Calculators and IconCounting and CupWriting in PreSchool and in Special Needs Education

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To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking preschool mathematics uncovers icon-counting in bundles less than ten implying recounting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, a calculator can predict recounting results before being carried out manually. By allowing overloads and negative numbers when recounting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary school only allowing ten-counting and on-top addition to take place.

Decreasing PISA Performance in spite of Increasing Research

Being highly useful to the outside world, math is a core part of education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise funding has increased witnessed by e.g. the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA results in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and sinificantly below the OECD average at 494. This got OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change' (OECD 2015).

Created to help students cope with the outside world, schools are divided into subjects that are described by goals and means with the outside world as the goal and the subjects as means. However, goal/means confusions might occur where the subject become the goal and the outside world a means.

A goal/means confusion is problematic since while there is one goal there are many means to be replaced if not leading to the goal, unless an ineffective means becomes a goal itself, leading to a new discussing about which means will best lead to this false goal; thus preventing looking for alternative means that would more effectively lead to the original goal. So we can ask: Does mathematics education build on a goal-means confusion seeing mathematics as the goal and the outside world as a means? Institutional skepticism might offer an answer.

Institutional Skepticism

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting patronizing choices as nature (Lyotard 1984).

Building on Kierkegaard, Nietzsche and Heidegger, Sartre defines existentiallism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). Kierkegaard was skeptical to institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone 'may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187). Inspired by Heidegger, Arendt divided dividing human activity into labor and work both focusing on the private sphere, and action focusing on the political, creating institutions to be treated with care to avoid the banality of evil by turning totalitarian (Arendt 1963).

Since one existence gives rise to many essence-claims, the existentialist distinction offers a perspective to distinguish between one goal and many means.

Mathematics as Essence

In ancient Greece the Pythagoreans used the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music now as independent knowledge areas, today mathematics is a common label for the two remaining activities, Geometry and Algebra replacing Greek Arithmetic (Freudenthal 1973).

Textbooks see mathematics as a collection of well-proven statements about well-defined concepts, defined 'from above' as examples from abstractions instead of 'from below' as abstractions from examples. The invention of the set-concept allowed mathematics to be self-referring. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by its inability to separate concrete examples from abstract essence.
And, as expected, teaching meaningless self-reference creates learning problems.

Mathematics as Existence

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry.

Meaning to reunite numbers in Arabic, Algebra contains four ways to unite as shown when writing out fully the total T = 354 = 3*B^2 + 5*B + 4*1 = 3 bundles of bundles and 5 bundles and 4 unbundled. Here we see that we reunite by using on-top addition, multiplication, power and next-to addition, called integration. So, with a human need to describe the physical fact Many, algebra was create as a natural science about Many.

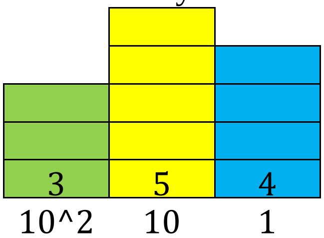


Figure 1: 354 = 3*10^2 + 5*10 + 4*1 shown as stacked bundles

To deal with Many, first we iconize, then we count by bundling and stacking. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in e.g. fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc. (Zybartas et al, 2005).

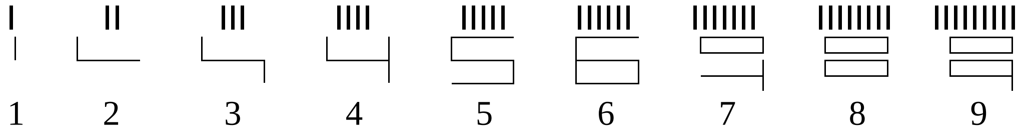
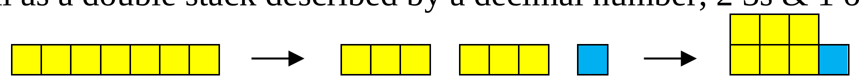


Figure 2: Digits as icons containing as many sticks as they represent

With ‘second order counting’ we bundle a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. The unbundled can be placed in a right single-cup; and in a left bundle-cup we trade the bundles, first with a thick stick representing a bundle glued together, then with a normal stick representing the bundle. The cup-contents is described by icons, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. In addition, we can also use plastic letters as B, C or D for the bundles.

