

ADDING MANY

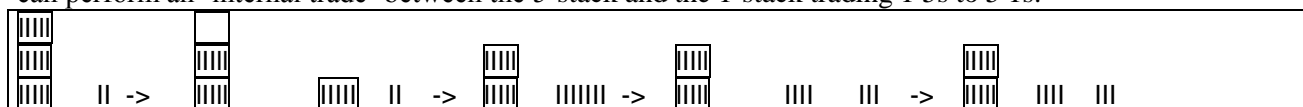
Questions	Answers
How to add stacks concretely? $T = 27 + 16 = 2\text{ten}7 + 1\text{ten}6 = 3\text{ten}13 = ?$	By restacking overloads predicted by the restack-equation $T = (T-b)+b$ $3\text{ten}13 = 3\text{ten}(13-10+10) = 3\text{ten}1\text{ten}3 = 4\text{ten}3 = 43$
How to add stacks abstractly?	Vertical calculation uses carrying. Horizontal calculation uses FOIL.

1 STACKS ARE SOLD

Question. How can we sell more from a stack than we have?

Answer. Create an overload by recounting and doing internal trade.

Example. From the stock $T = 3\text{ 5s} + 2\text{ 1s}$ we want to sell 3 1s, but we only have 2 1s in stock. However we can perform an 'internal trade' between the 5-stack and the 1-stack trading 1 5s to 5 1s:



$$3*5 + 2 = (3*5 - 1*5) + 1*5 + 2 = 2*5 + 7 = 2*5 + (7-3) + 3 = 2*5 + 4 + 3$$

After the matches we use cups and internal trade to write $T = 32 - 3 = 3)2)_{-3} = 3-1)5+2)_{-3} = 2)7)_{-3} = 2)4) = 24$

Or: $T = 32 - 3 = 3\text{five}2 - 3 = 2\text{five}1\text{five}2 - 3 = 2\text{five}7 - 3 = 2\text{five}4 = 24$

In case of tens we have $T = 32 - 3 = 3)2)_{-3} = 3-1)10+2)_{-3} = 2)12)_{-3} = 2)9) = 29$ ($-$ & \pm : outside - & +)

Or: $T = 32 - 3 = 3\text{ten}2 - 3 = 2\text{ten}1\text{ten}2 - 3 = 2\text{ten}12 - 3 = 2\text{ten}9 = 29$

Exercise1. Sell 3 from 41. Sell 34 from 421. Sell 342 from 4231. Count in fives. First use matches, then write.

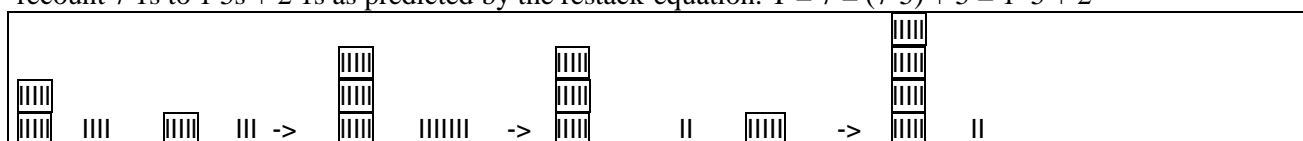
Exercise2. Sell 3 from 41. Sell 34 from 421. Sell 342 from 4231. Count in tens. First use matches, then write.

2 STACKS ARE BOUGHT

Question. How can stocks be added?

Answer. Remove the overload by recounting and doing internal trade.

Example. To the stock $T = 2\text{ 5s} + 4\text{ 1s}$ we add the stock $T' = 1\text{ 5s} + 3\text{ 1s}$. After adding the 1s we are able to recount 7 1s to 1 5s + 2 1s as predicted by the restack-equation: $T = 7 = (7-5) + 5 = 1*5 + 2$



$$2*5 + 4 + 1*5 + 3 = 3*5 + 7 = 3*5 + (7-5) + 5 = 4*5 + 2$$

After the matches we use cups and internal trade to write

$T = 24 + 13 = 2)4) \pm 1)3) = 3)7) = 3)7-5+5) = 3)5+2) = 3+1)2) = 4)2) = 42$

Or: $T = 24 + 13 = 2\text{five}4 + 1\text{five}3 = 3\text{five}7 = 3\text{five}1\text{five}2 = 4\text{five}2 = 42$

In case of tens we have $T = 24 + 17 = 2)4) \pm 1)7) = 3)11) = 3)11-10+10) = 3)10+1) = 3+1)1) = 4)1) = 41$

Or: $T = 24 + 17 = 2\text{ten}4 + 1\text{ten}7 = 3\text{ten}11 = 3\text{ten}1\text{ten}1 = 4\text{ten}1 = 41$

Exercise1. Add 3 to 24. Add 43 to 34. Add 241 to 444. Count in fives. First use matches, then write.

