EXISTENTIALISM IN MATHEMATICS AND ITS EDUCATION

FROM ESSENCE TO EXISTENCE

DON'T PREACH ESSENCE – TEACH EXISTENCE

THE ICME 13 PAPERS

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MATHECADEMY.NET

FROM ESSENCE TO EXISTENCE IN MATHEMATICS AND ITS EDUCATION

These papers, written for the ICME 13, the 13^{th} International Congress on Mathematics Education, ask how mathematics education would look like from the viewpoint of the existentialist principle 'existence precedes essence'.

Primary and secondary school textbooks show the essence of mathematics as four number sets and nine operations. The number sets are of natural, integer, rational and reel numbers, i.e. positive and negative counting numbers and fractions and roots and logs. Operations consist of addition and subtraction, multiplication and division, power and root and logarithm, presented in this order, together with differentiation and integration applied to 'stand-by-calculations' containing both numbers and letters also called expressions or functions. Together with numbers come lines, triangles, polygons and circles as well as statistics and probability.

As indicated by the 'irrelevance paradox' where increasing research funding gives decreasing PISA-results in the Scandinavian countries, mathematics is more than hard to many students having problems with especially fractions as shown by the 'the fraction paradox' where the textbook and the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles. Thus 'mathematics is hard' is seen as, not choice, but nature calling for more funding of teacher education.

However, letting existence precede essence shows a different picture. Here mathematics turns out to be a mere label for two activities, Algebra and Geometry, meaning to reunite numbers and to measure earth in Arabic and Greek. Now the only numbers are positive or negative decimal numbers with units occurring as units-numbers and per-numbers. As per-numbers, fractions are not numbers, but operators needing a number to become a number, and roots and logs are calculations, not numbers. While neglected almost totally in the essence version, counting totals gets priority over adding them. This leads to the opposite order of operations since counting means bundling, division, and stacking, multiplication, to be taken away, subtraction, to look for unbundle singles to be added next-to; and reported as a decimal number with a unit where the decimal point separates the bundles and the unbundles as e.g. T = 2.3 4s. Fractions only occur when counting in different physical units creates per-numbers as 3$ per 4 kg = 3$/4kg = 3/4 $/kg, that can be used to bridge the two units. Once counted, total can be added, on-top by changing the unit called proportionality, or next-to called integration. Also per-numbers are added by their area, i.e. by integration. Thus there are four ways to unite numbers, addition unites variable unit-numbers, multiplication unites constant unit-numbers, power unites constant per-numbers, and integration unites variable per-numbers. Once united, totals can be taken apart by the reverse operations subtraction, division, root and log, and differentiation. That counting should precede adding resonates with the original meaning of the word Algebra. Never adding without units transforms adding fractions to calculus.

The papers were written for the following Topic Study Groups 1, Early childhood mathematics education. 5, Activities for, and research on, students with special needs. 8, Teaching and learning of arithmetic and number systems. 12, Teaching and learning of geometry. 16, Teaching and learning of calculus. 27, Essence and Existence in Conflicting Cognitive Theories. 37, Mathematics curriculum development. 49, In-service education and professional development of primary mathematics teachers. 53, Philosophy of mathematics education.

This ICME13 was extra strict about submitting one paper only, so the Philosophy paper was chosen.

Allan Tarp, Aarhus, October 2015

PROPORTTIONALITY AND INTEGRATION IN PRESCHOOL THROUGH ICON-COUNTING AND NEXTTO-ADDITION

Allan Tarp

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To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking preschool mathematics uncovers icon-counting that by counting in numbers below ten allows recounting both in the same unit to create overloads or deficits, as well as in a different unit if added on-top, thus applying proportionality. Added next-to, icon-numbers are added by their area, i.e. by integration. These golden learning opportunities disappear when entering ordinary school only allowing ten-counting.

BACKGROUND

Institutionalized education typically has mathematics as a core subject in primary and secondary school. To evaluate the success of mathematics education, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015). This might be an effect of teaching a label, mathematics, instead of what it labels, geometry and algebra; and of hiding the Arabic meaning of algebra: to reunite variable and constant unit-numbers and per-numbers by addition, multiplication, power and integration, all allowed in preschool icon-counting before primary school institutionalizes the monopoly of ten-counting. Based upon existentialist philosophy this paper asks: What is existentialist preschool math and what difference will it make?

INSTITUTIONAL SKEPTICISM

The ancient Greek sophists recommended enlightenment to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, Symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting their patronizing choices as nature (Lyotard 1984).

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). Kierkegaard was skeptical towards institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone 'may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187). Inspired by Heidegger, Arendt divided human activity into labor and work both focusing on the private sphere, and action focusing on the political sphere creating institutions to be treated with care to avoid the banality of evil (Arendt 1963).

BUILDING A NATURAL SCIENCE ABOUT MANY

To deal with the physical fact Many, first we iconize, then we count by bundling and stacking. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in e.g. fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc..

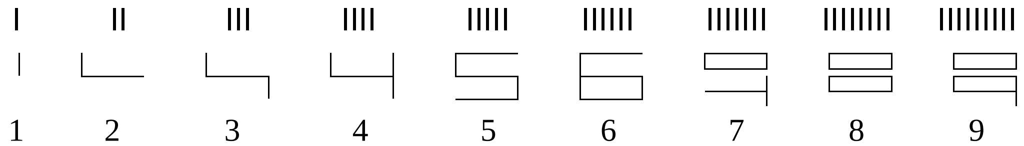

Fig. 1. Sticks or a folding rule used to create icons with as many sticks as they represent

With ‘second order counting’ we bundle in icon-bundles. A total T of 7 is bundled in 3s as T = 2 3s and 1. The unbundled we place in a right single-cup, and in a left bundle-cup we place two stick for the bundles. The cup-contents is described by icons, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Alternatively, we can also use plastic letters as B, C or D for the bundles.

IIIIIII → III III I → II) I) → 2)1) → 2.1 3s or BB I → 2B I

We live in space and in time. To include both when counting, we introduce two ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and show them on a ten-by-ten abacus in geometry-mode. Counting in time, we count bundles and report the result on a ten-by-ten abacus in algebra-mode with lines for bundled and unbundled.

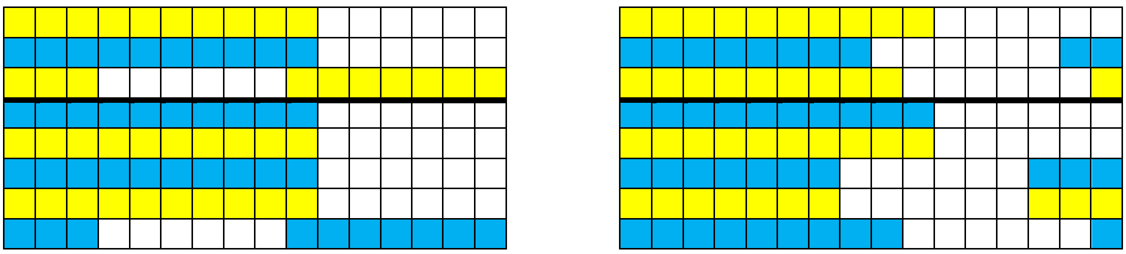

Fig. 2. Counting 7 in 3s on an abacus in geometry mode below and algebra mode above the rubber

To predict the counting result we use a calculator. A stack of 2 3s is iconized as 2x3 showing a jack used 2 times to lift the 3s. Taking away is iconized with ‘/3’ or ‘–3’ showing the broom or the trace when wiping away 3 several times or just once, called division and subtraction. Entering ‘7/3’, we ask the calculator ‘from 7 take away 3s’ and get the answer ‘2.some’. Entering ‘7 – 2x3’ we ask ‘from 7 take away 2 3s’ and get the answer 1 leftover. Thus the calculator predicts that 7 = 2.1 3s.

Re-counting in the Same Unit and in a Different Unit

Once counted, totals can be re-counted in the same unit or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3.2 2s or 2.4 2s. And 4.2s can be recounted as 5 2s less or short of 2; or as 6 2s less 4 thus leading to negative numbers.

T = 4 2s = 3.2 2s = 2.4 2s = 1.6 2s = 0.8 2s, or T = 4 2s = 5.-2 2s = 6.-4 2s = 7.-6 2s

To recount in a different unit means changing unit, called proportionality or linearity. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2.2 5s.

IIII IIII IIII → IIIII IIIII I I → 2) 2) 5s → 2.2 5s, or with C = B I, BBB → BB IIII → CC II

On an abacus in geometry mode, reserving the bottom line for the single 1s, a stack of 3 4s is moved from left to right on an abacus. The top bundle is changed to 1s on the single line and twice a stick is removed to enlarge the two 4-bundles to 5-bundles. So 3 4s can be recounted as 2.2 5s.

On an abacus in algebra mode, 3 beads are moved to the right on the bundle-line. Then one 4-bundle is changed to 4 1s on the bottom single-line. Moving 2 beads to the left on the single-line allows enlarging the 4s to 5s thus showing that 3 4s = 2.2 5s

Using a calculator to predict the result, we enter ‘3x4/5’ to ask ‘from 3 4s away 5s’ and get the answer ‘2.some’. We enter ‘3x4 – 2x5’ and get answer 2 leftovers. So we predict: T = 3 4s = 2.2 5s.

Adding On-top and Next-to

Once counted, totals can be added on-top or next-to. Asking ‘3 5s and 2 3s total how many 5s?’ we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s giving 1.1 s that added to the 3 5s gives a grand total of 4.1 5s. With letters: 3B + 2C = 3B III III = 4B I.

With sticks: IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4) 1) 5s → 4.1 5s

On an abacus in geometry mode, a stack of 3 5s is moved to the right and a stack of 2 3s is moved to the middle. Now, the 2 3s is changed to 6 1s on the bottom line allowing one additional 5s to be moved to the top of the stack of 5s to show the grand total is 4.1 5s.

On an abacus in algebra mode, the 3 5s become 3 beads on the 5-bundle line and the 2 3s become 2 beads on the 3-bundle line above. Again the 2 3s is changed to 6 1s on the bottom single-line allowing one additional bead to be added to the bundle-line to give the result 4.1 5s.

Using a calculator to predict the result we use a bracket before counting in 5s: Asking ‘(3x5 + 2x3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. So we predict: T = 3 5s + 2 3s = 4.1 5s.

Next-to addition of blocks means adding their areas, called integration. Asking ‘3 5s and 2 3s total how many 8s?’ sticks and letters gives the answer is 2.5 8s. With letters: 3B + 2C = BC BC III

With sticks: IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2) 5) 8s → 2.5 8s

On an abacus in geometry mode, a stack of 3 5s is moved to the right and a stack of 2 3s is moved to the middle. Now a 5-bundle is moved to the single line allowing the two stacks to be integrated as 8s, showing that the grand total is 2.5 8s. Likewise on an abacus in algebra mode.