IIIIIII → III III I → ▌▌) I) → II) I) → 2)1) → 2.1 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a decimal number, 2 3s & 1 or 2.1 3s.



We live in space and in time. To include both when counting, we can introduce two different ways of counting: counting in space, geometry-counting, and counting in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.

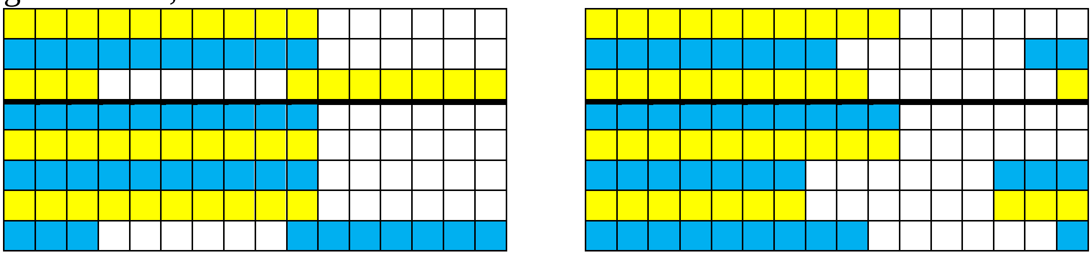


Figure 3: 7 counted in 3s on an abacus in geometry and algebra mode

To predict the counting result we can use a calculator. Building a stack of 2 3s is iconized as 2x3 showing a jack used 2 times to lift the 3s. As for the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once. Thus by entering ‘7/3’ we ask the calculator ‘from 7 we can take away 3s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 3s by asking ‘7 – 2x3’. From the answer ‘1’ we conclude that 7 = 2.1 3s. Showing ‘7 – 2x3 = 1’, a display indirectly predicts that 7 can be recounted as 2 3s and 1, or as 2.1 3s.

7 / 3	2.some
7 – 2 * 3	1

Re-counting in the Same Unit and in a Different Unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3.2 2s or as 2.4 2s. Likewise 4.2s can be recounted as 5 2s less 2; or as 6 2s less 4 thus leading to negative numbers:

Letters	Sticks	Calculator		T =
B B B B	II II II II			4.0 2s
B B B I I	II II II I I	4x2 – 3*2	2	3.2 2s
B B I I I I	II II I I I I	4*2 – 2*2	4	2.4 2s
B B B B B	II II II II II	4*2 – 5*2	-2	5.-2 2s
B B B B B B	II II II II II II	4*2 – 6*2	-4	6.-4 2s

Figure 4: Recounting 4 2s in the same unit creates overloads or deficits

To recount in a different unit means changing unit, called proportionality or linearity also. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2.2 5s.

IIII IIII IIII → IIIII IIIII I I → 2) 2) 5s → 2.2 5s

With letters, C = BI so that BBB → BB IIII → CC II

Using a calculator to predict the result we enter ‘3*4/5’ to ask ‘from 3 4s we take away 5s how many times?’ The calculator gives the answer ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be recounted as 2 5s and 2, or as 2.2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Once counted, totals can be added on-top or next-to. Asking ‘3 5s and 2 3s total how many 5s?’ we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s giving 1.1 s that added to the 3 5s gives a grand total of 4.1 5s. With letters: 3B + 2C = 3B III III = 4BI. With sticks:

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4) 1) 5s → 4.1 5s,

Using a calculator to predict the result we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2.5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2) 5) 8s → 2.5 8s

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2.5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing Adding On-top and Next-to

To reverse addition is also called backward calculation or solving equations. To reverse next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2.5 8s?’, using sticks will get the answer 2 3s:

IIIII IIIII IIIII III III ← IIIII III) IIIII III) IIIII ← 2) 5) 8s ← 2.5 8s

Using a calculator to predict the result the remaining is bracketed before counted in 3s. Adding the two stacks 2 3s and 3 5s next-to each other means multiplying before adding. Reversing integration means subtracting before dividing, as in the gradient formula y’ = dy/t = (y2 – y1)/t.

(2 * 8 + 5 – 3 * 5)/ 3	2
(2 * 8 + 5 – 3 * 5) – 2 * 3	0

Primary Schools use Ten-counting only

In primary school textbooks, numbers are counted in tens to be added, subtracted, multiplied and divided. This leads to questions as ‘3 4s = ? tens’. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount- and restack-formula is impossible since the calculator has no ten buttons. Instead it is programmed to give the answer directly in a special form that leaves out the unit and misplaces the decimal point one place to the right.