Exercise2. Add 8 to 24. Add 79 to 34. Add 879 to 444. Count in tens. First use matches, then write.

3 STOCKS ARE SPLIT

Question. How can stocks be split?

Answer. Create an overload by recounting and doing internal trade.

Example. The stock $T = 3\text{ 5s} + 4\text{ 1s}$ is split in two parts.

$$\text{VVV IIII} \quad \rightarrow \quad \text{VV} \quad \text{V} \quad \text{IIII} \quad \rightarrow \quad \text{VV IIIII IIII} \quad \rightarrow \quad \text{VV IIIII IIII} \quad \rightarrow \quad \text{VV IIII IIII I} \quad \rightarrow \quad \text{VIII VIII I}$$

$$34 = (3*V+4) = (3/2)*2*V+4 = 2*V + 1*V + 4 = 2*V+5+4 = 2*V+9 = 2*V+2*4+1 = 2*(V+4)+1 = 2*14+1$$

After the matches we use cups and internal trade to write

$$34 = 3)4) = 3/2)2)4) = 1*2+1)4) = 2*1)+5+4) = 2*1)9) = 2*1)9/2*2) = 2*1)4*2+1) = 2*1)2*4) \pm 1 = 2*14+1$$

$$\text{Or: } 34 = 3\text{five}4 = 2\text{five}1\text{five}4 = 2\text{five}9 = 2*1\text{five}2*4 + 1 = 2*1\text{five}4 + 1 = 2*14 + 1$$

In case of tens we write

$$34 = 3)4) = 3/2*2)4) = 1*2+1)4) = 1*2)+10+4) = 2*1)14) = 2*1)14/2*2) = 2*1)7*2) = 2*1)2*7) = 2*17$$

$$\text{Or: } 34 = 3\text{ten}4 = 2\text{ten}1\text{ten}4 = 2\text{ten}14 = 2*1\text{ten}2*7 = 2*1\text{ten}7 = 2*17$$

Remark. 8 counted in 2s: $8 = (8/2)*2 = 4*2 = 4\text{ 2s}$. 8 split in 2: $8 = (8/2)*2 = 4*2 = 2*4 = 2\text{ 4s}$. So division is used both when counting and when splitting.

Exercise1. Split 43 in 2. Split 43 in 3. Split 34 in 4. Split 43 in 7. Count in fives. First use matches, then write.

Exercise2. Split 43 in 2. Split 43 in 3. Split 34 in 4. Split 43 in 12. Count in tens. First use matches, then write.

4 STOCKS ARE MULTIPLIED

Question. How can stocks be multiplied?

Answer. Remove the overload by recounting and doing internal trade.

Example. The stock $T = 2\ 5s + 4\ 1s$ is bough three times.

$$\begin{array}{l} \text{VVVIII VVVIII VVVIII} \rightarrow \text{VV VV VV IIIIIIIIIII} \rightarrow \text{VVVVVV IIIII IIIII II} \rightarrow \text{VVVVV VVV II} \rightarrow \text{W VVV II} \\ 3*24 = 3*(2*V+4) = 3*2*V + 3*4 = 6*V + 5 + 5 + 2 = 5*V + 3*V + 2 = 1*W + 3*V + 2 = 132 \end{array}$$

After the matches we use cups and internal trade to write

$$T = 3*24 = 3*(2)4) = 6)12) = 6)12/5*5) = 6)2*5+2) = 6+2)2) = 8)2) = 8/5*5)2) = 1*5+3)2) = 1)3)2) = 132$$

$$\text{Or: } T = 3*24 = 3*2\text{five}4 = 6\text{five}12 = 6\text{five}2\text{five}2 = 8\text{five}2 = (1\text{five}3)\text{five}2 = 1\text{five}5\text{three}5\text{five}2 = 132$$

In case of tens we write

$$T = 3*24 = 3*(2)4) = 6)12) = 6)12/10*10) = 6)1*10+2) = 6+1)2) = 7)2) = 72$$

$$\text{Or: } T = 3*24 = 3*2\text{ten}4 = 6\text{ten}12 = 6\text{ten}1\text{ten}2 = 7\text{ten}2 = 72$$

Remark. $3*11 = 33$ both when counting in tens and fives; but $3*12 = 36$ or 41 .

