Mathematics Predicts
PreCalculus

Compendium & Projects

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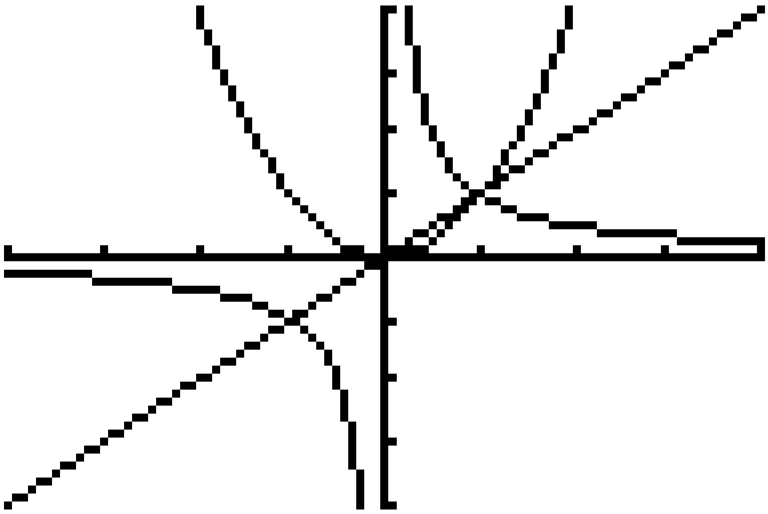
Version 0909



y = 1*x , y = x*x , y = 1/x

Contents

Mathematics Predicts

Mathematics
    Algebra
        Uniting and Splitting Numbers
    Geometry
        Measuring Earth
    Statistics
        Accounting Many
	Mathematics contains Algebra, Geometry and Statistics
	Algebra (calculation) can predict counting processes, both the end result and the parts.
	Geometry (earth measuring) can be used to calculate plane figures and spatial forms.
	Statistics (counting) is used for counting the actual size of different quantities.

Mathematics has two main fields, Algebra and Geometry, as well as Statistics.

Geometry means ’earth-measuring’ in Greek. Algebra means ’reuniting’ in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus together algebra and geometry give an answer to the fundamental human question: how do we split the earth on which we live and what it produces?

Originally human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus trade was only possible with those highlanders that had silver in their mountains.

The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire. The dark Middle Ages came when the Greek silver mines were emptied and the Arabs conquered the Spanish mines.

German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number system and algebra.

The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: A+B+C = 180 and a side law: a^2+b^2=c^2. Pythagoras generalized this findings by claiming: All is numbers.

This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don’t know Geometry. However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation.

The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has a n acceleration g; and the distance s and the time t are connected by the formula: s=½*g*t^2. However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middle men. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine.

The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move?

The church said ’among the stars’. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth.

But why do things fall? The church said: everything follows the unpredictable will of our metaphysical lord only attainable through belief prayers and church attendance. Newton objected: It follows its own will, a force called gravity that can be predicted by a formula telling how a force changes the motion, which made Newton develop change-calculations, calculus. So instead of obeying the church, people should enlighten themselves by knowledge, calculations and school attendance.

Brahe used his life to write down the positions of the planets among the stars. Kepler used these data to suggest that the sun is the center of the solar system, but could not prove it without sending up new planets. Newton, however, could validate his theory by different examples of falling and swinging bodies.

Newton’s discoveries laid the foundation of the Enlightenment period realizing that when an apple follows its own will and not that of a metaphysical patronizor, humans could do the same. Thus by enlightening themselves people could replace the double patronization of the church and the prince with democracy, which lead to two democracies, one in The US and one in France. Also formulas could be used to predict and therefore gain control over nature, using this knowledge to build an industrial welfare society based upon a silver-free economy emerging when the English replaced the import silk and pepper from the Far East with production of cotton in the US creating the triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton.

Calculations Predict

Calculations predict the total T. 2*4 calculation types are used to unite and split into four different types of numbers:

Uniting or splitting	Variable	Constant
Unit-numbers \$, kg, s	Plus + Minus -	Multiplication * Division /
Per-numbers \$/kg, \$/100\$, %	Integration ∑ ∫ Differentiation Δ	Power ^ Log or root √

a \$ and n \$ total T \$:	a+n = T
a \$ n times total T \$:	n*a = T
r % n times total T%:	(1+r)^n = 1+T
a1 kg at p1 \$/kg + a2 kg at p2 \$/kg total T \$: p1*a1+ p2*a2 = T:	∑ p*a = T

Algebra means re-uniting in Arabic and can be translated to predictions. Algebra thus predicts the result of uniting singles into totals or splitting totals into singles.

There are four ways of uniting numbers: addition (+), multiplication (*), power (^) and integration (∑ or ∫).

Addition + predicts the result of uniting variable singles:

2\$ and 3 \$ and 4 \$ total T \$: 2+3+4 = T

Multiplication * predicts the result of uniting constant singles:

2\$ + 2\$ + 2\$ + 2\$ + 2\$ = 5 times 2\$ = T, 5*2 = T

Job	Count	Predict
32\$ and 63 \$	1,2,…,95	T = 32+63
2\$ 36 times	2,4, …,72	T=72*2
20% 5 times	120, 144...	T=120%^5

Power ^ predicts the result of uniting constant percentages: 5 times 2% totals T%, 102%^5 = 1+T

Integration ∑ or ∫ predicts the result of uniting constant per-numbers:

2kg at 7\$/kg + 3kg at 8\$/kg totals T \$: 7*2 + 8*3 = T , ∑ \$/kg * kg = T , ∫ p*dx = T

Inverse or backward calculations predicts the result of splitting a Total into singles.

x+3 = 15	Question: Which number added to 3 gives 15?
x = 15-3	Prediction: 15-3 is the number that added to 3 gives 15. Test: 3+(15-3) = 15
Rule	**Plus-numbers move across as minus-numbers, and vice versa**

x*3 = 15	Question: Which number multiplied with 3 gives 15?
$x = \frac{15}{3}$	Prediction: $\frac{15}{3}$ is the number that added to 3 gives 15. Test: $3*\frac{15}{3} = 15$
Rule	**Multiplication-numbers move across as minus-numbers, and vice versa**

x^3 = 125	Question: Which number raised to power 3 gives 125?
$x = \sqrt[3]{125}$	Prediction: $\sqrt[3]{125}$ is the number that raised to power 3 gives 125. Test: $(\sqrt[3]{125})$^3= 125
Rule	**Exponent-numbers move across as reciprocal exponent-numbers, and vice versa**

3^x = 243	Question: 3 raised to which power gives 243?
$x = \frac{\log 243}{\log 3}$	Prediction: 3 raised to power $\frac{\log 243}{\log 3}$ gives 243. Test: 3^$\frac{\log 243}{\log 3} = 243$
Rule	**Base-numbers move across as logarithm-numbers, and vice versa**

A mixed calculation containing more calculations can be reduced to a single calculation by bracketing the stronger one.

T = 2+3*4 = 2+(3*4) , T = 2+3^4 = 2+(3^4) , T = 2*3^4 = 2*(3^4) Priority: 1. (), 2.^, 3. *, 4. +

A formula-table can be used to document the solving of an equation.

The unknown number	c = ?	T = a+b*c	*The formula*
The known numbers	a = 2	14 = 2+(3*c)	*From a mixed to a single calculation by bracketing the stronger*
	b = 3	$\frac{(14-2)}{3} = c$	*+ moves across as the opposite -, and * moves across as /*
	T = 14		*Bracket the calculation already present*
		4 = c	*Perform the calculation*
Tests	Test	14 = 2+3*4	'MATHSolver 0 = -14 + 2+3*x' gives 'x = 4'
		14 = 14 ☺	

Tasks

Find the unknown number in the formula. Make more with randM (3,1)

1. T = a+b*c
2. T = a+b/c
3. T = a*b^c
4. T = a+b^c
5. T = a-b*c
6. T = a-b/c
7. T = a/b^c
8. T = a-b^c

	T	b	a	c
I	60		12	20
II	60	1.5		20
III	60	1.5	12	

Formulas Predict

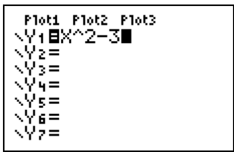
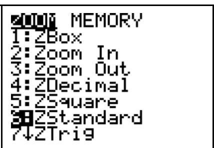
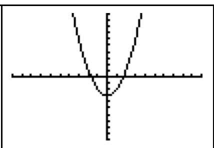
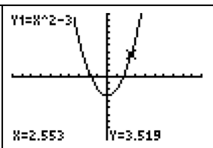
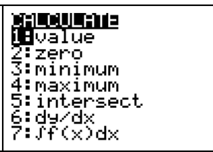
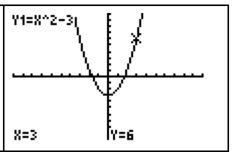
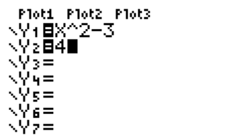
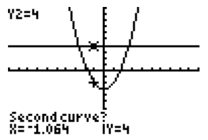
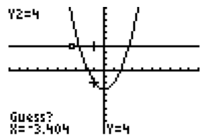
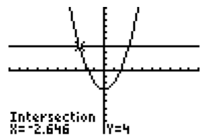
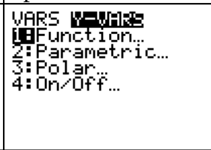
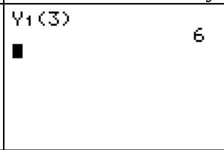
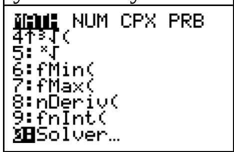
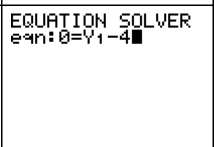
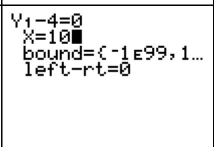
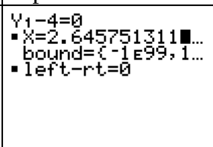
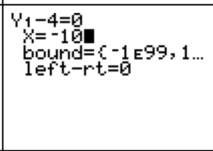
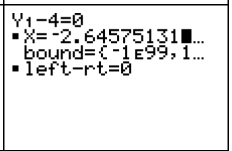
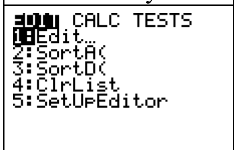
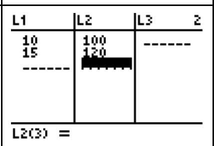
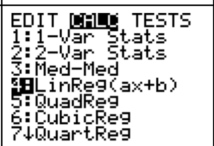
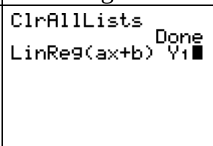
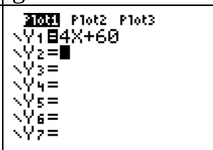
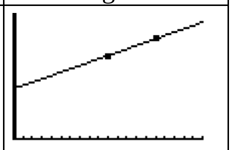
A formula contains a quantity y and a its calculation f, y = f(x,z,t) **An equation** is a formula with 1 unknown. An equation can be calculated or solved by finding the unknown. **A function** is a formula with 2 unknowns. A function can be tabled or graphed showing different scenarios: If x = a then y = f(a).	Purchase-formula: b $ + x kg at a $/kg totals y $: b + x * a = y Sharing-formula: b $ + a $ shared between x persons totals y $: b + a / x = y

A formula contains a quantity y and a its calculation f, y = f(x,z,t). Thus a formula might contain 2, 3, 4 or more variables. If the variables are replaced by fixed numbers, a formula is transformed into an equation or a function.

