Golden Learning Opportunities in Preschool

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Preschool allows rethinking mathematics outside the tradition of ordinary school. Seeing schooling as adapting the child to the outside world containing many examples of the natural fact Many, we can ask: How will mathematics look like if built as a natural science about Many? To deal with Many we count and add. The school counts in tens, but preschool also allows counting in icons. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. To add next-to means adding areas also called integration. So accepting icon-counting and adding next-to offers golden learning opportunities in preschool that are lost when ordinary school begins.

Math in Preschool – a Great Idea

Mathematics is considered one of the school's most important subjects. So it seems to be a good idea to introduce mathematics in preschool - provided we can agree upon what we mean by mathematics.

As to its etymology Wikipedia writes that the word mathematics comes from the Greek máthēma, which, in the ancient Greek language, means "that which is learnt". Later Wikipedia writes:

> In Latin, and in English until around 1700, the term mathematics more commonly meant "astrology" (or sometimes "astronomy") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. (http://en.wikipedia.org/wiki/Mathematics)

This meaning resonates with Freudenthal writing:

> Among Pythagoras' adepts there was a group that called themselves mathematicians, since they cultivated the four "mathemata", that is geometry, arithmetic, musical theory and astronomy. (Freudenthal 1973: 7)

Thus originally mathematics was a common word for knowledge present as separate disciplines as astronomy, music, geometry and arithmetic. This again resonates with the educational system in the North American republics offering courses, not in mathematics, but in its separate disciplines algebra, geometry, etc.

In contrast to this, in Europe with its autocratic past the separate disciplines called Rechnung, Arithmetik und Geomtrie in German were integrated to mathematics from grade one with the arrival of 'modern mathematics' wanting to revive the rigor of Greek geometry by defining mathematics as a collection of well-proven statements about well-defined concepts all defined as examples of the mother concept set.

Kline sees two golden periods, the Renaissance and the Enlightenment that both created and applied new mathematics by disregarding Greek geometry:

> Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

Furthermore, Gödel has proven that the concept of being well-proven is but a dream. And Russell's set-paradox questions the set-based definitions of modern mathematics by showing that talking about sets of sets leads to self-reference and contradiction as in the classical liar-paradox 'this sentence is false' being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$.

With no general agreement as to what mathematics is and with the negative effects of imposing rigor, preschool mathematics should disintegrate into its main ingredients, algebra meaning reuniting numbers in Arabic, and geometry meaning measuring earth in Greek; and both should be grounded in their common root, the natural fact Many. To see how, we turn to sceptical research.

Postmodern Contingency Research

Ancient Greece saw a controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that in a republic people must be enlightened about choice and nature to prevent being patronized by choices presented as nature. In contrast to this philosophers saw everything physical as examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become patronizors.

Enlightenment later had its own century that created two republics, an American and a French. Today the sophist warning is kept alive in the French republic in the postmodern sceptical thinking of Derrida, Lyotard, Foucault and Bourdieu warning against when categories, discourses, institutions and education become patronising by presenting their choices as nature (Tarp 2004).

Thus postmodern sceptical research discovers contingency, i.e. hidden alternatives to choices presented as nature. To make categories, discourses and institutions non patronizing they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget's principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

With only little agreement as to what mathematics is we ask: How will mathematics look like if built as a natural science about Many?

Building a Science about the Natural Fact Many

To deal with the natural fact Many we iconize and bundle. What could be called 'first order counting' bundles sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way icons are created for numbers until ten, the only number with a name, but without an icon.

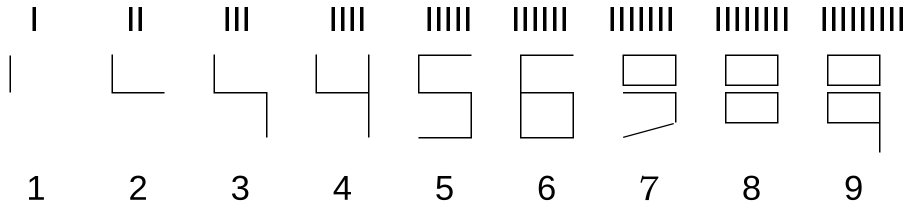

Figure 1: Icons contain as many sticks as they represent

What could be called 'second order counting' bundles in icon-bundles. So a total T of 7 1s can be bundled in 3s as T = 2 3s and 1, and placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Then the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III I -> III III) I) -> ▌▌) I) -> II) I) -> 2)1) -> 2.1 3s

Using squares or LEGO blocks or an abacus, the two 3-bundles can be stacked on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a decimal number.