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking ‘(3x5 + 2x3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2.5 8s.

Reversing Adding On-top and Next-to

Reversed addition is called backward calculation or solving equations. Reversed next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2.5 8s?’ sticks will get the answer 2 3s:

IIIII IIIII IIIII III III ← IIIII III IIIII III IIIII ← 2) 5) 8s ← 2.5 8s

On an abacus in geometry mode with 2 8s and 5 moved to the right, first 3 5s is moved to the left, then the remaining is recounted in 3s as 2 3s. Using a calculator to predict the result, the remaining is bracketed before counted in 3s.

Integration of 2 3s and 3 5s next-to each other means multiplying before adding. Reversed integration means subtracting before dividing, as in the gradient formula y' = dy/t = (y2 – y1)/t.

TESTING IN THE CLASSROOM

In Denmark, teacher education is kept outside the university, and preschool teachers do not exit. Although nationally recognized that in a multi-ethnic society an early start is important, local authorities show reluctance to expose preschool children to a mathematics that do not prepare them for ordinary school. In countries with preschools the same reluctance occurs when looking at the material published on the MATHeCADEMY.net. With home education it is different and the result of pilot projects are promising since most mathematics can be learned as 'One Digit Math' (Zybartas et al 2005).

CONCLUSION AND RECOMMENDATION

To implement its goal, an institution hires civil servants who will also guard it to protect their jobs and careers. To avoid the banality of evil, education should institutionalize only essence with existence behind it guaranteed by using institutional skepticism to uncover pure essence. In primary school, the monopoly of ten-counting and on-top addition present numbers as essence leaving out units and misplacing decimal point is hiding the existence behind, that totals can be counted by icon-bundles reported as decimal numbers with units to be added both on-top and next-to. Skipping the counting process entirely means missing the golden learning opportunities in

So to improve PISA results, existence should precede essence in preschool and primary school. So replace Vygotsky used to mediate essence with Piaget allowing grasping by grasping what exists: Totals that need to be counted in icons before being added both next-to and on-top. In a postmodern vocabulary: Maybe preschool mathematics needs to be deconstructed (MrAlTarp 2013).

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CALCULATORS AND ICONCOUNTING AND CUPWRITING IN PRESCHOOL AND IN SPECIAL NEEDS EDUCATION

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To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking preschool mathematics uncovers icon-counting that by counting in numbers below ten allows recounting both in the same unit to create overloads or deficits, as well as in a different unit if added on-top, thus applying proportionality. Added next-to, icon-numbers are added by their area, i.e. by integration. These golden learning opportunities disappear when entering ordinary school only allowing ten-counting.

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To deal with the physical fact Many, first we iconize, then we count by bundling and stacking. With 'first order counting' we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in e.g. fives.

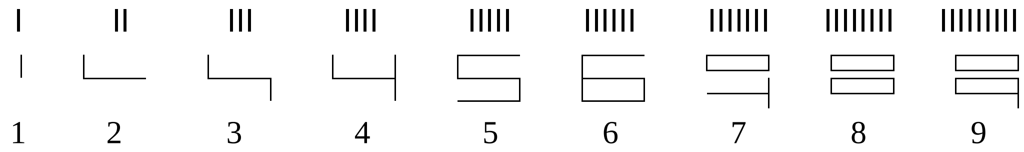

Fig. 1. Sticks or a folding rule used to create icons with as many sticks as they represent

With 'second order counting' we bundle in icon-bundles. A total T of 7 is bundled in 3s as T = 2 3s and 1. The unbundled we place in a right single-cup, and in a left bundle-cup we place two stick for the bundles. The cup-contents is described by icons, first using 'cup-writing' 2)1), then using 'decimal-writing' with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Alternatively, we can also use plastic letters as B, C or D for the bundles.

IIIIIII → III III I → II) I) → 2)1) → 2.1 3s or BB I → 2B I

We live in space and in time. To include both when counting, we introduce two ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and show them on a ten-by-ten abacus in geometry-mode. Counting in time, we count bundles and report the result on a ten-by-ten abacus in algebra-mode with lines for bundled and unbundled.

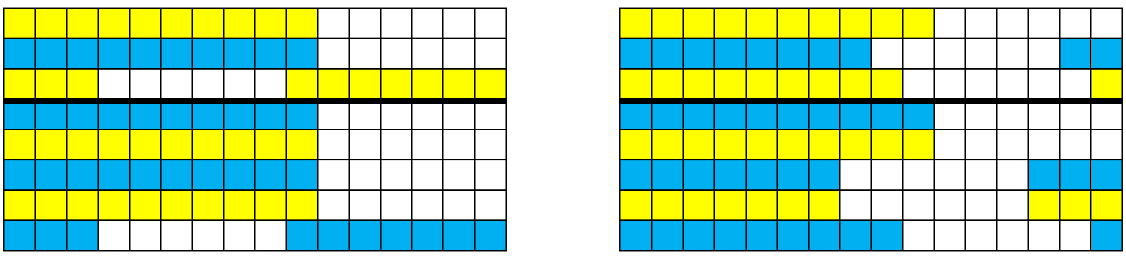

Fig. 2. Counting 7 in 3s on an abacus in geometry mode below and algebra mode above the rubber

To predict the counting result we use a calculator. A stack of 2 3s is iconized as 2x3 showing a jack used 2 times to lift the 3s. Taking away is iconized with '/3' or '–3' showing the broom or the trace when wiping away 3 several times or just once, called division and subtraction. Entering '7/3', we ask the calculator 'from 7 take away 3s' and get the answer '2.some'. Entering '7 – 2x3' we ask 'from 7 take away 2 3s' and get the answer 1 leftover. Thus the calculator predicts that 7 = 2.1 3s.

TESTED WITH A SPECIAL NEEDS LEARNER

A special needs learner taken out of her normal grade six class agreed to test the effects of using icon-counting, cup-writing, next-to addition and a calculator for number-prediction. As to the learner's initial level, when asked to add 5 to 3 she used the fingers to count on five times from three. To avoid previous frustrations from blocking the learning process, the word 'mathe-matics' was replaced by 'many-matics'. The material was 8 micro-curricula for preschool using activities with concrete material to obtain its learning goals in accordance with Piaget's principle 'greifen vor begrifen' (grasp to grasp) (MATHeCADEMY.net/ preschool).

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using two cups for the bundled and the unbundled reported with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and deficits. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eights, the learner reverses next-to addition. The micro-curricula M2-M8 used the recount- and restack formula on a calculator to predict the result.

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2) 1) 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	'2.7 5s is also how many 5s?' IIIII IIIII IIIIIII = V V V II = V V V V III 2)7) = 2+1)7-5) = 3)2) = 3+1)2-5) = 4)-3) So 2.7 5s = 3.2 5s = 4.-3 5s,	(2*5+7)/5 (2*5+7) -3*5 (2*5+7) -4*5	3.some 2 -3
M4	2 5s is how many 4s?' IIIII IIIII = IIII I IIII I = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2 * 4	2.some 2
M5	'2 5s and 4 3s total how many 5s?' IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2
M6	'2 5s and 4 3s total how many 8s?' IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 (2*5+4*3) – 2*8	2.some 6
M7	'2 5s and ? 3s total 4 5s?' IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1
M8	'2 5s and ? 3s total how 2.1 8s?' IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1

Fig. 3. A calculator predicts counting and adding results

One curriculum used silent education where the teacher demonstrates and guides through actions only, not using words; and one curriculum was carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator quickly took over the communication. Examples of statements are given below.

At the end the learner went back to her normal class where proportionality lessons created learning problems. The learner suggested renaming it to double-counting but the teacher insisted in following the textbook. However, observing that the class gradually took over the double-counting method, he finally gave in and allowed proportionality to be renamed and treated as double-counting. When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integration.

Thus icon-counting and a calculator for predicting recounting results allowed the learner to get to the goal 'Mastery of Many' by following an alternative to the institutionalized means that had become a stumbling block to her.

In the beginning the learner solved adding and subtraction problems by using the counting sequence forwards and backwards and she had given up with tables and division. With icon-counting, the order is turned around and the operations take on meanings rooted in activities: 7/3 now means 7 counted in 3s. 4*5 now means 4 5s. 7 – 2*3 now means to drag away a stack of 2 3-bundles from 7 to look for unbundled leftovers. Addition now comes in two versions, first next-to addition then on-top addition. In all cases a calculator predicts the result. Finally, double-counting in two physical units and recounting tens in icons allowed her to master proportionality and equations without following the traditionally road of institutionalized education. And performing and reversing next-to addition gave her an introduction to calculus way before this is included in the tradition.

CONCLUSION AND RECOMMENDATION

Institutionalized education sees mathematics, not as a means to an outside goal but as a goal in itself to be reached by hindering learners in learning to count; by insisting that only ten-counting is allowed; by using the word natural for numbers with misplaced decimal point and the unit left out; by reversing the natural order of the basic operations division, multiplication, subtraction and addition; and by neglecting activities as creating or removing overloads and double-counting.

To find how mathematics looks like if built as a natural science about its root, the physical fact Many, institutional skepticism has used the existentialist distinction between existence and essence to uncover 'ManyMatics' as a hidden alternative to the ruling tradition. Dealing with Many means bundling and counting in icons, and recounting when adding on-top or next-to thus introducing proportionality and calculus. Likewise reversing on-top or next-to addition leads to solving equations and differentiation. That totals must be counted before being added means introducing the operations division, multiplication, subtraction before addition. Consequently, a goal-means confusion in math education creates bad PISA-results. To respect its outside goal, mathematics education must develop mastery of Many by teaching mathematics as grounded ManyMatics, and not as self-referring 'MetaMatism', a mixture of 'MetaMatics' turning mathematics upside down by presenting concepts as examples of abstractions instead of as abstractions from examples, and 'MatheMatism' true inside a classroom but not outside where claims as '1+2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days. In short: Don't preach essence, teach existence.

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AS PER-NUMBERS, FRACTIONS ARE OPERATORS, NOT NUMBERS

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To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking fractions shows that fractions are not numbers but operators needing numbers to become numbers. This insight exposes adding fractions without units as 'mathematism' turning many learners away for mathematics.

BACKGROUND

Institutionalized education has mathematics as a core subject in primary and secondary school. To evaluate the educational success, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA studies and by the OECD report 'Improving Schools in Sweden' (OECD 2015). This might be an effect of teaching mathematics as 'mathematism' true inside but not outside classrooms as e.g. the 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles. Based upon existentialist philosophy this paper asks: What is an existentialist fraction and what difference will it make?

INSTITUTIONAL SKEPTICISM

The ancient Greek sophists recommended enlightenment to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive in American pragmatism, Symbolic Interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting their patronizing choices as nature (Lyotard 1984).