An equation is a formula with 1 unknown: y = 10 + 2*3, or 16 = b + 2*3, or 16 = 10 + a*3, or 16 = 10 + 2*x

An equation can be solved manually or by a calculator using MATHSolver. After using 'solve' the solution is tested by inserting all known numbers: 16 = 10 + 2*3 gives 16 = 16

A function is a formula with 2 (or more) unknowns: y = b + 2*3, or y = 10 + 2*x, or 16 = b + 2*x, or 16 = 10 + a*x. In a function one of the unknowns is isolated and entered on the calculators y-list. Thus x^2-y+3=0 gives y=x^2-3.

					
Formulas are put on the y-list	Always start with Standard Zoom	Choose Graph to graph	Choose Trace to see scenarios	Calc Value gives specific values	And is used for knownx/unknowny
					
Knowny/unknownx y is on the y-list	The intersecting curves marked	The cursor is close to the sol.	The procedure is repeated	VARS gives access to the Y-s	The known x is put after the Y
					
MATHSolver is used to find y's	CLEAR old and enter new	Enter a guess	Read the solution close to guess	Enter a new guess	Read the solution close to guess
					
From table to formula use STAT	Enter the table as lists	Choose a formula type	Add Y1 to bring formulas to y-list	Add Plot for visual control	Adjust window before graphing

Tasks: Find the question marks in three different ways: manually in a formula table, using graphs and using calculation.

1

x	y=3+2*x
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.7
?	-2.4
?	3.1
?	4.5

2

x	y=3-2*x
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.6
?	-2.5
?	3.2
?	4.6

3

x	y=x^2-4
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.8
?	-2.2
?	3.7
?	4.7

4

x	y=-x^2+5
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.2
?	-2.6
?	3.3
?	4.3

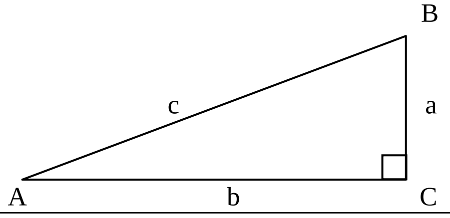
5	x	10	20	30	
	y	30	50		80
6	x	10	15	25	
	y	100	130		180
7	x	10	20	40	
	y	100	70		10

Ans:	a	b	Formula	y	x	T
lin	2	10	y = 10+2*x	70	35	
exp	1,052	18	y = 18*1,052^x	83,33	29,2	13,6
pow	0,737	5,5	y = 5,5*x^0,737	67,41	37,84	
lin	6	40	y = 40+6*x	190	23,33	
exp	1,054	59,17	y = 59,17*1,054^x	219,7	21,2	13,2
pow	0,647	22,54	y = 22,54*x^0,647	180,92	24,8	
lin	-3	130	y = 130+-3*x	10	40	
exp	0,965	142,86	y = 142,86*0,965^x	34,3	74,56	-19,4
pow	-0,515	327,02	y = 327,02*x^-0,515	49	877,72	

Trigonometry

Any land can be divided in triangles Any triangle can be divided into right-angled triangles	Two Greek Formula: A+B+C = 180 a^2 + b^2 = c^2 Three Arabic Formula: $\sin A = \frac{a}{c}$ $\cos A = \frac{b}{c}$ $\tan A = \frac{a}{b}$

A triangle is defines by 3 pieces. The rest can be predicted by formulas. The Greeks only found two formulas, so trigonometry first was developed when the Arabs added three extra formulas.

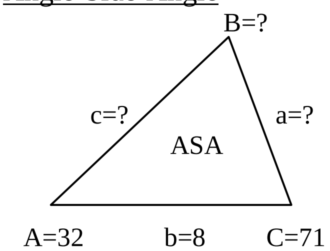


Greek formulas
A+B+C = 180 a^2 + b^2 = c^2 (Pyhtagoras)
Arabic formulas:
$\sin A = \frac{a}{c}$ (height in % of c) $\cos A = \frac{b}{c}$ (base in % af c) $\tan A = \frac{a}{b}$
A right-angled triangle can be seen as a rectangle divided by a diagonal.

In a non right-angled triangle the sine and cosine formulas have to be extended:

The Sine Rule	**The Cosine Rule (The Extended Pythagoras)**
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	a^2 = b^2 + c^2 - 2*b*c*cosA b^2 = a^2 + c^2 - 2*a*c*cosB c^2 = a^2 + b^2 - 2*a*b*cosC

Angle-Side-Angle

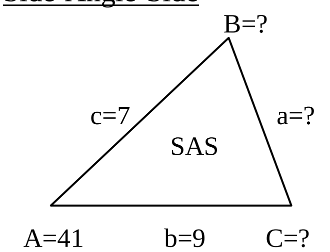


B = ?	A+B+C=180
A=32 C=71	B=180-A-C B=180-32-71 B=77

a = ?	$\frac{a}{\sin A} = \frac{b}{\sin B}$
b=8 A=32 B=77	$a = \frac{b*\sin A}{\sin B}$ $a = \frac{8*\sin 32}{\sin 77}$ a = 4.351

c = ?	$\frac{c}{\sin C} = \frac{b}{\sin B}$
b=8 C=71 B=77	$c = \frac{b*\sin C}{\sin B}$ $c = \frac{8*\sin 71}{\sin 77}$ c = 7.763

Side-Angle-Side



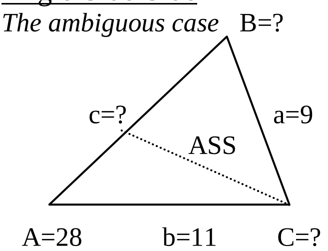
a = ?	a^2=b^2+c^2 -2*b*c*cosA
b=9 c=7 A=41	a^2=9^2+7^2 -2*9*7*cos41 $a = \sqrt{34.907}$ a = 5.908

B = ?	b^2=a^2+c^2 -2*a*c*cosB
b=9 c=7 a=5.9	$\cos B = \frac{a^2+c^2-b^2}{2*a*c}$ $\cos B = \frac{5.9^2+7^2-9^2}{2*5.9*7}$ B=cos-1(0.035)=88.0

C = ?	A+B+C=180
A=41 B=88	C=180-A-B B=180-41-88 B=51

Angle-Side-Side

The ambiguous case

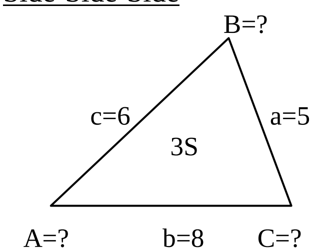


B = ?	$\frac{\sin B}{b} = \frac{\sin A}{a}$
b=11 A=28 a=9	$\sin B = \frac{b*\sin A}{a}$ $\sin B = \frac{11*\sin 28}{9}$ B=sin-1(0.574) $B = [\begin{matrix}35\\145\end{matrix}$

C = ?	A+B+C=180
A=28 B=35 or B=145	C=180-A-B C=180-28-35 C=117 or C=180-28-145 C=7

c = ?	$\frac{c}{\sin C} = \frac{a}{\sin A}$
a=9 A=28 C=117 or C=7	$c = \frac{a*\sin C}{\sin A}$ $c = \frac{9*\sin 117}{\sin 28}$ c = 17.081 or c = 2.336

Side-Side-Side



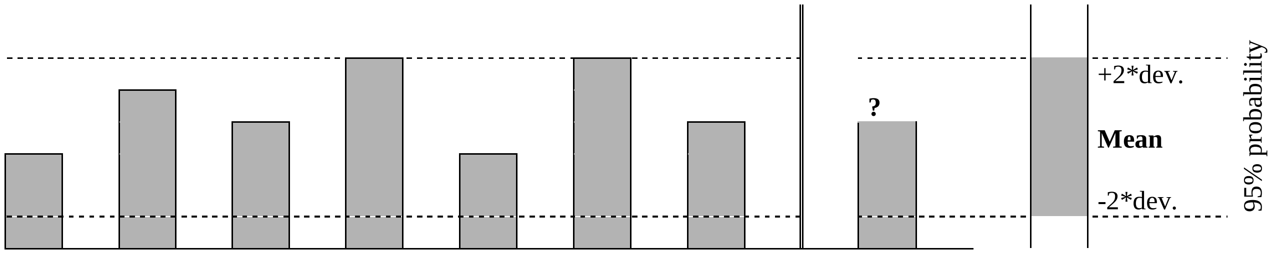
a=?	a^2=b^2+c^2 - 2*b*c*cosA
a=5 b=8 c=6	$\cos A = \frac{b^2+c^2-a^2}{2*b*c}$ $\cos A = \frac{8^2+6^2-5^2}{2*8*6}$ A=cos-1(0.781) A=38.6

b=?	b^2=a^2+c^2 - 2*a*c*cosB
a=5 b=8 c=6	$\cos B = \frac{a^2+c^2-b^2}{2*a*c}$ $\cos B = \frac{5^2+6^2-8^2}{2*5*6}$ B=cos-1(-0.05) B=92.9

C = ?	A+B+C=180
A=38.6 B=92.9	C=180-A-B C=180-38.8-92.9 C=48.5

Statistics, Stochastic Variation

Numbers may be predictable or unpredictable. Unpredictable numbers are also called random or stochastic numbers. Numbers that cannot be pre-dicted can often be post-dicted by setting up a statistics on their former behavior. A statistical table contains two columns, one with the numbers and one with their frequencies.
If arranged in increasing order:
The median = the middle observation, 1. (3.) quartile = the middle observation in the 1. (2.) half.
A histogram shows the frequencies
An ogive shows the cumulated frequencies from which the three quartiles can be read and reported on a box-plot



Statistics — Post-dicts mean and deviation

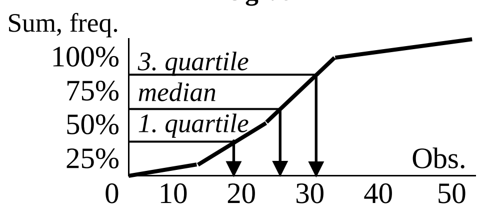
Probability — pre-dicts x = Mean ± 2* deviation

