Cupcounting and calculus in preschool and in special needs education

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To improve PISA results, institutional skepticism rethinks mathematics education to search for hidden alternatives to choices institutionalized as nature. Rethinking preschool and primary school mathematics uncovers cup-counting in bundles less than ten; as well as re-counting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, information and communications technology, a calculator can predict re-counting results before being carried out manually. By allowing overloads and underloads when re-counting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary classes only allowing ten-counting and on-top addition to take place.

Keywords: Numbers, numeracy, addition, calculus, elementary school mathematics.

Decreased PISA performance despite increased research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change' (OECD, 2015).

Created to help students cope with the outside world, schools institutionalize subjects as inside means to outside goals. To each goal there are many means, to be replaced if not leading to the goal; unless a means becomes a goal itself, thus preventing looking for alternative means that could lead to the real goal if difficult to access. So we can ask: Does mathematics education have a goal-means exchange seeing inside mathematics as the goal and the outside world as a means?

Once created as a means to solve an outside problem, not solving the problem easily becomes a means to necessitate the institution. So to avoid a goal/means exchange, an institution must be reminded constantly about its outside goal. Institutional skepticism is created to do precisely that.

Institutional skepticism

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by Plato philosophy presenting choices as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, symbolic interactionism and Grounded theory (Glaser & Strauss, 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget, 1970). In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnoses, and education all presenting patronizing choices as nature (Lyotard, 1984; Tarp, 2004).

Building on Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino, 2004, p. 344). Kierkegaard was skeptical to institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone 'may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino, 2004, pp. 186–187). Inspired by Heidegger, Arendt divided human activity into labor and work aiming at survival and reproduction, and action focusing on politics, creating institutions to be treated with utmost care to avoid the banality of evil by turning totalitarian (Arendt, 1963).

Since one existence gives rise to many essence-claims, the existentialist distinction between existence and essence offers a perspective to distinguish between one goal and many means.

Mathematics as essence

In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra (Freudenthal, 1973) both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite Many' in Arabic.

Then the invention of the concept SET allowed mathematics to be a self-referring collection of 'well-proven' statements about 'well-defined' concepts, i.e. as 'MetaMatics', defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract essence. Thus SET transformed grounded mathematics into a self-referring 'MetaMatism', a mixture of MetaMatics and 'MatheMatism' true inside a classroom but not outside where claims as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days. And, as expected, teaching numbers without units and meaningless self-reference creates learning problems.

Mathematics as existence

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry. Algebra contains four ways to unite as shown when writing out fully the total T = 342 = 3*B^2 + 4*B + 2*1 = 3 bundles of bundles and 4 bundles and 2 unbundled singles = 3 blocks. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reversing splitting operation. So, with a human need to describe the physical fact Many, algebra was created as a natural science about Many. To deal with Many, we count by bundling and stacking. But first we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written less sloppy. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc.

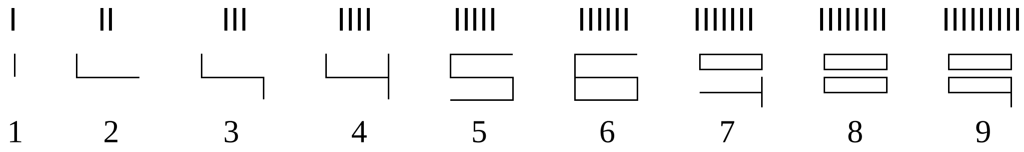
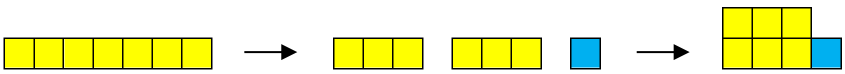


Figure 1: Digits as icons containing as many sticks as they represent

Holding 4 fingers together 2 by 2, a 3year old child will say ‘That is not 4, that is 2 2s. This inspires ‘cup-counting’ bundling a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1 where the bundles are placed in a bundle-cup with a stick for each bundle, leaving the unbundled outside. Then we describe by icons, first using ‘cup-writing’, T = 2)1, then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Moving a stick outside or inside the cup changes the normal form to overload or underload form. Also, we can use plastic letters as B and C for the bundles.

T = 7 = IIIIIII → III III I → II) I → 2)1 3s = 1)4 3s = 3)-2 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a cup-number or a decimal number, T = 7 = 2 3s & 1 = 2)1 3s = 2.1 3s.



We live in space and in time. To include both when counting, we introduce two different ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.

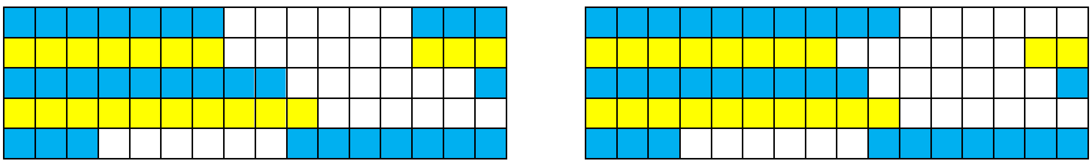


Figure 2: 7 counted in 3s on an abacus in geometry mode and in algebra mode

To predict the result we use a calculator. A stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s. As for the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once.