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism as holding that 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). Focusing on the three classical virtues Truth and Beauty and Goodness, Kierkegaard argued that to change from a person to a personality you must stop admiring essence created by others and instead realize your own existence through individual choice and action. Furthermore he showed fierce skepticism towards institutionalized Christianity in the form of Christendom, criticized also by Nietzsche for imprisoning people in moral serfdom until the coming of a 'redeeming man' whose isolation is not a flight from reality. Instead, 'it is only his absorption, immersion, penetration *into* reality, so that (..) he may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187).

Arendt carried Heidegger's work further by dividing human activity into labor and work both focusing on the private sphere, and action focusing on the political sphere creating institutions that should be treated with care to avoid the banality of evil by turning totalitarian. (Arendt 1963).

FRACTIONS AS ESSENCE

Textbooks see mathematics as a collection of well-proven statements about well-defined concepts, defined, not as abstractions from examples below, but as examples from abstractions above, made possible by the invention of the set-concept allowing mathematics to be self-referring. By looking at the set of sets not belonging to itself, Russell showed that self-reference let to the classical liar paradox 'this sentence it false' being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by its inability to separate concrete examples from abstract essence. That institutionalized education still teaches self-referring mathematics instead of grounded algebra and geometry can be seen as an example of symbolic violence used as an exclusion technique to keep today's knowledge nobility in power (Bourdieu 1977).

Set-based mathematics defines a fraction as an equivalence class in the set-product of ordered pairs of integers created by an equivalence relation making (a,b) equivalent to (c,d) if cross multiplication holds, $a \cdot b = c \cdot d$. This self-referring definition using numbers to define numbers prohibits fractions from being numbers according to Russell but not according to Zermelo–Fraenkel seeing fractions as a set of rational numbers to be taught in school alongside with the other number sets.

In textbooks, typically, fractions wait for all four basic operations to be introduced. Then unit fractions come in two versions. Geometric fractions are parts of pizzas or chocolate bars. And algebraic fractions are associated with simple division: 1/4 of the 12 apples is 12/4 apples. To find 4/5 of 20 apples, 1/5 of 20 is found by dividing with 5 giving the result by multiplying by 4.

Then it is time for decimals as tenths, and for percentages as hundredths. Then adding or removing common factors in the numerator and in the denominator introduces the idea of similar fractions. Then, in late primary or early middle school, addition of fractions is introduced, first with like, then with unlike denominators. Then decomposing a number into primes is introduced together with the lowest common multiple and the highest common factor to find the smallest common denominator when adding fractions with unlike denominators.

Then everything is repeated with numerical fractions replaced by algebraic fraction, first using monomials as (4abc)/(6ac) that are already factorized, then using polynomials as (4ab+8bc)/(3ab-6ac) that need to be factorized. Finally fractions enter into equations and functions.

FRACTIONS AS EXISTENCE

The Pythagoreans used the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas (Freudenthal 1973). With astronomy and music now as independent subjects, today only the two other activities remain both rooted as natural sciences about the physical fact Many, Geometry meaning to measure earth in Greek, and Algebra meaning to reunite numbers in Arabic and replacing Greek arithmetic.

Meeting Many we ask 'how many?' To answer we use counting and adding as uniting techniques. To count we bundle and stack, typically in ten-bundles, as seen when writing out a total T in its full form as a union of blocks, T = 345 = 3·B^2 + 4·B + 5·1, containing the four ways to unite: On-top addition unites variable numbers, multiplication unites constant numbers, power unites constant factors, and next-to addition, also called integration, unites variable blocks.

Counting a total T in b-bundles can be predicted by a recount formula T = (T/b) ·b saying 'from T, bs can be taken away T/b times'. To predict the result of counting 7 in 3s, a calculator shows that '7/3 = 2.some' and that '7 – 2·3 = 1' so the formula predicts the correct answer: 7 = 2 3s and 1.

The unbundled 1 can be counted in 3s as 1 = (1/3)·3 and placed on-top of the stack of 2 3s, giving the answer 7 = 2 1/3 3s; or it can be placed next-to the 3-stack as a stack of unbundled ones written as 7 = 2.1 3s using a decimal point to separate the bundles form the unbundled. Thus ordinary and decimal fractions are rooted in describing the leftovers when counting by bundling and stacking.

Fig. 1. Left: Counting 7 in 3s as 2 1/3 3s or as 2.1 3s. Right: Adding 1/2 + 2/3 as 3/5

If the 3 has the unit 2s then the leftover is 1 2s of 3 2s, i.e. 2 of 6 making 1/3 = 2/6. Likewise 1/3 = 3/9, so a fraction remains the same when changing unit above and below the fraction line.

Instituting ten as the only allowed bundle-size, primary school avoids fractions by leaving out the unit and misplacing the decimal point one place to the right when writing 23 instead of 2.3 tens.

In middle school different units occur when counting distances, time, value, etc. Double-counting a total in $ and in kg creates per-numbers as 3$ per 4 kg = 3$/4kg = 3/4 $/kg to bridge or change the units, called proportionality, by recounting $s in 3s and kgs in 4s:

T = 15$ = (15/3)·3$ = (15/3)·4kg = 20kg, and T = 12kg = (12/4)·4kg = (12/4)·3$ = 9$.

When parting a total in a given ratio, fractions also occur as per-numbers. That two persons A and B split 600$ in the ratio 1:2 means that for each 3$, A gets 1$ and B gets 2$, or that A and B gets 1 per 3 and 2 per 3. To find the two parts we recount the 600 in 3s as 600 = (600/3) ·3 = 200·3 so that A gets 1$ 200 times and B gets 2$ 200 times, thus parting the 600$ in 200$ and 400$ to A and B.

Per hundred is called percent. Again, to find 30% of 450$ or 30$ per 100$ of 450$, we recount 450$ in hundreds as 450$ = (450/100) ·100$ = 4.5·100$. So 30% of 450$ is 30$ 4.5 times = 135$.

Alternatively, we can see the hundred as a foreign currency # where the per-number 100# = 450$ means recounting the 30 in 100s: T = 30# = (30/100) ·100# = 30/100·450$ = 135$.

Likewise, to find 40$ in percent of 250$, the per-number 100# = 250$ means recounting the 40 in 250s: T = 40$ = (40/250) ·250$ = 40/250·100# = 16# or 16%.

As per-numbers, fractions must be multiplied to unit-numbers before being added as areas, called integration. Thus per-numbers and fractions are not numbers but operators needing a number to become a number. Adding 2kg at 1/2$/kg and 3kg @ 2/3$/kg gives (2+3)kg @ (1/2·2+2/3·3)/(2+3) $/kg or 5 kg @ 3/5 $/kg. We see that unit-numbers 2 and 3 add directly whereas the per-numbers 1/2 and 2/3 add as areas under the per-number graph that is piecewise constant in middle school.

In high school, the per-number graph p is locally constant, so the area comes from adding many area strips each with the area $p \cdot dx$ where dx is a small change in x. Adding many numbers is time-consuming unless they can be written as differences in which case the middle numbers cancel out so the sum becomes the difference between the end-number and the start-number. So to find the total area we want to write the single area $p \cdot dx$ as a difference: $p \cdot dx = dP$, or $dP/dx = p$. Thus to add fractions as areas we need a theory for change-fractions d/dx, called differential calculus.

TESTING IN THE CLASSROOM

In Denmark the passing limit in calculus has been lowered to about 30% correctness. Typically a class starts with a traditional fraction course where the fraction paradox produces many dropouts forced to stay in class so the school will not lose government funding. One class continued by presenting fractions as per-numbers added as areas. The students were surprised to have their side of the fraction paradox accepted and to hear about per-numbers for the first time. And going on with integral calculus before differential calculus made most students finish with good results. A report was sent to the teacher's journal and to the ministry of education but stayed unpublished.

CONCLUSION AND RECOMMENDATION

To implement its goal, an institution hires civil servants who will also guard it to protect their jobs and careers. To avoid the banality of evil, education should institutionalize only essence with existence behind it guaranteed by using institutional skepticism. Set-based fractions become pure essence when defined through self-reference as numbers without units instead of as operators. What has existence is leftovers when counting by bundling and stacking to be described as normal or decimal fractions, as well as per-numbers for bridging units when double-counting in two units. In a postmodern vocabulary: Maybe fractions needs to be deconstructed (MrAlTarp 2012).

To improve PISA results, education should institutionalize, not mathematism true only inside classrooms, nor the label mathematics, but its two remaining activities, geometry and algebra in its original meaning, to reunite variable and constant unit- and per-numbers by addition, multiplication, integration and power. Primary school should institutionalize counting in icons less than ten to make decimal numbers with a unit become natural numbers. Middle school should institutionalize fractions as per-numbers, both as operators to be added as areas. And high school should institutionalize integral calculus before differential calculus, not the other way around.

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FROM ESSENCE TO EXISTENCE IN GEOMETRY EDUCATION

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In mathematics and its education, the difference between essence and existence is seldom discussed although central to existentialist thinking. Thus we can ask: How will an existentialist geometry education look like? So we close the door to the library of the essence-claims of self-referring mathematics and go outside to rebuild geometry from its existing roots, blocks. A curriculum is designed so geometry can be rooted in its outside existence instead of in its inside essence claims.

BACKGROUND

Institutionalized education typically has mathematics as a core subject in primary and secondary school. To evaluate the success of mathematics education, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015). This may be caused by educational goal-means confusions. Geometry has a classical Euclidian and modern translation educational tradition both with learning problems coming from seeing themselves as goals and thus shadowing for its original outside goal, to measure earth, in accordance with its Greek meaning. Based upon existentialist philosophy this paper asks: What is existentialist geometry education?

INSTITUTIONAL SKEPTICISM

The ancient Greek sophists recommended enlightenment to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, Symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting their patronizing choices as nature (Lyotard 1984).

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). Kierkegaard was skeptical towards institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone 'may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187). Inspired by Heidegger, Arendt divided human activity into labor and work both focusing on the private sphere, and action focusing on the political sphere creating institutions to be treated with care to avoid the banality of evil (Arendt 1963).

GEOMETRY AS ESSENCE

In ancient Greece the Pythagoreans used the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music now as independent knowledge areas, today mathematics is a common label for the two remaining activities, Geometry and Algebra replacing Greek Arithmetic (Freudenthal 1973). The Pythagoreans chose the right triangle as their favorite form and discovered two laws, one about angles and one about sides. By adding extra laws Euclid created a geometry that still is taught today as classical geometry.

However, the invention of the set concept allowed mathematics to transform into a self-referring subject called New Math defining its concepts 'from above' as examples from abstractions instead of 'from below' as abstractions from examples. Now geometry was defined as an example of a vector space with point-sets that could be manipulated algebraically. This translation-geometry turned its back to Euclid by saying 'Euclid must go'.