1. Observations
x: 10, 12, 22, 12, 15, ...

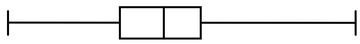
2. Grouping and counting frequencies

Observations x	Frequency h	Rel. Freq. p	Sum. freq. ∑p
0-10	3	3/40=0.075	0.075
10-20	12	0.300	0.375
20-30	18	0.450	0.825
30-50	7	0.175	1.000
Total	40	1.000	

Ogive



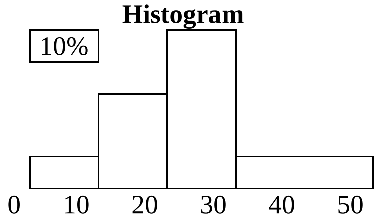
A Boxplot contains the median and two quartiles and the least and greatest observation



3. Mean or average: IF all the observations were the same ... however, the deviate

Observations x	Frequency h	Rel. Freq. p	Sum. freq. ∑p	Mean μ = ∑xi*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375
10-20	12	0.300	0.375	4.5
20-30	18	0.450	0.825	11.25
30-50	7	0.175	1.000	7
Total	40	1.000		23.1

Histogram



4. Variance, deviation: IF all the deviations were the same ...

Observations x	Frequency h	Rel. Freq. p	Sum. freq. ∑p	Mean μ = ∑xi*pi	Distance |xi - μ|	Variance v = ∑(xi-μ)^2*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375	|5-23.1|=18.13	18.13^2*0.075=24.64
10-20	12	0.300	0.375	4.5	8.13	19.80
20-30	18	0.450	0.825	11.25	1.88	1.58
30-50	7	0.175	1.000	7	16.88	49.83
Total	40	1.000		23.1		1 s^2 = 95.86

Deviation s = √95.86 = 9.8

5. Prediction: x = Mean ± 2*deviation =μ ± 2*s = 23.1 ± 19.6 *Confidence-interval* = [3.5 ; 42.7]

6. Using technology

On a GDC the interval midpoints are entered under STAT. Rel. frequency = freq/sum(freq). CumFreq = cumsum(freq).

Obs.	Freq	Rel.freq	CumFreq
0	2	.05	.050
1	5	.125	.175
2	9	.225	.400
3	12	.300	.700
4	8	.200	.900
5	4	.100	1.000

The different numbers can be calculated using 1-var statistics:
Mean m = 2.8
Standard deviation, s = 1.3
Confidence-interval = m ± 2*s = [0.2;5.4]
1. quartile = 2
Median = 3
3. quartile = 4

Polynomials and Calculus

0. degree polynomial tells the (initial) point	y = 5
1. degree polynomial tells the (initial) gradient or steepness	y = 5 + 2*x
2. degree polynomial tells the (initial) bending	y = 5 + 2*x + 0.3*x^2
3. degree polynomial tells the (initial) counter-bending	y = 5 + 2*x + 0.7*x^2 – 0.2*x^3
4. degree polynomial tells the (initial) counter-counter-bending	y = 5 + 2*x + 0.7*x^2 – 0.2*x^3 + 0.3*x^4

Arabic numbers are polynomials: 4352 = 4*10^3 + 3*10^2 + 5*10 + 2. General form: y = 4*x^3 + 3*x^2 + 5*x + 2

Polynomials with bending graphs (degree over 1) have some interesting points:

Turning points, either top-points (maximum) or bottom-points (minimum).

Intersections with the x-axis (zeros), with the y-axis (y-intercept), or with other graphs.

Intersecting other graphs (equations graphically), Intersecting vertical lines (tracing values).

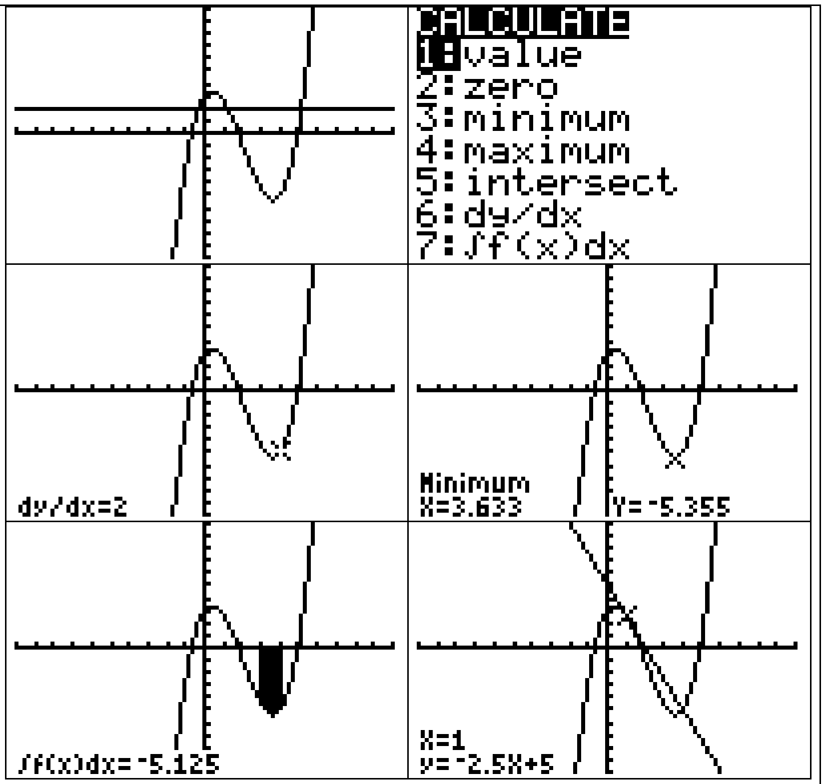
Shifting bending or curvature, where the bending changes its sign.

Tangent-point. A tangent is a straight line practically coinciding with the graph around the contact point, thus showing a scenario: this is how the graph would look like if the steepness stays constant.

If the curve graphs per-numbers, the total is found as the area under the per-number graph, i.e. by integration
If the curve graphs a Total, the per-numbers are found as the steepness of the total graph, i.e. by differentiation
Differentiation twice gives the bending, being positive when bending upwards and negative when bending downwards.
Finding the steepness (gradient, slope) formula is called differentiation. Finding the area under a curve is called integration. Together differentiation and integration are inverse operations called Calculus

Example: y = 0.5x^3-3x^2+2x+3

	Graphics	Formula
Intersecting the y-axis CALC value	y = 3	y1(0)
Intersecting the x-axis CALC zero	x = -0.694 x = 1.748 x = 4.946	Solve(0=Y1)
Intersecting y = 2 CALC intersection	x = -0.329 x = 1.181 x = 5.147	Solve(0=Y1-2)
Top CALC Maximum	x = 0.367 y = 3.355	MATH fMax(Y1,x,0,7)
Bottom CALC Minimum	x = 3.633 y = -5.355	MATH fMin(Y1,x,0,7)
Steepness in x = 4 CALC dy/dx	2	MATH nDeriv(Y1,x,4)
Area from 3 to 4 CALC Sf(x)dx	-5.125	MATH fnInt(Y1,x,3,4)
Tangent in x = 1 DRAW tangent x = 1	y=-2.5x+5	



Tasks

1. Repeat as above with y = 0.7x^3-4x^2+3x+4.	10. Find the cheapest pipe with double lid contain. 1 liter.
2. Repeat as above with y = -0.4x^3+2x^2-0.5x-3.	11. Find the cheapest cone without lid containing 1 liter.
3. Produce your own polynomials using randM(4,1).	12. Find the cheapest cone with lid containing 1 liter.
4. Produce your own polynomials using regression	13. Find the cheapest cone with double lid contain. 1 liter.
5. Find the cheapest box without lid containing 1 liter.	14. y1 is a polynomial of degree 0. If y1 is a Total, what is its per-number? If y1 is a per-number, what is its total?
6. Find the cheapest pipe without lid containing 1 liter.	15. As 14 with polynomials of degree 1.
7. Find the cheapest box with lid containing 1 liter.	16. As 14 with polynomials of degree 2.
8. Find the cheapest pipe with lid containing 1 liter.	17. As 14 with polynomials of degree 3.
9. Find the cheapest box with double lid containing 1 liter.	

Two equations With Two Unknowns; and Three

Two equations with two unknowns are solved manually, by intersection or by matrices	b \$ + 5kg at a \$/kg = 25 \$ b \$ + 8kg at a \$/kg = 34 \$	x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1 & 5\\1 & 8\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$

2 equations with 2 unknowns: The formula b \$ + 5kg at a \$/kg = 25\$ contains 2 unknowns and cannot be solved, unless we know another example of the same formula as e.g. b \$ + 8kg at a \$/kg = 34 \$.

Written as an equation system	Written as a matrix equation
x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1 & 5\\1 & 8\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$

Manually one variable is isolated in the first and inserted in the second equation: x=25-5*y, 25-5*y+8*y=34, y=3 & x=10.

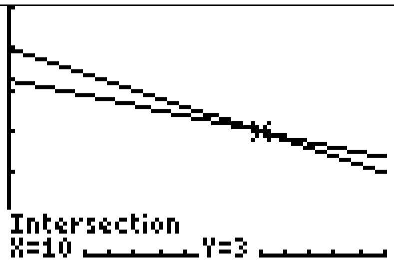
Using graphs, the y's are isolated and inserted into the y- editor.

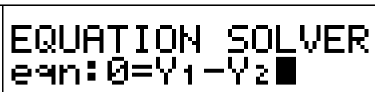
x + 5*y = 25 gives y = (25-x)/5 ,

x + 8*y = 34 gives y = (34-x)/8

The intersection point is found by 'Calc Intersection' to x = 10 and y = 3.

Also we can use Math Solver 0 = Y1 –Y2.





Matrix-solutions is found by entering the matrices into the matrix-editor as ml and mr (matrix-left & -right):

$\underline{V} = \begin{pmatrix}x\\y\end{pmatrix} = ?$	$\underline{ml} * \underline{V} = \underline{mr}$
$\underline{ml} = \begin{pmatrix}1 & 5\\1 & 8\end{pmatrix}$ $\underline{mr} = \begin{pmatrix}25\\34\end{pmatrix}$	$\underline{V} = \underline{ml}$^-1 * $\underline{mr}$ $\underline{V} = \begin{pmatrix}1 & 5\\1 & 8\end{pmatrix}$ ^-1 * $\begin{pmatrix}25\\34\end{pmatrix}$ $\underline{V} = \begin{pmatrix}10\\3\end{pmatrix}$
Test	$\begin{pmatrix}1 & 5\\1 & 8\end{pmatrix} * \begin{pmatrix}10\\3\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$ $\begin{pmatrix}25\\34\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$

$\underline{V} = \begin{pmatrix}x\\y\\z\end{pmatrix} = ?$	$\underline{ml} * \underline{V} = \underline{mr}$
$\underline{ml} = \begin{pmatrix}2 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix}$ $\underline{mr} = \begin{pmatrix}18\\-2\\16\end{pmatrix}$	$\underline{V} = \underline{ml}$^-1 * $\underline{mr}$ $\underline{V} = \begin{pmatrix}2 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix}$ ^-1 * $\begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\underline{V} = \begin{pmatrix}1\\2\\3\end{pmatrix}$
Test	$\begin{pmatrix}2 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix} * \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\begin{pmatrix}18\\-2\\16\end{pmatrix} = \begin{pmatrix}18\\-2\\16\end{pmatrix}$

3 equations with 3 unknowns cannot be solved graphically, but manually and by using matrices:

Written as an equation system	Written as a matrix equation
3*x + 5*y + 2*z = 19 x – z = -2 4*x – 3*y + 6*z = 16	$\begin{pmatrix}3 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix} * \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}19\\-2\\16\end{pmatrix}$

A matrix-solution is found by entering the matrices into the matrix-editor as ml and mr.