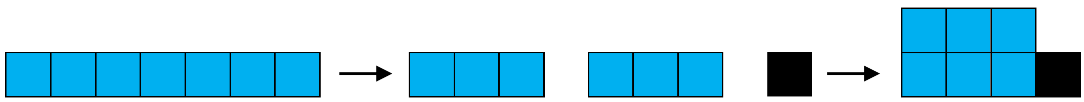

Figure 2: Seven 1s first becomes 2 3s & 1, and then 2x3 + 1 or 2.1 3s

With overloads also bundles can be bundled and placed in a new cup to the left. Thus in 6.2 3s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s:

III III) II) -> II)) II), or 6)2) = 2)0)2, or 6.2 3s = 20.2 3s.

Adding an extra cup to the right shows that multiplying with the bundle-size just moves the decimal point:

T = 2.1 3s = 2)1) -> 2)1)) = 21.0 3s

Operations iconize the bundling and stacking processes. Taking away 4 is iconized as – 4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 showing a 3 times lifting of the 4s. Placing a stack of 2 singles next to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. as 1 3^2-bundle.

Numbers and operations can be combined to calculations in formulas predicting the counting results. Counting a total T in bs can be predicted by a ‘recount-formula’ T = (T/b)*b telling that ‘From a total T, T/b times, b can be taken away’. Thus recounting a total T = 3 5s in 6s, the prediction says T = (3x5)/6 6s. Using a calculator we get the result ‘2.some’ where the some is found by dragging away the 2 6s, predicted by the ‘restack-formula’ T = (T–b) + b telling that ‘From a total T, T–b is left, when b is taken away and placed next-to’.

3 x 5 / 6	2.some
3 x 5 – 2 x 6	3

Figure 3: A calculator predicts that recounting 3 5s in 6s is 2.3 6s

The combined prediction T = 3 5s = 2 6s + 3 1s = 2.3 6 holds when tested:

IIIII IIIII IIIII -> IIIIII IIIIII III

Once counted, totals can be added on-top or next-to. To add on-top, the units must be the same, so one total must be recounted in the other total’s unit. Adding stacks with the same unit might create an overload forcing the sum to be recounted in the same unit. Adding totals next-to means adding the areas, which is also called integration. Again, a next-to addition of e.g. 4 3s and 1 5s can be predicted by a calculator using the recount- and restack-formulas.

(4 x 3 + 1 x 5) / 8	2.some
(4 x 3 + 1 x 5) – 2 x 8	1

Figure 4: A calculator predicts that adding 4 3s and 1 5s as 8s is 2.1 8s

Addition can be reversed by taking away what was added. If on-top addition created an overload that was removed it must be recreated in order to take away what was added. In next-to addition what is left, when what was added is taken away, must be recounted in the original unit. Reversed addition on-top is called subtraction and reversed addition next-to is called differentiation.

The tradition counts in tens only, which can be called third order counting.

Written in its full form, 354 = 3*10^2 + 5*10 + 4*1 becomes a sum of areas placed next-to each other, thus showing the four ways to unite numbers: Addition unites variable unit numbers, multiplication unites constant unit numbers, integration unites variable per-numbers, and power unites constant per-numbers.

De-uniting a total is predicted by the inverse operations, that are named subtraction, division, root and logarithm, and differentiation. Thus it makes good sense that algebra means reuniting in Arabic.

