Difference-Research Powering PISA Performance: Count and Multiply before you Add

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To explain 50 years of low performing mathematics education research, this paper asks: Can mathematics and education and research be different? Difference-research searching traditions for hidden differences provides an answer: Traditional mathematics, defining concepts from above as examples of abstractions, can be different by instead defining concepts from below as abstractions from examples. Also, traditional line-organized office-directed education can be different by uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks. And traditional research extending its volume of references can be different, either as grounded theory abstracting categories from observations, or as difference-research uncovering hidden differences to see if they make a difference. One such difference is: To improve PISA performance, Count and Multiply before you Add.

Keywords: PISA, mathematics education, calculus, early childhood, sociological imagination.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA performance decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

Other countries also experience low and declining PISA performance. And apparently research can do nothing about it. Which raises the question: Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? So, it is time to seek guidance by difference-research.

Difference-research Searching for Hidden Differences

Difference-research asks two questions: 'Can this be different – and will the difference make a difference?' If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

Natural sciences use difference-research to keep on searching until finding what cannot be different. Describing matter in space and time by weight, length and time intervals, they all seem to vary. However, including per-numbers will uncover physical constants as the speed of light, the gravitational constant, etc. The formulas of physics are supposed to predict nature's behavior. They cannot be proved as can mathematical formulas, instead they are tested as to falsifiability: Does nature behave different from predicted by the formula? If not, the formula stays valid until falsified.

Social sciences can also use difference-research; and since mathematics education is a social institution, social theory might be able to explain 50 years of unsuccessful research in mathematics education.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman (1990, p. 16) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.' Also, he talks about rationality as the base for social organizations:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called *goal displacement*. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

As an institution, mathematics education is a public organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false:

If $M = \{ A \mid A \notin A) \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

Thus, SET has transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘1 + 2 IS 3’ meet counter-examples as e.g. 1 week + 2 days is 9 days.

So looking back, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn’t mathematics just a name for knowledge about forms and numbers and operations? We all teach 3*8 = 24, isn’t that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence ‘The total is 3 8s’ we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers, operations and calculations. However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. And, being sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens, might speed up writing but might also slow down learning, together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, in Lincolns Gettysburg address, ‘Four scores and ten years ago’ shows that not all count in tens.

So, despite being presented as universal, many things can be different in mathematics, apparently having a tradition to present its choices as nature that cannot be different. And to uncover choice presented as nature is the aim of difference research.

A philosophical Background for Difference Research

Difference research began with the Greek controversy between two attitudes towards knowledge, called ‘sophy’ in Greek. To avoid hidden patronization, the sophists warned: Know the difference between nature and choice to uncover choice presented as nature. To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy. The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord using an unpredictable will to choose how the world behaves.

However, in the Renaissance difference research returns with Brahe, Kepler and Newton. Observations showed Brahe that planetary orbits are predictable in a way that did not falsify the church’s claim that the earth is the center of the universe. Kepler pointed to a different theory with the sun in the center. To falsify the Kepler theory a new planet had to be launched, which was impossible until Newton showed that planets and apples obey the same will, and a falling apple validates Kepler's theory.

As experts in sailing, the Viking descendants in England had no problem stealing Spanish silver on its way home across the Atlantic Ocean. But to get to India to exchange it for pepper and silk, the Portuguese fortification of Africa's cost forced them to take the open sea and navigate by the moon. But how does the moon move? The Church had one opinion, Newton had a different.

'We believe, as is obvious for all, that the moon moves among the stars,' said the Church; opposed by Newton saying: 'No, I can prove that the moon falls to the earth as does the apple.' 'We believe that when moving, things follow the unpredictable metaphysical will of the Lord above whose will is done, on earth as it is in heaven,' said the Church; opposed by Newton saying: 'No, I can prove they follow their own physical will, a force that is predictable because it follows a mathematical formula.' 'We believe, as Aristotle told us, that a force upholds a state,' said the Church; opposed by Newton saying: 'No, I can prove that a force changes a state. Multiplied with the time applied, the force's impulse changes the motion's momentum; and multiplied with the distance applied, the force's work changes the motion's energy.' 'We believe, as the Arabs have shown us, that to deal with formulas we use algebra,' said the Church; opposed by Newton saying: 'No, we need a different algebra of change which I will call calculus.'

By discovering a physical predictable will Newton inspired a sophist revival in the Enlightenment Century: With moons and apples obeying their own physical will instead of that of a metaphysical patronizer, once enlightened about the difference between nature and choice, humans can do the same and do without a double patronization by the Lord at the manor house and the Lord above. Thus, two Enlightenment republics were installed, one in North America in1776 and one in France in 1789.

The US still has its first republic showing skepticism towards philosophical claims by developing American pragmatism, symbolic interactionism and grounded theory; and by allowing its citizens to uncover and develop talents through daily lesson in self-chosen half-year blocks in secondary and tertiary education.

France now has its fifth republic turned over repeatedly by their German neighbors seeing autocracy as superior to democracy and supporting Hegel's anti-enlightenment thinking reinventing a metaphysical Spirit expressing itself through the history of different national people. To protect the republic, France established line-organized and office-directed elite schools, copied by the Prussia wanting to prevent democracy by Bildung schools meeting there criteria: The population must not be enlightened to prevent it asking for democracy as in France; instead a feeling of nationalism should be installed transforming the population into a people following the will of the Spirit by fighting other people especially the French; and finally the population elite should be extracted and receive Bildung to become a knowledge nobility for a new strong central administration to replace the inefficient blood nobility unable to stop democracy from spreading from France.

To warn against hidden patronization in institutions, France developed a post-structuralist thinking inspired by existentialist thinking (Tarp, 2016), especially as expressed in what Bauman (1992, p. ix). calls ‘the second Copernican revolution’ of Heidegger asking the question: What is ‘is’?

> Inquiry is a cognizant seeking for an entity both with regard to the fact that it is and with regard to its Being as it is. (Heidegger, 1962, p. 5)

Heidegger here describes two uses of ‘is’. One claims existence, ‘M is’, one claims ‘how M is’ to others, since what exists is perceived by humans wording it by naming it and by characterizing or analogizing it to create ‘M is N’-statements.

Thus, there are four different uses of the word ‘is’. ‘Is’ can claim a mere existence of M, ‘M is’; and ‘is’ can assign predicates to M, ‘M is N’, but this can be done in three different ways. ‘Is’ can point down as a ‘naming-is’ (‘M is for example N or P or Q or …’) defining M as a common name for its volume of more concrete examples. ‘Is’ can point up as a ‘judging-is’ (‘M is an example of N’) defining M as member of a more abstract category N. Finally, ‘is’ can point over as an ‘analogizing-is’ (‘M is like N’) portraying M by a metaphor carrying over known characteristics from another N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: ‘I am’ and ‘it is’ and ‘they are’; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the ‘I’ (Dasein) must create an authentic relationship to the ‘It’. However, this is made difficult by the ‘dictatorship’ of the ‘They’, shutting the ‘It’ up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one’s own Dasein completely into the kind of Being of ‘the Others’, in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the “they” is unfolded. (..) Discourse, which belongs to the essential state of Dasein’s Being and has a share in constituting Dasein’s disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

Inspired by Heidegger, the French poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu points out that society forces words upon you to diagnose you so it can offer curing institutions including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Derrida, 1991. Lyotard, 1984. Bourdieu, 1970. Tarp, 2012).