Thus by entering ‘7/3’ we ask the calculator ‘from 7 we can take away 3s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 3s by asking ‘7 – 2*3’. From the answer ‘1’ we conclude that 7 = 2)1 3s. Likewise, showing ‘7 – 2*3 = 1’, a display indirectly predicts that 7 can be recounted as 2 3s and 1, or as 2)1 3s.

7 / 3	2.some
7 – 2 * 3	1

A calculator thus uses a ‘recount-formula‘, T = (T/B)*B, saying that ‘from T, T/b times Bs can be taken away’; and a ‘restack-formula’, T = (T–B)+B, saying that ‘from T, T–B is left if B is taken away and placed next-to’. The two formulas may be shown by using LEGO blocks.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3)2 2s with an outside overload; or as 5)-2 2s with an outside underload thus leading to negative numbers:

Letters	Sticks	Total T =	Calculator	
B B B B	II II II II	4)0 2s	4*2 – 4*2	0
B B B I I	II II II I I	3)2 2s	4*2 – 3*2	2
B B B B B <u>B</u>	II II II II II <u>II</u>	5)-2 2s	4*2 – 5*2	-2

Figure 3: Recounting 4 2s in the same unit creates overload or underload

To recount in a different unit means changing unit, also called proportionality or linearity. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2)2 5s.

IIII IIII IIII → IIIII IIIII I I → 2)2 5s. With letters, C = BI so that BBB → BB IIII → CC II

A calculator can predict the result. Entering ‘3*4/5’ we ask ‘from 3 4s we take away 5s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be recounted as 2 5s and 2, or as 2)2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘6$ = ?kg’ we use the per-number to recount 6 in 2s: 6$ = (6/2)*2$ = 3*3kg = 9kg. And vice versa: Asking ‘?$ = 12kg’, the answer is 12kg = (12/3)*3kg = 4*2$ = 8$.

Once counted, totals can be added on-top or next-to.

Asking ‘3 5s and 2 3s total how many 5s?’ we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s as 1)1 5s that added to the 3 5s gives a total of 4)1 5s.

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4)1 5s. With letters: 3B + 2C = 3B III III = 4BI.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. Thus we get 4)1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2)5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2)5 8s → 2.5 8s

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2)5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing adding on-top and next-to

Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2)6 8s?’, using sticks will give the answer 2)1 3s:

IIIII IIIII IIIII III III I ← IIIII III) IIIII III) IIIIII ← 2)6 8s

Using a calculator to predict the result the remaining is bracketed before counted in 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration means subtracting before dividing, as shown in the gradient formula y’ = Δy/t = (y2 – y1)/t.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Primary schools use ten-counting only

In primary school numbers are counted in tens to be added, subtracted, multiplied and divided. This leads to questions as ‘3 4s = ? tens’. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount- and restack-formula above is impossible since the calculator has no ten button. Instead it is programmed to give the answer directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

3 * 4	12

Recounting icon-numbers in tens is called doing times tables to be learned by heart. So from grade 1, 3*4 is not 3 4s any more but has to be recounted in tens as 1.2 tens, or 12 in the short form.

Recounting tens in icons by asking ‘38 = ? 7s’ is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens, 0.3 7s must be written in fraction form as 3/7 and calculated as 0.428…, shown directly by the calculator, 38/7 = 5.428…

38 / 7	5.some
38 – 5 * 7	3

Without recounting, primary school labels the problem ‘38 = ? 7s’ as an example of a division, 38/7, which is hard to many, or as an equation ’38 = x*7’ to be postponed to secondary school.

Designing a micro-curriculum

With curriculum architecture as one of its core activities, the MATHeCADEMY.net was asked to design a micro-curriculum understandable and attractive to teachers stuck with division problems; and allowing special need students to return to their ordinary class. Two were designed.

In the ‘1 cup and 5 sticks’ micro-curriculum, 5 is cup-counted in 2s as 1)3 2s or 2)1 2s or 3)-1 2s to show that a total can be counted in 3 ways: overload, normal or underload with an inside and an outside for the bundles and singles. So to divide 336 by 7, 5 bundles are moved outside as 50 singles to recount 336 with an overload: 336 = 33)6 = 28)56, which divided by 7 gives 4)8 = 48.

Besides the ‘Cure Math Dislike by 1 cup and 5 sticks’, 8 extra micro-curricula were designed (mathecademy.net/preschool/) where cup-counting involves division, multiplication, subtraction and later next-to and on-top addition, in contrast to primary school that turns this order around and only allows on-top addition using carrying instead of overloads. Thus, if using cup-writing with overloads or underload instead of carrying, the order of operations can be turned around to respect that totals must be counted before being added.