Learning problems occur in both traditions. Many find it boring to prove obvious facts. And many find set-based definitions to abstract to give meaning. Furthermore, by looking at the set of sets not belonging to itself, Russell showed that self-reference let to the classical liar paradox 'this sentence it false' being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by its inability to separate concrete examples from abstract essence.

GEOMETRY AS EXISTENCE

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry, both rooted as natural sciences about the physical fact Many. Meeting Many we ask 'how many?' Counting and adding gives the answer as shown by the word algebra, meaning to reunite numbers in Arabic. We count by bundling and stacking as seen when writing a total T in its full form: $T = 345 = 3\cdot B^2 + 4\cdot B + 5\cdot 1$ to be presented in an algebraic form using place value and a geometrical form showing three blocks placed next to each other.

A total T of 7 is bundled in 3s as T = 2 3s and 1. The unbundled we place in a right single-cup, and in a left bundle-cup we place two stick for the bundles. The cup-contents is described by icons, first using 'cup-writing' 2)1), then using 'decimal-writing' with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. IIIIIII → III III I → 2)1) → 2.1 3s

We live in space and in time. To include both when counting, we introduce two ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and show them on a ten-by-ten abacus in geometry-mode. Counting in time, we count bundles and report the result on a ten-by-ten abacus in algebra-mode with lines for bundled and unbundled.

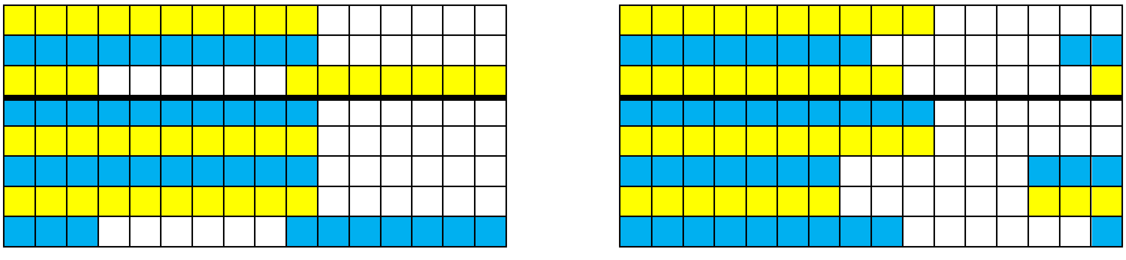

Fig. 1. Counting 7 in 3s on an abacus in geometry mode below and algebra mode above the rubber

BUILDING A CURRICULUM IN BLOCK-GEOMETRY

Thus, the root of geometry is the physical form called a block or a rectangle. It occurs when counting since all numbers are combinations of blocks. And it occurs all over in everyday life both inside and outside houses. This leads to the question how to design a curriculum in geometry using Lego 4 row blocks with the dimension 1cm by 3 cm as concrete material.

01. Measuring lengths. Having built a 2x3 cm block, a folding rule measures the horizontal base, the vertical height, i.e. the stack-size, and the diagonal, first on the physical block, then at a drawing on squared paper allowing drawing and counting or calculating the full stacks, or squares, for each side. Apparently the full stacks of the base and the height add up to that of the diagonal. This hypothesis is checked with a 3x3cm and a 4x3 cm block. To check further, four 2x3cm blocks are placed in a 5x5cm square by turning each a quarter turn and showing the diagonal with sticks that are also turned. Before removing the sticks we notices that the full square is the sum of the full diagonal stack and 4 half blocks; after, the same square can be combined by the full blocks of the base and the height and two blocks. Since 4 half blocks are the same as 2 full blocks, it follows by logical necessity that the full stacks of the short sides add up to the full stack of the long side. With the base and the height and the diagonal labeled b and h and d, this gives a Pythagorean formula b^2+c^2 = d^2 that can be used to predict the length of the diagonal in a block: In a 4x6 block the diagonal d can be predicted by the formula d^2 = 4^2 + 6^2 = 52. Her different numbers can be tested on a calculator, or we ca use a root-button created to predict the result, d = √52 = 7.2

02. Measuring turnings. Being halved by its diagonal, a block falls into to right triangles called so because turning the base to the height is called a right angle. To form a triangle the diagonal is turned away from the base line until it forms what could be called a 2 per 3 angle, a 2/3-angle. Using a protractor where half a turn is 180 degrees we see that a 2/3 angle has the measure 33.6 degrees. A calculator can predict this result by using the tan-button, tan(33.6) = 0.66 close to 2/3 = 0.67. Likewise we see that the diagonal in a 4x3 block is turned away from the base in a 4/3 angle that can be measured to 53.1 degrees, predicted by entering tan-1(4/3) on a calculator.

03. Predicting the three unknown sides and angles in a triangle. A folding rule can be used to create different triangles. To draw a triangle we need a side and two additional things. If none of the three angles exceeds a right angle, the triangle can be wrapped into a block and its unknown sides and angles can be predicted by calculating them in one of its outer right angles inside the block. A similar method can be used if one angle exceeds a right angle.

04. A half-circle can be fitted into 4 right triangles turned 180/4 degrees each time and where the heights add up to 4*tan(180/4) = 4 if the baseline, diameter, is 1. Likewise the heights of 100 triangles turned 180/100 degrees will add up to 100*tan(180/100) = 3.1426, which is close to the length of the half-circle, called pi, π, approximated even better by 1000*tan(180/1000) = 3.1416

05. A first step into coordinate geometry. In a coordinate system created by two rulers, a point is given by its horizontal and vertical distances from the starting point, called its coordinates. Three points gives a triangle where its ability to be wrapped into a block allows predicting its angles and sides from calculations on the outside right triangles.

06. A second step into coordinate geometry. Placing a 2x3 block next to a 5x3 block we see that the diagonals are turned in a 2/3 angle and a steeper 5/3angle. That question is when the two diagonals

intersect if allowed to continue out of the blocks. In the second block the distance between the two diagonals decreases with 5-2 = 3 per 3. So per 1 it will decrease by 1 reaching 0 in two steps. So the answer is that the two diagonal will meet after 3+2 = 5 steps.

07. A third step into coordinate geometry. To predict the height y of a 2/3 diagonal after x steps, we recount the x in 3s and get x = (x/3)*3 leading to the formula y = (x/3)*2, or y = (x/3)*2 + b if the height has an beginning value b.

08. In statistics, blocks can be used to show frequencies; and cumulated frequencies are shown by the height or by a traveling diagonal depending on the observations are grouped..

09. Including STEM, a projectile is send of in a 7x3 block followed by a 6x3 block etc. until impact. Its orbit is compared with the orbits from a 6x3 and an 8x3 block. This creates a spin-off problem: Can a formula predict the result of adding consecutive numbers? Likewise, on a racing filed drawn on a squared paper a car is allowed to change the height and the base by -1, 0 or 1 each time. A competition can be made to see who will reach the goal line in fewest blocks.

TESTED IN THE CLASSROOM

A school decided to include all three geometry traditions in their curriculum. To see the difference, three grade 5 classes began with different traditions. After finishing the mandatory problems the learners could choose between extra routine-building problems or to draw houses. In Euclidean geometry most learners chose to draw houses. In translation-geometry the boys typically chose to draw houses. In block-geometry very few chose to draw houses, apparently fascinated by the ability of a calculator to predict length and angles before drawing the triangles.

CONCLUSION AND RECOMMENDATION

Asking ‘what is existentialist geometry education?’ institutional skepticism has used the existence of blocks in counting and in every-day life to uncover block-geometry as a hidden alternative to the essence-claims of the two ruling traditions. To avoid goal-means confusions, education should institutionalize only essence with existence behind it guaranteed by using institutional skepticism to uncover pure essence. By transforming themselves from means Euclidean geometry and translation-geometry hide its physical roots, to measure earth. So please, don’t preach essence, teach existence.

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PRESCHOOL, MIDDLE SCHOOL AND HIGH SCHOOL CALCULUS

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To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking calculus uncovers a preschool calculus where icon-blocks add next-to, and a middle school calculus adding piecewise constant per-numbers, preparing for adding locally constant per-numbers in high school integral calculus.

BACKGROUND

Institutionalized education typically has mathematics as a core subject in primary and secondary school. To evaluate the success of mathematics education, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015). This might be an effect of teaching a label, mathematics, instead of what it labels, geometry and algebra; and of hiding the Arabic meaning of algebra: to reunite variable and constant unit-numbers and per-numbers by addition, multiplication, power and integration, all allowed in preschool icon-counting before primary school institutionalizes the monopoly of ten-counting. Based upon existentialist philosophy this paper asks: What is existentialist calculus and what difference will it make?

INSTITUTIONAL SKEPTICISM

The ancient Greek sophists recommended enlightenment to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, Symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions. In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting their patronizing choices as nature (Lyotard 1984).

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). Focusing on the three classical virtues Truth, and Beauty and Goodness, Kierkegaard left truth to the natural sciences and argued that to change from a person to a personality the individual should stop admiring essence created by others and instead realize their own existence through individual choices and actions. Furthermore he showed violent resistance against institutionalized Christianity in the form of Christendom, seen also by Nietzsche as imprisoning people in moral serfdom. To break out he hoped that someday we will see a '*redeeming* man (..) whose isolation is misunderstood by the people as if it were flight *from* reality – while it is only his absorption, immersion, penetration *into* reality, so that (..) he may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187).

Arendt carried Heidegger's work further by dividing human activity into labor and work both focusing on the private sphere, and action focusing on the political sphere creating institutions that should be treated with care to avoid the banality of evil by turning totalitarian. (Arendt 1963).

CALCULUS AS ESSENCE

Textbooks see mathematics as a collection of well-proven statements about well-defined concepts, defined from above as examples from abstractions instead of from below as abstractions from examples. The invention of the set-concept allowed mathematics to be self-referring. By looking at the set of sets not belonging to itself, Russell showed that self-reference let to the classical liar paradox 'this sentence it false' being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by its inability to separate concrete examples from abstract essence. That institutionalized education still teaches self-referring mathematics instead of grounded algebra and geometry can be seen as an example of 'symbolic violence' used as an exclusion technique to keep today's knowledge nobility in power (Bourdieu 1977).

In set-based mathematics, differential calculus precedes integral calculus. First a function is defined as an example of a set-relation where first component-identity implies second-component identity. Then the limit concept is used to define a continuous and a differentiable function as well as its derivative. Theorems are presented for the derivative of specific functions and for their combinations. Applications include natural science and function monotony. Integral calculus begins by defining a primitive and presenting the theorem that the area-function to a function f is a primitive to f. This allows presenting the main theorem of calculus: A definite integral of a function is found by subtracting the values of a primitive evaluated in the two endpoints.

CALCULUS AS EXISTENCE

The Pythagoreans used the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas (Freudenthal 1973). With astronomy and music now as independent subjects, today only the two other activities remain both rooted as natural sciences about the physical fact Many, Geometry meaning to measure earth in Greek, and Algebra meaning to reunite numbers in Arabic and replacing Greek arithmetic.

