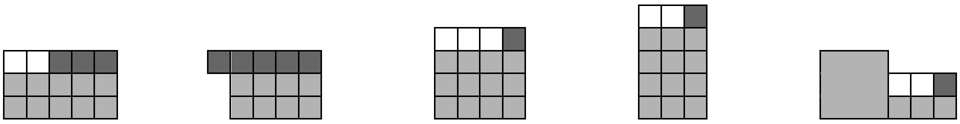
A14. We recount 2 3/5 5s in 6s, 7s, 8s, and 9s:

T = 2 3/5 5s = 2 1/6 6s = 1 6/7 7s = 1 5/8 8s = 1 4/9 9s



2 3/5 5s = 2 1/6 6s = 1 6/7 7s = 1 5/8 8s = 1 4/9 9s

A15. We recount 2.3 5s in 4s and 3s:



2 3/5 5s = 2 5/4 4s = 3 1/4 4s = 4 1/3 3 = 11 1/3 3s

3 5s = 2.3 5s + 2 4 4s = 3.1 4s + 3 12 3s = 11.1 3s + 2

B. Inventing Operations To Predict Counting Results

In this section we combine numbers with operations to predicting formulas.

B1. The rest formula: From 7 sticks we' take away 3' to be placed next-to.

T = I I I I I I I -> I I I I I I I or 7 = (7-3) + 3 = 4 + 3 or T = (T– b) +b

B2. A calculator predicts. We calculate 8 – 5 on a calculator and perform the action

Prediction: 8 – 5 = 3. Test: I I I I I I I I -> I I I I I I I I or 8 = (8 – 5) + 5 = 3 + 5

B3. The recount formula: From 8 sticks we 'take away 2s' to be placed next-to.

T = I I I I I I I I -> II II II II or 8 = 8/2 2s = (8/2) * 2 = 4 * 2 or T = (T/b) * b = T/b bs

B4. A calculator predicts. We calculate 6/2 on a calculator and perform the action

Prediction: 6/2 = 3. Test: I I I I I I -> II II II or 6 = (6/2) * 2 = 3 * 2

B5. Predicting leftovers: We calculate 7/2 on a calculator and perform the action

Prediction: 7/2 = 3.r and 7 = (7 – 3*2) + 3*2 = 1 + 3*2 = 3.1 2s

Test: I I I I I I I -> II II II I or T = 3.1 2s

C. Solving Equations By Restacking And Rebundling

In this section we look at reversed calculations also called equations and observe that the natural way to solve equations is to move numbers to the opposite side with opposite sign.

C1. We solve the equation x + 5 = 9 by restacking and perform the action

x + 5 = 9 = (9 – 5) + 5, so x = 9 –5 = 4 . Test: **X** I I I I I = I I I I I I I I I = I I I I I I I I I

C2. We solve the equation x * 2 = 6 by rebundling and perform the action

x * 2 = 6 = (6/2)*2, so x = 6/2 = 3 . Test: **XX** = I I I I I I = II II II = III III since 3 2s = 2 3 s.

C3. We solve the equation 2*x + 1 = 7 by restacking and rebundling

2*x + 1 = 7 = (7 – 1) +1 = 6 + 1, so x*2 = 6 = (6/2)*2 = 3*2, so x = (7-1)/2 = 6/2 = 3.

D. Splitting Stacks

In this section we look at splitting stacks by taking away, or selling, a part.

D1. From a stack of 4.2 5s is sold 1.4 5s. What is left?

Transform 4.2 5s into cup-writing	T = 4.2 5s = 4) 2)
Change icons into sticks	T = IIII) II)
Change 1 5s into 5 1s	T = III) IIIII II)
Remove the 1.4 5s	T = II) III) + I) IIII)
Go back to icons and decimals	T = 2.3 5s + 1. 4 5s
Test the result by adding	T = 3.7 5s = 4. 2 5s

D2. Splitting, using numbers.

Transform 4.2 5s into cup-writing	T = 4.2 5s = 4) 2)
Change 1 5s into 5 1s	T = 4-1) 5+2) = 3) 7)
Remove the 1.4 5s	T = 2) 3) + 1) 4)
Go back to decimals	T = 2.3 5s + 1.4 5s
Test the result by adding	T = 3.7 5s = 4. 2 5s

D3. From a stack of 42 (4.2 tens) is sold 14 (1.4 tens). What is left?

Transform 42 into cup-writing	T = 42 = 4) 2)
Change 1 tens into 10 1s	T = 4-1) 10+2) = 3) 12)
Remove the 14 = 1.4 tens	T = 2) 8) + 1) 4)
Go back to decimals	T = 2.8 tens + 1.4 tens = 28 + 14
Test the result by adding	T = 2)8) + 1)4) = 3)12) = 4)2)

E. Adding Stacks

Once counted, stacks can be added on-top or next-to. Adding on-top means the units must be the same. Adding next-to integrates the units.

E1. To a stack of 2.3 5s is bought 1.4 5s. What is the Total?

Transform 2.3 5s and 1.4 5s into cup-writing	2.3 5s + 1.4 5s = 2)3) + 1)4)
Change icons into sticks	T = I I) I I I) + I) I I I I)
Adding 1.4 5s to the 2.3 5s gives 3.7 5s	T = I I I) I I I I I I I)
Change 1 5s into 5 1s	T = I I I) IIIII I I) -> I I I I) I I)
Write down the addition result	2.3 5s + 1.4 5s = 3.7 5s = 4.2 5s

E2. Adding using numbers. To a stack of 2.3 5s is bought 3.2 4s. What is the total in 4s?

Recount the 2.3 5s in 4s	T = (2*5+3)/4 *4 = 3.1 *4
Add 3.1 4s and 3.2 4s	T = 3.1 4s + 3.2 4s = 6.3 4s
Recount the result	T = 12.3 4s

E3. Add the two stacks 2.3 5s and 3.2 4s as 9s (integration).

Recount the 2.3 5s in 9s	T = (2*5+3)/9 *9 = 1.4 *9
Recount 3.2 4s in 9s	T = (3*4+2)/9 *9 = 1.5 *9
Perform the addition	T = 1.4 9s + 1.5 9s = 2.9 9s
Recount the result	T = 3.0 9s

F. Adding Or Removing Cups

In this section we see that adding and removing cups to the right means multiplying or dividing with the bundle-number, which just moves the decimal point one place.

Multiply 3.2 5s with 5 by adding a cup to the right	3)2) -> 3)2)) = 3)2) *5
Divide 14.0 5s with 5 by removing a cup to the right	1)4)) -> 1)4) = 1)4)) / 5

G. Recounting In Different Physical Units Creates Per-Numbers

In this section we see how per-numbers as 2$/5kg = 2/5 $/kg are used to relate physical units.

G1. With 2$/5kg, what does 40 kg cost?

We simply recount 40 in 5s: T = 40kg = (40/5)*5kg = (40/5)*2$ = 16$

G2. With 2$/5kg, how much can 12 $ buy?

We simply recount 12 in 2s: T = 12$ = (12/2)* 2$ = (12/2)* 5kg = 30kg

G3. Transform the per-number 2/5 u/u to percent %.

We simply recounts 100 in 5s: T = 100u = (100/5)*5u -> (100/5)*2u = 40u, so 2/5 = 40/100.

G4. Find 20% of 350$.

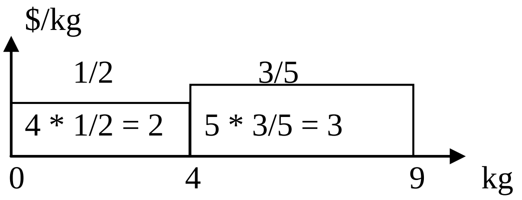
We recount the 350 in 100s: T = 350$ = (350/100)*100$ -> (350/100)*20$ = 70$.

H. adding per-numbers

In this section we see that their areas add variable per-numbers, thus rooting integration.

H1. 4 kg at 1/2 $kg + 5kg at 3/5 $/kg = 9 kg at ? $/kg. The answer is given by a graph.

4kg at 1/2 $/kg = 4 * 1/2 = 2 5kg at 3/5 $/kg = 5 * 3/5 = 3	
9kg at ? $/kg = 9 * ? = 5 ? = 5/9	



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313. Posters at ICME 12

These are the posters presented at the ICME12. First the poster for the general poster session. Then posters for the Topic Study Groups 7, 13, 21, 24, 26 and 37

Mathematism: Adding Without Units, Poster Session

Which is Correct, A or B?

I	A. 2 + 3 = 5	B. 2 * 3 = 6
II	A. Teacher: What is 1/2+2/3? No. The correct answer is 1/2 + 2/3 = 3/6 + 4/6 = 7/6	B. The students: 1/2 + 2/3 = (1+2)/(2+3) = 3/5 But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 out of 6 cokes?
III	A. 2 3s + 4 5s = 3.2 8s	B. 2 3s + 4 5s = 5.2 5s

1. Students and teachers normally agree on answering 'A, A,?'. After filling out a ballot and seeing the correct answer 'B, B, A', a discussion may follow on the difference between grounded mathematics rooted in observations and ungrounded 'mathematism' true in the library but not in the laboratory. Thus '2*3 = 6' is natural correct since 3 is a unit, and 2 3s can be recounted as 6 1s, whereas '2+3 = 5' may be political correct in a library, but has countless counterexamples in the laboratory: 2 weeks + 3 days = 17 days etc. Likewise, real-life fractions are per-numbers to be multiplied with their totals to becomes unit-numbers before being added: 2 kg at 3/4 $/kg + 5 kg at 6/7 $/kg = 7 kg at (3/4*2+6/7*5)/7 $/kg. The argument 'but 2+3 IS 5' can be met with questions as 'is mathematics a science or a religion?' and 'should mathematics education teach how to math or how to totalize?'

2. Geometry and algebra means to measure earth and to re-unite in Greek and Arabic. The basic question 'what is the total?' shows that both are grounded in the natural fact Many. To deal with Many, we count and add. 1st order counting rearranges sticks into icons with five sticks in the 5-icon etc. 2nd order counting bundles in icon-bundles, e.g. T = 3 4s = 3*4;and 3rd order counting bundles in ten-bundles. Both resulting in decimal numbers with units, using a decimal point to separate the bundles and the unbundled: T = 3.4 5s, T = 6.7 tens = 67. Counting in 4s means removing 4s, i.e. divide by 4. So when counting in b-bundles, the process and product can be reported as T = T/b bs = (T/b)*b, the recount-formula.

3. Once counted, totals can be added. Added horizontally 'next-to', 2 3s and 4 5s gives a total of T = T1+T2 = 2 3s + 4 5s = 3.2 8s. Adding vertically 'on-top', recounting makes the units the same: T1 = 2 3s = 1.1 5s making T = T1+T2 = 1.1 5s + 4 5s = 5.1 5s; or T2 = 4 5s = 6.2 3s and T = 8.2 3s. Thus changing units when adding on-top roots proportionality. And juxtaposing stacks when adding next-to roots integration, both practiced in first grade. So 2nd order counting offers many golden learning possibilities that 3rd order ten-counting prevents.

Recounting is the Root of Grounded Mathematics, TSG 7

Mathematics: A Natural Science About The Natural Fact Many

To deal with Many we totalise by bundling and stacking. 1.order counting bundles sticks into icons with five sticks in the icon 5, thus making 5 1s into 1 5s. 3.order counting bundles in tens needing no icon since 10 means 1 bundle. 2.order counting bundles in icon-bundles. Thus a total of 7 can be bundled in 3s as T = 2 3s and 1. When placed in a right single-cup and in a left bundle-cup, 3 sticks is traded first to a thick stick gluing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using ‘cup-writing’ 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III) I) -> ▌▌) I) -> 2)1) -> 2.1 3s or 2 1/3 3s if placed next-to or on-top a stack.

Once counted, totals can be added on-top or next-to. Adding on-top, the units must be the same using the recount-formula T = T/b bs = (T/b)*b saying that T/b times bs can be taken away from T. Thus 3 7s can be recounted in 5s as T = (3*7/5)*5 = 4.1 5s. Changing units roots proportionality. Next-to addition of 2 3s and 3 4s as areas to 2.4 7s roots integration.

To find 3/5 of 20 we recount 20 in 5s: T = 20 = (20/5)*5 = 4*5 = 5*4 = 5 fours.

So 3/5 of 20 = 3/5 of 5 fours = 3 4s = 3*4 = 12. Or the short version: 3/5 of 20 = 3 * (20/5).

Prices are per-numbers as 5$/3kg or 5/3 $/kg used to shift units by recounting:

T = 12 kg = (12/3)*3kg = (12/3)*5$ = 20$ and T = 40$ = (40/5)*5$ = (40/5)*3kg = 24 kg

Thus 2.order icon-counting allows fractions and decimals as well as proportionality and integration to be introduced and practised from grade one making mathematics easy for all. And as to tables, 3*8 is 3 8s and only 24 if recounted in tens. Also 24 can be recounted as 3 8s.

However, the tradition skips 1. and 2.order counting and goes directly to 3.order ten-counting claiming that 10 IS the follower of 9 in spite of the fact that 10 is the follower of 4 when counting in 5s; and forces 2.4 tens to be written as 24 leaving out the unit and misplacing the decimal point.

With ten as THE bundle-size, recounting is impossible to do and to predict by formulas since asking ‘3 8s = ? tens’ leads to T = (3*8/ten)*ten that cannot be calculated. Now recounting in tens is called multiplication, 3*8 = 24 = 2 tens + 4 ones, thus transforming multiplication into division. Also ten-counting prevents changing units and adding next-to.

Silencing 2.order recounting makes mathematics difficult to learners. So 3.order ten-counting should be postponed to after recounting has rooted mathematics.

Calculus Grounded in Per-numbers Added Next-to, TSG 13

Primary School Calculus Is Rooted In Next-To Stack Addition

Mathematics is a natural science dealing with the natural fact Many by bundling and stacking. 1.order counting bundles sticks into icons with five sticks in the 5-icon making 5 1s to 1 5s. 3.order counting bundles in tens needing no icon since 10 means 1 bundle. 2.order counting bundles in icon-bundles. Thus a total of 7 can be bundled in 3s as T = 2 3s & 1 = 2.1 3s

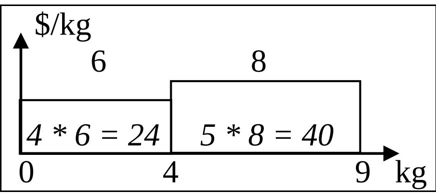
Once counted, totals can be added on-top or next-to. With on-top addition the units must be the same using the recount-formula T = T/b bs = (T/b)*b saying that T/b times bs can be taken away from T. Thus 3 7s can be recounted in 5s as T = (3*7/5)*5 = 4.1 5s. Changing units roots proportionality. Next-to addition of 2 3s and 3 4s as areas to 2.4 7s roots integration.

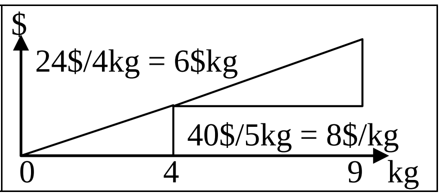
Middle School Calculus Is Rooted In Adding Per-Numbers

Recounting creates fractions when e.g. counting 3 1s in 5s as T = 3 = (3/5)*5 = 0.3 5s, which shows that 3/b = 0.3 for any bundle-size b; and 'per-numbers' when a quantity is recounted first as 2$ then as 5kg thus containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg.

In primary school addition next-to adds stacks by integrating bundles. Now integration adds per-numbers when adding two recounted quantities, asking e.g. 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg. This question can be answered by using a table or a graph.

4kg at 6$/kg	= 4 * 6 = 24
5kg at 8$/kg	= 5 * 8 = 40
9kg at ?$/kg	= 9 * ? = 64
	? = 64/9





Using a graph we see that integration means finding the area under a per-number graph; and opposite that the per-number is found as the gradient on the total-graph.

Hign School Calculus Is Adding Footnotes

Middle school per-numbers are piecewise constant. In high school per-numbers are locally constant, continuous. Still, the area under its graph adds variable per-numbers f by summing up area strips f*dx. This area can be found by a graphical calculator, or by integration using that the sum of many small changes gives a total change: $\int dF = \Delta F = F2 - F1$, using technology or the head to solve the change equation dF = f*dx.

Contingency Research Discovers Hidden Math, TSG 21

Mathematics From Below Or Metamatism From Above

Enlightenment mathematics was as a natural science exploring the natural fact Many by grounding its abstract concepts in examples, and by using the lack of falsifying examples to validate its theory.

But after abstracting the set-concept, mathematics was turned upside down to modern mathematics or ‘metamatism’, a mixture of ’meta-matics’ defining its concepts as examples of abstractions, and ‘mathema-tism’ true in the library, but not in the laboratory, as e.g. 2+3 = 5, which has countless counterexamples: 2m+3cm = 203 cm, 2weeks+3days = 17 days etc.

To avoid becoming metamatism, mathematics must return to its roots, Many, guided by a contingency research looking for hidden alternatives to choices presented as nature.

Avoiding hidden patronization is the root of scepticism looking for hidden contingency to choices presented as nature, and expressed in the two Enlightenment republics in American pragmatism and grounded theory, and in French poststructural thinking of Derrida, Lyotard, Foucault and Bourdieu warning against hidden patronization by the terms deconstruction, postmodernism, pastoral power of institionalized disciplines, and education as symbolic violence using mathematics to monopolize the knowledge capital for a knowledge nobility.

Hidden Mathematics In Primary And Middle School

Mathematics is a natural science dealing with the natural fact Many by bundling and stacking. 1.order counting bundles sticks into icons with five sticks in the 5-icon making 5 1s to 1 5s. 3.order counting bundles in tens needing no icon since 10 means 1 bundle. 2.order counting bundles in icon-bundles. Thus a total of 7 can be bundled in 3s as T = 2 3s & 1 = 2.1 3s.

Once counted, totals can be added on-top or next-to. At on-top addition the units must be the same, using the recount-formula T = T/b bs = (T/b)*b saying that T/b times bs can be taken away from T. Thus 3 7s can be recounted in 5s as T = (3*7/5)*5 = 4.1 5s. Changing units roots proportionality. Next-to addition of 2 3s and 3 4s as areas to 2.4 7s roots integration.

Recounting creates fractions when e.g. counting 3 1s in 5s as T = 3 = (3/5)*5 = 0.3 5s, which shows that 3/b = 0.3 for any bundle-size b; and ‘per-numbers’ when a quantity is recounted first as 2$ then as 5kg thus containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg.

Primary school integration adds stacks horizontally.

Middle school integration adds per-numbers coming from recounting quantities, asking e.g.

4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg.

Again the answer is an area, this time under a per-number graph.

Using contingency research to unhide hidden differences makes a big difference to mathematics education.

Recounting Roots Secondary Mathematics, TSG 24

Recounting is the Root of Primary Mathematics

Mathematics is a natural science dealing with the natural fact Many by bundling and stacking.

1.order counting bundles sticks into icons with five sticks in the 5-icon making 5 1s to 1 5s.

3.order counting bundles in tens needing no icon since 10 means 1 bundle.

2.order counting bundles in icon-bundles. Thus a total of 7 ones can be bundled in 3s as T = 2 3s & 1 = 2.1 3s.

Once counted, totals can be added on-top or next-to. With on-top addition the units must be the same using the recount-formula T = T/b bs = (T/b)*b stating that T/b times bs can be taken away from T. Thus 3 7s can be recounted in 5s as T = (3*7/5)*5 = 4.1 5s. Changing units roots proportionality. Next-to addition of 2 3s and 3 4s as areas to 2.4 7s roots integration.

