Core Papers

2017-2018

final version

Allan.Tarp@MATHeCADEMY.net

CONTENTS

Core papers 2017-18

This selection contains four papers written in 2017 and 2018. The total collection of papers can be found at

http://mathecademy.net/2017-math-articles/

http://mathecademy.net/2018-articles-on-math-education/

01. The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants.

This article is due to be published in the next number 34 of Philosophy of Mathematics Education Journal. The abstract says that Swedish educational shortages challenge traditional mathematics education offered to migrants. Mathematics could be taught in its simplicity instead of as 'meta-matsim', a mixture of 'meta-matics' defining concepts as examples of inside abstractions instead of as abstractions from outside examples; and 'mathe-matism' true inside classrooms but seldom outside as when adding numbers without units. Rebuilt as 'many-matics' from its outside root, Many, mathematics unveils its simplicity to be taught in a STEM context at a 2year course providing a background as pre-teacher or pre-engineer for young migrants wanting to help rebuilding their original countries.

02. Addition-free migrant-math rooted in STEM re-counting formulas

A short version of the article above was sent to the Topic Working Group 26 on STEM mathematics at the CERME 11 conference. It was rejected as a paper, so it was redrawn.

The abstract says that a curriculum architect is asked to avoid traditional mistakes when designing a curriculum for young migrants that will allow them to quickly become STEM pre-teachers and pre-engineers. Typical multiplication formulas expressing re-counting in different units suggest an addition-free curriculum. To answer the question 'How many in total?' we count and re-count totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a re-count formula as a core formula in all STEM subjects.

03. Mastering Many by Counting, Re-counting and Double-counting before Adding On-top and Next-to

This article was published in the Journal of Mathematics Education, March 2018, 11(1), 103-117.

The abstarct says that observing the quantitative competence children bring to school, and by using difference-research searching for differences making a difference, we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

04. A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many

This article was accepted at the conference ICMI Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunitie, in Tsukuba Japan, 26-30 November 2018. The abstract says that mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory, we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum.

05. Counting before Adding, The Child's Own Twin Curriculum, Count & ReCount & DoubleCount before Adding NextTo & OnTop

This is a Power Point Presentation made from the article above.

Allan Tarp, Aarhus, November 2018

THE SIMPLICITY OF MATHEMATICS DESIGNING A STEM-BASED CORE MATH CURRICULUM FOR OUTSIDERS AND MIGRANTS

Swedish educational shortages challenge traditional mathematics education offered to migrants. Mathematics could be taught in its simplicity instead of as 'meta-matsim', a mixture of 'meta-matics' defining concepts as examples of inside abstractions instead of as abstractions from outside examples; and 'mathe-matism' true inside classrooms but seldom outside as when adding numbers without units. Rebuilt as 'many-matics' from its outside root, Many, mathematics unveils its simplicity to be taught in a STEM context at a 2year course providing a background as pre-teacher or pre-engineer for young migrants wanting to help rebuilding their original countries.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

Other countries also experience declining PISA results. Since mathematics education is a social institution, social theory might be able to explain 50 years of unsuccessful research in mathematics education.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16). A wish to uncover unnoticed alternatives motivates a 'difference-research' (Tarp, 2017) asking two questions: 'Can this be different – and will the difference make a difference?' If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

Natural sciences use difference-research to keep on searching until finding what cannot be different. Describing matter in space and time by weight, length and time intervals, they all seem to vary. However, including per-numbers will uncover physical constants as the speed of light, the gravitational constant, etc. The formulas of physics are supposed to predict nature's behavior. They cannot be proved as can mathematical formulas, instead they are tested as to falsifiability: Does nature behave different from predicted by the formula? If not, the formula stays valid until falsified.

Social sciences also use difference-research beginning with the ancient Greek controversy between two attitudes towards knowledge, called ‘sophy’ in Greek. To avoid hidden patronization, the sophists warned: Know the difference between nature and choice to uncover choice presented as nature. To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy. The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord deciding world behavior.

Today’s democracies implement common social goals through institutions with means decided by parliaments. As to rationality as the base for social organizations, Bauman says:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called *goal displacement*. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

As an institution, mathematics education is a public organization with a ‘rational action in which the end to be achieved is clearly spelled out’, apparently aiming at educating students in mathematics, ‘The goal of mathematics education is to teach mathematics’. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn’t the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here the invention of the concept SET created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A)\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about shapes and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers, operations and calculations. However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. And, being sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens, might speed up writing but might also slow down learning, together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, in Lincolns Gettysburg address, 'Four scores and ten years ago' shows that not all count in tens.

To get an answer to the questions 'What is mathematics?' and 'How is mathematics education improved?' we might include philosophy in the form of what Bauman calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'? (Bauman, 1992, p. ix).

> Inquiry is a cognizant seeking for an entity both with regard to the fact <u>that</u> it is and with regard to its Being <u>as</u> it is. (Heidegger, 1962, p. 5)

Heidegger here describes two uses of 'is'. One claims existence, 'M is', one claims 'how M is' to others, since what exists is perceived by humans wording it by naming it and by characterizing or analogizing it to create 'M is N'-statements.

Thus, there are four different uses of the word 'is'. 'Is' can claim a mere existence of M, 'M is'; and 'is' can assign predicates to M, 'M is N', but this can be done in three different ways. 'Is' can point down as a 'naming-is' ('M is for example N or P or Q or …') defining M as a common name for its volume of more concrete examples. 'Is' can point up as a 'judging-is' ('M is an example of N') defining M as member of a more abstract category N. Finally, 'is' can point over as an 'analogizing-is' ('M is like N') portraying M by a metaphor carrying over known characteristics from another N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

In France, Heidegger inspired the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu, pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Lyotard, 1984. Bourdieu, 1970. Chomsky et al, 2006).

From a Heidegger view a sentence contains two things: a subject that exists, and the rest that might be gossip. So, to discover its true nature hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its predicate-prison. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young migrants wanting to help rebuilding their original countries.

So, to restore its authenticity, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many

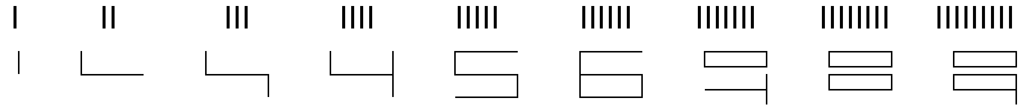
As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep the balance and to store sounds assigned to what we grasped with our forelegs, now freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into two languages, a word-language and a number-language.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair'. Observing the existence of many chairs, we ask 'how many in total?' and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, 'the total is 3 chairs' or, if counting legs, 'the total is 3 fours', abbreviated to 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is an operation'. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

With 2017 as the 500year anniversary for Luther’s 95 theses, we can choose to describe meeting Many in theses.

01. Using a folding ruler, we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the four icon, and five sticks in the five icon, etc. Counting in 5s, the counting sequence is 1, 2, 3, 4, Bundle, 1-bundle-1, etc. This shows, that the bundle-number does not need an icon. Likewise, when bundling in tens. Instead of ten-1 and ten-2 we use the Viking numbers eleven and twelve meaning 1 left and 2 left in Danish.



1 2 3 4 5 6 7 8 9

02. Transforming four ones to a bundle of 1 4s allows counting with bundles as a unit. Using a cup for the bundles, a total can be ‘bundle-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘bundle-writing’ to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s. Next, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Or, with tens: T = 234 = 23B4 = 2BB3B4.

03. Recounting in the same unit by creating or removing overloads or underloads, bundle-writing offers an alternative way to perform and write down operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

04. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved when counting by bundling and stacking. Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication, iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate T = 3 4s, we

discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.

05. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. The 'recount formula' T = (T/B)*B says that 'from T, T/B times B can be taken away' as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the 'restack formula' T = (T–B)+B says that from T, T–B is left when B is taken away and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. The recount or proportionality formula turns out to a very basic formula. It turns up in proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right-angled triangles, and in calculus as dy = (dy/dx)*dx = y'*dx when counting steepness on a curve.

06. Wanting to recount a total in a new unit, we discover that a calculator can predict the result when bundling and stacking and dragging away the stack. Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Also we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.

07. Wanting to recount a total in tens, we discover that a calculator predicts the result directly by multiplication, only leaving out the unit and misplacing the decimal point. Thus, asking T = 3 7s = ? tens, the calculator predicts: T = 21 = 2.1 tens. Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged. With 5 as half of ten, and 8 as ten less 2, a 10x10 multiplication table can be reduced to a 3x3 table including the numbers 2, 3 and 4. Thus, 4*8 = 4*(ten less 2) = 4ten less 8 = 32; 5*8 = half of 8ten = 4ten = 40; 7*8 = (ten less 3)*(ten less 2) = tenten, less 3ten, less 2ten, plus 6 = 56.

Wanting to recount a total from tens to icons, we discover this as another example of recounting to change the unit. Thus, asking T = 3 tens = ? 7s, the calculator predicts: First 30/7 = 4.some; then 30 – (4*7) = 2; and finally T = 30 = 4.2 7s. Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged.

08. Using the letter u for an unknown number, we can rewrite the recounting question '? 7s = 3tens' as 'u*7 = 30' with the answer 30/7 = u since 30 = (30/7)*7, officially called to solve an equation. Here we discover a natural way to do so: Move a number to the opposite side with the opposite calculation sign. Thus, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, obtained again by moving a number to the opposite side with the opposite calculation sign.

09. Once counted, totals can be added, but addition is ambiguous. Thus, with two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to have the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s =

(2*3 + 4*5)/8 * 8 = 3.2 8s. So next-to addition geometrically means adding areas, and algebraically it means combining multiplication and addition. Officially, this is called integration, a core part of traditional mathematics in high school and at the first year of university.

10. Also we discover that addition and other operations can be reversed. Thus, in reversed addition, 8 = u+2, we ask: what is the number u that added to 2 gives 8, which is precisely the formal definition of u = 8–2. And in reversed multiplication, 8 = u*2, we ask: what is the number u that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root, the factor-finder, and logarithm, the factor-counter, as u = 3√20 and u = log3(20). Again we solve the equation by moving to the opposite side with the opposite calculation sign. Reversing next-to addition we ask 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer u, from the terminal total T we remove the initial total T1 before we count the rest in 5s: u = (T–T1)/5 = ΔT/5, using Δ for the difference or change. Letting subtraction precede division is called differentiation, the reverse operation to integration letting multiplication precede addition.

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

12. Economy is based upon investing money and expecting a return that might be higher or lower than the investment, e.g. 7$ per 5$ or 3$ per 5$. Here when double-counting in the same unit, per-numbers become fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. Thus, to find 3 per 5 of 20, or 3/5 of 20, as before we just recount 20 in 5s and replace 5 with 3, T = 20 = (20/5)*5 giving (20/5)*3 = 12.

To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s and replace 5 with 3: T = 100 = (100/5)*5 giving (100/5)*3 = 60. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly, but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise when adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

13. Halved by its diagonal, a rectangle splits into two right-angled triangles. Here the angles are labeled A and B and C at the right angle. The opposite sides are labeled a and b and c.

The height a and the base b can be counted in meters, or in diagonals c creating a sine-formula and a cosine-formula: a = (a/c)*c = sinA*c, and b = (b/c)*c = cosA*c. Likewise, the height can be recounted in bases, creating a tangent-formula: a = (a/b)*b = tanA*b

As to the angles, with a full turn as 360 degrees, the angle between the horizontal and vertical directions is 90 degrees. Consequently, the angles between the diagonal and the vertical and horizontal direction add up to 90 degrees; and the three angles add up to 180 degrees.

An angle A can be counted by a protractor, or found by a formula. Thus, in a right-angled triangle with base 4 and diagonal 5, the angle A is found from the formula cosA = a/c = 4/5 as cos-1(4/5) = 36.9 degrees.

The three sides have outside squares with areas a^2 and b^2 and c^2. Turning a right triangle so that the diagonal is horizontal, a vertical line from the angle C split the square c^2 into two rectangles. The rectangle under the angle A has the area (b*cosA)*c = b*(cosA*c) = b*b = b^2. Likewise, the rectangle under the angle B has the area (a*cosB)*c = a*(cosB*c) = a*a = a^2. Consequently c^2 = a^2 + b^2, called the Pythagoras formula.

This allows finding a square-root geometrically, e.g. x = √24, solving the quadratic equations x^2 = 24 = 4*6, if transformed into a rectangle. On a protractor, the diameter 9.5 cm becomes the base AB, so we have 6units per 9.5cm. Recounting 4 in 6s, we get 4units = (4/6)*6units = (4/6)*9.5 cm = 6.33 cm. A vertical line from this point D intersects the protractor's half-circle in the point C. Now, with a 4x6 rectangle under BD, BC will be the square-root √24, measured to 4.9, which checks: 4.9^2 = 24.0.

A triangle that is not right-angled transforms into a rectangle by outside right-angled triangles, thus allowing its sides and angles and area to be found indirectly. So, as in right-angled triangles, any triangle has the property that the angles add up to 180 degrees and that the area is half of the height times the base.

Inside a circle with radius 1, the two diagonals of a 4sided square together with the horizontal and vertical diameters through the center form angles of 180/4 degrees. Thus the circumference of the square is 2*(4*sin(180/4)), or 2*(8*sin(180/8)) with 8 sides instead. Consequently, the circumference of a circle with radius 1 is 2*π, where π = n*sin(180/n) for n large.

14. A coordinate system coordinates algebra with geometry where a point is reached by a number of horizontally and vertically steps called the point's x- and y-coordinates.

Two points A(xo,yo) and B(x,y) with different x- and y-numbers will form a right-angled change-triangle with a horizontal side Δx = x-xo and a vertical side Δy = y-yo and a diagonal distance r from A to B, where by Pythagoras r^2 = Δx^2 + Δy^2. The angle A is found by the formula tanA = Δy/Δx = s, called the slope or gradient for the line from A to B. This gives a formula for a non-vertical line: Δy/Δx = s or Δy = s*Δx, or y-yo = s*(x-xo). Vertical lines have the formula x = xo since all points share the same x-number.

In a coordinate system three points A(x1,y1) and B(x2,y2)and C(x3,y3) not on a line will form a triangle that packs into a rectangle by outside right-angled triangles allowing indirectly to find the angles and the sides and the area of the original triangle.

Different lines exist inside a triangle: Three altitudes measure the height of the triangle depending on which side is chosen as the base; three medians connect an angle with the middle of the opposite side; three angle bisectors bisect the angles; three line bisectors bisect the sides and are turned 90 degrees from the side. Likewise, a triangle has two circles; an outside circle with its center at the intersection point of the line bisectors, and an inside circle with its center at the intersection point of the angle bisectors.

Since Δx and Δy changes place when turning a line 90 degrees, their slopes will be $\Delta y/\Delta x$ and $-\Delta x/\Delta y$ respectively, so that s1*s2 = -1, called reciprocal with opposite sign.

Geometrical intersection points are predicted algebraically by solving two equations with two unknowns, i.e. by inserting one into the other. Thus with the lines y = 2*x and y = 6-x, inserting the first into the second gives 2*x = 6-x, or 3*x = 6, or x = 2, which inserted in the first gives y = 2*2 = 4, thus predicting the intersection point to be (x,y) = (2,4). The same answer is found on a solver-app; or using software as GeoGebra.

Finding possible intersection points between a circle and a line or between two circles leads to a quadratic equation x^2 + b*x + c = 0, solved by a solver. Or by a formula created by two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter round clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution formula is x = -b/2 ±√((b/2)^2 – c).

Finding a tangent to a circle at a point, its slope is the reciprocal with opposite sign of the radius line.

15. A formula predicts a total before counting it. A formula typically contains both specified and unspecified numbers in the form of letters, e.g. T = 5+3*x. A formula containing one unspecified number is called an equation, e.g. 26 = 5+3*x, to be solved by moving to opposite side with opposite calculation sign, (26-5)/3 = x. A formula containing two unspecified numbers is called a function, e.g. T = 5+3*x. An unspecified function containing an unspecified number x is labelled f(x), T = f(x). Thus f(2) is meaningless since 2 is not an unspecified number. Functions are described by a table or a graph in a coordinate system with y = T = f(x), both showing the y-numbers for different x-numbers. Thus, a change in x, Δx, will imply a change in y, Δy, creating a per-number $\Delta y/\Delta x$ called the gradient.

16. In a function y = f(x), a small change x often implies a small change in y, thus both remaining ‘almost constant’ or ‘locally constant’, a concept formalized with an ‘epsilon-delta criterium’, distinguishing between three forms of constancy. y is ‘globally constant’ c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is ‘piecewise constant’ c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is ‘locally constant’ c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in the latter case written as dy/dx. Formally, local constancy and linearity is called continuity and differentiability.

17. As to change, a total can change in a predictable or unpredictable way; and predictable change can be constant or non-constant.

Constant change comes in several forms. In linear change, T = b + s*x, s is the constant change in y per change in x, called the slope or the gradient of its graph, a straight line. In exponential change, T = b*(1+r)^x, r is the constant change-percent in y per change in x, called the change rate. In power change, T = b*x^p, p is the constant change-percent in y per change-percent in x, called the elasticity. A saving increases from two sources, a constant $-amount per month, c, and a constant interest rate per month, r. After n months, the saving has reached the level C predicted by the formula C/c = R/r. Here the total interest rate after n months, R, comes from $1+R = (1+r)^n$. Splitting the rate r = 100% in t parts, we get the Euler number $e = (1+100\%/t)^t = (1+1/t)^t$ if t is large.

Also the change can be constant changing. Thus in $T = c + s*x$, s might also change constantly as $s = c + q*x$ so that $T = b + (c + q*x)*x = b + c*x + q*x^2$, called quadratic change, showing graphically as a line with a curvature, a parabola.

If not constant but still predictable, we have a change formula $\Delta T/\Delta x = f(x)$ or $dT/dx = f(x)$ in the case of interval change or local change. Such an equation is called a differential equation which is solved by calculus, adding up all the local changes to a total change being the difference between the end and start number: $T2-T1 = \Sigma\ \Delta T = \int dT = \int f(x)*dx$. Thus, with $dT/dx = 2*x$, $T2-T1 = \Delta(x^2)$. Change formula come from observing that in a block, changes Δb and Δh in the base b and the height h impose on the total a change ΔT as the sum of a vertical strip $\Delta b*h$ and a horizontal strip $b*\Delta h$ and a corner $\Delta b*\Delta h$ that can be neglected for small changes; thus $d(b*h) = db*h + b*dh$, or counted in T's: $dT/T = db/b + dh/h$, or with $T' = dT/dx$, $T'/T = b'/b + h'/h$. Therefore $(x^2)'/x^2 = x'/x + x'/x = 2/x$, giving $(x^2)' = 2*x$ since $x' = dx/dx = 1$.

18. Unpredictable change can be exemplified by throwing a dice with two results: winning, +1, if showing 4 or above, and losing, 0, if showing 3 or below. Throwing a dice 5 times thus have 6 outcomes, winning from 0 to 5 times. The outcome is called an unpredictable or stochastic or random number or variable. Per definition, random numbers cannot be pre-dicted, instead they can be 'post-dicted' using statistics and probability.