Meeting Many we ask 'how many?' Counting and adding gives the answer as shown by the word algebra, meaning to reunite numbers in Arabic. We count by bundling and stacking as seen when writing a total T in its full form: T = 345 = 3·B^2 + 4·B + 5·1 where the bundle B typically is ten. This shows the four ways to unite: On-top addition unites variable numbers, multiplication constant numbers, power constant factors, and next-to addition, also called integration, unites variable blocks.

As indicated by its name, uniting can be reversed to split a total into parts predicted by the reversed operations, subtraction and division and root & logarithm and differentiation. Likewise, a total can be presented in two forms, an algebraic using place value to separate the singles from the bundles and from the bundle-bundles, and a picture showing three blocks placed next to each other.

Thus the root of calculus is next-to addition of blocks which takes place in preschool, in middle school and in high school, but not in primary school where blocks can only be added on-top because of ten-counting monopoly. But before this, preschool allows icon-counting in units below ten by bundling and stacking resulting in a decimal number with the bundles-size as the unit and with a decimal point to separate the unbundled form the unbundled. Thus a given total might be counted as 3 7s and recounted as 2.5 8s or 4.1 5s. Once counted, totals can be added on-top and next-to. To add 2 3s and 4 5s on-top the units must be the same giving 5.1 5s since 2 3s can be recounted to 1.1 5s. Added next-to as 8s means adding areas, also called integration giving the result is 3.2 8s.

Primary school allows counting in one unit only, tens: But middle school allows several units when counting distances, time, mass, value, etc. The need for changing unit creates per-numbers as 3$/4kg serving as bridges when recounting $s in 3s or kgs in 4s:

15$ = (15/3)·3$ = (15/3)·4kg = 20kg.

Like fractions, per-numbers are not numbers but operators needing a number to become a number. To add, per-numbers must be multiplied to unit-numbers, thus adding as areas, called integration.

Adding 2kg @ 3$/kg and 4kg @ 5$/kg gives (2+4)kg @ (3*2+5*4)/(2+4) $/kg. We see that unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add as areas under the per-number graph that is piecewise constant in middle school.

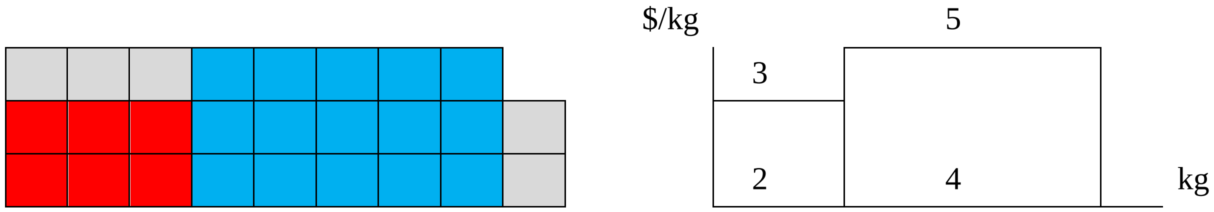

Table 1. Left: Adding 2 3s and 4 5s to 3.2 8s. Right: Adding 2kg @ 3$/kg and 4kg @ 5$/kg as areas

In high school, the per-number graph p is locally constant, so the area comes from adding many area strips with the area p*dx where dx is a small change in x. Adding many numbers is time-consuming unless they are differences:

The numbers n1, n2, n3, …, n20 generates the differences (n2-n1), (n3-n2),…, (n20-n19) adding up to n20 – n1 since the middle numbers cancel out. So to find the total area we try to write the single area p*dx as a difference or a change: p*dx = dP, or dP/dx = p. Thus we need to develop a theory for finding d/dx of functions, called differential calculus - and not the other way around. To do so we need a formal definition of constancy of a variable y:

y is *constant* c if the distance between y and c is arbitrarily small, or in other words, if for all critical distances ε, the distance between y and c is less than ε.	V ε > 0 : I y – c I < ε
y is *piecewise constant* c_A, if an area A exists where y is constant c_A:	Э A, V ε > 0 : I y – c_A I < ε in A
y is *locally constant* c_o, if for all critical distances ε there is an area A where y is constant c_o	V ε > 0, Э A : I y – c_o I < ε in A

Likewise we call the relation between two variables y and x piecewise or locally linear if the change per-number $\Delta y/\Delta x$ is piecewise or locally constant. Continuous and differentiable are other words for locally constant and locally linear.

TESTING IN THE CLASSROOM

In Denmark the passing limit in calculus has been lowered to about 30% correctness. Typically a class starts with a traditional fraction course including 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles. This produces many dropouts forced to stay in class so the school will not lose government funding. One class continued by presenting fractions as per-numbers added as areas. The students were surprised to have their side of the fraction paradox accepted and to hear about per-numbers for the first time. And going on with integral calculus before differential calculus made most students finish with good results. A report was sent to the teacher's journal and to the ministry of education but stayed unpublished.

CONCLUSION AND RECOMMENDATION

To implement its goal, an institution hires civil servants who will also guard it to protect their jobs and careers. To avoid the banality of evil, education should institutionalize only essence with existence behind it guaranteed by using institutional skepticism to uncover pure essence. In calculus, what has existence is next-to addition of icon-counted totals in preschool and of areas created by changing per-numbers to unit-numbers by multiplication in middle school.

So to improve PISA results, textbooks should include calculus in preschool and middle school; and in high school integral calculus should be presented before and as motivation for its reverse, differential calculus; and as a solution to the question: How to add locally constant per-numbers? In a postmodern vocabulary: Maybe calculus needs to be deconstructed (MrAlTarp 2012).

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ESSENCE AND EXISTENCE IN CONFLICTING COGNITIVE THEORIES

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To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking cognitive theories uncovers a hidden conflict between Piaget and Vygotsky, having monopoly in constructivist thinking in North America and in Europe respectively. Seeing existence as preceding essence, existentialist thinking recommends replacing Vygotskyan with Piagetian theory to improve European PISA results.

BACKGROUND

Institutionalized education typically has mathematics as a core subject in primary and secondary school. To evaluate the success of mathematics education, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015). This might be an effect of goal-means confusions. An educational subject always has an outside goal to be reached by inside means. Institutionalized as mandatory, a means might become an inside goal hiding the original outside goal. Existentialist distinction between existence and essence allows institutional skepticism to tell outside from inside goals. So this paper asks: will Piagetian and Vygotskyan theories both serve to improve European PISA results?

INSTITUTIONAL SKEPTICISM

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, Symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting patronizing choices as nature (Lyotard 1984).

Building on Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). Kierkegaard was skeptical towards institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone 'may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187). Inspired by Heidegger, Arendt divided human activity into labor and work both focusing on the private sphere, and action focusing on the political sphere creating institutions to be treated with care to avoid the banality of evil by turning totalitarian (Arendt 1963).

EDUCATION ACCORDING TO VYGOTSKY

By statements as ‘The true direction of the development of thinking is not from the individual to the social, but from the social to the individual’ and ‘What a child can do today with assistance, she will be able to do by herself tomorrow’ and ‘A word devoid of thought is a dead thing, and a thought unembodied in words remains a shadow’ Vygotsky points to what he see as important in the learning process, the presence of a teacher able and willing to mediate social knowledge to the learner expected to use language to discuss it with the teacher and with peers.

As to education Vygotsky points out that ‘Pedagogy must be oriented not to the yesterday, but to the tomorrow of the child's development. Only then can it call to life in the process of education those processes of development which now lie in the zone of proximal development.’

EDUCATION ACCORDING TO PIAGET

As to teaching Piaget says ‘Teaching means creating situations where structures can be discovered’ and ‘When you teach a child something you take away forever his chance of discovering it for himself’. Piaget does not see the goal of education to connect the learners to what counts as today’s knowledge, rather he asks: ‘Are we forming children who are only capable of learning what is already known? Or should we try to develop creative and innovative minds, capable of discovery from the preschool age on, throughout life?’

As to understanding Piaget says that’ Experience precedes understanding’ and that ‘Each time one prematurely teaches a child something he could have discovered himself, that child is kept from inventing it and consequently from understanding it completely.’ And as to learning, Piaget says ‘Equilibrium is the profoundest tendency of all human activity’ and ‘The essential functions of the mind consist in understanding and in inventing, in other words, in building up structures by structuring reality’ and ‘Accommodation of mental structures to reality implies the existence of assimilatory schemata apart from which any structure would be impossible’.

VYGOTSKY AND PIAGET, IN ACCORDANCE OR IN CONFLICT?

Vygotsky sees education as consisting of social knowledge with a teacher creating a ladder to connect it to the learner’s zone of proximal development (ZPD). Piaget, on the other hand, is skeptical towards overemphasizing contemporary knowledge and sees experiencing as more important than teaching, which might hinder the learners in getting an understanding of what they are supposed to learn. Instead Piaget sees education as providing experiences that invite learners to assimilate them to existing schemata or to accommodate these to the experiences, and to discuss their experiences with peers and the teacher.

As to discussing goal-means confusions in education, Vygotsky sees the social knowledge as the goal without discussing if instead it might be a means hindering access to an outside goal. The job of the teacher is to identify the learner’s ZPD to connect it to the goal by a ladder. And the job of learner is to accept this ladder to avoid being excluded socially and from social knowledge. Thus the lacking skepticism towards the background of the social knowledge makes it impossible to have an existentialist discussion on a goal-means confusion based on a Vygotsky point of view. In contrast to this, Piaget explicitly points to outside existence as goals and warns against impressing upon the learner descriptions that do not build upon experience based understandings.

A CLASS ON PROPORTIONALITY

At the MATHeCADEMY.net four in-service teachers working two and two were asked to write an essay reflecting on planning a lesson on proportionality from a Vygotsky and Piaget viewpoint.

The first pair followed Vygotsky in accepting fully the textbook knowledge defining proportionality as a function f(x) = c*x with a proportionality constant c. Their job was to build a ladder from this to the learner's ZPD. They discussed using a deductive or an inductive method. A deductive method would present the textbook definition to be explored algebraically and geometrically through examples, and finally to be applied to examples from trade and physics. An inductive way would show a table relating two variables s and t and asking for patterns if combining the numbers with standard operations. The hope being that the learners cold see that the ratio between the two numbers was a constant c thus allowing setting up the formula s/t = c. Then transforming this formula to s = c*t could lead to the formal definition f(x) = c*x.

As to potential problems they realized, that by defining proportionality as an example of a function, this concept should already be in the learner's ZPD, together with the concepts of fractions and solving simple equations if choosing the inductive method. This might be the case if the class was a self-chosen half-year block as in a North America block-organized school. However, Europe's line-organized Bildung schools use compulsory classes where learners are forced to follow their class for years even if their motivation and understanding is on a low level, which creates a huge variety in ZPDs. So they chose the deductive method hoping to persuade the learners that since proportionality was a core part of social Bildung they were wise to learn it, if necessary by heart, to avoid being excluded socially and from its many social and technical and economical applications.