Recounting is the Root of Secondary Mathematics

Recounting creates fractions when e.g. counting 3 1s in 5s as T = 3 = (3/5)*5 = 0.3 5s, which shows that 3/b = 0.3 for any bundle-size b; and ‘per-numbers’ when a quantity is recounted first as 5$ then as 3kg thus containing 5$ per 3kg, or 5$/3kg or 5/3 $/kg used to shifting units:

T = 12 kg = (12/3)*3kg = (12/3)*5$ = 20$ and T = 40$ = (40/5)*5$ = (40/5)*3kg = 24 kg

Primary school’s next-to addition roots integration. Now integration adds per-numbers when adding two recounted quantities, asking e.g. 4 kg at 6$/kg + 5kg at 8$/kg = 9 kg at ? $/kg. Again the answer comes by finding the area, this time under the per-number graph.

Geometry is grounded in its Greek meaning, earth-measuring: All forms can be split into right-angled triangles, where the relationship between the angle and the side can be expressed by recounting the sides, a = a/c*c = sinA*c, b = b/c*c = cosA*c, a = a/b*b = tanA*b, thus creating the percentage numbers sinA, cosA and tanA.

Recounting solves equations by moving numbers across: x*4 = 20 = (20/4)*4, so x = 20/4.

A formula containing x, y = f(x), relates two variables y and x. A change in x will give a change in y that recounted in x-changes provides gradient numbers f´: dy = dy/dx*dx = f´*dx

Middle school per-numbers are piecewise constant. In high school per-numbers are locally constant, continuous. Still, the area under its graph adds variable per-numbers f by summing up area strips f*dx. This area can be found by a graphical calculator, or by integration using that the sum of many small changes gives a total change: ∫dF = ΔF = F2 – F1, using technology or the head to solve the change equation dF = f*dx.

Recounting and Adding Next-To Roots Mathematics, TSG 26

Recounting is the Root of Primary Mathematics

Mathematics is a natural science dealing with the natural fact Many by bundling and stacking. 1.order counting bundles sticks into icons with five sticks in the 5-icon making 5 1s to 1 5s. 3.order counting bundles in tens needing no icon since 10 means 1 bundle. 2.order counting bundles in icon-bundles. Thus a total of 7 can be bundled in 3s as T = 2 3s & 1 = 2.1 3s.

Rewriting total as T = 345 = 3*10^2 + 4*10 + 5*1 shows numbers have units; that totalising means adding bundles; and there are the four ways of uniting bundles: +, *, ^ and integration.

Recounting means accepting that a given total is described by different numbers depending on the unit: T = 3 4s = 2.2 5s = 2 6s = 1.5 7s etc, and T = 8 kg = (8/2)*2kg = 4 (2kg) etc.

Once counted, totals can be added on-top or next-to. With on-top addition the units must be the same, using the recount-formula T = T/b bs = (T/b)*b stating that T/b times bs can be taken away from T. Thus 3 7s is recounted in 5s as T = (3*7/5)*5 = 4.1 5s. Changing units roots proportionality. Next-to addition of 2 3s and 3 4s as areas to 2.4 7s roots integration.

Counting & Adding in Time & Space: The Cats Approach to Math

MATH**e**CADEMY.net contains grounded mathematics organized in activities where primary and secondary teachers learns ‘CATS’ through educational questions Q and answers A. Examples of Counting C1, Adding A1, Time T1 and Space S1 from primary school:

Q: How to recount 7 in 3s: T = 7 = ? 3s. A: 7 = (7/3)*3 = 2*3+1*1= 2)1) = 2.1*3 = 2 1/3*3 Q: How to recount 6kg in $: T= 6kg = ?$. A: If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$

Q: How to add stacks concretely? T = 27 + 16 = 2 ten 7 + 1 ten 6 = 3 ten 13 = ?.

A: By restacking overloads predicted by the ‘restack-equation’ T = (T–b) + b: T = 27 + 16 = 2ten 7 + 1ten 6 = 3ten 13 = 3ten 1ten 3 = 4ten 3 = 43, or T = 2)7) + 1)6) = 3)13) = 4)3) = 43.

Q: How can counting & adding be reversed? A: By reversed calculation: move a number to the opposite side of the equation sign with opposite calculation sign: if x+3 = 8, then x = 8-3.

Q: Counting ? 3s and adding 2 gave 14. A: x * 3 + 2 = 14 is reversed to x = (14 – 2)/3.

Q: How to count the plane and spatial properties of stacks, boxes and round objects?

A: By using a ruler, a protractor and a triangular shape; by the 3 Greek Pythagoras’s: mini, midi & maxi; and by the 3 Arabic recount-formulas: a = a/c = sin A *c, b = b/c = cos A *c, and a = a/b = tan A *b.

Is Mathematics Education a Mental Gas Chamber? TSG 37

Social Theory Can Unmask Mathematics as Metamatism

Social theory has human interaction as its main focus. As to communication, Berne's concepts Parent, Adult and Child reflect the social fact that interaction between human beings can be patronized and non-democratic, or it can be non-patronized and democratic.

Newton discovered that objects fall by self-determination and not by patronization. This created the Enlightenment 1700-century with two democracies, in the US and in France. US still has its first republic based upon pragmatism seeing humans as agents creating social structures via symbolic interactionism, and creating social knowledge via Grounded Theory resonating with the principles of natural learning expressed by Piaget.

In Europe counter-enlightenment reinstalled metaphysical patronization in the form of Hegel's Spirit expressing itself through a people with a history. Marxism developed Hegel thinking into critical theory in the Frankfurt school infiltrating the 1968 student revolt so that EU's Bildung universities could carry on protecting its Hegel-based patronization.

Wanting to protect its republic against patronization, France developed post-structuralism inspired by Nietzsche's opposition against Hegel, and by Heidegger's question 'what is IS?'

Derrida's 'logocentrism' warns against patronizing words installing what they label and recommends that such word be deconstructed.

Lyotard's 'postmodern' warns against sentences expressing political instead of natural correctness.

Foucault's 'pastoral power' warns against institutionalized disciplines disciplining itself and its object.

Bourdieu sees society as fields where people fight for capital; and sees mathematics as specially suited to perform the symbolic violence that monopolizes the knowledge capital for a knowledge nobility.

Bauman points out that by following authorized routines modernity can create both gas turbines and gas chambers.

Arendt shows how in institutionalised societies patronization might become totalitarian, thus reintroducing evil actions this time rooted not in a devil but in the sheer banality of just following orders.

Does education create mental concentration camps with forced classes, and mental gas chambers when replacing mathematics rooted in the natural fact Many with 'metamatism', a mixture of 'metamatics' defining concept as examples of abstractions instead of as abstractions from examples; and 'matematism' with claims that are true in a library but not in a laboratory as 2+3 = 5, contradicted e.g. by 2 weeks + 3 days = 17 days?

314. Three Teacher Taboos in Mathematics Education

As a mathematics teacher I face Three Teacher Taboos:

- As to mathematics: shall I preach self-referring metamatism, or mathematics grounded in the outside world?
- As to education: shall I choose a line-organized talent impeding school, or a block-organized talent developing school?
- As to research: shall I seek guidance in self-referring discourse protection from monastery-like universities, or in grounded contingency research from Internet academies?

Teacher Taboo I: LIB-grounded Metamatism or LAB-grounded Mathematics?

The first of the Three Teacher Taboos forces me to make a choice: shall I follow the textbook and preach metamatism, or shall I enlighten the outside world and teach how to totalize the natural fact Many? My orders are to teach mathematics. However, the textbook I am given is not on mathematics but on 'metamatism', a mixture of 'metamatics' and 'mathematism'.

Metamatics defines its concepts from above as examples of abstractions instead of, as they were created, from below as abstractions from examples. Thus the textbook chooses metamatics when defining a function as an example of the abstraction 'set-product', when historically it was defined as an abstraction from examples to label calculations containing both constant and variable numbers.

Mathematism is true in a library, but not in a laboratory, as e.g. the statement '2+3=5' that has countless counterexamples: 2weeks + 3day = 17 days, 2m + 3cm = 203cm, etc. In contrast, the statements '2*3=6'stating that 2 3s can be recounted as 6 1s, is true both in the LIB and in the LAB. Likewise, adding fractions is mathematism: I am supposed to teach that 1/2 + 2/3 = 7/6 even if the students prove that when counting cokes, 1 of 2 bottles plus 2 of 3 is 3 of 5 bottles and not 7 of 6.

I thought teaching was about enlightening natural correctness from the outside world, and not about preaching political correctness from a textbook.

Teacher Taboo II: Line-organized Symbolic Violence or Block-organized Enlightenment?

Also the second of the Three Teacher Taboos forces me to make a choice: I see that two different forms of secondary education exist. The North American republics have block-organized education receiving the learner with a 'welcome, you have a talent, and it is my job together with you to uncover and develop your talent with daily lessons in self-chosen half-year blocks. If you succeed you get a 'good job' and an opportunity to chose more blocks in the same subject. If not, you receive a 'good try' in appreciation of the courage it takes to give it a try, and you will be offered alternative blocks in other subjects.'

The European non-republics have line-organized office directed education allowing continuation to only the high marks at the end-exam that cannot be retaken, and that might even be oral. Line-organized education concentrate learners in age-determined classes where all are force-fed with the same subject blocks so that only learners with an academic family background get the high marks that give access to the offices in the highly institutionalized state apparatus, much like the mandarin system of ancient China, and labeled 'symbolic violence' by French sociologist Bourdieu. Concentration also applies to the universities offering countless uncoordinated office-directed lines making transfer almost impossible: once a teacher, always a teacher; in contrast to block organized colleges where you can always add extra blocks to your degree in the case of changing job. With the Nuremberg Verdicts in 1946 in mind I find it difficult to accept a job where all I have to do is to follow orders.

Teacher Taboo III: Guidance by Self-referring Discourse Protection or by Grounded Theory?

Also the third of the Three Teacher Taboos forces me to make a choice: I see that mathematics education research articles is referring to other research articles and very seldom to the actual classroom problems that makes it hard to be a mathematics teacher. And if they do, the articles are more interested in analyzing the problems than in offering alternatives that might solve the problems. Of course I could apply for a master degree, but it is my impression that it also contains self-referring articles on mathematics education research. What I need are articles that enlighten the hidden alternatives to the choices presented as nature in mathematics education, and that allow me to observe and report what happens in the classroom when trying out these hidden alternatives.

Teacher Taboos Lifted by the MATHeCADEMY.net

Googling ‘grounded mathematics’ leads to the web-based MATHeCADEMY.net that free of charge allows teachers to learn about mathematics as a natural science investigating the natural fact Many.

All of a sudden mathematics makes sense as the art of totalizing. To deal with Many we totalize, first by counting, then by adding. First order counting rearranges sticks in icons so that there are five sticks in the five-icon etc. Second order counting bundles in icons: The total is e.g. two threes, T = 2 3s. Finally, third order counting bundles in tens, the only number with a name but without an icon, which makes 10 a cognitive bomb that should be postponed until after having learned ‘One Digit Mathematics’.

The counting result can be presented as a geometric stack with the 2 3-bundles on top of each other, T = 2*3; or as an algebraic number using a decimal point to separate the bundles from the unbundled, T = 2.0 3s = 1.2 4s as predicted by the re-count formula T = (T/b)*b and the re-stack formula T = (T-b)+b.

Once counted, the totals can be added in two different ways, horizontal geometric addition next-to where 4 3s + 2 5s becomes 2.6 8s; and vertical algebraic addition on-top where the units must be the same so that 4 3s + 2 5s becomes 7.1 3s or 4.2 5s since 4 3s can be recounted to 2.2 5s, and 2 5s to 3.1 3s. And where overloads leads to bundling bundles so 3 threes = 1 three-three and T = 7.1 3s = 21.1 3s.

Addition next-to means adding areas, thus rooting integration. Addition on-top means changing units, thus rooting proportionality, which uses per-numbers 2$/3kg to find the price of 12 kg by recounting in 3s: T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Instead of counting on, operations are invented to use calculations to predict adding results. Thus 3+5 predicts the result of counting on 5 times from 3. The calculation 3*5 predicts the result of adding 3 5 times, and 3^5 predicts the result of multiplying with 3 5 times.

Inverse operations predict the result of the reversed processes. Thus the splitting 8 = 5+x is predicted by x = 8–5, the splitting 8 = 5*x is predicted by x = 8/5, the splitting 8 = 5^x is predicted by x = log5(8), and the splitting 8 = x^5 is predicted by x = 5√8. Renaming reversed calculation to equations, the natural way to solve equations is seen to be: move numbers across with inverse signs.

Reversing geometric addition next-to is differentiation since the question T1 + x bs = T2 is solved by recounting T2-T1 = ΔT in bs, ΔT = (ΔT/b)*b, so x = ΔT/b = (T-T1)/b.

Adding on-top and next-to thus provide four ways to add: Plus adds unlike unit-numbers: 3 cokes and 5 cokes = (3+5) cokes, multiplication adds like unit-numbers: 3 cokes 5 times = (3*5) cokes; power adds like per-numbers: 12% 5 times = 76% since 1.12^5 = 1.76; and integration adds unlike per-numbers: 2 kg at 3 $/kg + 4 kg at 5 $/kg = 6 kg at (2*3+4*5)/6 $/kg.

Seeing mathematics as a natural science investigating the natural fact Many, i.e. as Manyology, allows teaching how to totalize Many instead of teaching how to math, which is meaningless since math is not an action-word, a verb. By entering a formula's left hand side as Y1 and the right hand side as Y2, technology use graphs to illustrate formulas and solve equations by 'solve 'Y1-Y2 = 0'.

By replacing metamatism by Manyology, I can finally follow the order of article 26 in the Universal Declaration of Human Rights: 'Everyone has the right to education. Education shall be directed to the full development of the human personality. And higher education shall be equally accessible to all on the basis of merit.' Now there are no losers anymore and learning is so fast that it leaves time for the quantitative literature and its three genres: fact predicting predictable quantities, fiction predicting unpredictable quantities and fiddle predicting qualities. Replacing line-organized education with block-organized would help my even more.

315. To Math or to Totalize, That is the Question

Question: How to educate math users, math teachers, and math teacher educators?

Answer: By learning, not how to math, since math is not an action word, but how to deal with the natural fact Many by totalizing; in short, by becoming competent in counting and adding in time and space.

The MATHeCADEMY.net is free for users and for franchise takers. It offers Internet PYRAMIDeDUCATION to teachers and educators wanting to learn about mathematics as a natural science investigating the natural fact Many through the CATS approach, Count&Add in Time&Space, building upon five principles.

Principle I. Education Means Adapting Learners to the Outside World

To adapt to the outside world reptiles use genes, mammals also use parental care, and humans also use language, developed when motion-freed forelegs created additional brain capacity to keep the balance and to store the sounds associated with what was grasped with the forelegs, thus supplying humans with a language to share information through communication that, if institutionalized, is called schooling or education, typically containing primary education for children, secondary education for teenagers and tertiary education for adults.

You adapt to the outside world by choosing proper actions. A typical action as ‘Peter eats apples’ is described by a three-term sentence with a subject, a verb and an object. Thus mathematics education should be described in this ways. The learner is the subject, the object is the natural fact Many, and the verb is what we do to deal with Many, we totalize by counting and adding expressing the total as a formula, e.g. T = 345 = 3*10^2 + 4*10 + 5 showing three of the four ways to add: +, * and ^. Totalizing can also be called algebra by using the Arabic word for reuniting.

Not being a verb, the school subject ‘mathematics’ should be renamed to ‘totalizing by counting and adding’. If not, mathematics might turn into ‘metamatism’, a mixture of ‘metamatics’ and ‘mathematism’ where metamatics defines its concepts from above as examples of abstractions instead of, as they were created, from below as abstractions from examples; and where mathematism is true in a library, but not in a laboratory, as e.g. the statement ‘2+3 = 5’ that has countless counterexamples: 2weeks + 3day = 17 days, 2m + 3cm = 203cm, etc. In contrast, the statements ‘2*3 = 6’, stating that 2 3s can be recounted as 6 1s, is true both in the LIB and in the LAB.

Principle II. All Three Ways of Counting are Accepted

The tradition only accepts third order counting in ten-bundles: T = 47 = 4 tens + 7.

First order counting rearranges sticks into icons with five sticks in the five-icon, etc. This shows the difference between numbers and words. Numbers are universal and show the degree of many it represents, whereas words are local sounds associated with things or actions. Thus numbers can produce natural correctness, and words can produce political correctness.

Second order counting bundles in icon-bundles, e.g. T = 4 5s. The counting result can be represented by a geometric stack with the 4 5-bundles on top of each other, T = 4*5; or by an algebraic number using a decimal point to separate the bundles from the unbundled, T = 4.0 5s. 3 leftovers may be placed either next to the stack of 5-bundles in a stack of 1-bundles, which can be written as 4.3 5s; or on top of the stack of 5-bundles counted as 5s, i.e. as 3 = (3/5)*5 thus giving a total of T = 4 3/5 5s. So for any bundle-number, 3/b = 0.3

Thus a calculator can predict the counting result by two formulas, the recount-formula T = (T/b)*b, saying that from T the bundle b can be taken away T/b times, and the restack-formula T = (T-b)+b, saying that from T a bundle b can be taken away and placed next-to T. Thus recounting 4 5s in 7s is predicted to be T = ((4*5)/7)*7 = 2.6 7s since T = (4*5-2*7) + 2*7 = 6 + 2*7.

Principle III. Both Horizontal Addition Next-to and Vertical Addition On-top are Accepted

Once counted, totals can be added in two different ways. With horizontal geometric addition next-to, 4 3s + 2 5s becomes 2.6 8s; and with vertical algebraic addition on-top, the units must be the same so that 4 3s + 2 5s becomes 7.1 3s or 4.2 5s since 4 3s can be recounted to 2.2 5s, and 2 5s to 3.1 3s. Overloads lead to also bundling bundles so 3 threes = 1 three-three and T = 7.1 3s = 21.1 3s.

Addition next-to means adding areas, thus rooting integration. Addition on-top means changing units, thus rooting proportionality, which uses per-numbers as 2$/3kg to find the price of 12 kg by recounting in 3s: T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Adding on-top and next-to provides four ways to add numbers: Plus adds unlike unit-numbers: 3 cokes and 5 cokes = (3+5) cokes, multiplication adds like unit-numbers: 3 cokes 5 times = (3*5) cokes; power adds like per-numbers: 12% 5 times = 76% since 1.12^5 = 1.76; and integration adds unlike per-numbers: 2 kg at 3 $/kg + 4 kg at 5 $/kg = 6 kg at (2*3+4*5)/6 $/kg.

Principle IV. With Second Order Counting and Next-to Addition, the Math Core is not Enlarged only Refined from K to 12

The core of mathematics says that Many is totalized by 3 ways of counting and 4 ways of adding.

Instead of counting on, operations are invented to use calculations to predict adding results. Thus 3+5 predicts the result of counting on 5 times from 3. The calculation 3*5 predicts the result of adding 3 5 times, and 3^5 predicts the result of multiplying with 3 5 times.

Inverse operations predict the result of the reversed processes. Thus the splitting 8 = 5+x is predicted by x = 8–5, the splitting 8 = 5*x is predicted by x = 8/5, the splitting 8 = 5^x is predicted by x = log5(8), and the splitting 8 = x^5 is predicted by x = 5√8. Renaming reversed calculation to equations, the natural way to solve equations is seen to be: move numbers across with inverse signs.

Reversing geometric addition next-to is differentiation since the question T1 + x bs = T2 is solved by recounting T2-T1 in bs, T2-T1 = ((T2–T1)/b)*b, so x = (T2-T1)/b. In right-angled triangles, trigonometry recounts the short sides in the long side: a = (a/c)*c = sinA*c, and b = (b/c)*c = cosA*c.

