Fifty Years of Research without Improving Mathematics Education, Why?

An academic essay by Allan.Tarp@MATHeCADEMY.net, February 2017

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by the PISA studies organised by the Organisation for Economic Co-operation and Development, OECD, showing a low level and a continuing decline in many countries. Thus, to help the former model country Sweden, OECD wrote a critical 2015 report 'Improving Schools in Sweden, an OECD Perspective': "PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life."

Researchers in mathematics education meet in different fora. On a world basis, the International Congress on Mathematical Education, ICME, meets each four year. And on a European basis, the Congress of the European Society for Research in Mathematics Education, CERME, meets each second year.

At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is "Only one: to improve, mathematics education should ask, not what to do, but what to do differently." Thus, to be successful, research should not study problems but look for hidden differences that might make a difference. Research that is skeptical towards institutionalized traditions could be called difference research or contingency research or Cinderella research making the prince dance by looking for hidden alternatives outside the ruling tradition. The French thinker Lyotard calls it 'paralogy' inventing dissension to the reigning consensus. Difference research scarcely exists today since it is rejected at conferences for not applying or extending existing theory that is able to produce new researchers and to feed a growing appliance industry, but unable to reach its goal, to improve mathematics education.

To elaborate, mathematics education research is sterile because its three words are not well defined.

As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist: a continental European education serving the nation's need for public servants though multi-year compulsory classes and lines at the secondary and tertiary level; and a North American education aiming at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers.

As to research, academic articles can be written at a master level applying or exemplifying existing theories, or at a research level questioning them. Just following ruling theories is especially problematic in the case of conflicting theory as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible respectively.

Consequently, you cannot know what kind of mathematics and what kind of education has been studied, and you cannot know if research is following ruling traditions or searching for new discoveries. So, seeing education as an institutional help to children and youngsters master outside phenomena leads to the question: What outside phenomena roots mathematics?

The Outside Roots of Mathematics

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep balance and to store sounds assigned to what we grasped with our forelegs, thus providing the holes in the head with our two basic needs, food for the body and information for the brain.

The sounds developed into languages. In fact, we have two languages, a word-language and a number-language. Children learn to talk and to count at home. Then, as an institution, school takes over and teaches children to read and to write and to calculate.

The word language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair'. Observing the existence of many chairs, we ask 'how many totally?' and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, 'the total is 3 chairs' or, if counting legs, 'the total is 3 fours', which we abbreviate to 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is an operation'.

And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

And since we master outside phenomena through actions, learning the word-language means learning actions as how to listen, to read, to write and to speak. Likewise, learning the number-language means learning actions as how to count and to add. You cannot learn how to math, since math is not an action word, it is a label as is grammar. Thus, mathematics may be seen as the grammar of the number-language.

Using the phrasing 'the number-language is an application of mathematics' implies that then 'of course mathematics must be taught and learned before it can be applied'. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

So, one way of improving mathematics education is to respect that language comes before meta-language. Which was also the case in continental Europe before the arrival of the 'New Math' that turned mathematics upside down to become a 'meta-matics' presenting its concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically and which would present mathematics as 'many-matics', a natural science about Many.

Thus, Euler defined a function as a common name for calculations with unspecified numbers, in contrast to calculations without that could be calculated right away without awaiting numbers to be specified. Defining all concepts as examples of the mother concept set, New Math turned a function into an example of a set-product where first-component identity implies second-component identity, which learners heard as 'bublibub is an example of bablibab'.

Before New Math, Germanic countries taught counting and reckoning in primary school. Then the lower secondary school taught algebra and geometry, which are also action words meaning to reunite totals and to measure earth in Arabic and in Greek. 50 years ago, New Math made all these activities disappear. This means that what research has studied is problems coming from teaching how to math. So, one alternative presents itself immediately: Forget about New Math and, once again, teach mathematics as rooted in numbers and reckoning and reuniting totals and measuring earth.

Re-rooting mathematics resonates with its historic origin as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many.

As to New Math, its idea of deriving definitions from the mother concept set leads to meaningless self-reference as in the classical liar paradox ‘This sentence is false’, being true if false and false if true. This was shown by Russell looking at the set of sets not belonging to itself. Here a set belongs to the set if it doesn’t, and does not belong if it does.

To avoid self-reference, Russell created a hierarchical type theory in which fractions could not be numbers if defined by numbers as done by New Math defining fractions as equivalence classes in a set of number-pairs. Insisting that fractions are numbers, New Math invented a new set-theory that by mixing sets and elements also mixes concrete examples and their abstract names, thus mixing concrete apples that can feed humans and the word ‘apple’ that cannot. By mixing things and their names, New Math and its meta-matics ceases to be a language about the real world. Still, it has entered universities worldwide as the only true version of mathematics.