The second pair followed Piaget in looking for an outside goal rooting the concept. On a picture of three baskets of apples they placed tags saying '3kg, 5$', '7kg, ?$', and '?kg, 10$'. Earlier the class had met the recount-formula, T = (T/b)*b, used to re-count in a different unit, e.g. recounting 2 3s in 4s by saying T = (2x3/4)*4 = 1 4s + 2. By introducing the word 'double-counting' they hoped the learners could assimilate the problem to what they already knew by recounting the kgs in 3s and the $s in 5s: 10$ = (10/5)*5$ = (10/5)*3kg = 6 kg, or $ = ($/kg)*kg rooting concepts as a 'per-number' 5$ per 3 kg = 5$/3kg = 5/3 $/kg and y = c*x as a new version of the recount-formula. This would allow the learners to build an understanding through experiencing and without too much teaching.

THE VYGOTSKY-PIAGET CONFLICT FROM A LUHMANNIAN VIEWPOINT

The German sociologist Luhmann sees communication as the core activity in a learning process and says that 'Humans cannot communicate; not even their brains can communicate; not even their conscious minds can communicate. Only communication can communicate'. To Luhmann, a learner and the classroom are self-referring systems having each other as surrounding environments from which complexity can be imported to reduce outside and increase inside complexity.

In the Vygotsky class, the classroom textbook communication choose to, not import complexity, but to export a threat to provoke the learner to import outside textbook concepts, not to reduce complexity, but to avoid social exclusion. In the Piaget class, outside examples allows the classroom and learner systems to communicate about how they use self-reference to reduce the complexity of the outside examples thus allowing the learners and the teacher to apply or to differentiate their existing complexity.

CONCLUSION AND RECOMMENDATIONS

This paper asked: will Piagetian and Vygotskyan theories both serve to improve EU PISA results?

To Vygotsky, teaching is improved when teachers can identify a learner's zone of proximate development to construct a ladder connecting it to the textbook that by being socially approved cannot be questioned. Making the textbook a mandatory inside goal opaque for the original outside goal creates an inflexibility that makes a goal-means discussion difficult. Likewise, Vygotsky does not discuss the challenges of being a European teacher in a line-organized education with compulsory classes where learners by being forced to stay together over the years develop a huge variety of positive and negative ZPDs caused by individual motivation and ability differences.

To Piaget, teaching is improved by limiting itself and by supplying or replacing the textbook with examples of the outside goal allowing learners to experience to create individual schemata to be accommodated by further experiences and by dialogues with peers and with the teacher. Having chosen daily lessons in self-chosen half-year blocks as the structure for secondary and tertiary education, North America favors Piaget when discussing constructivist learning as an alternative to the dominant textbook tradition.

To Luhmann, improving teaching means improving communication between the two self-referring classroom systems, the teaching system and the learner system, having each other as environment. To improve, both systems must be willing to learn by importing complexity from each other. Consequently, teachers should be able to see a textbook opaque to its outside goal as a dead and non-emerging system, and that development means that leaners and teacher communicate about the complexities of the outside goal in their mutual environment.

From an existentialist viewpoint distinguishing between existence and essence there is a danger that a textbook reflects only essence. Seeing the textbook as the goal, Vygotskyan theory has difficulties discussing goal-means confusions; in opposition to Piagetian theory pointing out that too much teaching will prevent this discussion. Luhmannian theory makes the same point by saying that teachers should cooperate with the learner to develop understandings of their common outside goal.

Seeing existence as preceding essence, an existentialist recommendation to European education will be: To improve PISA results, replace line- with block-organized education, and replace Vygotskyan dominance with a combination of Piaget and Luhmann.

In short: Teachers, don't preach essence, teach existence - and learn about it at the same time.

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FROM ESSENCE TO EXISTENCE IN CURRICULUM DEVELOPMENT

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To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking primary mathematics uncovers icon-counting in numbers below ten, thus allowing recounting in a different unit or in the same unit to create overloads or deficits; and when added on-top or next-to roots proportionality and integration. These golden learning opportunities disappear if the curriculum allows ten-counting only.

BACKGROUND

Institutionalized education typically has mathematics as a core subject in primary and secondary school. To evaluate the success of mathematics education, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015). This might be an effect of goal-means confusions. An educational subject always has an outside goal to be reached by inside means. Seen as mandatory, a means becomes an inside goal that might hinder learners to reach the original outside goal. Distinguishing existence from essence allows existentialist philosophy to distinguish outside from inside goals. So this paper asks: What is existentialist mathematics curriculum - and what difference will it make?

INSTITUTIONAL SKEPTICISM

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, Symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting their patronizing choices as nature (Lyotard 1984).

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BUILDING A NATURAL SCIENCE ABOUT MANY: MANY-MATICS

To deal with Many, first we iconize, then we count by bundling and stacking. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in e.g. fives.

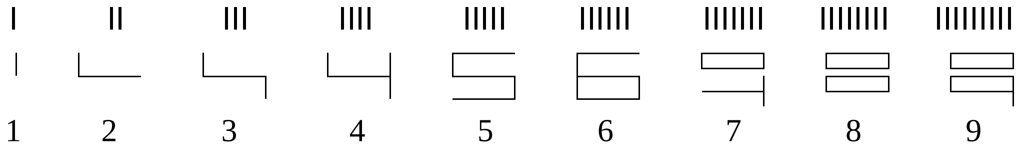

Fig. 1. Sticks or a folding rule used to create icons with as many sticks as they represent

With ‘second order counting’ we count in icon-bundles. A total T of 7 is bundled in 3s as T = 2 3s and 1. The unbundled we place in a right single-cup, and in a left bundle-cup we place two stick for the bundles. The cup-contents is described by icons, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Alternatively, we can use plastic letters as B, C or D for the bundles.

IIIIIII → III III I → II) I) → 2)1) → 2.1 3s or BB I → 2B I

We live in space and in time. To include both when counting, we introduce two ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and show them on a ten-by-ten abacus in geometry-mode. Counting in time, we count bundles and report the result on a ten-by-ten abacus in algebra-mode with lines for bundled and unbundled.

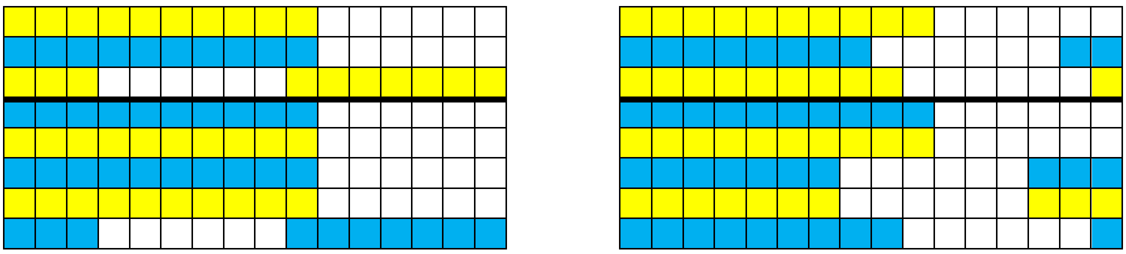

Fig. 2. Counting 7 in 3s on an abacus in geometry mode below and algebra mode above the rubber

To predict the counting result we use a calculator. A stack of 2 3s is iconized as 2x3 showing a jack used 2 times to lift the 3s. Taking away is iconized with ‘/3’ or ‘–3’ showing the broom or the trace when wiping away 3 several times or just once, called division and subtraction. Entering ‘7/3’, we ask the calculator ‘from 7 take away 3s’ and get the answer ‘2.some’. Entering ‘7 – 2x3’ we ask ‘from 7 take away 2 3s’ and get the answer 1 leftover. Thus the calculator predicts that 7 = 2.1 3s.

7 / 3	2.some
7 – 2 * 3	1

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3.2 2s or as 2.4 2s. Likewise 4.2s can be recounted as 5 2s less 2; or as 6 2s less 4 thus leading to negative numbers.

To recount in a different unit means changing unit, called proportionality. Asking ‘3 4s is how many 5s?’ we can use letters or sticks to see that 3 4s becomes 2.2 5s. With letters, C = BI so that BBB → BB IIII → CC II . With sticks: IIII IIII IIII → IIIII IIIII I I → 2) 2) 5s → 2.2 5s.

Once counted, totals can be added on-top or next-to. Asking '3 5s and 2 3s total how many 5s?' we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s giving 1.1 s that added to the 3 5s gives a grand total of 4.1 5s. With letters: 3B + 2C = 3B III III = 4BI. With sticks: IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4) 1) 5s → 4.1 5s,

Since 3*5 is an area, adding next-to means adding areas, called integration. Asking '3 5s and 2 3s total how many 8s?' we can use sticks, or letters, to get the answer 2.5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2) 5) 8s → 2.5 8s

Different units occur when counting value, weight, etc. Double-counting a total T in $ and in kg creates per-numbers as 3$ per 4 kg = 3/4 $/kg bridging units by recounting $s in 3s and kgs in 4s:

T = 15$ = (15/3)·3$ = (15/3)·4kg = 20kg, and T = 12kg = (12/4)·4kg = (12/4)·3$ = 9$.

DESIGNING A CURRICULUM IN MANY-MATICS FOR PRE- OR PRIMARY SCHOOL

The findings let to designing eight micro-curricula for preschool using activities with concrete material to obtain its learning goals in accordance with Piaget's principle 'greifen vor begrifen' (grasp to grasp) (MATHeCADEMY.net/ preschool).

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using two cups for the bundled and the unbundled reported with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and deficits. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other. In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eights, the learner reverses next-to addition. The micro-curricula M2-M8 uses the recount- and restack-formula on a calculator to predict the result. Examples and calculator predictions are shown below:

M2	7 1s is how many 3s? I I I I I I I → III III I → 2) 1) 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	'2.7 5s is also how many 5s?' IIIII IIIII IIIIIII = V V V II = V V V V III 2)7) = 2+1)7-5) = 3)2) = 3+1)2-5) = 4)-3) So 2.7 5s = 3.2 5s = 4.-3 5s,	(2*5+7)/5 (2*5+7) -3*5 (2*5+7) -4*5	3.some 2 -3
M4	2 5s is how many 4s?' IIIII IIIII = IIII I IIII I = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2 * 4	2.some 2
M5	'2 5s and 4 3s total how many 5s?' IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2
M6	'2 5s and 4 3s total how many 8s?' IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 (2*5+4*3) – 2*8	2.some 6
M7	'2 5s and how many 3s total 4 5s?' IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1

M8	'2 5s and how many 3s total how 2.1 8s?' IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(4*5 – 2*5)/3 3.some (4*5 – 2*5) – 3*5 1

Fig. 3. A calculator predicts counting and adding results

Part of a curriculum can use silent education where the teacher demonstrates and guides through actions only, not using words; another part can be carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator are expected quickly to take over the communication.