Allowing second order counting and next-to addition, all four ways are introduced in grade one. Consequently both primary and secondary teachers should be educated in the same way.

Principle V. Learning How to Deal with Many is Self-instructing if Using PYRAMIDeDUCATION

In PYRAMIDeDUCATION, 8 learners are organized in 2 teams of 4 students choosing 3 pairs and 2 instructors by turn. The coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation.

The coach helps the instructors when correcting the count&add problems. In each pair each learner corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Having finished the course, each learner will ‘pay’ by coaching a new group of 8 learners.

The learning principles are ‘grip&grasp’, and ‘learn from gossip’. In primary school children as mammal offspring learn by doing. This means that learning has to come through the hands, ‘greifen vor begreifen’, both as objects you can grip and as actions you can perform.

Thus in primary school learning takes place, not by reading, but by gripping and moving.

And in secondary school texts have the form of gossip since young people learn by listening, but the sentences need to be gossip with known subjects, so they tell you something new about something you already knew. This means that abstract concepts must be presented as abstractions from examples and not as examples of abstractions, i.e. from below and not from above.

VI. Learning How to Deal with Many Through 4x2 Educational Questions

The website MATHeCADEMY.net contains grounded mathematics organized in activities where learners learn ‘CATS’, Count & Add in Time & Space, guided by 4x2 educational questions Q and answers A. The four study units CATS1 are for primary school and the four study units CATS2 are for secondary school.

Count C1

Q: How to count Many? **A:** By bundling and stacking the total T predicted by T = (T/b)*b

Q: How to recount 8 in 3s: T = 8 = ? 3s = ?*3. **A:** 8 = (8/3)*3 = 2*3+2*1= 2)2) = 2.2*3 = 2 2/3*3

Q: How to recount 6kg in $: T = 6kg = ?$. **A:** If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$

Q: How to count in standard bundles? **A:** Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2 + 2*B + 3

Add A1

Q: How to add stacks concretely? T = 27 + 16 = 2 ten 7 + 1ten6 = 3 ten 13 = ?.

A: By restacking overloads predicted by the ‘restack-equation’ T = (T–b) + b:

T = 27 + 16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43.

Q: How to add stacks abstractly? **A:** Vertical addition uses carrying. Horizontal addition uses FOIL.

Time T1

Q: How can counting & adding be reversed? **A:** By calculating backwards, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign.

Q: Counting ? 3s and adding 2 gave 14. **A:** x * 3 + 2 = 14 is reversed to x = (14 – 2)/3.

Q: Can all calculations be reversed? **A:** Yes. x + a = b is reversed to x = b – a, x * a = b is reversed to x = b/a, x ^ a = b is reversed to x = a√b, a ^ x = b is reversed to x = loga(b).

Space S1

Q: How to count the plane and spatial properties of stacks, boxes and round objects?

A: By using a ruler, a protractor and a triangular shape; by the 3 Greek Pythagoras’, mini, midi & maxi; and by the 3 Arabic recount-equations: sin A = a/c, cos A = b/c, tan A = a/b.

Count C2

Q: How to count possibilities? **A:** By using the numbers in Pascal’s triangle.

Q: How to predict unpredictable numbers? **A:** If a 'post-diction' gives the average 8.2 with deviation 2.3, the 'pre-diction' gives the confidence interval 8.2 ± 2*2.3 with 95% probability.

Add A2

Q: What is a per-number? **A:** Per-numbers occur when counting, when pricing and when splitting.

Q: How to add per-numbers? **A:** The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b, where a = (T2-T1)/b.

Time T2

Q: How to predict the terminal number when the change is constant?

A: By constant change-equations: If Ko = 30 and a = 2, then K7 = Ko+a*n = 30+2*7 = 44.

If Ko = 30 and r = 2%, then K7 = Ko*(1+r)^n = 30*1.02^7 = 34.46

Q: How to predict the terminal number when the change is variable, but predictable?

A: By a variable change-equation: If Ko = 30 and dK/dx = K', then K – 30 = ∫dK = ∫K'dx

Space S2

Q: How to predict the position of points and lines? **A:** By using a coordinate-system: If Po(x,y) = (3,4) and dy/dx = 2, then P1(8,y) = P1(x+dx, y+dy) = P1(3+(8-3),4+2*(8-3)) = (8,14)

Quantitative Literature

Q: What is quantitative literature? **A:** Quantitative literature is stories about Many in time and space.

Q: Does quantitative literature share the 3 different genres: fact, fiction and fiddle?

A: Yes. In formulas as T = c*p we need to know what quantities are described to determine the truth-value of the formula's prediction. It turns out that both word-statements and number-statements share the same genres: fact, fiction and fiddle. Fact-models predict predictable quantities. Fiction-models predict unpredictable quantities. Fiddle-models predict qualities.

VII. Replacing Metamatism with Totalizing makes Losers Users.

Nothing is easier than making learners drop out by teaching metamatics defining concepts upside down, and mathematism obviously incorrect in the outside world. But it makes more sense instead to teach Totalizing by Counting and Adding, since losers then become users, adapted to deal with the root of mathematics, the natural fact Many.

316. Discussion Group 6 on Postmodern Mathematics, Theses 1-7

"To be able to participate in this discussion group, could you give me a short introduction to your view on postmodern thinking in mathematics education?"

Aim & Rationale

The goal of the discussion group is to elucidate the multiplicity of the subject of mathematics: to explore and share how postmodern perspectives offer new ways of seeing mathematics, teachers and learners. The two key themes are

Theme 1: Perspectives of mathematics as having multiple dimensions / components

Theme 2: Multiple-self perspectives of the human subject (teacher/ learner/ researcher)

These will be used as organizing themes for the two sessions

Postmodernism rejects a single authoritative way of seeing mathematics, teachers and learners, for each can be seen and interpreted in multiple ways. Mathematics can be seen as axiomatic and logical leading to indubitable conclusions, but it can also be seen as intuitive and playful, open-ended, with surprises and humour, as evidenced in popular mathematical images and cartoons. Additionally it can be seen in its applications in science, information and communication technologies, everyday life and ethnomathematics. All of these dimensions are part of what makes up mathematics and they all co-exist successfully.

It is also important to recognize that all human subjects have multiple selves and that we all (mathematicians, teachers and learners) have access to different selves: authoritative knowers, researchers, learners, appreciators and consumers of popular and other cultures, as well as having non-academic selves. Thus mathematics teachers can be seen as epistemological authorities in the classroom as well as co-explorers of unfamiliar realms both mathematical and cultural, and as ring-masters in the mathematical circus. Students can be seen as receivers of mathematical knowledge, but also as explorers and interpreters of mathematical and cultural realms that can be related to mathematics.

All of these perspectives and selves are resources we can use to enhance the teaching and learning of mathematics, but many are currently overlooked or excluded.

The aim of the Discussion Group is to raise and discuss these ideas and explore and generate examples relevant to classroom practices.

Papers and resources will be available on-line before the conference so that participants can prepare themselves and so that presentations can be kept short and most of the time is devoted to discussion. Examples will also be distributed in hard copy in the sessions.

Key Questions

* What is postmodern thinking in mathematics and mathematics education? What is new or different about it and what are the implications for research in mathematics education?

* Given a postmodern multiple-perspectives view of mathematics what illuminations and surprises can be found for mathematics and its teaching and learning in multidisciplinary sources including: history of mathematics, ethnomathematics, science, information and communication technology, art works, stories, cartoons, films, jokes, songs, puzzles, etc.?

* How might the new emphases and differences foregrounded by postmodern perspectives impact in the primary and secondary mathematics classrooms? What concrete examples serve to illustrate these differences?

* How can a multiple-selves view of the human subject be reflected in the mathematics classroom and in mathematics teacher education? How can a multiple-selves view of the teacher facilitate teacher education?

I. Postmodernism Means Skepticism toward Hidden Patronization

1. A republic knows only parent/child patronization - and hidden patronization. Skepticism toward hidden patronization may be termed postmodernism according to Lyotard's 'Simplifying to the extreme, I define postmodern as incredulity toward metanarratives'.

2. In the first republic the ancient Greek sophists expressed skepticism: to prevent patronization by choices presented as nature the people must be enlightened to tell choice from nature. The philosophers rejected skepticism saying that all physical are examples of metaphysical forms only visible to philosophers educated at Plato's academy, who consequently should patronize the people.

3. The Christian church transformed academies into monasteries, again turned into academies by the Reformation, where Luther expressed skepticism toward the patronization of the Roman Church. Descartes expressed skepticism with his 'Dubito, ergo sum'. Brahe grounded his skepticism in observations that allowed Newton to replace a political correct geocentric theory with a natural correct heliocentric theory.

4. Newton's theory avoids skepticism by being falsifiable: the moon behaves like an apple; both follow their own will creating predictable orbits when falling.

5. Discovering physical will, Newton grounded the Enlightenment century where Locke's rejection of Hobbes' patronization inspired Kant's 'dare to know' as well as French thinkers: when falling apples obey their own will, people could do the same and replace patronization with voting. Two republics were formed, an American still having its first republic, and a French now having its fifth.

6. In the American republic, skepticism toward philosophical patronization is expressed in pragmatism, grounded theory and block-organized education allowing everybody to uncover and develop their individual talent; in contrast to the patronizing line-organized education in the EU.

7. In the French republic, Derrida, Lyotard, Foucault and Bourdieu express skepticism against hidden patronization occurring in words, sentences, institutions and education. Derrida suggests deconstructing patronizing words that construct instead of label what they name. Lyotard suggests that research should be paralogy, inventing dissensus to the ruling consensus. Foucault shows how a discourse claiming correctness, a discipline, disciplines itself and its subject, thus forcing false identities upon humans, seeking cure at correcting institutions copying the pastoral power of the Christian church. And Bourdieu sees education as symbolic violence reserving society's knowledge capital for a privileged knowledge nobility; and sees mathematics as especially useful to that end.

8. So skeptical thinking is as old as the republic. Today skepticism is called postmodernism since, inspired by the success of natural science to replace faith with certainty, modernism sees science as synonymous with correctness. Postmodernism accepts natural science as valid, but questions the social and human sciences; as well as the institution called 'mathematics education' by asking if 'mathematics' and 'education' is based upon political or natural correctness.

II. Deconstructed, Mathematics is a Natural Science Counting and Adding Many

1. The words geometry and algebra show the actions that are labeled mathematics: in Greek, geometry means measuring earth; and in Arabic, algebra means reuniting. So the root of mathematics is the natural fact Many. To deal with Many, we count and add.

2. ‘First order counting’ arranges sticks in icons with five sticks in the 5-icon, etc. ‘Second order counting’ bundles in icon-bundles. Here 5-bundling needs no 5-icon: 1, 2, 3, 4, 10 meaning 1 bundle and 0 unbundled. ‘Third order counting’ bundles in tens, consequently ten needs no icon.

3. A total of 7 bundled in 3s, T = 2 3s & 1, can be placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle if placed in the left bundle-cup. Now the cup-contents can be described by icons, first using ‘cup-writing’ 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

T = IIIIIII = III III I = III III) I) = ▌▌) I) = II) I) = 2)1) = 2.1 3s

A total of 3 4s can be recounted: T = 3 4s = 6 2s = 4 3s = 2.2 5s = 2 6s = 1.5 7s = 1.4 8s = 1.3 9s = 1.2 tens = 12, if written in a sloppy way leaving out the unit and misplacing the decimal point.

4. Once counted, totals can be added. Added horizontally ‘next-to’, 2 3s and 4 5s gives a total of T = T1+T2 = 2 3s + 4 5s = 3.2 8s. Addition vertically ‘on-top’, recounting makes the units the same:

T1 = 2 3s = 1.1 5s, or T2 = 4 5s = 6.2 3s. So T = T1+T2 = 5.1 5s or T = 8.2 3s.

5. Recounting is predicted by a ‘re-bundle’ formula T = (T/b)*b, e.g. T = 8 = (8/4)*4 = 2*4 = 2 4s. The formula says that T can be counted in bs by taking away bs T/b times.

The ‘re-stack’ formula, T = (T-b)+b, says that T can be split into two stacks, b and T-b.

Together the two formulas allow a calculator to predict the result when recounting e.g. 2 7s in 4s:

T = (2*7)/4*4 = 3.? 4s, and T = (2*7 – 3*4) + 3*4 = 2 + 3*4, so T = 2 7s = 3.2 4s

6. Recounting changes units, so recounting is the root of proportionality as shown by T = (T/b)*b. Thus with 2$ per 3 kg, recounting 6$ in 2s gives T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg.

7. Where vertical addition is the root of proportionality, horizontal addition is the root of integration since per-numbers are added by their areas: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at (3*2+5*4)/6 $/kg.

8. Rewriting T = 456 as 4*10^2 + 5*10 + 6 shows three ways of uniting numbers, +, * and ^. These operations all predict counting results: 3+5 predicts the result of counting on 5 times from 3; 3*5 predicts the result of adding 3 5 times, and 3^5 predicts the result of multiplying with 3 5 times.

9. Reversing uniting parts to a total splits a total into parts. Inverse operations predict the result: x = 8-3 predicts the answer to the splitting question ‘3+x = 8’, x = 15/3 predicts the answer to the splitting question ‘3*x = 15’; x = √16 predicts the answer to the splitting question ‘x^2 = 16’; and x = log2(32) predicts the answer to the splitting question ‘2^x = 32’. Calling splitting problems ‘solving equations’ we see that moving numbers across with inverse calculation signs solves equations.

10. Algebra’s reuniting project contains four ways to add numbers: Multiplication and addition add like and unlike unit numbers; and power and integration add like and unlike per-numbers.

11. A formula tells a story about a calculation and what it calculates, e.g. T = a+b*c. With one unknown, a formula is an equation to be solved. With two unknowns, a formula is a curve to be graphed on a graphical calculator, that also use 'solve Y1-Y2 = 0' to solves any equation having its left hand side entered as Y1 and its right hand side as Y2.

12. The word- and the number-language share the same genres when describing the real world: fact, fiction and fiddle. Thus 'cost = price*quantity' predicts predictable quantities, 'cost = rate*capital' predicts unpredictable quantities, and 'risk = utility*probability' predicts unpredictable qualities.

III. Block-organized Education Develops Talents, Line-organized Drowns Talents

1. The planet Earth has a distance from its energy-providing star that allows water to be present in all three states: solid, liquid and vaporous. The size of the planet allows it to hold on to an atmosphere. This again allows energy to be stored in three cycles: as currents in the sea, as evaporation and rain between the sea and the atmosphere, and as wind in the atmosphere. These cycles allow the creation of cells organizing themselves as green plants storing energy; and as grey animals consuming green energy. Animals come in three forms all needing information to survive. Reptiles are born with routines. Mammals feed its offspring so it can adapt before becoming parents. Humans has an additional brain to keep the balance on two feet, which allows the forefeet to become graspers and allows sounds associated with the grasped to be stored as words in the human brain, thus creating a language to share information; meaning that humans can adapt both before and after becoming sexually mature. Adaptation or learning takes place in everyday life and in special intuitions called education.

2. Most nations have three levels of education: primary education for children, secondary education for adolescents and tertiary education for adults. After primary education two different educational systems exist.

3. Block-organized education takes place in the North-American republics. Its task is to uncover and develop the learner's individual talents through daily lessons in self-chosen half-year blocks. If successful, the learner can chose more blocks of the same kind. If not successful, the learner receives a 'thanks for trying' before trying blocks from alternative subjects. At the last year in secondary school the learner can try out tertiary blocks. As adult, a learner can add extra blocks to their tertiary degree in the case of loosing or changing job.

4. Line-organized education takes place in the EU having an autocratic structure with a strong central administration at the time of creating a public school system. Unable to use the blood nobility for administration, the school system was designed to select the population elite and prepare it for posts as civil servants centrally or locally. Thus the education contains lines leading to different posts. As democracies the EU countries kept the autocratic central administration and its 'knowledge nobility' protecting the line-organization to secure its own reproduction.

5. In line-organized education, the population's access to knowledge is made difficult by multi-year fixed classes based upon biological age instead of half-year classes based upon subject curiosity or talent. Fixed classes force boys to learn together with girls that mentally are two years older, with the consequence that girls are over-represented at tertiary educations. Fixed classes prevent daily lessons resulting in poor learning unless the family arranges daily lessons. Fixed classes also force teachers to teach other subjects than their special subjects resulting in poor teaching and a high proportion of substitute teaching.

6. To provide learners with a number-language to deal with everyday quantities, block-organized mathematics education should teach how to count and add Many. To prevent learners from developing quantitative competence and to reserve the knowledge capital for the knowledge

nobility, line-organized mathematics education should present mathematics as not rooted in but applicable to the natural fact Many.

IV. Post-modernity is When Electrons Carry both Information and Energy

1. To survive, humans have holes in the head to supply the body with energy and information. Like other mammals, humans were gathers and hunters in the beginning.

2. When informed about how to make iron, humans were able to produce artificial hands, tools, enabling agriculture, pre-modernity.

3. When informed about how to make energy, humans were able to produce artificial muscles, motors; and to combine tools and motors to machines, modernity. Early modernity used steam to carry energy forcing production of goods to be centralized. Late modernity used electrons allowing decentralized production.

4. When informed about how to use electrons to also carry information, human were able to produce artificial brains, computers; and to combine machines and computers to robots, post-modernity. With robots, many humans again become gather-hunters gathering food in supermarkets and hunting information on multichannel TVs. Using electrons, also production of knowledge can be decentralized allowing the huge central universities of modernity to be transformed into smaller universities placed locally and in cyberspace.

5. In post-modernity, information technology, IT, gives humans instant access to information, especially information about contingency, i.e. the hidden alternatives to the traditions of modernity. Thus often IT transforms nature back into choice needing information and debate before being decided. Giddens describes post-modernity as a post-traditional reflexive modernity where humans create their self-identity as a biographical narrative build upon meaning and authenticity.

6. Being skeptical toward traditions, authenticity seekers

- will ask for half year introductions to different subject areas. Thus postmodern education must be organized as self-chosen half year block instead of as forced multi year lines

- will reject a self-referring mathematics. Thus mathematics must respect its nature as ‘Manyology’ rooted in and not only applicable to the outside world

- will not compete but cooperate with IT. Thus graphical calculators must be introduced as early as possible in primary school together with modeling where humans formulate questions to the technology and evaluate the answers given.

7. When will the post-postmodern occur? Ontologically, ‘postmodernity’ labels the creation of the artificial human: tools as artificial hands, motors as artificial muscles, and computers as artificial brains. Together they form a robot, the artificial human being that can supply real human beings with the food and information they need to survive. With no additional things on the human body to invent, post-postmodernity will never come. Epistemologically, ‘postmodernism’ labels skepticism toward hidden patronization presenting choice as nature. According to this, post-postmodernism will mean skepticism toward skepticism, i.e. a return to patronization. However, the creation of IT makes this less likely. So the post-postmodern will never occur. What comes, are new labels allowing the postmodern to develop as a social order in its own right. E.g. by using research to uncover hidden contingency and by using IT to report on the multiple choices in everyday life.

V. Post-modern Research Simply Discovers Dissensus

1. Natural learning through individual experimenting is made possible by mammals feeding their offspring until it has offspring of its own. Humans use two legs to move and two to grasp. Associating sounds to the grasped develops language enabling humans to also learn by sharing information in the form of words combined to statements about subjects, either do-statements reporting actions or is-statements stating verdicts. Do-statements may be tested; is-statements are decided by voting as in a courtroom or by patronization. Piaget sees adaption as schemata accommodating the individual to the world, and assimilating the world to the individual.