So, to improve its education, mathematics should stop teaching top-down meta-matics from above and begin teaching bottom-up many-matics from below instead.

Rethinking Mathematics from Below

To improve it we must rethink mathematics. To rethink we seek guidance by one of the greatest thinkers of the 20th century, Heidegger, being very influential within existentialist thinking and French skeptical post-structural thinking.

Heidegger holds that to exist fully means to establish an authentic relationship to the things around us. To allow a thing to open its ‘Wesen’ and escape its gossip-prison created by reigning essence-claims we must use constant questioning. So, returning to the fundamental goal of education, preparing humans for what is outside, we must keep on asking to the Wesen of the root of the number language, the physical fact Many, and allow Many to escape from its New Math gossip, ‘Gerede’.

With 2017 as the 500year anniversary for Luther’s 95 theses, we can describe meeting Many in theses.

1. Using a folding ruler we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent.

2. Using a cup for the bundles we discover that a total can be ‘cup-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘cup-writing’ to report cup-counting, T = 5 = 2]1 2s = 1]3 2s = 3]-1 2s. Likewise, when counting in tens, T = 37 = 3]7 tens = 2]17 tens = 4]-3 tens. Finally, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3]1 2s = 1]1]1 2s. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3]1 2s = 3.1 2s.

3. Using recounting a total in the same unit by creating or removing overloads or underloads, we discover that cup-writing offers an alternative way to perform and write down operations:

T = 65 + 27 = 6]5 + 2]7 = 8]12 = 9]2 = 92 ; and T = 65 – 27 = 6]5 – 2]7 = 4]-2 = 3]8 = 38

T = 7* 48 = 7* 4]8 = 28]56 = 33]6 = 336 ; and T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48

4. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved in counting by bundling and stacking. To count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. Showing 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, dividing and subtracting a multiple, the calculator predicts that 7 = 2]1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack

of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate e.g. T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.

5. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. A 'recount formula' T = (T/B)*B saying that T/B times B can be taken away from T, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and a 'restack formula' T = (T–B)+B saying that T–B is left when B is taken away from T and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Thus we discover the nature of formulas: formulas predict.

6. Wanting to recount a total in a new unit, we discover that again a calculator can predict the result by bundling and stacking and dragging away the stack:

T = 4 5s = ? 6s. First (4*5)/6 = 3.some. Then (4*5) – (3*6) = 2. Finally T = 4 5s = 3.2 6s

Also, we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.

7. Wanting to recount a total in tens, we discover that a calculator can predict the result directly by multiplication. Only, the calculator leaves out the unit and misplaces the decimal point:

T = 3 7s = ? tens. Answer: T = 21 = 2.1 tens

Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.

And wanting to recount a total from tens to icons, we discover that this again is an example of recounting to change the unit:

T = 3 tens = ? 7s. First 30/7 = 4.some. Then 30 – (4*7) = 2. Finally T = 30 = 4.2 7s

Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged

8. Using the letter u for an unknown number, we can rewrite recounting from tens, e.g. 3 tens = ? 7s, as 30 = u*7 with the answer 30/7 = u. Officially this is called to solve an equation, so here we discover a natural way to do so: Move a number to the opposite side with the opposite sign. The equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, thus predicted by the restack-formula as 8 = (8–2)+2. Thus, the equation 8 = u + 2 has the solution is 8–2 = u, again moving a number to the opposite side with the opposite sign.

9. Once counted, totals can be added. But we discover that addition is not well defined. With two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to get the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. Thus next-to addition geometrically means to add areas, and algebraically it means to combine multiplication and addition. Officially this is called integration, a core part of traditional mathematics in high school and at the first year of university.

10. Also we discover that addition can be reversed. Thus, the equation above restacking 8 by moving 2, 8 = u+2, can also be read as reversed addition: u is the number that added to 2 gives 8, which is precisely the formal definition of u = 8–2. So, we discover that subtraction is reversed addition. And, again we see that the equation u+2 = 8 is solved by u = 8–2, i.e. by moving to the opposite side with the opposite sign. Likewise, the equation recounting 8 in 2s, 8 = u*2, can be read as reversed multiplication: u is the number that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2? So, we discover that division is reversed multiplication. And, again we see that the

equation u*2 = 8 is solved by u = 8/2, i.e. by moving to the opposite side with the opposite sign. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root and logarithm as u = 3√20 and u = log3(20). So, again we solve the equations by moving to the opposite side with the opposite sign. Reversing next-to addition, we can ask e.g. 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer, first we remove the initial total T1, then we count the rest in 5s: u = (T–T1)/5. Combining subtraction and division in this way is called differentiation. By observing that this is reversing multiplication and addition we discover that differentiation is reversed integration.