Thus the outcome '0,0,0,1,1' can be described by three numbers. The mode is 0 since this number has the highest frequency, 3 per 5, or 3/5. The median is 0 since this is the middle number when aligned in increasing order. The mean u is the fictional number had all numbers been the same: $u*5 = 0+0+0+1+1$ with the solution $u = 2/5 = 0.4$. With the outcome '0,0,1,1,1', the mode and median and mean is 1 and 1 and 3/5 = 0.6.

To find the three numbers if the experiment is repeated many times, we look at a 'possibly tree'. The first toss has two results, win or lose, both occurring ½ of the times. Likewise, with the following tosses: After two tosses we have three outcomes: 2 wins, 1 win and 0 wins. Here 2 wins and 0 wins occur half of half of the times, i.e. with a probability ¼. 1 win occurs twice, as win-lose or as lose-win, both with a probably of ¼, so the total probability for 1 win is 2*1/4 = ½. Continuing in this way we find that with 5 tosses there are 6 outcomes, winning from 0 to 5 times with the probabilities $½^5$ a certain number of times: 1, 5, 10, 10, 5, 1. By calculations we find that the mode is 2 and 3, and that the median and the mean is 2.5, also found by multiplying the number of repetitions with the probability for winning.

A spreadsheet random generator can show examples of other outcomes.

19. A sphere may be distorted into a cup. Even if distorted, a rectangle will still divide a sphere into an inside and an outside needing a bridge to be connected. And a sphere with a bridge may be distorted into a cup with a handle or into a donut. Distortion geometry is called topology, useful when setting up networks, thus able to prove that connecting three houses with water, gas and electricity is impossible without a bridge.

20. As qualitative literature, also quantitative literature has three genres, fact and fiction and 'fiddle', used when modeling real world situations. Fact is 'since-then' calculations using numbers and formulas to quantify and to predict predictable quantities as e.g. 'since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20'. Fact models can be trusted once the numbers and the formulas and the calculation has been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter. Fiction is 'if-then' calculations using numbers and formulas to quantify and to predict unpredictable quantities as e.g. 'if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20'. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made. Fiddle models is 'what-then' models using numbers and formulas to quantify and to predict unpredictable qualities as e.g. 'since a graveyard is cheaper than a hospital, then a bridge across the highway is too costly.' Fiddle models should be rejected and relegated to a qualitative description.

Meeting Many in a STEM Context

Having met Many by itself, now we meet Many in time and space in the present culture based upon STEM, described by OECD as follows:

> The New Industrial Revolution affects the workforce in several ways. Ongoing innovation in renewable energy, nanotech, biotechnology, and most of all in information and communication technology will change labour markets worldwide. Especially medium-skilled workers run the risk of being replaced by computers doing their job more efficiently. This trend creates two challenges: employees performing tasks that are easily automated need to find work with tasks bringing other added value. And secondly, it propels people into a global competitive job market. (..) In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles, mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves. Nitrates and carbon-dioxide and water is waste for grey cells, but food for green cells producing proteins and carbon-hydrates and oxygen as food for the grey cells in return.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. To master plane or spatial shapes, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent.

So, a core STEM curriculum could be about cycling water. Heating transforms it from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we must build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

A short World History

When humans left Africa, some went west to the European mountains, some went east where the fertile valleys in India supplied everything except for silver from the mountains. Consequently, rich trade took place sending pepper and silk west and silver east. European culture flourished around the silver mines, first in Greece then in Spain during the Roman Empire. Then the Vandal conquest of the mines brought the dark middle age to Europe until silver was found in the Harz valley (Tal in German leading to thaler and dollar), transported through Germany to Italy. Here silver financed the Italian Renaissance, going bankrupt when Portugal discovered a sea route to India enabling them to skip the cost of Arab middlemen. Spain looked for a sea route going west and found the West Indies. Here there was neither pepper nor silk but silver in abundance e.g. in the land of silver, Argentine. On their way home, slow Spanish ships were robbed by sailing experts, the Vikings descendants living in England, now forced to take the open sea to India to avoid the Portuguese fortification of Africa's coast.

In India, the English found cotton that they brought to their colonies in North America, but needing labor they started a triangle-trade exchanging US cotton for English weapon for African slaves for US cotton. In the agricultural South, a worker was a cost to be minimized, but in the industrial North a worker was a consumer needed at an industrial market. During the civil war, no cotton came to England that then conquered Africa to bring the plantations to the workers instead. Dividing the world in closed economies kept new industrial states out of the world market that it took two world wars to open for free competition.

Nature Obeys Laws, but from Above or from Below?

In the Lord's Prayer, the Christian Church says: 'Thy will be done, on earth as it is in heaven'. Newton had a different opinion.

As experts in sailing, the Viking descendants in England had no problem stealing Spanish silver on its way across the Atlantic Ocean. But to get to India to exchange it for pepper and silk, the Portuguese fortification of Africa's cost forced them to take the open sea and navigate by the moon. But how does the moon move? The church had one opinion, Newton meant differently.

'We believe, as is obvious for all, that the moon moves among the stars,' said the Church, opposed by Newton saying: 'No, I can prove that the moon falls to the earth as does the apple.' 'We believe that when moving, things follow the unpredictable metaphysical will of the Lord above whose will is done, on earth as it is in heaven,' said the Church, opposed by Newton saying: 'No, I can prove they follow their own physical will, a physical force, that is predictable because it follows a mathematical formula.' 'We believe, as Aristotle told us, that a force upholds a state,' said the Church, opposed by Newton saying: 'No, I can prove that a force changes a state. Multiplied with the time applied, the force's impulse changes the motion's momentum; and multiplied with the distance applied, the force's work changes the motion's energy.' 'We believe, as the Arabs have shown us, that to deal with formulas you just need ordinary algebra,' said the Church, opposed by Newton saying: 'No. I need to develop a new algebra of change which I will call calculus.'

Proving that nature obeys its own will and not that of a patronizor, Newton inspired the Enlightenment century realizing that if enlightened we don't need the double patronization of the physical Lord at the Manor house and the metaphysical Lord above. Citizens only need to inform themselves, debate and vote. Consequently, to enlighten the population, two Enlightenment republics were created, in the US in 1776 and in France in 1789. The US still have their first republic allowing its youth to uncover and develop their personal talent through daily lessons in self-chosen half-year blocks, whereas the Napoleon wars forced France and the rest of continental Europe to copy the Prussians line-organized education forcing teenagers to follow their year-group and its schedule, creating a knowledge nobility (Bourdieu, 1970) for public offices, and unskilled workers, good for yesterday's industrial society, but bad for today's information society where a birth rate at 1.5 child per family will halve the population each 50 years since (1.5/2)^2 = 0.5 approximately.

Counting and DoubleCounting Time, Space, Matter, Force and Energy

Counting time, the unit is seconds. A bundle of 60seconds is called a minute; a bundle of 60minutes is called an hour, and a bundle of 24hours is called a day, of which a bundle of 7 is called a week. A year contains 365 or 366days, and a month from 28 to 31days.

Counting space, the international unit is meter, of which a bundle of 1000 is called a kilometer; and if split becomes a bundle of 1000millimeters, 100centimeters and 10decimeters. Counting squares, the unit is 1 square-meter. Counting cubes, the unit is 1cubic-meter, that is a bundle of 1000cubic-decimeters, also called liters, that split up as a bundle of 1000milliliters.

Counting matter, the international unit is gram that splits up into a bundle of 1000milligrams and that unites in a bundle of 1000 to 1 kilogram, of which a bundle of 1000 is called 1tons.

Counting force and energy, a force of 9.8Newton will lift 1 kilogram, that will release an energy of 9.8Joule when falling 1meter.

Cutting up a stick in unequal lengths allows the pieces to be double-counted in liters and in kilograms giving a per-number around 0.7kg/liter, also called the density.

A walk can be double-counted in meters and seconds giving a per-number at e.g. 3meter/second, called the speed. When running, the speed might be around 10meter/second. Since an hour is a bundle of 60 bundles of 60seconds this would be 60*60meters per hour or 3.6 kilometers per hour, or 3.6km/h.

A pressure from a force applied to a surface can be double-counted in Newton and in square meters giving a per-number Newton per square-meter, also called Pascal.

Motion can be double-counted in Joules and seconds producing the per-number Joule/second called Watt. To run properly, a bulb needs 60Watt, a human needs 110Watt, and a kettle needs 2000Watt, or 2kiloWatt. From the Sun the Earth receives 1370Watt per square meter.

Warming and Boiling water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

Heating 1000gram water 80degrees in 167seconds in a 2000Watt kettle, the per-number will be 2000*167/80Joule/degree, creating a double per-number 2000*167/80/ 1000Joule/degree/gram or 4.18Joule/degree/gram, called the specific heat of water.

Producing 100gram steam in 113seconds, the per-number is 2000*113/100Joule/gram or 2260J/g, called the heat of evaporation for water.

Letting Steam Work

A water molecule contains two Hydrogen and one Oxygen atom weighing 2*1+16 units. A collection of a million billion billion molecules is called a mole; a mole of water weighs 18 gram. Since the density of water is roughly 1000gram/liter, the volume of 1000moles is 18liters. Transformed into steam, its volume increases to more than 22.4*1000liters, or an increase factor of 22,400liters per 18liters = 1244 times. The volume should increase accordingly. But, if kept constant, instead the inside pressure will increase.

Inside a cylinder, the ideal gas law, p*V = n*R*T, combines the pressure, p, and the volume, V, with the number of moles, n, and the absolute temperature, T, which adds 273degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

So, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus decreasing and increasing the pressure by letting steam in and out of the two holes. The leaving steam the is visible on steam locomotives. In the third generation used in power plants, two cylinders, a hot and a cold, connect with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back to the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical power to industries and homes.

An Electrical circuit

To work properly, a 2000Watt water kettle needs 2000Joule per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per charge-unit.

Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm.

Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So, the 120Watt bulb only receives half of the energy of the 60Watt bulb.

How high up and how far out

A ping-pong ball is sent upwards. This allows a double-counting between the distance and the time to the top, 5meters and 1second. The gravity decreases the speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second.

To find its initial speed we turn the gun 45degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5meters and 5meters and 0,71 seconds. From a folding ruler we see, that now the speed is split into a vertical and a horizontal part, both reducing it with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed by the formula: Horizontal distance to the top = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

The vertical distance is halved, but the vertical speed changes from 9.92 to 9.92*0.707 = 7.01 only. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the distance four times.

How many turns on a steep hill

On a 30degree hillside, a 10degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2.

In the triangle ACD, the angle A is 10degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2.

In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370meter and 740meter up the hillside.

Dissolving material in water

In the sea, salt is dissolved in water. The tradition describes the solution as the number of moles per liter. A mole of salt weighs 59gram, so recounting 100gram salt in moles we get 100gram = (100/59)*59gram = (100/59)*1mole = 1.69mole, that dissolved in 2.5liter has a strength as 1.69moles per 2.5liters or 1.69/2.5 moles/liters, or 0.676moles/liter.

The Simplicity of Mathematics

Meeting Many, we ask 'How many in total?' To answer, we count and add. To count means to use division, multiplication and subtraction to predict unit-numbers as blocks of stacked bundles, but also to recount to change unit, and to double-count to get per-numbers bridging the units, both rooting proportionality.

Adding thus means uniting unit-numbers and per-numbers, but both can be constant or variable, so to predict, we need four uniting operations: addition and multiplication unite variable and constant unit-numbers; and integration and power unite variable and constant per-numbers. As well as four splitting operations: subtraction and division split into variable and constant unit-numbers; and differentiation and root/logarithm split into variable and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite. And it appears in Arabic numbers written out fully as T = 456 = 4 bundles-of-bundles & 5 bundles & 6 unbundled, showing all four uniting operations, addition and multiplication and power and next-to addition of stacks; and showing that the word-language and the number-language share the same sentence form with a subject and a verb and a predicate or object.

Shapes can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

So, in principle, mathematics is simple and easy and quick to learn if institutionalized education wants to do so; however, to preserve and expand itself, the institution might want instead to hide the simplicity of mathematics by leaving out the subject and the verb in the number-language sentences; and by avoid counting to hide the block-nature of numbers as stacked bundles in order to impose linear place-value numbers instead; and by reversing the natural order of operations by letting addition precede subtraction, preceding multiplication, preceding division; and by hiding the double nature of addition by silencing next-to addition; and by silencing per-numbers and present fractions as numbers instead of operators needing numbers to become numbers; and by adding fractions without units to hide the true nature of integration as adding per-numbers by their areas; and by postponing trigonometry to after ordinary geometry and coordinate geometry; and by forcing equations to be solved by obeying the commutative and associative laws of abstract algebra; and by hiding that a function is but another name for a number-language sentence; and by forcing differential calculus to precede integral calculus.

Discussion: How does Traditional MatheMatics differ from ManyMatics

But in the end, how different is traditional mathematics from ManyMatics? As their base they have Set and Many, but isn't that just two different words for the same? Not entirely. Many exists in the world, it is physical, whereas Set exists in a description, it is meta-physical. Thus, traditional mathematics defines its concepts top-down as examples, whereas ManyMatics defines its concepts bottom-up as abstractions. Still, the concepts might be the same, at least when taught? But a comparison uncovers several differences between the Set-derived tradition and its alternative grounded in Many.

The tradition sees the goal of mathematics education as teaching numbers and shapes and operations. In numbers, digits are symbols like letters, ordered according to a place value system, seldom renaming '234' to '2tentens 3tens 4'. There are four kinds of numbers: natural and integers and rational and real. The natural numbers are defined by a successor principle making them one dimensional placed along a number line given the name 'cardinality'. The integers are defined as equivalence classes in a set of ordered number-pairs where (a,b) is equivalent to (c,d) if a+d = b+c. Likewise, the rational numbers are defined by (a,b) being equivalent to (c,d) if a*d = b*c. Finally, the real numbers are defined as limits of number sequences.

The alternative sees the goal of mathematics education as teaching a number-language describing the physical fact Many by full sentences with the total as the subject, e.g. T = 2*3, thus having the same structure as the word-language, both having a language level describing the world, and a meta-language level describing the language. Digits are icons containing as many sticks as they represent if written less sloppy. Numbers occur when counting Many by bundling and stacking produces a block of bundles and unbundled, using bundle- or decimal-writing to separate the inside bundles from the outside unbundled. The bundle-number, typically ten, does not need an icon since it is counted as '1 bundle'. Thus, a natural number is a decimal number with a unit, illustrated geometrically as a row of blocks containing the unbundled, the bundles, the bundle of bundles etc. Counting includes recounting in the same unit to create overload or underload, as well as recounting in another unit, especially in and from tens. Double-counting in different units gives per-numbers and fractions; however, these are not numbers but operators needing a number to become a number. A diagonal divides a block into two like right-angled triangles where the base and the altitude can be recounted in diagonals or in each other. Real numbers as √2 are calculations with as many decimals as needed, since a single can always be seen as a bundle of parts.

The tradition sees operations in a number set as mappings from a set-product into the set. Addition is the basic operation allowing number sets to be structured with an associative and a commutative and a distributive law as well as a neutral element and inverse elements. Addition is defined as repeating the successor principle, and multiplication is defined as repeated addition. Subtraction and division is defined as adding or multiplying inverse numbers. Standard algorithms for operations are introduced using carrying. Electronical calculators are not allowed when learning the four basic operations. The full ten-by-ten multiplication tables must be learned by heart.

The alternative sees operations as icons describing the counting process. Here division is an uphill stroke showing a broom wiping away the bundles; multiplication is a cross showing a lift stacking the bundles into a block, to be dragged away to look for unbundled singles, shown by a horizontal track called subtraction. Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other. To add on-top, the blocks must be recounted in the same unit, thus grounding proportionality. Next-to addition means adding areas, thus grounding integration. Reversed adding on-top or next-to grounds equations and differentiation. A calculator is used to predict the result by two formulas, a recount-formula T = (T/B)*B, and a restack-formula T = (T-B)+B. A multiplication table shows recounting from icons to tens, and is used when recounting from tens to icons introduces equations as reversed calculations. When recounting a total to or from tens, increasing the base means decreasing the altitude, and vice versa. As to multiplication, the commutative law says that the total stays unchanged when turning over a 3 by 4 block to a 4 by 3 block. The associative law says that the total stays unchanged when including or excluding a factor from the unit, T = 2*(3*4) = (2*3)*4. The distributive law says that before adding, recounting must provide a common unit to bracket out, T = 2 3s + 4 5s = 1.1 5s + 4 5s = (1.1 + 4) 5s.

The tradition sees fractions as rational numbers to which the four basic operations can be applied. Thus, fractions can be added without units by finding a common denominator after splitting the numerator and the denominator into prime factors. Fractions are introduced after division, and is followed by ratios and percentages and decimal numbers seen as examples of fractions.

The alternative sees fractions as per-numbers coming from double-counting in the same unit. As per-numbers, fractions are operators needing a number to become a number, thus added by areas, also called integration. Double-counting is introduced before addition. With factors as units, splitting a number in prime factors just means finding all possible units.

After working with number sets, the tradition introduces working with letter sets and polynomial sets to which the four basic operations can be applied once more observing that only like terms can be added, but not mentioning that this is because it means the unit is the same. The alternative sees letters as units to bracket out during addition or subtraction, and that when multiplied or divided gives a composite unit.

The tradition sees an equation as an open statement expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*x = 8	an open statement
(2*x)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(x*2)*(1/2) = 4	since multiplication is commutative
x*(2*(1/2)) = 4	since multiplication is associative
x*1 = 4	by definition of an inverse element
x = 4	by definition of a neutral element

As to the equation 2 + 3*x = 14, the same procedure as above is carried out twice, first with addition then with multiplication.

The alternative sees an equation as another name for a reversed calculation, to be reversed once more by recounting. Thus in the equation ‘2*x = 8’, recounting some 2s in 1s resulted in 8 1s, which recounted back into 2s gives 2*x = 8 = (8/2)*2, showing that x = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the opposite side & sign method.

The equation 2 + 3*x = 14, can be seen in two ways. As reversing a next-to addition of the two blocks, thus solved by differentiation, first removing the initial block and then recounting the rest in 3s: x = (14-2)/3 = 4. Or as a walk that multiplying by 3 and then adding by 2 gives 14,

x *(*3→)* 3*x *(+2→)* 3*x+2 = 14.

Reversing the walk by subtracting 2 and dividing by 3 gives the initial number:

x = 4 = (14-2)/3 *(←/3)* 14-2 *(←-2)* 14

The answer is tested by once more walking forward, 3*4 + 2 = 12 + 2 = 14.

The tradition sees a quadratic equation x^2 + b*x + c = 0 as a pure algebraic problem to be solved, first by factorizing, then by completing the square, and finally by using the solution formula.

The alternative sees solving a quadratic equation as a problem combining algebra and geometry, where a square with the sides x+b/2 creates fives areas, x^2 and b/2*x twice and c and (b^2/4-c) where the first four disappear and leaves (x+b/2)^2 to be the latter, b^2/4-c.