2. Natural research creates collective learning by using the method of natural learning: Words label reliable do-observations forming do-statements to be tested for validity. Is-statements are valid if accepted by other researchers. This self-reference makes Glaser warn against using predefined categories when creating Grounded Theory, the natural research method of the American republics. In contrast to this, patronizing research practice 'discourse-protection' by insisting that new research is based on existing research. Being skeptical toward hidden patronization, Lyotard recommends that postmodern research should be 'paralogy' inventing dissensus to the ruling consensus thus again making research discovery, searching for hidden alternatives, contingency.

3. Modernism and modernity is based upon the success of the natural science method: categories are induced from reliable data, and category relations are validated by trying to falsify deduced predictions. This method cannot be used in human and social studies. Still they call themselves sciences claiming that interpretation be recognized as a scientific method. This problematic claim is illustrated by the 'pencil paradox': Placed between a ruler and a dictionary, a '17 cm pencil' can falsify 17 by pointing to 15, but unable to point to words it cannot falsify 'pencil'; thus numbers form do-statements that can be tested while words form is-statements that only can be decided by voting or by patronization.

4. Information is shared systematically through education. Primary education informs children about words and numbers using patronization. Secondary education informs adolescents about nature and society. The North American republics organize secondary information as daily lessons in self-chosen half-year blocks including periodically tests. Patronizing countries force learners to receive secondary information in multi-year lines with a final test in the end that cannot be retaken.

5. The Enlightenment created the republic and its individual talent developing education. The German autocracy sent in an army to terminate the French republic. However, German mercenaries were no match to French conscripts only to aware of the feudal consequence of loosing. Instead Berlin created a strong central administration to be served by the elite from at a line-organized 'bildung' education created with three specific goals: to prevent democracy, the population must be kept unenlightened; instead nationalism should be installed so that it sees itself as a people wanting to fight other people, especially the French democracy. Furthermore, the population elite should be selected and 'builded' as a new knowledge nobility to replace the old blood nobility unable to stop democracy from spreading. As democracies, the EU still keeps its strong central administrations protecting its line-organized education with its discourse protecting universities.

6. Postmodern research has found many examples of dissensus in mathematics education: Natural numbers are not '23', but '2.3 tens' including both the unit and a decimal point separating bundles from unbundled. The first natural operation is division since counting in 5s means taking away 5s. Numbers can be added both vertically and horizontally. Integration adds per-numbers. Defining $x = 8/2$ as the answer to $x*2 = 8$ reveals the natural way to solve equations: move over and change sign. When saving, $A/a = R/r$, telling that the total and the monthly capital has the same ratio as the total and the monthly interest, which is easily proven: Putting the monthly interest $r*(a/r)$ of a/r \$ into an account makes this a saving account containing the total interest $R*a/r = A$. Also postmodern

research can tell natural correctness from political correct ‘mathematism’: 2*3 = 6 since 2 3s can be recounted as 6 1s, whereas 2+3 = 5 may be correct in a library, but has countless counterexamples in the laboratory: 2 weeks + 3 days = 17 days, 2 m + 3 cm = 203 cm etc.

7. Postmodern research is published at the MATHeCADEMY.net and used in its free web-based PYRAMIDeDUCATION allowing teachers to learn about mathematics as ‘Manyology’, a natural science investigating the natural fact Many. Postmodern researchers meet yearly in Berlin to design classroom experiments. To paraphrase Cohen: ‘First we took Manhattan, now we take Berlin.’

VI. 12 Mistakes Mystify Modern Mathematics

Math-Mistake 1: Teaching Both Numbers & Letters as Symbols

Numbers rearrange strokes to show different degrees of Many. Letters just install chosen sounds.

Math-Mistake 2: Teaching 2digit Numbers before Decimal Numbers

Counting by bundling, T = 7 = 2.1 3s = 2)1), makes 1-digit mathematics a natural learning method. Using 2-digit numbers directly makes 10 a cognitive bomb: 10 = 1 bundle = 5 when counting in 5s.

Math-Mistake 3: Teaching Fractions too late & before Decimals

Recounting 2 4s in 5s creates 3 leftovers that may be placed next to the bundle as a decimal, or on top of the bundles as a fraction; or: T = 2 4s = 1.3 5s = 1 3/5 5s. Thus 3/5 fives = 0.3 fives.

Math-Mistake 4: Teaching Addition before Division

Counting in 5-bundles is predicted by the ‘recount-formula’ using division: T = (T/5)*5.

Math-Mistake 5: Cup-writing would make Overflow Easy

Add: 45 † 17 = 4)5) † 1)7) = 5)12) = 6)2) = 62. Subtract: 45 – 17 = 4)5) – 1)7) = 3)_2) = 2)8) = 28

Multiply: 6 * 47 = 6 * 4)7) = 24)42) = 28)2) = 282. Divide: 42/3 = 4)2)/3 = 1)12:3) = 1)4) = 14

Math-Mistake 6: Teaching Fractions instead of PerNumbers

Per-numbers is the second natural root of fractions: 2$ per 3 kg = (2$)/(3kg) = 2/3 $/kg

Math-Mistake 7: Teaching Proportionality instead of reCounting

A quantity counted in one unit can be recounted in another unit using per-numbers:

With 4kg/5$ or 4/5 kg/$, 4kg = 5$. So recounting 12 in 4s, 12kg = (12/4)*4kg = 3*5$ = 15$.

Math-Mistake 8: Adding Numbers & Fractions without Units

Adding numbers without units leads to ‘MatheMatism’, true in the library but not in the laboratory.

‘2†3 = 5’ is mathematism since 2m†3cm = 203cm, etc. ‘2*3 = 6’ is mathematics since 2 3s is 6 1s.

Fraction paradox: who is right?

Teacher: 1/2†2/3 = 7/6. Students: 1/2 of 2 cokes † 2/3of 3 cokes = 3/5 of 5 cokes, not 7 of 6 cokes.

Math-Mistake 9: Teaching Killer-Equations instead of Grounded Equations

‘2†3*x=14’ is grounded in ‘2$ plus 3kg at ? $/kg total 14$’. Killer-equation: $2 \dagger \frac{3 - 4x}{5x - 6} = 7x - \frac{8}{9x}$

Math-Mistake 10: Solving Equations by Neutralizing instead of Reversed Calculation

Forward: x *-(*3)->* 3*x *-(†2)->* 3*x†2 = 17. Reversed 17 *<-(†2)-* (17-2) *<-(*3)-* (17-2)/3 = x

Math-Mistake 11: Teaching Geometry before Trigonometry

Geometry: earth-measuring. Earth splits into triangles, triangles split into right-angled triangles.

Math-Mistake 12: Teaching Calculus instead of Adding PerNumbers
Primary School Calculus. Integration: 2 3s † 4 5s = ? 8s . Differentiation: 2 3s † ? 5s = 4 8s

Middle School Calculus. Integration: 2 kg at 3 $/kg † 4 kg at 5 $/kg = 6 kg at (3*2†5*4)/6 $/kg

High School Calculus. Integration: 5 seconds at 2m/s increasing to 6 m/s = $\int_0^5 (2 \dagger \frac{6-2}{5} x)\, dx$

Moral: the Natural fact Many is the Root of, not an Application of Mathematics

VII. Stop Just Following Orders: Teach How to Reckon, not How to Math
1. Replacing gather-hunter culture with agriculture implies a need for numbers to name the different degrees of many. Numbers can be found by counting or more quickly by calculating. This created the craft of reckoning (Rechnung in German) and in many countries reckoning was part of school until replaced by mathematics in the mid 1900s.

2. The Greek and Roman problem with numbers was solved by Hindu-Arabic numbers bundling totals in tens and replacing the natural description '3.2 tens' with the shorter '32' leaving out the unit and misplacing the separator between bundles and unbundled one place to the right. Leaving out units is useful in business, but in education two digit numbers without units create learning problems; and leads to 'mathematism', true in a library but not in a laboratory, as the statement '2 + 3 = 5' that is falsified in the laboratory: 2 weeks + 3 days = 17 days, 2 m + 3 cm = 203 cm etc.

3. The Enlightenment period treated mathematics as a natural science. Grounded in the natural fact Many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Using the concept set, modern mathematics turned Enlightenment mathematics upside down to a deductive 'metamatics' that by defining its concepts as examples of abstractions becomes entirely self-referring. However, self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox 'this statement is false' inspired Russell to formulate a paradox about the set of sets not belonging to itself: 'If M = A│AA), then MM MM.' Instead of returning to its root, reckoning, mathematics mixes mathematism and metamatics to 'metamatism' creating big learning problems. But why do teachers keep on teaching false statements thus creating a teaching/learning paradox 'Teach mathematics so it will not be learned'? Social theorists offer a simple explanation: Teachers just follow orders.

4. Bauman points out that by following authorized routines, modernity creates both gas turbines and gas chambers. Analyzing the latter, Arendt shows how in institutionalized societies patronization might become totalitarian, thus reintroducing evil actions this time rooted not in a devil but in the sheer banality of just following orders.

5. Foucault sees mathematics as just another discourse suppressing competing discourses to obtain monopoly. Once in power, it becomes a discipline that disciplines itself and its subject by issuing identity orders: Numbers must not contain units and separators. The operation order is addition, subtraction, multiplication and division. A formula is a function, which is a many-one set-relation. These orders discipline the discipline itself so that algebra meaning 'reuniting' in Arabic changes identity to 'pattern recognition' thus hiding the four ways of uniting: Addition/multiplication unite unlike/like unit-numbers, and integration/power unite unlike/like per-numbers. Also the total, the subject of mathematics, is disciplined by first being allowed to occur when introducing formulas. Mathematics has its root, the natural fact Many, replaced by the self-referring concept set to secure absolute discourse protection. And finally, action-words as count, add, unite and reckon is replaced by the verdict-word math allowing a huge patronizing pyramid institutions to be build called

Mathematics, using its discourse to legitimize diagnoses to be cured at an institution called education, which is a parallel institution to the other discourse based institutions as the hospital, the prison, the barrack etc. using human disciplines to install diagnoses as uneducated, unhealthy, unsocial, uncoordinated etc.; institutions with jobs and advancement to those willing to obey orders.

6. For learners to obtain a number-language education must de-throne its metamatism-discourse by reestablishing mathematics as ‘Manyology’, a natural science rooted in the natural fact Many. This is a single job in republics already using block-organized education; but a double job in countries also needing to change its autocratic line-organized education protected by a ruling mandarin class defending its knowledge privilege, and by self-referring discourse protecting universities.

7. Luther de-throned the Roman Church by promising the monarchs the land of the monasteries. A reformation of mathematics education will promise governments that boys can change identity from unskilled dropouts to engineers boosting the national economy in the future knowledge economy.

317. Postmodern Skepticism toward Mathematics and Education and Research

The cornerstones of modern society is research and education, especially in the two basic languages, the word-language assigning words and sentences to qualities, and the number-language assigning numbers and calculations to quantities. And as an important institution, mathematics education is equipped with its own research to make it successful. Still the problems in mathematics education seem to grow with the number of research articles. This irrelevance paradox makes postmodern skepticism ask: Are mathematics, education and research what they claim to be? Or are they choices that presented as nature install patronization to be unmasked by postmodern contingency research?

The Background

The modern world began when natural science replaced belief with certainty by using the outside world to provide reliable data, and to falsify deduced predictions to test their validity. This enlightening method created the Enlightenment century wanting to replace autocratic patronization with democratic voting. Hence schools were created as enlightenment institutions with special focus on the three basic Rs: how to Read, how to wRite and how to Reckon. However, today reckoning enlightenment is called mathematics education supplied with its own research and facing huge learning problems as e.g. formulated in 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371). What went wrong? Let us ask research.

Scriptures Root Patronizing Research

In ancient Greece the Pythagoreans formed a closed society to discover metaphysical laws in the physical world. They found three. Halving the length of a vibrating string makes the tone go up one octave; and other proportions also create harmony. In a triangle, two angles can be chosen freely, but a metaphysical law binds the third. Likewise with the sides in a right angled triangle. From their discoveries the Pythagoreans formulated the thesis 'all is numbers'. Plato generalized this by saying 'all is forms', meaning that physical phenomena are examples of metaphysical forms only visible to the philosophers educated at the Plato academy, and who consequently should be accepted as social patronizors (Russell, 1945).

The Christian church took over the idea of metaphysical patronization, but changed the academies to monasteries with corridors housing cells with writers commenting on the Holy Scriptures, and calling their interpretations research using a hermeneutic method.

Using a physical method to discover natural correctness, natural science created the modern university taking over the monastery. However, hermeneutics stayed so today two kinds of descriptions claim to be scientific, one using numbers and the other using words; in spite of the fact that placed between a ruler and a dictionary, a stick can point to 15 but not to 'pencil'.

To postmodern skepticism, natural sciences produce enlightening natural correctness, while social and human sciences produce patronizing political correctness.

Skepticism Roots Postmodern Contingency Research

Skepticism towards hidden patronization is the root of postmodern thinking as formulated in Lyotard's definition 'Simplifying to the extreme, I define postmodern as incredulity toward metanarratives (Lyotard, 1984: xxiv).'

In the first republic, the ancient Greek sophists expressed skepticism: to prevent patronization by choices presented as nature, the people must be enlightened to tell choice from nature, i.e. to tell political correctness from natural. So authority should come from voting instead of 'Gewalt', claiming that by nature ordinary people, like children, do not possess 'Mündigkeit'; a continental concept that the lack of autocracy never made relevant in the English language.

In the Renaissance, Brahe grounded his skepticism in physical observations allowing Newton to replace a political correct geocentric theory with a natural correct heliocentric theory.

Discovering physical will, Newton inspired French Enlightenment thinking: if falling apples obey their own will, then people could do the same and replace patronization with voting. Two republics were formed, an American and a French.

The US still has its first republic using pragmatism and grounded theory to exert skepticism towards philosophical patronization. The French has its fifth republic, repeatedly turned over by German neighbors, which has created a French skeptical poststructuralist thinking that warns against patronization hidden in words, sentences, institutions and education.

Derrida thus uses the term 'deconstruction' to warn against patronizing words installing instead of labeling what they name. Lyotard uses the term 'postmodern' to warn against patronizing sentences stating political instead of natural correctness. Foucault uses the term 'pastoral power' to warn against the human disciplines using self-reference to discipline themselves and their subject by using is-statements to install diagnoses to be cured by normalizing institutions using the methods of the same human disciplines. And Bourdieu uses 'symbolic violence' to label patronizing education giving monopoly of knowledge capital to a knowledge nobility; and sees mathematics as especially useful to that end. (Tarp, 2004)

Based upon this French warning against hidden patronization, a research paradigm can be created called postmodern 'contingency research' unmasking hidden patronization by finding alternatives to choices presented as nature. Categories and discourses are non-patronizing if grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the US enlightenment democracy; and resonating with Piaget's principles of natural learning, saying that individuals adapt to the outside world by creating schemata to assimilate the outside world or to be accommodated to the outside world (Piaget, 1970); both resonating with the inductive method used by natural science to produce natural correctness.

Natural And Political Rooted Education

To adapt to the outside world reptiles use genes, mammals also use parental care, and humans also use language, developed when motion-freed forelegs created additional brain capacity to keep the balance and to store the sounds associated with what was grasped with the forelegs, thus supplying humans with a language to share information through communication that, if institutionalized, is called schooling or education, typically containing primary education for children, secondary education for teenagers and tertiary education for adults.

In primary education, children learn basic communication skills, i.e. reading, writing and reckoning, and obtain a general knowledge about nature and society in time and space. Here it makes good sense to concentrate children of the same age in the same class.

In a republic with few institutions as the American, you earn your living from your talents uncovered and developed through daily lessons in self-chosen half-year blocks in secondary and tertiary schools. Block-organized education creates a flexible workforce that adapt quickly to changes in technology, e.g. when modernity, created by the invention of artificial muscles that combined with tools became machines to do the hard physical jobs, changed to postmodernity created by the invention of artificial brains, computers, that combined with machines become robots to perform routine information and production jobs.

In more autocratic states, primary education was sufficient background for being an unskilled or skilled worker. But, exercising its Gewalt through institutions, the state needed skilled officials with secondary and tertiary exams leading directly to office. Institutionalization thus makes line-organized office-preparing education a rational choice if only 10% of a youth population continue in secondary and tertiary education, all wanting to qualify for an office. But line-organization becomes problematic when 95% need a secondary education in the information economy of postmodernity where robots have taken over many traditional jobs.

As office preparation line-organized education teaches learners to follow orders, which may be OK in industry controlled externally by the correctness of the market, but might become problematic in institutions controlled internally by political correctness. Bauman and Arendt point out that, by following orders, modernity can create both gas turbines and gas chambers. Following authorized routines, institutions cannot know if they produce benefit or crime against humanity as shown in the Nuremberg Trials (Bauman, 1989; Arendt, 2005).

Using line-organized education to educate a whole population involves non-democratic patronization. The learners are concentrated in age-determined classes, and forced to choose its multi-year block-combination with exams in the end that cannot be retaken. The use of KZ-like techniques to practice forced classes and forced feeding prevents learners from leaving unsuccessful subjects, instead they are forced to leave education entirely, thus having their personal talents left unnourished. And the inflexibility of line-organization forces people to stay so long in education that the EU reproduction rate now is 1.5 children per family, reducing the EU population to 10% in 200 years. Although efficient in an autocracy wanting to exercise state Mündigkeit, line-organized education is a disaster in a democracy accepting adult Mündigkeit. However, EU finds it hard to learn the Nuremberg lesson: Don’t exaggerate institutionalization, and don’t use forced concentration for crowd-control.

Selfreference Changes Mathematics Into Metamatism

Although created as abstractions from external examples, the set-concept allowed mathematical concepts to be internally defined as examples of abstractions. But a self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox ‘this statement is false’ inspired Russell to formulate a paradox about the set of sets not belonging to itself: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$ (Kline, 1972).

To avoid this paradox, a set-theory was invented not distinguishing between elements and sets, thus being meaningless by mixing examples and abstractions: you can eat an example of an apple, but not the abstraction ‘apple’. Also Peano axioms were invented using a follower principle to prove that 1+1 = 2 is a natural correct statement. However, in a laboratory, 1 week + 1 day = 8 days, and 1 is not well defined since 1 threes = 3 ones and 1 fives = 5 ones etc.

Thus two terms can be coined: In ‘meta-matics’ definitions are examples of abstractions instead of abstractions from examples; and ‘mathe-matism’ produces statements true in a library, but not in a laboratory, as e.g. ‘1+1 = 2’. Together, meta-matics and mathe-matism form ‘meta-matism’ preaching politically correct, but ungrounded concepts and statements.

The Natural Fact Many Makes Mathematics A Natural Science

The natural fact Many occurs in time and space as repetition and multiplicity. To deal with Many, we totalize by counting and adding. Counted by bundling, a total T becomes a number of unbundled, bundled, bundles of bundled etc. Thus a natural number includes a unit and a decimal point to separate the bundled from the unbundled, e.g., T = 9 1s = 2.1 4s. This way of counting allows mathematics to be learned with 1digit numbers alone (Zybartas et al, 2005).

Once counted, totals can be united, called algebra using an Arabic word. United next-to, the units are integrated: 2 3s + 4 5s = 3.2 8s. United on-top, the units must be changed to be the same: 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s. Changing units is called proportionality. In the case of overloads, also bundles are bundled, e.g. making 5 fives to 1 five-fives: 5.1 5s = 10.1 5s.

Instead of counting on, operations are invented to predict the result by calculations. Thus 3+5 predicts the result of counting on 5 times from 3. The calculation 3*5 predicts the result of adding 3 5 times, and 3^5 predicts the result of multiplying with 3 5 times.

Uniting on-top and next-to provide four ways to add: Plus adds unlike unit-numbers: 3 cokes and 5 cokes = (3+5) cokes, multiplication adds like unit-numbers: 3 cokes 5 times = (3*5) cokes; power adds like per-numbers: 12% 5 times = 76% since 1.12^5 = 1.76; and integration adds unlike per-numbers: 2 kg at 3 $/kg + 4 kg at 5 $/kg = 6 kg at (2*3+4*5)/6 $/kg.

