Diagnozing

P

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ISA
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In MatheMatics education:

Teach ManyMatics	- a natural science about Many
Not MatheMatism	- true inside but not outside the classroom
Nor MetaMatics	- defining concepts as examples of abstractions instead of as abstractions from examples

The MATHeCADEMY.net 2015

Diagnozing Poor PISA Performance

Decreased PISA Performance in spite of Increased Research
Being highly useful to the outside world, math is a core part of education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise funding has increased witnessed by e.g. the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA levels in mathematics decrease from 509 in 2003 to 478 in 2012, far below the OECD average at 494. This made OECD write a report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Created to help students cope with the outside world, schools are divided into subjects that are described by goals and means with the outside world as the goal and the subjects as the means. However, a goal/means confusion might occur where the subjects become the goals and the outside world a means.

A goal/means confusion is problematic since while there is only one goal there are many means that can be replaced if not leading to the goal, unless an ineffective means becomes a goal itself, leading to a new discussing about which means will best lead to this false goal; thus preventing looking for alternative means that would more effectively lead to the original goal.

So we can ask: Does mathematics education build on a goal-means confusion seeing mathematics as the goal and the outside world as a means? Or, how would mathematics look like if built as a means for proper real world actions?

The three papers below constitute a 'Catania Trilogy' written for 13th International Conference of The Mathematics Education for the Future Project in Catania, Sicily, September 2015.

Conclusion: Goal/Means Confusion leads to Poor PISA Performance
Increased research has let to decreasing PISA math results as In Sweden caused by a goal/means confusion. Grounded as a means to an outside goal, mathematics becomes a natural science about the physical fact Many. This ManyMatics differs from the school's MetaMatism, mixing MetaMatics, defining its concepts as examples from internal abstractions, with MatheMatism, true inside but not outside the classroom.

Content

Count in Icons before Tens, then Add NextTo before OnTop

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Abstract

Preschool allows rethinking mathematics outside the tradition of ordinary school. Seeing schooling as adapting the child to the outside world containing many examples of Many, we can ask: How will mathematics look like if built as a natural science about physical fact Many? To deal with Many we count and add. The school counts in tens, but preschool allows counting in icons also. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. Adding next-to means adding areas, also called integration. So icon-counting and next-to addition offers golden learning opportunities in preschool that is lost in ordinary school allowing only ten-counting to take place.

Math in Preschool – a Great Idea

Mathematics is considered one of the school's most important subjects. So it seems a good idea to introduce mathematics in preschool - provided we can agree upon what we mean by mathematics.

As to its etymology Wikipedia writes that the word mathematics comes from the Greek máthēma, which, in the ancient Greek language, means "that which is learnt". Later Wikipedia writes:

> In Latin, and in English until around 1700, the term mathematics more commonly meant "astrology" (or sometimes "astronomy") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. (http://en.wikipedia.org/wiki/ Mathematics)

This meaning resonates with Freudenthal writing:

> Among Pythagoras' adepts there was a group that called themselves mathematicians, since they cultivated the four "mathemata", that is geometry, arithmetic, musical theory and astronomy. (Freudenthal 1973: 7)

Thus originally mathematics was a common word for knowledge present as separate disciplines as astronomy, music, geometry and arithmetic.

This again resonates with the educational system in the North American republics offering courses, not in mathematics, but in its separate disciplines algebra, geometry, etc.

In contrast to this, in Europe with its autocratic past the separate disciplines called Rechnung, Arithmetik und Geomtrie in German were integrated to mathematics from grade one with the arrival of the 'new math' wanting to revive the rigor of Greek geometry by defining mathematics as a collection of well-proven statements about well-defined concepts all being examples of the mother concept set.

Kline sees two golden periods, the Renaissance and the Enlightenment that both created and applied mathematics by disregarding Greek geometry:

> Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

Furthermore, Gödel has proven that the concept of being well-proven is but a dream. And Russell's set-paradox questions the set-based definitions of modern mathematics by showing that talking about sets of sets will lead to self-reference and contradiction as in the classical liar-paradox 'this sentence is false' being false if true and true if false:

If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$.

Without an agreement as to what mathematics is and with the negative effects of imposing rigor, preschool math should disintegrate into its main ingredients, algebra meaning reuniting numbers in Arabic, and geometry meaning measuring earth in Greek; and both should be grounded in their common root, the physical fact Many. To see how, we turn to skeptical research.

Postmodern Contingency Research

Ancient Greece saw a knowledge controversy between the sophists and the philosophers. The sophists warned that in a republic people must be enlightened about choice and nature to prevent being patronized by choices presented as nature. In contrast to this skepticism philosophers saw the physical as examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should be allowed to patronize.

Enlightenment later had its own century, the 18th, that created two republics, an American and a French. Today the sophist warning against hidden patronization is kept alive in the French republic in the postmodern skeptical thinking of Derrida, Lyotard, Foucault and Bourdieu warning against patronizing categories, discourses, institutions and education presenting their choices as nature (Tarp 2004).

Thus postmodern skeptical research discovers contingency, i.e. hidden alternatives to choices presented as nature. To make categories, discourses and institutions non patronizing they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment republic, the American; and resonating with Piaget's principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

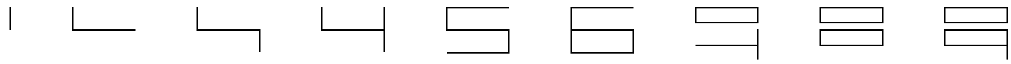
To look for patronization hidden in the words, truths and discourses of math education we ask: How will mathematics look like if built, not as a self-referring science about its own invention Set, but as a natural science about the physical fact Many; and how can this affect early childhood education?

The answers will be presented in papers and in YouTube videos (2013).

Building a Natural Science about Many

To deal with the physical fact Many, first we iconize, then we count by bundling. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in e.g. fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc..