The tradition sees a function as an example of a relation between two sets where first-component identity implies second-component identity. And it gives the name 'linear function' to f(x) = a*x+b even if this is an affine function not satisfying the linear condition f(x+y) = f(x)*f(y), as does the proportionality formula f(x) = a*x.

The alternative sees a function as a name for a formula containing two unspecified numbers or variables, typically x and y. Thus, a function is a fiction showing how the y-numbers depends on the x numbers as shown in a table or by a graph.

The tradition sees proportionality as an example of a function satisfying the linear condition. The alternative sees proportionality as a name for double-counting in different units creating per-numbers.

The tradition sees geometry to be introduced in the order: plane geometry, coordinate geometry and trigonometry.

The alternative has the opposite order. Trigonometry comes first grounded in the fact that halving a block by its diagonal allows the base and the altitude to be recounted in diagonals or in each other. This also allows a calculator to find pi from a sine formula. Next comes coordinate geometry allowing geometry and algebra to always go hand in hand so that algebraic formula can predict intersection points coming from geometrical constructions.

The tradition has quadratic functions following linear functions, both examples of polynomials.

The alternative sees affine functions as one example of constant change coming in five forms: constant y-change per x-change, constant y-percent-change per x-change, constant y-percent-change per x-percent-change, constant y-change per x-change together with constant y-percent-change per x-change, and finally constantly changing y- change.

The tradition sees logarithm as defined as the integral of the function y = 1/x.

The alternative sees logarithm and root combined both solving power equations. Thus a^x = b gives x = loga(b); and x^a = b gives x = a√b. This shows the logarithm as a factor-counter and the root as a factor-finder.

The tradition sees differential calculus as preceding integral calculus, and the gradient y' = dy/dx is defined algebraically as the limit of Δy/Δx for Δx approaching 0, and geometrically as the slope of a tangent being the limit position of a secant with approaching intersection points. The limit is defined by an epsilon-delta criterium.

The alternative sees calculus as grounded in adding blocks next-to each other. In primary school calculus occurs when performing next-to addition of 2 3s and 4 5s as 8s. In middle school calculus occurs when adding piecewise constant per-numbers, as 2m at 3m/s plus 4m at 5m/s. In high school calculus occurs when adding locally constant per-numbers, as 5seconds at 3m/s changing constantly to 4m/s. Geometrically, adding blocks means adding areas under a per-number graph. In the case of local constancy this means adding many strips, made easy by writing them as differences since many differences add up to one single difference between the terminal and initial numbers, thus showing the relevance of differential calculus. The epsilon-delta criterium is a straight forward way to formalize the three ways of constancy, globally and piecewise and locally, by saying that constancy means an arbitrarily small difference.

Conclusion

With 50 years of research, mathematics education should have improved significantly. Its lack of success as illustrated by OECD report 'Improving Schools in Sweden' made this paper ask: Applying sociological imagination when meeting Many without having predicates forced upon it by traditional mathematics, can we design a STEM-based core math curriculum aimed at making migrants pre-teachers and pre-engineers in two years?

This depends on what we mean by mathematics. And, looking back, mathematics has meant different things through its long history, from a common label for knowledge to today's 'meta-matism' combining 'meta-matics' defining concepts by meaningless self-reference, and 'mathe-matism' adding numbers without units thus lacking outside validity. So, inspired by Heidegger's 'always question sentences, except for its subject' we returned to the original Greek meaning of mathematics: Knowledge about Many by itself and in time and space.

Observing Many by itself allows rebuilding mathematics as a 'many-matics', i.e. as a natural science about the physical fact Many, where counting by bundling leads to block-numbers that recounted in other units leads to proportionality and solving equations; where recounting sides in triangles leads to trigonometry; where double-counting in different units leads to per-numbers and fractions, both adding by their areas, i.e. by integration; where counting precedes addition taking place both on-top and next-to involving proportionality and calculus; where using a calculator to predict the counting result leads to the opposite order of operations: division before multiplication before subtraction before next-to and on-top addition; and where calculus occurs in primary school as next-to addition, and in middle and high school as adding piecewise and locally constant per-numbers; and where integral calculus precedes differential calculus.

With water cycles fueled by the sun and run by gravity as exemplary situations, STEM offers various examples of Many in space and time since science and technology and engineering basically is about double-counting physical phenomena in different units.

The designed STEM-based core math curriculum has been tested in parts with success at the educational level in Danish pre-university classes. It might also be tested on a research level if it becomes known through publishing, i.e., if it will be accepted at the review process. It will offer a sociological imagination absent from traditional research seen by many teachers as useless because of its many references.

Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality used to collect reliable data and to test the validity of its findings by falsification attempts.

In a Swedish context, obsessive self-referencing has been called the 'irrelevance of the research industry' (Tarp, 2015, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics education and research are two examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead thus marginalizing or forgetting its original outside goal.

So, if a society as Sweden really wants to improve mathematics education, extra funding might just produce more researchers more eager to follow inside traditions than solving outside problems. Instead funding should force the universities to arrange curriculum architect compositions to allow alternatives to compete as to creativity and effectiveness, thus allowing the universities to rediscover their original outside rational goals and to change its routines accordingly.

A situation described in several fairy tales; the Sleeping Beauty hidden behind the thorns of routines becoming rituals until awakened by the kiss of an alternative; and Cinderella making the prince dance, but only found when searching outside the established nobility.

References

Bauman, Z. (1990). *Thinking Sociologically*. Oxford, UK: Blackwell.

Bauman, Z. (1992). *Intimations of Postmodernity.* London, UK: Routledge.

Bauman, Z. (2014). *What Use is Sociology.* Cambridge, UK: Polity.

Bourdieu, P. (1970). *Reproduction in Education, Society and Culture*, London: Sage.

Chomksky, N. & Foucault, M. (2006). *The Chomsky-Foucault Debate on Human Nature.* New York: The New Press.

Chomsky-Foucault debate on Human Nature, www.youtube.com/watch?v=3wfNl2L0Gf8.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht-Holland: D. Reidel Publishing Company.

Glaser, B. G. & Strauss, A. L. (1967). *The Discovery of Grounded Theory.* New York: Aldine de Gruyter

Han, S., Capraro, R. & Capraro MM. (2014). How science, technology, engineering, and mathematics (STEM) project-based learning (PBL) affects high, middle, and low achievers differently: The impact of student factors on achievement. *International Journal of Science and Mathematics Education. 13* (5), 1089-1113.

Hargreaves, D.H. (1996). *Teaching as a Research-based Profession: Possibilities and Prospects*. Cambridge: Teacher Training Agency Lecture.

Heidegger, M. (1962). *Being and Time.* Oxford, UK: Blackwell.

Lyotard, J. (1984). *The postmodern Condition: A report on Knowledge.* Manchester, UK: Manchester University Press.

Mills, C. W. (1959). *The Sociological Imagination.* UK: Oxford University Press

Negt, O. (2016). *Soziologische Phantasie und exemplarisches Lernen: Zur Theorie und Praxis der Arbeiterbildung.* Germany: Steidl.

OECD. (2015a). *Improving Schools in Sweden: An OECD Perspective.* Retrieved from: www.oecd.org/edu/school/improving-schools-in-sweden-an-oecd-perspective.htm.

OECD. (2015b). *OECD Forum 2015.* Retrieved from http://www.oecd.org/forum/oecdyearbook/we-must-teach-tomorrow-skills-today.htm.

Piaget, J. (1969). *Science of Education of the Psychology of the Child.* New York: Viking Compass

Tarp, A. (2015). *The MADIF Papers 2000-2016. Ten papers written for the biannual MADIF conference arranged by the Swedish Mathematics Education Research Seminar.* Retrieved from: www.mathecademy.net/papers/madif-papers/.

Tarp, A. (2017). *Math Ed & Research 2017. Retrieved from http*://mathecademy.net/2017-math-articles/.

ADDITION-FREE MIGRANT-MATH ROOTED IN STEM RE-COUNTING FORMULAS

A curriculum architect is asked to avoid traditional mistakes when designing a curriculum for young migrants that will allow them to quickly become STEM pre-teachers and pre-engineers. Typical multiplication formulas expressing re-counting in different units suggest an addition-free curriculum. To answer the question 'How many in total?' we count and re-count totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a re-count formula as a core formula in all STEM subjects.

Keywords: STEM, migrant, elementary school mathematics, curriculum, PISA.

Decreased PISA performance despite increased research

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise has funding, see e.g. Swedish Centre for Mathematics Education. Yet, despite extra research and funding, decreasing Swedish PISA result caused OECD to write the report "Improving Schools in Sweden" (2015a) describing its school system as "in need of urgent change" since "more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life." (p. 3).

To find an unorthodox solution we pretend that a university in southern Sweden, challenged by numerous young male migrants, arranges a curriculum architect competition: "Theorize the low success of 50 years of mathematics education research; and derive from this theory a STEM based core curriculum allowing young migrants to return as STEM pre-teachers and pre-engineers."

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to help migrants and how to improve schools in Sweden and elsewhere.

Social theory looking at mathematics education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16).

As to institutions, of which mathematics education is an example, he talks about rational action "in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)". He then points out that "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement (p. 84)."

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since such a goal statement is meaningless by its self-reference. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'. Here the invention of the concept Set created a Set-based 'meta-matics', self-referential defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. But, then Russell looked at the set of sets not belonging to itself. Here a set belongs only if it does not: if $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. Thus pointing out that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false.

In this way, Set changed grounded classical mathe-matics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

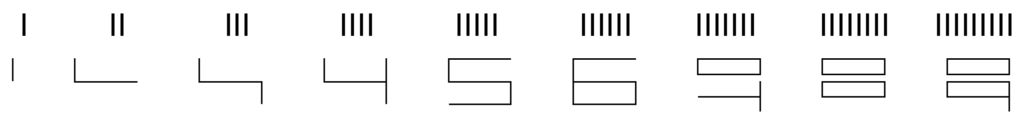
Difference research looks at mathematics education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' (Tarp, 2017) is searching for hidden differences making a difference. An additional inspiration comes from existentialist philosophy described by Sartre (2007, p. 20) as holding that "Existences precedes essence". So, to avoid a goal displacement in math education, difference-research asks: How will math look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a mathematics core curriculum based upon examples of Many in a STEM context (Lawrenz et al, 2017). So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many creates a 'count-before-adding' curriculum

Meeting Many, we ask "How many in Total?" To answer, we total by counting to create number-language sentences as e.g. T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence; and connecting the outside total T with its inside predicate 2 3s (Tarp, 2018b). Rearranging many 1s into one symbol with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:



1 2 3 4 5 6 7 8 9

Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, bundles of 2s and 2 of them. This inspires ‘bundle-counting’, re-counting a total in icon-bundles to be stacked as bundle- or block-numbers, which can be re-counted and double-counted before being processed by on-top and next-to addition, direct or reversed. Thus, a total T of 5 1s is re-counted in 2s as T = 2 2s & 1; described by ‘bundle-writing’, T = 2B1 2s; or by ‘decimal-writing’, T = 2.1 2s, where, with a bundle-cup, a decimal point separates the bundles inside from the outside unbundled singles; or by ‘deficit-writing’, T = 3B-1 2s = 3.-1 2s = 3 bundles less 1 2s.

So, to count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator thus predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: entering ‘5/2’ on a calculator gives ‘2.some’, and ‘5 – 2x2’ gives ‘1’, so T = 5 = 2B1 2s. The unbundled can be placed next-to or on-top the stack thus rooting decimals and fractions.

The re-count formula occurs all over. With proportionality: y = c*x; in trigonometry as sine, cosine and tangent: a = (a/c)*c = sinA*c and b = (b/c)*c = cosA*c and a = (a/b)*b = tanA*b; in coordinate geometry as line gradients: Δy = Δy/Δx = c* Δx; and in calculus as the derivative, dy = (dy/dx)*dx = y’*dx. In economics, the re-count formula is a price formula: $ = ($/kg)*kg, $ = ($/day)*day, etc.

Re-counting in the same unit or in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 2B1 2s as 1B3 2s with an outside ‘overload’; or as 3B-1 2s with an outside ‘underload’ thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48; or 336 = 35B-14, so 336/7 = 5B-2 = 48. Re-counting in a different unit means changing unit, also called proportionality. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2B2 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’. The answer, ‘2.some’, predicts that the singles come from taking away 2 5s, now asking ‘3*4 – 2*5’. The answer, ‘2’, predicts that 3 4s can be re-counted in 5s as 2B2 5s or 2.2 5s.

Re-counting to and from tens

Asking ‘3 4s = ? tens’ is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer directly as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right. Re-counting from tens to icons by asking ‘35 = ? 7s’ is called solving an equation x*7 = 35. It is easily solved by re-counting 35 in 7s: x*7 = 35 = (35/7)*7. So x = 35/7, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘T = 6$ = ?kg’, we re-count 6 in the per-number 2s: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

A short curriculum in addition-free mathematics

01. To stress the importance of bundling, the counting sequence can be: 01, 02, …, 09, 10, 11 etc.; or 01, 02, 03, 04, 05, Ten less 4, T-3, T-2, T-1, Ten, Ten and 1, T and 2, etc.

02. Ten fingers can be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers can be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables can be formulated as re-counting from icon-bundles to tens and use underload counting after 5: T = 4*7 = 4 7s = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 can be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

06. Solving proportional equations as 3*x = 12 can be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the ‘opposite side & sign’ method.

07. Proportional tasks can be done by re-counting in the per-number: With 3$/4kg, 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and 18$ = (18/3)*3$ = (18/3)*4kg = 24kg.

08. Fractions and percentages are per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$, where 60$ = (60/3)*3$ then gives (60/3)*2$ = 40$.

09. Trigonometry can precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sinA*c, and a = (a/b)*b = tanA*b.

10. Counting by stacking bundles into adjacent blocks leads to the number formula or bundle formula called a polynomial: T = 456 = 4*BundleBundle + 5*Bundle + 6*single = 4*B^2 + 5*B + 6*1. In its general form, the number formula T = a*x^2 + b*x + c contains the different formulas for constant change: T = a*x (proportionality), T = a*x^2 (acceleration), T = a*x^c (elasticity) and T = a*c^x (interest rate); as well as T = a*x+b (linearity, or affinity, strictly).

11. Predictable change roots pre-calculus (if constant) and calculus (if changing). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

12. Integral calculus can precede differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers. So, both per-numbers and fractions must be multiplied by the units before being added as the area under the per-number graph.

Meeting Many in a STEM context

OECD (2015b) says: ‘In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.’ STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature’s behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots will help transforming nature into human necessities.

Nature as heavy things in motion

To meet, we must specify space and time in a nature consisting of heavy things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature's three main factors, matter and force and motion, like the three social factors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water into electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to dissolving matter in water; to the trajectory of a ball pulled down by gravity; to put steam and electrons to work in a power plant creating an electrical circuit transporting energy from a source to many consumers.

In nature, we count heaviness in kilograms, space in meters and time in seconds. Heavy things in motion have a momentum = mass*velocity, a multiplication formula as most STEM formulas expressing re-counting by per-numbers: kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter, where force is the per-number change in momentum per second. Thus, STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Warming and boiling water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number 410/70/1.4 Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is 316/0.14 kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving material in water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so re-counting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

Building batteries with water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H2O, and carbon dioxide CO2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH4, the energy comes from using 4 Os to burn it.

Technology and engineering: letting steam and electrons produce and distribute energy

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units. Thus a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, p*V = n*R*T, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g. steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to industries and homes.

An electrical circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per charge-unit. Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere. To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120 Watt bulb, because of proportionality the latter needs twice the current, and consequently half the resistance of the former.

How high up and how far out

An inclined gun sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total sped with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

Compared with the horizontal, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

Adding addition to the curriculum

Once counted as block-numbers, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being re-counted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations where the question 2 3s + ? 4s = 5 7s leads to differentiation: ? = (5*7 – 2*3)/4 = ΔT/4. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular.

The number formula T = 456 = 4*B^2 + 5*B + 6*1 shows there are four ways to unite numbers: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers (Tarp, 2018b).

Conclusion and recommendation

This paper argues that 50 years of unsuccessful mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views lead to different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying "To master Many, counting and re-counting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating." A proposal for two separate twin-curricula in counting and adding is found in Tarp (2018a). Thus, the simplicity of mathematics as expressed in a 'count-before-adding' curriculum allows replacing line-numbers with block-numbers; and allows learning core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM pre-teachers or pre-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017). Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

References

Bauman, Z. (1990). *Thinking sociologically.* Oxford, UK: Blackwell.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht-Holland: D. Reidel Publishing Company.

Glaser, B. & Strauss, A. (1967). *The discovery of grounded theory.* New York: Aldine de Gruyter.

Lawrenz, F., Gravemeijer, K., & Stephan, M. (2017). Introduction to this Special Issue. *International Journal of Science and Mathematics Education, Vol. 15, Issue 1 Supplement,* pp. 1-4.

Mills, C. (1959). *The sociological imagination.* Oxford, UK: Oxford University Press.

OECD. (2015a). *Improving schools in Sweden: An OECD Perspective.* Retrieved from: www.oecd.org/edu/school/improving-schools-in-sweden-an-oecd-perspective.htm.

OECD. (2015b). *OECD Forum 2015.* Retrieved from www.oecd.org/forum/oecdyearbook/ we-must-teach-tomorrow-skills-today.htm.

Piaget, J. (1969). *Science of education of the psychology of the child.* New York: Viking Compass.

Russell B. (1945). *A history of western philosophy.* New York: A Touchstone Book.

Sartre, J.P. (2007). *Existentialism is a humanism.* CT. Yale University Press.

Tarp, A. (2017). *Math ed & research 2017.* Retrieved from //mathecademy.net/2017-math-articles/.

Tarp, A. (2018a). *Math ed & research 2018.* Retrieved from //mathecademy.net/2018-math-articles/.

Tarp, A. (2018b). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, March 2018, Vol. 11, No. 1,* pp. 103-117. http://www.educationforatoz.net/images/Allan_Tarp-JME-2018-1.pdf.

MASTERING MANY BY COUNTING, RE-COUNTING AND DOUBLE-COUNTING BEFORE ADDING ON-TOP AND NEXT-TO

Observing the quantitative competence children bring to school, and by using difference-research searching for differences making a difference, we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

Keywords: numbers, operations, proportionality, calculus, early childhood

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed e.g. by the International Congress on Mathematics Education taking place each four year since 1969. However, despite 50 years of research, many countries still experience poor results in the Programme for International Student Assessment (PISA). In the former model country Sweden this caused the Organisation for Economic Co-operation and Development (OECD) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change' since 'more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.' (OECD, 2015, p. 3).

Mathematics thus seems to be hard by nature. But, with mathematics and education as social institutions, a different answer, by choice, may come from sociology, having imagination as a core part as pointed out by Mills (1959). Bauman (1990) agrees when talking about organizations:

> Sociological thinking is, one may say, a power in its own right, an *anti-fixating* power. It renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. (p.16) **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (p.79) Last but not least, the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right: the new end against which the organization tends to measure the rationality of its performance (p.84).