CONCLUSION AND RECOMMENDATIONS

Institutionalized education sees mathematics, not as a means to an outside goal, but as a goal in itself to be obtained by curricula that, instead of teaching counting before adding, skip counting and represents totals by place value numbers to be added and subtracted and multiplied using carrying.

Using the existentialist distinction between existence and essence, institutional skepticism has uncovered a hidden alternative to the ruling tradition, 'ManyMatics', a natural science about the physical fact Many. Mastery of Many comes from counting and adding. We count by bundling and stacking reported by cup-writing and decimal numbers with a unit as an alternative to the ruling place value tradition. Once counted, a total can be recounted in the same unit by creating or removing overloads or deficits. This makes cup-writing an alternative to carrying. Also, totals can be recounted in another unit predicated by a calculator using a recount- and a restack-formula. Recounting in tens create the times tables, and recounting tens in icon-bundles uncovers recounting as the root of equations. Finally, recounting in two physical units creates per-numbers.

With Mastery of Many as its goal these findings motivate the creation of an alternative algebra curriculum for preschool and primary school that is grounded in its outside goal by allowing the learner to count before they add on-top and next-to, to write numbers using cup-writing and as decimal numbers with units, to use an abacus in both algebra and geometry-mode, to predict the result of recounting in the same or in another unit by a recount- and a restack-formula on a calculator, to create and remove overloads or deficits when adding or subtracting or multiplying, to recount digits in 3s when dividing by 3, to reverse adding on-top and next-to, to use per-numbers when recounting in physical units, and to use integration and differentiation when doing forward or reversed next-to addition of per-numbers or icon-numbers.

In short: Don't preach essence, teach existence.

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FROM ESSENCE TO EXISTENCE IN MATH TEACHER EDUCATION

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In mathematics and its education, the difference between essence and existence is seldom discussed although central to existentialist thinking. Thus we can ask: How will an existentialist mathematics teacher education look like? So we close the door to the library of self-referring mathematics and go outside to rebuild mathematics from its roots, the physical fact Many. Likewise, we can ask if mathematics is learned by being rooted in its outside existence or in its inside essence claims.

BACKGROUND

Institutionalized education typically has mathematics as a core subject in primary and secondary school. To evaluate the success of mathematics education, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report ‘Improving Schools in Sweden’ (OECD 2015). This might be an effect of teaching a label, mathematics, instead of what it labels, geometry and algebra; and of hiding the Arabic meaning of algebra: to reunite variable and constant unit-numbers and per-numbers by addition, multiplication, power and integration, all allowed in preschool icon-counting before primary school institutionalizes the monopoly of ten-counting. Based upon existentialist philosophy this paper asks: What is existentialist mathematics teacher education?

INSTITUTIONAL SKEPTICISM

The ancient Greek sophists recommended enlightenment to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, Symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget’s principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting their patronizing choices as nature (Lyotard 1984).

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers ‘existence precedes essence, or (..) that subjectivity must be the starting point’ (Marino 2004: 344). Kierkegaard was skeptical towards institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone ‘may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.’ (Marino 2004: 186-187). Inspired by Heidegger, Arendt divided human activity into labor and work both focusing on the private sphere, and action focusing on the political sphere creating institutions to be treated with care to avoid the banality of evil (Arendt 1963).

MATHEMATICS AS ESSENCE

In ancient Greece the Pythagoreans used the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music now as independent knowledge areas, today mathematics is a common label for the two remaining activities, Geometry and Algebra replacing Greek Arithmetic (Freudenthal 1973).

Textbooks see mathematics as a collection of well-proven statements about well-defined concepts, defined from above as examples from abstractions instead of from below as abstractions from examples. The invention of the set-concept allowed mathematics to be self-referring. By looking at the set of sets not belonging to itself, Russell showed that self-reference let to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by its inability to separate concrete examples from abstract essence. That institutionalized education still teaches self-referring mathematics instead of grounded algebra and geometry can be seen as an example of ‘symbolic violence’ used as an exclusion technique to keep today’s knowledge nobility in power (Bourdieu 1977).

MATHEMATICS AS EXISTENCE

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry, both rooted as natural sciences about the physical fact Many.

Meeting Many we ask ‘how many?’ Counting and adding gives the answer as shown by the word algebra, meaning to reunite numbers in Arabic. We count by bundling and stacking as seen when writing a total T in its full form: T = 345 = 3·B^2 + 4·B + 5·1 where the bundle B typically is ten. This shows the four ways to unite: On-top addition unites variable numbers, multiplication constant numbers, power constant factors, and next-to addition, also called integration, unites variable blocks. As indicated by its name, uniting can be reversed to split a total into parts predicted by the reversed operations, subtraction and division and root & logarithm and differentiation. Likewise, a total can be presented in two forms, an algebraic using place value to separate the singles from the bundles and from the bundle-bundles, and a picture showing three blocks placed next to each other.

So a grounded mathematics is about counting and adding in time and space, CATS. The website MATHeCADEMY.net contains grounded mathematics organized in activities where the learner learns ‘CATS’, guided by educational questions and answers. The study units CATS1 are for primary school and the study units CATS2 are for secondary school.

Section Counting C1 looks at ways to count Many. Spatial multiplicity is representing temporal repetition through sticks and strokes. A multiplicity of sticks can be rearranged in icons so that there are four sticks in the icon 4 etc. Then a given total T can be counted in e.g. 4s by repeating the process ‘fromT take away 4’, which can be iconized as ‘T–4 ’; where the repeated process ‘from T take away 4s’ can be iconized as ‘T/4’. This makes it possible to predict the counting-result through a calculation using the ‘recount-equation’ T = (T/b)*b. Leftovers are stacked as 1s creating a stock T = 2*3 + 2*1. The stacks can be placed in two cups, a left bundle-cup and a right single-cup, and described by cup-writing T = 2)2), or decimal-writing including the unit T = 2)2) = 2.2 3s = 2.2*3; or the leftovers can be counted in 3s and added on top of the 3-stack: T = 2 2/3 * 3 = 2 2/3 3s.

Changing units is another example of a recounting where a given total is double-counted in two different units e.g. T = 4$ = 5kg producing a per-number 4$/5kg = 4/5 $/kg. Thus to answer the question ' 7kg = ?$' we just have to recount the 7 in 5s: T = 7kg = (7/5)*5kg = (7/5)*4$ = 5 3/5$.

The number ten has a name but no icon, since the bundle-size is not used: Counting in 5s, 5 1s = 1 5s = 1 bundle. Before introducing ten as the standard-bundle and leaving out the units, 2.4 tens = 24, the core of mathematics can be leaned by using 1digit numbers alone (Zybartas et al 2005).

Section Adding A1 looks at how stacks can be added by removing overloads that often appears when one stack is placed on top of another stack. The overload leads to 'internal trade' between two stacks where a stack of 10 1s is rebundled and restacked as 1 10-bundle. The result can be predicted by a calculation on paper using either a vertical way of writing the stacks using carrying to symbolize the internal trade; or using a horizontal way of writing the stacks using the FOIL-principle (First, Outside, Inside, Last). In both cases the overload can be restacked predicted by the restack-equation (T–b) + b, and recounted predicted by the recount-equation T = (T/b)*b.

Section Time T1 looks at formulas, the sentences of the number-prediction language. Containing two unknown variables, a formula becomes a function to be tabled and graphed. Containing one unknown variable, a formula becomes an equation to be solved by reversing the calculations, moving numbers from the forward-calculation side to the backward-calculation side reversing their signs: x*3 + 2 = 14 is reversed to x = (14-2)/3. This forward and backward calculation method gives a new perspective on the classical quantitative literature consisting of word-problems.

Section Space S1 looks at how to describe plane properties of stacks as areas and diagonals by the 3 Greek Pythagoras', mini, midi & maxi; and by the 3 Arabic recount-equations: sin A = a/c, cos A = b/c and tan A = a/b. A circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: C = 2*r*(n*sin(180/n)) = 2*r*π for n sufficiently big. Finally we look at how to describe spatial properties of solids such as surfaces and volumes by formulas and by a 2-dimensional representation of 3-dimensional shapes.

Section Counting C2 looks at numbers that change unpredictably e.g. in surveys. Through counting we can set up a frequency-table accounting for the previous behavior of the numbers. From this table their average level and their average change can be calculated. From this we can predict that with a 95% probability, future numbers will occur within an interval determined by the average level and double the average change. Counting the numbers of wins when repeating a game with winning probability p, is another example of an unpredictable number, called a stochastic variable.

Section Adding A2 looks at how to add per-numbers by transforming them to totals. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b. 2days at 6$/day + 3days at 8$/day = 5days at 7.2$/day. And 1/2 of 2 cans + 2/3 of 3 cans = 3 of 5 cans = 3/5 of 5 cans. Repeated and reversed addition of per-numbers leads to integration and differentiation: T2 = T1 + a*b; T2 - T1 = a*b; $\Delta T = \Sigma a*b = \int y*dx$; and T2 = T1 + a*b; T2 – T1 = a*b; a = (T2 – T1)/b = $\Delta T/\Delta b$ = dy/dx

Section Time T2 looks at how a stack changes in time by adding a constant number, or by a constant percent where adding 5% means changing 100% to 105%, i.e. multiplying with 105% = 1.05. If related by a formula y = f(x), a x-change Δx will effect a y-change Δy that can be recounted in the x-change as $\Delta y = (\Delta y/\Delta x)*\Delta x$, or dy = (dy/dx)*dx = y'*dx in the case of micro-changes. If a stack y changes by adding variable predictable numbers dy, summing up the single y-changes gives the total y-change, i.e. the terminal y2 minus the initial y1: $\int dy = \int y' dx = y2 - y1$.

Section Space S2 looks at how to predict the position of points and lines and geometrical figures and graphs using a coordinate system; and how to use the new calculation technology such as computers and calculators to calculate a set of numbers, vectors, and a set of vectors, matrices.

In literature, qualitative word-statements and quantitative number-statements share the same genres. Predicting predictable quantities, fact-models must be tested for errors. Predicting unpredictable quantities, fiction-models must be supplemented with scenarios built on alternative assumptions. Predicting qualities, fiddle-models must be expelled from the number- to the word-language.

LEARNING AS ESSENCE AND EXISTENCE

Constructivist learning theory contains a European social Vygotskian and a North American radical Piagetian version believing learning takes place through guidance or exposure respectively. To let existence precede essence, Vygotsky must go and Piaget used for external exposure to Many.