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII

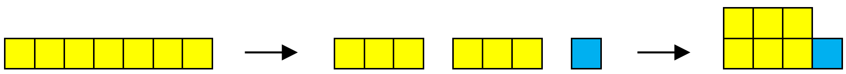


1 2 3 4 5 6 7 8 9

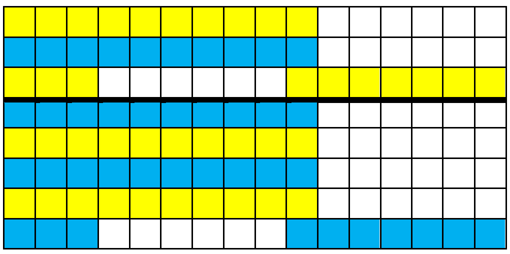
With ‘second order counting’ we bundle a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. The unbundled can be placed in a right single-cup, and in a left bundle-cup we trade the bundles, first with a thick stick representing a bundle glued together, then with a normal stick representing the bundle. The cup-contents is described by icons, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Alternatively, we can also use plastic letters as B, C or D for the bundles.

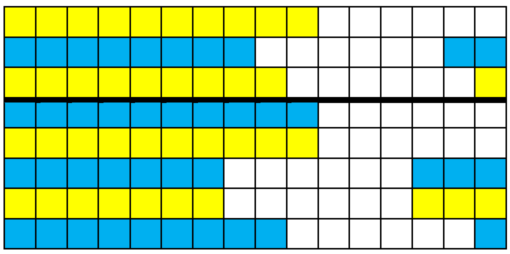
IIIIIII → III III I → ▌▌) I) → II) I) → 2)1) → 2.1 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a decimal number.



We live in space and in time. To include both when counting, we can introduce two different ways of counting: counting in space, geometry-counting, and counting in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.





To predict the counting result we can use a calculator. Building a stack of 2 3s is iconized as 2x3 showing a jack used 2 times to lift the 3s. As to the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once. Thus by entering ‘7/3’ we ask the calculator ‘from 7 we can take away 3s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 3s by asking ‘7 – 2x3’. From the answer ‘1’ we conclude that 7 = 2.1 3s. Showing ‘7 – 2x3 = 1’, a display indirectly predicts that 7 can be recounted as 2 3s and 1.

7 / 3	2.some
7 – 2 x 3	1

Re-counting in the Same Unit and in a Different Unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3.2 2s, 2.4 2s. Likewise 4.2s can be recounted as 5 2s less or short of 2; or as 6 2s less 4 thus leading to negative numbers:

Letters	Sticks	Calculator		T =
B B B B	ǂ ǂ ǂ ǂ			4.0 2s
B B B I I	ǂ ǂ ǂ I I	4x2 – 3x2	2	3.2 2s
B B I I I I	ǂ ǂ I I I I	4x2 – 2x2	4	2.4 2s
B B B B B	ǂ ǂ ǂ ǂ ǂ	4x2 – 5x2	-2	5.2 2s
B B B B B B	ǂ ǂ ǂ ǂ ǂǂ	4x2 – 6x2	-4	6.4 2s

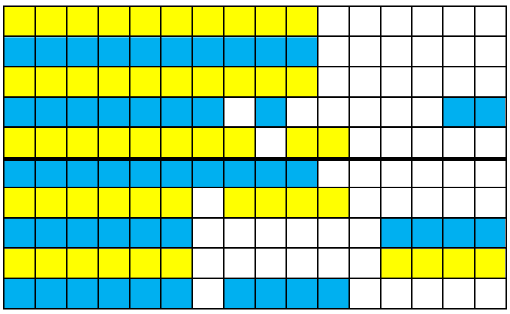
To recount in a different unit means changing unit, called proportionality or linearity also. Asking '3 4s is how many 5s?' we can use sticks or letters to see that 3 4s becomes 2.2 5s.

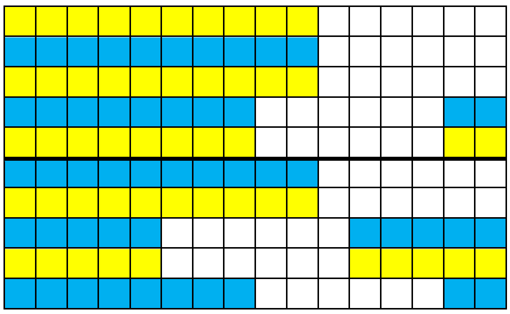
IIII IIII IIII → IIIII IIIII I I → 2) 2) 5s → 2.2 5s

or with C = BI, BBB → BBIIII → CCII

Using geometry-counting on an abacus, reserving the bottom line for the single 1s, a stack of 3 4s is moved from left to right on an abacus. The top bundle is changed to 1s in the single line and twice a stick is removed to enlarge the two 4-bundles to 5-bundles. This shows that '3 4s can be recounted as 2.2 5s.'

Using algebra-counting, 3 beads are moved to the right on the bundle-line. Then one 4-bundle is changed to 4 1s on the single-line. Moving 2 beads to the left on the single-line allows enlarging the 4s to 5s thus showing that 3 4s = 2.2 5s





Using a calculator to predict the result we enter '3x4/5' to ask 'from 3 4s we take away 5s how many times?' The calculator gives the answer '2.some'. To find the leftovers we take away the 2 5s and ask '3x4 – 2x5'. Receiving the answer '2' we conclude that T = 3 4s = 2.2 5s.