Exercise1. Do $2*34$. Do $3*34$. Do $4*34$. Do $7*34$. Count in fives. First use matches, then write.

Exercise2. Do $2*34$. Do $3*34$. Do $4*34$. Do $12*34$. Count in tens. First use matches, then write.

5 STOCKS ARE ACCOUNTED FOR

Question. How can we account for selling and buying?

Answer. By ledger accounts.

Example. From the stock $4)2)$ we sell $S: 2)3)$ and buy $B: 1)3)$. We set up two ledger accounts, one for the bundles, and one for the ones to account for the IN (debit) and OUT (credit) of the internal and external trade.

Bundle = 5

	Fives		TOTAL	Ones		TOTAL
	IN	OUT		IN	OUT	
Stock			4			2
Internal trade		1		5		
External trade S: 2)3)		2			3	

Stock			1			4
External trade B: 1)3)	1			3		
Internal trade	1				5	

Bundle = 10

	Tens		TOTAL	Ones		TOTAL
	IN	OUT		IN	OUT	
Stock			4			2
Internal trade		1		10		
External trade S: 2)3)		2			3	

Stock			1			9
External trade A: 1)3)	1			4		
Internal trade	1				10	

Exercise1. From the stock $3)1)$ we sell $S: 2)3)$ and add $A: 2)4)$. Count in fives. First use matches, then write.

Exercise2. From the stock $3)1)$ we sell $S: 2)3)$ and add $A: 2)4)$. Count in sevens. First use matches, then write.

Exercise3. From the stock $3)1)$ we sell $S: 2)3)$ and add $A: 2)4)$. Count in tens. First use matches, then write.

6 TWO STOCK ACCOUNTS I

Question. How can we account for selling and buying between two traders?

Answer. By ledger accounts.

Example. The TwoTrader-game: Two traders A and B throw two dices in turn, a red and a white. Red means bundles, white means ones and throwing means buying. The dices are reduced so that $4 \rightarrow 1$, $5 \rightarrow 2$ and $6 \rightarrow 3$ in case of 5-bundles. We set up two ledger accounts for each trader, one for the bundles, and one for the ones to account for the IN and OUT of the internal and external trade.

Bundle = 5 (red,white)	Fives			Ones			A	B	Fives			Ones		
	IN	OUT	TOT	IN	OUT	TOT			IN	OUT	TOT	IN	OUT	TOT
A(2,3)			4			2					4			2
Internal										1			5	
External	2			3					2				3	
Internal	1				5									
B(3,1)			7			0					1			4
Internal		1		5									1	
External		3			1				3				5	
Internal									1					
Stock			3			4					5			0

Bundle = 10 (red,white)			Tens			Ones			Tens			Ones		
IN	OUT	TOT	IN	OUT	TOT	A	B	IN	OUT	TOT	IN	OUT	TOT	
A(5,4)			4			2			8			2		
Internal									1		10			
External	5		4					5				4		
Internal														
B(4,3)			9			6			2			8		
Internal														
External		4		3				4			3			
Internal								1				10		
Stock			5			3			7			1		

Exercise1. Continue the TwoTrader-game. Count in fives. First use matches, then write.

Exercise2. Continue the TwoTrader-game. Count in tens. First use matches, then write.

Exercise3. Repeat the TwoTrader-game. Count in sixes. First use matches, then write.

7 TWO STOCK ACCOUNTS II

Question. How can we simplify the account for the selling and buying between two traders?

Answer. By using many-digit numbers instead of one-digit numbers, and carrying instead of internal trade.

Example. The same example as in 6.

