GOOD, BAD & EVIL MATHEMATICS - TALES OF TOTALS, NUMBERS & FRACTIONS

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Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA result made OECD (2015) write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.’ (p. 3).

This may prove that, by its very nature, mathematics is indeed hard to learn. On the other hand, since mathematics education is a social institution, social theory may provide a different reason.

Social Theory Looking at Mathematics Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking ‘renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now’ (p. 16).

Mathematics education is an example of ‘rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)’. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference.

So, inspired by sociology we can ask the ‘Cinderella question’: ‘as an alternative to the tradition, is there is a different way to the goal of mathematics education, mastery of Many?’

In short, could there be different kinds of mathematics? And could it be that among them, one is good, and one is bad, and one is evil? In other words, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in

Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics'.

Here the invention of the concept Set created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false:

If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

Good and Bad and Evil Mathematics

The existence of three different versions of mathematics, many-matics and meta-matics and mathe-matism, allows formulating the following definitions:

Good mathematics is absolute truths about things rooted in the outside world. An example is $T = 2*3 = 6$ stating that a total of 2 3s can be re-counted as 6 1s. So good mathematics is tales about totals, and how to count and unite them.

Bad mathematics is relative truths about things rooted in the outside world. An example is claiming that 2+3 = 5, only valid if the units are the same, else meeting contradictions as 2weeks + 3days = 17days. So bad mathematics is tales about numbers without units.

Evil mathematics talks about something existing only inside classrooms. An example is claiming that fractions are numbers, and that they can be added without units as claiming that 1/2 + 2/3 = 7/6 even if 1 red of 2 apples plus 2 reds of 3apples total 3reds of 5 apples and not 7reds of 6apples. So bad mathematics is tales about fractions as numbers.

Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks the grounded theory question: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a 'Count-before-Adding' Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create number-language sentences, *T* = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to use as units when counting:

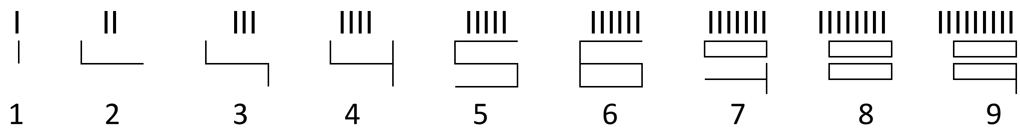

Figure 1. Digits as icons containing as many sticks as they represent

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total *T* we take away bundles *B* thus rooting and iconizing division as a broom wiping away the bundles. Stacking the bundles roots and iconizes multiplication as a lift stacking the bundles into a block. Moving the stack away to look for unbundled singles roots and iconizes subtraction as a trace left when dragging the block away. A calculator predicts the counting result by a 're-count formula' *T = (T/B)*B* saying that 'from *T*, *T/B* times, *B* can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so *T* = 7 = 2*B*1 3s.

Placing the unbundled singles next-to or on-top of the stack of 3s roots decimals and fractions:

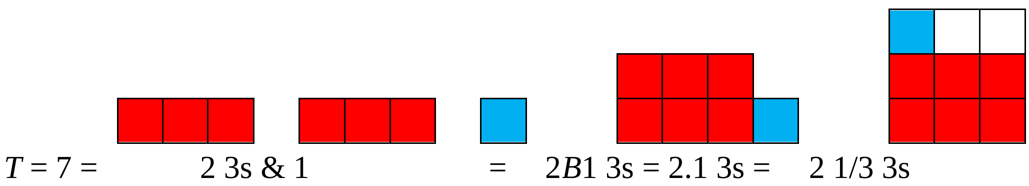

Figure 2. Re-counting a total of 7 in 3s, the unbundled single can be placed in three different ways

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, *T* = 42 = ? 7s = *u**7, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations, and differential calculus:

T = 2 3s + ? 4s = 5 7s gives differentiation: ? = (5*7 – 2*3)/4 = Δ*T*/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers *sine*, *cosine* and *tangent*. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the *y*-change in the *x*-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A Short Version of a Curriculum in Good Mathematics, Grounded Many-matics

01. To stress the importance of bundling, the counting sequence should be: 01, 02, …, 09, 10,11 etc.

02. The ten fingers should be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers should be re-counted in three ways (standard, and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables should be formulated as re-counting from icon-bundles to tens and use overload counting after 5: T = 4 7s = 4*7 = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 should be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

06. Solving proportional equations as 3*x = 12 should be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks should be done by re-counting in the per-number: With 3$/4kg, T = 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and T = 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages should be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$, so T = 60$ = (60/3)*3$ gives (60/3)*2$ = 40$

09. Integral should precede differential calculus and include adding both piecewise and locally constant per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at (2*3+4*5)$/(2+4)kg thus showing that per-numbers and fractions are added with their units as the area under the per-number graph.