4 equations with 4 unknowns, 5 equations with 5 unknowns etc. Like 3 equations with 3 unknowns.

Equation systems for skill building can be generated by 'randM(3,3)' and 'randM(3,1).

Tasks. Solve the equation systems

1. 4x - 1*y = -9 4x - 4*y = 0 2. 4x + 2*y = 16 5x - 3*y = -2 3. 7x + 4*y = -1 -3x + 2*y = 19 4. 2x - 5*y = 16 3x - 4*y = 17	5. -7*x - 3*y - 7*z = 3 -1*x - 5*y + 1*z = -13 9*y - 5*z = 36 6. 4*x + 3*y + 7*z = 81 5*x + 3*y + 1*z = 54 2*x + 9*y + 5*z = 57 7. 2*x + 3*y - 1*z = - 6 5*x + 3*y - 4*z = -15 2*x – 2*y + 5*z = 40	8. 2*x + 5*y - 1*z + 9t = 118 1*x + 1*y - 9*z - 5t = -88 - 3*y + 7*z + 5t = -51 -3*x + 5*y + 2*z - 5t = -10 9. -6*x - 1*y + 8*z + 8t = 129 -2*x + 2*y - 5*z + 7t = 60 8*x + 6*y + 3*z + 3t = -40 -7*x - 4*y - 8*z - 4t = 12

Letter Calculation, Transposing Formulas

Change the T-formulas to a-formulas, b-formulas and c-formulas, and vice versa.

	T		**a**	**b**	**c**
1	$T = a + b\cdot c$		$a = T - b\cdot c$	$b = \frac{T-a}{c}$	$c = \frac{T-a}{b}$
2	$T = a - b\cdot c$		$a = T + b\cdot c$	$b = \frac{a-T}{c}$	$c = \frac{a-T}{b}$
3	$T = a + \frac{b}{c}$		$a = T - \frac{b}{c}$	$b = (T-a)\cdot c$	$c = \frac{b}{T-a}$
4	$T = a - \frac{b}{c}$		$a = T + \frac{b}{c}$	$b = (a-T)\cdot c$	$c = \frac{b}{a-T}$
5	$T = (a + b)\cdot c$		$a = \frac{T}{c} - b$	$b = \frac{T}{c} - a$	$c = \frac{T}{a+b}$
6	$T = (a - b)\cdot c$		$a = \frac{T}{c} + b$	$b = a - \frac{T}{c}$	$c = \frac{T}{a-b}$
7	$T = \frac{a+b}{c}$		$a = T\cdot c - b$	$b = T\cdot c - a$	$c = \frac{a+b}{T}$
8	$T = \frac{a-b}{c}$		$a = T\cdot c + b$	$b = a - T\cdot c$	$c = \frac{a-b}{T}$
9	$T = \frac{a}{b+c}$		$a = T\cdot (b+c)$	$b = \frac{a}{T} - c$	$c = \frac{a}{T} - b$
10	$T = \frac{a}{b-c}$		$a = T\cdot (b-c)$	$b = \frac{a}{T} + c$	$c = b - \frac{a}{T}$
11	$T = \frac{a}{b} + c$		$a = (T-c)\cdot b$	$b = \frac{a}{T-c}$	$c = T - \frac{a}{b}$
12	$T = \frac{a}{b} - c$		$a = (T+c)\cdot b$	$b = \frac{a}{T+c}$	$c = \frac{a}{b} - T$
13	$T = a\cdot b^{c}$		$a = \frac{T}{b^{c}}$	$b = \sqrt[c]{\frac{T}{a}}$	$c = \frac{\log(\frac{T}{a})}{\log b}$
14	$T = \frac{a}{b^{c}}$		$a = T\cdot b^{c}$	$b = \sqrt[c]{\frac{a}{T}}$	$c = \frac{\log(\frac{a}{T})}{\log b}$
15	$T = (a\cdot b)^{c}$		$a = \frac{\sqrt[c]{T}}{b}$	$b = \frac{\sqrt[c]{T}}{a}$	$c = \frac{\log T}{\log(a\cdot b)}$
16	$T = (\frac{a}{b})^{c}$		$a = \sqrt[c]{T}\cdot b$	$b = \frac{a}{\sqrt[c]{T}}$	$c = \frac{\log T}{\log(\frac{a}{b})}$
17	$T = (a + b)^{c}$		$a = \sqrt[c]{T} - b$	$b = \sqrt[c]{T} - a$	$c = \frac{\log T}{\log(a+b)}$
18	$T = (a - b)^{c}$		$a = \sqrt[c]{T} + b$	$b = a - \sqrt[c]{T}$	$c = \frac{\log T}{\log(a-b)}$
19	$T = a + b^{c}$		$a = T - b^{c}$	$b = \sqrt[c]{T-a}$	$c = \frac{\log(T-a)}{\log b}$
20	$T = a - b^{c}$		$a = T + b^{c}$	$b = \sqrt[c]{a-T}$	$c = \frac{\log(a-T)}{\log b}$
21	$T = a^{(b+c)}$		$a = \sqrt[(b+c)]{T}$	$b = \frac{\log T}{\log a} - c$	$c = \frac{\log T}{\log a} - b$
22	$T = a^{(b-c)}$		$a = \sqrt[(b-c)]{T}$	$b = \frac{\log T}{\log a} + c$	$c = b - \frac{\log T}{\log a}$

Homework

1. In the triangle ABC, C is 90, A=42, c=5. Find the rest.
2. In the triangle ABC, C is 90, A=34, a=6. Find the rest.
3. In the triangle ABC, C is 90, A=28, b=7. Find the rest.
4. In the triangle ABC, C is 90, a=5, c=7. Find the rest.
5. In the triangle ABC, C is 90, b=4, c=7. Find the rest.
6. In the triangle ABC, C is 90, a=4, b=5. Find the rest.
7. In the triangle ABC, A is 32.6, b=4.6, c=5.2. Find the rest.
8. In the triangle ABC, A is 34.8, b=5.6, a=7.2. Find the rest.
8. In the triangle ABC, A is 42.6, B=74.6, c=6.2. Find the rest.
10. In the triangle ABC, A is 34.8, C=54.6, a=5.2. Find the rest.

11. (all lin, exp & pow)

x	y
2	10
7	15
9	?
?	30

12

x	y
3	8
7	12
9	?
?	28

13

x	y
1	20
5	30
9	?
?	80

14

x	y
10	80
20	62
30	?
?	30

15

x	y
12	64
18	42
25	?
?	24

16

x	y
3	50
12	28
20	?
?	10

17. In 1993 there was 420 $. In 1998 there was 630 $. In 2005 there was ? $. In ? there was 950 $. Linear and exponential and power change.
18. In 1994 there was 520 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 1250 $. Linear and exponential and power change.
19. In 1992 there was 920 $. In 1996 there was 730 $. In 2005 there was ? $. In ? there was 450 $. Linear and exponential and power change.
20. In 1994 there was 720 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 250 $. Linear and exponential and power change.
21. A capital had 753 $. increased with 20% 4 times and became ? $. What is the doubling-time?
22. A capital had 956 $. decreased with 25% 5 times and became ? $. What is the half-time?
23. A capital had 486 $. increased with 30% ? times and became 2345.83 $. What is the doubling-time?
24. A capital had 324 $. decreased with 35% ? times and became 25.88 $. What is the half-time?
25. A capital had 743 $. increased with ?% 4 times and became 2854.32 $. What is the doubling-time?
26. A capital had 896 $. decreased with ?% 5 times and became 45.09 $. What is the half-time?
27. A capital had ? $. increased with 50% 6 times and became 2423.83 $. What is the doubling-time?
28. A capital had ? $. decreased with 55% 7 times and became 2.45 $. What is the half-time?

31. (Polynomial regr.)

x	y
2	10
7	30
9	35
12	?
?	30
?	turn

32

x	y
3	8
7	5
11	12
9	?
?	28
?	turn

33

x	y
1	20
5	30
7	35
9	?
?	10
?	turn

34

x	y
10	60
20	120
30	30
40	70
50	?
?	80
?	turn

35

x	y
12	74
18	22
20	43
25	41
30	?
?	34
?	turn

36

x	y
3	9
12	28
15	8
17	14
20	?
?	10
?	turn

41. (Mean, ogive. & boxplot)

Obs	Freq
0-10	6
10-20	9
20-30	12
30-40	15
40-50	6

42

Obs	Freq
0-10	50
10-20	20
20-30	10
30-40	20
40-50	30

43

Obs	Freq
0-10	16
10-20	29
20-30	52
30-40	25
40-50	16

44

Obs	Freq
0-10	16
10-20	29
20-30	32
30-40	45
40-50	56
50-60	66

45

Obs	Freq
0-10	12
10-20	56
20-30	42
30-40	13
40-50	73
50-60	25

46

Obs	Freq
0-10	23
10-20	45
20-30	25
30-40	12
40-50	86
50-60	23
60-70	45

51. Solve the equation 2+3*(1+x)^4 = 20
52. Solve the equation 4+5*(1+x)^6 = 30
53. Solve the equation 40-3*(1-x)^4 = 20
54. Solve the equation 50-4*(1-x)^5 = 10
55. Transpose the equations $T= d-e$, $T = d - \frac{e}{f}$, $T = d - \frac{e-f}{g}$
56. Transpose the equations $T = \frac{d}{e}$, $T = \frac{d}{e} - f$, $T = \frac{d-e}{f} - g$

Project Forecasting

Problem: How to set up a forecast assuming constant growth?

	A mathematical model		
Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real World Problem
A capital is assumed to grow constantly. From two data sets we would like to establish a forecast predicting the capital at a certain time and when a certain level will be reached.