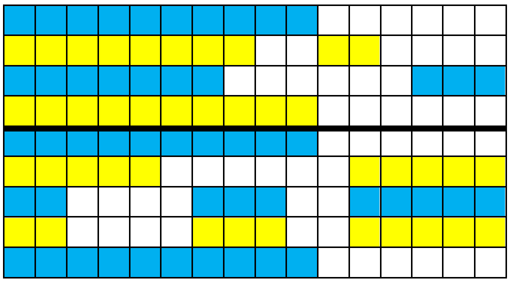
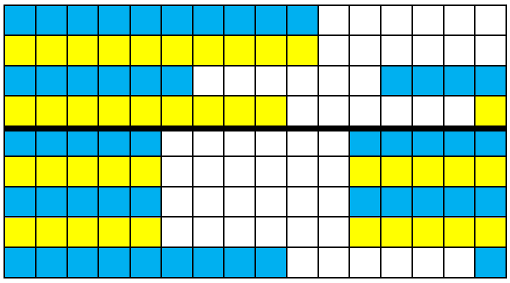
3 x 4 / 5	2.some
3 x 4 – 2 x 5	2

Adding On-top and Next-to

Once counted, totals can be added on-top or next-to. Asking ‘3 5s and 2 3s total how many 5s?’ we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s giving 1.1 s that added to the 3 5s gives a grand total of 4.1 5s. With letters: 3B + 2C = 3B III III = 4BI. With sticks:

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4) 1) 5s → 4.1 5s,

On an abacus in geometry mode a stack of 3 5s is moved to the right and a stack of 2 3s is moved to the middle. Now, the 2 3s is changed to 6 1s on the bottom line allowing one additional 5s to be moved to the top of the stack of 5s to show the grand total is 4.1 5s. Using algebra mode, the 3 5s become 3 beads on the bundle line and the 2 3s become 2 beads on the line above. Again the 2 3s is changed to 6 1s on the bottom line allowing one additional bead to be added to the bundle-line to give the result 4.1 5s

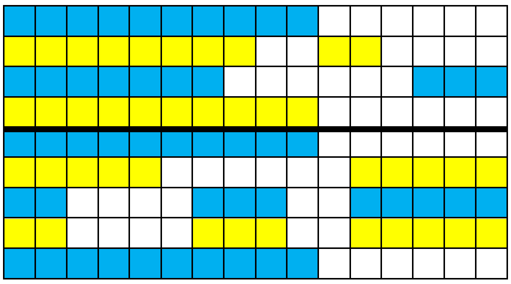
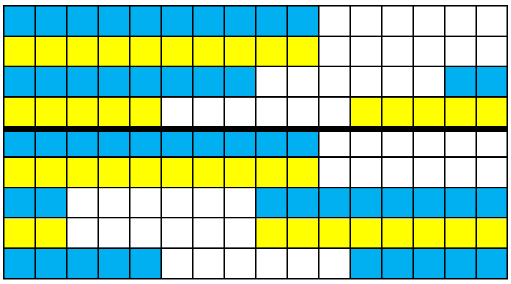
Using a calculator to predict the result we use a bracket before counting in 5s: Asking ‘(3x5 + 2x3)/5’, the answer is 4.some. Taking away 4 5s leaves 1.

(3 x 5 + 2 x 3)/ 5	4.some
(3 x 5 + 2 x 3) – 4 x 5	1

To add next-to means adding areas called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks or letters to see that the answer is 2.5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2) 5) 8s → 2.5 8s

On an abacus in geometry mode a stack of 3 5s is moved to the right and a stack of 2 3s is moved to the middle. Now a 5-bundle is moved to the single line allowing the two stacks to be integrated as 8s, showing that the grand total is 2.5 8s. Likewise when using algebra mode.

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking ‘(3x5 + 2x3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2.5 8s.

(3 x 5 + 2 x 3)/ 8	2.some
(4 x 5 + 2 x 3) – 2 x 8	5

Reversing Adding On-top and Next-to

To reverse addition is called backward calculation or solving equations also. To reverse next-to addition is called reversed integration or differentiation. Asking '3 5s and how many 3s total 2.5 8s?' sticks will get the answer 2 3s:

IIIII IIIII IIIII III III ← IIIII III IIIII III IIIII ← 2) 5) 8s ← 2.5 8s

On an abacus in geometry mode with 2 8s and 5 moved to the right, first 3 5s is moved to the left, then the remaining is recounted in 3s as 2 3s. Using a calculator to predict the result the remaining is bracketed before counted in 3s.

(2 x 8 + 5 – 3 x 5)/ 3	2
(2 x 8 + 5 – 3 x 5) – 2 x 3	0

Adding the two stacks 2 3s and 3 5s next-to each other means performing multiplication before adding. Reversing integration means performing subtraction before division, as in the gradient formula y' = dy/t = (y2 – y1)/t.

Conclusion

To find how mathematics looks like if built as a natural science about Many, and how this could affect early childhood education, postmodern contingency research has uncovered a 'ManyMatics' as a hidden alternative to the ruling tradition in mathematics. Dealing with Many means bundling and counting in icons, and recounting when adding on-top or next-to thus introducing proportionality and calculus. Likewise reversing on-top or next-to addition leads to solving equations and differentiation. That totals must be counted before being added means introducing the operations division, multiplication, subtraction before addition. These golden learning opportunities must be realized in preschool since they are lost from grade one, where the monopoly of ten-counting and the opposite order of operations prevent both from happening. Furthermore, here grounded ManyMatics is replaced by 'MetaMatism', a mixture of 'MetaMatics' turning mathematics upside down by presenting concepts as examples of abstractions instead of as abstractions from examples, and 'MatheMatism' true inside a classroom but not outside where claims as '1+2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

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Truth, Beauty and Goodness in Mathematics Education

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Abstract
In math education we can ask how education can lead to mathematics. But we could also ask how mathematics could lead to general educational goals as the three classical virtues: Truth, Beauty and Goodness. To do so, math must change from a self-referring MetaMatism true inside but not outside the classroom to a grounded ManyMatics, a natural science about Many, with numbers as blocks and with algebra as the art of reuniting numbers.

Goals and Means in Mathematics Education

Mathematics education is a core part of a school and is described by goals and means. Typically, mathematics is the goal with assessment focusing on the degree to which it has been learned. As means, different kinds of education are considered: Should the main emphasis be on teaching with high quality in teacher training and textbooks? Or should the main emphasis be on learning with focus on constructivism be it social or radical?

Once a means has been chosen education can begin, hopefully resulting in leading to the goals. However, PISA studies show that student performances are decreasing e.g. in the former model country Sweden seeing its mathematics levels decrease from 509 in 2003 to 478 in 2012 far below the OECD average at 494. This made OECD write a report describing the Swedish school system as being in need of urgent change (OECD, 2015).