From a Heidegger view a sentence contains two things: a subject that exists, and the rest that might be gossip. So, to discover its true nature hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its ‘predicate-prison’. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a different mathematics curriculum, e.g. one based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help rebuilding their original countries.

The philosophical and sociological background for difference research may be summed up by the Heidegger warning: In sentences, trust the subject but question the rest since it might be gossip. So, to restore its authenticity, we now return to the original subject in Greek mathematics, the physical fact Many, and use Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep the balance and to store sounds assigned to what we grasped with our forelegs, now freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into two languages, a word-language and a number-language. The 'pencil-paradox' observes that placed between a ruler and a dictionary, a pencil can itself point to its length but not to its name. This shows the difference between the two languages, the word-language is for opinions, the number-language is for prediction.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair'. Observing the existence of many chairs, we ask 'how many in total?' and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, 'the total is 3 chairs' or, if counting legs, 'the total is 3 fours', abbreviated to 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is an operation'. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

Thus, we can ask: What happens if looking at mathematics differently as a number-language? Again, difference-research might provide an answer.

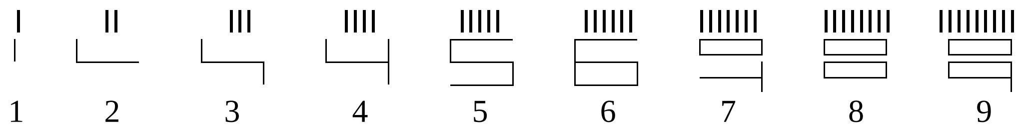
Examples of Difference-research

To prevent that mathematics becomes a meta-language that can be applied to describe and solve real-world problems, we must be careful with our language. Although it seems natural to talk about mathematics and its applications, this includes the logic that 'of course mathematics must be learned before it can be applied'. Which is equivalent to saying 'of course a grammar must be learned before it can be applied to describe a language'. This would lead to widespread illiteracy if applied to the word-language. And 'grammar before language' might be the cause of several problems in mathematics education. Of course, the subject must exist before the sentences can be made about it. So differences typically come from respecting that the number-language comes before its grammar and after meeting and experiencing the subject of its sentences, the total, describing the physical fact Many.

Digits as icons

A class of beginners, e.g. preschool or year 1 or migrants, is stuck in the traditional introduction of digits as symbols like letters. Some confuse the symbols, some have difficulties writing them, some can’t see why ten is written 10, some ask why eleven and twelve is not called ten-1 and ten-2.

Here a difference is to use a folding ruler to discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the four-icon, and five sticks in the five-icon, etc. Counting in 5s, the counting sequence is 1, 2, 3, 4, Bundle, 1-bundle-1, etc. This shows, that the bundle-number does not need an icon. Likewise, when bundling in tens. Instead of ten-1 and ten-2 we use the Viking numbers eleven and twelve meaning ‘1 left’ and ‘2 left’ in Danish, understood that the ten-bundle has already been counted.



Will this difference make a difference? In theory, yes, since rearranging physical entities into icons, e.g. five cars into in a five-icon, makes the icons physically before being formally written down. In his genetic epistemology, Piaget expresses a ‘greifen-vor begreifen’ principle, grasping physically before mentally. Thus, going from unordered cars to cars ordered into an icon to writing down the icon includes three of the four parts of his stage theory, the preoperational and the concrete operational and the formal operational stage. In practice, it works on a pilot study level thus being ready for a more formal study.

Counting sequences in different forms

A class of beginners have problems with the traditional introduction of the counting sequence and the place value system. Some count ‘twenty-nine, twenty-ten, twenty-eleven’. Some mix up 23 and 32.

Here a difference is to count a total of a dozen sticks in fives using different counting sequences: ‘1, 2, 3, 4, bundle, 1-bundle-1, …, 2 bundles, 2-bundles-1, 2-bundles-2’. Or ’01, 02, 03, 04, 10, 11, …, 22’. Or ‘.1, .2, .3, .4, 1., 1.1, …, 2.2’. Or ‘1, 2, bundle less 2, bundle less 1, bundle, bundle&1, bundle&2, 2bundle less 2, 2bundle less 1, 2bundles, 2bundles&1, 2bundles&2.’

Using a cup for the bundles, a total can be ‘cup-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘cup-writing’ to report cup-counting, T = 5 = 2]1 2s = 1]3 2s = 3]-1 2s. Likewise, when counting in tens, T = 37 = 3]7 tens = 2]17 tens = 4]-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, shows that a natural number is a decimal number with a unit: T = 3]1 2s = 3.1 2s; and t = 3]1 tens = 3.1 tens = 31 if leaving out the unit and misplacing the decimal point

Will this difference make a difference? In theory, yes, since counting by taking away bundles and placing one stick in a cup per bundle again combines the three operational parts of Piaget’s stage-theory allowing the learner to see, that a number has three parts: a unit, and some bundles inside the cup, and some unbundled outside. In practice, it works on a pilot study level thus being ready for a more formal study.

Multiplication tables made simpler

A class is stuck in multiplication tables. Some add instead of multiplying, some tries to find the answer by repeated addition, some just give random answers, and some have given up entirely to learn the tables by heart.

Here a difference is to see multiplication as a geometrical stack or block that recounted in tens increases its width and therefore decreases its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21.

Another difference is to begin by reducing the full ten-by-ten table to a small 2-by-2 table containing doubling and tripling, using that 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc.

Thus, beginning with doubling visualized by LEGO bricks, T = 2 6s = 2*6 = 2*(5&1) = 10&2 = 12, or T = 2*6 = 2*(10-4) = 20-8 = 12. And T = 2 7s = 2*7 = 2*(5&2) = 10&4 = 14, or T = 2*7 = 2*(10-3) = 20 – 6 = 14. Doubling then can be followed by halving that by counting in 2s will introduce a recount-formula T = (T/B)*B saying that T/B times B may be taken away from T: So when halving 8, 8 = (8/2)*2 = 4 2s, and 9 = (9/2)*2 = (<u>8&1</u>/2)*2 = (4&1/2)*2 = 4 & ½ 2s.

As to tripling, T = 3*7 = 3*(10-3) = 30 – 9 = 21.

Proceeding with factors after 2 and 3, 2-by-2 Medieval multiplication squares can be used to see that e.g. T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Division using cup-writing and recounting

A class is stuck in short and long division. Some subtract instead of dividing, some invent their own algorithms typically time-consuming and often without giving the correct answers, some give up because they never learned the multiplication tables.

Here a difference is to talk about 8/2 as ‘8 counted in 2s’ instead of as ‘8 divided between 2’; and to rewrite the number as ‘10 or 5 times less something’ and use the results from a multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

As to long division, here a difference is to combine renaming numbers using bundle names, e.g. sixty-five as 6ten5, with cup-writing allowing recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48.

Once cup-writing is introduced, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3]1 2s = 1]1]1 2s. Or, with tens: T = 234 = 23]4 = 2]3]4.

Thus, by recounting in the same unit by creating or removing overloads or underloads, cup-writing offers an alternative way to perform and write down all operations.