Inverse operations predict the result of reversing uniting. Thus the splitting 8 = 5+x is predicted by x = 8–5, the splitting 8 = 5*x is predicted by x = 8/5, the splitting 8 = 5^x is predicted by x = log5(8), and 8 = x^5 is predicted by x = 5√8. Calling reversed calculation equations, the natural way to solve equations is: move to the opposite side with opposite sign.

A formula shows how a total is calculated, e.g. T = 2+3*4. Variable numbers are unknown and written as letters, e.g. y = 2+3*x. This transforms a formula into a function, using tables and graphs (and a graphical calculator) to describe corresponding numbers (Tarp, 2009).

Conclusion

From a postmodern skeptical viewpoint, what is called mathematics may turn out to be metamatism rooting its concepts in internal abstractions instead of in external examples, and producing statements that are true in a library but not in a laboratory. And what is called education may turn out to be office preparation in line-organized forced classes adapting learners to internal political correctness, instead of talent development in self-chosen half-year blocks adapting learners to external natural correctness. And what is called research may turn out to be orthodoxy forcing the outside world to assimilate, instead of accommodating itself to the outside world. Consequently, mathematics education must choose between adapting to internal political correctness or external natural correctness. If choosing to adapt the learner to the natural fact Many, the diagnose-word mathematics must be replaced by action-words as totalize, count, add, reckon, triangulate etc. Likewise, post-primary education must be organized as daily lessons in self-chosen half-year blocks to allow learners to try different approaches to different subjects. As to research, instead of disciplining itself and its subject with political correctness, it should produce natural correctness by searching for hidden contingency uncovering alternatives to patronizing choices presented as nature.

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318. Manuscript to a YoyTube Video on Postmodern Math Education

In this video Paul Ernest and Allan Tarp discuss 8 questions: What is meant by postmodern? What is meant by modern? What is the root of postmodern thinking? Who is the most important postmodern thinker? What is mathematics? What is postmodern mathematics? What is postmodern research?

Link http://youtu.be/ArKY2y_ve_U

Bo: Welcome to this discussion on Postmodern Mathematics education. My name is Bo. And welcome to our two guests. Welcome to Paul.

Paul: Thank you Bo. I am really looking forward to this discussion.

Bo: And welcome to Allan.

Allan: Thank you Bo, so am I.

Bo: I will ask you eight questions. The first question is: what is meant by postmodern?

Paul: As I see it, postmodernism means the rejection of a single all-encompassing metanarrative – whether it be Freudianism, Marxism, Logicism, Radical Constructivism, Enactivism, even Bourbakianism. Instead it means acceptance of multiple perspectives offering new ways of seeing mathematics, teachers and learners. Thus it is important to recognize that all human subjects have multiple selves and that we all (mathematicians, teachers and learners) have access to different selves: authoritative knowers, researchers, learners, appreciators and consumers of popular and other cultures, as well as having non-academic selves.

Allan: It seems to me that we must distinguish between post-modernism and post-modernity. Post-modernity is what we do with our hands, i.e. how we act in the world. And post-modernism is what we do with our head, i.e. how we think about the world. To simplify, post-modernity is the social condition that was created by I,T, information technology. And postmodernism is skepticism toward hidden patronization.

Paul: I would agree that postmodernism is a conceptual position. Rather it is several positions because we should distinguish cultural postmodernism (in art, architecture, music, design and fashion) from philosophical postmodernism. This is more about the rejection of a single all-encompassing theoretical metanarrative. I see post-modernity to be the epoch when postmodern ideas are current. But this is semantic and I accept your distinction between the theory and practice of the postmodern

Bo: The second question is: ’What is meant by modern?’

Paul: To me modern thinking began with Descartes, who puts forward a logical master plan to provide certain and indubitable foundations for all of knowledge. This begins with a small basis of clear and distinct ideas, and then deduces all subsequent truths by the clear rules of logic. This plan was modeled on the axiomatic geometry of Euclid, already two thousand years old, which Hobbes called the only true science bestowed on humankind.

Allan: To me, modernity means the social condition created by the invention of the artificial muscle, the motor; and the combination of the motor and tools to machines. And modernism means the transition from belief to certainty, provided by the natural sciences, producing real world knowledge by inducing categories from observations; and validating theories, by trying to falsify deduced predictions.

Bo: The third question is: What is the root of postmodern thinking?

Paul: As I see it, Lyotard, is one of the first to use of the term 'postmodernism' with reference to philosophical discourse in his book The Postmodern Condition. Lyotard considers all of human knowledge to consist of narratives, whether it is in the traditional narrative forms, such as literature, or in the scientific disciplines. Each disciplined narrative has its own legitimation criteria, which are internal, and which develop to overcome or engulf contradictions. However the roots of postmodernism can be traced to Nietzsche too. When he said ‘God is dead’ he meant that the days of all absolutes were over – and absolutes as whet underpins modernism.

Allan: To me, postmodernism means Skepticism toward hidden patronization according to Lyotard’s statement: ‘Simplifying to the extreme, I define postmodern as, incredulity, toward metanarratives’. Skepticism is as old as the republic, beginning in the ancient Greek republic with the sophists, and continuing in the Enlightenment Century. Today skepticism is expressed in the two Enlightenment republics, in the American with pragmatism and grounded theory; and in the French with the post-structural thinking of Derrida, and Lyotard, and Foucault, and Bourdieu.

Bo: The fourth question is: Who is the most important postmodern thinker?

Paul: To me, the most important theorist is Wittgenstein. Wittgenstein made the transition from modern to postmodern thinking in his two books. First in Tractatus Logico-Philosophicus he tries to finish the modern project by showing how the outside world has created our language to represent it. Then in Philosophical Investigations he turns around and shows how in return it is language games that construct the outside world. Thus with his own person and his own work Wittgenstein is the first to realize that the world is not creating language, but created by language. But this language use – in what he terms language games – is based in everyday living in what he terms ‘forms of life’.

Allan: To me, the most important theorist comes from the threatened republic, the French. Here I will point to Foucault and his statement when discussing human nature with Chomsky coming from the unthreatened republic, the American. Foucault says: "It seems to me, that the real political task in a society, such as ours, is to criticize the working of institutions, which appear to be both neutral and independent; to criticize them in such a manner, that the political violence, which has always exercised itself obscurely through them, will be unmasked, so that one can fight them. "

Bo: Could you please elaborate?

Paul: Wittgenstein says, mathematical foundations are quite irrelevant to the continued healthy practice of mathematics, both pure and applied. Wittgenstein offers a powerful social vision of mathematics. One of his key contributions is to recognize the social basis of certainty, that following a rule in mathematics or logic does not involve logical compulsion. Instead it is based on the tacit or conscious decision to accept the rules of a 'language game' which are grounded in pre-existing social 'forms of life'. Wittgenstein's importance is to show that that the 'certainty' and 'necessity' of mathematics are the result of social processes of development, and that all knowledge including that in education presupposes the acquisition of language in meaningful, already existing, social contexts and interactions.

Allan: To me, Foucault shows how human disciplines discipline themselves and their subject. This forces false identities upon humans, who then seek cure at correcting institutions that copy the pastoral power of the Christian church. Furthermore, the institution called education might instead be a place for symbolic violence, that monopolizes society's knowledge capital for a privileged knowledge nobility. Also, institutions are run by people that follow authorized routines, which can create both gas turbines and gas chambers. Following orders might be OK in industry since it is controlled from below by the natural correctness of the market: sell or die. But it may become problematic in institutions that are controlled from above by a political correctness: conform or die. Thus institutionalized patronization might become totalitarian, reintroducing evil actions, rooted not in a devil, but in the sheer banality of just following orders.

Paul: I agree with Allan that Foucault is a very important thinker. His originality lies in his historical analyses of power, institutions and identity, and his rejection of essences underlying everything – from persons and identities, academic subjects and knowledge to names, concepts and ideas. He is usually called a post-structuralist, but post-structuralists share much with postmodernists – most notably the rejection of single monolithic structural theories to explain anything – from society to language.

Bo: The fifth question is: What is mathematics?

Paul: To me, mathematics is what mathematicians do. Mathematics is a language game, or rather a set of language games and forms of life. Mathematics is taught to allow students to take part in some of these language games; because it can be applied in many real world situations and because it has great social and personal power. To get a deeper understanding, again we should listen to Wittgenstein. Mathematics is a multiplicity of practices – Wittgenstein calls it the 'motley of mathematics'. Mathematics provides the entry ticket to many of these practices.

Allan: To me, mathematics is not an action-word, it is a verdict-word that labels or installs what it names. To see, if it labels or installs something, we must ask, which actions are named mathematics? Many, is a natural fact. To deal with Many, we count and add, in short we reckon. Consequently, there is a need for education in reckoning, also called algebra, which in Arabic means to re-unite.

Bo: The sixth question is: What is postmodern mathematics?

Paul: To me, Postmodernism rejects a single authoritative way of seeing mathematics, teachers and learners, for each can be seen and interpreted in multiple ways. Mathematics can be seen as axiomatic and logical leading to indubitable conclusions, but it can also be seen as intuitive and playful, open-ended, with surprises and humor, as evidenced in popular mathematical images and cartoons. Additionally it can be seen in its applications in science, information and communication technologies, everyday life and ethnomathematics. All of these dimensions are part of what makes up mathematics and they all co-exist successfully.

Allan: To me, postmodern mathematics raises the question: Is mathematics education what it says, education in mathematics - Or is it something else, like symbolic violence in meta-matism, which is a mixture of meta-matics, that turns mathematics upside down, by defining concepts as examples of abstractions, instead of as abstractions from examples; and of mathe-matism that is true in a library, but not in a laboratory: To exemplify, the statement , 2 times 3 is 6, is matte-matics, since 2 threes can be re-counted as 6 ones. Whereas the statement, 2 plus 3 is 5, is mathe-matism, since it has countless counterexamples as e.g. 2 weeks plus 3 days is 17 days.

Bo: The seventh question is: What is postmodern education?

Paul: To me, postmodern education means accepting the diversity in learners' background and interests, in learning material and situations, and in teacher personalities and ethnic background. Allowing playfulness and surprise to enter into education is also necessary. I think mathematics plays a small but significant part in postmodern education. What I would like to see is a truly responsive education system that has a real personal face. Every student should have a relationship with a teacher or other mentor who finds out what the student loves and can achieve real success at. Whether it be some sport, model making, dance, or academic study and creativity, such as in mathematics, it is the responsibility of education to help the student find their own bliss and success. The area that students experience this success in doesn't really matter. Once they have success, enjoyment and self-confidence unleashed by their manifested talent, I believe students can go on to succeed in many other areas of study and life' including mathematics. So the flexibility and truly individualistic element of education is what would make it postmodern, and a great education.

Allan: To me postmodern education means replacing the forced classes and the forced subject combinations of line-organized education aiming at preparing for public offices, with the self-chosen half-year blocks of block-organized education aiming at uncovering and developing the individual talent of the learners.

Bo: The last question is: What is postmodern research?

Paul: To me, all forms of research can fit under the umbrella of postmodernism since it allows for different methods, meanings and interpretations. As it stands research in the interpretative paradigm, is more open to multiple meanings since it accepts that both the world and its description are social constructions in the end. So postmodern research fits less well with the scientific research paradigm with its assumption of one true reality. However, no methods are ruled out by postmodern research, be they qualitative, quantitative or mixed.

Allan: To me, postmodern research was created by the ancient Greek sophists saying: we must enlighten ourselves to tell nature from choice, to avoid being patronized by choices presented as nature. So postmodern research is a search for hidden alternatives to patronizing choices presented as nature.

Bo: Thank you Paul and Allan. It was a nice debate, wasn't it? I learned that you have more in common than differences, although you do differ on some issues.

319. Manuscript to a YoyTube Video on A Postmodern Deconstruction of World History

This YouTube video on postmodern deconstruction describes world history as the history of trade. First eastern lowland pepper and silk was traded with western highland silver, then eastern cotton was moved west and traded with northern industrial products, and finally electrons replaced the silver and cotton economy with an information economy.

Link http://youtu.be/xQAdrI_CvyY
Screens: ..\Manuscripts and Screens\World_History.ppt

Screen 1 & 2

Bo: Welcome to the MATHeCADEMY.net Channel. And welcome to our series on postmodern deconstruction. My name is Bo. Today we look at world history where we address the question "World History is made by personalities - or is it?" And welcome to our guest, Allan, who uses postmodern deconstruction in his work.

Allan: Thank you Bo.

Bo: Allan, what does deconstruction mean to you?

Allan: To me, deconstruction means what it says, destruction and reconstruction. It is the method that is used by postmodern skeptical thinking that dates back to the ancient Greek sophists. The sophists warned against patronization that is hidden in choices that are presented as nature. So to avoid hidden patronization, false nature must be unmasked as choice. And to deconstruct then means to discover or to invent alternatives to choices that are presented as nature.

Bo: And what does postmodern mean to you?

Allan: It seems to me that we must distinguish between post-modernism and post-modernity. Post-modernism is what we do with our head, i.e. how we think about the world. And post-modernity is what we do with our hands, i.e. how we act in the world. To simplify, postmodernism is skepticism toward hidden patronization. And post-modernity is the social condition that was created by IT, information technology.

Screen 3

Bo: Thank you, Allan. Do you have a short answer to today's question?

Screen 4

Allan: To me, the short answer is that world History is made, not by personalities, but by silver and cotton and electrons. First silver created wealth from pepper and silk. Then cotton created welfare from machines. Finally electrons spread out productivity first and then information.

Bo: So it all began with silver?

Screen 5

Allan: Yes Bo. But first threatened apes in Africa transformed into humans that developed technology to master the surrounding threats. We have holes in the head to supply the stomach with food and the brain with information. Animals must do this individually. Humans share both. When the warm and humid golf stream was sucked away from Africa to the north by the ice cap on the North Pole, the rainforest in Sahara disappeared. The lack of trees for rescue forced some apes to run away on two legs. This forced their brain to grow to keep the balance. Also it freed the forelegs transforming the fore-toes to fingers that can be used to hunt for food, to gather food, and to share food. And associating sounds with what you have in your hand developed a language to share information. So these apes slowly transformed into humans. And equipped with physical and mental graspers, humans developed their four cultures.

Bo: What do you mean by the four cultures?

Screen 6

Allan: The first culture was a hunter-gatherer culture using the hands to hunt and gather food as just mentioned. But then information about matter, especially about iron, allowed humans to develop artificial hands, i.e. tools. Tools replaced hunter-gatherer culture with agriculture, where humans themselves control nature's production of food. Then information about energy allowed humans to develop artificial muscles, motors. And combining the artificial hand with the artificial muscle, i.e. combining the tools with motors, created the machines replacing agriculture with industry by taking over the hard physical work. Finally, information about information allowed humans to develop an artificial brain, a computer. And combining the artificial hand with the artificial muscle with the artificial brain, i.e. combining tools with motors and with computers, created the artificial human, the robot, replacing industry with information culture by taking over physical and mental routine work.

Bo: But, Allan, where does the silver come in?

Screen 7

Allan: Well Bo. Transformed from apes to humans, people left Africa. Some went to the western highlands, but most went to the eastern lowlands made fertile by rivers. Lowland products as pepper and silk from India and China were traded with silver from Europe's highlands. The first silver mines were found outside Athens financing the Greek culture for one hundred years until they were emptied.

Bo: And then the Romans took over?

Screen 8

Allan: Yes, the Romans found silver in Spain, financing the Roman Empire until the mines were taken over by Vandals from the north settling in southern Spain, in Andalusia, the land of the Vandals. That made the Arabs very angry. In the trade between Europe and India, the Arabs were the middlemen making fortunes. The Vandals didn't care about pepper and silk so the trade stopped. This made the Arabs take the silver mines themselves together with North Africa.

Bo: So the lack of silver created the dark middle age?

Screen 9

Allan: Precisely! But then came the rebirth, the renaissance, in Italy, financed by German silver found in the silver dale in the Harz, giving name to today's dollar. However, silver only travels a short distance before nightfall where it must be protected by a strong castle. This split Germany and Italy up into hundreds of small principalities all deriving wealth from the silver stream. Additional wealth was created when Roman numbers were replaced with Arabic numbers. With Roman numbers you can add, but you cannot multiply. And multiplication allowed the Italians to set up banks creating wealth from interest.

Bo: But banks can go bankrupt?

Allan: Yes. And the Italian banks did when the Portuguese found the seaway to India south of Africa, with gold mines on the way. Here there were no Arab middlemen to increase the cost. So cheap Portuguese pepper forced Italy out of business. Spain had its own silver-mines; and once the Arabs were pushed out of Andalusia, Spain tried to find another sea route to India by going west.

Bo: And here they found the West Indies?

Screen 10 & 11

Allan: Indeed they did. But there was no pepper and no silk. In return there was lot of silver, e.g., in the land of silver, Argentine. The silver then was sailed home to Spain and used to finance religious wars in Europe, helping the Catholic Church to try to win back northern Europe lost to the Reformation.

Bo: Allan, you also talked about cotton?

Screen 12

Allan: Yes, Bo. In England they had no silver, but being of Viking descent they knew how to sail, and they easily robbed the slow Spanish silver boats returning over the Atlantic. However, to get to India the Brits had to go on open sea to avoid the Portuguese fortifications on Africa's coast. Once in India they discovered that cotton was much cheaper than silk. So they brought back cotton plants to be planted in the southern states of their North American colonies. However, to grow cotton you also need a lot of labor. So the Brits had to buy labor in Africa to be sailed to North America as slaves.

Bo: So stolen silver was traded for stolen bodies?

Screen 13

Allan: Not exactly. The Brits exchanged slaves for guns produced by machines originally created to transform cotton into clothes. This new economy was a triangular trade. It was not based upon silver but upon exchange: Cotton for guns, guns for slaves, and slaves for cotton. And once machines were created to produce clothes, other machines were invented to produce other forms of goods in huge quantities, thus creating the foundation for a welfare society.

Bo: But machines also need a lot of labor?

Screen 14

Allan: They do. And the American civil war was about labor. Cotton is not made to be consumed by the workers, machine products are. So the agricultural South wanted low-paid workers to produce raw material to industrial states. And the industrial North wanted well-paid workers to consume the surplus production of the machines. In the end industry won over agriculture. So to keep up a supply of cheap cotton, the Brits moved the cotton to Africa and transformed Africa into colonies, thus creating a closed economy with the motherland by supplying it with cheap raw material and by consuming the surplus production of its industry.

Bo: It sounds like imperialism?

Screen 15 & 16

Allan: Which it was. It split the world up in three closed economies: An American, a British, and a French. And two nasty European civil wars, World War one and World War two, had to be fought before finally world trade was set free, so that not only Germany and Japan, but all countries can compete on equal terms.

Bo: Allan, you also talked about electrons?

Screen 17 & 18

Allan: Yes Bo. Things can be moved by our muscles; and by the artificial muscles, i.e. motors. Transforming water to steam produces pressure to move things by using a steam engine to deliver energy to motors in factories. However, production must take place close to the steam engine, so people had to move in to huge cities as Chicago, Birmingham, the Ruhr district, etc. But then a discovery was made: Metal contains tiny particles called electrons only visible when lightening. Connecting different metals will make electrons flow in a connecting wire, thus creating an electrical current. Winded up, a current becomes a coil that acts like an artificial magnet able to move other magnets. Steam doesn't travel long distances, but electrons do, so their moving ability allowed machines to be set up all over the world, thus spreading out productivity.

Bo: So using electrons to carry energy prevented large concentrations of population?

Allan: It did. Now people could work with machines wherever they lived.