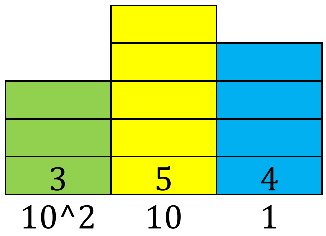

Figure 5: The number 354 = 3*10^2 + 5*10 + 4*1 shown as stacks

Comparing Manyology and the Tradition

Using postmodern contingency research we have discovered a natural science about Many that can be called Manyology and that allows us to deal with Many by counting and adding: First we count in icons, then in icon-bundles allowing a total to be written in a natural way as a decimal number with a unit where the decimal point separates the bundles from the unbundled. To add on-top and next-to we change the unit by recounting, predicted by a recount- and a restack-formula. Written out fully as stacked bundles, numbers show the four ways to unite: on-top and next-to addition, multiplication, and power. And to reverse addition we need inverse operations (Zybartas et al 2005), (YouTube), (Tarp 2014).

Counting Many by cup-writing and as stacked bundles contains the core of the mathematical sub-disciplines algebra and geometry. However there are fundamental differences between Manyology and traditional mathematics.

In the first an icon contains as many sticks or strokes as it represents, in the second an icon is just a symbol. In the first a natural number is a decimal number with a unit using the decimal point to separate bundles and unbundled; in the second a natural number hides the unit and misplaces the decimal point one place to the right.

The first presents operations as icons with the natural order division, multiplication, subtraction and two kinds of addition, on-top and next-to; the second presents operations as symbols; the order is the opposite; and next-to addition is neglected.

The first uses a calculator for number prediction. The second neglects it. The first allows counting in icons, the second only allows counting in tens.

With ten as THE bundle-size, recounting becomes irrelevant and impossible to predict by a calculator since asking '3 8s = ? tens' leads to T = (3x8/ten) tens that cannot be entered. Now the answer is given by multiplication, 3x8 = 24 = 2 tens + 4 1s, thus transforming multiplication into division. Likewise adding next-to is neglected and adding on-top becomes THE way to add.

Furthermore the tradition changes mathematics into 'metamatism', a combination of 'meta-matics' and 'mathema-tism' where metamatics turns mathematics upside down by presenting concepts as examples of abstractions instead of as abstractions from examples, thus insisting that numbers are examples of sets in one-to-one correspondence; and where mathematism allows addition without units, thus presenting ‘1+2=3’ as a natural fact in spite of its many counterexamples as 1 week + 2 days = 9 days, 1 m + 2 cm = 102 cm etc.

Thus the goal of a preschool curriculum should be the golden learning opportunities coming from icon-counting and next-to addition since they both disappear when traditional metamatism suppresses Manyology from day one in school. So Manyology is an example of postmodern paralogy described by Lyotard to be a dissension to the ruling consensus (Lyotard 1984, 61).

The Traditional Preschool Mathematics

At the twelfth International Congress on Mathematical Education, ICME 12, the topic study group on Mathematics education at preschool level contains two interesting contributions from Sweden (http://www.icme12.org/sub/tsg/ tsg_last_ view.asp?tsg_param=1). The second discusses the content knowledge needed for preschool teachers to guide mathematical learning; and the first discusses the difficulties trying to categorize children behaviour according to the revised preschool curriculum in Sweden from 2011, inspired by five categories claimed by Bishop to constitute mathematics (Bishop 1988).

The five categories are counting, i.e. the use of a systematic way to compare and order discrete phenomena; locating, i.e. exploring one’s spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means; measuring, i.e. quantifying qualities for the purposes of comparison and ordering; designing, i.e. creating a shape or design for an object or for any part of one’s spatial environment; and playing, i.e. devising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by.

Bishop’s five activities reminds of Niss’ eight competencies: thinking mathematically; posing and solving mathematical problem; modelling mathematically ; reasoning mathematically; representing mathematical entities; handling mathematical symbols and formalisms; communicating in, with, and about mathematics; and making use of aids and tools (Niss 2003). Both define mathematics with action words. Bishop uses general words whereas Niss is caught in self-reference by including the term mathematics in its own definition.