T = 65 + 27 = 6]5 + 2]7 = 8]12 = 9]2 = 92
T = 65 – 27 = 6]5 – 2]7 = 4]-2 = 3]8 = 38
T = 7* 48 = 7* 4]8 = 28]56 = 33]6 = 336
T = 7* 48 = 7* 5]-2 = 35]-14 = 33]6 = 336
T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48
T = 338 /7 = 33]8 /7 = 28]58 /7 = 4]8 + 2/7= 48 2/7

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Proportionality as double-counting creating per-numbers

A class stuck in proportionality. Nearly all find the $-number for 12kg at a price of 2$/3kg but some cannot find the kg-number for 16$. Here a difference is to see the price as a per-number, 2$ per 3kg, bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Will this difference make a difference? In theory, yes, since proportionality is translated to a basic physical activity of counting and recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Fractions and percentages as per-numbers

A class is stuck in fractions. Rewriting fractions by shortening or enlarging, some subtract and add instead of dividing and multiplying; and some add fractions by adding numerators and denominators.

Here a difference is to see a fraction as a per-number coming from double-counting in the same unit, 3/5 = 3$ per 5$, or as percentage 3% = 3/100 = 3$ per 100%. Thus 2/3 of 12 is seen as 2$ per 3$ of 12$ that recounts in 3s as 12$ = (12/3)*3$ giving (12/3)*2$ = 8$ of the 12$. So 2/3 of 12 is 8. Other examples are found in economy investing money and expecting a return that might be higher or lower than the investment, e.g. 7$ per 5$ or 3$ per 5$.

The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6.

To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100$ in 5s and replace 5$ with 3$: T = 100$ = (100/5)*5$ giving (100/5)*3$ = 60$. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%.

As per-numbers, also fractions are operators needing a number to give a number: a half is always a half of something as shown by the recount-formula T = (T/B)*B = T/B Bs. So also fractions must have units to be added.

If the units are different, adding fractions means finding the average fraction. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as the tradition teaches.

Taking fractions of the same quantity makes the unit the same, assumed to be already bracketed out, so that T = a/b + c/d really means T = (a/b + c/d) of (b*d). Thus adding 2/3 and 4/5 it is implied that the fractions are taken of the same total 3*5 = 15 that is bracketed out, so the real question is 'T = 2/3 of 15 + 4/5 of 15 = ? of 15, giving T = 10 + 12 = 22 = (22/15)*15 when recounted in 15s.

Thus, adding fractions is ambiguous. If taken of the same total, 2/3 + 4/5 is 22/15; if not, the answer depends on the totals: 2/3 of 3 + 4/5 of 5 is (2+4)/(3+5) of 8 or 6/8 of 8, and 2/3 of 3 + 4/5 of 10 is 10/13 of 13, thus providing three different answers, 22/15 and 6/8 and 10/13, to the question '2/3+4/5 = ?'

Hiding the ambiguity of adding fractions makes mathematics 'mathe-matism' true inside but seldom outside classrooms.

As to algebraic fractions, a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

As when adding fractions, adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly, but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So adding fractions as the area under a piecewise constant per-number graph becomes 'middle school integration' later to be generalized to high school integration finding the area under a locally constant per-number graph. Thus calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level. n practice, it works on a pilot study level thus being ready for a more formal study.

Will this difference make a difference? In theory, yes, if first performing double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand.

Equations as walking or recounting

A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. when equations are composite or with a reverse sign in front of the unknown.

Here a difference is to use the definitions of reverse operations to establish the basic 'OSS'-rule for solving equations, 'move to the Opposite Side with the opposite Sign'. Thus, in the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise, with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12).

As to 2+3*u = 14, a difference is to see it as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn't have a reverse sign. This gives 3*u = 14 – 2. Since u doesn't have a reverse sign, 3 moves to the opposite side where a bracket tells that this must be calculated first:

u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4 since 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14.

Another difference is to see 2+3*u =14 as a walk, first multiplying u by 3 then adding 2 to give 14. To get back to u we reverse the walk by performing the reverse operations in reverse order. Thus, first subtracting 2 and then dividing by 3 gives u = (14-2)/3 = 4, checked by repeating the walk now with a known staring number: 4*3+2 = 14. Seeing an equation as a walk motivates using the terms ‘forward and backward calculation sides’ for 2+3*u and 14 respectively.

With 25 – u = 14, u moves to the opposite side to have its reverse sign reversed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5 giving 40 = 5*u; 40/5 = u; 8 = u. Alternatively, recounting twice gives 40 = (40/u)*u = 5*u, and 40 = (40/5)*5, consequently u = 40/5.

Pure letter-formulas build routine as e.g. ‘transform the formula T = a/(b-c) so that all letters become subjects.’ When building a routine, students often have fun singing:

“Equations are the best we know / they’re solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don’t want to stop.”

Another difference is to introduce equations the foist year in primary school as another name for recounting from tens to icons, e.g. asking ‘How many 9s are 45’ or ‘u*9 = 45’ giving u = 45/9 since recounting 45 in 9s, the recount formula gives 45 = (45/9)*9, again showing the OppositeSide&Sign rule.

Likewise, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, again obtained by moving a number to the opposite side with the opposite calculation sign.

Will this difference make a difference? In theory, yes, since equations are related to something concrete, walking or recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Geometry and algebra, always together, never apart

A class is stuck in geometry. Some mix up definitions, some find the theorems to abstract to understand, some find proofs difficult and hard to remember, some find geometry boring.

Here a difference is to use a coordinate system to coordinate geometry and algebra so they go hand in hand always and never apart, thus using algebra to predict geometrical intersection points, and vice versa, to use intersection points to solve algebraic equations. Both in accordance with the Greek meaning of mathematics as a common label for algebra and geometry.

In a coordinate-system a point is reached by a number of horizontally and vertically steps called the point’s x- and y-coordinates. Two points A(xo,yo) and B(x,y) with different x- and y-numbers will form a right-angled change-triangle with a horizontal side Δx = x-xo and a vertical side Δy = y-yo and a diagonal distance r from A to B, where by Pythagoras r^2 = Δx^2 + Δy^2. The angle A is found by the formula tanA =

$\Delta y/\Delta x = s$, called the slope or gradient for the line from A to B. This gives a formula for a non-vertical line: $\Delta y/\Delta x = s$ or $\Delta y = s*\Delta x$, or $y-yo = s*(x-xo)$. Vertical lines have the formula $x = xo$ since all points share the same x-number.

In a coordinate system three points A(x1,y1) and B(x2,y2)and C(x3,y3) not on a line will form a triangle that packs into a rectangle by outside right triangles allowing indirectly to find the angles and the sides and the area of the original triangle.

Different lines exist inside a triangle: Three altitudes measure the height of the triangle depending on which side is chosen as the base; three medians connect an angle with the middle of the opposite side; three angle bisectors bisect the angles; three line bisectors bisect the sides and are turned 90 degrees from the side. Likewise, a triangle has two circles; an outside circle with its center at the intersection point of the line bisectors, and an inside circle with its center at the intersection point of the angle bisectors.

Since Δx and Δy changes place when turning a line 90 degrees, their slopes will be $\Delta y/\Delta x$ and $-\Delta x/\Delta y$ respectively, so that $s1*s2 = -1$, called reciprocal with opposite sign.

As mentioned, geometrical intersection points are predicted algebraically by equating formulas. Thus with the lines $y = 2*x$ and $y = 6-x$, equating formulas gives $2*x = 6-x$, or $3*x = 6$, or $x = 2$, which inserted in the first gives $y = 2*2 = 4$, thus predicting the intersection point to be $(x,y) = (2,4)$. The same answer is found on a solver-app; or using software as GeoGebra.