2. The Mathematical Problem
We set up a table showing the capital to two differnt times. X are years, y is 1000 $

x	y = ?
2	10
5	30
8	?
?	60

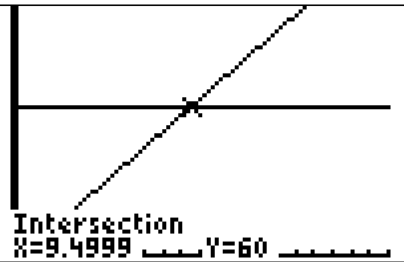
1. Linear ++ growth y = a*x + b	x: +1, y: +a (gradient, slope)
2. Exponential +* growth y = a*b^x = a*(1+r)^x	x: +1, y: + r% (interest rate, b=1+r)
3. Power ** growth y = a*x^b	x: +1%, y: + r% (elasticity)

3. Solution to the Mathematical Problem
First we find the y-formulas using regression. We enter the table as lists L1 and L2 und STAT.
'LinReg Y1' produces a linear model transferred to the y-list as Y1
'ExpReg Y1' produces an exponential model transferred to the y-list as Y1
'PowerReg Y1' produces a power model transferred to the y-list as Y1

Linear growth

y = ?	y = 6.667*x – 3.333
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 6.667*8 – 3.333 = 50
Test	Trace x = 8 gives y = 50

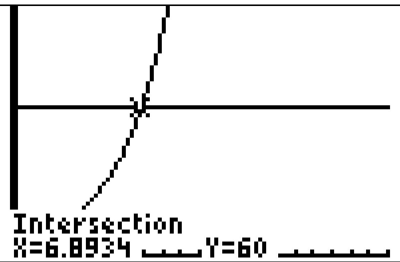
x = ?	y = 6.667*x – 3.333
y = 60	60 = (6.667*x)– 3.333 60 + 3.333 = 6.667*x 63.333/6.667 = x 9.5 = x
Test1	60 = 6.667*9.5– 3.333 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 9.5
Test3	CALC Intersection with y2=60 gives x = 9.5



Exponential growth

y = ?	y = 4.807 * 1.442^x
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.807 * 1.442^8 = 89.9
Test	Trace x = 8 gives y = 89.9

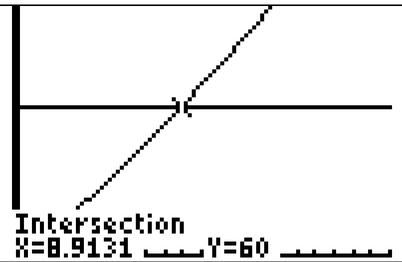
x = ?	y = 4.807 * 1.442^x
y = 60	60 = 4.807 * (1.442^x) 60/4.807 = 1.442^x log(60/4.807)/log1.142 = x 6.89 = x
Test1	60 = 4.807 * 1.442^6.89 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 6.89
Test3	CALC Intersection with y2=60 gives x = 6.89



Power growth

y = ?	y = 4.356*x^1.199
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.356*8^1.199 = 52.7
Test	Trace x = 8 gives y = 52.7

x = ?	y = 4.356*x^1.199
y = 60	60 = 4.356*(x^1.199) 60/4.356 = x^1.199 1.199√(60/4.356) = x 8.91 = x
Test1	60 = 4.356*8.91^1.199 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 8.91
Test3	CALC Intersection with y2=60 gives x = 8.91



4. Solution to the Real World Problem
We see that forecast can be made by using technology's regression lines.
The forecasts give different answers to the same questions since different forms of growth is assumed.
Linear growth assumes that the gradient is constant
Exponential growth assumes that the interest rate is constant
Power growth assumes that the elasticity is constant

Project Distance to a Far-away Point

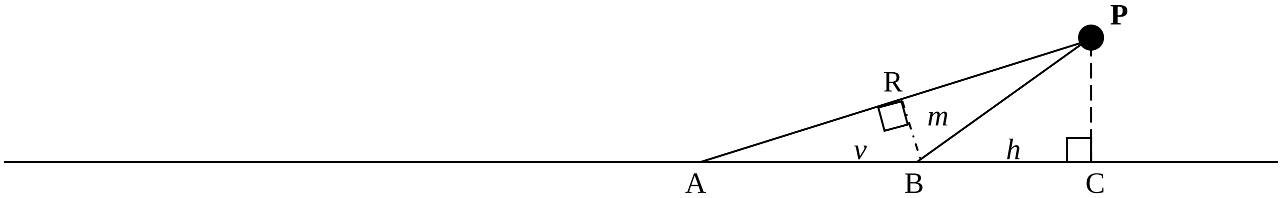
Problem: How to determine the distance to an inaccessible distant point??

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real World Problem

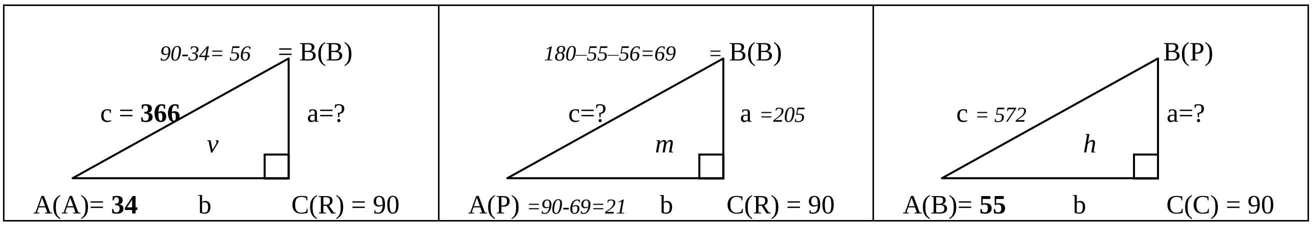
From a given baseline we want to determine the distance to a far-away inaccessible point P.



2. The Mathematical Problem

From a known baseline AB we measure the angles A and B to the inaccessible point P.
From the three right angled triangles ABR, BRP and BCP we calculate RB, BP as well as the distance PC.

Measurements: AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees



3. Solution to the Mathematical Problem

We set up three formula tables

Triangle ABR

a = ?	$\sin A = \frac{a}{c}$
A = 34 c = 366	$\sin 34 = \frac{a}{366}$ sin34 * 366 = a 205 = a
Test1 ☺	$\sin 34 = \frac{205}{366}$ 0.559 = 0.560
Test2 ☺	Math Solver $0 = \frac{x}{366} - \sin 34$ gives x = 205

Triangle PBR

c = ?	$\sin A = \frac{a}{c}$
A = 21 a = 205	$\sin 21 = \frac{205}{c}$ c * sin21 = 205 $c = \frac{205}{\sin 21}$ c = 572
Test1 ☺	$\sin 21 = \frac{205}{572}$ 0.358 = 0.358
Test2 ☺	Math Solver $0 = \frac{205}{x} - \sin 21$ gives x = 572

Triangle PBC

a = ?	$\sin A = \frac{a}{c}$
A = 55 c = 572	$\sin 55 = \frac{a}{572}$ sin55 * 572 = a 469 = a
Test1 ☺	$\sin 55 = \frac{469}{572}$ 0.819 = 0.820
Test2 ☺	Math Solver $0 = \frac{x}{572} - \sin 55$ gives x = 469

4. Solution to the Real World Problem

By using trigonometry we are able to determine the distance to the inaccessible point P to 469 cm.

Project the Bridge

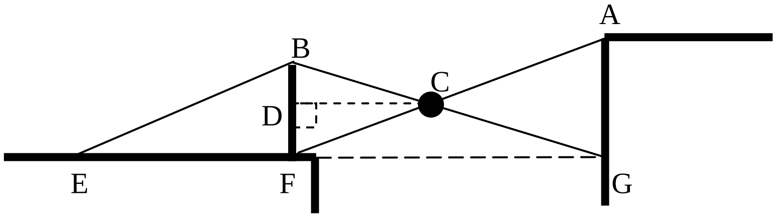
Problem: How to determine the dimensions of a bridge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

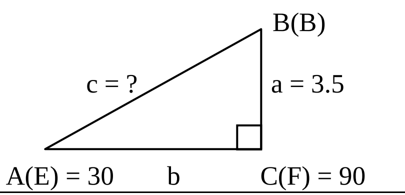
1. The Real World Problem

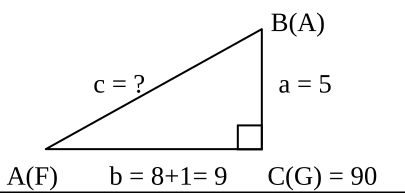
Over a canyon a suspension bridge made of steel i fastened to the cliff and to a vertical upright. We want to determine the length of the 3 beams as well as the welding point. The left fixing angle must be 30 degrees.



2. The Mathematical Problem

From the right angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between the lines BG and FA.
Measurements: angle FEB = 30 degrees, FB = 3.5m, FG = 8m + 1m = 9m and AG = 5m.





I a coordinate system with F as zero the following coordinates emerge: F: (0,0) and A: (9,5), as well as B: (0,3.5) and G: (9,0). Usinmg linear regression we determine the equations for the lines FA and BG.

3. Solution to the Mathematical Problem

We set up formula tables

Triangle EFB

c = ?	sinA = $\frac{a}{c}$
A = 30 a = 3.5	$\sin30 = \frac{3.5}{c}$ sin30*c = 3.5 c = 3.5/sin30 c = 7.0
Test1 ☺	$\sin30 = \frac{3.5}{7}$ 0.5 = 0.5
Test2 ☺	Math Solver $0 = \frac{3.5}{x} - \sin30$ gives x = 7

Triangle FGA og GFB

c = ?	$a^2 + b^2 = c^2$
a = 5 b = 9	$5^2 + 9^2 = c^2$ √(106)= c 10.30 = c
Test1 & Test2	
c = ?	$a^2 + b^2 = c^2$
a = 3.5 b = 9	$3.5^2 + 9^2 = c^2$ √(93.25) = c 9.66 = c

Lines BG and FA

BG: ?	y = ax + b
Test	y = – 0.389x + 3.5 Found by LinReg L1, L2, Y1 Trace x=0 gives 3.5 Trace x=9 gives 0 StatPlot fits

Likewise we find
FA: ? y = 0.556x

Calc Intersection gives
x = 3.71 and y = 2.06

In the triangle FDC, DC = 3.71 and FD = 2.06
Pythagoras gives: FC = √(3.71^2 + 2.06^2) = 4.24

In the triangle BDC, DC = 3.71 and FD = 3.6-2.06 = 1.54
Pythagoras gives: BC = √(3.71^2 + 1.54^2) = 4.02

4. Solution to the Real World Problem

Using trigonometry we have found the lengths of the three steel beams as EB = 7.00 m, FA = 10.30 m and BG = 9.66

The welding point is determined by FC = 4.24 m and BC = 4.02 m.

As an extra control the bridge can be drawn and build by pipe cleaners in the ration 1:100.

Project Golf

Problem: How to hit a golf hole behind a hedge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real World Problem

From a position on a 2 meter high flat hill we want to send a golf ball over a 3 meter hedge 2 meter away on the hill to hit a hole situated 12 meters away at level zero.

What is the orbit of the ball? How high is the ball at the distance 10 meters? When does the ball have a height of 6 meters? How high does the ball go? What is the direction of the ball in the beginning, at 10 meters distance and at the impact?

2. The Mathematical Problem

We set up a table with the length x and the height y having the domain 0<x<12.