Bo: Allan, you also talked about information?

Screen 19 & 20

Allan: Yes, Bo. With electrons a lamp can have two states; it can be on and off. This can be used to represent numbers. Counting takes place in bundles, thus 456 means 4 ten-bundles of ten-bundles, and 5 ten-bundles and 6 unbundled ones. However, instead of counting in ten-bundles we can also count in two-bundles. Thus 5 can be bundled as 4 and 1, i.e. as 1 two-bundle of two-bundles, and no two-bundles, and 1 unbundled ones, which is the same as the number 101. So we can use electrons to carry information about numbers, and also to perform computations on these numbers, thus creating a computer as an artificial brain. And since physical and mental routine jobs can be described in numbers, computers can control machines to perform routine jobs. And a computer together with a machine becomes an artificial human, a robot.

Bo: But Allan, if the robots take over the production then what should people do?

Screen 21

Allan: This is today's big challenge. If not met, people will again be hunter-gatherers. In the morning they will hunt entertaining information on the internet and on the TV. And in the afternoon they will gather fast food at the super-market.

Bo: What a nightmare. I hope it will not come through.

Allan: It will not if robots taking over routine jobs make humans enlighten themselves especially about how hunter-gatherers can create a rich day without having to work.

Bo: But after the Enlightenment 18th century everybody is enlightened today?

Screen 22

Allan: They would have been had the Enlightenment century continued. But counter-enlightenment used forced line-organized education to stop Enlightenment from spreading from the two Enlightenment republics, the American and the French. However there is hope. But first Europe must replace its line-organized office-preparing education with North American's block-organized education that can uncover and develop the students' individual talents through daily lessons in self-chosen half year blocks.

Bo: But what happens if Europe keeps on to its line-organized education?

Allan: Then the problem will solve itself since forcing students to stay at a line produces many dropouts that have to go back to start if changing to another line. So instead of reproducing, Europe's students change lines. This makes Europe's educational system an exterminator that by reducing the birth rate to 1.5 child per family will wipe out Europe's population over 200 years.

Screen 23

Bo: So Allan, next time maybe we should deconstruct education?

Allan: That would be a very good idea, Bo.

Screen 24

Bo: Thank you Allan, for sharing with us your view on the question: World history is created by personalities – or is it? Next time at the MATHeCADEMY.net channel we will look at deconstruction of education. Again we will ask: Education must be line organized – or must it?

320. Manuscript to a YoyTube Video on Deconstruction of Fractions

This YouTube video on deconstruction in mathematics education connects fractions to its root, the leftovers when performing icon-counting. To deal with Many, we total by bundling in icon-numbers less than ten, or in tens needing no icon as the standard bundle. When bundling in 5s, 3 leftovers becomes 0.3 5s or 3/5 5s, thus leftovers root both decimal fractions and ordinary fractions.

Link http://www.youtube.com/watch?v=PtRuk0EWmaQ
Screens: [..\Manuscripts and Screens\ScreensDeconstructFractions.ppt](..\Manuscripts and Screens\ScreensDeconstructFractions.ppt)

Fractions Grounded as Decimals, or 3/5 as 0.3 5s.

Screen 1 & 2

Bo: Welcome to the MATHeCADEMY.net Channel. And welcome to our series on postmodern deconstruction in mathematics education. My name is Bo. Today we address the question “Fractions IS hard - or is it?” And welcome to our guest, Allan, who uses postmodern deconstruction in his work.

Allan: Thank you Bo.

Bo: Allan, what does deconstruction mean to you?

Allan: To me, deconstruction means what it says, destruction and reconstruction. It is the method that is used by postmodern skeptical thinking that dates back to the ancient Greek sophists. The sophists warned against patronization that is hidden in choices that are presented as nature. So to deconstruct means to unmask false nature by finding hidden alternatives to choices presented as nature.

Bo: And what does postmodern mean to you?

Screen 3

Allan: Post-modernism is what we do with our head, i.e. how we think about the world. And post-modernity is what we do with our hands, i.e. how we act in the world. To simplify, postmodernism is skepticism toward traditions hiding patronization by presenting its choices as nature. And post-modernity is the society that is created by IT, information technology.

Bo: Thank you, Allan. Do you have a short answer to today’s question?

Allan: To me, the short answer to the question “Fractions IS hard - or is it” is that fractions is not hard by nature, but by choice. Fraction is made difficult by its missing link to the root of mathematics, the natural fact Many.

Bo: So, Allan, what is the root of fractions?

Screen: 4

Allan: Well Bo. To deal with Many, we count by bundling. We can bundle in icon-numbers, or we can bundle in tens, needing no icon since it has been chosen as the standard bundle-number. When bundling in 5s, 3 leftovers becomes 0.3 5s or 3/5 5s, thus leftovers root both decimal fractions and ordinary fractions.

Bo: Allan, do you have a short answer to how to make fractions easy?

Allan: Yes, Bo. Simply teach first order icon-creation and second order icon-counting instead of skipping both and go directly to third order ten-counting.

Bo: Allan, can you please specify?

Screen 5

Allan: Certainly, Bo. Mathematics is a natural science about the natural fact Many. To deal with many, we total, i.e. we ask the question “how many?” And the answer is given by the total. That is why we use the word algebra that means to unite in Arabic. Thus all mathematics statements should begin with its subject, the total, and specify what the total is. The first step is to represent the total in three ways, by physical sticks, by graphical strokes, and by spoken words. Thus four things can be represented by four sticks on a table; and by four strokes on a paper; and by the word four. Now we can write that T equals stroke stroke stroke stroke.

Bo: That is what the Romans did, isn’t it?

Allan: Indeed it is. But the Arabs went one step further by using icons. They united the four strokes into one single symbol consisting of four strokes. Thus they transformed four ones into one fours.

Bo: But isn’t four ones and one fours the same?

Screen 6

Allan: Well Bo. Four ones means that you count in ones, so that one is the unit. Whereas one fours means that you count in fours, so that four is the unit. And that you count by bundling and stacking the total in fours. If we write the digits in a less sloppy way, we can see that there are five sticks and strokes in the five-icon, and six strokes in the six-icon, etc. So first order counting bundles sticks into icons, so that there are five sticks in the 5-icon etc. And second order counting bundles the total in icon-bundles, where third order counting bundles in tens.

Screen 7

Bo: But why do the icons stop at ten?

Screen 8

Allan: We count by bundling, but we never use the bundle-icon. If we count or bundle in fives, we count one, two, three, four, bundle, one bundle one, one bundle two etc. Or in a shorter way: 1, 2, 3, 4, 10, 11, 12 etc. Thus we do not use the five-icon. Likewise, ten does not need an icon when counting in tens. In this way ten is the only number with its own name but without its own icon. This makes ten a cognitive bomb since ten is the follower of nine while one zero is the follower of four when counting in fives. So ten is not one zero by nature, but by choice of the bundle-number.

Bo: Allan, what do you mean with leftovers?

Screen 9 & 10

Allan: Well, Bo. First we count the sticks or strokes by bundling them in e.g. fours. Then we can place the bundles in a left bundle-cup, and the unbundled in a right single-cup. We don’t have to place the physical bundles. For each bundle we just place a stick in the bundle-cup. Now we can use the icons to write the total using a decimal point to separate the bundles from the unbundled. In this way a total of 6 1s can be counted in 4s as 1 4s and 2 1s, and written as 1.2 4s. So a natural number includes a decimal point to separate the bundles from the unbundled, and a unit. Thus a total of 7 1s can be recounted in 5s as 1.2 5s, or as 1.3 4s, or as 2.1 3s, etc. If counted in tens, 7 1s become 0.7 tens. However, we only write 7 leaving out the unit and misplacing the decimal point one place to the right. This may be OK in business, but it creates learning problems in school.

Bo: So decimal numbers come before ordinary fractions.

Screen 11

Allan. They do if we represent Many by sticks or strokes. But they are created at the same time if we represent Many by blocks. Here the bundles can be placed on-top of each other as a stack. As to the unbundled we must make a choice. We can place them next-to the stack, or we can place them on-top of the stack. If placing the unbundled next-to the stack we might have 2 fives and 3 ones, which can be written with decimals as 2.3 5s. Placing the unbundled on-top of the stack we have to count the 3 unbundled in 5s, and the recount formula then gives the result that 3 = (3/5) 5s = (3/5)*5. So preferring decimal to fractions is a question of taste. In all case the two forms is united by the fact that 0.3 5s is the same as 3/5 of five.

Bo: What do you mean with the recount-formula?

Screen 12

Allan: We saw that digits are icons that contain the number of strokes they represent. Likewise, operations are also icons that show the counting processes they represent. Taking away 4 is iconized as a horizontal stroke showing the trace left when dragging away the 4. Taking away 4 many times, i.e., taking away 4s, is iconized as an up-hill stroke showing the broom sweeping away the 4s. Placing a stack of 4 singles next-to another stack is iconized as a cross showing the juxtaposition of the two stacks. And building up a stack of 3 4s is iconized as an up-hill cross showing a 3 times lifting of the 4s.

Bo: So 3/4 means 3 counted in 4s?

Screen 13

Allan: Precisely! Counting in fours means to repeat taking away four, i.e. dividing the total by four. So the counting result can be predicted by a recount-formula saying that the total T can be bundled in bs T/b times. So T is (T/b) bs, which is the same as (T/b) times b. Thus the recount-formula predicts that recounting a total of 8 1s in 4s gives 8/4 of the 4s, which is 2 4s.

Bo: So recounting is done by de-bundling and re-bundling?

Screen 14

Allan: That is one option. So to recount 4 5s in 6s manually, first we must count up the 4 5s. Then we must debundle them in 1s. And finally we must rebundle them in 6s. This is a long and tiresome job. However, if we use the recount-formula and a calculator, we can predict the result to be 3 6s and a rest. And the rest is found by removing the 3 6s from the 4 5s. So the result of recounting can be predicted by a calculator using division and subtraction.

Bo: From the recount-formula it seems as if division and multiplication come before addition and subtraction?

Screen 15

Allan: Indeed they do. After recounting is predicted by a rebundle-formula using division and multiplication, we can predict the unbundled by a restack-formula using subtraction and addition. So the natural order of operations is division, multiplication, subtraction, and in the end addition. This is in contrast to the tradition that reverses the natural order, which creates yet more learning problems.

Bo: Allan, how about adding fractions?

Screen 16

Allan: OK, Bo. Let us begin with adding numbers. What would you say is most correct, saying that 2+3 is five or saying that 2 times 3 is 6?

Bo: To me they are both correct, but if I should choose I would say that 2+3 = 5 is most correct.

Allan: Well, I think we should distinguish between grounded mathe-matics that is rooted rooted in observations and ungrounded 'mathe-matism' that is true in the library but not in the laboratory. Thus '2*3 = 6' is natural correct since it is grounded in the fact that with 3 as a unit, 2 3s can be recounted as 6 1s. In contrast to this saying that '2+3 = 5' may be political correct in a library, but has countless counterexamples in the laboratory: 2 weeks + 3 days = 17 days etc.

Bo: But to add fractions we must teach how to find the common denominator?

Screen 17

Allan: We must indeed if we want to teach mathematism instead of mathematics. The fraction paradox will illustrate the difference. A teacher asks the class: What is 1/2+2/3? The class answers that 1/2 + 2/3 = (1+2)/(2+3) = 3/5. The teacher then says: No. The correct answer is 1/2 + 2/3 = 3/6 + 4/6 = 7/6. To this the class asks: But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 out of 6 cokes? The point is that all numbers have units and you can only add if the units are the same. 2/3 does not exist in itself; it will always be 2/3 of something as demonstrated in the recount formula.

Bo: But the recount formula only shows that 2/3 of 3 is two. How about 2/3 of 15?

Screen 18

Allan: Well, you just recount 15 1s in 3s as (15/3) 3s i.e. as 5 threes. So 2/3 of 15 is the same as 2/3 of 3 5 times , that is ten.

Bo: Allan, cam you briefly sum up your view on fractions.

Allan: I will be glad to do so, Bo. Fractions did not create themselves, fractions are rooted in the root of mathematics, the natural fact Many. Counting Many in bundles, leftovers might be placed next to the bundle-stack, described as decimals or on top described as ordinary fractions. Thus both decimal fractions and ordinary fractions come in naturally in grade one. However, the tradition insists that counting only takes place in tens. And instead of using the natural way to represent numbers as e.g. 3.2 tens, the tradition insists that the unit is removed and that the decimal point is misplaced. In this way decimal fractions are hidden until they are introduced in middle school as special fractions, having as denominator the number ten or powers of ten. Likewise, ordinary fractions are postponed to middle school and defined as a special case of division. So what is easy by nature is made difficult by somebody's choice. And you might ask: What is the purpose in making an easy thing difficult?

Bo: A good question, Allan, let us return to that later.

Allan: A good idea, Bo.

Bo: Thank you Allan, for sharing with us your view on the question: Fractions is hard – or is it? Next time at the MATHeCADEMY.net channel we will look at deconstruction of pre-calculus. Again we will ask: pre-calculus is hard – or must it?

321. Manuscript to a YoyTube Video on a Postmodern Deconstruction of of PreCalculus

This YouTube video on deconstruction in mathematics education connects preCalculus to its root, the natural fact Many. To deal with Many, we total. Some totals are constant, some change in space or time. The change might be predictable, or not. Pre-calculus describes predictable constant change. Calculus describes predictable changing change.

Link: http://www.youtube.com/watch?v=3C39Pzos9DQ
Screens: ..\Manuscripts and Screens\ScreensDropuotRyan.ppt

Screen 1 & 2

Bo: Welcome to the MATHeCADEMY.net Channel. And welcome to our series on postmodern deconstruction in mathematics education. My name is Bo. Today we address the question "PreCalculus IS hard - or is it?" And welcome to our guest, Allan, who uses postmodern deconstruction in his work.

Allan: Thank you Bo.

Bo: Allan, what does deconstruction mean to you?

Screen 3

Allan: To me, deconstruction means what it says, destruction and reconstruction. It is the method that is used by postmodern skeptical thinking that dates back to the ancient Greek sophists. The sophists warned against patronization that is hidden in choices that are presented as nature. So to deconstruct means to unmask false nature by finding alternatives to choices presented as nature.

Bo: And what does postmodern mean to you?

Allan: Post-modernism is what we do with our head, i.e. how we think about the world. And post-modernity is what we do with our hands, i.e. how we act in the world. To simplify, postmodernism is skepticism toward traditions hiding patronization by presenting its choices as nature. And post-modernity is the social condition that was created by IT, information technology.

Bo: Thank you, Allan. Do you have a short answer to today's question?

Allan: To me, the short answer to the question "PreCalculus IS hard - or is it" is that preCalculus is not hard by nature, but by choice. PreCalculus is made difficult by its missing link to the root of mathematics, the natural fact Many.

Bo: So, Allan, what is the root of preCalculus?

Screen 4

Allan: Well Bo. To deal with Many, we total. Some totals are constant, some change. Some changes are constant, some change. Pre-calculus is about constant change. Calculus is about changing change. And statistics is about unpredictable change.

Bo: Allan, do you have a short answer to how to make pre-calculus easy?

Screen 5 & 6

Allan: Yes Bo, I have three stories about formulas: How they predict; how they change to equations and functions; and how they model real world problems. First we look at the two natural examples of constant change. When saving at home the change-number is constant, and adding a number a to b x times gives the number y = b+a*x. When saving in a bank the change-percent is constant, and so is the change-multiplier since adding 5% to b means multiplying b with 105%. And multiplying b with a x times produces the number y = b*a^x. This gives the two basic formulas for constant change.

Bo: And what will be your second step?

Screen 7 & 8

Allan: Now we look at formulas with 1 or 2 unknown number, also called equations and functions. Equations we solve by reversed calculation, i.e., we isolate the unknown number by moving the known numbers to the opposite side with opposite sign. Functions we graph by setting up tables for related values of the two unknowns, x and y. Or we use a cheap graphical calculator as TI-82. Here we simply enter left hand side as Y1 and the right hand side as Y2 and find the unknown by pressing solve 0 =Y1-Y2; or by finding the intersection between the two graphs, which is where the left and right hand side numbers are equal.

Bo: And what will your third step then be?

Allan: Finally, to connect mathematics to its root, the natural fact Many, we model real world tables with formulas by using the calculator's regression facility. In this way real world problems are solved by cooperation between humans and technology: Humans ask the question and evaluate the answer giving by technology.

Bo: OK, Allan, now for the details. With many dropouts, what will you do in your first class?

Screen 9

Allan: Since mathematics is a natural science about Many, I will ask the class the question 'How Many?' e.g. 'how Many fingers are there in this room?' Then I will ask the class to write out a total of 456 as we say it, i.e. as T = 4 hundreds and 5 tens and 6, which is the same as T = 4 ten-tens and 5 tens and 6 ones. Now we see, that we count by bundling in tens, ten-tens etc. And we see that all numbers carry units: hundreds, tens, ones. Also we see that a number is really a formula that is called a polynomial being a foreign word for many terms. A polynomial shows that there are four different ways to unite numbers: By multiplication as 5*10, by power as 10^2, and by addition, which normally is written as 4+5. But here the units are different, transforming the hundreds and the tens into rectangular blocks that cannot be added on-top, but must be added next-to each other. And adding next-to is integration.

Bo: So there are only four ways to unite numbers?

Screen 10

Allan: Precisely! There are four ways to unite numbers since there are four kinds of numbers in the world: Constant and changing unit- and per-numbers. Unit-numbers carry single-units as meters or seconds, and per-numbers carry double-units as meter/second or meter/100meter = %. So pre-calculus deals with the first three ways to unite numbers: addition, multiplication and power. And calculus deals with the fourth way to unite numbers, which is integration.

Bo: Are you going to introduce the traditional names, algebra and geometry?

Screen 11

Allan: Yes I am, since algebra is the Arabic word for reuniting, i.e. uniting numbers into totals and splitting totals into single numbers; and since geometry means earth measuring in Greek, which is done by splitting it up into triangles. And any triangle can be split into right-angled triangles, which can be seen as a rectangle halved by its diagonal. So a right-angled triangle has three angles and the three sides: the base, the height and the diagonal. And the goal of trigonometry is to relate angles and sides with formulas as sine, cosine and tangent expressing one side in percentage of another, thus sinus to angle A is the height in percent of the diagonal.

Bo: And what do you say when the class asks: Why learn mathematics?

Screen 12

Allan: I will point out that we have two languages to describe the world, a word-language and a number-language, and mathematics is really just another word for our number-language. The difference is that the word-language expresses opinions, whereas the number-language predicts.

Bo: Allan, can you please specify?

Screen 13

Allan: Certainly, Bo. Operations are invented to predict numbers. Thus 3 plus 5 predicts the result of adding 1 to 3 5 times. And 3 times 5 predicts the result of adding with 3 5 times. And 3 to the power of 5 predicts the result of multiplying with 3 5 times. And adding 2 3s and 4 5s as areas predicts the number of 8s. So calculations predict the total resulting from a uniting process. A formula combines operations. And a graphical calculator is a typewriter for the number-language used to set up tables for formulas, and to set up formulas from tables. So mathematics is useful because formulas predict.

Bo: So students only learn to solve equations on a calculator?

Screen 14 & 15

Allan: No. Equations can also be solved by reversed calculation. In a formula we can go two ways. Going forward we can ask: two plus three gives what? Going backwards we ask: two plus what gives five, which we write as the equation 2 + x gives 5. Instead of trying out alternatives we can predict the result by inventing an inverse operation to addition, called subtraction or minus. This allows the answer to be predicted by the calculation five minus two. So by definition five minus two is the number x that added to two gives five. So, an equation is just another word for reversed calculation. And since the equation 2 + x gives 5 is solved by the number x equal 5 minus 2, we see that solving an equation means isolating the unknown by moving numbers to the opposite side with the opposite sign.