Finding possible intersection points between a circle and a line or between two circles leads to a quadratic equation $x^2 + b*x + c = 0$, solved by a solver. Or by a formula created by two x-by-(x+k) playing cards placed on top of each other with the bottom left corner at the same place and the top card turned a quarter round clockwise. This creates 4 areas combining to $(x + k)^2 = x^2 + 2*k*x + k^2$. With $k = b/2$ this becomes $(x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c$ since $x^2 + b*x + c = 0$. Consequently the solution formula is $x = -b/2 \pm\sqrt{(b/2)^2 – c}$.

Finding a tangent to a circle at a point, its slope is the reciprocal with opposite sign of the radius line.

Will this difference make a difference? In theory, yes, since coordinating geometry and algebra gives equations a geometrical form and allows geometrical situations to be predicted by equations. In practice, it works on a pilot study level thus being ready for a more formal study.

Trigonometry as right triangles with sides mutually recounted

A class is stuck in trigonometry. Some find the ratios to abstract to understand, some mix up the formulas, some find the algebra difficult to use.

A difference is to introduce trigonometry as blocks halved in two by its diagonal, making a rectangle split into two right triangles. Here the angles are labeled A and B and C at the right angle. The opposite sides are labeled a and b and c.

The height a and the base b can be counted in meters, or in diagonals c creating a sine-formula and a cosine-formula: $a = (a/c)*c = \sin A*c$, and $b = (b/c)*c = \cos A*c$. Likewise, the height can be recounted in the base, creating a tangent-formula: $a = (a/b)*b = \tan A*b$

As to the angles, with a full turn as 360 degrees, the angle between the horizontal and vertical directions is 90 degrees. Consequently, the angles between the diagonal and the vertical and horizontal direction add up to 90 degrees; and the three angles add up to 180 degrees.

An angle A can be counted by a protractor, or found by a formula. Thus, in a right triangle with base 4 and diagonal 5, the angle A is found from the formula cosA = a/c = 4/5 as cos-1(4/5) = 36.9 degrees.

The three sides have outside squares with areas a^2 and b^2 and c^2. Turning a right triangle so that the diagonal is horizontal, a vertical line from the angle C splits the square c^2 into two rectangles. The rectangle under the angle A has the area (b*cosA)*c = b*(cosA*c) = b*b = b^2. Likewise, the rectangle under the angle B has the area (a*cosB)*c = a*(cosB*c) = a*a = a^2. Consequently c^2 = a^2 + b^2, called the Pythagoras formula.

This allows finding a square-root geometrically, e.g. x = $\sqrt{24}$, solving the quadratic equations x^2 = 24 = 4*6, if transformed into a rectangle. On a protractor, the diameter 9.5 cm becomes the base AB, so we have 6units per 9.5cm. Recounting 4 in 6s, we get 4units = (4/6)*6units = (4/6)*9.5 cm = 6.33 cm. A vertical line from this point D intersects the protractor's half-circle in the point C. Now, with a 4x6 rectangle under BD, BC will be the square-root $\sqrt{24}$, measured to 4.9, which checks: 4.9^2 = 24.0.

A triangle that is not right-angled transforms into a rectangle by outside right triangles, thus allowing its sides and angles and area to be found indirectly. So, as in right triangles, any triangle has the property that the angles add up to 180 degrees and that the area is half of the height times the base.

Inside a circle with radius 1, the two diagonals of a 4sided square together with the horizontal and vertical diameters through the center form angles of 180/4 degrees. Thus the circumference of the square is 2*(4*sin(180/4)), or 2*(8*sin(180/8)) with 8 sides instead. Consequently, the circumference of a circle with radius 1 is 2*π, where π = n*sin(180/n) for n large.

Will this difference make a difference? In theory, yes, since in Greek, geometry means to measure earth, typically by dividing it into triangles, again divided into right triangles, which can be seen as rectangles halved by their diagonals; and recounting totals in new units leads directly to mutual recounting the sides in a right triangle, which leads on to a formula for calculating pi. Furthermore, the many applications of trigonometry might increase the motivation for learning more geometry where coordinate geometry uses right triangles to increase any triangle to a rectangle with horizontal and vertical sides. In practice, it works on a pilot study level thus being ready for a more formal study.

PreCalculus as constant change

A class is stuck in precalculus. Some find the function concept to abstract to understand, some sees f(2) as a variable f multiplied by 2, some cannot make sense of roots and logarithm. The tradition defines a function top-down from above as a set-relation where first-component identity implies second component identity.

A difference is to return to the original Euler-meaning of a function defining it bottom-up from below as a name for a formula containing specified and unspecified numbers. And to see a formula as the core concept of mathematics respecting that, whatever it means, in the end mathematics is but a means to an outside goal, a number-language.

As a number-language sentence, a formula contains both specified and unspecified numbers in the form of letters, e.g. T = 5+3*x. A formula containing one unspecified number is called an equation, e.g. 26 = 5+3*x, to be solved by moving to opposite side with opposite calculation sign, (26-5)/3 = x. A formula containing two unspecified numbers is called a function, e.g. T = 5+3*x. An unspecified function containing an unspecified number x is labelled f(x), T = f(x). Thus f(2) is meaningless since 2 is not an unspecified number. Functions are described by a table or a graph in a coordinate system with y = T = f(x), both showing the y-numbers for different x-numbers. Thus, a change in x, Δx, will imply a change in y, Δy, creating a per-number Δy/Δx called the gradient of the formula.

As to change, a total can change in a predictable or unpredictable way; and predictable change can be constant or non-constant.

Constant change comes in several forms. In linear change, T = b + s*x, s is the constant change in y per change in x, called the slope or the gradient of its graph, a straight line. In exponential change, T = b*(1+r)^x, r is the constant change-percent in y per change in x, called the change rate. In power change, T = b*x^p, p is the constant change-percent in y per change-percent in x, called the elasticity. A saving increases from two sources, a constant $-amount per month, c, and a constant interest rate per month, r. After n months, the saving has reached the level C predicted by the formula C/c = R/r. Here the total interest rate after n months, R, comes from the formula 1+R = (1+r)^n. Splitting the rate r = 100% in t parts, we get the Euler number e = (1+100%/t)^t = (1+1/t)^t if t is large.

Also the change can be constant changing. Thus in T = c + s*x, s might also change constantly as s = c + q*x so that T = b + (c + q*x)*x = b + c*x + q*x^2, called quadratic change, showing graphically as a bending line, a parabola.

The difference seeing functions as predicting number-language sentences also suggests that functions in the form of formulas should be introduced from the first class of mathematics to predict counting results by a calculator, allowing the basic operations to be introduced as icons showing the three tasks involved when counting by bundling and stacking. Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication, iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. To also bundle bundles, power is iconized as a cap, e.g. 5^2, indicating the number of times bundles themselves have been bundled. Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other. To add on-top, the blocks must be recounted in the same unit, thus grounding proportionality. Next-to addition means adding areas, thus grounding integration.

Reversed adding on-top or next-to grounds equations and differentiation. Also, the four basic operations uncover the original meaning of the word algebra, meaning ‘to reunite’ in Arabic: Addition unites unlike numbers, multiplication unites like numbers into blocks, power unites like factors, and integration unite unlike blocks.

Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2]1 3s = 2.1 3s, using a cup or a decimal point to separate the ‘inside’ bundles from the ‘outside’ unbundled. This prediction holds at a manual counting:

T = 7 = I I I I I I I = III III I = 2 3s & 1.