It is a general opinion that the goal of mathematics education is to learn mathematics. However, this goal is self-referring. So maybe traditional mathematics has a goal displacement hiding a different more fruitful way to the outside goal, to master Many as it occurs in space and time?

Difference-research

To find differences we use ‘Difference-research’ (Tarp, 2018a) searching for differences making a difference, thus containing two parts: finding a difference, and testing it to see if it makes a difference. This paper focuses on the first part in order to find differences that can be tested to create a background for a possible paradigm shift (Kuhn, 1959).

Difference-research builds on sociological imagination; and on the skeptical thinking of the ancient Greek sophists warning against choice presented as nature. Thus disagreeing with Plato seeing choice as an illusion since the physical is but examples of meta-physical forms visible only to philosophers educated at his academy, later by Christianity turned into monasteries before being changed back again by the Reformation. In the Renaissance, this created the skeptical thinking of natural science, which rooted the Enlightenment century with its two republics, the American and the French (Russell, 1945).

Where France now has its fifth republic, the USA still has its first with skepticism as pragmatism and symbolic interactionism and grounded theory. To protect its republic, France has developed a skepticism inspired by the German thinker Heidegger, seen by Bauman as starting ‘the second Copernican revolution’ by asking: What is ‘is’? (Bauman, 1992, p. ix).

Heidegger (1962) sees three of our seven basic is-statements as describing the core of Being: ‘I am’ and ‘it is’ and ‘they are’; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the ‘I’ must create an authentic relationship to the ‘It’. However, this is made difficult by the ‘dictatorship’ of the ‘They’, shutting the ‘It’ up in a predicate-prison of idle talk, gossip.

Heidegger thus uses existentialist thinking, described by Sartre (Marino, 2004) as holding that ‘existence precedes essence’ (p. 22). In France, Heidegger inspired the poststructuralist thinking pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Foucault, 1995; Lyotard, 1984; Tarp, 2016).

Difference-research tells what can be different from what cannot. From a Heidegger view, an is-sentence contains two things: a subject that exists and cannot be different, and a predicate that can and that may be gossip masked as essence, provoking ‘the banality of Evil’ (Arendt, 1963) if institutionalized. So, to discover its true nature, we need to meet the subject, Many, outside the predicate-prison of traditional mathematics. We will use Grounded Theory (Glaser and Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1970) from a personal to a social level, to allow Many create its own categories and properties. In this way, we can see if our observations can be assimilated to traditional mathematics or will suggest it be accommodated.

Our Two Languages with Word- and Number-Sentences

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: ‘This is a chair’. As does the number-language assigning numbers instead: ‘the 3 chairs each have 4 legs’, abbreviated to ‘the total is 3 fours’, or ‘$T = 3$ 4s’ or ‘$T = 3*4$’. Unfortunately, the tradition hides the similarity between word- and number-sentences by leaving out the subject and the verb by just saying '3*4 = 12'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is an auxiliary verb'. Likewise, the sentence '$T = 3*4$' leads to a meta-sentence ''*' is a commutative operation'.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead its self-referring set-based form has turned mathematics into a grammar labeling its outside roots as 'applications', used as means to dim the impeding consequences of teaching a grammar before its language.

So, using full sentences including the subject and the verb in number-language sentences is a difference to the tradition; as is teaching language before grammar.

Mathematics, Rooted in Many, or in Itself

The Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

However, 50 years ago the set-concept created a self-referring 'New Math' or 'meta-matics' with concepts defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. And neglecting that Russell, by looking at the set of sets not belonging to itself, showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

So, to find a difference we now return to the Greek origin to meet Many openly to uncover a 'Many-matics' as a natural science about Many.

Meeting Many, Children use Block-numbers to Count and Share

How to master Many can be observed from preschool children. Asked 'How old next time?', a 3year-old child will say 'Four' and show 4 fingers; but will react strongly if held together 2 by 2, 'That is not 4, that is 2 2s.'

Children thus describes what exists in the world: bundles of 2s, and 2 of them. So, what children bring to school is 2-dimensional block-numbers, illustrated geometrically by LEGO blocks, together with some quantitative competence. Children thus love re-counting 5 sticks in 2s in various ways as 1 2s & 3, as 2 2s & 1, and as 3 2s less 1.

Sharing nine cakes, four children take one by turn saying 'I take 1 of each 4'. With 1 left they might say 'let's count it as 4'. Thus, children share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts.

Children quickly observe the difference between a ‘stack-number’ as 6 = 3 2s or 2 3s, and a prime number as 3, serving only as a bundle-number by always leaving singles if stacked.

Finally, by turning and splitting 2-dimensional or 3-dimensional blocks, children see their commutative, distributive and associative properties as self-evident: of course, 2 3s is the same as 3 2s; and 6 3s can be split in 4 3s and 2 3s; and 2 3*4s is the same as 2*3 4s.

Meeting Many Openly

Many exists in space and time as multiplicity and repetition. Meeting Many we ask: ‘how many in total?’ To answer, we count and add. We count by bundling and stacking as seen when writing out fully the total *T* = 456 = 4**B*^2 + 5**B* + 6*1 showing three stacks or blocks added next-to each other: one with 4 bundles of bundles, one with 5 bundles, and one with 6 unbundled singles. Typically, we use ten as the bundle-size, formally called a base.

Digits occur by uniting e.g. five ones to one fives, rearranged as an icon with five strokes if written less sloppy. As the bundle-size, ten needs no icon when counted as 10, one bundle and no unbundled. Then follow eleven and twelve coming from Danish Vikings counting ‘one left’ and ‘two left’.

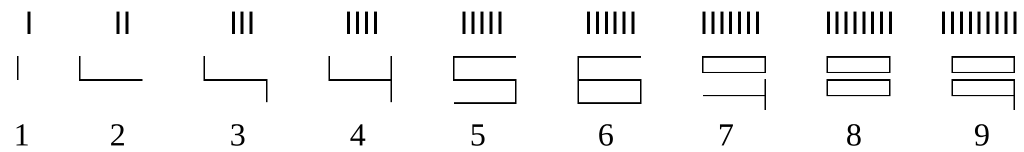


Figure 1. *Digits as icons with as many sticks as they represent.*

Counting by Bundling

We count in several ways. Some gather-hunter cultures count ‘one, two, many’. Agriculture needs to differentiate degrees of Many and typically bundles in tens. To include the bundle, we can count ‘0Bundle1, 0*B*2, 0*B*3,…, 1*B*, 1*B*1, 1*B*2’, etc.; or ‘0.1 tens, 0.2 tens’, etc., using a decimal point to separate the bundles from the unbundled singles. To signal nearness to the bundle we can count ‘1, 2, …, 7, bundle less 2, bundle less 1, bundle’, etc. Thus a number always contains three numbers: a number of bundles, a number of singles, and a number for the bundle-size.

Bundle-counting, we ask e.g. ‘A total of 7 is how many 3s?’ Using blocks, we stack the 3-bundles on-top of each other. The single can be placed next-to, or on-top counted in 3s. Thus, the result of counting 7 in 3s, *T* = 2 3s & 1, can be written as *T* = 2*B*1 3s using ‘bundle-writing’, and as *T* = 2.1 3s using ‘decimal-writing’, and as *T* = 2 1/3 3s using ‘fraction-writing’.

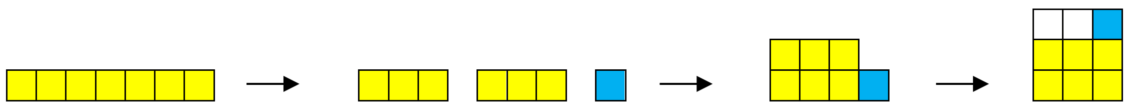


Figure 2. *Seven bundle-counted as 2Bundle1 3s, as 2.1 3s, and as 2 1/3 3s.*

Bundle-counting in Space and Time

We include space and time by using ‘geometry-counting’ in space, and ‘algebra-counting’ in time. Counting in space, we stack the bundles and report the result on an abacus in ‘geometry-mode’. Here the total 7 is on the below bar with 1 unbundled and a block with 2 bundles on the bars above.

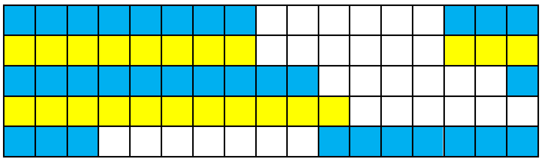


Figure 3. *Seven bundle-counted as 2B1 3s on an abacus in geometry-mode.*

Counting in time, we count the bundles and report the result on an abacus in 'algebra-mode'. Here the total 7 is on the below bar with 1 unbundled and the number of bundles on the bars above.

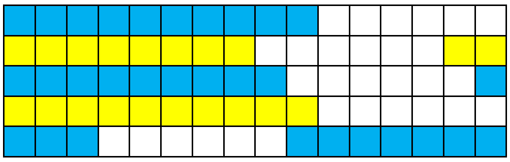


Figure 4. *Seven bundle-counted as 2.1 3s on an abacus in algebra-mode.*

A Calculator Predicts Counting-results

Iconizing the counting processes also, a calculator can predict a counting-result. A stack of 2 3s is iconized as 2x3 (or 2*3) showing a lift used 2 times to stack the 3s. As for taking away, subtraction shows the trace left when taking away just once, and division shows the broom wiping away several times.

So, entering '7/3' we ask the calculator 'from 7, 3s can be taken away how many times?' The answer is '2. some'. To find the leftover singles we take away the stack of 2 3s by asking '7 – 2*3'. From the answer '1' we conclude that 7 = 2*B*1 3s. Showing '7 – 2*3 = 1', a display indirectly predicts that 7 can be re-counted as 2 3s and 1, or as 2*B*1 3s or 2.1 3s.

7 / 3	2.some
7 – 2 * 3	1

Figure 5. *A calculator predicts how 7 re-counts in 3s as 2.1 3s.*

A calculator thus uses a 're-count formula', *T* = (*T*/*B*)**B*, saying that 'from *T*, *T/B* times, *B*s can be taken away'; and a 're-stack formula', *T* = (*T*–*B*)+*B*, saying that 'from *T*, *T*–*B* is left, if *B* is taken away and placed next-to'. The formulas may be illustrated by LEGO blocks. The re-count formula introduces early algebra (Kieran, Pang, Schifter and Ng, 2016) from grade one; and it occurs all over mathematics and science as proportionality formulas. Likewise, the early use of a calculator shows the importance of mathematics as a language for prediction.

Cup-Counting Allows Re-Counting in the Same Unit

Cup-counting uses a cup when bundle-counting e.g. 7 in 3s. For each bundle we place a stick inside the cup, leaving the unbundled singles outside.

T = 7 = I I I I I I I → ~~III~~ ~~III~~ I → [▌▌] I → 2B1 3s = 2.1 3s

One stick moves outside the cup as a bundle of 1s, that moves back inside as 1 bundle. This will change the 'normal' form to an 'overload', or to an 'underload' leading to negative numbers that may be used freely in childhood even if adults abstain from doing so:

T = 7 = I I I I I I I → ~~III~~ I I I I → [▌] I I I I → 1*B*4 3s = 1.4 3s

T = 7 = I I I I I I I → ~~III~~ ~~III~~ ~~III~~ ~~II~~ → [▌▌▌] ~~II~~ → 3*B*-2 3s = 3.-2 3s

Re-Counting in a Different Unit

Re-counting in a different unit means changing units, also called proportionality. Re-counting 3 4s in 5s, the re-count formula and a calculator predict the result 2 5s & 2 by entering ‘3*4/5’ and taking away the 2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Figure 6. *A calculator predicts how 3 4s re-counts in 5s as 2.2 5s.*

Re-Counting from Icons to Tens

A calculator has no ten-button. Instead, to re-count an icon-number as 3 4s in tens, it gives the result 1.2 tens directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

3 * 4	12

Figure 7. *A calculator predicts how 3 4s re-counts in tens as 1.2 tens.*

Re-counting from icons to tens, 3 4s is a geometrical block that increases its base. Therefore, it must decrease its height to keep the total unchanged.

Re-counting in tens is called multiplication tables to be learned by heart. However, the ten-by-ten table can be reduced to a 4-by-4 table since 5 is half of ten and 6 is ten less 4, and 7 is ten less 3 etc. Thus *T* = 4*7 = 4 7s that re-counts in bundles of tens as

T = 4*7 = 4*1*B*-3 tens = 4*B*-12 tens = 3*B*-2 tens = 2*B*8 tens = 28

Such results generalize to algebraic formulas as *a*(b – c) = a*b – a*c*.

Re-Counting from Tens to Icons

Re-counting from tens to icons will decrease the base and increase the height. The question ‘38 is ? 7s’ is called an equation ’38 = *u**7’, using the letter *u* for the unknown number. An equation is easily solved by recounting 38 in 7s, thus providing a natural ‘to opposite side with opposite sign’ method as a difference to the traditional ‘do the same to both sides’ method.

*u**7 = 38 = (38/7)*7	so	*u* = 38/7 = 5 3/7

Figure 8. *An equation solved by re-counting, the OppositeSide&Sign method.*

Once Counted, Totals Can be Added On-Top or Next-To

To add on-top by asking ‘3 5s and 2 3s total how many 5s?’, the units must be the same. So, 2 3s must be re-counted in 5s as 1*B*1 5s that added to the 3 5s gives 4*B*1 5s.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is ‘4. Some’. Taking away 4 5s leaves 1. So again, we get the result 4*B*1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Figure 9. *A calculator predicts how 3 5s and 2 3s re-counts in 5s as 4.1 5s.*

To add next-to by asking '3 5s and 2 3s total how many 8s?', we add by areas, called integral calculus. With blocks we get the answer 2*B*5 8s. Using a calculator to predict the result, we use a bracket before counting in 8s: Asking '(3*5 + 2*3)/8', the answer is '2. Some'. Taking away 2 8s leaves 5. So again, we get the result 2*B*5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Figure 10. *A calculator predicts how 3 5s and 2 3s re-counts in 8s as 2.5 8s.*

Reversing Adding On-Top and Next-To

Reversed addition may be called backward calculation or solving equations. Reversing next-to addition may be called reversed integration or differentiation. Asking '3 5s and how many 3s total 2*B*6 8s?', using blocks gives the answer 2*B*1 3s. Using a calculator to predict the result, the remaining is bracketed before counting in 3s.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Figure 11. *A calculator predicts how 2.6 8s re-counts in 3 5s and 2.1 3s.*

Adding or integrating two areas next-to each other means multiplying before adding. Reversed integration, i.e. differentiation, then means subtracting before dividing, as shown by the gradient formula $y' = \Delta y/t = (y2 - y1)/t$.

Double-Counting in Two Units Creates Per-Numbers and Proportionality

Double-counting the same total in two units is called proportionality, which produces 'per-numbers' as e.g. 2$ per 5kg, or 2$/5kg, or 2/5 $/kg.

To answer the question '*T* = 6$ = ?kg' we use the per-number to re-count 6 in 2s, that many times we have 5kg: *T* = 6$ = (6/2)*2$ = (6/2)*5kg = 3*5kg = 15kg. And vice versa: Asking '*T* = 20kg = ?$', the answer is *T* = 20kg = (20/5)*5kg = (20/5)*2$ = 4*2$ = 8$.

A total can be double-counted in colored blocks of different values, e.g. 1 red per 3 blues. Here, a total of 10 blues re-counts as *T* = 7*b* & 1*r* = 4*b* & 2*r* = 1*b* & 3*r*. Likewise, a total of 3 reds re-counts as *T* = 3*b* & 2*r* = 6*b* & 1*r* = 9*b*. Placed next to each other, this introduces a primitive coordinate system.

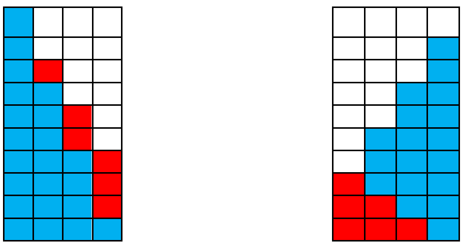


Figure 12. *10 blues left, and 3 reds right, re-counted in combinations.*

Double-Counting in the Same Unit Creates Fractions as Per-Numbers

Double-counting a total in the same unit, per-numbers take the form of fractions, e.g. as 3$ per 5$ = 3/5; or percentages as 3$ per 100$ = 3/100 = 3%.

Thus, to find 3$ per 5$ of 20$, or 3/5 of 20, we re-count 20 in 5s as 20 = (20/5)*5 = 4*5. Now we have two options. Seeing 20 as 4 5s, 4 times we get 3, i.e. 4*3 = 12; and seeing 20 as 5 4s, we get 3 4s, i.e. 3*4 = 12.

Likewise, to find what 3$ per 5$ is in percent, i.e. per 100, we re-count 100 in 5s as 100 = (100/5)*5 = 20*5. Again, we have two options. Seeing 100 as 20 5s, 20 times we get 3, i.e. 20*3 = 60; and seeing 100 as 5 20s, we get 3 20s, i.e. 3*20 = 60. So, 3 per 5 gives 60 per 100 or 60%.

Including or removing units will enlarge or reduce fractions:

4/6 = 4 3s/6 3s = 4*3/6*3 = 12/18

4/6 = 2*2/3*2 = 2 2s/3 2s = 2/3

Adding Per-numbers Roots Integral Calculus before Differential Calculus

Adding 2kg at 3$/kg and 4kg at 5$/kg, the 'unit-numbers' 2 and 4 add directly, but the per-numbers 3 and 5 must be multiplied first, thus creating areas. So per-numbers and fractions are not numbers, but operators needing numbers to become numbers. Per-numbers thus add by the areas under the per-number graph, here being 'piecewise constant'.

Asking '3 seconds at 4m/s increasing steadily to 5m/s', the per-number is 'locally constant'. This concept is formalized by an '*epsilon-delta* criterion' seeing three forms of constancy: *y* is 'globally constant' *c* if, for any positive number *epsilon*, the difference between *y* and *c* is less than *epsilon.* And *y* is 'piecewise constant' *c* if an interval-width *delta* exists such that, for any positive number *epsilon*, the difference between *y* and *c* is less than *epsilon* in this interval. Interchanging *epsilon* and *delta* makes *y* 'locally constant' or continuous. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in the latter case written as $dy/dx = y'$.

With locally constant per-numbers, the area under the graph splits up into countless strips that add easily if written as differences since the middle terms then will disappear, leaving just the difference between the end- and start-values. Thus, adding areas precedes and motivates differential calculus.

Using Letters and Functions for Unspecified Numbers and Calculations

At the language level we can set up a calculation with an unspecified number *u*, e.g. *T* = 2 + ? = 2 + *u*. Also, at the meta-language level we can set up an unspecified formula with an unspecified number *u*, written as *T* = *f(u).*

With one unspecified number, a formula becomes an equation as 8 = 2**u*; with two, a formula becomes a function as T = 2**u*; and with three, a formula becomes a surface as T = 2**u* + 2**w*.

Although we can write it, *T* = *f*(2) is meaningless since 2 is not an unspecified number. When specified, a function can be linear or exponential, but it cannot be a number or increase. A total can increase, but the way it does so cannot. Mixing language and meta-language creates meaningless sentences as 'the predicate ate the apple'.