Bo: Does the opposite side and sign method also apply to the other operations?

Screen 16

Allan: Indeed it does. Since 6/2 is defined as the number x that multiplied with two gives six, we see that the equation x*2= 6 again is solved by moving the number two to the opposite side with opposite sign, i.e., by x = 6/2. Likewise, the equation x to the power of 3 gives 8 is solved by moving the exponent 3 to the opposite side as the third root; and the equation 2 to which power will give eight is solved by moving the base 2 to the opposite side as the 2-based logarithm. Finally the last of the four basic operations, integration, is moved to the opposite side as its opposite operation, differentiation. Again we see that calculations predict numbers. There is even a Hymn to Equations.

Bo: A hymn?

Screen 17

Allan: Equations are the best we know. They're solved by isolation. But first the brackets must be placed around multiplication. We change the sign and take away, and only x itself will stay. We just keep on moving, we never give up. So feed us equations, we don't want to stop.

Bo: So equations can be solved both manually and by using a calculator?

Screen 18

Allan Yes. And to report the work we can use a formula table, i.e., a 2x2 formula table where the first column is for the numbers with the unknown number above the line and the known numbers below. The second column is for the calculations with the predicting formula above the line being transformed to an equation by inserting the known numbers below the line. Now the equation can be solved using the head to move the known numbers to the opposite side with opposite sign, or using algebra to solve the equation Y1-Y2 = 0; or by using geometry to identify intersection points. To check if the solution is correct, a test is performed inserting the numbers to see if the left-side number is equal to the right-side number.

Bo: So now the class can use formulas for modeling?

Screen 19 & 20

Allan: Yes, mathematical modeling has four steps: First humans ask a question to a table. Then technology uses regression to transform a table to a formula. Now technology provides the answer to the question. And finally humans translate these answers to the real world to be evaluated. To exemplify: If we have 10$ after 4 days and 16$ after 9 days, then what do we have after 12 days, and when do we have 30$, assuming tat the change will be constant in number or in percent.

Bo: But with different formulas, how does the calculator know which to choose?

Screen 21

Allan: Humans must make the choice between the different change formulas. A two-line table only has one change, so as regression formula you can choose between constant change-number and constant change-percent. In a three-line table there are two changes. So here we can use a polynomial of degree two that is graphed as a bending line called a parabola, having a turning point that can be found by technology. Likewise, a four-line table can be modeled by a degree three polynomial having a double parabola as graph that might have two turning points.

Bo: Allan, can you give examples?

Screen 22

Allan: Certainly Bo. The project 'Food versus Population' looks at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2-line table regression can produce two formulas: with x counting years after

1850, the population is modeled by Y1= 815*1.013^x and the food by Y2= 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

Bo: I guess all tables are two line tables at the pre-calculus level?

Screen 23

Allan: Not necessarily. The project 'Fundraising' finds the revenue assuming all students will accept a free ticket, that 100 students will buy a 20$ ticket and that no one will buy a 40$ ticket. From this 3-line table the demand is modeled by a degree 2 polynomial Y1 = .375*x^2 – 27.5*x + 500. Thus the revenue formula is the product of the price and the demand, Y2 = x*Y1. Graphical methods shows that the maximum revenue will be 2688 $ at a ticket price of 12$.

Bo: So you also an introduction to calculus?

Screen 24

Allan: I will by using many-line tables. In the project 'Driving with Peter' his velocity is measured five times. A 5-line table can be modeled by the degree 4 polynomial which is graphed as a triple parabola with three potential turning points. This model can answer questions, as when was Peter accelerating? And what distance did Peter travel in a given time interval? Graphical methods shows that a minimum speed is attained after 14.2 seconds; and that Peter traveled 115 meters from the 10^{th} to the 15^{th} second.

Bo: So calculus is about finding turning points?

Screen 25

Allan: Geometrically yes. Algebraically, calculus is about adding per-numbers. Calculus occurs first time in grade one when adding two totals that are counted, not in tens, but in icons less than ten. Thus if we want to add one fourth and two threes we can do it in two different ways. We can add them on-top as 'hard algebraic addition'. Then the units must be the same, so we recount the 2 3s to 1.2 4s. Thus the total is 2.2 4s. Or, we can add them next-to each other as 1.3 7s. And adding next-to each other as geometrical stacks or blocks is in fact integration.

Bo: But adding blocks and adding per-numbers, is that the same?

Screen 26 & 27

Allan. As a matter of fact it is since per-numbers must be transformed to blocks before adding them next-to each other. Thus to add 2kg at 3$/kg and 4 kg at 5$/kg we can add the unit-numbers to 6 kg. But we cannot directly add the per-numbers. To add the per-numbers, we transform them to unit-numbers by multiplication. This gives the area under the per-number graph. Thus the 6 kg cost 2 times 3$ plus 4 times 5$, which is 26$. So the total per-number is 26$ per 6kg. This shows that per-numbers are added by their areas, i.e. by integration. Here the per-number was piecewise constant jumping from 3 $/kg to 5 $/kg. However, the same procedure applies if the per-number is locally constant, continuously changing form one to another.

Bo: Allan, I see that in 2001 you wrote an article called Qualitative and Quantitative literature has three genres: fact, fiction and fiddle. What do you mean by this?

Screen 28

Allan: Well Bo. Fact models quantify and predict predictable quantities, as e.g. ‘What is the area of the walls in this room?’ In this case the calculated answer of the model is what is observed. Hence calculated values from a fact models can be trusted. A fact model can also be called a since- then model or a room-model. Most models from basic science and economy are fact models.

Bo: And what is a fiction?

Allan: Fiction models quantify and predict unpredictable quantities, as e.g. ‘My debt will soon be paid off at this rate!’ Fiction models are based upon assumptions and its solutions should be supplemented with predictions based upon alternative assumptions or scenarios. A fiction model can also be called an if-then model or a rate-model. Models from basic economy assuming variables to be constant or predictable by a linear formula are fiction models.

Bo: And fiddle models, what is that?

Allan: Fiddle models quantify and predict unpredictable qualities: ‘Is the risk of this road high enough to cost a bridge?’ Many risk-models are fiddle models. The basic risk model says: Risk = Consequence * probability. Statistics can provide probabilities for casualties, but if casualties are quantified, it is much cheaper to stay in a cemetery than in a hospital, pointing to the solution ‘no bridge’. Fiddle models should be rejected asking for a word description instead of a number description.

Screen 29

Bo: Thank you, Allan, for sharing with us your view on the question ‘precalculus IS hard – or is it?” In these two sessions we heard about the eight missing links of mandarin mathematics. Next time at the MATHeCADEMY.net channel we will look at deconstruction of calculus. Again we will ask: ‘Calculus is hard – or is it?’

322. Manuscript to a YoyTube Video ona Postmodern Deconstruction of Mathematics Education

This YouTube video on deconstruction in mathematics education describes how natural mathematics is made difficult by removing eight links to its roots, the natural fact Many. The missing links make mathematics a privilege to a mandarin class wanting to monopolize public offices. Reopening the eight missing links will make mathematics easy and accessible to all.

Session I: Primary school, http://youtu.be/sTJiQEOTpAM

Session II: Secondary school, http://youtu.be/MItYFL-3JnU

Screens: ..\Manuscripts and Screens\ScreensMisLin.ppt

The Eight Missing Links of Mandarin Mathematics

Screen 2 & 3

Bo: Welcome to the MATHeCADEMY.net Channel. And welcome to our series on deconstruction in mathematics education. My name is Bo. Today we address the question “Mathematics IS hard - or is it?” We begin with primary school. And welcome to our guest, Allan, who uses deconstruction in his work.

Allan: Thank you, Bo.

Bo: Allan, what does deconstruction mean to you?

Allan: Well Bo. To me, deconstruction means what it says, destruction and reconstruction. It is the method that is used by postmodern skeptical thinking that dates back to the ancient Greek sophists. The sophists warned against patronization that is hidden in choices that are presented as nature. So to avoid hidden patronization, false nature must be unmasked as choice. And to deconstruct then means to discover or to invent alternatives to choices that are presented as nature.

Bo: Thank you, Allan. Do you have a short answer to today’s question?

Allan: To me, the short answer to the question “Mathematics IS hard - or is it” is that mathematics is not hard by nature, but by choice. Mathematics is made difficult by its missing links.

Bo: And what do you mean by a missing link?

Screen 4

Allan: I use the term missing link to indicate that mathematics has lost its links to its root, the natural fact Many. However, if these links are established again, then mathematics will once more become a science about Many. And accepting it as a natural science will make mathematics easy and accessible to all, instead of difficult and accessible only to few.

Bo: So Allan, what is the first missing link?

Screen 5

Allan: Well Bo, the first missing link is the total. To deal with many, we total, i.e. we ask the question “how many?” And the answer is given by the total. That is why we use the word algebra that means to unite in Arabic. Thus all mathematics statements should begin with its subject, the total, and specify what the total is. The first step is to represent the total in three ways, by physical sticks, by graphical strokes, and by spoken words. Thus four things can be represented by four sticks on a table; and by four strokes on a paper; and by the word four. Now we can write that T equals stroke stroke stroke stroke.

Bo: That is what the Romans did, isn't it?

Allan: Indeed it is. But the Arabs went one step further by using icons. They united the four strokes into one single symbol consisting of four strokes. Thus they transformed four ones into one fours.

Bo: But isn't four ones and one fours the same?

Screen 6

Allan: Well Bo. Four ones means that you count in ones, so that one is the unit. Whereas one fours means that you count in fours, so that four is the unit. And that you count by bundling and stacking the total in fours. Thus rearranging sticks and strokes into icons is the second missing link. If we write the digits in a less sloppy way, we can see that there are five sticks and strokes in the five-icon, and six strokes in the six-icon, etc.

Screen 7

Bo: But why do the icons stop at ten?

Screen 8

Allan: We count by bundling, but we never use the bundle-icon. If we count or bundle in fives, we count one, two, three, four, bundle, one bundle one, one bundle two etc. Or in a shorter way: 1, 2, 3, 4, 10, 11, 12 etc. Thus we do not use the five-icon. Likewise, ten does not need an icon when counting in tens. In this way ten is the only number with its own name but without its own icon. This makes ten a cognitive bomb since ten is the follower of nine while one zero is the follower of four when counting in fives. So ten is not one zero by nature, but by choice of the bundle-number.

Bo: So what is the third missing link?

Screen 9

Allan: The third missing link is cup-writing. First we count the sticks or strokes by bundling them in e.g. fours. Then we can place the bundles in a left bundle-cup, and the unbundled in a right single-cup. We don't have to place the physical bundles. For each bundle we just place a stick in the bundle-cup. Now we can use the icons to write the total using a decimal point to separate the bundles from the unbundled. In this way a total of 6 1s can be counted in 4s as 1 4s and 2 1s, and written as 1.2 4s. So a natural number includes a decimal point to separate the bundles from the unbundled, and a unit.

Bo: And what is the fourth missing link?

Screen 10

Allan: The fourth missing link is recounting, or changing the unit. Thus a total of 7 1s can be recounted in 5s as 1.2 5s, or as 1.3 4s, or as 2.1 3s, etc. If counted in tens, 7 1s become 0.7 tens. However, we only write 7 leaving out the unit and misplacing the decimal point one place to the right. This may be OK in business, but it creates learning problems in school.

Bo: I guess that recounting is done by de-bundling and re-bundling?

Screen 11

Allan: Yes. Re-counting can be done manually by de-bundling and re-bundling. However, counting in fours means to repeat taking away four, i.e. dividing the total by four. So the counting result can be predicted by a recount-formula saying that the total T can be bundled in bs T/b times. So T is (T/b) bs, which is the same as (T/b) times b. Thus the recount-formula predicts that recounting a total of 8 1s in 4s gives 8/4 of the 4s, which is 2 4s.

Bo: So recounting is another word for shifting units?

Screen 12

Allan: Yes. To shift the unit means to recount. The tradition calls it proportionality or linearity. Shifting unit is the core of mathematics and easy to learn as recounting. However, the tradition avoids recounting. Instead it insists that counting only takes place in tens, that the decimal point is misplaced and that the unit is left out and. Thus what is called natural numbers are in reality highly unnatural. Many learning problems may disappear by respecting that, of course, any natural number has a unit and a decimal point. And by postponing counting in tens as long as possible in grade one.

Bo: And then what is the fifth missing link?

Screen13 & 14

Allan: The fifth missing link is predicting numbers by calculations. To recount 4 5s in 6s manually, first we must count up the 4 5s. Then we must debundle them in 1s. And finally we must rebundle them in 6s. This is a long and tiresome job. However, if we use the recount-formula and a calculator, we can predict the result to be 3 6s and a rest. And the rest is found by removing the 3 6s from the 4 5s. So the result of recounting can be predicted by a calculator using division and subtraction. Predicting recount results shows the very essence of mathematics. Mathematics is our language for number-prediction. Experiencing its predicting ability yourself may remove many motivation problems.

Bo: Allan, you say that mathematics is a language?

Screen 15

Allan: Yes Bo. We describe the world by words and numbers, so we have two languages. Our word-language combines letters to words and words to statements that describe the world. Our number-language combines digits to numbers and numbers and operations to formulas that predict the world. However, a language also has a meta-language, a grammar. A language is like a two level house, a language-house: At the lower level the language describes the outside world. And at the upper level the grammar describes the language. Meeting a language before its grammar makes it easy to learn, as in the case of the word-language. Meeting a language after its grammar makes it difficult to learn, as in the case of the number-language, which claims that its grammar, mathematics, must be learned before it can be applied.

Bo: But clearly, to apply mathematics, you must first learn mathematics?

Allan: Here we are seduced by our words. If we talk about rooting instead of applying, the logic is the opposite: Of course the root comes before what it roots. And the roots of mathematics can be seen in the names algebra and geometry meaning earth-measuring in Greek and uniting in Arabic. So presenting a grammar before its language is not nature, but choice.

Bo: Allan, can you give more examples on how mathematics predicts?

Screen 16

Allan: Certainly, Bo. As a matter of fact, operations are invented to predict numbers. Thus 3+5 predicts the result of counting-on 5 times from 3. And 3 times 5 predicts the result of adding with 3 5 times. And 3 to the power of 5 predicts the result of multiplying with 3 5 times. Furthermore, we saw that digits are icons that contain the number of strokes they represent. Likewise, operations are also icons that show the counting processes they represent.

Bo: Can you please specify that?

Screen 17

Allan: I would like to. Taking away 4 is iconized as a horizontal stroke showing the trace left when dragging away the 4. Taking away 4 many times, i.e., taking away 4s, is iconized as an up-hill stroke showing the broom sweeping away the 4s. Placing a stack of 4 singles next-to another stack is iconized as a cross showing the juxtaposition of the two stacks. And building up a stack of 3 4s is iconized as an up-hill cross showing a 3 times lifting of the 4s.

Bo: From the recount-formula it seems as if division and multiplication come before addition and subtraction?

Screen 18

Allan: Indeed they do. After recounting is predicted by a rebundle-formula using division and multiplication, we can predict the unbundled by a restack-formula using subtraction and addition. So the natural order of operations is division, multiplication, subtraction, and in the end addition. This is in contrast to the tradition that reverses the natural order, which creates yet more learning problems. And that brings us to the sixth missing link, that operations, and especially addition, can be both soft and hard. This link is missing both in primary school and in secondary school.

Bo: OK, Allan. So next time we will hear about that. Can you shortly mention what the other missing links are?

Allan: Yes Bo. The 7. and 8. missing links are reversed calculations and per-numbers.

Screen 19 & 20

Bo: Thank you, Allan. This time we have heard about five missing links: The Total, from sticks to icons, cup-writing, recounting and predicting calculations. Next time on the MATHeCADEMY.net channel we will look at deconstruction of secondary mathematics and hear about the three remaining missing links.

Session II: Secondary school

Screen 21 & 22

Bo: Welcome to the MATHeCADEMY.net Channel. And welcome to our series on deconstruction in mathematics education. My name is Bo. Today we address the question “Mathematics IS hard -

or is it?" Last time we looked at primary school. This time we look at secondary school. And welcome to our guest, Allan, who uses deconstruction in his work.

Allan: Thank you Bo.

Bo: Allan, can you please repeat what deconstruction means to you?

Allan: Well Bo. To me, deconstruction means what it says, destruction and reconstruction. It is the method that is used by postmodern skeptical thinking that dates back to the ancient Greek sophists. The sophists warned against patronization that is hidden in choices that are presented as nature. So to avoid hidden patronization, false nature must be unmasked as choice. And to deconstruct then means to discover or to invent alternatives to choices that are presented as nature.

Bo: Thank you, Allan. And can you please repeat your short answer to today's question?

Allan: To me, the short answer to the question "Mathematics IS hard - or is it" is that mathematics is not hard by nature, but by choice. Mathematics is made difficult by its missing links.

Bo: And what do you mean by a missing link?

Screen 23

Allan: I use the term missing link to indicate that mathematics has lost its link to its root, the natural fact Many. However, if this link is established again, then mathematics will once more become a science about Many. And accepting it as a natural science will make mathematics easy and accessible to all, instead of difficult and accessible only to few.

Bo: Will you please repeat the five missing links in primary school?

Screen 24

Allan: I will be glad to. The first missing link is the total: We only write numbers and always exclude the total. The second missing link is the missing transition from sticks to icons, showing that there are five sticks in the five-icon, etc. The third missing link is not using cup-writing to report the result of counting Many by bundling, and placing the bundles and the unbundled in separate cups, thus separating them with a decimal point and including the bundle-size as a unit. The fourth missing link is the missing use of re-counting when changing units. And the fifth missing link is the missing emphasis on mathematics as a number-predicting language.

Bo: Thank you, Allan. And that brings us to the remaining three missing links?

Screen 25

Allan: Yes Bo. The sixth missing link is the difference between soft and hard operations. Let us write the total 456 as we say it, i.e., as four hundreds, five tens and six ones. We see that we count by bundling in tens; and that we total by uniting bundles of bundles, bundles and unbundled. Thus the total is not a number but a calculation that adds three stacks: a stack of four ten-bundles of ten-bundles, and a stack of five ten-bundles and a stack of six unbundled ones. However, the three stacks are not added on-top of each other; they are added next-to each other. So in what we can call 'hard algebraic addition' stacks are added on-top of each other, while in what we can call 'soft geometric addition' stacks are added next-to each other. And adding next-to each other as geometrical stacks is in fact integration. Also we see that all numbers carry units: ones, tens, and tens-tens, also called hundreds. And finally we see the four ways we unite numbers: we add, we multiply, we power, and we integrate.

Bo: I now see why we use the Arabic word for uniting, algebra. But Allan, can you please specify a little?

Screen 26

Allan: Certainly, Bo. If we want to add one fourth and two threes we can do it in two different ways. We can add them on-top as 'hard algebraic addition'. Then the units must be the same, so we recount the 2 3s to 1.2 4s. Thus the total is 2.2 4s. Or, we can add them next-to each other as 1.3 7s.

Bo: But Allan, four times five is still twenty, isn't it?

Screen 27 & 28 & 29

Allan: Again, Bo, looking at the number 456 we see that five times ten means five tens. Likewise, four times five just means four fives, which may be, but doesn't have to be, recounted in 6s as 3.2 6s, etc., or in tens as 2.0 tens. So 'soft geometric multiplication' repeats adding on-top, while 'hard algebraic multiplication' recounts in tens, making 2 times 3 sevens not 6 sevens, but 4 tens and 2 ones = 4.2 tens = 42.

Bo: And what is the seventh missing link?