Thus a calculator can predict a counting result by describing the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, i.e. with two formulas. The ‘recount formula’ T = (T/B)*B says that ‘from T, T/B times B can be taken away’ as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the ‘restack formula’ T = (T–B)+B says that ‘from T, T–B is left when B is taken away and placed next-to’, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. Wanting to recount a total in a new unit, the two formulas can predict the result when bundling and stacking and dragging away the stack. Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Recounting a total in a new unit means changing unit, also called proportionality or linearity, a core concept in mathematics at school and at university level. Thus the recount formula turns up in proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right triangles, and in calculus as dy = (dy/dx)*dx = y’*dx when counting steepness on a curve by recounting a vertical change in a horizontal.

Will this difference make a difference? In theory, yes, since describing mathematics as the grammar of the number-language is a powerful metaphor uncovering the real outside goal of mathematics education, to develop a number-language having the same sentence structure as the word-language, which will demystify the nature of mathematics to many students. In practice, it works on a pilot study level thus being ready for a more formal study.

Calculus as adding locally constant per-numbers

A class is stuck in calculus. Some find the limit concept too abstract. Some find the applications too artificial. For some, their hate to differential calculus prevents them from learning integral calculus.

Here a difference is to postpone differential calculus till after integral calculus is presented as a means to add piecewise or locally constant per-numbers by their areas. Thus, when adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 add by their areas as 3*2 + 5*4, meaning that per-numbers add by the area under the per-number-graph. With a piecewise constant per-number this mean a small number of area strips to add. But seeing a non-constant per-number as locally constant it means adding a huge amount of area strips, only possible if we can rewrite the strips as differences since the disappearance of the middle terms makes many differences add up to one single difference between the terminal and initial number. This of course makes rewriting a formula as a difference highly interesting, thus motivating a study of differential calculus. Thus, with the area strip 2*x*dx written as d(x^2), summing up the strips gives a single difference:

T2-T1 = Δ(x^2) = Σ ΔT = ∫ dT = ∫f(x)*dx = ∫2*x*dx .

Change formula come from observing that in a block, changes Δb and Δh in the base b and the height h impose on the total a change ΔT as the sum of a vertical strip Δb*h and a horizontal strip b*Δh and a corner Δb*Δh that can be neglected for small changes; thus d(b*h) = db*h + b*dh, or counted in T's: dT/T = db/b + dh/h, or with T' = dT/dx, T'/T = b'/b + h'/h. Therefore (x^2)'/x^2 = x'/x + x'/x = 2/x, giving (x^2)' = 2*x since x' = dx/dx = 1.

As to the limit concept, a difference is to rename it to 'local constancy': In a function y = f(x) a small change x often implies a small change in y, thus both remaining 'almost constant' or 'locally constant', a concept formalized with an 'epsilon-delta criterium', distinguishing between three forms of constancy. y is 'globally constant' c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is 'piecewise constant' c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is 'locally constant' c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio Δy/Δx can be globally, piecewise or locally constant, in the latter case written as dy/dx. Formally, local constancy and linearity is called continuity and differentiability.

Finally, calculus allows presenting the core of the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Will this difference make a difference? In theory, yes, since presenting it as adding piecewise or locally constant per-numbers will ground integral calculus in meaningful real-world problems. Likewise, observing the enormous advantage in adding differences gives a genuine motivation for differential calculus that is lost if insisting that it comes before integral calculus. In practice, it works on a pilot study level thus being ready for a more formal study.

How Different is the Difference?

Difference research uses sociological imagination to revive the ancient sophist warning: Know nature from choice to discover choice presented as nature. Thus, true and false nature are separated by asking the tradition: Can this be different, and will the difference make a difference? Witnessed by 50 years of sterility, mathematics education research is a natural place to see if difference-research, DR, will make a difference.

The tradition says, 'To obtain its goal, to learn mathematics, mathematics education must teach mathematics!' DR objects, 'No, to obtain its goal, mastery of Many, mathematics is a means to be replaced by another means if not leading to the goal, e.g. by ‘Many-matics’, defining its concepts from below as abstractions from examples instead of from above as examples of abstractions as does the traditional ‘meta-matics’.

The tradition says, ‘The core of mathematics is to operate on numbers!’ DR objects, ‘No, the core of mathematics is number-language sentences describing how totals are counted and recounted before being added; and having the same sentence structure as the word-language: a subject, a verb and a predicate.’

The tradition says, ‘Digits must be taught as symbols like letters!’ DR objects, ‘No, digits are icons containing as many strokes as they represent.’

The tradition says, ‘To describe cardinality, numbers must be taught as a one-dimensional number-line!’ DR objects, ‘No, numbers are two-dimensional blocks counting a total in stacks of bundles and unbundled singles.’

The tradition says, ‘Natural numbers must be taught as a place value system and ten-bundling is silently understood!’ DR objects, ‘No, numbers should be taught using cup-writing to separate inside bundles from outside singles, making a natural number a decimal number with a unit. And ten-counting should be postponed until icon-counting and re-counting in the same and in a different unit has been experienced’.

The tradition says, ‘There are four kinds of numbers, natural and integer and rational and real numbers!’ DR objects, ‘No, a number is a positive or negative decimal number with a unit. Rational numbers are per-numbers, i.e. operators needing a number to become a number; and real numbers are calculations to deliver as many decimals as wanted.’

The tradition says, ‘Operations must be taught as functions from a set-product to the set supplying it with a structure obeying associative, commutative and distributive laws as well as neutral and inverse elements allowing equations to be solved by neutralization!’ DR objects, ‘Operations are icons showing the three processes of counting, bundling and stacking and removing stacks to look for unbundled singles; and adding stacks or blocks on-top or next-to. Solving equations is another word for reversing the processes by re-bundling or re-stacking’

The tradition says, ‘The natural order of teaching operations is addition before subtraction before multiplication before division allowing fractions to be introduced as rational numbers to which the same operations can be applied!’ DR objects, ‘No, since totals must be counted before they can be added, the natural order is the opposite: first division to take away bundles many times, then multiplication to stack the bundles, then subtraction to take away the stack once to look for unbundled singles, and finally addition in its two versions, on-top and next-to. And counting also implies recounting in the same or another unit, to and from tens, and double-counting producing per-numbers as operators needing numbers to become numbers, thus being added by their areas, i.e. by integration.’

The tradition says, ‘Calculators should not be allowed before all four operations are taught and learned!’ DR objects, ‘Calculators should be used from the start to predict counting and recounting results.’

The tradition says, ‘Operations must be taught using carrying!’ DR objects, ‘No, operations should be taught using cup-writing allowing totals to be recounted with overloads or underloads.’

The tradition says, ‘Multiplication tables must be learned by heart!’ DR objects, ‘No, multiplication tables describe recounting from icon-bundles to ten-bundles; geometrically seen as changing a block by increasing the width and decreasing the height to keep the total unchanged; and algebraically sees as doubling or tripling totals written with an overload or an underload.’

The tradition says, ‘Division is difficult and must be taught using constructivism to allow learners invent their own algorithms!’ DR objects, ‘No, division should be taught as recounting from ten-bundles to icon-bundles using cup-writing and recounting in the same unit to benefit from the multiplication tables.’

The tradition says, ‘Arithmetic comes before geometry, and they must be held apart until the introduction of the coordinate system!’ DR objects, ‘No, arithmetic should be seen as algebra kept together with geometry all the time and from the beginning, where numbers are a collection of blocks as well as a collection of numbers in cups; where recounting and multiplication means changing block-sizes as well as changing cup-numbers; and where addition means adding blocks as well as cup-numbers.’