Bundle = 5 (red,white)			TOT			A			B			TOT		
IN	OUT	TOT	IN	OUT	TOT	IN	OUT	TOT	IN	OUT	TOT	IN	OUT	TOT
A(2,3)			12						32					
Internal									5	10				
External	23									23				
Internal	10			5										
B(3,1)			40						4					
Internal														
External		5		10					31					
Internal									10	5				
Stock			4						40					
Bundle = 10 (red,white)			TOT			A			B			TOT		
A(5,4)			42						82					
Internal									10	10				
External	54									54				
Internal														
B(4,3)			96						28					
Internal														
External		43							43					
Internal									10	10				
Stock			53						71					

Exercise1. Continue the TwoTrader-game. Count in fives. First use matches, then write.

Exercise2. Continue the TwoTrader-game. Count in tens. First use matches, then write.

Exercise3. Repeat the TwoTrader-game. Count in sixes. First use matches, then write.

8 TWO STOCK ACCOUNTS III

Question. What is the simplest way to account for the selling and buying between two traders?

Answer. By using number signs instead of ledger accounts, i.e. by replacing the medieval 3|5 by 3-5.

Example. The same example as in 6.

Bundle = 5 (red,white)			TOTAL T			A			B			TOTAL T		
CHANGE ΔT	TOTAL T	TOTAL T	CHANGE ΔT	TOTAL T	TOTAL T	CHANGE ΔT	TOTAL T	TOTAL T	CHANGE ΔT	TOTAL T	TOTAL T	CHANGE ΔT	TOTAL T	
	42	42		42	42		42	42		42	42		42	
A(2,3)	+23	70	-23	47	70	-23	24	14		14	14		14	
B(3,1)	-31	34	+31	65	34	+31	96	50		50	50		50	
Bundle = 10 (red,white)			TOTAL T			A			B			TOTAL T		
	42	42		42	42		42	42		42	42		42	
A(5,4)	+54	96	-54	42	96	-54	38	28		28	28		28	
B(4,3)	-43	53	+43	96	53	+43	140	71		71	71		71	

Exercise1. Continue the TwoTrader-game. Count in fives. First use matches, then write.

Exercise2. Continue the TwoTrader-game. Count in tens. First use matches, then write.

Exercise3. Repeat the TwoTrader-game. Count in sixes. First use matches, then write.

9 STOCKS MAY CHANGE

Question. In what way can a total stock change?

Answer. Predictably by a change equation $\Delta T = +2$. Unpredictably by throwing a dice $\Delta T = 1, 2$ or 3

Example2. Subtraction

Verbal: T = 263 - 87 = 2 tenten 6 ten 3 - 8 ten 7 = 2 tenten -2 ten -4 = 2 tenten -3 ten 6 = 1 tenten 7 ten 6

<p>T = 263 - 87</p> <p>= 2*B^2 + 6*B + 3*1 - (8*B + 7*1)</p> <p>= 2*B^2 + 6*B + 3*1 - 8*B - 7*1</p> <p>= 2*B^2 + (6*B - 8*B) + (3*1 - 7*1)</p> <p>= 2*B^2 + (6 - 8)*B + (3 - 7)*1</p> <p>= 2*B^2 - 2*B - 4*1</p> <p>= (2-1+1)*B^2 - 2*B - 4*1</p> <p>= (1+1)*B^2 - 2*B - 4*1</p> <p>= 1*B^2 + 1*B^2 - 2*B - 4*1</p> <p>= 1*B^2 + 10*B - 2*B - 4*1</p> <p>= 1*B^2 + (10 - 2)*B - 4*1</p> <p>= 1*B^2 + 8*B - 4*1</p> <p>= 1*B^2 + (8-1+1)*B - 4*1</p> <p>= 1*B^2 + (7+1)*B - 4*1</p> <p>= 1*B^2 + 7*B + 1*B - 4*1</p> <p>= 1*B^2 + 7*B + 1*10 - 4*1</p> <p>= 1*B^2 + 7*B + 10*1 - 4*1</p> <p>= 1*B^2 + 7*B + (10 - 4)*1</p> <p>= 1*B^2 + 7*B + 6*1</p> <p>= 176</p>	<p><i>By restacking</i></p> <p>restacking</p> <p><u>underload!</u></p> <p>restacking</p> <p>recounting</p> <p>restacking</p> <p>restacking</p> <p>restacking</p> <p>recounting</p>	<table border="1"> <thead> <tr> <th>B^2</th> <th>B</th> <th>1</th> </tr> </thead> <tbody> <tr> <td colspan="3"><i>Borrowing:</i></td> </tr> <tr> <td>-1</td> <td>+10</td> <td>-1</td> </tr> <tr> <td>2</td> <td>6</td> <td>3</td> </tr> <tr> <td></td> <td>8</td> <td>7</td> </tr> <tr> <td>1</td> <td>7</td> <td>6</td> </tr> <tr> <td colspan="3"><i>Adding from below:</i></td> </tr> <tr> <td>2</td> <td>6</td> <td>3</td> </tr> <tr> <td></td> <td>8</td> <td>7</td> </tr> <tr> <td>1</td> <td>7</td> <td>6</td> </tr> <tr> <td>1</td> <td>1</td> <td></td> </tr> <tr> <td colspan="3"><i>Deficits:</i></td> </tr> <tr> <td>2</td> <td>6</td> <td>3</td> </tr> <tr> <td></td> <td>8</td> <td>7</td> </tr> <tr> <td>2</td> <td>-2</td> <td>-4</td> </tr> <tr> <td>2-1</td> <td>+10-2-1</td> <td>+10-4</td> </tr> <tr> <td>1</td> <td>7</td> <td>6</td> </tr> </tbody> </table>	B^2	B	1	<i>Borrowing:</i>			-1	+10	-1	2	6	3		8	7	1	7	6	<i>Adding from below:</i>			2	6	3		8	7	1	7	6	1	1		<i>Deficits:</i>			2	6	3		8	7	2	-2	-4	2-1	+10-2-1	+10-4	1	7	6
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Example3. Multiplication