10. Trigonometry should precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. $a = (a/c)*c = \sin A*c$.

Good and Bad Mathematics

Today's tradition begins with arithmetic telling about line-numbers, processed by four basic operations, later extended with negative numbers and rational numbers and reel numbers. Algebra then repeats it all with letters instead. Geometry begins with plane geometry followed by coordinate geometry and trigonometry later. Functions are special set-products, and differential calculus precedes integral calculus.

In general, we see mathematics as truths about well-defined concepts. So we begin by discussing what can be meant by good and bad concepts.

Good and Bad Concepts

As an example, let us look at a core concept in mathematics, a calculation. To differentiate between $y = 2*3$ and $y = 2*x$, around 1750 Euler defined the concept 'function' as a calculation containing unspecified numbers. Later, around 1900, set-based mathematics defined a function as an example of a set-product where first component identity implies second component identity.

So where the former is a bottom-up definition of a concept as an abstraction from examples, the latter is a top-down definition of a concept as an example of an abstraction.

Since examples are in the world and since Russell warned that by its self-reference the set-concept is meaningless, we can label bottom-up and top-down definitions good and bad concepts respectively.

Good and Bad Numbers

Good numbers should reflect that our number-language describes a total as counted in bundles and expressing the result in a full sentence with subject and verb and predicate as in the word-language, as e.g. $T = 2$ 3s. These are the numbers that children bring to school, two-dimensional block-numbers that contain three different number-types: a 'unit-number' for the size, a 'bundle-number' and a 'single-number' for the number of bundles and unbundled singles. Totals then are written in bundle-form or in decimal-form with a unit where a bundle-B or a decimal point separates the inside bundles from the outside singles, as e.g. T = 3*B*2 tens = 3.2 tens.

Good numbers are flexible to allow a total to be re-counted in a different unit; or in the same unit to create an overload or underload to make calculations easier, as e.g. T = 3*B*2 tens = 2*B*12 tens = 4*B*-8 tens. Good numbers are shown in two ways: an algebraic with bundles, and a geometrical with blocks. Good numbers also tell that eleven and twelve come from the Vikings saying 'one left' and 'two left'.

Bad numbers do not respect the children's own two-dimensional block-numbers by insisting on one-dimensional line-numbers be introduced as names along a line without practicing bundling. Numbers follow a place value system with different places for the ones, tens, hundreds, and thousands; but seldom renaming them as bundles, bundle of bundles, and bundles of bundles of bundles.

Good and Bad Counting

A good counting sequence includes bundles in the names, as e.g. 01, 02, …, Bundle, 1*B*1, etc.; or 0Bundle1, 0*B*2, etc. Another sequence respects the nearness of a bundle by saying 0*B*6, 1*B*less3, 1*B*-2, etc.

Good counting lets counting and re-counting and double-counting precede addition; and allows the re-count formula to predict the counting-result; and it presents the symbols for division, multiplication and subtraction as icons coming from the counting process, thus introducing the operations in the opposite order.

Bad counting neglects the different forms of counting by going directly to adding, thus not respecting that totals must be counted before they can be added.

Bad counting treats numbers as names thus hiding their bundle nature by a place value system. This leads some to count 'twenty-ten' instead of 'thirty', and to confuse 23 and 32.

Good and Bad Addition

Good addition waits until after totals have been counted and re-counted in the same and in a different unit, to and from tens, and double-counted in two units to create per-numbers bridging the units. Likewise, good addition respects its two forms: on-top rooting proportionality since changing the units might be need; and next-to rooting integral calculus by being added by the areas.

Bad addition claims it priority as the fundamental operation defining the others: multiplication as repeated addition, and subtraction and division as reversed addition and multiplication. It insists on being the first operation being taught. Numbers must be counted in tens. Therefore there is no need to change or mention the unit; nor is there a need to add next-to as twenties.

Bad addition does not respect that in block-numbers as T = 2*B*3 4s, the three digits add differently. Unit-numbers, as 4, only add if adding next-to. Bundle-numbers, as 2, only add if the units are the same; else re-counting must make them so. Single-numbers, as 3, always add, but might be re-counted because of an overload.

Good and Bad Subtraction

Good subtraction sees its sign as iconizing the trace left when dragging away a stack to look for unbundled singles, thus leading on to division as repeated subtraction moving bundles away. It does not mind taking too much away and leaving an underload, as in 3*B*2 - 1*B*5 = 2*B*-3.