However, both exceed in numbers vastly the two activities of Manyology, counting and adding, so sceptical thinking could ask: Since the numbers of activities alone makes it almost impossible for teachers and children to learn, is there a hidden patronizing agenda in these longs lists since just two activities or competences are needed to deal with the natural fact Many? And is it mathematics or metamatism these lists define?

To illustrate the issue we now look at the web-based training of in-service teachers at the MATHeCADEMY.net using ‘pyramid-education’.

Micro-Curricula at the MATHeCADEMY.net

The MATHeCADEMY.net sees mathematics as Manyology, the natural science about the natural fact Many. It teaches teachers to teach this natural science about Many to learners by allowing both teachers and learners to learn mathematics through investigations guided by educational questions and answers.

Seeing counting and adding as the two basic competences needed to deal with Many, it uses a CATS method, Count & Add in Time & Space, in a Count&Add laboratory where addition predicts counting-results, thus making mathematics a language for number-prediction. The website contains 2x4 study units with CATS1 for primary school and CATS2 for secondary school.

In pyramid-education 8 in-service teachers are organized in 2 teams of 4 teachers, choosing 3 pairs and 2 instructors by turn. The Academy coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach helps the instructors to correct the count&add problems. In each pair each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair. Having finished the course, each in-service teacher will 'pay' by coaching a new group of 8 in-service teachers.

Five plus Two Learning Steps

The in-service teachers learn in the same way as their students by carrying out five learning steps: to do, to name, to write, to reflect and to communicate. For a teacher two additional steps are added: to design and to carry out a learning experiment, while looking for examples of cognition, both existing recognition and new cognition. To give an example, wanting children to learn that 5 is an icon with five sticks, the steps could be:

Do: take 5 sticks and arrange them next to each other, then as the icon 5.

Say: a total of five sticks is rearranged as the number icon 5, written as T=5.

Reflect. That five sticks is called five is old cognition. It is new cognition that five sticks can be rearranged as a 5-icon and that this contains the number of sticks it represents.

Communicate. Write a postcard: 'Dear Paul. Today I was asked to take out five sticks and rearrange them as a 5-icon. All of a sudden I realized the difference between the icon 5 and the word five, the first representing what it describes and the second representing just a sound. Best wishes'.

Design an experiment: I will help Michael, who has problems understanding 2digit numbers. Once he tries to build a number symbol for ten, eleven and twelve, he will realize how smart it is to stop inventing new symbols and instead begin to double-count bundles and unbundled. So I design an experiment asking the children to build the first twelve number-icons by rearranging sticks.

Carry out the experiment: It is my impression that constructing the number icon for ten was what broke the ice for Michael. It seems as if it enabled Michael to separate number-names from number-icons, since it made him later ask 'Why don't we say one-ten-seven instead of seventeen? It would make things much easier.' This resonates with what Piaget writes:

> Intellectual adaptation is thus a process of achieving a state of balance between the assimilation of experience into the deductive structures and the accommodation of those structures to the data of experience (Piaget 1970: 153-154).

Designing a Micro-Curriculum so Michael Learns to Count

This 5-lesson micro-curriculum uses activities with concrete material to obtain its learning goals. In lesson 1 Michael learns to use sticks to build the number icons up to twelve, and to use strokes to draw them, thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy.

In lesson 2 Michael learns to count a given total in 1s and in 4s; and to count up a given total containing a specified numbers of 1s or of 4s.

Lesson 3 repeats lesson 2, now counting in 3s.

Lesson 4 combines lesson 2 and 3, now counting in 1s, 3s and 4s.

In lesson 5 Michael learns to recount in 4s a total already counted in 3s, both manually and by using a calculator; and vice versa.

As concrete materials anything goes in lesson 1. The other lessons will use fingers, sticks, pegs on a pegboard, beads on an abacus, and LEGO blocks.

Another 5-lesson micro-curriculum could make Michael learn to add on-top and next-to to be able to answer questions like 2 3s + 4 5s = ? 3s = ?5s = ?8s. This will not be discussed further here.