Verbal: T = 8*246 = 8 * 2tenten4ten6 = 16tenten32ten48 = 16tenten36ten8 = 19tenten6ten8 = 1tententen9tenten6ten 8 = 1968

<p>T = 8*246</p> <p>= 8*(2*B^2 + 4*B + 6*1)</p> <p>= 8*2*B^2 + 8*4*B + 8*6*1</p> <p>= 16*B^2 + 32*B + 48*1</p> <p>= (16-10+10)*B^2 + (32-30+30)*B + (48-40+40)*1</p> <p>= (6+B)*B^2 + (2+3*B)*B + (8+4*B)*1</p> <p>= 6*B^2 + 1*B*B^2 + 2*B + 3*B*B + 8*1 + 4*B*1</p> <p>= 1*B^3 + 6*B^2 + 3*B^2 + 2*B + 4*B + 8*1</p> <p>= 1*B^3 + (6+ 3)*B^2 + (2 + 4)*B + 8*1</p> <p>= 1*B^3 + 9*B^2 + 6*B + 8*1</p> <p>= 1968</p> <p>T = 8*246</p> <p>= 8*(2*B^2 + 4*B + 6*1)</p> <p>= 8*2*B^2 + 8*4*B + 8*6*1</p> <p>= 16*B^2 + 32*B + 48*1</p> <p>= (16/B*B)*B^2 + (32/B*B)*B + (48/B*B)*1</p> <p>= (1*B+6)*B^2 + (3*B+2)*B + (4*B+8)*1</p> <p>= 1*B*B^2 + 6*B^2 + 3*B*B + 2*B + 4*B*1 + 8*1</p> <p>= 1*B^3 + 6*B^2 + 3*B^2 + 2*B + 4*1*B + 8*1</p> <p>= 1*B^3 + (6+ 3)*B^2 + (2 + 4)*B + 8*1</p> <p>= 1*B^3 + 9*B^2 + 6*B + 8*1</p> <p>= 1968</p>	<p><i>By restacking</i></p> <p><u>overload!</u></p> <p>restacking</p> <p>recounting</p> <p>restacking</p> <p>recounting</p> <p><u>overload!</u></p> <p>recounting</p> <p>restacking</p> <p>restacking</p>	<table border="1"> <thead> <tr> <th>B^3</th> <th>B^2</th> <th>B</th> <th>1</th> </tr> </thead> <tbody> <tr> <td colspan="4"><i>Carrying:</i></td> </tr> <tr> <td>8*</td> <td>3</td> <td>4</td> <td></td> </tr> <tr> <td>8*</td> <td>2</td> <td>4</td> <td>6</td> </tr> <tr> <td>1</td> <td>9</td> <td>6</td> <td>8</td> </tr> <tr> <td colspan="4"><i>Overloads:</i></td> </tr> <tr> <td>8*</td> <td>2</td> <td>4</td> <td>6</td> </tr> <tr> <td></td> <td>16</td> <td>32</td> <td>48</td> </tr> <tr> <td>0+1</td> <td>6+3</td> <td>2+4</td> <td>8</td> </tr> <tr> <td>1</td> <td>9</td> <td>6</td> <td>8</td> </tr> <tr> <td colspan="4"><i>The harmonica:</i></td> </tr> <tr> <td>8*</td> <td>243</td> <td>= ?</td> <td></td> </tr> <tr> <td>8*</td> <td>243</td> <td>= 1944</td> <td></td> </tr> <tr> <td></td> <td>200</td> <td>1600</td> <td></td> </tr> <tr> <td></td> <td>40</td> <td>320</td> <td></td> </tr> <tr> <td></td> <td>3</td> <td>24</td> <td></td> </tr> </tbody> </table>	B^3	B^2	B	1	<i>Carrying:</i>				8*	3	4		8*	2	4	6	1	9	6	8	<i>Overloads:</i>				8*	2	4	6		16	32	48	0+1	6+3	2+4	8	1	9	6	8	<i>The harmonica:</i>				8*	243	= ?		8*	243	= 1944			200	1600			40	320			3	24	
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Example4. Division