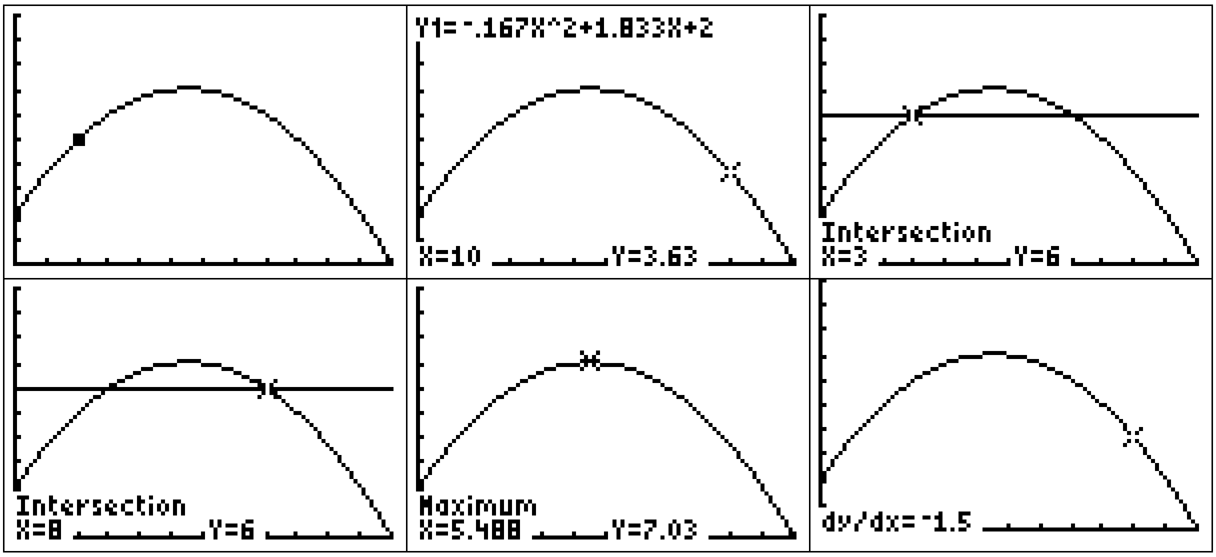
Length x	Height y	Direction v
0	2	?
2	5	
12	0	?
10	?	?
?	6	

3. Solution to the Mathematical Problem

We insert the table as lists L1 and L2. # data-sets allows a 2nd degree polynomial, quadratic regression, which produces the formula y= -0.167x^2 + 1.833x + 2, which is transferred to Y1.

Now the questions asked can be answered using formula tables and a calculator for graphing or calculating. The Y-number can be found by CALC Value, the x-number by CALC Intersection, the maximum by CALC Maximum, and the gradient by CALC dy/dx.

To determine the direction angle v we use the gradient formula: Tan v = gradient = dy/dx.



y = ?	y = y1
x=10	y = y1(10) = 3.667
Test	Trace x=10 gives y=3.67

x = ?	y = y1
y = 6	Math solver 0 = y1 - 6 gives x =3 & x = 8
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC Intersection gives x = 3 and 8

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx ≈ 0 at x = 5.5

v = ?	tan v = dy/dx
x = 0	tan v = 1.833 v = tan-1(1.833) v = 61.4

v = ?	tan v = dy/dx
x = 12	tan v = -2.167 v=tan-1(-2.167) v =- 65.2

v = ?	tan v = dy/dx
x =10	tan v = -1.5 v = tan-1(-1.5) v = -56.3

4. Solution to the Real World Problem

The orbit of the ball is a parabola. The height of the ball at the distance 10 meters is 3.67 meters? At the distances 3 meters and 8 meters the ball has a height of 6 meters. The ball goes to the maximum height 7.04 meters? The direction of the ball in the beginning, at 10 meters distance and at the impact are 61.4 grader, -65.2 grader and -56.3 grader.

Project Driving

Problem: How far and how did Peter drive?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real World Problem

When driving, the velocity 100 km/t is 100*1000/(60*60) = 27.8 m/s. A camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s og 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

2. The Mathematical Problem

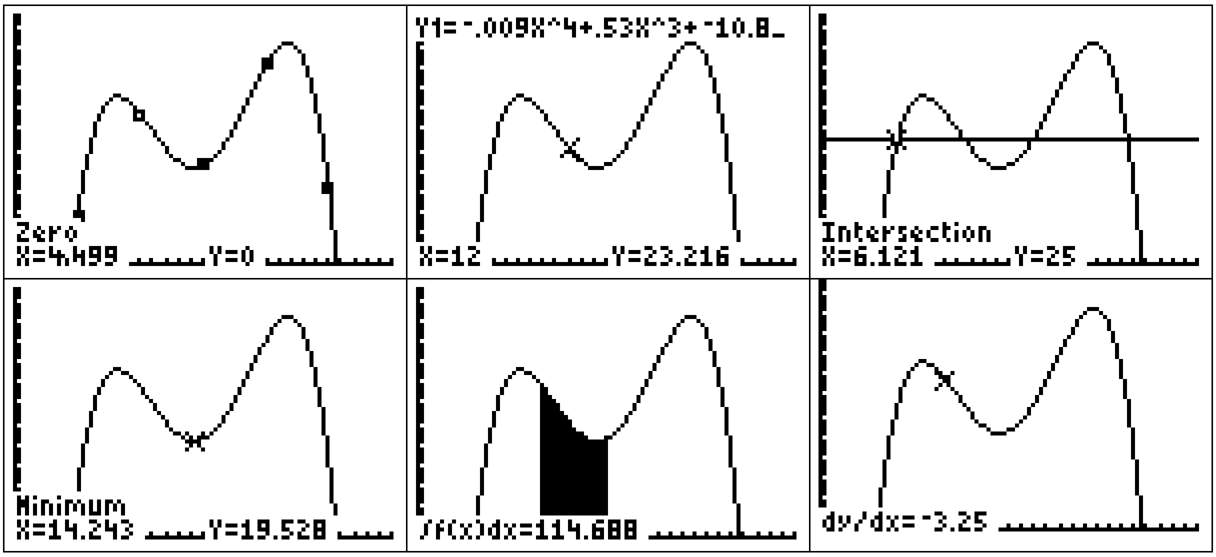
We set up a table showing time x and velocity y. The domain of the table is taken to be 0<x<30.

Time x sec	Velocity y m/s	Accel. dy/dx
5	10	?
10	30	?
15	20	?
20	40	?
25	15	?

3. Solution to the Mathematical Problem

On TI-84 the table is entered as the lists L1 og L2. 5 data sets allow quartic regression (a 4. degree polynomial with a 3-fold parabola) providing the formula y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235 placed as Y1. No the question asked can be answered using formula tables, or using technology, i.e. graphical readings or calculations.

Starting and ending points are found using 'CALC Zero'. Y-numbers are found using 'TRACE'. X-numbers are found using 'CALC Intersection'. Maximum and minimum are found with 'CALC Maximum/Minimum'. The total meter-number is obtained by summing up the m/s*s = ∫Y1dx. Acceleration is found by the gradient 'CALC dy/dx'.



y = ?	y = y1
x=12	y = y1(12) = 3.667
Test	TRACE x = 12 gives y = 23.216

x = ?	y = y1
y = 25	MATH Solver 0 = y1 - 25 gives x =6.12 and …
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC intersection gives x = 6.12, 11.44, 16.86 and 24.47

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx = 0 at x = 5.5

4. Solution to the Real World Problem

The driving began after 4.50 sec. and ended after 25.62 sec. After 12 sec the velocity was 23.2 m/s. And it was 25m/s after 6.12 sec, 11.44 sec, 16.86 sec and 24.47 sec. Acceleration took place in the time-intervals (4.50; 8.19) and (14.24; 21.74). Deceleration in the intervals (8.19;14.24) and (21.74;25.62). Max-velocity was 44.28 m/s = 159 km/t. after 21.7 sec. In the time-intervals (5;10), (10;15), (15;20) and (20;25) the distance traveled was 142.8 m, 114.7 m, 142.8 m and 189.7 m. The acceleration in the beginning of these time-intervals were 17.75, -3.25, 1.25, 4.25, -21.25 m/s^2. The total distance traveled was 597.4 m.

Project Vine Box

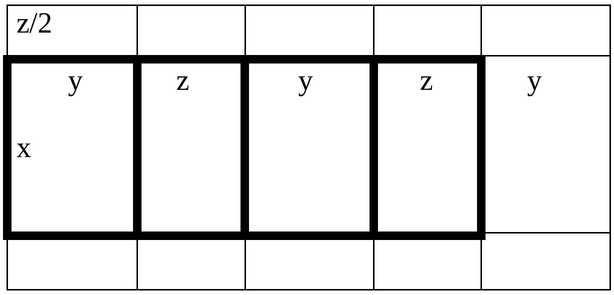
Problem: What are the dimensions of a 3 liters vine bag with the least surface area?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real World Problem

Vine is sold in bottles or in boxes. A 3 liter bag will be constructed by cutting out a piece of cardboard.



2. The Mathematical Problem

The cardboard dimensions are called x, y & z all in dm. We express the volume V and the Surface S as formulas:

V = x*y*z = 3,   S = x*(3y+2z) + 2*z/2*(3y+2z)

3. Solution to the Mathematical Problem

We expand the S-formula: S = x*(3y+2z) + 2*z/2*(3y+2z) = 3xy + 2xz + 3yz + 2z^2

We now insert z = 3/(x*y) so that S only depends on two variables x and y:

S = 3xy + 2xz + 3yz + 2z^2 and z = 3/(x*y) gives $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2}$

Scenario A. We assume that y should be half the length of x: y = 0.5*x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 1.5x^2 + \frac{21}{x} + \frac{72}{x^4}$, which gives $\frac{dS}{dx} = 3x - \frac{21}{x^2} - \frac{288}{x^5} = 0$ for x = 2.4

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

x = 2.4 and S = 19.56, so y = 0.5*x = 0.5*2.4 = 1.2, and z = 3/(2.4*1.2) = 1.0

Scenario B. We assume that y should be the same length of x: y = x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 3x^2 + \frac{15}{x} + \frac{18}{x^4}$, which gives $\frac{dS}{dx} = 6x - \frac{15}{x^2} - \frac{72}{x^5} = 0$ for x = 1.7

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

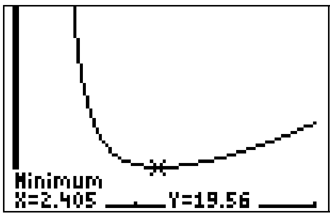
x = 1.7 and S = 19.65, so y = x = 1.7, and z = 3/(1.7*1.7) = 1.0

Scenario C. We assume that y should be double the length of x: y = 2*x. This restriction is inserted:

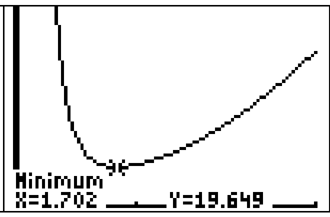
$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 6x^2 + \frac{12}{x} + \frac{4.5}{x^4}$, which gives $\frac{dS}{dx} = 12x - \frac{12}{x^2} - \frac{18}{x^5} = 0$ for x = 1.2

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

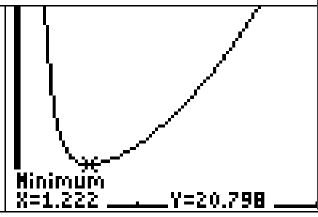
x = 1.2 and S = 20.80, so y = 2x = 2*1.2 = 2.4 , and z = 3/(1.2*2.4) = 1.0



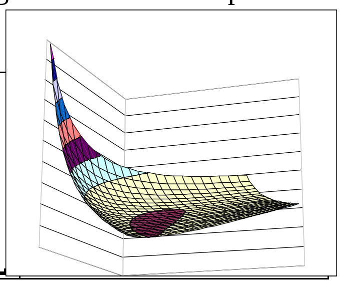
Scenario A



Scenario B



Scenario C


3 dim. in Excel

4. Solution to the Real World Problem

We see that the minimum surface area is a little above 19 dm^2. Using an Excel-spreadsheet we can determine the optimal solution to be x = 2.1 and y = 1.4 and z = 1.0, giving a minimum surface area at 19.47 dm^3.