Bad subtraction sees its sign as a mere symbol; and sees itself as reversed addition; and doesn't mind subtracting numbers without units.

Good and Bad Multiplication

Good multiplication sees its sign as iconizing a lift stacking bundles. It sees 5*7 as a block of 5 7s that may or may not be re-counted in tens as 3.5 tens or 35; and that has the width 7 and the height 5 that, if recounted in tens, must widen it width and consequently shorten its height. Thus, it always sees the last factor as the unit.

Good multiplication uses flexible numbers when re-counting in tens by multiplying, as e.g. *T* = 6*8 = 6*(ten-2) = (ten-4)*8 = (ten-4)*(ten-2). This allows reducing the ten by ten multiplication table to a five by five table.

Bad multiplication sees its sign as a mere symbol; and insists that all blocks must be re-counted in tens by saying that 5*7 IS 35. It insists that multiplication tables must be learned by heart.

Good and Bad Division

Good division sees its sign as iconizing a broom wiping away the 2s in *T* = 8/2. It sees 8/2 as 8 counted in 2s; and it finds it natural to be the first operation since when counting, bundling by division comes before stacking by multiplication and removing stacks by subtraction to look for unbundled singles.

Bad division sees its sign as a mere symbol; and teaches that 8/2 means 8 split between 2 instead of 8 counted in 2s. Bad division accepts to be last by saying that division is reversed multiplication; and insists that fractions cannot be introduced until after division.

Good and Bad Calculations

Good calculations use the re-count formula to allow a calculator to predict counting-results.

Bad calculations insist on using carrying so that the result comes out without overloads or underloads.

Good and Bad Proportionality

Good proportionality is introduced in grade 1 as re-counting in another unit predicted by the re-count formula. It is re-introduced when adding blocks on-top; and when double-counting in two units to create a per-number bridging the units by becoming a proportionality factor.

Bad proportionality is introduced in secondary school as an example of multiplicative thinking or of a linear function.

Good and Bad Equations

Good equations see equations as reversed calculations applying the opposite operations on the opposite side thus using the 'opposite side and sign' method in accordance with the definitions of opposite operations: 8-3 is the number x that added to 3 gives 8; thus if $x+3 = 8$ then $x = 8-3$. Likewise with the other operations.

Good equations sees equations as rooted in re-counting from tens to icons, as e.g. 40 = ? 8s, leading to an equation solved by re-counting 40 in 8s: $x*8 = 40 = (40/8)*8$, thus $x = 40/8 = 5$.

Bad equations insist that the group definition of abstract algebra be used fully or partwise when solving an equation. It thus sees an equation as an open statement expressing identity between two number-names. The statements are transformed by identical operations aiming at neutralizing the numbers next to the unknown by applying commutative and associative laws.

$2*x = 8$	an open statement about the identity of two number-names
$(2*x)*(1/2) = 8*(1/2)$	½, the inverse element of 2, is multiplied to both names
$(x*2)*(1/2) = 4$	since multiplication is commutative
$x*(2*(1/2)) = 4$	since multiplication is associative
$x*1 = 4$	by definition of an inverse element
$x = 4$	by definition of a neutral element

Figure 3. Solving an equation using the formal group definition from abstract algebra

Good and Bad Pre-calculus

Good pre-calculus shows that the number-formula, $T = 345 = 3*BB + 4*B + 5*1 = 3*x\text{^}2 + 4*x + 5$, has as special cases the formulas for constant linear, exponential, elastic, or accelerated change: $T = b*x+c$, $T = a*n\text{^}x$, $T = a*x\text{^}n$, and $T = a*x\text{^}2 + b*x + c$. It uses 'parallel wording' by calling root and logarithm 'factor-finder' and 'factor-counter' also. It introduces integral calculus with blending problems adding piecewise constant per-numbers, as e.g. 2kg at 3 $/kg plus 4kg at 5$/kg. It includes modeling examples from STEM areas (Science, Technology, Engineering, Mathematics)

Bad pre-calculus introduces linear and exponential functions as examples of a homomorphism satisfying the condition f(x#y) = f(x)$f($y$). It includes modeling from classical word problems only.

Good and Bad Calculus

Good calculus begins with primary school calculus, adding two blocks next-to each other. It also includes middle school calculus adding piecewise constant per-numbers, to be carried on as high school calculus adding locally constant per-numbers.

It motivates the epsilon-delta definition of constancy as a way to formalize the three forms of constancy: global, piecewise and locally. It shows series with single changes and total changes calculated to realize that many single changes sum up as one single change, calculated as the difference between the end- and start-values since all the middle terms disappear.