3 * 4	12

Recounting icon-numbers in tens is called doing times tables to be learned by heart. So from grade 1, 3*4 is not 3 4s any more but has to be recounted in tens as 1.2 tens, or 12 in the abbreviated form.

Recounting tens in icons by asking ‘38 = ? 7s’, is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens 0.3 7s must be written in fraction form as 3/7 and calculated as 0.428…, shown directly by the calculator, 38/7 = 5.428…

38 / 7	5.some
38 – 5 * 7	3

Without icon-counting, primary school labels the problem ‘38 = ? 7s’ as an example of an equation ’38 = x*7’ to be postponed to secondary school.

Where icon-counting involves division, multiplication, subtraction and later next-to and on-top addition, primary school turns this order around and only allows on-top addition using carrying instead of overloads. Using cup-writing with overloads or deficits instead of carrying, the order of operations can be turned around to respect, that totals must be counted before being added.

	Carry	Cup-writing	Words
Add	1 4 5 1 7 6 2	4) 5) 1)7) 5)12) 6)2) = 62	4 ten 5 1 ten 7 5 ten 12 5 ten 1 ten 2 6 ten 2 = 62
Subtract	1 4 5 1 7 2 8	4) 5) 1)7) 3)-2) 2)10-2) 2)8) = 28	4 ten 5 1 ten 7 3 ten less2 2 ten 8 = 28
Multiply	4 2 6 * 7 1 8 2	7 * 2) 6) 14)42) 18) 2) = 182	7 times 2 ten 6 14 ten 42 14 ten 4 ten 2 18 ten 2 = 182
Divide	2 4 rest 1 3| 7 3 6 1 3 1 2 1	7)3) counted in 3s 6)13) 6)12) + 1 2 3s)4 3s) + 1 24 3s + 1 73 = 24*3 + 1	7ten3 6ten 13 6ten12 + 1 3 times 2ten4 + 1 3 times 24 + 1

Figure 5: Cup-writing with overloads and deficits instead of carrying

As to addition, subtraction and multiplication, carrying occurs indirectly as an overload to be removed or created by recounting in the same unit. As to division the recounting is guided by the 3-tables showing which numbers should occur in the cups and how much to move to the next cup or outside.

Tested with a Special Needs Learner

A special needs learner taken out of her normal grade six class agreed to test the effects of using icon-counting, cup-writing, next-to addition and a calculator for number-prediction. As to the learner's initial level, when asked to add 5 to 3 she used the fingers to count on five times from three. To avoid previous frustrations from blocking the learning process, the word 'mathe-matics' was replaced by 'many-matics'. The material was 8 micro-curricula for preschool using activities with concrete material to obtain its learning goals in accordance with Piaget's principle 'greifen vor begrifen' (grasp to grasp) (MATHeCADEMY.net/ preschool).

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using two cups for the bundled and the unbundled reported with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and deficits. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eights, the learner reverses next-to addition. The micro-curricula M2-M8 used the recount- and restack formula on a calculator to predict the result:

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2) 1) 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	'2.7 5s is also how many 5s?' IIIII IIIII IIIIIII = V V V II = V V V V III 2)7) = 2+1)7-5) = 3)2) = 3+1)2-5) = 4)-3) So 2.7 5s = 3.2 5s = 4.-3 5s,	(2*5+7)/5 (2*5+7) -3*5 (2*5+7) -4*5	3.some 2 -3
M4	2 5s is how many 4s?' IIIII IIIII = IIII I IIII I = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2 * 4	2.some 2
M5	'2 5s and 4 3s total how many 5s?' IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2

M6	'2 5s and 4 3s total how many 8s?' IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 2.some (2*5+4*3) – 2*8 6
M7	'2 5s and ? 3s total 4 5s?' IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 3.some (4*5 – 2*5) – 3*5 1
M8	'2 5s and ? 3s total how 2.1 8s?' IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(4*5 – 2*5)/3 3.some (4*5 – 2*5) – 3*5 1

Figure 6: A calculator predicts counting and adding results

One curriculum used silent education where the teacher demonstrates and guides through actions only, not using words; and one curriculum was carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator quickly took over the communication. Examples of statements are given below.