The tradition says, ‘Proportionality must be postponed until functions have been introduced!’ DR objects, ‘No, as another name for changing units, proportionality occurs from the beginning as recounting in another unit; and is needed when adding on-top and next-to. And reoccurring when double-counting creates per-numbers as bridges between physical units.’

The tradition says, ‘Fractions must be introduced first as parts of something then as numbers by themselves!’ DR objects, ‘No, created by double-counting in the same unit, fractions are per-numbers and as such operators needing a number to become a number.’

The tradition says, ‘Prime-factorizing must precede adding fractions by finding a common denominator!’ DR objects, ‘No, prime-factorizing comes with recounting to another unit to find the units allowing a total to be recounted fully without any unbundled singles. And fractions should be added as operators, i.e. by integrating their areas.’

The tradition says, ‘Equations must be taught as statements about equivalent number-names, solved by the neutralizing method obeying associative, commutative and distributive laws!’ DR objects, ‘No, equations occur when recounting totals from tens to icons, and when reversing on-top and next-to addition.’

The tradition says, ‘A function must be taught as an example of a set-relation where first-component identity implies second-component identity!’ DR objects, ‘No, a function should be taught as a formula with two unspecified numbers thus respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. Later polynomials can be introduced as the number-formula containing the different formulas for constant change: T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x.’

The tradition says, ‘Linear functions must be taught before quadratic functions!’ DR objects, ‘No, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d.’

The tradition says, ‘Quadratic equations must be solved by factorizing before introducing the solution formula!’ DR objects, ‘No, when solving the quadratic equation x^2+b*x+c = 0, algebra and geometry should go hand in hand to show that inside a square with the sides x+b/2, the equation makes three rectangles disappear leaving only (b/2)^2-c, allowing possible roots to be found and used in factorization if necessary.’

The tradition says, ‘Differential calculus must be taught before integral calculus since the integral is defined as the anti-derivate.’ DR objects, ‘No, integral calculus comes before differential calculus. In primary school, next-to addition means multiplying before adding when asking e.g. T = 2 3s + 4 5s = ? 8s’, while reversing the question by asking 2 3s + ? 5s = 6 8s, or T1 + ? 5s = T, leads to differential calculus subtracting before dividing to get the answer (T-T1)/5. In middle school, fractions and per-numbers add by their areas, i.e. by integration. And in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant predictable change.’

The tradition says, ‘The epsilon-delta definition is essential in order to understand real numbers and calculus and must be learned by heart!’ DR objects, ‘No, it needs not be learned by heart. With units, it can be grounded in formalizing three ways of constancy; globally constant needing only the epsilon, piecewise constant with delta before epsilon, and locally constant with epsilon before delta.’

The tradition says, ‘Statistics and probability must be taught separately!’ DR objects, ‘No, they should be taught together aiming at pre-dicting unpredictable numbers by intervals coming from ‘post-dicting’ their previous behavior.’

In continental Europe, the tradition says, ‘Education means preparing for offices in the public or private sector. Hence the necessity of line-organized education with forced year-group classes in spite of the fact that teenage girls are two years ahead of the boys in personal development. Of course, boys and dropouts are to pity, but they all had the chance.’ North American republics object: ‘No, Education means uncovering and developing the learner’s individual talents through daily lessons in self-chosen practical or theoretical half-year blocks together with a person teaching only one subject and praising the learner for having a talent or for having courage to test it.’

In mathematics education, the tradition says, ‘Education means connecting learners to the canonical correctness through scaffolding from the learner’s zone of proximal development as described in social constructivism by Bruner and Vygotsky.’ DR objects, ‘No, education means bringing outside phenomena inside a classroom to be assimilated or accommodated by the learners thus respecting that in a sentence, the subject is objective but the rest might be subjective as described in radical constructivism by Piaget and Grounded Theory and Heidegger existentialism.

In mathematics education, the tradition says, ‘Research means applying or extending existing theory.’ DR objects, ‘No, where master level work means applying existing theory, research level means questioning existing theory, e.g. by asking if it could be different.’

How to Improve PISA Performance

PISA performance (Tarp, 2015a) can be improved in three ways: by a different macro-curriculum from class one, by remedial micro-curricula when a class is stuck, and by a STEM-based core-curriculum for outsiders.

Improving PISA performance means improving mathematics learning which can be done by observing three basic facts about our human and mammal and reptile brains.

The human brain needs meaning, so what is taught must be a meaningful means to a meaningful outside goal, mastery of Many; thus mathematics must be taught as ‘Many-matics’ in the original Greek sense as a common name for algebra and geometry both grounded in an motivated by describing Many in time and space; and not as ‘meta-matism’ mixing ‘meta-matics’, defining concepts from above as examples of internal abstractions instead of from below as abstractions from external examples, with ‘mathe-matism’, true inside but seldom outside classrooms as adding numbers without units.

The mammal brain houses feelings, positive and negative. Here learning is helped by experiencing a feeling of success from the beginning, or of suddenly mastering or understanding something difficult.

The reptile brain houses routines. Here learning is facilitated by repetition and by concreteness: With mathematics as a text, its sentences should be about subjects having concrete existence in the world, and having the ability to be handled manually according to Piagetian principle ‘through the hand to the head’.

Also, we can observe that allowing alternative means than the tradition makes it not that difficult to reach the outside goal, mastery of many. Meeting Many, we ask ‘How many in total?’ To get an answer we count and add. We count by bundling and stacking and removing the stack to look for unbundles leftovers. This gives the total the geometrical form of a collection of blocks described by digits also having a geometrical nature by containing as many sticks as they represent. Counting also includes recounting in the same or in a new unit; or double-counting to produce per-numbers. Once counted, totals can be united or split, and with four kinds of numbers, constant and variable unit-numbers and per-numbers, there are four ways to unite: addition, multiplication, power and integration; and four ways to split: subtraction, division, root/logarithm and differentiation.

Thus, the best way to obtain good PISA performance is to replace the traditional SET-based curriculum with a different Many-based curriculum from day one in school, and to strictly observe the warning: Do not add before totals are counted and recounted – so multiplication must precede addition. However, this might be a long-term project. To obtain short-term improvements, difficult parts of a curriculum where learners often are stuck might be identified and replaced by an alternative remedial micro-curriculum designed by curriculum architecture using difference-research and sociological imagination. Examples can be found in the above chapter ‘Examples of difference-research’.

Finally, in the case of teaching outsiders as migrants or adults or dropouts with no or unsuccessful educational background, it is possible to design a STEM-based core curriculum as described above allowing the outsiders become pre-teachers and pre-engineers in two years. Thus, applying sociological imagination when meeting Many without predicates forced upon it, allows avoiding repeating the mistakes of traditional mathematics.

The Tradition’s 3x3 mistakes

Choosing learning mathematics as the goal of teaching mathematics has serious consequences. Together with being set-based this makes both mathematics education and mathematics itself meaningless by self-reference. Here a difference is to accept that the goal of teaching mathematics is mastering Many by developing a number-language parallel to the word-language; both having a meta-language, a grammar, that should be taught after the language to respect that the language roots the grammar instead of being an application of it; and both having the same sentence structure with a subject and a verb and a predicate, thus saying ‘T = 2*3’ just saying ‘2*3’ instead of just ‘2*3’.