This motivates the introduction of differential calculus as the ability to rewrite a block *h*dx* as a difference *dy, dy/dx = h*; and where the changes of block with sides f and g leads on to the fundamental formula of differential calculus, (f*g)'/(f*g)= f'/f + g'/g, giving (x^n)'/x^n = n*1/x, or (x^n)' = n*x^(n-1).

Bad calculus introduces differential calculus before integral calculus that is defined as anti-differentiation where the area under *h* is a primitive to *h*; and it introduces the epsilon-delta criterion without grounding it in different kinds of constancy.

Good and Bad Modeling

Good modeling is quantitative literature or number-stories coming in three genres as in word stories: Fact, fiction and fiddle. Fact and fiction are stories about factual and fictional things and actions. Fiddle is nonsense like ‘This sentence is false’ that is true if false, and vice versa.

Fact models, also called ‘since-then’ or ‘room’ models, quantify quantities and predict predictable quantities: “What is the area of the walls in this room?”. Since the prediction is what is observed, fact models can be trusted. Fiction models, also called ‘if-then’ or ‘rate’ models, quantify quantities but predict unpredictable quantities: “My debt is gone in 5 years at this rate!”. Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based on alternative assumptions. Fiddle models, also called ‘then-what’ or ‘risk’ models, quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. (Tarp, 2017).

Bad modeling does not distinguish between the three genres but sees all models as approximations.

Good and Bad Geometry

Good geometry lets trigonometry precede plane geometry that is integrated with coordinate geometry to let algebra and geometry go hand in hand to allow formulas predict geometrical intersection points.

Bad geometry lets plane geometry precede coordinate geometry that precedes trigonometry.

Evil Mathematics

Evil mathematics talks about something existing only inside classrooms. Fractions as numbers and adding fractions without units are two examples. The tradition presents fractions as rational numbers, defined as equivalence classes in a set product created by the equivalence relation R, where

(a,b) R (c,d) if $a*d = b*c$.

Grounded in double-counting in two units, fractions are per-numbers double-counted in the same unit, as e.g. 3$ per 5$ or 3 per 5 or 3/5. Both are operators needing a number to become a number. Both must be multiplied to unit-numbers before adding, i.e. they add by their areas as in integral calculus.

Shortening or enlarging fractions is not evil mathematics. They could be called 'footnote mathematics' since they deal with operator algebra seldom appearing outside classrooms. They deal with re-counting numbers by adding or removing common units: to shorten, 4/6 it is re-counted as 2 2s over 3 2s giving 2/3. To be enlarged, both take on the same unit so that 2/3 = 2 4s over 3 4s = 8/12.

Educating teachers, it is evil to silence the choices made in mathematics education. Instead, teachers should be informed about the available alternatives without hiding them in an orthodox tradition. Especially the difference between good and bad mathematics should be part of a teacher education.

Good and Bad Education

When children become teenagers, their identity work begins: 'Who am I; and what can I do?' So good education sees its goal as allowing teenagers to uncover end develop their personal talent through daily lessons in self-chosen practical or theoretical half-year blocks with teachers having only one subject, and praising the students for their talent or for their courage to try out something unknown.

Bad education sees its goal as selecting the best students for offices in the private or public sector. It uses fixed classes forcing teenagers to follow their age-group despite the biological fact that girls are two years ahead in mental development.

Good and Bad Research

Good research searches for truth about things that exist. It poses a question, and choose a methodology to transform reliable data into valid statements. Or it uses methodical skepticism to unmask choice masked as nature.

Bad research is e.g. master level work applying instead of questioning existing research. Or journalism describing something without being guided by a question.

With these three research genres, peer-review only works inside the same genre.

Conclusion and Recommendation

This paper used difference-research to look for different ways to the outside goal of mathematics education, mastery of Many. By meeting Many outside the present self-referring set-based tradition three ways were found, a good, and a bad, and an evil. Good mathematics respects the original tasks in Algebra and Geometry, to reunite Many and to measure earth. By identifying a hidden alternative, good mathematics creates a paradigm shift (Kuhn, 1962) that opens up a vast field for new research seeing mathematics as a many-matics, i.e. as a natural science about Many (cf. Tarp, 2018).

In short, we need to examine what happens if we allow children to keep and develop the quantitative competence they bring to school, two-dimensional block-numbers to be recounted and double-counted before being added on-top or next-to; and reported with full number-language sentences including both a subject that exists, and a verb, and a predicate that may be different.

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