Verbal: T = 253/8 = 2tenten5ten3 / 8 = 25ten3 / 8 = 3ten + 1ten3 / 8 = 3ten + 13 / 8 = 3ten + 1 + 5/8 = 31 5/8

<p>T = 253</p> <p>= 2*B^2 + 5*B + 3*1</p> <p>= 2*B*B + 5*B + 3*1</p> <p>= 20*B + 5*B + 3*1</p> <p>= (20 + 5)*B + 3*1</p> <p>= 25*B + 3*1</p> <p>= 25/8*8*B + 3*1</p> <p>= (3*8 + 1)*B + 3*1</p> <p>= 3*8*B + 1*B + 3*1</p> <p>= 3*B*8 + B*1 + 3*1</p> <p>= 30*8 + (10+3)*1</p> <p>= 30*8 + (13/8*8)*1</p> <p>= 30*8 + (1*8+5)*1</p>	<p>recounting</p> <p>restacking</p> <p>recounting</p> <p>restacking</p> <p>recounting</p> <p>restacking</p> <p>recounting</p>	<p><i>Finding the remainder:</i></p> <table border="1"> <tr> <td>8</td> <td>253</td> <td>31</td> </tr> <tr> <td></td> <td>24</td> <td></td> </tr> <tr> <td></td> <td>13</td> <td></td> </tr> <tr> <td></td> <td>8</td> <td></td> </tr> <tr> <td></td> <td>5</td> <td></td> </tr> </table> <p><i>Reversed harmonica:</i></p> <table border="1"> <tr> <td>8*</td> <td>?</td> <td>= 253</td> </tr> <tr> <td>8*</td> <td></td> <td>= 253</td> </tr> <tr> <td>30</td> <td>240</td> <td></td> </tr> <tr> <td></td> <td>13</td> <td></td> </tr> </table>	8	253	31		24			13			8			5		8*	?	= 253	8*		= 253	30	240			13	
8	253	31																											
	24																												
	13																												
	8																												
	5																												
8*	?	= 253																											
8*		= 253																											
30	240																												
	13																												

$= 30*8 + 1*8*1 + 5*1$	restacking	$8^* \quad = 253$
$= 30*8 + 1*1*8 + 5*1$	restacking	$30 \quad \underline{240}$
$= (30+1)*8 + 5*1$	restacking	$1 \quad \underline{13}$
$= 31*8 + 5*1$	recounting	$\quad \underline{8}$
$= 31*8 + 5*1/8*8$	restacking	$\quad +5$
$= 31*8 + 5/8*8$		$8^* \quad 31 \quad +5 = 253$
$= (31 + 5/8)*8$		$30 \quad \underline{240}$
$= 31 \ 5/8 * 8$		$1 \quad \underline{13}$
T = 253 = 253/8*8 = 31 5/8 * 8		$\quad \underline{8}$
so 253/8 = 31 5/8		$\quad +5$