(Graphing $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2}$ does not give a curve but a surface as shown on the above Excel-file.)

Revision Problems Using TI-84

1.	$\begin{array}{\|c\|c\|} \hline x & y = ? \\ \hline 3 & 12 \\ \hline 7 & 16 \\ \hline 10 & ? \\ \hline ? & 40 \\ \hline \end{array}$	Answer the question marks in case of a linear model. Answer the question marks in case of an exponential model. What is the doubling time? Answer the question marks in case of a power model.
2.	$\begin{array}{\|c\|c\|} \hline x & y = ? \\ \hline 3 & 12 \\ \hline 7 & 16 \\ \hline 10 & 18 \\ \hline 15 & ? \\ \hline ? & 10 \\ \hline \end{array}$	Answer the question marks in case of a quadratic model. Find maxima or minima. Find the equation for the tangent line in x=2. Find the gradient formula. Find the gradient number in x = 5 Find the area formula Find the area number from x= 1 to x = 6 Find the intersection points with the line y = 3 + 2x
3.	$\begin{array}{\|c\|c\|} \hline x & y = ? \\ \hline 3 & 12 \\ \hline 7 & 16 \\ \hline 10 & 14 \\ \hline 12 & 18 \\ \hline 15 & ? \\ \hline ? & 30 \\ \hline \end{array}$	Answer the question marks in case of a cubic model. Find maxima and minima. Find the equation for the tangent line in x=2. Find the gradient formula. Find the gradient number in x = 5 Find the area formula Find the area number from x= 1 to x = 6 Find the intersection points with the line y = 3 + 2x
4.	3x + 4y = 15 & 5x – 6y = 12	Solve the simultaneous equations
5.	Given two points in a coordinate system P(2,4) and Q(6,10)	Find the midpoint of the line PQ. Find the equation for the line through P and Q Find the equation for the normal line to PQ passing through P Find the angle between PQ and the x-axis. Find the distance between P and Q Find the distance from the line PQ to the point S(8,1) Find the equation for the circle through P and Q and with the midpoint of PQ as centre. Find the intersection point between the circle and the line y = 12-2x
6.	Let X be a normal random variable with mean m = 100 and standard deviation d = 12	$P(X<89) = ?$ $P(X>108) = ?$ $P(93<X<109) = ?$
7.	X counts the numbers of wins in 100 repetitions of a game with 65% winning chance.	$P(X<70) = ?$ $P(X\leq 60) = ?$ $P(X\geq 58) = ?$ $P(63<X\leq 72) = ?$
8.	sin(3x) = 0.4, $0\leq x\leq 2\pi$ con(½x) = -0.3, $0\leq x\leq 2\pi$ tan(2x) = 0.7, $0\leq x\leq 2\pi$	Find the solutions: *Remember to adjust the window* Find the solutions: Find the solutions:
9.	A = 40, b = 7, C = 90	Find a, B and c.
10.	a = 4, c = 7, C = 90	Find A, B and b.
11.	A = 40, b = 7, C = 68	Find a, B and c.
12.	A = 40, b = 7, c = 6.8	Find a, B and C.
13.	A = 40, b = 7, a = 6.2	Find c, B and C.
14.	a = 4, b = 7, c = 6.8	Find A, B and C.
15.	$T = \frac{d}{e-f} + g$	Transpose the T-formula to a d-, e-, f-, and g-formula
16.	The capital 785 increased with 2.7% 5 times and became ?	Find the answer Find the corresponding doubling time.
17.	The capital 785 increased with 2.7% ? times and became 980	Find the answer Find the corresponding doubling time.
18.	The capital 785 increased with ?% 5 times and became 980	Find the answer Find the corresponding doubling time.
19.	-21	As 16-18, but with $ instead of %

Problem 1. *Linear model*

Equation:	y=ax+b
	y=x+9, found by Stat, Calc, LinReg
Test	y1(3) = 12 ☺

y=?	y=x+9
x=10	y=19 found by y1(10)
Test	y=19 found by CalcValue ☺

x=?	y=x+9
y=40	x=31, found by Math, Solver 0=y1-40
Test	y1(31) = 40 ☺

Exponential model

Equation:	y=a*b^x
	y=9.671*1.075^x, found by Stat, Calc, ExpReg
Test	y1(3) = 12 ☺

y=?	y=9.671*1.075^x
x=10	y=19.853 found by y1(10)
Test	y=19.853 found by CalcValue ☺

x=?	y=9.671*1.075^x
y=40	x=19.740, found by Math, Solver 0=y1-40
Test	y1(19.740) = 40 ☺

Doubling time T = log2/logb = log2/log1.075 = 9.6

Power model

Equation:	y=a*x^b
	y=8.264*x^0.340 found by Stat, Calc, PwrReg
Test	y1(3) = 12 ☺

y=?	y=8.264*x^0.340
x=10	y=18.060 found by y1(10)
Test	y=18.060 found by CalcValue ☺

x=?	y=8.264*x^0.340
y=40	x=104.024 found by Math, Solver 0=y1-40
Test	y1(104.024) = 40 ☺

Problem 2. *Quadratic model*

Equation:	y=a*x^2+b*x+c
	y=-0.048x^2+1.476x+8 found by Stat, Calc, QuadReg
Test	y1(3) = 12 ☺

y=?	y=-0.048x^2+1.476x+8
x=15	y=19.429 found by y1(15)
Test	y=19.429 found by Graph, Calc, Value ☺

x=?	y=-0.048x^2+1.476x+8
y=10	x=1.420 or 29.580 found by Math, Solver 0=y1-10
Test	y1(1.420) = 10 y1(29.580) = 10 ☺

Maximum:	y=-0.048x^2+1.476x+8
	(x,y) = (15.500,19.140) found by Graph, Calc, Maximum
Test	dy/dx = 0 for x = 15.5 y1(15.5)= 19.14 ☺

Tangent in x=2	y=-0.048x^2+1.476x+8
x=2	y=1.286x + 8.190 found by Graph, Draw, Tangent

Gradient formula	y=-0.048x^2+1.476x+8
	y'= -0.095*x + 1.476, found by TI89
Test	∫y'dx=-0.048x^2+1.476x found by TI89 ☺

Gradient number:	y=-0.048x^2+1.476x+8
x=5	dy/dx = 1 for x=5 found by Graph, Calc, dy/dx
Test	1, found by Math, nDeriv ☺

Area formula:	y=-0.048x^2+1.476x+8
x=2	∫ydx = -0.016*x^3 + 0.738*x^2 + 8.000*x found by TI89
Test	d(∫ydx)/dx = -0.048x^2+1.476x+8 found by TI89 ☺

Area number:	y=-0.048x^2+1.476x+8
	$\int_1^6 ydx$ = 62.421, found by Graph, Calc, ∫f(x)dx
Test	62.421, found by Math, fnInt ☺

Intersection points	y = -0.048x^2+1.476x+8 and y = 3+2x (y1 = y3)
	(x,y) = (-17.130,-31.260) and (x,y) = (6.130, 15. 260), found by Math, Solver 0=y1-y3 and y1(-17.130) = -31.260 etc.
Test	tested by Graph, Calc, Intersect ☺

Problem 3. *Cubic model*

Equation:	y=a*x^3+b*x^2+c*x+d
	y=0.086x^3-1.952x^2 +13.752x-14, found by Stat, Calc, CubicReg
Test	y1(3) = 12 ☺

y=?	y=0.086x^3-1.952x^2 +13.752x-14
x=15	y=42.286 found by y(15)
Test	y=42.286 found by Graph, Calc, Value ☺

x=?	y=0.086x^3-1.952x^2 +13.752x-14
y=30	x=13.885 found by Math, Solver 0=y1-30
Test	y1(13.885) = 30 ☺

Maximum Minimum:	y=0.086x^3-1.952x^2+13.752x-14
	Max: (x,y) = (5.552, 16.841) found by Graph, Calc, Maximum Min: (x,y) = (9.634, 13.925) found by Graph, Calc, Minimum
Test	dy/dx = 0 for x = 5.552 y1(5.552)= 16.841 dy/dx = 0 for x = 9.643 y1(9.643)= 13.925 ☺

Tangent in x=2	y=0.086x^3-1.952x^2+13.752x-14
x=2	y = 6.971x – 7.562 found by Graph, Draw, Tangent

Gradient formula	y=0.086x^3-1.952x^2+13.752x-14
	y'= 0.257*x^2 – 3.905*x + 13.752, found by TI89
Test	∫y'dx=0.086x^3-1.952x^2+13.752x found by TI89 ☺

Gradient number:	y=0.086x^3-1.952x^2+13.752x-14
x=5	y'(5) = 0.657 found by Graph, Calc, dy/dx
Test	0.657, 1 found by Math, nDeriv ☺

Area formula:	y=0.086x^3-1.952x^2+13.752x-14
x=2	∫ydx = 0.021*x^4 – 0.651*x^3+6.876*x^2+14*x found by TI89
Test	d(∫ydx)/dx = 0.086x^3-1.952x^2+13.752x-14 found by TI89 ☺

Area number:	y=0.086x^3-1.952x^2+13.752x-14
	$\int_1^6 ydx = 58.496$, found by Graph, Calc, ∫f(x)dx
Test	58.496, 62.421 found by Math, fnInt ☺

Intersection points with y=3+2x: (x,y) = (2.129,-7.259) and (x,y) = (6.657, 16.315) and (x,y) = (13.991, 30.981) found by Math, Solver 0=y1-y3 , tested by Graph, Calc, Intersect.