A general number-formula as e.g. $T = a*x^2 + b*x + c$ is called a polynomial. It shows the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a*n}$ $T/n = a$
Per-numbers m/s, $/kg, $/100$ = %	$\boldsymbol{T = \int a*dn}$ $dT/dn = a$	$\boldsymbol{T = a^n}$ $\sqrt[n]{T} = a \quad \log_a T = n$

Figure 13. *The 'algebra-square' shows the four ways to unite or split numbers.*

The number-formula contains the formulas for constant change:

$T = b*x$ (proportional)

$T = b*x + c$ (linear)

$T = a * x^n$ (elastic)

$T = a * n^x$ (exponential)

$T = a*x^2 + b*x + c$ (accelerated)

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to 'post-dict' what we cannot 'pre-dict'.

Combining linear and exponential change by n times depositing $a$$ to an interest rate r%, we get a saving $A$$ predicted by a simple formula, $A/a = R/r$, where the total interest rate R is predicted by the formula $1+R = (1+r)^n$.

The formula and the proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, $r*a/r = a$, and its own interest amount, thus containing a saving A that is the total interest amount $R*a/r$, which gives $A/a = R/r$.

Trigonometry before Geometry

The tradition introduces plane geometry before coordinate geometry and trigonometry. A difference is the opposite order with trigonometry first since halving a block by its diagonal allows the base and the height to be re-counted in the diagonal or in each other to create the per-numbers sine, cosine, tangent and gradient:

height = (height/base)*base = tangent*base = gradient*base.

This allows a calculator to find *pi* from a formula: $\pi = n$*tan(180/n) for n sufficiently large; and it allows to predict an angle A form its base b and height a by reversing the formula tan $A = a/b$.

Integrating plane and coordinate geometry allows geometry and algebra to always go hand in hand. In this way solving algebraic equations predicts intersection points in geometrical constructions, and vice versa.

Testing a Many-matics Micro-curriculum

A '1 cup and 5 sticks' micro-curriculum can be designed to help a class stuck in division. The intervention begins by bundle-counting 5 sticks in 2s, using the cup for the bundles. The results, 1*B*3 2s and 2*B*1 2s and 3*B*-1 2s, show that a total can be counted as an inside number of bundles, and an outside number of singles; and written in three ways: overload and normal and underload.

So, to divide 336 by 7, we move 5 bundles outside as 50 singles to re-count 336 with an overload: 336 = 33*B*6 = 28*B*56, which divided by 7 gives 4*B*8 = 48. With multiplication, singles move inside as bundles: 7* 4*B*8 = 28*B*56 = 33*B*6 = 336. 'Is it that easy?' is a typical reaction.

Algebra before Arithmetic may now be Possible

Introducing algebra before arithmetic was central to the New Math idea and to the work of Davidov (Schmittau, 2004). Introducing algebra as generalized arithmetic, the book 'Early Algebra' describes how 'a fourth-grade USA class is investigating what happens to the product of a multiplication expression when one factor is increased by a certain amount.' (Kieran et al, 2016, p.17). The investigation begins with an example showing that 7*3 = 21, and 7*5 = 35, and 9*3 = 27.

In a first-grade class working with block-numbers with the bundle as the unit, the answer would be: 7*3 is 7 3s, and 7*5 is 7 5s, and 9*3 is 9 3s. So 7*5 means that 7 2s is added next-to 7 3s. Re-counted in tens this will increase the 2*B*1 tens with 1*B*4 tens to 3*B*5 tens. Likewise, 9*3 means that 2 3s is added on-top of 7 3s. Re-counted in tens this will increase the 2*B*1 tens with 0*B*6 tens to 2*B*7 tens.

Adding 2 to both numbers means adding additional 2 2s. Re-counted in tens this will increase the 2*B*1 tens with 1*B*4 tens and 0*B*6 tens and additional 0*B*4 tens to 4*B*5 tens.

Counting 7 as 9 less 2, and 3 as 5 less 2, will decrease the 9 5s with 2 5s and 2 9s. Only now we must add the 2 2s that was removed twice, so (9-2)*(5-2) = 9*5 - 9*2 - 2*5 + 2*2 as shown on a western ten by ten abacus as a 9 by 5 block. This roots the algebraic formula $(a - b)*(c - d) = a*c - a*d - b*c + b*d$.

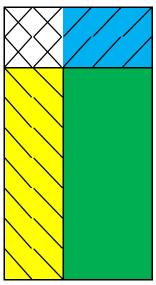


Figure 14. *An abacus shows that 7*3 = (9-2)*(5-2) = 9*5 - 9*2 - 2*5 + 2*2.*

Later follows a discussion on solving equations (pp. 25-29). In a first-grade class working with block-numbers with the bundle as the unit, solving the equation $3x + 9 = 5x + 1$, the geometrical answer would be: to the left we have a block of 3*B*9 *x*s, and to the right we have a block of 5*B*1 *x*s. Removing 3 bundles and 1 single from both, we get $8 = 2x$. Re-counting 8 in 2s we get $2*x = 8 = (8/2)*2$, so $x = 8/2 = 4$.

The algebraic answer would be similar: to the left we have 3 bundles inside and 9 singles outside the bundle-cup, and to the right we have 5 bundles inside and 1 single outside. Removing 3 bundles from the inside and 1 single from the outside, we get $8 = 2x$. Re-counting 8 in 2s we get $2*x = 8 = (8/2)*2$, so $x = 8/2 = 4$.

Using block-numbers instead of line-numbers thus allows introducing algebra before arithmetic since with the re-count formula, counting and re-counting and double-counting precede addition.

Conclusion and Recommendation

Among the many research articles on counting and arithmetic, only few deal with block-numbers (Zybartas and Tarp, 2005). Dienes (2002), the inventor of Multi-base blocks, has similar ideas when saying (p. 1):

> The position of the written digits in a written number tells us whether they are counting singles or tens or hundreds or higher powers. (..) My contention has been, that in order to fully understand how the system works, we have to understand the concept of power. (..) In school, when young children learn how to write numbers, they use the base ten exclusively and they only use the exponents zero and one (namely denoting units and tens), since for some time they do not go beyond two digit numbers. So neither the base nor the exponent are varied, and it is a small wonder that children have trouble in understanding the place value convention.

Instead of talking about bases and higher powers, working with icon-bundles and bundles of bundles will avoid that 'neither the base nor the exponent are varied'. By seeing bundles as existence and bases as essence, block-numbers differ from Dienes' multi-base blocks that seem to have set-based mathematics as the goal, and blocks as a means.

Set, however, changed mathematics from a bottom-up Greek 'Many-matics' into today's self-referring top-down 'meta-matism', a mixture of 'meta-matics' with concepts defined top-down instead of bottom-up, and 'mathe-matism' with statements true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

So, mathematics is not hard by nature but by choice. And yes, a different way exists to its outside goal, mastery of Many. Still, it teaches line-numbers as essence to be added without units and without being first bundle-counted and re-counted and double-counted. By neglecting the existence of block-numbers and re-counting, it misses the golden learning opportunities from introducing formulas, proportionality, calculus and equations in early childhood education through its grounded alternative, Many-matics.

Consequently, let us welcome 'good' 2-dimensional block-numbers and drop 'bad' 1-dimensional line-numbers and 'evil' fractions (Tarp, 2018b). Let us bundle-count and re-count and double-count before adding on-top and next-to. Let us use full sentences about how to count and (re)unite totals. And, let difference-research use sociological imagination to design a diversity of micro-curricula (Tarp, 2017) to test if Many-matics makes a difference by fulfilling the 'Mathematics for All' dream.

Let existence precede essence in mathematics education also. So, think things.

References

Arendt, H. (1963). *Eichmann in Jerusalem, a report on the banality of evil.* London: Penguin Books.

Bauman, Z. (1990). *Thinking sociologically*. Oxford, UK: Blackwell.

Bauman, Z. (1992). *Intimations of postmodernity.* London, UK: Routledge.

Dienes, Z. (2002). *What is a base?* Retrieved from www.zoltandienes.com/academic-articles.

Foucault, M. (1995). *Discipline & punish.* New York: Vintage Books.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht-Holland: D. Reidel Publishing Company.

Glaser B. G. & Strauss A. L. (1967). *The discovery of grounded theory.* New York: Aldine de Gruyter.

Heidegger, M. (1962). *Being and time.* Oxford, UK: Blackwell.

Kieran, C., Pang J. S., Schifter, D. & Ng, S. F. (2016). *Early algebra: research into its nature, its learning, its teaching (ICME 13 topical surveys)*. Hamburg: Springer Open.

Kuhn T.S. (1962). *The structure of scientific revolutions.* Chicago: University of Chicago Press.

Lyotard, J. (1984). *The postmodern condition: a report on knowledge.* Manchester, UK: Manchester University Press.

Marino, G. (2004). *Basic writings of existentialism.* New York: Modern Library.

Mills, C. W. (1959). *The sociological imagination.* UK: Oxford University Press.

Piaget, J. (1970). *Science of education of the psychology of the child.* New York: Viking Compass.

Russell B. (1945). *A history of western philosophy.* New York: A Touchstone Book.

Schmittau, J. & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of V.V. Davydov. *The Mathematics Educator, 8* (1), 60-87.

Tarp, A. (2016). From essence to existence in mathematics education. *Philosophy of Mathematics Education Journal, 31.*

Tarp, A. (2017). *Math Ed & Research 2017.* Retrieved 4.4.2018 from http://mathecademy.net/2017-math-articles/.

Tarp, A. (2018a). A Heidegger view on how to improve mathematics education. *Philosophy of Mathematics Education Journal, 33.*

Tarp, A. (2018b). Good, bad & evil mathematics - tales of totals, numbers & fractions. *Flexibility in Mathematics Education: Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education, Taipei, Taiwan, May 7-11, 2018.* Forthcoming.

The Organisation for Economic Co-operation and Development (OECD). (2015). *Improving schools in Sweden: an OECD perspective.* Retrieved from http:// www.oecd.org/edu/school/ improving-schools-in-sweden-an-oecd-perspective.htm.

Zybartas, S. & Tarp, A. (2005). One-digit mathematics. *Pedagogika (78/2005).* Vilnius, Lithuania.

A TWIN CURRICULUM SINCE CONTEMPORARY MATHEMATICS MAY BLOCK THE ROAD TO ITS EDUCATIONAL GOAL, MASTERY OF MANY

Mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory, we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum.

POOR PISA PERFORMANCE DESPITE FIFTY YEARS OF RESEARCH

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in math education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 years since 1969. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to significantly below the OECD average in 2012, causing OECD (2015) to write the report ‘Improving Schools in Sweden’. Likewise, math dislike seems to be widespread in high performing countries also. With mathematics and education as social institutions, grand theory may explain this ‘irrelevance paradox’, the apparent negative correlation between research and performance.

GRAND THEORY

Ancient Greece saw two forms of knowledge, ‘sophy’. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Christianity eagerly took over metaphysical patronage and changed the academies into monasteries. The sophist skepticism was revived by Brahe and Newton, insisting that knowledge about nature comes from laboratory observations, not from library books (Russell, 1945).

Newton’s discovery of a non-metaphysical changing will spurred the Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. Two republics arose, in the United States and in France. The US still has its first Republic, France its fifth, since its German-speaking neighbors tried to overthrow the French Republic again and again.

In North America, the sophist warning against hidden patronization lives on in American pragmatism and symbolic interactionism; and in Grounded Theory, the method of natural research resonating with Piaget’s principles of natural learning. In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructural thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnosed cures, and education; all may hide patronizing choices presented as nature (Lyotard, 1984).

Within philosophy itself, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that ‘existence precedes essence’, exemplified by the Heidegger-warning: In a sentence, trust the subject, it exists; doubt the predicate, it is essence coming from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber was the first to theorize the increasing goal-oriented rationalization that de-enchant the world and create an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking “renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now” (p. 16). But he also formulates a warning (p. 84): “The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right”. Which may lead to ‘the banality of evil’ (Arendt, 1963).

As to what we say about the world, Foucault (1995) focuses on discourses about humans that, if labeled scientific, establish a ‘truth regime’. In the first part of his work, he shows how a discourse disciplines itself by only accepting comments to already accepted comments. In the second part he shows how a discourse disciplines also its subject by locking humans up in a predicate prison of abnormalities from which they can only escape by accepting the diagnose and cure offered by the ‘pastoral power’ of the truth regime. Foucault thus sees a school as a ‘pris-pital’ mixing the power techniques of a prison and a hospital: the ‘pati-mates’ must return to their cell daily and accept the diagnose ‘un-educated’ to be cured by, of course, education as defined by the ruling truth regime.

Mathematics, stable until the arrival of SET

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught ‘reckoning’ in primary school and ‘arithmetic’ and ‘geometry’ in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here a wish for exactness and unity created a SET-derived ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox ‘this sentence is false’, being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

SET thus transformed classical grounded ‘many-matics’ into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside a classroom where adding numbers without units as ‘1 + 2 IS 3’ meets counter-examples as e.g. 1week + 2days is 9days.

Proportionality illustrates the variety of mastery of Many and of quantitative competence

Proportionality is rooted in questions as “2kg costs 5$, what does 7kg cost; and what does 12$ buy?”

Europe used the ‘Regula de Tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, the divide. So first we ask, Q1: ‘2kg cost 5$, 7kg cost ?$’ to get to the answer (7*5/2)$ = 17.5$. Then we ask, Q2: ‘5$ buys 2kg, 12$ buys ?kg’ to get to the answer (12*2)/5$ = 4.8kg.

Then, two new methods appeared, ‘find the unit’, and cross multiplication in an equation expressing like proportions or ratios:

Q1: 1kg costs 5/2$, so 7kg cost 7*(5/2) = 17.5$. Q2: 1$ buys 2/5kg, so 12$ buys 12*(2/5) = 4.8kg.
Q1: 2/5 = 7/x, so 2*x = 7*5, x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, so 5*x = 12*2, x = (12*2)/5 = 4.8.

SET chose modeling with linear functions to show the relevance of abstract algebra’s group theory: Let us define a linear function $f(x) = c*x$ from the set of kg-numbers to the set of $-numbers, having as domain $DM = \{x\in R \mid x>0\}$. Knowing that $f(2) = 5$, we set up the equation $f(2) = c*2 = 5$ to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: $c*2 = 5$, $(c*2)*½ = 5*½$, $c*(2*½) = 5/2$, $c*1 = 5/2$, $c = 5/2$. With $f(x) = 5/2*x$, the inverse function is f-1(x) = 2/5*x. So with 7kg, $f(7) = 5/2*7 = 17.5$$; an with 12$, f-1(12) = 2/5*12 = 4.8kg.

In the future, we simply ‘re-count’ in the ‘per-number’ 2kg/5$ coming from ‘double-counting’ the total T. Q1: T = 7kg = (7/2)*2kg = (7/2)*5$ = 17.5$; Q2: T = 12$ = (12/5)*5$ = (12/5)*2kg = 4.8kg.

Grand theory looks at mathematics education

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologically, how we perceive and verbalize it. University mathematics holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB+4*B+5*1, resonating with what children bring to school, e.g. T = 2 5s.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? University mathematics holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian psychology.

Sociologically, a Weberian viewpoint would ask if SET is a rationalization of Many gone too far leaving Many de-enchanted and the learners in an iron cage. A Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, contemporary university mathematics has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word ‘mathematics’ must go from goal descriptions. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a pris-pital, a Foucaultian viewpoint, would ask, first which structure to choose, European line-organization forcing a return to the same cell after each hour, day and month for several years; or the North American block-organization changing cell each hour, and changing the daily schedule twice a year? Next, as prisoners of a ‘the goal of math education is to learn math’ discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g. by examining and developing the existing mastery children bring to school?

Meeting Many, children bundle in block-numbers to count and share

How to master Many can be learned from preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, ‘That is not four, that is two twos’, thus describing what exists, and with units: bundles of 2s, and 2 of them.

Children also use block-numbers when talking about Lego bricks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3 extra’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, placing them next-to each other, they typically say ‘2 7s and 3 extra’.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 4s &3, etc. And children don’t mind writing a total of 7 using ‘bundle-writing’ as *T* = 7 = 1*B*5 = 2*B*3 = 3*B*1 = 4*B*~~1~~; or even as 1*BB*3 or 1*BB*1*B*1. Also, children love to count in 3s, 4s, and in hands.

Sharing 9 cakes, 4 children take one by turn saying they take 1 of each 4. Taking away 4s roots division as counting in 4s; and with 1 left they often say “let’s count it as 4”. Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. And they smile when seeing that entering ‘9/4’ allows a calculator to predict the sharing result as 2 1/4; and when seeing that entering ‘2*5/3’ will predict the result of sharing 2 5s between 3 children.

Children thus master sharing, taking parts and splitting into parts before division and counting- and splitting-fractions is taught; which they may like to learn before being forced to add without units.

So why not develop instead of rejecting the core mastery of Many that children bring to school?

A typical contemporary mathematics curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting ‘(ten and) 1 left’, ‘(ten and) 2 left’. Never children are asked to use full number-language sentences, *T* = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1*B*14 tens is not allowed. Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

A QUESTION GUIDED COUNTING CURRICULUM

The question guided re-enchantment curriculum in counting could be named 'Mastering Many by counting, recounting and double-counting'. The design is inspired by Tarp (2018). It accepts that while eight competencies might be needed to learn university mathematics (Niss, 2003), only two are needed to master Many (Tarp, 2002), counting and uniting, motivating a twin curriculum. The corresponding pre-service or in-service teacher education can be found at the MATHeCADEMY.net. Remedial curricula for classes stuck in contemporary mathematics can be found in Tarp (2017).

Q01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

Q02, counting sequences: "How to count fingers?" Here the learning opportunity is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H". Follow-up activities could be counting things.

Q03, icon-counting: "How to count fingers by bundling?" Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported in a number-language sentence with subject, verb and predicate: *T* = 5 = 1Bundle3 2s = 2*B*1 2s = 3*B*-1 2s = 1*BB*1 2s, called an 'inside bundle-number' describing the 'outside block-number'. A western abacus shows this in 'outside geometry space-mode' with the 2 2s on the second and third bar and 1 on the first bar; or in 'inside algebra time-mode' with 2 on the second bar and 1 on the first bar. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, calculator-prediction: "How can a calculator predict a counting result?" Here the learning opportunity is to see the division sign as an icon for a broom wiping away bundles: 5/2 means 'from 5, wipe away bundles of 2s'. The calculator says '2.some', thus predicting it can be done 2 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes the trace left when dragging away the block to look for unbundled singles. By showing '5-2*2 = 1' the calculator indirectly predicts that a total of 5 can be recounted as 2B1 2s. An additional learning opportunity is to write and use the 'recount-formula' *T* = (*T*/*B*)***B* saying "From *T*, *T*/*B* times *B* can be taken away." This proportionality formula occurs all over mathematics and science. Follow-up activities could be counting cents: 7 2s is how many fives and tens? 8 5s is how many tens?