	Carry	Cup-writing	Words
Add	1 4 5 1 7 6 2	4)5 1)7 5)12 6)2 = 62	4 ten 5 1 ten 7 5 ten 12 5 ten 1 ten 2 6 ten 2 = 62
Subtract	1 4 5 1 7 2 8	4)5 1)7 3)-2 2)10-2 = 2)8 = 28	4 ten 5 1 ten 7 3 ten less2 2 ten 8 = 28
Multiply	4 2 6 * 7 1 8 2	7 * 2)6 14)42 18) 2 = 182	7 times 2 ten 6 14 ten 42 14 ten 4 ten 2 18 ten 2 = 182
Divide	2 4 rest 1 3 | 7 3 6 1 3 1 2 1	7)3 counted in 3s 6)13 6)12 + 1 2 3s)4 3s + 1 24 3s + 1 73 = 24*3 + 1	7ten3 6ten 13 6ten12 + 1 3 times 2ten4 + 1 3 times 24 + 1

Figure 4: Cup-writing with overloads and underloads instead of carrying

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using a cup for the bundles; and reporting first with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and underloads. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other. In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eighths, the learner reverses next-to addition. Finally, the learner sees how double-counting creates per-numbers.

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2)1 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	‘2.7 5s is also how many 5s?’ IIIII IIIII IIIIIII = V V V II = V V V V III 2)7 = 2+1)7–5 = 3)2 = 3+1)2–5 = 4)-3 So 2.7 5s = 3.2 5s = 4.-3 5s	(2*5+7)/5 (2*5+7) – 3*5 (2*5+7) – 4*5	3.some 2 -3
M4	2 5s is how many 4s?’ IIIII IIIII = IIII I IIII I = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2*4	2.some 2
M5	‘2 5s and 4 3s total how many 5s?’ IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2
M6	‘2 5s and 4 3s total how many 8s?’ IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 (2*5+4*3) – 2*8	2.some 6
M7	‘2 5s and ? 3s total 4 5s?’ IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1
M8	‘2 5s and ? 3s total how 2.1 8s?’ IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(2*8+1 – 2*5)/3 (2*8+1 – 2*5) – 2*8	2.some 1

Figure 5: A calculator predicts counting and adding results

One curriculum used silent education where the teacher demonstrates and guides through actions only, not using words; and one curriculum was carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator quickly took over the communication. For further details watch www.youtube.com/watch?v=IE5nk2YEQIA.

After these micro-curricula a learner went back to her grade 6 class where proportionality created learning problems. The learner suggested renaming it to double-counting but the teacher insisted on following the textbook. However, observing that the class took over the double-counting method, he finally gave in and allowed proportionality to be renamed and treated as double-counting. When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integration.

Thus cup-counting and a calculator for predicting recounting results allowed the learner to reach the outside goal, mastering Many, by following an alternative to the institutionalized means that because of a goal-means exchange had become a stumbling block to her; and performing and reversing next-to addition introduced her to and prepared her for later calculus classes.

Literature on cup-counting

No research literature on cup-counting was found. Likewise, it is not mentioned by Dienes (1964).

Conclusion and recommendation

As to theory, two genres exist; a master genre exemplifying existing theory, and a research genre developing new theory by including a question and a theoretical guidance to a valid answer based upon analyzing reliable data. To avoid indifference, this paper addresses the OECD report 'Improving schools in Sweden' by asking if mathematics education might have a goal-means exchange. As theoretical guidance, institutional skepticism allows using the existentialist existence-versus-essence distinction to distinguish outside goals from inside means, which leads to asking when mathematics is respectively existence and essence. Analyzing traditional math shows that by being set-based and by adding numbers without units, its concepts and statements are unrooted and little applicable to the outside world, thus being primarily essence. Then grounded theory helps showing how mathematics looks like if grounded in its physical root, Many. To tell the difference, two names are coined, 'ManyMatics' versus 'MetaMatism' mixing 'MetaMatics' defining concepts as examples of abstractions instead of as abstractions from examples, with 'MatheMatism' valid only inside classrooms. To validate its findings and again to avoid indifference, the paper includes a classroom test of a micro curriculum described in details to allow it to be tested in other classrooms. Its originality should welcome the paper for publishing since no literature on ManyMatics exists.

So, if a research conference fails to accept the paper for presentation or as a poster, an extra exchange can be added to help solving the paradox that the Swedish problems occur despite increased research and funding: Neglecting a genre analysis might exchange the master and research genres with the consequence that peer-review becomes unable to accept groundbreaking new paradigms. Such research conferences include master papers that, although career promoting, are unable to uncover alternative, hidden ways to guide solving problems in mathematics education.

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