CONCLUSION AND RECOMMENDATIONS

To implement its goal, an institution hires civil servants who will also guard it to protect their jobs and careers. To avoid the banality of evil, education should institutionalize only essence with existence behind it guaranteed by using institutional skepticism to uncover pure essence. In pre- and in-service education teachers should meet mathematics as essence deduced from above to understand the content of textbooks; and mathematics as existence induced from below, i.e. as a mere label for Algebra and Geometry, both created as natural science about the physical fact Many.

Knowing nature from choice allows a teacher to avoid the banality of evil and to carry out action research in the classroom to balance the irrelevance paradox of traditional research. By experiencing proportionality and integration as golden learning opportunities in icon-counting and next-to addition, teachers see that neglecting icon-counting and the actual order of operations is not nature, but choice that can be changed. By realizing that fractions are per-numbers, i.e. not numbers but operators, needing unit-numbers to be added by their areas, teachers see that integration comes before integration in both middle and high school calculus. In a postmodern vocabulary: Maybe teacher education in mathematics needs to be deconstructed (MrAlTarp 2012).

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FROM ESSENCE TO EXISTENCE IN MATHEMATICS EDUCATION

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In mathematics and its education, the difference between essence and existence is seldom discussed although central to existentialist thinking. So we can ask: What will an existentialist mathematics education look like? Thus we close the door to the library with today's self-referring mathematics and go outside to rebuild mathematics from its roots, the physical fact Many. Likewise, we can ask if mathematics is learned by exposure to its outside roots or to its inside essence claims.

BACKGROUND

Institutionalized education typically has mathematics as one of its core subjects in primary and secondary school. To evaluate the success OECD arranges PISA studies on a regular basis. Here increasing funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015).

As to the content of education, sociology offers understandings of schools and teacher education, psychology of learning, and philosophy of textbooks. Focusing upon existentialist philosophy this paper asks: What will an existentialist mathematics and education look like? The purpose is not to replace one tradition with another but to uncover hidden alternatives to choices presented as nature.

EXISTENTIALISM

The Pythagoreans labeled their four knowledge areas by a Greek word for knowledge, mathematics. With astronomy and music now as independent areas, today mathematics is a common label for the two remaining activities, geometry meaning to measure earth in Greek, and Algebra meaning to reunite numbers in Arabic and replacing Greek arithmetic (Freudenthal 1973).

The Greeks used the word 'sophy' meaning knowledge for men of knowledge, the sophists and the philosophers, disagreeing on the nature of knowledge. Seeing democracy with information and debate and choice as the natural state-form, the sophists emphasized knowing nature from choice to prevent patronization by choices presented as nature. Seeing autocracy patronized by themselves as the natural state-form, the philosophers saw choice as an illusion since to them physical existence was but examples of metaphysical essence only visible to the philosophers educated at the Plato academy having as entrance sign 'Let no one ignorant of geometry enter.'

Today, the sophist skepticism towards false is-claims is carried on by French post-structuralism and by the existentialism of Kierkegaard, Nietzsche, Heidegger and Sartre, defining existentialism as holding that 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). In Denmark, a heritage allowed Kierkegaard to publish whatever he wrote. At the end, shortage forced him to shift to flying papers when rebelling against institutionalized Christianity in the form of Christendom. Focusing on the three classical virtues Truth, and Beauty and Goodness, Kierkegaard left truth to the natural sciences, and argued that to change from a person to a personality the individual should stop admiring beauty created by others and instead realize their own existence through individual choices. Of course, angst is a consequence when fearing to choose the bad instead of the good, and death might follow, but so will forgiveness and resurrection to a new life in real existence, as promised by Christianity in the Holy Communion.

In Germany, Nietzsche saw institutionalized Christendom as the creator of moral is-statements that prevented individuals from realizing their true existence through individual choices and action. To end this serfdom he hoped that someday we will see a '*redeeming* man (..) whose isolation is misunderstood by the people as if it were flight *from* reality – while it is only his absorption, immersion, penetration *into* reality, so that (..) he may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187).

Likewise in Germany, Heidegger saw that to avoid traditional essence-claims, is-statements must be replaced by has-statements so that being is characterized by what it has, 'Dasein'. Arendt carried his work further by dividing human activity into labor and work focusing on the private sphere and action focusing on the political sphere thus accepting as the first philosopher political action as a worthy human activity creating institutions that should be treated with care to avoid 'the banality of evil' if turning totalitarian (Arendt 1998).

MATHEMATICS AS ESSENCE

Within mathematics, the existentialist distinction is shown by the function concept, defined by Euler as labeling the existence of calculations combining known and unknown numbers, and today defined as set-relations where first component-identity implies second-component identity thus becoming pure essence through self-reference. The set-concept transformed mathematics to 'meta-matics', a self-referring collection of well-proven statements about well-defined concepts, defined as examples from internal abstractions instead of as abstractions from external examples. Looking at the set of sets not belonging to itself allowed Russell to show that self-reference leads to the classical liar paradox 'this sentence it false' being false if true and true if false:

If $M = \{ A \mid A \notin A) \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by its inability to separate outside examples from inside abstractions. That institutionalized education ignores this can be seen as an example of 'symbolic violence' used to protect the privileges of today's 'knowledge nobility' (Bourdieu 1977).

Behind colorful illustrations, self-referring metamatics is taught through a gradual presentation of different number types, natural numbers and integers and rational and real numbers, together with the four basic operations, addition and subtraction and multiplication and division, where especially division and letter fractions create learning problems. Equations are introduced as equivalent number names to be changed by identical operations. In pre-calculus polynomial functions are introduced as a basis for calculus presenting differential calculus before integral calculus.

MATHEMATICS AS EXISTENCE

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, geometry and algebra, both rooted as natural sciences about the physical fact Many.

The root of geometry is the standard form, a rectangle, that halved by a diagonal becomes two right-angled triangles where the sides and the angles are connected by three laws, A+B+C = 180, a^2+b^2 = c^2 and sinA = a/c. Being filled from the inside by such triangles, a circle with radius r gets the circumference 2·π·r where π = n·sin(180/n) for n sufficiently large.

Meeting Many we ask 'how many?' Counting and adding gives the answer. We count by bundling and stacking as seen when writing a total T in its full form: T = 345 = 3·B^2 + 4·B + 5·1 where the bundle B typically is ten. This shows the four ways to unite: On-top addition unites variable numbers, multiplication constant numbers, power constant factors, and next-to addition, also called integration, unites variable blocks. As indicated by its name, uniting can be reversed to split a total into parts predicted by the reversed operations, subtraction and division and root & logarithm and differentiation. Likewise, a total can be presented in two forms, an algebraic form using place values to separate the singles from the bundles and the bundle-bundles, and a geometrical form showing three blocks placed next to each other.

Although presented as essence, ten-bundling is a choice. To experience its existence and the root of core mathematics as proportionality and linearity, Many should be bundled in icon-bundles below ten to allow a calculator to predict the result of shifting units: Asking e.g. 'T = 2 3s = ? 4s' the answer is predicted by two formulas, a recount-formula T = (T/b)·b telling that from a total T, bs can be taken away T/b times, and a restack-formula T = (T-b)+b telling that from a total T, b can be taken away and placed next-to. First T = (2·3)/4 gives 1.some. Then T = 2·3 - 1·4 gives 2. Si the prediction is T = 2 3s = 1 4s & 2 = 1.2 4s. Thus with icon-counting a natural number is a decimal number with a unit where the decimal point separates the singles from the bundled.

With physical units, the need for changing units creates per-numbers as 3$/4kg serving as bridges when recounting $s in 3s or kgs in 4s: 15$ = (15/3)·3$ = (15/3)·4kg = 20kg. As per-numbers, fractions are not numbers but operators needing a number to become a number. To add, per-numbers must be multiplied to unit-numbers, thus adding as areas, called integration.

So relinking it to its root, Many, allows today's 'mandarin mathematics' to escape from its present essence-prison. For details, see the 2012 videos from MrAlTarp.

LEARNING AS ESSENCE AND EXISTENCE

Constructivist learning theory contains a European social Vygotskian and a North American radical Piagetian version believing learning taking place through guidance or exposure respectively. The question now is what is to be learned.

Here Vygotsky accepts the ruling essence-claims about the outside fact Many even if self-reference makes them meaningless. Learning is seen as adapting to them and teaching as developing the learner's mind in their direction using outside artefacts as means. Piaget sees learning as a means to adapt to the outside world, and sees teaching as asking guiding questions to outside existence brought inside the classroom to allow learners construct individual schemata to be accommodated through exposure and communication. So to let existence precede essence, Piaget is useful to mediate learning through inside exposure to outside existence. Vygotsky is useful if accepting that outside existence can lead to competing inside essence-claims. However, its lacking skepticism towards the ruling claim involves a high risk for mediating the banality of evil.

INSTITUTIONALIZED EDUCATION AS ESSENCE AND EXISTENCE

Two versions of post-primary education exist, one letting national administration define its essence, the other letting individual talents define its existence. To get Napoleon out of Berlin, a European line-organized office-directed education was created that concentrate teenagers in age-groups and force them to follow the same schedule. To meet the international norm that 95% of an age-group finishes high school, dropout rates are lowered by low passing grades and by strict retention policy.

In the North American republics middle and high schools teachers teach their major subject in their own classroom where they welcome teenagers with recognition: 'Inside, you carry a talent and it is our mutual task to uncover and develop your personal talent through daily lessons in self-chosen half-year blocks. If successful I say 'Good job, you have a talent, you need more'. If not I say 'Good try, you have courage to try uncertainty, now try something else that might be your talent.'

CONCLUSION

An existentialist view replacing essence with existence exposes today's mathematics as pure essence with little existence behind. What has existence is Many waiting to be united by bundling and stacking into a decimal number with a unit presented geometrically or algebraically as a row of blocks or digits. Thus mathematics exists as geometry measuring forms divided into triangles, and as algebra reuniting numbers by four uniting techniques, addition, multiplication, power and integration each with a corresponding reversed splitting technique. So concepts should present themselves as created, not by self-reference as examples from abstractions above, but as abstractions from examples below. And statements should be held true when not falsified. In short, mathematics should be taught and learned as 'many-matics', not as 'metamatism', a mixture of metamatics and 'mathematism' true inside but not outside the classroom as e.g. 'the fraction paradox' where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

As to learning, mediating the ruling essence should be replaced by guided exposure to the roots of mathematics, the physical fact Many, thus replacing Vygotsky with Piaget. And institutionalized education using camps to concentrate teenagers in age-groups obliged to follow forced schedules should be labeled as such allowing mathematics education to avoid the banality of evil. Christianity's Holy Communion offers forgiveness to individuals, not to institutions. Instead institutionalized force should be limited to provide teenagers with daily lessons in self-chosen half-year blocks to uncover and develop their individual talent, as would be the case if the North American Enlightenment republics replaced essence with existence in algebra and geometry.

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