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

12. Observing that a quantity may be double-counted in the same unit, we discover that per-numbers may take the form of fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. Thus, to find 3 per 5 of 20, 3/5 of 20, we just recount 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3. To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. Thus, calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

Writing out a total T as we say it, T = 345 = 3*ten*ten + 4*ten + 5*1, shows a number as blocks united next-to each other. Also, we see algebra's four ways to unite numbers: addition, multiplication, repeated multiplication or power, and block-addition also called integration. Which is precisely the core of mathematics: addition and multiplication together with their reversed operations subtraction and division in primary school; and power and integration together with their reversed operations root, logarithm and differentiation in secondary school. Including the units, we see there can only be four ways to unite numbers: addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers.

How School Teaches Mathematics

Before addressing how school guides children on their way to mastering Many let us look at the number-language children bring to school. Asking a three-year old child "how old will you next time?" the answer is four with four fingers shown. But displaying four fingers held together two and two will prompt an immediate protest: "No, that is not four, that is two twos!"

So, children come to school with two-dimensional 'block-numbers' all carrying a unit, corresponding to LEGO-bricks that stack as 1, 2, 3 or more 4s. Thus, by combining geometry and algebra in their shapes and knobs, they are an excellent basis for connecting the starting point, children's block-numbers, with the final goal, the Arabic numbers also being a collection of blocks of 1s, tens, ten-tens etc.

To emphasize that we count by bundling and stacking, the school could tell children that eleven and twelve is a special 'Viking-way' to say ten-1 and ten-2. Then they probably would count '2ten9, 3ten, 3ten1' instead of saying 'ten-and-twenty' and risk being diagnosed with dyscalculia. In Danish, eleven and twelve mean 'one left' and 'two left', implying that the ten-bundle has been counted already. And, except from some French additions because of the Norman conquest, English is basically Anglish, a dialect from Harboøre on the Danish west coast where the ships left for Angland.

Now let us see how school prepare children and youngsters to meet Many by offering them what is called mathematics education. Again, we use the form of theses.

1. School could respect the origin of the word mathematics as a mere name for algebra and geometry both grounded in the physical fact Many and created to go hand in hand. Instead, school teaches mathematics as a self-referring 'meta-matics' defining concepts as examples of abstractions, and not as abstractions from examples. Likewise, school teaches algebra and geometry separately.

2. School could respect that a digit is an icon containing as many sticks as it represents. Instead, school presents numbers as symbols like letters. Seldom it tells why ten does not have an icon or why ten is written as 10; and seldom it tells why ten1 and ten2 is called eleven and twelve.

3. School could follow the word-language and use full sentences 'The total is 3 4s or T = 3 4s or T = 3*4'. Instead, by only saying '3', school removes both the subject and the unit from number-language sentence, thus indicating that what children should learn is not a number-language but a one-dimensional number system claimed to be useful later when meeting life's two-dimensional numbers.

4. School could develop the two-dimensional block-numbers children bring to school and are supposed to leave school with. Instead, school teaches its one-dimensional line-numbers as names for the points along a number line, using a place-value system. Seldom numbers are written out as we say them with the unit ones, ten, ten-tens, etc. Seldom a three-digit number is taught as a short way to report three countings: of ones, of bundles, and of bundles of bundles. Seldom tens is called bundles; seldom hundreds is called ten-tens or bundles of bundles.

5. School could respect that a number is a horizontal union of vertical blocks of 1s, bundles, bundles of bundles etc., and that counting-on means going up one step in the 1-block until we reach the bundle level where a bundle of 1s is transformed into 1 extra bundle making the bundle block go up 1 while the 1-block falls back to zero; and school could respect that a natural number is a decimal number with a unit. Instead school represents numbers by a horizontal number-line, where counting-on means moving one step to the right and where a natural number is presented without unit and with a misplaced decimal point.

6. School could respect that totals must be counted and sometimes recounted in a different unit before being added. Instead, without first teaching counting, school teaches addition from the beginning regardless of units, thus transforming addition to mere counting-on. Seldom school teaches real on-top and next-to addition respecting the units.

7. School could respect that also operations are icons showing the three basic counting activities: division as bundling, multiplication as stacking the bundles, and subtraction as removing the stack to look for unbundled singles; and school could respect the natural order of operations: division before multiplication before subtraction before addition. Instead school reverses this order without respecting that addition has two meanings, on-top and next-to, or that division has two meanings, counted in and split between.

8. School could respect that 3*8 means 3 8s that may or may not be recounted in tens. Instead school insists the 3*8 IS 24 and asks children to learn the multiplication tables by heart. Seldom the geometrical understanding is included showing that recounting in tens means the stack increases its width and therefore must decrease its height to leave the total unchanged.