Q05, unbundled as decimals, fractions or negative numbers: “Where to put the unbundled singles?” Here the learning opportunity is to see that with blocks, the unbundled occur in three ways. Next-to the block as a block of its own, written as *T* = 7 = 2.1 3s, where a decimal point separates the bundles from the singles. Or on-top as a part of the bundle, written as *T* = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q06, prime or foldable units: “Which blocks can be folded?” Here the learning opportunity is to examine the stability of a block. The block *T* = 2 4s = 2*4 has 4 as the unit. Turning over gives *T* = 4*2, now with 2 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a ‘prime unit’ and 4 a ‘foldable unit’, 4 = 2 2s. So, a block of 3 2s cannot be folded, whereas a block of 3 4s can: *T* = 3 4s = 3 * (2*2) = (3*2) * 2. A number is called even if it can be written with 2 as the unit, else odd.

Q07, finding units: “What are possible units in *T* = 12?”. Here the learning opportunity is that units come from factorizing in prime units, *T* = 12 = 2*2*3. Follow-up activities could be other examples.

Q08, recounting in another unit: “How to change a unit?” Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. *T* = 3 4s = ? 5s, the recount-formula will say *T* = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

Q09, recounting from tens to icons: “How to change unit from tens to icons?” Here the learning opportunity is that asking ‘*T* = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter *u* for the unknown number, *u**8 = 24. This is easily solved by recounting 24 in 8s as 24 = (24/8)*8 so that the unknown number is *u* = 24/8 attained by moving 8 to the opposite side with the opposite sign. Follow-up activities could be other examples of recounting from tens to icons.

Q10, recounting from icons to tens: “How to change unit from icons to tens?” Here the learning opportunity is that if asking ‘T = 3 7s = ? tens’, the recount-formula cannot be used since the calculator has no ten-button. However, it is programmed to give the answer directly by using multiplication alone: *T* = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: *T* = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Follow-up activities could be other examples of recounting from icons to tens.

Q11, double-counting in two units: “How to double-count in two different units?” Here the learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To answer questions we just recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa, asking ‘?$ = 12kg’, the answer is T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Follow-up activities could be numerous other examples of double-counting in two different units since per-numbers and proportionality are core concepts.

Q12, double-counting in the same unit: "How to double-count in the same unit?" Here the learning opportunity is that when double-counted in the same unit, per-numbers take the form of fractions, 3$ per 5$ = 3/5; or percentages, 3 per hundred = 3/100 = 3%. Thus, to find a fraction or a percentage of a total, again we just recount in the per-number. Also, we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Follow-up activities could be other examples of double-counting in the same unit since fractions and percentages are core concepts.

Q13, recounting the sides in a block. "How to recount the sides of a block halved by its diagonal?" Here, in a block with base b, height a, and diagonal c, mutual recounting creates the trigonometric per-numbers: $a = (a/c)*c = \sin A*c$; $b = (b/c)*c = \cos A*c$; $a = (a/b)*b = \tan A*b$. Thus, rotating a line can be described by a per-number a/b, or as tan$A$ per 1, allowing angles to be found from per-numbers. Follow-up activities could be other blocks e.g. from a folding ruler.

Q14, double-counting in STEM (Science, Technology, Engineering, Math) multiplication formulas with per-numbers coming from double-counting. Examples: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter) * square-meter = pressure * square-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg) * kg = heat * kg; gram = (gram/mole) * mole = molar mass * mole; Δ momentum = (Δ momentum/sec) * sec = force * sec; Δ energy = (Δ energy/ meter) * meter = force * meter = work; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp; dollar = (dollar/hour)*hour = wage*hour.

Q15, navigating. "Avoid the rocks on a squared paper". Four rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock? Predict and measure the angles on the journey.

A QUESTION GUIDED UNITING CURRICULUM

The question guided re-enchantment curriculum in uniting could be named 'Mastering Many by uniting and splitting constant and changing unit-numbers and per-numbers'.

A general bundle-formula $T = a*x^2 + b*x + c$ is called a polynomial. It shows the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Q21, next-to addition: "With $T1$ = 2 3s and $T2$ = 4 5s, what is $T1+T2$ when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus. Follow-up activities could be other examples of next-to addition.

Q22, reversed next-to addition: “If $T1 = 2$ 3s and $T2$ add next-to as $T = 4$ 7s, what is $T2$?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Follow-up activities could be other examples of reversed next-to addition.

Q23, on-top addition: “With $T1 = 2$ 3s and $T2 = 4$ 5s, what is $T1+T2$ when added on-top as 3s; and as 5s?” Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Follow-up activities could be other examples of on-top addition.

Q24, reversed on-top addition: “If $T1 = 2$ 3s and $T2$ as some 5s add to $T = 4$ 5s, what is $T2$?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again is called differential calculus. An underload is removed by recounting. Follow-up activities could be other examples of reversed on-top addition.

Q25, adding tens: “With $T1 = 23$ and $T2 = 48$, what is $T1+T2$ when added as tens?” Again, recounting removes an overload: $T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71$; or $T = 236 + 487 = 2BB3B6 + 4BB8B7 = 6BB11B13 = 6BB12B3 = 7BB2B3 = 723$.

Q26, subtracting tens: “If $T1 = 23$ and $T2$ add to $T = 71$, what is $T2$?” Again, recounting removes an underload: $T2 = 71 – 23 = 7B1 – 2B3 = 5B\text{-}2 = 4B8 = 48$; or $T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB\text{-}3B\text{-}1 = 4BB7B\text{-}1 = 4BB6B9 = 469$. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, showing that -(-1) = +1.

Q27, multiplying tens: “What is 7 43s recounted in tens?” Here the learning opportunity is that also multiplication may create overloads: $T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301$; or $27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161$, solved geometrically in a 2x2 block.

Q28, dividing tens: “What is 348 recounted in 6s?” Here the learning opportunity is that recounting a total with overload often eases division: $T = 348 /6 = 3BB4B8 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58$.

Q29, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform into unit-number by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining constancy by epsilon and delta.

Q30, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Q31, finding common units: “Only add with like units, so how to add T = 4ab^2 + 6abc?”. Here units come from factorizing:

$T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b) + 3*c*(2*a*b) = 2b+3c\ 2ab\text{s}.$

CONCLUSION

A math education curriculum must make a choice. Shall it teach the ontology or the epistemology of Many? Shall it mediate the contemporary university discourse where the set-concept has transformed classical bottom-up ‘many-matics’ into a self-referring top-down ‘meta-matism’; or shall it develop the mastery of Many already possessed by children? Shall it teach about numbers or how to number? To allow choosing between a mediating and a developing curriculum, we need an alternative to the present curriculum, unsuccessfully trying to mediate contemporary university mathematics. So, Luther has a point arguing that reaching a goal is not always helped by institutional patronization. Grand theory thus has an answer to the ‘irrelevance paradox’ of mathematics education research: Accepting the child’s own epistemology will avoid a goal displacement where a litany of self-referring university mathematics blocks the road to its outside educational goal, mastery of Many.

References

Arendt, H. (1963). *Eichmann in Jerusalem, a report on the banality of evil.* London: Penguin Books.

Bauman, Z. (1990). *Thinking sociologically.* Oxford, UK: Blackwell.

Foucault, M. (1995). *Discipline & punish.* New York: Vintage Books.

Freudenthal, H. (1973). *Math as an educational task.* Dordrecht-Holland: D. Reidel Publishing Company.

Lyotard, J. (1984). *The postmodern condition: a report on knowledge.* Manchester: Manchester Univ. Press.

Marino, G. (2004). *Basic writings of existentialism.* New York: Modern Library.

Mills, C. W. (1959). The sociological imagination. UK: Oxford University Press.

Niss, M. (2003). *Mathematical competencies and the learning of mathematics: the Danish KOM project.* Retrieved from http://www.math.chalmers.se/Math/Grundutb/ CTH/mve375/1112/docs/KOMkompetenser.pdf.

OECD. (2015). *Improving schools in Sweden: an OECD perspective.* Retrieved from: www.oecd.org/edu/ school/improving-schools-in-sweden-an-oecd-perspective.htm.

Russell, B. (1945). *A history of western philosophy.* New York: A Touchstone Book.

Tarp, A. (2002, 2017). The ‘KomMod report’, a counter report to the ministry’s competence report. In Tarp, A. *Math ed & research 2017.* Retrieved from http://mathecademy.net/2017-math-articles/.

Tarp, A. (2017). *Math ed & research 2017.* Retrieved from http://mathecademy.net/2017-math-articles/.

Tarp, A. (2018). Mastering Many by counting, recounting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, March 2018, 11*(1), 103-117.

A NEW CURRICULUM - BUT FOR WHICH OF THE 3X2 KINDS OF MATHEMATICS EDUCATION

An essay on observations and reflections at the ICMI study 24 curriculum conference

As part of institutionalized education, mathematics needs a curriculum describing goals and means. There are however three kinds of mathematics: pre-, present and post-'setcentric' mathematics; and there are two kinds of education: multi-year lines and half-year blocks. Thus, there are six kinds of mathematics education to choose from before deciding on a specific curriculum; and if changing, shall the curriculum stay within the actual kind or change to a different kind? The absence of federal states from the conference suggests that curricula should change from national multi-year macro-curricula to local half-year micro-curricula; and maybe change to post-setcentric mathematics.

COHERENCE AND RELEVANCE IN THE SCHOOL MATHEMATICS CURRICULUM

The International Commission on Mathematical Instruction, ICMI, has named its 24th study "School mathematics Curriculum Reforms: Challenges, Changes and Opportunities". Its discussion document has 5 themes among which theme B, "Analysing school mathematics curriculum for coherence and relevance" says that "All mathematics curricula set out the goals expected to be achieved in learning through the teaching of mathematics; and embed particular values, which may be explicit or implicit."

So, to analyze we use the verb 'cohere' and the predicate 'relevant' when asking: "to what does this curriculum cohere and to what is it relevant?" As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.'

We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that 'to cohere' means 'to hold together firmly as parts of the same mass'. As to synonyms for cohere, it lists: 'accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.' And as to antonyms, it lists: 'differ (from), disagree (with).'

In the same dictionary, the word 'relevant' means 'having significant and demonstrable bearing on the matter at hand'. As to synonyms for relevant, it lists: 'applicable, apposite, apropos, germane, material, pertinent, pointed, relative.' And as to antonyms, it lists: 'extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.'

If we accept the verb 'apply' as having a meaning close to the predicate 'relevant', we can rephrase the above analysis question using verbs only: "to what does this curriculum cohere and apply?"

Seeing education metaphorically as bridging an individual start level for skills and knowledge to a common end level described by goals and values, we may now give a first definition of an ideal curriculum: “To apply to a learning process as relevant and useable, a curriculum coheres to the start and end levels for skills and knowledge.”

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?
- How to make the end level cohere to goals and values expressed by the society?
- How to make the end level cohere to goals and values expressed by the learners?
- How to make the bridge cohere to previous and following bridges?
- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

GOALS AND VALUES EXPRESSED BY THE SOCIETY

In her plenary address about the ‘OECD 2030 Learning Framework’, Taguma shared a vision:

> The members of the OECD Education 2030 Working Group are committed to helping every learner develop as a whole person, fulfil his or her potential and help shape a shared future built on the well-being of individuals, communities and the planet. (..) And in an era characterised by a new explosion of scientific knowledge and a growing array of complex societal problems, it is appropriate that curricula should continue to evolve, perhaps in radical ways (p. 10).

Talking about learner agency, Taguma said:

> Future-ready students need to exercise agency, in their own education and throughout life. (..) To help enable agency, educators must not only recognise learners’ individuality, (..) Two factors, in particular, help learners enable agency. The first is a personalised learning environment that supports and motivates each student to nurture his or her passions, make connections between different learning experiences and opportunities, and design their own learning projects and processes in collaboration with others. The second is building a solid foundation: literacy and numeracy remain crucial. (p. 11)

By emphasizing learner’s individual potentials, personalised learning environment and own learning projects and processes, Taguma seems to indicate that flexible half-year micro-curricula may cohere better with learners’ future needs than rigid multi-year macro-curricula. As to specifics, numeracy is mentioned as one of the two parts of a solid foundation helping learners enable agency.

DIFFERENT KINDS OF NUMERACY

Numeracy, however, is not that well defined. Oxford Dictionaries and Merriam-Webster agree on saying ‘ability to understand and work with numbers’; whereas the private organization National Numeracy (nationalnumeracy.org.uk) says ‘By numeracy we mean the ability to use mathematics in everyday life’.

The wish to show usage was also part of the Kilpatrick address, describing mathematics as bipolar:

> I want to stress that bipolarity because I think that’s an important quality of the school curriculum and every teacher and every country has to deal with: how much attention do we give to the purer side of mathematics. The New Math thought that it should be entire but that didn’t work really as well as people thought. So how much attention do we give to the pure part of mathematics and how much to the applications and how much do we engage together. Because it turns out if the applications are well-

chosen and can be understood by the children then that helps them move toward the purer parts of the field. (p. 20)

After discussing some problems caused by applications in the curriculum, Kilpatrick concludes:

> If we stick with pure mathematics, with no application, what students cannot see, "when will I ever use this?", it's not surprising that they don't go onto take more mathematics. So, I think for self-preservation, mathematicians and mathematics educators should work on the question of: how do we orchestrate the curriculum so that applications play a good role? There is even is even a problem with the word applications, because it implies first you do the mathematics, then you apply it. And actually, it can go the other way. (p. 22)

So, discussing what came first, the hen or the egg, applications or mathematics, makes it problematic to define numeracy as the ability to apply mathematics since it gives mathematics a primacy and a monopoly as a prerequisite for numeracy. At the plenary afterwards discussion, I suggested using the word 're-rooting' instead of 'applying' to indicate that from the beginning, mathematics was rooted in the outside world as shown by the original meanings of geometry and algebra: 'to measure earth' in Greek and 'to reunite' in Arabic.

MATHEMATICS THROUHG HISTORY

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy, seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra. And in Europe, Germanic countries taught 'reckoning' in primary school and 'arithmetic' and 'geometry' in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here a wish for exactness and unity created a 'setcentric' 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

Setcentrism thus changed classical grounded 'many-matics' into a self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meets counter-examples as e.g. 1week + 2days is 9days.

The introduction of the setcentric New Mathematics created different reactions. Inside the United States it was quickly abandoned with a 'back-to-basics' movement. Outside it was implemented at teacher education, and in schools where it gradually softened. However, it never retook its original form or name, despite, in contrast to 'mathematics', 'reckon' is an action-word better suited to the general aim of education, to teach humans to master the outside world through appropriate actions.

DIFFERENT KINDS OF MATHEMATICS

So, a curriculum must choose between a pre-, a present, and a post-setcentric mathematics as illustrated by an example from McCallum's plenary talk. After noting that "a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships" (p. 4), McCallum asked the audience: "What is the difference between 5/3 and 5÷3".

Pre-setcentric mathematics will say that 5/3 is a number on the number-line reached by taking 5 steps of the length coming from dividing the unit in 3 parts; and that 5÷3 means 5 items shared between 3.

Present setcentric mathematics will say that 5/3 is a rational number defined as an equivalence class in the product set of integers, created by the equivalence relation (*a*,*b*) eq. (*c*,*d*) if cross-multiplication holds, *a*x*d* = *bxc*; and, with 1/3 as the inverse element to 3 under multiplication, 5÷3 should be written as 5x1/3, i.e. the as the solution to the equation 3x*u* = 5, found by applying and thus legitimizing abstract algebra and group theory; thus finally saying goodbye to the Renaissance use of a vertical line to separate addends from subtrahends, and a horizontal line to separate multipliers from divisors.

Post-setcentric mathematics (Tarp, 2018) sees setcentric mathematics as meta-matism hiding the original Greek meaning og mathematics as a science about Many. In this 'Many-math', 5/3 is a per-number coming from double-counting in different units (5$/3kg), becoming a fraction with like units (5$/3$ = 5/3). Here per-numbers and fractions are not numbers but operators needing a number to become a number (5/3 of 3 is 5, 5/3 of 6 is 10); and 5÷3 means 5 counted in 3s occurring in the 'recount-formula' recounting a total *T* in bundles of 3s as *T* = (*T*/3)x3, saying 'from *T*, *T*/3 times, 3 can be taken away'. This gives flexible numbers: *T* = 5 = 1B2 3s = 1.2 3s = 1 2/3 3s = 2B-1 3s = 2.-1 3s, introduced in grade one where bundle-counting and re-counting in another unit precedes adding, and where recounting from tens to icons, *T* = 2.4 tens = ? 6s, leads to the equation *T* = *u*x6 = 24 = (24/6)x6 solved by recounting. In post-setcentric mathematics, per-numbers, fractions, ratios and proportionality melt together since double-counting in two units gives per-numbers as ratios, becoming fractions with like units. And here proportionality means changing units using the recount-formula to recount in the per-number: With 5$/3kg, "how much for 20$?" is found by re-counting 20 in 5s: *T* = 20$ = (20/5)x5$ = (20/5)x3kg = 12 kg. Likewise if asking "how much for 15 kg?"

DIFFERENT KINDS OF EDUCATION

As to education, from secondary school there is a choice between multi-year lines and half-year blocks. At the discussion after the Kilpatrick plenary session I made a comment about these two educational systems, which mas a lady from the United States say I was misinforming since in the states Calculus required a full year block. Together with other comments in the break, this made me realize that internationally there is little awareness of these two different kinds of educational systems. So here is another example of what the Greek sophists warned against, choice masked as nature.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

At the conference, the almost total absence of federal states as Germany, Canada, the United States and Russia seems to indicate that the problems reside with multi-year national curricula, becoming rigid traditions difficult to change. While federal competition or half-year blocks creates flexibility through an opportunity to try out different curricula.

Moreover, as a social institution involving individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlighten teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching one subject only in their own classrooms.

To protect its republic against its German speaking neighbors, France created elite schools, criticized today for exerting hidden patronization. Bourdieu thus calls education ‘symbolic violence’, and Foucault points out that a school is really a ‘pris-pital’ mixing power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings.

DIFFERENT KINDS OF COMPETENCES

As to competences, new to many curricula, there are at least three alternatives to choose among. The European Union recommends two basic competences, acquiring and applying, when saying that “Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge.”

At the conference two alternatives notions of competences were presented. In his plenary address, Niss recommended a matrix with 8 competences per concept (p. 73). In his paper, Tarp (pp. 317-324) acknowledged that 8 competences may be needed if the goal of mathematics education is to learn present setcentric university mathematics; but if the goal is to learn to master Many with post-setcentric mathematics, then only two competences are needed: counting and adding, rooting a twin curriculum teaching counting, recounting in different units and double-counting before adding.

MAKING THE LEARNING ROAD MORE PASSABLE

Once a curriculum is chosen, the next question is to make its bridge between the start and end levels for skills and knowledge more passable. Here didactics and pedagogy come in; didactics as the captain choosing the way from the start to the end, typically presented as a textbook leaving it to pedagogy, the lieutenants, to take the learners through the different stages.