Example5. Multiplying many-digit numbers

Verbal: T = 42*68 = 4ten2*6ten8 = 24tente(32+12)ten16 = 24tente44ten 16 = 24tente45ten6 = 28tente5 ten 6 = 2 tentente8tente5ten 6 = 2856

T = 42*68		<i>Multiplying double stacks T = 42 * 68</i>
$= 42 * (6*B + 8*1)$	restacking	$2 \quad \begin{array}{ c c } \hline 60 & 8 \\ \hline 120 & 16 \\ \hline I & L \\ \hline \end{array}$
$= 42*6*B + 42*8*1$		$40 \quad \begin{array}{ c c } \hline 2400 & 320 \\ \hline F & O \\ \hline \end{array}$
$= (4*B + 2*1)*6*B + (4*B + 2*1)*8*1$	restacking	<i>Medieval or diagonal multiplication</i>
$= (4*B*6*B + 2*1*6*B) + (4*B*8*1 + 2*1*8*1)$	restacking	$2 \quad \begin{array}{ c c } \hline 6 & 8 \\ \hline 1 & 2 & 1 & 6 \\ \hline 2 & 4 & 3 & 2 \\ \hline \end{array}$
$= 24*B^2 + 12*B + 32*B + 16*1$	restacking	$4 \quad \begin{array}{ c c } \hline 2 & 8 & 5 & 6 \\ \hline \end{array}$
$= 24*B^2 + (12 + 32)*B + 16*1$	restacking	<i>Multiplying using carrying</i>
$= 24*B^2 + 44*B + 16*1$	restacking	$2 \quad \begin{array}{ c c c c } \hline 6 & 8 & & \\ \hline 1 & 3 & 6 & \\ \hline 2 & 7 & 2 & 0 \\ \hline 2 & 8 & 5 & 6 \\ \hline \end{array}$
$= (24/B*B)*B^2 + (44/B*B)*B + (16/B*B)*1$	recounting	<i>Spreadsheet</i>
$= (2*B+4)*B^2 + (4*B+4)*B + (1*B+6)*1$	restacking	$40 \quad \begin{array}{ c c } \hline 60 & 8 \\ \hline 2400 & 320 \\ \hline 2 & 16 \\ \hline \end{array} \quad \begin{array}{l} 2720 \\ 136 \\ \hline 2856 \end{array}$
$= 2*B^3 + 4*B^2 + 4*B^2 + 4*B + 1*B + 6*1$	restacking	
$= 2*B^3 + (4 + 4)*B^2 + (4 + 1)*B + 6*1$	restacking	
$= 2*B^3 + 8*B^2 + 5*B + 6*1$		
$= 2856$		
<i>FOIL: First+Outside+Inside+Last</i>		
T = 42*68		
$= (40+2)*(60+8)$		
$= 2400+320+120+16$		
$= 2856$		
T = 54^2		
$= (50+4)*(50+4)$		
$= 2500 + 200 + 200 + 16$		
$= 2916$		

Example6. Dividing many-digit numbers the medieval way

T = 2856 = 42*?		$2 \quad \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
?: 4*5 = 20, 4*6 = 24, 4*7 = 28		$4 \quad \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
! : 4*6 = 24		$2 \quad \begin{array}{ c c } \hline 6 & \\ \hline 1 & 2 & 1 & 6 \\ \hline 2 & 4 & 3 & 2 \\ \hline \end{array}$
2*6 = 12		$4 \quad \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
1+4+? = 8, ? = 3		$2 \quad \begin{array}{ c c } \hline 6 & 8 \\ \hline 1 & 2 & 1 & 6 \\ \hline 2 & 4 & 3 & 2 \\ \hline \end{array}$
We choose 2 expecting to carry because of 8		$4 \quad \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
?: 4*7 = 20, 4*8 = 32		$2 \quad \begin{array}{ c c } \hline 6 & 7 \\ \hline 1 & 2 & 2 & 1 \\ \hline 2 & 4 & 2 & 8 \\ \hline \end{array}$
7 is a possibility		$4 \quad \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
?: 4*7 = 28		
! : 4*8 = 32		
Answer: D = 2856 = 42*? = 42*68		

Exercise1. Do 376+87, 376-87, 6*376, 376/5, 26*37, 326/21 (verbal calculation)

Exercise2. Do 376+87, 376-87, 6*376, 376/5, 26*37, 326/21 (horizontal calculation)