9. School could respect that basic calculations become understandable by recounting a total in the same unit to create or remove over- or underloads. Instead school does not allow over- and underloads and insists on using specific algorithms with a carry-technique.

10. School could respect that proportionality is just another word for per-numbers coming from double-counting, and that per-numbers are operators that need a number to become a number. Instead school renames per-numbers to fractions, percentages and decimal numbers and teach them as

numbers that can be added without considering the unit, and teaches proportionality as an example of a linear function, which isn't linear since the b in y = a*x+b makes it an affine function instead.

11. School could respect that equations are just another name for reversed calculation rooted in recounting tens in icons and solved by moving to the opposite side with the opposite sign. Instead school teaches equations as statements expressing equivalence between two different number-names to be solved by performing the same operation to both sides aiming at using the laws of abstract algebra to neutralize the numbers next to the unknown.

12. School could respect that integrating means adding non-constant per-numbers to be taught in primary school as next-to addition of block-numbers, and in middle school as mixture tasks; and respect that reversed integration is called differentiation made relevant since adding many differences boils down to one single difference between the end- and start-number. Instead school neglects primary and middle school calculus; and it teaches differentiation before integration, that is reduced to finding an antiderivative to the formula to be integrated. Seldom continuity and differentiability are introduced as formal names for local constancy and local linearity. Seldom the units are included to make clear that per-numbers are integrated, and that differentiation cerates per-numbers.

How School Could Teach Mathematics

Seeing the goal of mathematics education as preparing students for meeting Many, doing so in a Heideggerian gossip-free space offers many differences to be tried out and studied. Again, we use a list form.

1. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to teach cup-counting, recounting in the same and in a different unit, calculator prediction, on-top and next-to addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

2. A class is stuck in addition. Here a difference is to introduce recounting in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6]5 + 2]7 = 8]12 = 8+1]12-10 = 9]2 = 92.

3. A class is stuck in subtraction. Here a difference is to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6]5 – 2]7 = 4]-2 = 4-1]-2+10 = 3]8 = 38.

4. A class is stuck in multiplication. Here a difference is to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4]8 = 28]56 = 28+5]56-50 = 33]6 = 336.

5. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Another difference is to reduce the full ten-by-ten table to a small 4-by-4 table since 5 is half of ten and 6 is ten less 4, 7 is ten less three etc. Thus T = 4*7 = 4 7s that recounts in tens as T = 4*7 = 4*(10-3) = 40 – 12 = 28; and T = 6*9 = (10–4)*(10–1) = 100 – 40 – 10 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

6. A class is stuck in short division. Here a difference is to rewrite the number as '10 or 5 times less something' and use the results from the small 4-by-4 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

7. A class is stuck in long division. Here a difference is to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33]6 /7 = 33-5]6+50 /7 = 28]56 /7 = 4]8 = 48.

8. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

9. A class is stuck in fractions. Here a difference is to see a fraction as a per-umber and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s

as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of 'mathe-matism' true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes 'middle school integration' later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of reverse operations to establish the basic rule for solving equations 'move to the opposite side with the opposite sign': In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn't have a reverse sign. This gives 3*u = 14 – 2. Since u doesn't have a reverse sign, 3 moves to the other side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. 'transform the formula T = a/(b-c) so that all letters become subjects.' A hymn can be created: "Equations are the best we know / they're solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don't want to stop."

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to 'pre-

dict' a number, instead statistics can 'post-dict' its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5^{th} root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio Δy/Δx can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the

children’s own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math’s meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead ‘meta-matism’, a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: ‘What is this? What does it do?’; and ‘How many in total? How many if we change the unit?’

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the ‘FREE 1day SKYPE Teacher Seminar: Cure Math Dislike’ where, in the morning, a power point presentation ‘Curing Math Dislike’ is watched, and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a ‘CupCount before you Add booklet’ to experience proportionality and calculus and solving equations as golden learning opportunities in cup-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 student teachers.

The material for primary and secondary school has a short question-and-answer format.

The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial:

T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, with at least two alternative meanings for all three words, at least 2*2*2 i.e. 8 different forms of mathematics education research exist. The past 50 years has shown the little use of the present form applying theory to study meta-matics taught in compulsory multi-year classes or lines. So, one alternative presents itself directly as an alternative for future studies: to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many, and to teach it in self-chosen half-year block at the secondary and tertiary level; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining and trying to understand the misery coming from teaching meta-matism in compulsory classes. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research must search for differences and test if they make a difference, not in compulsory classes, but with daily lessons in self-chosen half-year blocks. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a sentence, the subject exists, but the sentence about it may be gossip; so, stop teaching essence and start experiencing existence.

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