Activity	Examples of statements
Icon-creation with a folding rule	Oh that's where the digits come from.
Icon-counting	So that means that 3*5 is 3 5s and not a tables-question?
Recounting in the same unit	That is the same as changing coins or getting back change.
Recounting in a different unit	Wow, a calculator can predict the result before I carry it out. Can I please keep this calculator?
Adding on-top	Oh, I see, balconies are not allowed
Adding next-to	This is like building with Lego blocks
Reversed adding on-top	Well, you just take away what was added and then count in 3s
Reversed adding next-to	Take away and count, again.
Recounting icon-numbers in tens	Hey, you just have to enter 3*4 to recount in tens.
Recounting tens in icon-numbers	So recounting in icons is just another word for solving equations?
Removing overloads with addition and multiplication 35+47 = 7)12) = 8)2) = 82 3 * 58 = 15)24) = 17)4) = 174	Hey, its fun to trade bundles for singles and vice versa.
Creating overloads with subtraction 35-17 = 2)15)-1)7) = 1)8) = 18, or 35-17 = 3)5)-1)7) = 2)-2) = 1)8) = 18	Why didn't the teacher teach me this method the first time?

Creating overloads with division 86/3 = ?; 86 = 28*3 + 2 since 8)6)= 6)26)= 6)24)+2= 2 3s)8 3s)+2	Now I see why tables are useful. They find the contents of the cups.
Creating per-numbers as bridges when double-counting in 2 different physical units With 3$/4kg, 5kg = (5/4)*4kg = (5/4)*3$ = 3.75$ 5$ = (5/3)*3$ = (5/3)*4kg = 6.67kg	OK, so recounting dollars in kgs is just like recounting 3s in 5s, isn't it? And again, we just use the calculator to predict the answer.

Figure 7: Examples of comments

At the end the learner went back to her normal class where proportionality lessons created learning problems. The learner suggested renaming it to double-counting but the teacher insisted in following the textbook. However, observing that the class gradually took over the double-counting method, he finally gave in and allowed proportionality to be renamed and treated as double-counting.

When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integration.

Thus icon-counting and a calculator for predicting recounting results allowed the learner to get to the goal, mastery of Many, by following an alternative to the institutionalized means that had become a stumbling block to her.

In the beginning the learner solved adding and subtraction problems by using the counting sequence forwards and backwards and she had given up with tables and division. With icon-counting, the order is turned around and the operations take on meanings rooted in activities: 7/3 now means 7 counted in 3s. 4*5 now means 4 5s. 7 – 2*3 now means to drag away a stack of 2 3-bundles from 7 to look for unbundled leftovers. Addition now comes in two versions, first next-to addition then on-top addition. In all cases a calculator predicts the result. Finally, double-counting in two physical units and recounting tens in icons allowed her to master proportionality and equations without following the traditionally road of institutionalized education. And performing and reversing next-to addition gave her an introduction to calculus way before this is included in the tradition.

Conclusion and Recommendation

Institutionalized education sees mathematics, not as a means to an outside goal but as a goal in itself to be reached by hindering learners in learning to count; by insisting that only ten-counting is allowed; by using the word natural for numbers with misplaced decimal point and the unit left out; by reversing the natural order of the basic operations division, multiplication, subtraction and addition; and by neglecting activities as creating or removing overloads and double-counting.

To find how mathematics looks like if built as a natural science about its root, the physical fact Many, institutional skepticism has used the existentialist distinction between existence and essence to uncover 'ManyMatics' as a hidden alternative to the ruling tradition. Dealing with Many means bundling and counting in icons, and recounting when adding on-top or next-to thus introducing proportionality and calculus. Likewise reversing on-top or next-to addition leads to solving equations and differentiation. That totals must be counted before being added means introducing the operations division, multiplication, subtraction before addition.

Consequently, mathematics education suffers from a goal-means confusion to be removed to improve PISA-results. To respect its outside goal, mathematics education must develop mastery of Many by teaching mathematics as grounded ManyMatics, and not as self-referring 'MetaMatism', a mixture of 'MetaMatics' turning mathematics upside down by presenting concepts as examples of abstractions instead of as abstractions from examples, and 'MatheMatism' true inside a classroom but not outside where claims as '1+2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

In short: Don't preach essence, teach existence.

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