The didactical choices must answer general questions from grand theory. Thus, philosophy will ask: shall the curriculum follow the existentialist recommendation, that existence precedes essence? And psychology will ask: shall the curriculum follow Vygotsky mediating institutionalized essence, or Piaget arranging learning meetings with what exists in the outside world? And sociology will ask: on which ethical grounds are children and teenagers retained to be cured by institutionalized education?

COLONIZING OR DECOLONIZING CURRICULA

The conference contained two plenary panels, the first with contributors from France, China, The Philippines and Denmark, almost all from the northern hemisphere; the second with contributors from Chile, Australia, Lebanon and South Africa, almost all from the southern hemisphere. Where the first panel talked more about solutions, the second panel talked more about problems.

In the first panel, France and Denmark represented some of the world's most centralized states with war-time educational systems dating back to the Napoleon era, which in France created elite-schools to protect the young republic from the Germans, and in Germany created the Humboldt Bildung schools to end the French occupation by mediating nationalism, and to sort out the population elite for jobs as civil servants in the new central administration; both just replacing the blood-nobility with a knowledge-nobility as noted by Bourdieu. The Bildung system latter spread to most of Europe.

Not surprisingly, both countries see university mathematics as the goal of mathematics education ('mathematics is what mathematicians do'), despite the obvious self-reference avoided by instead formulating the goal as e.g. learning numerical competence, mastery of Many or number-language. Seeing mathematics as the goal, makes mathematics education an example of a goal displacement (Bauman) where a monopoly transforms a means into a goal. A monopoly that makes setcentric mathematics an example of what Habermas and Derrida would call a 'center-periphery colonization', to be decentered and decolonized by deconstruction.

Artigue from France thus advocated an anthropological theory of the didactic, ATD, (p. 43-44), with a 'didactic transposition process' containing four parts: scholarly knowledge (institutions producing and using the knowledge), knowledge to be taught (educational system, 'noosphere'), taught knowledge (classroom), and learned available knowledge (community of study).

> The theory of didactic transposition developed in the early 1980s to overcome the limitation of the prevalent vision at the time, seeing in the development of taught knowledge a simple process of elementarization of scholarly knowledge (Chevallard 1985). Beyond the well-known succession offered by this theory, which goes from the reference knowledge to the knowledge actually taught in classrooms (..), ecological concepts such as those of niche, habitat and trophic chain (Artaud 1997) are also essential in it.

Niss from Denmark described the Danish 'KOM Project' leading to eight mathematical competencies per mathematical topic (pp. 71-72).

> The KOM Project took its point of departure in the need for creating and adopting a general conceptualisation of mathematics that goes across and beyond educational levels and institutions. (..) We therefore decided to base our work on an attempt to define and characterise mathematical competence in an overarching sense that would pertain to and make sense in any mathematical context. Focusing - as a consequence of this approach - first and foremost on the *enactment* of mathematics means attributing, at first, a secondary role to mathematical content. We then came up with the following definition of

> mathematical competence: Possessing *mathematical competence* – mastering mathematics – is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve actual or potential mathematical challenges of any kind. In order to identify and characterise the fundamental constituents in mathematical competence, we introduced the notion of mathematical competencies: A *mathematical competency* is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve a certain kind of mathematical challenge.

Some of the consequences by being colonized by setcentrism was described in the second panel.

In his paper 'School Mathematics Reform in South Africa: A Curriculum for All and by All?' Volmink from South Africa Volmink writes (pp. 106-107):

> At the same time the educational measurement industry both locally and internationally has, with its narrow focus, taken the attention away from the things that matter and has led to a traditional approach of raising the knowledge level. South Africa performs very poorly on the TIMSS study. In the 2015 study South Africa was ranked 38th out of 39 countries at Grade 9 level for mathematics and 47th out of 48 countries for Grade 5 level numeracy. Also in the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ), South Africa was placed 9th out of the 15 countries participating in Mathematics and Science – and these are countries which spend less on education and are not as wealthy as we are. South Africa has now developed its own Annual National Assessment (ANA) tests for Grades 3, 6 and 9. In the ANA of 2011 Grade 3 learners scored an average of 35% for literacy and 28% for numeracy while Grade 6 learners averaged 28% for literacy and 30% for numeracy.

After thanking for the opportunity to participate in a cooperative effort on the search of better education for boys, girls and young people around the world, Oteiza from Chile talked about 'The Gap Factor' creating social and economic differences. A slide with the distribution of raw scores at PSU mathematics by type of school roughly showed that out of 80 points, the median scores were 40 and 20 for private and public schools respectively. In his paper, Oteiza writes (pp. 81-83):

> Results, in national tests, show that students attending public schools, close to de 85% of school population, are not fulfilling those standards. How does mathematical school curriculum contribute to this gap? How might mathematical curriculum be a factor in the reduction of these differences? (..) There is tremendous and extremely valuable talent diversity. Can we justify the existence of only one curriculum and only one way to evaluate it through standardized tests? (..) There is a fundamental role played by researchers, and research and development centers and institutions. (..) How do the questions that originate in the classroom reach a research center or a graduate program? "*Publish or perish*" has led our researchers to publish in prestigious international journals, but, are the problems and local questions addressed by those publications?"

The Gap Factor is also addressed in a paper by Hoyos from Mexico (pp. 258-259):

> The PISA 2009 had 6 performance levels (from level 1 to level 6). In the global mathematics scale, level 6 is the highest and level 1 is the lowest. (..) It is to notice that, in PISA 2009, 21.8% of Mexican students do not reach level 1, and, in PISA 2015, the percentage of the same level is a little bit higher (25.6%). In other words, the percentage of Mexican students that in PISA 2009 are below level 2 (i.e., attaining the level 1 or zero) was 51%, and this percentage is 57% in PISA 2015, evidencing then an increment of Mexican students in the poor levels of performance. According to the INEE, students at levels 1 or cero are susceptible to experiment serious difficulties in using mathematics and benefiting from new educational opportunities throughout its life. Therefore, the challenges of an adequate educational attention to this population are huge, even more if it is also considered that approximately another fourth of the total Mexican population (33.3 million) are children under 15 years of age, a population in priority of attention".

As a comment to Volminks remark "Another reason for its lack of efficacy was the sense of scepticism and even distrust about the notion of People's Mathematics as a poor substitute for the "real mathematics"" (p. 104), and inspired by the sociological Centre–Periphery Model for colonizing, by post-colonial studies, and by Habermas' notion of rationalization and colonization of the lifeworld by the instrumental rationality of bureaucracies, I formulated the following question in the afterwards discussion: "As former colonies you might ask: Has colonizing stopped, or is it still taking place? Is there an outside central mathematics that is still colonizing the mind? What happens to what could be called local math, street math, ethno-math or the child's own math?"

CONCLUSION AND RECOMMENDATIONS

Designing a curriculum for mathematics education involves several choices. First pre-, present and post-setcentric mathematics together with multi-year lines and half-year blocks constitute 3x2 different kinds of mathematics education. Combined with three different ways of seeing competences, this offers a total of 18 different ways in which to perform mathematics education at each of the three educational levels, primary and secondary and tertiary, which may even be divided into parts.

Once chosen, institutional rigidity may hinder curriculum changes. So, to avoid the ethical issues of forcing cures from self-referring diagnoses upon children and teenagers in need of guidance instead of cures, the absence of participants from federal states might be taken as an advice to replace the national multi-year macro-curriculum with regional half-year micro-curricula. At the same time, adopting the post version of setcentric mathematics will make the curriculum coherent with the mastery of Many that children bring to school, and relevant to learning the quantitative competence and numeracy desired by society.

And, as Derrida says in an essay called 'Ellipsis' in 'Writing and Difference': "Why would one mourn for the centre? Is not the centre, the absence of play and difference, another name for death?"

POSTSCRIPT: MANY-MATH, A POST-SETCENTRIC MATHEMATICS FOR ALL

As post-setcentric mathematics, Many-math, can provide numeracy for all by celebrating the simplicity of mathematics occurring when recounting the ten fingers in bundles of 3s:

T = ten = 1B7 3s = 2B4 3s = 3B1 3s = 4B-2 3s. Or, if seeing 3 bundles of 3s as 1 bundle of bundles,

T = ten = 1BB0B1 3s = 1*B^2 + 0*B + 1 3s, or T = ten = 1BB1B-2 3s = 1*B^2 + 1*B – 2 3s.

This number-formula shows that a number is really a multi-numbering of singles, bundles, bundles of bundles etc. represented geometrically by parallel block-numbers with units. Also, it shows the four ways to unite: on-top addition, multiplication, power and next-to addition, also called integration. Which are precisely the four ways to unite constant and changing unit- and per-numbers numbers into totals as seen by including the units; each with a reverse way to split totals. Thus, addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the Algebra Square', also showing that equations are solved by moving to the opposite side with opposite signs.

Operations **unite**/ *split* Totals in	Changing	Constant

Unit-numbers m, s, kg, $	***T = a + n*** *T – n = a*	***T = a*n*** *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	***T = ∫a*dn*** *dT/dn = a*	***T = a^n*** *n√T = a logaT = n*

An unbundled single can be placed on-top of the block counted in 3s as T = 1 = 1/3 3s, or next-to the block as a block of its own written as T = 1 = .1 3s Writing T = ten = 3 1/3 3s = 3.1 3s = 4.-2 3s thus introduces fractions and decimals and negative numbers together with counting.

The importance of bundling as the unit is emphasized by counting: 1, 2, 3, 4, 5, 6 or bundle less 4, 7 or B-3, 8 or B-2, 9 or B-1, ten or 1 bundle naught, 1B1, …, 1B5, 2B-4, 2B-3, 2B-2, 2B-1, 2B naught.

This resonates with 'Viking-counting': 1, 2, 3, 4, hand, and1, and2, and3, less2, less1, half, 1left, 2left. Here '1left' and '2left' still exist as 'eleven' and 'twelve', and 'half' when saying 'half-tree', 'half-four' and 'half-five' instead of 50, 70 and 90 in Danish, counting in scores; as did Lincoln in his Gettysburg address: "Four scores and seven years ago …"

Counting means wiping away bundles (called division iconized as a broom) to be stacked (called multiplication iconized as a lift) to be removed to find unbundled singles (called subtraction iconized as a horizontal trace). Thus, counting means postponing adding and introducing the operations in the opposite order of the tradition, and with new meanings: 7/3 means 7 counted in 3s, 2x3 means stacking 3s 2 times. Addition has two forms, on-top needing recounting to make the units like, and next-to adding areas, i.e. integral calculus. Reversed they create equations and differential calculus.

The recount-formula, T = (T/B)*B, appears all over mathematics and science as proportionality or linearity formula:

- Change unit, T = (T/B)*B, e.g. T = 8 = (8/2)*2 = 4*2 = 4 2s
- Proportionality, $ = ($/kg)*kg = price*kg
- Trigonometry, a = (a/c)*c = sinA*c, a = (a/b)*b = tanA*b, b = (b/c)*c = cosA*c
- STEM-formulas, meter = (meter/sec)*sec = speed*sec, kg = (kg/m^3)*m^3 = density*m^3
- Coordinate geometry, Δy = (Δy/Δx)*Δx = m*Δx
- Differential calculus, dy = (dy/dx)*dx = y' * dx

The number-formula also contains the formulas for constant change: *T = b*x* (proportional), *T = b*x + c* (linear), *T = a*x^n* (elastic), *T = a*n^x* (exponential), *T = a*x^2 + b*x + c* (accelerated).

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics 'post-dicting' what we cannot be 'pre-dicted'.

THE GENERAL CURRICULUM CHOICES OF POST-SETCENTRIC MATHEMATICS

Making the curriculum bridge cohere with the individual start levels in a class is obtained by always beginning with the number-formula, and with recounting tens in icons less than ten, e.g. T = 4.2 tens = ? 7s, or u*7 = 42 = (42/7)*7, thus solving equations by moving to opposite side with opposite sign. And by always using full number-language sentences with a subject, a verb and a predicate as in the word language, e.g. T = 2*3. This also makes the bridge cohere to previous and following bridges.

Making the end level cohere to goals and values expressed by the society and by the learners is obtained by choosing mastery as the end goal, not of the inside self-referring setcentric construction of contemporary university mathematics, but of the outside universal physical reality, Many.

Making the bridge passable is obtained by choosing Piagetian psychology instead of Vygotskyan.

FLEXIBLE NUMBERS MAKE TEACHERS FOLLOW

Changing a curriculum raises the question: will the teachers follow? Here, seeing the advantage of flexible numbers makes teachers interested in learning more about post-setcentric mathematics:

Typically, division creates problems to students, e.g. 336/7. With flexible numbers a total of 336 can be recounted with an overload as T = 336 = 33B6 = 28B56, so 336/7 = 28B56 /7 = 4B8 = 48; or with an underload as T = 336 = 33B6 = 35B-14, so 336/7 = 35B-14 /7 = 5B-2 = 48.

Flexible numbers ease all operations:

T = 48*7 = 4B8*7 = 28B56 = 33B6 = 336

T = 92 – 28 = 9B2 – 2B8 = 7B-6 = 6B4 = 64

T = 54 + 28 = 5B4 + 2B8 = 7B12 = 8B2 = 82

To learn more about flexible numbers, a group of teachers can go to the MATHeCADEMY.net designed to teach teachers to teach MatheMatics as ManyMatics, a natural science about Many, to watch some of its YouTube videos. Next, the group can try out the "Free 1day Skype Teacher Seminar: Cure Math Dislike by ReCounting" where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally, and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows after the coffee break.

To learn more, a group of eight teachers can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for secondary school. For modelling, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where the 8 teachers form 2 teams of 4, choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers. The material mediates learning by experimenting with the subject in number-language sentences, i.e. the total T. Thus, the material is self-instructing, saying “When in doubt, ask the subject, not the instructor”.

The material for primary and secondary school has a short question-and-answer format. The question could be: “How to count Many? How to recount 8 in 3s? How to count in standard bundles?” The corresponding answers would be: “By bundling and stacking the total T, predicted by T = (T/B)*B. So, T = 8 = (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s = 3.-1 3s. Bundling bundles gives multiple blocks, a polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4*B^2 + 5*B + 6*1.”

References

ICMI study 24 (2018). School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities. *Pre-conference proceedings*. Editors: Yoshinori Shimizu and Renuka Vithal.

Tarp, A. (2018). Mastering Many by Counting, Recounting and Double-counting before Adding On-top and Next-to. *Journal of Mathematics Education, March 2018, Vol. 11*(1), 103-117.

Tarp, A. (submitted). Examples of Goal Displacements in Mathematics Education, Reflections from the CTRAS 2017 Conference. *Journal of Mathematics Education.*

COUNTING BEFORE ADDING, A PPP FOR THE ARTICLE ON A TWIN CURRICULUM

COUNTING before ADDING

The Child's Own Twin Curriculum

Count & ReCount & DoubleCount

before Adding NextTo & OnTop

master many with manyMath

Allan.Tarp@MATHeCADEMY.net, November 2018

MATHeCADEMY

Background: Our two language houses

The **WORD language** assigns words in sentences with The **NUMBER language** assigns numbers instead with	• a subject • a verb • a predicate

Both languages have a META-language, a grammar, describing the language, that is learned before the grammar. But does mathematics respect teaching language before grammar?

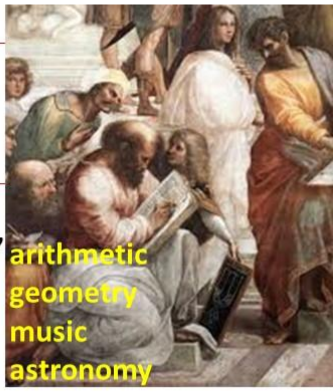
	WORD language	NUMBER language
META-language, grammar	*'is' is a verb*	*'x' is an operation*
Language	This is a chair	T = 3x4
	Mother tongue	Mathematics

WORLD

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 2

MATHeCADEMY

How well defined is mathematics?



In ancient Greece, Pythagoras used mathematics, meaning knowledge, as a common label for four descriptions of Many by itself & in space & time:

Together they formed the '**quadrivium**' recommended by Plato as a general curriculum after the '**trivium**' consisting of grammar & logic & rhetoric.

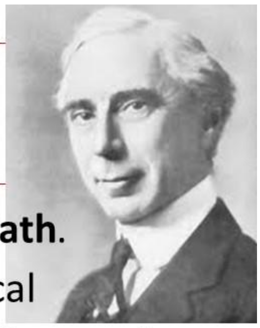
Geometry & algebra are both grounded in Many as shown by names:

Geometry means to measure earth in Greek

Algebra means to reunite numbers in Arabic

MATHeCADEMY

Modern mathematics, MetaMath



Around 1900, **SET** made math a self-referring **MetaMath**.

But Russell saw that self-reference leads to the classical liar paradox 'this sentence is false', being false if true & opposite:

"Let M be the set of sets not belonging to itself, $M = \{A \mid A \notin A\}$.

Then $M \in M \Leftrightarrow M \notin M$. Forget about sets. Use type theory instead. So, by self-reference, fractions cannot be numbers."

Mathematics: "Forget about Russell, he is not a mathematician. We just institutionalize fractions as so-called rational numbers."

MATHeCADEMY

Institutions as thorns protecting Sleeping Beauties

- Weber on institutionalization: Rationalized too far, mathematics may become an **iron cage** that **disenchants** its subject.
- Bauman on self-reference: „The ideal model of action subjected to rationality as the supreme criterion contains a danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right“.
- Arendt: Just following orders may lead to '**the banality of evil**'.
- Sartre on existentialism: „**Existence precedes essence**“.

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 5

MATHeCADEMY

Three curriculum choices to make

What is the goal of mathematics education?

1. To learn institutionalized mathematics, or
2. To learn to master what exists, Many

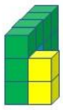
What is the core means?

1. To learn about numbers without units, or
2. To learn how to number with units

1, -2, 3/4, √5, π, e

T = 3B2 = 3.2 = 4.-2 4s

What are numbers?



1. One-dimensional line-numbers without units, or
2. Two-dimensional block-numbers with units

Choosing 1 may have caused 50 years of less successful math education research.

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 6

MATHeCADEMY

Different curricula: MetaMath or ManyMath

What is the goal of mathematics education?

1. To learn mathematics (self reference pointing up, Vygotsky theory)
2. To learn to Master Many (external reference pointing down, Piaget theory)

What is a core means?

1. To learn about numbers (operations on specified and unspecified numbers)
2. To learn how to number (number-language sentences about counting & adding totals)

What are numbers?

1. 1D line-numbers (integer, natural, rational, real, place value system)
2. 2D bundle-numbers (constant & changing unit-numbers & per-numbers)

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 7

MATHeCADEMY

Why teach children if they already know?

With education curing un-educatedness, we ask:

To CURE, be SURE

1. The diagnosed is not already cured
2. The diagnose is not self-referring: *teach math to learn math*

Core Questions:

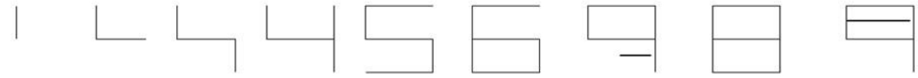
- What Mastery of Many does the child have already?
- What could be a <u>ChildCenteredCurriculum</u> in Quantitative Competence?