Lesson 1, Building and Drawing Number Icons

On the floor the children place six hula hoop rings next to each other as six different lands: empty-land, 1-land, 2-land, 3-land, 4-land and 5-land shown by the corresponding number of chopsticks on a piece of paper outside the ring.

Each child is asked to find a thing to place in 1-land, and to explain why. Then they are asked to turn their thing so it has the same direction as the chopstick. Finally the group walks around the room and points out examples of 'one thing' always including the unit, e.g. 1 chair, 1 ball, etc.

In the same way each child is asked to find a thing to place in 2-land. The instructor shows how the two chopsticks can be rearranged to form one 2-icon. The children are asked to pick up two sticks and do the same; and to draw many examples of the 2-icon on a paper discussing with the instructor why the 2-icon on the wall is slightly different from the ones they draw. Now the children are asked to rearrange their 2s in 2-land so they have the same form as the 2-icon. And again the group walks around the room and points out examples of ‘two things’ that is also called ‘one pair of things’.

This is now repeated with 3-land where three things are called one triplet.

Before going on to 4-land the instructor asks the children to do the same with empty-land. Since the empty-icon cannot be made by chopsticks the instructor ask for proposals for an empty-icon hoping that one or more will suggest the form of the ring, i.e. a circle. And again the group walks around the room to try to locate examples of ‘no things’ or zero things.

Now the activity is repeated with 4-land. Here the instructor asks the children to suggest an icon for four made by four sticks. When summing up the teacher explains that the adults have rejected the square since it reminds too much of a zero, so the top stick is turned and placed below the square to the right. Here the children are asked to rearrange their 4s in 4-land so they have the same form as a square, and as the 4-icon. And again the group walks around the room and points out examples of ‘four things’ that is also called ‘a double pair’.

Now the activity is repeated with 5-land. Here the instructor asks the children to suggest an icon for five made by 5 sticks. When summing up the teacher explains that the adults have decided to place the five stick in an s-form. When walking around the room to point out examples a discussion is initiated if ‘five things’ is the same as a pair plus a triplet, and as a double pair plus one.

This activity can carry on to design icons for the numbers from six to twelve realizing that the existing icons can be recycled if bundling in tens.

Observing and Reflecting on Lesson 1

Having designed a micro-curriculum, the in-service teacher now carries it out in a classroom looking for examples of recognition and new cognition.

One teacher noticed the confusion created by asking the children to bring things to empty-land. It disappeared when one child was asked what he had just put into the ring and answered no elephant. Now all of the children were eager to put no cars, no planes etc. into the ring.

Later the teacher witnessed children discussing why the 3-icon was not a triangle, and later used the word four-angle for the square. Also this teacher noticed that some children began to use their fingers instead of the chopsticks.

Under the walk around the room a fierce discussion about cheating broke out when a child suggested that clapping his hand three times was also an example of three things. Its not, another child responded. It is. No its not! Why not? Because you cannot bring it to 3-land! Let’s ask the teacher!

After telling about space and time, children produced other examples as three knocks, three steps, three rounds around a table, three notes. Other children began to look at examples of threes at their own body soon finding three fingers, three parts on a finger, and three hands twice when three children stood side by side and the middle one lent out his two hands to his neighbours.

Conclusion

To find which mathematics can be treated in preschool, postmodern contingency research uncovered Manyology as a hidden alternative to the ruling tradition. Dealing with the natural fact Many means counting in icons, and recounting when adding on-top or next-to thus introducing linearity and calculus. However, these golden learning opportunities are lost when entering grade one, where the monopoly of ten-counting prevents both from happening; and furthermore grounded mathematics is replaced with metamatism when introducing one-to-one corresponding sets and when teaching that 1+2 IS 3. So maybe someone should tell the governments that in a republic the educational system must not present choice as nature. Instead governments should accept the historic fact that long, long ago the antique collective name mathematics was split up into independent disciplines. So instead of teaching mathematics, schools should prepare for the outside world by teaching the two competences needed to deal with the natural fact Many, to count and to add. Consequently, the golden learning opportunities in preschool mathematics should enter ordinary school instead of being suppressed by it.

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