Screen 30

Allan: The seventh missing link is reversed calculation. Going forward we ask: two plus three gives what? Going backwards we ask: two plus what gives five? Instead of trying out alternatives we can predict the result by inventing an inverse operation to addition, called subtraction or minus. This allows the answer to be predicted by the calculation five minus two. So by definition five minus two is the number x that added to two gives five.

Bo: But isn't 2 + x = 5 an equation?

Screen 31

Allan: Well, an equation is just another word for reversed calculation. And since the equation 2 + x gives 5 is solved by the number x equal 5 minus 2, we see that solving an equation means isolating the unknown by moving numbers to the opposite side with the opposite sign.

Bo: Does this also apply to the other operations?

Screen 32

Allan: Indeed it does. Since 6/2 is defined as the number x that multiplied with two gives six, we see that the equation x*2= 6 again is solved by moving the number two to the opposite side with opposite sign, i.e., by x = 6/2. Likewise, the equation x to the power of 3 gives 8 is solved by moving the exponent 3 to the opposite side as the third root. And the equation 2 to which power will give eight is solved by moving the base 2 to the opposite side as the 2-based logarithm. Finally the last of the four basic operations, integration, is moved to the opposite side as its opposite operation, differentiation. Again we see that calculations predict numbers.

Bo: So what is the eighth missing link?

Screen 33 & 34

Allan: The last missing link is per-numbers. Per-numbers only occur as percentages meaning per hundred. However, recounting also gives meaning to per five, per seven, etc. So returning from sticks and strokes to the real world, all totals have physical units: meters, kilograms, hours, dollars etc. As an example we can look at coffee where 5 kg might be recounted as 8 $. This means that the quantity-price relation can be described by the per-number 8 $ per 5 kg, or 8 per 5 $ per kg. To find the cost of 20 kg we juts recount 20 in 5s by saying T = 20 kg = (20/5) times 5 kg = (20/5) times 8$, which is 32 $.

Bo: So proportionality is basically about per-numbers?

Screen 35 & 36

Allan: Precisely! And so is integration. Integration just means adding per-numbers. To add 2kg at 3$/kg and 4 kg at 5$/kg we can add the unit-numbers to 6 kg. But we cannot directly add the per-numbers. To add the per-numbers, we transform them to unit-numbers by multiplication. This gives the area under the per-number graph. Thus the 6 kg cost 2 times 3$ plus 4 times 5$, which is 26$. So the total per-number is 26$ per 6kg. This shows that per-numbers are added by their areas, i.e. by integration, i.e. by adding next-to each other.

Bo: So Allan, you think reopening these eight links will change mathematics from hard to easy?

Screen: MANY: Count & Add; Count: decimal-numbers with units; Add on-top: Recount to shift unit; Add next-to: Integration; Add per-numbers by areas; Reversed Calculation = Equation; Move to opposite side with opposite sign.

Screen 37

Allan: Yes, Bo, I do. Mathematics becomes easy when presented as a natural science grounded in the natural fact Many. To deal with Many, we simply count and add. First we total Many by counting bundles and unbundled, using cup-writing to report counting results as decimal numbers with units. Once counted, totals can be added on-top or next-to. If added on-top, the units must be the same. So a total might have to be recounted to shift its unit. Thus recounting is the root of proportionality. And adding next-to is the root of integration, also occurring when adding per-numbers that come from recounting totals in different physical units. Finally, any calculation can be reversed, and the unknown number can be predicted by opposite operations. So to solve an equation you simply move numbers to the opposite side with opposite sign.

Bo: Allan, it seems so simple?

Allan: And Bo, it is simple. However, if you remove the eight links to its root, the natural fact Many, mathematics is transformed from a natural science into what might be called mandarin mathematics. That is why the MATHeCADEMY.net is created so teachers can learn about and teach mathematics as a natural science about Many.

Bo: What do you mean by mandarin mathematics?

Screen 38

Allan: It is interesting to compare the Korean and the Chinese alphabet. The Korean was made to allow as many as possible to read and write; and the Chinese had the opposite goal. It was created by a mandarin class possessing public jobs that they wanted their children to inherit. Consequently the mandarins invented an alphabet so difficult that only mandarin children had the time and support to learn it before the exams. The French thinker Bourdieu says that Europe might have get rid of its autocratic kings but their administrations still remain, and they still manned by a mandarin class, that use mathematics as what he calls symbolic violence to ensure that the mandarin children will inherit the public offices.

Bo: But why is mathematics special useful for this purpose?

Allan: The word language is learned when growing up, in contrast to the number-language. So by removing the eight links to its root, the natural fact Many, the mandarins have succeeded in reserving mathematics for their children. Other means are forced classes with variable time tables to prevent daily lessons from providing a serious learning.

Bo: Do you know of any mandarin class that deliberately tries to make math difficult?

Screen 39

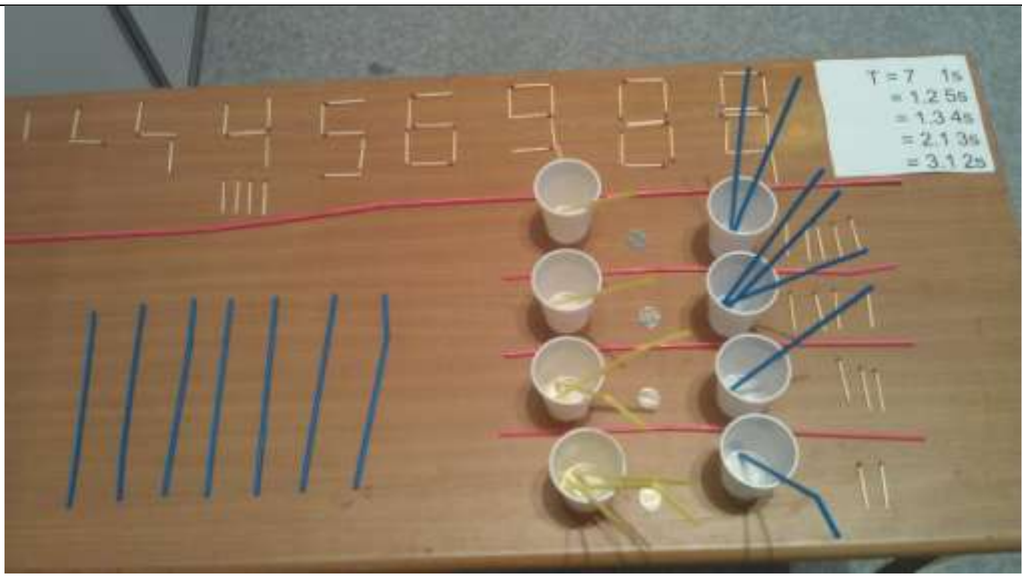
Allan: There doesn't have to be one as demonstrated by the French thinker, Foucault. Foucault shows how once a certain way of talking about things, a discourse, has established itself as a discipline, it will discipline not only itself, but also its subject. Meaning, that once mandarin mathematics with its eight missing links has been established as the dominating discourse, it is impossible to talk or teach outside this discourse. And to stay in office, mandarins must follow orders given by the discourse. And in this way they reproduce the discourse. Also the subject of mathematics, the total, is forced to silence both itself and its nature as a decimal number with a unit. Instead the discourse declares that unnatural numbers without a unit and with a misplaced decimal point should be called natural. This creates huge learning problems. But it serves well the hidden agenda of the mandarins in the public offices.

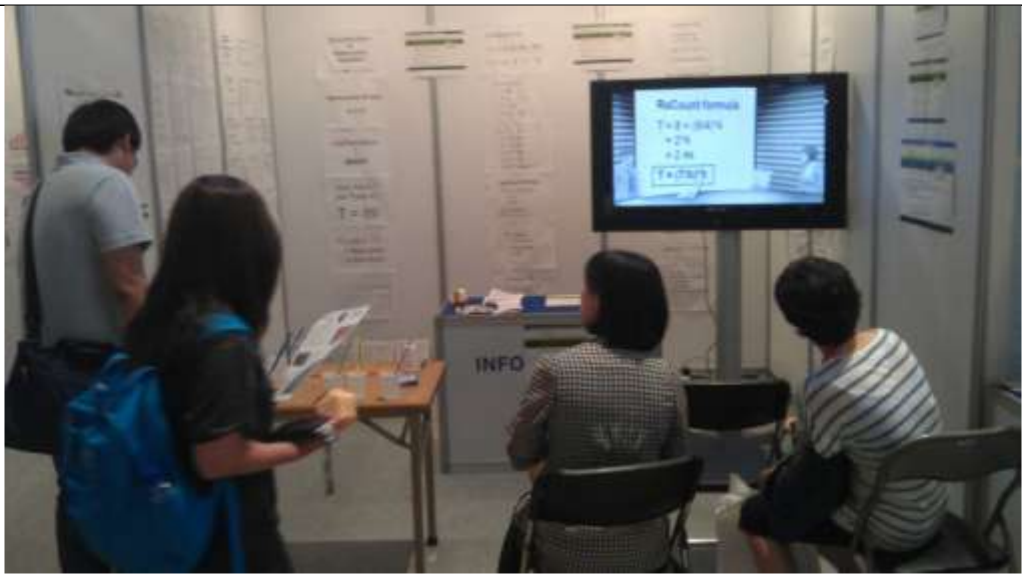
Screen 40 & 41

Bo: Thank you, Allan, for sharing with us your view on the question 'mathematics IS hard – or is it?" In these two sessions we heard about the eight missing links of mandarin mathematics. Next time on the MATHeCADEMY.net channel we will look at fractions. Again we will ask: 'Fractions is hard – or is it?'

323. The MATHeCADEMY.net Booth

To share information about free web-based teacher education on mathematics as a natural science about Many, the MATHeCADEMY.net had a booth in the exhibition area.





324. The MATHeCADEMY.net Folder

S2 | *Forms and positions can be predicted when placed in a grid*

In an a*b stack the upper right corner-point P has the coordinates P(b,a). A form consists of points P(x,y):

A line:	y = a*x+b
A vertical line:	x = c
A circle with radius r:	(x/r)^2 + (y/r)^2 = 1
An ellipse with radius a and b:	(x/a)^2 + (y/b)^2 = 1
A hyperbola:	y*x = a
A parabola:	y = a*x^2 + b*x + c

If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = (x+Δx,y+Δy) = ((8-3)+3,4+2*(8-3)) = (8,14).

Computers can calculate a set of numbers (vectors) and a set of vectors (matrices).

QL | *Quantitative literature: Fact, fiction & fiddle*

Quantitative literature occurs within algebra (reuniting), geometry (earth-measuring), economy, and science where it became the foundation of modern time by its ability to predict the behaviour in space and time of a physical mass.

There are 3 genres:
- Fact is a 'since calculation', quantifying what is quantitative and calculating what can be calculated: 'Since the price is 4 $/kg, the cost of 6 kg is 6*4 $'.
- Fiction is an 'if calculation', quantifying what is quantitative and calculating what cannot be calculated: 'If the income per year is 4 mio$, then the income the next 6 years will be 6*4 = 24 mio$'.
- Fiddle is a 'so-what calculation', quantifying what is not quantitative: 'If the consequence 'broken leg' C is 2 mio$, and if its probability S is 30%, then the risk R will be R = C*S = 2*0.3 = 0.6 mio$'.

Three kinds of action can be taken towards QL:

Facts are checked; fiction is supplemented with scenarios; and fiddle is rejected and referred from the number-language to the qualitative word-language.

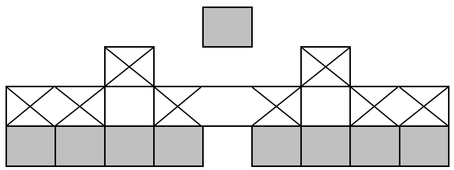
LAB-based PYRAMIDeDUCATION

MANY exists in time and space. When studied in a LAB, Many becomes the source of 8 CATS tales: Counting & Adding in Time & Space

Patronizing mathematics places the authority in the library from which concepts are defined & statements proved. Natural mathematics places the authority in the laboratory, where mathematics arises from 2*4 tasks in counting and adding in time and space. In this way the learner is educated, not by books, but by the natural fact Many. Primary school mathematics is learned through educational sentence-free meetings with the sentence subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic 'grasp-to-grasp' learning. Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic 'gossip-learning': Thank you for telling me something I don't know about something I know.

In PYRAMIDeDUCATION 8 teachers form 2 teams choosing 3 pairs and 2 instructors by turn. Instructing the rest of their team the instructors consult the coach. Each pair works together to solve count&add assignments and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation.
The coach assists the instructors in correcting the count& add assignments. In each pair each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

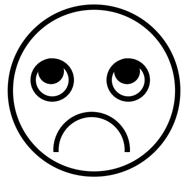
1 coach
2 instructors
3 pairs
2 teams

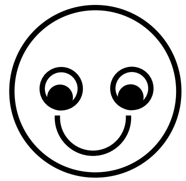


8 teachers and 1 coach

SET
↓
Patronizing-Math

Natural-Math
↑
MANY





Mathematics

A **Patronizing** Science From Above
or
A **Natural** Science From Below

The **CATS** Approach:
Count & **A**dd in **T**ime & **S**pace

Add	**Variable**	**Constant**
Unit-numbers	**T = a + n** *T-n = a*	**T = a*n** *T/n = a*
Per-numbers	**ΔT = ∫a*dn** *dT/dn = a*	**T = a^n** *$^n\sqrt{}$T=a , log_aT=n*

Found by Anti-Patronizing Contingency Research
Discovering Alternatives to Choices Presented as Nature

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C1 *Multiplicity is counted in stacks using the recount-prediction: T = (T/b)*b*

Multiplicity occurs as extension in space and time:
Put the finger on the pulse, and add a stroke for each beat, multiplicity in time is then transferred to multiplicity in space through an iconization.
The strokes can be arranged in different ways:
1. Next to each other: | | | | | | |
2. Connected in a number-icon (4 strokes in 4)
3. Bundled and stacked: | | | | | | -> ┼┼┼ ┼┼┼ -> 2 3s = 2*3

Multiplicity can be re-stacked or re-counted by changing the bundle-size: T = 3 7s = 3*7 = ?*4 = ? 4s
We count in 4s by taking away 4s. The process 'taking away 4s' may be iconized as '/4' and worded as 'counted or divided in 4s'. So the **recount-formula** says **T = (T/4)*4**. T/4 is the counter, 4 the unit.
The answer can be counted, or predicted by calculating:
T=3 7s=3*7=(3*7/4)*4=5*4 + 1=5*4 + ¼*4=(5 ¼)*4
Multiplication is a standard rebundling in tens and 1s:
T = 3*6 = 3 6s or T = 3*6 = 18 = 1*10 + 8*1.
Rebundling in tens leads to decimals and percentages:
T = 3 6s = 3*6 = (3*6/10)*10 = (1 8/10)*10 = 1.8*10
T = 3 6s = 3*6 = 18 = (18/100)*100 = 18%*100
Sugar can be bundled in kilos, litres, dollars and %.
A rebundling can change the bundle type:
2 kg = 5 $ = 6 litres = 100 %, T = 7 kg = ?
T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $
T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres
T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %
P = 5% = (5/100)*100% = (5/100)*2 kg = 0.1 kg
Also bundles can be bundled and stacked in bundles-of-bundles, bundles & unbundled:
T=234=2 bundles-of-bundles+3 bundles+4 unbundled.
In short a given multiplicity can always be rearranged as a multiple stack (a polynomial):
T = 2345 = 2+3+4+5 = 2*B^3 + 3*B^2 + 4*B + 5*1.
T = 2 8s. The 'ten-bundler' counts 'bundle + 6' i.e. 16, and the 'twelve-bundler' counts 'bundle + 4' i.e. 14.
So T = $(16)_{10} = (14)_{12}$.

A1 *Adding stacks gives overloads removed by the restack-prediction T = (T-b)+b*

T = 38+29 = 3ten8 + 2ten9 = 5ten17 = ?
An overload can be restacked and rebundled. First it is restacked by taking away 10 1s: T = 17 = (17-10)+10 = 7+10. Then the 10 1s is rebundled to 1 10s and added to the 10s as in book-keeping:
T = 38+29 = 3t8+2t9 = 5t17 = (5+1)t(17-10) = 6t7= 67
T = 38+29=3ten8+2ten9=5ten17=5ten1ten7 =6ten7=67
The process 'take away 4' is iconized as '-4' , 'minus 4'. So the **restack-formula** says **T = (T-b)+b**. The answer can be counted, or predicted by calculating.
Repeating adding stacks might lead to a big overload:
T=4*18 = 4*(1ten8) = 4ten32 =4ten3ten2 = 7ten2 = 72

T1 *A calculation can be analysed & reversed: x*3+2=14 -> (x*3)+2=14 -> x=(14-2)/3*

A forward calculation 4*3 = ? can be reversed to a backward calculation (equation) ?*3 = 12 or x*3 = 12
Also the repeated calculations 4*3+2 can be reversed:

		*3		+2	
forward:	x	——>	x*3	——>	(x*3)+2
back:	4	<——	12	<——	14
		/3		-2	

Forward and backward calculation may be walked along the floor, or arranged in columns in a 2x2 calculation-table as the move&change-method: Change the calculation-sign when moving to the other side.

a	**= ?**	T	= b + (a*n)		
T	= 80	T-b	= a*n		
b	= 20	(T-b)/n	= a		
n	= 5	(80-20)/5	= a = 12		
Test:		*80*	*= 20+12*5*	*= 80*	☺

S1 *Stacks in space can be reshaped, or made round a*b=(a*b/c)*c =√(a*b)^2; a=a/b*b=tanA*b*

An area can be divided, first in polygons, then in triangles and finally in right triangles, half-stacks. An a*b stack has area a*b. Its diagonal c can be found by reversing Pythagoras a^2+b^2=c^2. Its angle A can be found by reversing tanA= a/b, recounting a in b's.
A square with the diameter d has a circumference c= d*4*sin(180/4). A circle with the diameter d has a circumference c= d*n*sin(180/n)= d*π, where n-> ∞.

C2 *Unpredictable change can be predicted by the average level and average change*

Numbers may change unpredictably as e.g. in surveys. Through counting, a table is set up accounting for the frequency of the different numbers. From this the average level and the average change are calculated.
These averages can then be used in a prediction saying that with 95% probability that future numbers lies within an interval determined by the average level and change: T = Taverage ± 2*ΔTaverage
Counting the different possibilities in a repeated win-or-loose game leads to the Pascal triangle.

A2 *Per-numbers must be transformed to totals before being added*

The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b
2days @ 6$/day + 3days @ 8$/day = 5days @ 7.2$/day
2days @ 6% + 5days @ 8% = 7days @ 7.4%
1/2 of 2 cans + 2/3 of 3 cans = 3 of 5 cans = 3/5 of 5 cans
1/2 of 4 cans + 2/3 of 3 cans = 4 of 7 cans = 4/7 of 7 cans
Repeated and reversed addition of per-numbers leads to
Integration: T2=T1+a*b; T2-T1= +a*b; ΔT= ∑a*b= ∫y*dx
Differentiation: T2=T1+a*b; a= (T2-T1)/b= ΔT/Δb= dy/dx

T2 *Stacks in time may change in a constant or in a variable way*

A stack may be constant or variable. A variable stack has a level T and a change ΔT = T*final*-T*initial*
The change of the stack T=c*b can be a number or a %. ΔT = Δc*b + c*Δb (+Δc*Δb), or ΔT/T ≈ Δc/c + Δb/b
The change might be a constant or variable:

number	ΔT = a	T = b + a*x
percent	ΔT = r*T	T = b * (1+r)^x
number&%	ΔT = r*T+a	T/a=R/r, 1+R=(1+r)^x
increasing nu.	ΔT = b+a*x	T = 1/2*a*x^2+b*x+c
predictable	dT = y*dx	T = b + ∫ y*dx, y = T'