This goal displacement seeing mathematics as the goal of mathematics education leads to 3x3 specific mistakes in primary, middle and high school:

In primary school, numbers are presented as 1dimensional line numbers written according to a place value convention; instead of accepting that our Arabic numbers like the numbers children bring to school are 2dimensional block numbers. Together with cup-counting and cup-writing this gives an understanding that a number really is a collection of numbers counting what exists in the world, first inside bundles and outside unbundled singles, later a collection of unbundled and bundles and bundles of bundles etc.

Furthermore, school skips the counting process and goes directly to adding numbers without considering units; instead of exploiting the golden learning opportunities in counting and recounting in the same or in another unit, and to and from tens. This would allow multiplication to be taught and learned before addition by accepting that 4*7 is 4 7s that maybe recounted in tens as T = 4 7s = 2.8 tens = 28, to be checked by recounting 28 back to 7s, T = 28 = (28/7)*7 = 4*7 = 4 7s, using the recount-formula reappearing in proportionality, trigonometry and calculus. And giving division by 7 the physical meaning of counting in 7s.

Finally, addition only includes on-top addition of numbers counted in tens only and using carrying, a method that neglects the physical fact that adding or subtracting totals might crate overloads or underloads to be removed by recounting in the same unit. And neglecting the golden learning opportunities that on-top addition of numbers with different unit roots proportionality, and that next-to addition roots integration, that reversed roots differentiation thus allowing calculus to be introduced in primary school.

In middle school, fractions are introduced as numbers that can be added without units thus presenting mathematics as ‘mathematism’ true inside but seldom outside classrooms. Double-counting leading to per-numbers is silenced thus missing the golden learning opportunities that per-numbers give a physical understanding of proportionality and fractions, and that both per-numbers and fractions as operators need numbers to become numbers that as products add as areas, i.e. by integration.

Furthermore, equations are presented as open statements expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*u = 8	an open statement about two equivalent number-names
(2*u)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(u*2)*(1/2) = 4	since multiplication is commutative
u*(2*(1/2)) = 4	since multiplication is associative
u*1 = 4	by definition of an inverse element
u = 4	by definition of a neutral element

The alternative sees an equation as another name for reversing a calculation that stops because of an unknown number. Thus the equation ‘2*u = 8’ means wanting to recount 8 in 2s: 2*u = 8 = (8/2)*2, showing that u = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the ‘opposite side&sign’ method. A method that allows the equation ‘20/u = 5’ to be solved quickly by moving across twice; 20 = 5*u’ and 20/5 = u’, or more thoroughly by recounting 20 = (20/u)*u = 5*u = (20/5)*5 = 4*5, so u = 4.

Finally, middle school lets geometry precede coordinate geometry, again preceding trigonometry; instead of respecting that in Greek, geometry means to measure earth, which is done by dividing it into triangles again divided into right triangles. Consequently, trigonometry should come first as a mutual recounting of the sides in a right triangle. And geometry should be part of coordinate geometry allowing solving equations predict intersection points and vice versa, thus experiencing repeatedly that the strength of mathematics is the fact that formula predict.

In high school, a function is presented as an example of a set-relation where first-component identity implies second-component identity; and the important functions are polynomials with linear functions preceding quadratic functions; instead of respecting that a function is a name for a formula with two unspecified numbers, again respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. As to polynomials, they should be introduced as the number-formula containing the different forms of formulas for constant change, T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x. Consequently, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d and parallel to the other examples of constant change. Thus emphasizing the double nature of formulas that the can predict both level and change.

Furthermore, differential calculus is presented before integral calculus, presenting an integral as an antiderivative; instead of postponing differential calculus until after integral calculus is presented as adding locally constant per-numbers, i.e. as a natural continuation of adding fractions as piecewise constant per-numbers in middle school and next-to addition of blocks in primary school. Only in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant change.

Finally, high school presents algebra as a search for patterns, instead of celebrating the fact that calculus completes the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Uniting/ *splitting*	Variable	Constant
Unit-numbers	T = a + n *T – a = n*	T = a * n *T/n = a*
Per-numbers	T = ∫ a dn *dT/dn = a*	T = a^n, *loga(T) = n* *n√T = a*

Remedial Curricula

A remedial micro-curriculum might be relevant whenever learning problems are observed. Since you never get a second chance to create a first impression, especially remedial curricula in primary school are important to prevent mathematics dislike.

Thus, as described above in the chapter 'examples of difference-research', in primary school, problems might be eased by

- with digits, using a folding ruler to observe that a digit contains as many sticks or strokes as it represents if written in a less sloppy way.
- with counting sequence, using sequences that shows the role of bundling when counting to indicate that a given total as e.g. seven can be named in different ways: 7, .7, 0.7, bundle less 3, ½bundle&2, etc.
- with recounting, using a cup and 5 sticks to experience that at total of 5 can be recounted in 2s in three ways: with an overload, normal, or with an underload: T = 5 = 1]3 2s = 2]1 2s = 3]-1 2s, or T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s if using decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles.

- when learning multiplication tables, letting 3*7 mean 3 7s recounted in tens, i.e. a block that when increasing its width must decrease its height to keep the total unchanged.
- when learning multiplication tables, beginning by doubling and halving and tripling; and to recount numbers using half-ten and ten as e.g. 7 = half-ten&2 = 10less3 so that 2 times 7 is 2 times half-ten&2 = ten&4 = 14, or 2 times 10less3 = 20 less 6 = 14.
- when multiplying, using cup-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 7* 48 = 7* 4]8 = 28]56 = 33]6 = 336, or T = 7* 48 = 7* 5]-2 = 35]-14 = 33]6 = 336
- when dividing, using cup-writing to create overloads or underloads according to the multiplication table, as e.g. T = 336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 48
- when subtracting, using cup-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 – 27 = 6]5 – 2]7 = 4]-2 = 3]8 = 38
- when adding, using cup-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 + 27 = 6]5 + 2]7 = 8]12 = 9]2 = 92

In middle school, problems might be eased by keeping algebra and geometry together and by re-describing

- proportionality as double-counting in different units leading to per-numbers
- fractions as per-numbers coming from double-counting in the same unit
- adding fractions as per-numbers by their areas, i.e. by integration
- solving equations as reversing calculations by moving to the opposite side with the opposite calculation sign

In high school, problems might be eased by re-describing

- functions as number-language sentences, i.e. formulas becoming equations or functions with 1 or 2 unspecified numbers
- calculus as integration preceding differentiation
- integration as adding locally constant per-numbers
- pre-calculus, calculus and statistics as pre- or post-dicting constant, non-constant and non-predictable change

A Macro STEM-based Core Curriculum

A macro-curriculum (Tarp, 2017) was designed as an answer to a fictitious curriculum architect contest set up by a Swedish university wanting to help the increasing number of young male migrants coming to Europe each year: ‘The contenders will design a STEM-based core mathematics curriculum for a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries.’

The design was built upon two assumptions. The curriculum goal is mastery of Many in a STEM context for learners with no background. As to STEM, OECD writes:

> The New Industrial Revolution affects the workforce in several ways. Ongoing innovation in renewable energy, nanotech, biotechnology, and most of all in information and communication technology will change labour markets worldwide. Especially medium-skilled workers run the risk of being replaced by computers doing their job more efficiently. This trend creates two

> challenges: employees performing tasks that are easily automated need to find work with tasks bringing other added value. And secondly, it propels people into a global competitive job market. (..) In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules, called heat; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles and mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves. Nitrates and carbon-dioxide and water is waste for grey cells, but food for green cells producing proteins and carbon-hydrates and oxygen as food for the grey cells in return.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is cup-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. To master plane or spatial forms, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent. So, mastery of Many means counting and recounting and adding and reversing addition and describing change and spatial shapes.