Problem 4

Solutions: $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3.632\\1.027\end{pmatrix}$, found by A*B=C, B=A^-1*C, where A = $\begin{pmatrix}3 & 4\\5 & -6\end{pmatrix}$ and B = $\begin{pmatrix}x\\y\end{pmatrix}$ and C = $\begin{pmatrix}15\\12\end{pmatrix}$

Tested by A*B=C : A*B = $\begin{pmatrix}3 & 4\\5 & -6\end{pmatrix} * \begin{pmatrix}3.632\\1.027\end{pmatrix} = \begin{pmatrix}15\\12\end{pmatrix}$ = C ☺

Problem 5

Midpoint:	$(x,y) = (\frac{x1+x2}{2}, \frac{y1+y2}{2})$
x1=2 x2=6 y1=4 y2=10	$(x,y) = (\frac{2+6}{2}, \frac{4+10}{2})$ (x,y) = (4,7)
Test	Tested geometrically ☺

Gradient PQ:	$a = \frac{y2-y1}{x2-x1}$
x1=2 x2=6 y1=4 y2=10	$a = \frac{10-4}{6-2}$ a= 3/2 a = 1.5
Test	Tested geometrically ☺

Line PQ:	y = y1 + a*(x – x1)
a =1.5 x1=2 y1=4	y = 4 + 1.5*(x – 2) y = 1.5*x + 1
Test	Tested geometrically ☺

Gradient perpend.:	c*a = -1
a=3/2	c = -2/3 found by Math, Solver 0 = c*3/2+1
Test	Tested geometrically ☺

Normal:	y = y1 + a*(x – x1)
a=-2/3 x1=2 y1=4	y = 4 + -2/3*(x – 2) y = -2/3*x + 5.333
Test	Tested geometrically ☺

Distance PQ	$d = \sqrt{(x2-x1)^2 + (y2-y1)^2}$
x1=2 x2=6 y1=4 y2=10	$d = \sqrt{(6-2)^2 + (10-4)^2}$ d = 7.21
Test	Tested geometrically ☺

Distance point-line	$d = \frac{\lvert y1 - a*x1 - b\rvert}{\sqrt{1 + a^2}}$
a=1.5 b=1 x1=8 y1=1	$d = \frac{\lvert 1 - 1.5*8 - 1\rvert}{\sqrt{1 + 1.5^2}}$ d = 6.66
Test	Tested geometrically ☺

Circle equation	$(x-c1)^2 + (y - c2)^2 = r^2$
r=½*7.21 r=3.61 c1=4 c2=7	$(x -4)^2+(y - 7)^2 = 3.61^2$ $(x -4)^2+(y - 7)^2 = 13.03$
Test	Tested geometrically ☺

Intersection	$(x-c1)^2 + (y - c2)^2 = r^2$ and y = 12-2x
r =½*7.21 =3.61 c1 =4 c2 =7	(x,y) = (1.30,9.40) and (4.30,3.40) found by Math, Solver $0 = (x-4)^2 + (12-2x-7)^2 – 3.61^2$
Test	Tested geometrically ☺

Angle: tan(v) = a, a=3/2 ; v = 56.31 found by Math, Solver 0 = tanv - 3/2, v>0 and v<90. Tested geometrically ☺

Problem 6	Problem 7
p(X<115) = 0.894, found by normalCdf(1EE-99,115,100,12)	p(X<70) = 0.827, found by binomCdf(100,0.65,0,69)
p(X<89) = 0.180, found by normalCdf(1EE-99,89,100,12)	p(X≤60) = 0.172, found by binomCdf(100,0.65,0,60)
p(X>108) = 0.253, found by normalCdf(108,1EE99,100,12)	p(X≥58) = 0.941, found by binomCdf(100,0.65,58,100)
p(93<X<109) = 0.494, found by normalCdf(93,109,100,12)	p(63<X≤72) = 0.571, found by binomCdf(100,0.65,64,72)

Problem 8

x=?	sin(3x) = 0.4
	x =0.137, or 0.910, or 2.232 or 3.004 or 4.326 or 5.099 found by Math, Solver 0=y1-0.4
Test	tested by Graph, Calc, Intersect ☺

x=?	cos(½x) = -0.3
	x =3.745 found by Math, Solver 0=y1+0.3
Test	tested by Graph, Calc, Intersect ☺

x=?	tan(2x) = 0.7
	x =0.305, or 1.876, or 3.447 or 5.018 found by Math, Solver 0=y1-0.7
Test	tested by Graph, Calc, Intersect ☺

Problem 9

a = ?	tan A = a/b
A =40 b=7	a = 5.874 found by Math, Solver 0=a/7-tan40
Test	tan40 = 5.874/7 0.839 = 0.839 ☺

c = ?	cos A = b/c
A =40 b=7	c = 9.138 found by Math, Solver 0=7/c-cos40
Test	cos40 = 7/9.138 0.766 = 0.766 ☺

B = ?	A + B = 90
A = 40	B = 50 found by Math, Solver 0=40+B-90
Test	50+40 = 90 90 = 90 ☺

Problem 10

b = ?	a^2 + b^2 = c^2
a =4 c=7	b = 5.745 found by Math, Solver 0=4^2+b^2-7^2
Test	4^2 + 5.745^2 = 7^2 49 = 49 ☺

A = ?	sinA = a/c
a =4 c=7	A =34.85 found by Math, Solver 0=4^2+b^2-7^2
Test	Sin34.85 = 4/7 0.571 = 0.571 ☺

B = ?	A + B = 90
A = 34.85	B = 55.15 found by Math, Solver 0=34.85+B-90
Test	34.85+55.15 = 90 90 = 90 ☺

Problem 11

B = ?	A+B+C=180
A =40 C =68	B = 72 found by Math, Solver 0=40+B+68-180
Test	40+72+68=180 180= 180 ☺

a = ?	a/sinA = b/sinB
A =40 B =72 b = 7	a =4.731 found by Math, Solver 0=a/sin40- 7/sin72
Test	4.731/sin40 = 7/sin72 7.360 = 7.360 ☺

c = ?	c/sinC = b/sinB
C =68 B =72 b = 7	c =6.824 Math, Solver 0=c/sin68 - 7/sin72
Test	6.824/sin68 = 7/sin72 7.360 = 7.360 ☺

Problem 12

a = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
A =40 c =6.8 b = 7	a =4.724 found by Math, Solver $0=a^2-6.8^2-7^2+2*6.8*7*\cos 40$
Test	$4.724^2=6.8^2+7^2–2*6.8*7*\cos 40$ 22.316= 22.316 ☺

B = ?	a/sinA = b/sinB
A =40 b = 7 a=4.724	B = 72.3 found by Math, Solver 0=4.724/sin40- 7/sinB
Test	4.724/sin40 = 7/sin72.3 7.348 = 7.348 ☺

C = ?	A+B+C=180
A =40 B =72.3	C = 67.7 found by Math, Solver 0 = 40+72.3+C-180
Test	40+72.3+67.7=180 180= 180 ☺

Problem 13

B = ?	a/sinA = b/sinB
A =40 a =6.2 b = 7	B = 46.53 or B = 133.47 found by Math, Solver 0 = 6.2/sin40 - 7/sinB
Test	6.2/sin40=7/sin46.53=7/sin133.47 9.645 = 9.645 = 9.645 ☺

C = ?	A+B+C=180
A =40 B =46.53 or B=133.47	C = 93.47or C = 6.53 found by Math, Solver 0 = 40+B+C-180
Test	40+46.53+93.47=180 180= 180 ☺

c = ?	a/sinA = c/sinC
A =40 a=6.2 C =93.47, C=6.53	c = 9.628 or C = 1.097 found by Math, Solver 0 = 6.2/sin40 - c/sinC
Test	6.2/sin40=9.628 /sin93.47=9.628 /sin6.53 9.645 = 9.645 = 9.645 ☺

Problem 14

A = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
a = 4 c =6.8 b = 7	A =33.66 found by Math, Solver $0 = 4^2-6.8^2-7^2+ 2*6.8*7*\cos A$
Test	$4^2 =6.8^2+7^2–2*6.8*7*\cos 33.66$ 16 = 16 ☺

B = ?	$b^2 = a^2+c^2–2*a*c*\cos B$
a = 4 c = 6.8 b = 7	B =75.91 found by Math, Solver $0 = 7^2-4^2-6.8^2+2*6.8*4*\cos B$
Test	$7^2=4^2+6.8^2–2*6.8*4*\cos 75.91$ 49 = 49 ☺

C = ?	A+B+C=180
A =33.66 B =75.91	C = 70.43 found by Math, Solver 0 = 33.66+75.91+C-180
Test	33.66+75.91+70.43=180 180= 180 ☺

Problem 15

d=?	$T = \frac{d}{e-f} + g$
	$T = \left(\frac{d}{(e-f)}\right) + g$ d = (e-f)*(T-g)
Test	$T= \frac{(e-f)*(T-g)}{e-f} +g = T$

e=?	$T = \frac{d}{e-f} + g$
	$T = \left(\frac{d}{(e-f)}\right) + g$ (T-g)(e-f) = d $e = \frac{d}{T-g} + f$
Test	$T= \frac{d}{\frac{d}{T-g} + f - f} +g =T$

f=?	$T = \left(\frac{d}{(e-f)}\right) + g$
	(T-g)(e-f) = d $e = \frac{d}{T-g} + f$ $e - \frac{d}{T-g} = f$
Test	$T= \frac{d}{e- e - \frac{d}{T-g}} +g =T$

g=?	$T = \frac{d}{e-f} + g$
	$T = \left(\frac{d}{(e-f)}\right) + g$ $T - \left(\frac{d}{(e-f)}\right) = g$
Test	$T= \frac{d}{e-f} +T - \left(\frac{d}{(e-f)}\right)=T$

Problems 16-18

y = ?	y = a*b^x
a=785 b=1.027 x=5	y = 785*1.027^5 y = 896.85

T = log(2)/log(1.027) = 26.0

x = ?	y = a*b^x
a=785 b=1.027 y=980	x = 8.3 found by Math, Solver 0=785*1.027^x - 980
Test	980 = 785*1.027^8.3 980 = 980 ☺

T = log(2)/log(1.027) = 26.0

b = ?	y = a*b^x
a=785 y=980 x=5	b = 1.045 = 1 + 4.5% found by Math, Solver 0=785*b^5 - 980
Test	980 = 785*1.045^5 980 = 980 ☺

T = log(2)/log(1.045) = 15.7

Problems 19-21

y = ?	y = a*x + b
b=785 a= 2.7 x=5	y = 2.7*5 + 785 y = 798.5

x = ?	y = a*x + b
b=785 a= 2.7 y=980	x = 72.2 found by Math, Solver 0 = 2.7*x+785 - 980
Test	980 = 2.7*72.2 + 785 = 980 ☺

a = ?	y = a*x + b
b=785 y=980 x=5	a = 39 found by Math, Solver 0 = a*5+785 - 980
Test	980 = 39*5 + 785 = 980 ☺