Increased funding of mathematics education research in the period seems to have made the situation even worse. So to change the situation, unorthodox methods must be used by e.g. turning the goal and means discussion around and ask: How can mathematics contribute to general educational goals?

As to general educational goals Howard Gardner, known for his theory on MI, multiple intelligences, writes

> In my book *The Disciplined Mind*, published in 1999, I put forth a simple educational agenda: To help students understand, and act, on the basis of what is true, what is beautiful and what is good. I believed – and still believe – in that agenda. (Gardner 2001, xiv)

From this we can ask: how can mathematics be a means leading to the goal of implementing the three classical virtues Truth, Beauty and Goodness?

Truth in Mathematics

As to mathematics, its strength comes from including only well-defined concepts and well-proven statement, and from being highly applicable to the outside world. However, the declining PISA performance in many countries leads to ask: Is it mathe-matics that is taught or ‘meta-matism’, a mixture of ‘meta-matics’ and ‘mathe-matism’?

MetaMatics is mathematics that uses self-reference to define its concepts top-down as examples of abstractions instead of using its historically roots to define its concepts bottom-up as abstractions from examples. Originally Euler defined a function as a common name for calculations containing numbers and letters. The invention of the abstraction Set turned this upside down so that today a function is defined as an example of a many-to-one set relation.

MatheMatism is mathematics that is true inside but not necessarily outside the classroom. Thus the statement ‘2+3 = 5’ is not true with different units, e.g. 2 weeks + 3 days = 17 days. The statement ‘2x3 = 6’ is always true since here 3 is the unit. Likewise with fractions where 1 empty bottle of 2 added to 2 empty bottles of 3 totals 3 empty bottles of 5 and not 7 empty bottles of 6. So to teach mathematics instead of mathematism we must always include the units as shown when writing out numbers fully: 345 = 3x100 + 4x10 + 5x1.

In the questionnaire below, teacher-answers marked with xs differ from the correct answers marked with dots. This and textbook definitions of functions show that what schools teach is indeed metamatism, not mathematics.

This is true	Always	Never	Sometimes
2+3 = 5	x		●
2x3 = 6	x ●		
1/2+2/3 = 3/5		x	●
1/2+2/3 = 7/6	x		●

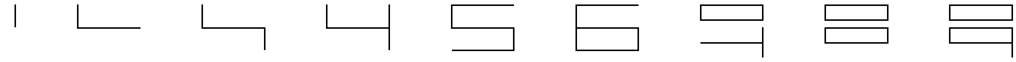
To bring back truth to mathematics it must be rebuilt from its original roots.

Building a Natural Science about Many

The core of mathematics is geometry and algebra, meaning to measure earth in Greek and to reunite numbers in Arabic. This shows that the root of mathematics is the physical fact ‘Many’ as it occurs in space and time.

To deal with Many, first we iconize, then we count by bundling. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. We create icons until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle one, one bundle two etc..

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



1 2 3 4 5 6 7 8 9

With ‘second order counting’ we bundle a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. So we place 2 sticks in a left bundle-cup and the unbundled we place in a right single-cup.

Writing the total in ‘algebra-form’, the cup-content is described by an icon, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T= 2.1 3s.

Alternatively, we can use plastic letters as B, C or D for the bundles.

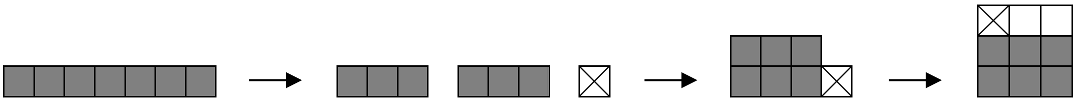
IIIIIII → III III I → II) I) → 2)1) → 2.1 3s or BBI → 2BI

A calculator can predict the counting result. To count in 3s we take away 3s, iconized as '/3' showing the broom wiping away the 3s several times. Building a stack of 2 3s we iconize as 2x3 showing a jack used to lift the 3s. And the trace coming from taking away the stack of 2 3s to look for unbundled is iconized as '-2x3'. These three operations are called division, multiplication and subtraction respectively.

Entering '7/3' the answer is '2.some'. To find the unbundled we take away the 2 3s by asking '7 – 2x3'. From the answer '1' we conclude that 7 = 2.1 3s.

7 / 3	2.some
7 – 2 x 3	1

Writing the total in 'geometry-form we use squares or LEGO blocks or an abacus to stack the 2 3-bundles on-top of each other with an additional stack of unbundled 1s next-to or on-top, thus describing the total as a decimal number 2.1 3s, or as a fraction number 2 1/3 3s counting the unbundled in 3s.



T = 7 = 2 3s & 1 = 2.1 3s = 2 1/3 3s

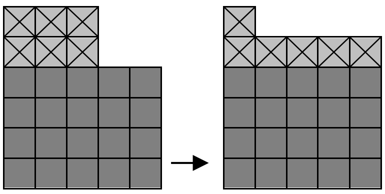
DoubleCounting creates PerNumbers Connecting Units

A physical quantity can be counted in different units. With 4kg = 5$ we have the 'per-number' 4kg/5$ = 4/5 kg/$. To shift from one physical unit to another we simply use the per-number to recount in a new number unit. To change unit is called proportionality, which is one of the core concepts in mathematics.

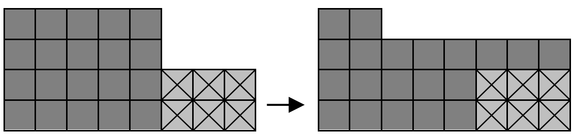
7 kg = ? $	8 $ = ? kg
7 kg = (7/4) x 4 kg = (7/4) x 5 $ = 8.75 $	8 $ = (8/5) x 5 $ = (8/5) x 4 kg = 6.4 kg

Adding Totals

Once Counted, totals can be added on-top or next-to. To add on-top, the units must be changed to be the same, typically by recounting one total in the other's unit. Adding next-to is called integrating areas.