A STEM-based core curriculum can be about cycling water. Heating transforms it from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

Teaching Differences to Teachers

A group of teachers wanting to bring difference-research findings to the classroom might want first to watch some YouTube videos at the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many.

Then to try out the ‘Free 1day SKYPE Teacher Seminar: Cure Math Dislike by 1 cup and 5 sticks’ where, in the morning, a power point presentation ‘Curing Math Dislike’ is watched and discussed locally, and at a Skype conference with a coach. After lunch the group tries out a ‘CupCount before you Add booklet’ to experience proportionality and calculus and solving equations as golden learning opportunities in cup-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and

S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in the three genres of quantitative literature, fact and fiction and fiddle. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Being a Difference-researcher

In mathematics education, difference-research can be used by teachers observing problems in the classroom, or by teacher-researchers splitting their time between academic work at a university and intervention research in a classroom. Or by full-time researchers cooperating with teachers both using difference-research, the teacher to observe problems, the researcher to identify differences, working out a different micro-curriculum together to be tested by the teacher and reported by the researcher conducting a pretest-posttest study.

Thus, a typical difference-researcher begins as an ordinary teacher observing learning problems in his classroom and wondering if he could teach differently. Personally, in a precalculus class I taught linear and exponential functions by following the textbook order presenting them as examples of functions, again presented as examples of relations between two sets assigning one and only one element in one set to each element in the other set. I realized that by defining concepts as examples of abstractions instead of as abstractions from examples, I basically taught that 'bublibub is an example of bablibab' which some learners just memorized while others refused to learn before I gave them some applications. Talking about the difference between saving at home and in a bank, some asked me: Instead of calling it linear and exponential functions, why don't you just call it change by adding and by multiplying since that is what it is?'

So here the students themselves invented a difference that makes sense since historically, functions came after calculus. And the difference made two differences. Nobody had problems with learning about change by adding and by multiplying. And the Ministry of Education followed my suggestion to replace functions with variables instead of making pre-calculus non-compulsory, which was the plan because of the high number of low marks.

So one way to become a difference-teacher is to combine elements from action learning and action research and intervention research and design research. First you identify a difference, then you design a micro-curriculum, then you teach it to learn what difference the difference makes, then you learn from reporting and discussing it internally with colleagues. After having repeated this cycle of teaching and reporting the difference, the difference and the difference it makes in a posttest or a pretest-posttest setting is reported externally to teacher magazines or to conferences or to research journals.

Research is an institution supposed to produce knowledge to explain nature and improve social conditions. But as an institution, research risks a goal displacement if becoming self-referring. This raises two questions: Can a teacher produce research, and can research produce teaching? (Hammersley, 1993, p. 215). Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality used to collect reliable data and to test the validity of its findings by falsification attempts.

Often sociological imagination (see e.g. Zybartas et al, 2005) seems to be absent from traditional research seen by many teachers as useless because of its many references. In a Swedish context, this has been called the 'irrelevance of the research industry' (Tarp, 2015b, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying in a YouTube debate with Chomsky on Human nature:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics education and research are two examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead thus marginalizing or forgetting its original outside goal.

Conclusion

With 50 years of research, mathematics education should have improved significantly. Its lack of success as illustrated by OECD report ‘Improving Schools in Sweden’ made this paper ask: Apparently half a century’s research in mathematics education has not prevented low and declining PISA performance. Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? Seeking guidance by difference-research searching traditions for hidden differences that make a difference, the answer is: Yes, mathematics can be different, education can be different, and research can be different.

Looking back, mathematics has meant different things through its long history, from a common label for knowledge in ancient Greece to today’s ‘meta-matism’ combining ‘meta-matics’ defining concepts by meaningless self-reference, and ‘mathe-matism’ adding numbers without units thus lacking outside validity. So, looking for a difference to traditional set-based meta-matism, one alternative is the original Greek meaning of mathematics: Knowledge about Many in time and space.

Observing Many, allows rebuilding mathematics as a ‘many-matics’, i.e. as a natural science about the physical fact Many, where counting by bundling and stacking leads to block-numbers that recounted in other units leads to proportionality and solving equations; where recounting sides in triangles leads to trigonometry; where double-counting in different units leads to per-numbers and fractions, both adding by their areas, i.e. by integration; where counting precedes addition taking place both on-top and next-to involving proportionality and calculus. And where using a calculator to predict the counting result leads to the opposite order of operations: division before multiplication before subtraction before next-to and on-top addition.

Observing classes in continental Europe and in North America shows that education can be line-organized with forced year-group classes aiming at fulfilling the nation’s need for officials for the public or private sector; or education can be block-organized with self-chosen half-year classes aiming at uncovering and developing the learner’s individual talent. In mathematics education, the tradition sees learning mathematics as the goal of teaching mathematics and defines its concepts from above as examples of abstractions, part of the ruling canonical correctness, to be reached by learners through scaffolding. Here a difference is to accept that concepts historically arose from below as abstractions from examples, thus allowing new concepts to connect to existing.

Observing conference proceedings, shows that research papers may instead be master level papers applying instead of questioning existing theory and aiming at explaining instead of solving educational problems. Here a difference is difference-research searching traditions for hidden differences that make a difference.

So yes, as to mathematics education research, all three components can be different. Bottom-up many-matics can replace top-down meta-matism. In teenage education, daily lessons in self-chosen half-year blocks can replace periodic lessons in forced year-group lines. And, searching for useable differences can replace attempts at understanding the lack of understanding non-understandable self-reference.

Consequently, PISA performance may increase instead of decrease, and Swedish schools might improve dramatically by respecting that education means preparing learners for the outside world, brought inside to change the classroom from a library with self-referring textbooks to be learned by hart into a laboratory allowing the learner to meet the educational subject directly instead of indirectly through textbook 'gossip'. And by avoiding a goal displacement seeing mathematics as the goal for mathematics education, thus hiding the real goal, a number-language about Many in time and space.

To teach many-matics instead of meta-matism, big-scale in-service teacher training is needed, e.g. through the MATHeCADEMY.net, designed to teach teachers to teach mathematics as a natural science about Many by the CATS-approach, Count & Add in Time & Space, using PYRAMIDeDUCATION, where learners learn by being taught by the subject directly instead of indirectly by a sentence.

So, if a society as Sweden really wants to improve mathematics education, extra funding should force its universities to arrange curriculum architect contests to allow differences to compete as to imagination, creativity and effectiveness, thus allowing universities to rediscover their original external goal and to change their internal routines accordingly. A situation described in several fairy tales: The Beauty Sleeping behind the thorns of routines becoming rituals; and Cinderella making the prince dance, but only found when searching outside the canonical correctness.

With 2017 as the 500 anniversary of Luther's 95 theses, the recommendation of difference-research to mathematics education research could be the following theses:

- To master Many, count and multiply before you add
- Counting and recounting give block-numbers and per-numbers, not line-numbers
- Adding on-top and next-to roots proportionality and integration, and equations when reversed
- Beware of the conflict between bottom-up enlightening and top-down forming theories.
- Institutionalizing a means to reach a goal, beware of a goal displacement making the institution the goal instead
- To cure, be sure, the diagnose is not self-referring
- In sentences, trust the subject but question the rest

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