Exercise3. Do 376+87, 376-87, 6*376, 376/5, 26*37, 326/21 (vertical calculation)

10 CODED STOCKS ARE MULTIPLIED

Question. How can two double-stacks be multiplied? $(a+b)*(c+d) = ?$

Answer. By the FOIL-method: $(a+b)*(c+d) = a*c$ (First) + $a*d$ (Outside) + $b*c$ (Inside) + $b*d$ (Last)

Example1. $T = (4x+3y)*(6x+7y) = ?$ and $(4x+3y)*? = 24x^2 + 46xy + 21y^2$

Multiplying stocks horizontally

FOIL: First+Outside+Inside+Last

$$\begin{aligned} T &= (4x+3y)*(6x+7y) \\ &= 4*x*6*x + 4*x*7*y + 3*y*6*x + 3*y*7*y \\ &= 24x^2 + 28xy + 18xy + 21y^2 \\ &= 24*x^2 + 46*xy + 21*y^2 \end{aligned}$$

Multiplying special stocks

The square of a two-term sum:

$$\begin{aligned} T &= (a+b)^2 \\ &= (a+b)*(a+b) \\ &= (a*a + b*a) + (a*b + b*b) \\ &= a^2 + 2*a*b + b^2 \end{aligned}$$

Example:

$$13^2 = (10+3)*(10+3) = 10*10 + 2*3*10 + 3^2 = 169$$

The square of a two-term difference:

$$\begin{aligned} T &= (a-b)^2 \\ &= (a-b)*(a-b) \\ &= (a-b)*a - (a-b)*b \\ &= a*a - b*a - (a*b - b*b) \\ &= a^2 - a*b - a*b + b^2 \\ &= a^2 - 2*a*b + b^2 \end{aligned}$$

Example: $9^2 = (10-1)*(10-1) = 10*10 - 2*1*10 + 1^2 = 81$

Remark. We see that $+*- = -$; and $-*- = +$; and that numbers moving outside a negative parenthesis have their signs changed.

A two-term sum multiplied with their difference

$$\begin{aligned} T &= (a+b)*(a-b) \\ &= (a+b)*a - (a+b)*b \\ &= (a*a + b*a) - (a*b + b*b) \\ &= a^2 + a*b - a*b - b^2 \\ &= a^2 - b^2 \end{aligned}$$

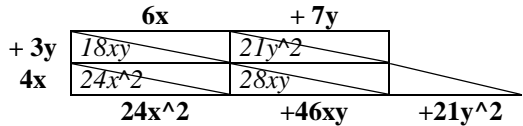
Example: $9^2 = (10-1)*(10-1) = 10*10 - 2*1*10 + 1^2 = 81$

Multiplying stocks vertically

Method 1 (stacking) $T = (4x+3y)*(6x+7y)$

	6x	7y
3y	18xy	21y ²
4x	24x ²	28xy

Method 2: Medieval calculation



	a	b
b	ab	b ²
a	a ²	ab

	a-b	b
b	ab	b ²
a-b	(a-b) ²	ab

	a	+b
b	ab	b ²
a-b	(a-b) *	(a+b)

Exercise1. Do $(4x+5y)*(6x+8y)$. Do $(4x+2y)*(9x+7y)$.

Exercise2. Do $(x+2y)^2$, $(x-2y)^2$ and $(x+2y)*(x-2y)$. Do $(3x+2y)^2$, $(3x-2y)^2$ and $(3x+2y)*(3x-2y)$.

11 ADDING WITH AN ABACUS

Question. How to add on an 1&4-abacus?

Answer. By recounting both in 2s and 5s.

An abacus is a double-cup calculation board counting e.g. both ones and fives: $8 = 1]3$, $37 = 3)1]2$

Horizontal:

			I		I	I		I	II		I	I	
		III	II	+		III	=	III	IIII	=	IIII		

$37 + 58 = 3) 1]2 + 1]0) 1]3 = 1]3) 2]5 = 1]4) 1]0 = 95$

Vertical:

37 ->	3	1	2
58 ->	1	0	1 3
		1	3 2 5
95 <-	1	4	1 0

Exercise1. Do $28+46$. Do $86-58$. Etc.

Exercise2. Do $3*46$. Do $58/4$. Do $24*36$. Etc.