OnTop: 4 5s + 2 3s = 5.1 5s



Next-to: 4 5s + 2 3s = 3.2 8s

NextTo addition is used when adding piecewise constant per-numbers:

4 kg at 5 $/kg + 2 kg at 3 $/kg = (4x5 + 2x3) $ = Σ (per-number x quantity)

Or when adding locally constant (continuous) per-numbers:

6 kg at 5 $/kg decreasing to 3 $/kg = $\int_0^6 (5 + \frac{3-5}{6} u)\, du$

Reversing Addition, or Solving Equations

Reversing addition we ask e.g. ‘2+? = 8’. With the restack-formula **T= (T-b)+b** we can restack 8 as (8-2)+2 to get the answer 8-2. Reversing multiplication we ask e.g. ‘2x? = 8’. With the re-count formula **T= (T/b)xb** we can recount 8 as (8/2)x2 to get the answer 8/2. We see that solving equations means moving numbers to the opposite side with opposite sign.

OnTop		NextTo
2 + ? = 8 = (8-2) + 2	**2 x ? = 8** = (8/2) x 2	**2 3s + ? 5s = 3 8s**
? = 8-2	? = 8/2	? = (3 8s – 2 3s)/5

Reversing adding next-to we ask e.g.‘2 3s + ? 5s = 3 8s’. To find what was added we take away the 2 3s and count the rest in 5s. Combining subtraction and division in this way is called reversed integration or differentiation.

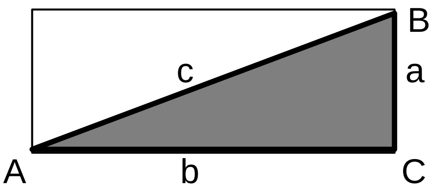
The Algebra Project: the Four Ways to Add

Meaning ‘to re-unite’ in Arabic, the ‘Algebra-square’ shows that with variable and constant unit-numbers and per-numbers there are four ways to unite numbers into a total, all present when writing 345 as 3x10^2 + 4x10 + 5x1; and that there are five ways to split up a united total.

Uniting/*splitting*	Variable	Constant
Unit-numbers	**T = a + n**, *T – a = n*	**T = a x n**, *T/n = a*
Per-numbers	**T = ∫ a dn**, *dT/dn = a*	**T = a^n**, *$\log_a(T) = n$*, *$\sqrt[n]{T} = a$*

Geometry: Measuring Earth divided into HalfBlocks

Geometry means earth-measuring in Greek. The earth can be divided in triangles that can be divided in right triangles that can be seen as blocks halved by their diagonals thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras theorem a^2 + b^2 = c^2; and connected with the angles by formulas recounting the sides in diagonals:

a = (a/c) x c = sin A x c b = (b/c) x c = cos A x b *a = (a/b) x b = tan A x b*	

Different answers to the same Questions

Asking the same questions Q, ‘ManyMatics’ and ‘MetaMatics’ gives different answers A1 and A2

Q: Digits? A1: ‘Icons, different from letters’. A2: ‘Symbols like letters’.

Q: Count? A1: ‘Count in icons before in tens’. A2: ‘Only count in tens’.

Q: Natural numbers? A1: ‘2.3 tens’. A2: ‘23’.

Q: Fractions? A1: ’Per-numbers needing a number to produce a number’. A2: ‘Rational numbers’.

Q: Per-numbers? A1: ‘Double-counting’. A2: ‘Not accepted’.

Q: Operations? A1: ‘Icons for the counting processes’. A2: ‘Mappings from a set-product to a set’.

Q: Order of operations? A1: ‘/, x, - , +’. A2: ‘+, - , x, /’.

Q: Addition? A1: ‘On-top and next-to’. A2: ‘On-top only’.

Q: Integration? A1: ‘Preschool: Next-to addition; Middle school: Adding piece-wise constant per-numbers. High school: Adding locally constant per-numbers’. A2: ‘Last year in high school, only for the few’.

Q: A formula? A1: ‘A stand-by calculation with numbers and letters’. A2: ‘An example of a function that is an example of a relation in a set-product where first component identity implies second component identity’.

Q: Algebra? A1: ‘Re-unite constant and variable unit-numbers and per-numbers’. A2: ‘A search for patterns’.

Q: The root of Mathematics? A1: ‘The physical fact Many’. A2: ‘The metaphysical invention Set’.

Q: Concepts? A1: ‘Abstraction from examples’. A2: ‘’Example of abstractions’.

Q: An equation? A1: ‘A reversed operation’. A2: ‘An example of an equivalence relation between two number-names’.

Can Education be Different?

From secondary school, continental Europe uses line-organized education with forced classes and forced schedules making teenagers stay together in age-groups even if girls are two years ahead in mental development.

The classroom belongs to the class. This forces teachers to change room and to teach several subjects outside their training in lower secondary school.

Tertiary education is also line-organized preparing for offices in the public or private sector. This makes it difficult to change line in the case of unemployment, and it forces the youth to stay in education until close to 30 making reproduction fall to 1.5 child per family so the European population will be reduced to 10% in 200 years.

Alternatively, North America uses block-organized education saying to teenagers: “Welcome, you carry a talent! Together we will uncover and develop your personal talent through daily lessons in self-chosen half-year blocks.” If successful the school will say ‘good job, you have a talent, you need more’. If not, the school will say ‘good try, you have courage, now try something else’. The classroom belongs to the teacher teaching only one subject and helped by daily lessons to adapt quickly to learner differences.

Likewise, college is block-organized to be tested already in high school and easy to supplement with additional blocks in the case of unemployment.

At the age of 25, most students have an education, a job and a family with three children to secure reproduction: one for mother, one for father and one for the state.

So different education forms might not all lead to Truth, Beauty and Goodness.

Conclusion: Blocks in Mathematics Education, Please

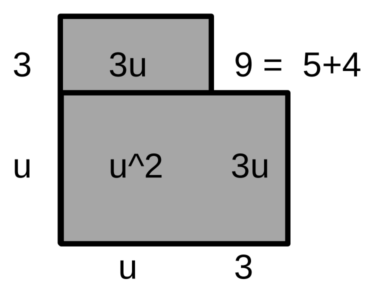
We asked: How can mathematics be a means leading to the goal of implementing the three classical virtues Truth, Beauty and Goodness? The answer is very simple: Blocks in mathematics and in education, please.

Blocks will bring Truth and Goodness back if mathematics will

- respect the nature of numbers as integrated blocks
- replace self-referring meta-matics and falsified mathe-matism with grounded many-matics presenting mathematics as a natural science about the physical fact Many
- make geometry grounded in blocks halved by their diagonals
- bring back algebra to its original Arabic meaning: to reunite constant and variable unit-numbers and per-numbers.

Blocks will bring Truth and Goodness to education that uncovers and develops a teenager's individual talent through daily lessons in self-chosen half-years blocks made possible when replacing line- with block-organization.

Blocks will bring Beauty to the streets with Block-Art posters showing how algebra and geometry work nicely together:



2 cards solve quadratic equations

u^2 + 6u + 5 = 0

(u + 3)^2	= u^2 + 6u + 5 + 4 = 0 + 4 = 4
u + 3	= ± 2
u	= -3 ± 2
u	= -1 and u = -5

Recommendation: Mathematics, Unmask Yourself, Please

Mathematics, in Greek you mean 'knowledge' and you were chosen as a common label for 4 activities: Music, Astronomy, Geometry & Arithmetic (Freudenthal 1973: 7). Today only 2 activities remain: Geometry and Algebra. Then Set transformed you from a natural science about the physical fact Many to a self-referring metaphysical subject, MetaMatism, combining MetaMatics and MatheMatism (MrAlTarp YouTube videos 2013).

So please, unmask your true identity, and tell us how you would like to be presented in education: MetaMatism for the few - or ManyMatics for the many.

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PerNumbers replace Proportionality, Fractions & Calculus

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Abstract
Increased research can lead to decreasing PISA math results as In Sweden. A goal/means confusion might be the cause. Grounded as a means to an outside goal, mathematics becomes a natural science about the physical fact Many. This ManyMatics differs from the school's MetaMatism, mixing MetaMatics, defining its concepts as examples from internal abstractions, with MatheMatism, true inside but not outside the class. Replacing proportionality, fractions and calculus with per-numbers will change math from goal to means.

Decreasing PISA Performance, a Result of a Goal/Means Confusion?

Being highly useful to the outside world has made mathematics a core of education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise funding has increased witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA level in mathematics decrease from 509 in 2003 to 478 in 2012, far below the OECD average at 494. This has made OECD write a report describing the Swedish school system as being 'in need of urgent change' (OECD, 2015).

Created to enable students cope with the outside world, schools consist of subjects that are described by goals and means with the outside world as the goal and the subjects as the means. However, a goal/means confusion might occur where the subjects become the goals and the outside world a means.

A goal/means confusion is problematic since while there is only one goal there are many means that can be replaced if not leading to the goal, unless an ineffective means becomes a goal itself, leading to a new discussing about which means will best lead to this false goal; thus preventing looking for alternative means that would more effectively lead to the original goal.

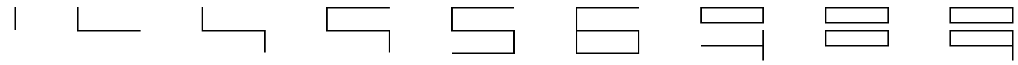
So we can ask: Does mathematics education build on a goal-means confusion seeing mathematics as the goal and the outside world as a means? For a grounded answer (Glaser 1967) we reformulate the question: How will mathematics look like if built as a means for proper real world actions?

Mathematics is not an action word itself, but so are its two main activities, geometry and algebra, meaning to measure earth in Greek, and to reunite numbers in Arabic. Thus mathematics is an answer to the two basic questions of mankind: How to divide the earth we live on, and the many goods it produces? (Tarp 2012). So what we really ask is: Which actions will enable us to deal with the physical fact Many as it exists in space and in time?

Mathematics as a Natural Science about Many

To deal with Many we count and add. To count we stack icon-bundles. To iconize five we bundle five ones to one fives to be rearranged as one five-icon 5 with five sticks if written in a less sloppy way. We create icons until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle one, one bundle two etc.

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



1 2 3 4 5 6 7 8 9

With Icons we count by bundling a total in icon-bundles. Thus a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. Now we place two sticks in a left bundle-cup and one stick in a right single-cup to write the total in ‘algebra-form’. Here the cup-content is described by an icon, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T= 2.1 3s.

Alternatively, we can use the plastic letters, B for a bundle and C for a bundle of bundles.

IIIIIII → III III I → II) I) → 2)1) → 2.1 3s or BBI → 2BI

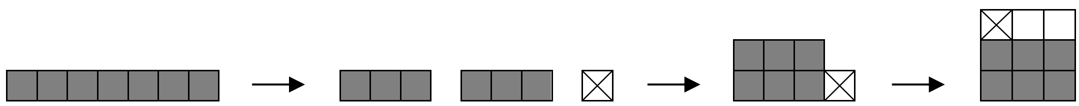
A calculator can predict the counting result. To count in 3s we take away 3s, iconized as ‘/3’ showing the broom wiping away the 3s several times. Building a stack of 2 3s we iconize as ‘2x3’ showing a jack used to lift the 3s. And ‘-2x3’ iconizes the trace coming from taking away 2 3s to look for unbundled. These three operations are called division, multiplication and subtraction respectively.

Entering ‘7/3’ the answer is ‘2.some’. To find the unbundled we take away the 2 3s by asking ‘7 – 2x3’. From the answer ‘1’ we conclude that 7 = 2.1 3s.

7 / 3	2.some
7 – 2 x 3	1

Thus a total T is counted in 3s by taking away 3 T/3 times. This can be written as a ‘re-count formula’ T = (T/3)x3 or as T = (T/b)xb if re-counting T in bs. Taking away a stack b to be placed next-to the unbundled T-b can be written as a ‘re-stack formula’ T= (T-b)+b.

To write the total in ‘geometry-form’ we use squares or LEGO blocks or an abacus to stack the 2 3-bundles on-top of each other with an extra stack of unbundled 1s next-to or on-top, thus describing the total as a decimal number 2.1 3s, or as a fraction number 2 1/3 3s counting the unbundled 1 in 3s.



T = 7 = 2 3s & 1 = 2.1 3s = 2 1/3 3s

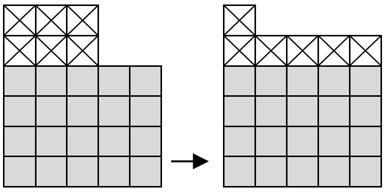
DoubleCounting creates PerNumbers Bridging Units

A physical quantity can be counted in different units, e.g. as 4kg or as 5$. This creates the ‘per-number’ 4kg/5$ = 4/5 kg/$. To shift from one unit to another we simply recount in the part of the per-number that has the same unit:

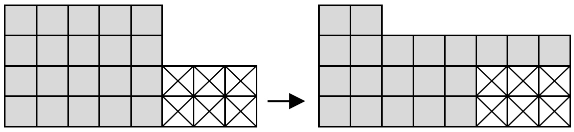
7 kg = ? $	8 $ = ? kg
7 kg = (7/4) x 4 kg = (7/4) x 5 $ = 8.75 $	8 $ = (8/5) x 5 $ = (8/5) x 4 kg = 6.4 kg

Adding Totals

Once Counted, totals can be added on-top or next-to. To add on-top, the units must be changed to be the same, typically by recounting one total in the other total’s unit. Adding next-to is called integrating areas.



OnTop: 4 5s + 2 3s = 5.1 5s



Next-to: 4 5s + 2 3s = 3.2 8s

Next-to addition is also used when adding piecewise constant per-numbers:

4 kg at 5 $/kg + 2 kg at 3 $/kg = (4x5 + 2x3) $ = Σ (per-number x quantity)

Or when adding locally constant (continuous) per-numbers:

6 kg at 5 $/kg decreasing to 3 $/kg = $\int_0^6 (5 + \frac{3-5}{6} u)\, du$

Global, piecewise and local constancy all express the fact that y is a constant k if the distance between the two can be made arbitrarily small:

- y is globally constant k if V ε > 0 : I y – k I < ε.
- y is piecewise constant k_C if Э C so V ε > 0 : I y – k_C I < ε inside C.
- y is locally constant yo if V ε > 0 Э C : I y – yo I < ε inside C.

Reversing Addition, or Solving Equations

Reversing addition, we ask e.g. ‘2+? = 8’. Restacking 8 as (8-2)+2 we get the answer 8-2. Reversing multiplication, we ask e.g. ‘2x? = 8’. Recounting 8 in 2s as (8/2)x2 we get the answer 8/2. We see that solving equations means moving numbers to the opposite side with the opposite sign.

OnTop		NextTo
2 + ? = 8 = (8-2) + 2	**2 x ? = 8** = (8/2) x 2	**2 3s + ? 5s = 3 8s**
? = 8-2	? = 8/2	? = (3 8s – 2 3s)/5

Reversing adding next-to, we ask e.g. ‘2 3s + ? 5s = 3 8s’. To find what was added we take away the 2 3s and count the rest in 5s. Combining subtraction and division in this way is called reversed integration or differentiation.

The Algebra Project: the Four Ways to Add

Meaning ‘to re-unite’ in Arabic, the ‘Algebra-square’ shows that with variable and constant unit-numbers and per-numbers there are four ways to unite numbers into a total and five ways to split a total: addition/subtraction unites/splits-into variable unit-numbers, multiplication/division unites/splits-into constant unit-numbers, power/root&log unites/splits-into constant per-numbers and integration/differentiation unites/slits into variable per-numbers.

Uniting/*splitting*	Variable	Constant
Unit-numbers	**T = a + n**, *T – a = n*	**T = a x n**, *T/n = a*
Per-numbers	**T = ∫ a dn**, *dT/dn = a*	**T = a^n**, *log*a*(T) = n, n√T = a*

School only counts in tens writing 2.3 tens as 23 thus leaving out the unit and misplacing the decimal point. So icon-counting must take place in preschool.

Writing 345 as 3x10^2 + 4x10 + 5x1, i.e. as areas placed next-to each other, again shows that there are four ways to unite, and that all numbers have units.

ManyMatics versus MatheMatism and MetaMatics

Built as a natural science about the physical fact Many, mathematics becomes ManyMatics dealing with Many by counting and adding as shown by the Algebra-square and in accordance with the Arabic meaning of algebra.

With counting and adding Many as outside goal, a proper means would teach icon-counting and on-top and next-to addition in grade one. However, only ten-counting occurs. And addition takes place without including units claiming that 2+3 IS 5 in spite of counterexamples as 2 weeks + 3 days = 17 days. So what is taught in primary school is not ManyMatics leading to proper actions to deal with Many, but what could be called 'MatheMatism' true inside but not outside a class thus making itself a goal not caring about outside world falsifications.

A counting result can be predicted by a re-count and a re-stack formula. So formulas as means to real world number-prediction should be a core subject in secondary school. However, here a formula is presented as an example of a function, again being an example of a set-relation where first-component identity implies second-component identity. So what is taught in primary school is not ManyMatics leading to the ability to predict numbers, but what could be called 'MetaMatics' presenting concepts from the inside as examples of abstractions instead of from the outside as abstractions from real world examples; and becoming 'MetaMatism' when mixed with MatheMatism.

So yes, a goal-means confusion exits in mathematics education seeing MetaMatsm as the goal and real world as applications and means; and claiming that 'of course mathematics must be learned before it can be applied'. To lift this confusion the outside world must again be the goal and ManyMatics the means. Testing examples will show if this can turnaround the PISA-results.

Proportionality or Linearity

Linearity is a core concept in mathematics, defined by MetaMatism as a function f obeying the criterion f(x+y) = f(x)*f(y). The function f(x) = a*x is linear since f(x+y) = a*(x+y) = a*x + a*y = f(x) + f(y). This 'proportionality function' is applied to the outside world when solving a '3&4&5-problem': 'If 3 kg cost 4 $ then 5 kg cost ? $'. Asking '5 kg = ? $' shows that the '3&4&5-problem' is an example of a more general 'change-unit problem' as e.g. '5 £ = ? $'.

Historically, the outside goal 'to change-units' has created different means. The Middle Ages taught 'Regula Detri', the rule of three: The middle number is multiplied with the last number and then divided by the first number.

The industrial age introduced a two-step rule: First go to the unit by dividing 4 by 3, then multiply by 5. Having learned how to solve equations in secondary school, a proportion can be set up equalizing two ratios: 3/4 = 5/u. Now cross-multiplication leads to the equation 3xu = 4x5 with the solution u = 4x5/3.

As shown above, the per-number 3$/4kg offers a fifth alternative finding the answer by recounting 5 in 3s: T = 5$ = (5/3)x3 $ = (5/3)x4 kg.

So the action ‘to change unit’ can be attained by five different means, all to be part of teacher education in order to create a turnaround in the PISA results.

Fractions

Defining everything as examples of sets, MetaMatism sees fractions as what is called ‘rational numbers’, defined as equivalence sets in the set-product of ordered pairs of integers created by an equivalence relation making (a,b) equivalent to (c,d) if cross multiplication holds: axd = bxc.

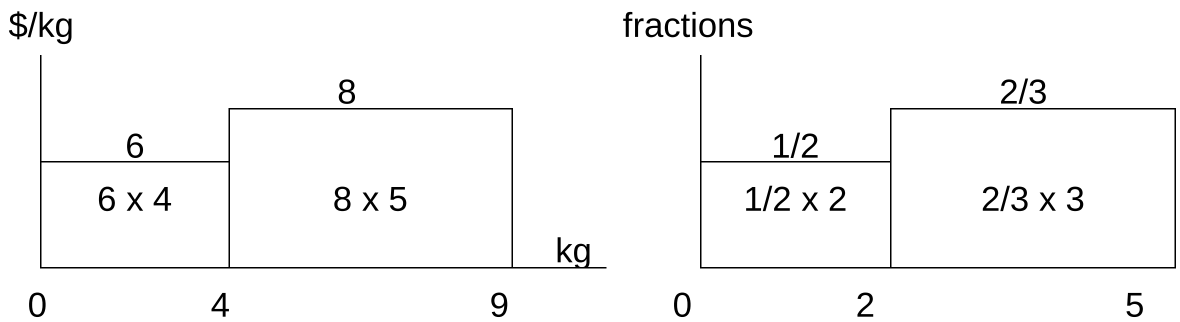
In primary school fractions come after division, the last of the basic operations. Unit fractions come in geometry as parts of pizzas or chocolate bars; and in algebra as parts of a total: 1/4 of the 12 apples is 12/4 apples. To find 4/5 of 20, first 1/5 of 20 is found by dividing with 5 and then the result is multiplied by 4. Then it is time for decimals as tenths, and percentages as hundredths. Then similar fractions occur when adding or removing common factors in the numerator and the denominator.

When including units, fractions respect the outside goal ‘to divide something’. Excluding units, adding fractions becomes MateMatism as shown by the ‘fraction paradox’: 1/2 + 2/3 is 7/6 inside a classroom, but can be 3/5 outside where 1 red of 2 apples plus 2 red of 3 total 3 red of 5 and certainly not 7 of 6.

From outside examples, per-numbers become fractions, 4kg/5$ = 4/5 kg/$. And, as per-numbers, fractions add by integrating the areas under their graph:

4kg at 6$/kg + to 5kg at 8$/kg = 9 kg at (6x4 + 8x5)/9 $/kg.

2 of which 1/2 + 3 of which 2/3 = 5 of which (1/2 x 2 + 2/3 x 3)



Integration

Adding variable per-numbers by integrating blocks, integration is one of the four ways to add as shown by the Algebra-square. So, integration should not be postponed to late secondary school but be part of primary school when adding icon-blocks next-to and when integrating areas under fraction graphs.

Also, integration should be taught before differentiation and before functions, since what we integrate (and differentiate into) is per-numbers, not functions.

Conclusion

To see if mathematics education has a goal/means confusion we asked: How will mathematics look like if built as a means for proper real world actions? Or more precisely: Which actions will enable us to deal with the physical fact Many as it exists in space and in time?

To deal with Many, first we count, then we add. But first rearranging Many create icons. Counted in icon-bundles a total transforms into a stack of unbundled, bundles, bundles of bundles etc., i.e. into a decimal number with a unit. The basic operations, / and x and –, iconize the three counting operations: to take away bundles, to stack bundles and to take away a stack. Double-counting in different units create per-numbers used to bridge the units.

Once counted, totals can be added on-top or next to; and addition can be reversed by inventing reverse operations as shown in the Algebra-square.

Constructed as abstractions from the physical fact Many, ManyMatics prevents a goal/means confusion in mathematics education seeing the outside world as applications of MetaMatism, a mixture of MetaMatics defining concepts as examples from abstractions instead of as abstractions from examples, and MatheMatism with statements that are true inside but not outside a classroom.

Recommendations

So, to improve PISA results, mathematics education must teach actions enabling students to deal with the physical fact Many. Making mathematics a means and the outside world the goal prevents a goal/means confusion to occur. Consequently, mathematics education must teach ManyMatics abstracted from the outside world as a natural science about Many. And it must reject self-referring MateMatism containing concepts based internally instead of externally, and neglecting outside falsification of inside correctness.

In primary school, recounting in different icons should precede adding on-top and next-to. And double-counting create the per-numbers allowing the two units to be bridged without waiting for proportionality. To avoid nonsense, fractions must be added as per-numbers by integrating areas thus introducing primary school calculus as the fourth way to unite numbers. In this way everybody will be able to deal with Many by applying the full Algebra-square.

The MATHeCADEMY.net is designed to teach teachers to teach mathematics as ManyMatics as illustrated by its many MrAlTarp videos on YouTube.

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