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 8

Creating icons: | | | | → |||| → 4 →

Children love making number-icons of cars, dolls, spoons, sticks.
Counting in ones means naming the different degrees of Many.
Changing four ones to one fours creates a 4-icon with four sticks.
An icon contains as many sticks as it represents, if written less sloppy.
Once created, icons become units to use when counting in bundles.

one	two	three	four	five	six	seven	eight	nine
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII

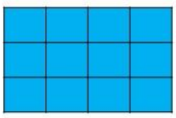


1	2	3	4	5	6	7	8	9

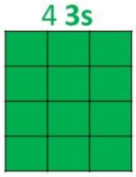
Children see Many as bundles with units

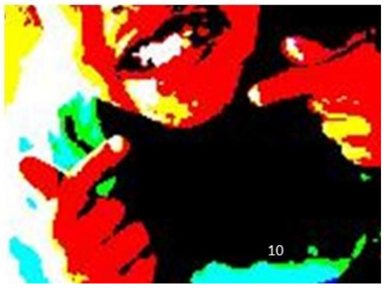
- “How old next time?” A 3year old says “Four” showing 4 fingers: | | | |
- But, the child reacts strongly to 4 fingers held together 2 by 2: || ||
- “That is not four, that is two twos”
- The child describes what exists, and with units: bundles of 2s, and 2 of them
- The block 3 **4s** has two numbers: 3 (the counting-number) and **4** (the unit-number)
- Children also use bundle-numbers with Lego blocks:

3 **4s**



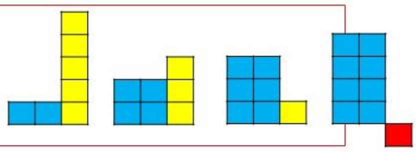
4 **3s**





MATHeCADEMY

To count Many, children bundle



- Children are flexible when re-counting a Total of 7 sticks in **2s**:

I I I I I I I ● ǂǂ I I I I I ● ǂǂ ǂǂ I I I ● ǂǂ ǂǂ ǂǂ I ● ǂǂ ǂǂ ǂǂ ǂǂ

T = 7 = 1 **2s** &5 = 2 **2s** &3 = 3 **2s** &1 = 4 **2s** less 1

- And children don't mind writing a total of 7 using 'bundle-writing':

T = 7 = 1**B**5 = 2**B**3 = 3**B**1 = 4**B** ~~1~~, or even as

T = 7 = 1**BB**3 = 1**BB**1**B**1 = 2**BB** ~~1~~

- Also, children love to count in **3s**, **4s**, and in **hands**:

Thus, a number is a multi-counting of bundles as units

(…, bundles-of-bundles, bundles, unbundled)

IIII I I
T = 7 = 1 **5s** & 2
T = 7 = 1**B**2 **5s**

MATHeCADEMY

Counting bundles gives a number formula

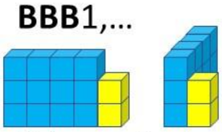
Children have fun when counting bundles, bundles of bundles, etc.:

With ten-bundles: 01, 02, …, 09, **Bundle**,

B1, **B**2,…, 9**B**8, 9**B**9, **BundleBundle**,

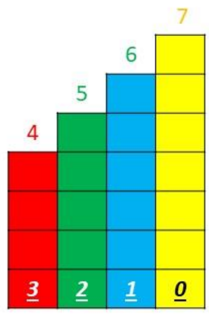
BB1,…, 2**BB**3**B**4,…, 9**BB**9**B**9, **BundleBundleBundle**,

BBB1,…



With blocks turned to hide the units behind:

B is marked with 1, **BB** with 2, **BBB** with 3, etc., singles with 0.



Later, this is a number formula T = 4567 = 4**BBB**5**BB**6**B**7 = 4x**B**^3 + 5x**B**^2 + 6x**B** + 7

MATHeCADEMY

Counting ten fingers & counting in tens

Children have fun when flexibly counting ten fingers in different ways:

- The Roman way: 01, 02, 03, HandLess1, **HAND**, Hand1, H2, H3, 2H-1, 2H, 2H1, 2H2
- The Viking way: 01, 02, 03, 04, HALF, 06, 07, less2, less1, **FULL**, 1left, 2left
- The modern way: 01, 02,…, 09, **ten**, ten1, ten2,…, 9ten8, 9ten9, **tenten**, tenten1,…, 2tenten3ten4,…, 9tenten9ten9, **tententen**, tententen1,…

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 13

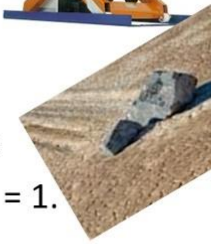
MATHeCADEMY

Division, multiplication & subtraction as icons also

'From 9 take away **4s**' we write 9/4
iconizing the sweeping away by a broom, called division.

'2 times stack **4s**' we write 2x4
iconizing the stacking up by a lift called multiplication.

'From 9 take away 2 **4s**' to look for un-bundled we write 9 – 2x4
iconizing the dragging away by a trace called subtraction.



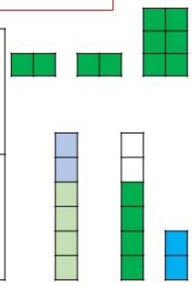
So counting includes division and multiplication and subtraction:

Finding the bundles: 9 = 9/4 **4s**. Finding the un-bundled: 9 – 2x4 = 1.

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 14

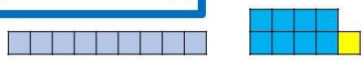
Counting creates two counting formulas

ReCount **T = (T/B) x B**	from a total **T**, **T/B** times, **Bs** is taken away and stacked on-top
ReStack **T = (T–B) + B**	from a stack **T**, **T–B** is left when **B** is taken away and placed next-to



With formulas, a calculator can **predict** the counting-result 9 = 2**B**1 **4s**

9/4	**2.some**
9 – 2x4	**1**



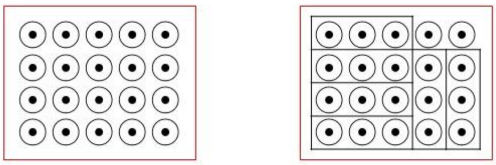
*As sentences of the number language, **formulas predict***

To share Many, children take away bundles predicted by division, multiplication and subtraction

2 preschoolers share 6 cakes by taking away **2s** from 6, thus rooting division as counting in **2s**. ⊙⊙⊙⊙⊙⊙ → ⊙⊙⊙⊙ ⊙⊙ → ⊙⊙ ⊙⊙ ⊙⊙

- They smile when seeing that entering ‘6/2’ allows a calculator to predict that they can take cakes 3 times.
- And when seeing that ‘4x5/3’ predicts that 3 children can take cakes 6 times (or 6 cakes 1 time) when sharing 4 rows of 5 cakes.
- And when seeing that ‘4x5-6x3’ predicts that 2 will be left.

6/2	3
4x5/3	6.some
4x5-6x3	2



MATHeCADEMY

Question Guided Counting Curriculum

A question guided re-enchanting COUNTING curriculum could be named Mastering Many by counting, re-counting & double-counting.

- The design accepts that while 8 competences might be needed to learn university mathematics, only 2 are needed to Master Many: COUNTING & ADDING, motivating a twin curriculum.
- The corresponding pre-service or in-service question guided teacher education can be found at the MATHeCADEMY.net.
- Remedial micro-curricula for classes stuck in traditional mathematics can be found there also.

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 17

MATHeCADEMY

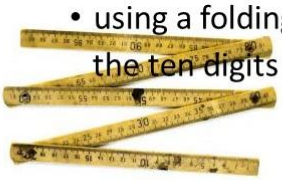
Q01, icon making I I I I → IIII → 4

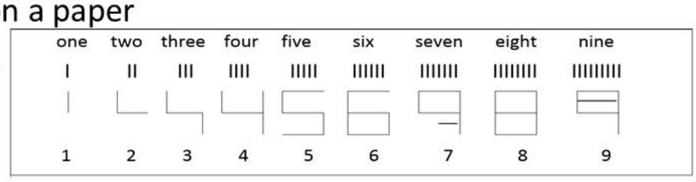
“The digit-icon 4 seems to be have four sticks. Does this apply to all digit-icons?”

We can change many ones to one icon with as many sticks or strokes as it represents, if written in a less sloppy way.

Follow-up activities could be:

- rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.
- rearranging sticks on a table or on a paper
- using a folding ruler to construct the ten digits as icons





MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 18

MATHeCADEMY

Q02, counting sequences I

"How to count fingers?"

- Using **5s** as the bundle-size, five fingers can be counted as
  "01, 02, 03, 04, **Bundle**"
- And ten fingers can be counted as
  "01, 02, **Bundle** less2, **Bundle** -1, **Bundle**"
  "**Bundle**&1, **B**&2, 2**B** less2, 2**B**-1, 2**B**".

Follow-up activities could be counting the fingers in **3s** and **4s** and **7s**:

T = ten = 1**B**3 **7s** = 2**B**2 **4s** = 3**B**1 **3s** = 1**BB**1 **3s**.

MATHeCADEMY

Q02, counting sequences II

Counted as 1**B**, the bundle-number needs no icon. So counting a dozen cakes we say:

in	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙
4s	01	02	03	**B**	1**B**1	1**B**2	1**B**3	2**B**	2**B**1	2**B**2	2**B**3	3**B**
7s	01	02	03	04	05	06	**B**	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5
tens	01	02	03	04	05	06	07	08	09	**B**	1**B**1	1**B**2

The number names, eleven and twelve come from 'one left' and 'two left' in Danish, (en / twe levnet), again showing that counting takes place by taking away bundles.

Q03, bundle-counting in icon-units I

“How to count by bundling?”

Five fingers can be bundle-counted in pairs or triplets, allowing both an overload and an underload; and reported in a number-language sentence with a subject & a verb & a predicate as e.g. T = 2 **3s.**

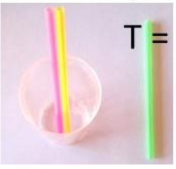
IIIII • ~~II~~ III • ~~II~~ ~~II~~ I • ~~II~~ ~~II~~ ~~II~~ • ~~II~~ ~~II~~ I

T = 5 = 1Bundle3 **2s** = 2B1 **2s** = 3B-1 **2s** = 1BB1 **2s**

- Cup- & decimal-writing separates inside bundles from outside singles:

T = 5 = 1]3 **2s** = 2]1 **2s** = 3]-1 **2s** = 1]]0]1 **2s**

T = 5 = 1.3 **2s** = 2.1 **2s** = 3.-1 **2s** = 10.1 **2s**



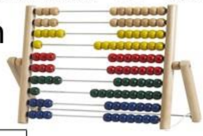
Likewise, if counting in ten-bundles: T = 57 = 5**B**7 = 4**B**17 = 6**B**-3 **tens**

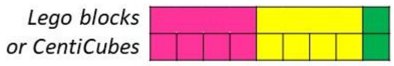
Q03, bundle-counting in icon-units II

To count 9 in **4s**, we may bundle in a cup with 1 stick per bundle.

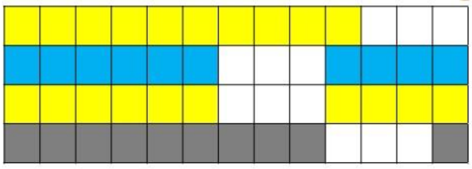
9 = IIIIIIIII = ~~IIII~~ ~~IIII~~ I = **II]**I = 2]1 **4s** = 2**B**1 **4s** = 2.1 **4s**

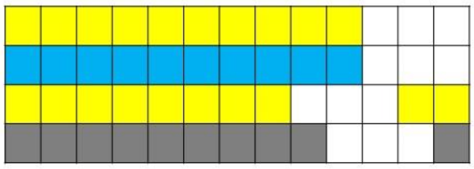
We may report with cup-, bundle- or decimal-writing, or on a western **ABACUS** in





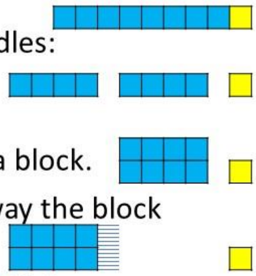
Outside geometry mode or Inside algebra mode





Q04, calculators predict

"Can a calculator predict a counting result?"



We may see the division sign as an icon for a broom wiping away bundles:
9/4 means 'from 9, wipe away bundles of 4s'.

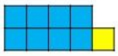
- The calculator says '2.some', thus predicting it can be done 2 times.
  Now the multiplication sign iconizes a lift stacking the bundles into a block.
- Finally, the subtraction sign iconizes the trace left when dragging away the block to look for unbundled singles.
- With '9-2x4 = 1' the calculator predicts that 9 can be recounted as 2B1 **4s**.

9/4	**2.some**
9 – 2x4	**1**

Q04, counting creates 2 counting formulas

ReCount **T = (T/B) x B**	from a total **T**, **T/B** times, **Bs** is taken away and stacked
ReStack **T = (T–B) + B**	from a total **T**, **T–B** is left, when **B** is taken away and placed next-to

As sentences of the number language, **Formulas Predict:**
Predicting that **T** = 9 = 2.1 **4s:**



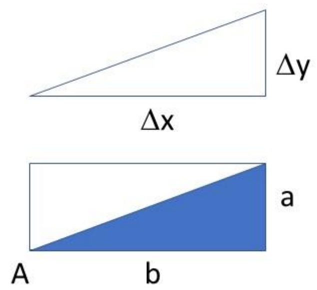
9/4	**2.some**
9 – 2x4	**1**

MATHeCADEMY

Q04, the recounting formula is a core formula

T = (T/B)*B saying 'from T, T/B times, Bs can be taken away', is all over:

Proportionality	y = k * x
Linearity	Δy = (**Δy/Δx**) * Δx = **m** * Δx
Local linearity	dy = (**dy/dx**) * dx = **y'** * dx
Trigonometry	a = (**a/b**) * b = **tanA** * b
Trade	$ = (**$/kg**) * kg = **price** * kg
Science	meter = (**meter/second**) * second = **velocity** * second



MATHeCADEMY

Q05, unbundled as decimals or negatives or fractions

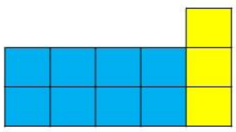
0.3 **4s** or 0.-1 **4s** or 3/4 **4s**

"Where to put the unbundled singles?"

When counting by bundling, the unbundled singles can be placed

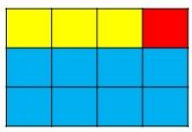
NextTo the block — counted as a block of **1s**

OnTop of the block — counted as a bundle / counted in bundles



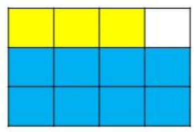
T = 2B3 **4s** = 2.3 **4s**

A decimal number



T = 3B-1 **4s** = 3.-1 **4s**

A negative number



T = 2 3/4 **4s**

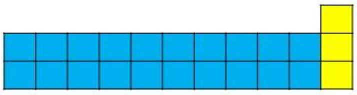
A fraction

MATHeCADEMY

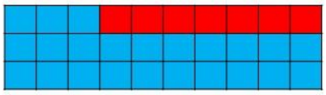
Q05, counting in tens

“Where to put the unbundled singles with tens?”

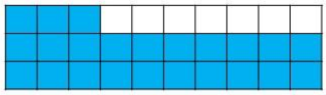
Counting in tens, an outside Total of 2 **tens** & 3 can be described inside as T = 23 if leaving out the unit, or as



T = 2.3 **tens**



T = 3.-7 **tens**

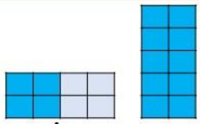


T = 2 3/10 **tens**

MATHeCADEMY

Q06, prime & foldable bundle-units

“When can blocks be folded in like bundles?”



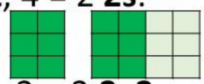
The block T = 2 **4s** = 2 x 4 has 4 as the bundle-unit.

Turning over gives T = 4 **2s** = 4 x 2, now with 2 as the bundle-unit.

4s can be folded in another bundle as 2 **2s**, whereas 2s cannot.

(1 is not a bundle, nor a unit since a bundle-of-bundles stays as 1).

We call 2 a **prime bundle-unit** and 4 a **foldable bundle-unit**, 4 = 2 **2s**.



A block of 3 **2s** cannot be folded.

A block of 3 **4s** can be folded: T = 3 **4s** = 3 x (2 x 2) = (3 x 2) x 2 = 2 **3x2s**.

A number is called **even** if it can be written with 2 as the unit, else **odd**.

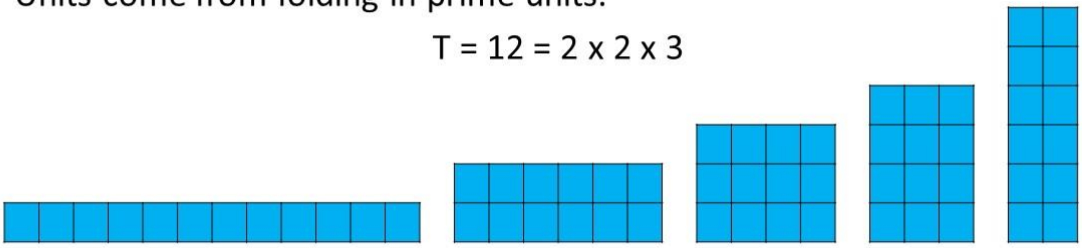
MATHeCADEMY

Q07, finding possible units

"What are possible units in T = 12?"

Units come from folding in prime units:

T = 12 = 2 x 2 x 3



T = 12 **1s** = 2 **2x3s** = 3 **2x2s** = 2x2 **3s** = 2x3 **2s**

MATHeCADEMY

Q08, recounting in a different unit

"How to change a unit?"

The recount-formula allows changing the unit.

Asking T = 3 **4s** = ? **5s**, the recount-formula gives T = 3 **4s** = (3x4/5) **5s**.

Entering 3x4/5, the answer '2.some' shows that a block of 2 **5s** can be taken away.

With 3x4–2x5, the answer '2' shows that 3 **4s** can be recounted as 2B2 **5s** or 2.2 **5s**.

3 **4**s = IIII IIII IIII = IIII I III II II = IIIII IIIII II = 2B2 **5**s = 2.2 **5s**

3x4/5	**2.**some
3x4 – 2x5	**2**

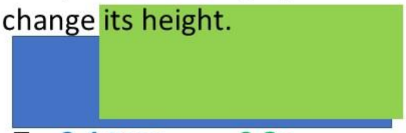
Change Unit = Proportionality

MATHeCADEMY

Q09, recounting from tens to icons

"How to change unit from tens to icons?"

Asking 'T = 2.4 **tens** = 24 = ? **8s**', we just recount 24 in **8s**:
T = 24 = (24/8)x8 = 3x8 = 3 **8s.**

Formulated as an **equation** we use ***u*** for the unknown number, ***u***x8 = 24. Recounting 24 in 8s shows that ***u*** is 24/8 attained by moving 8 **to opposite side - with opposite sign**	To keep its size, a block changing its unit must also change its height.  T = 2.4 **tens** = 3 **8s**

u x 8 = 24 = (24/8)x8
u = 24/8 = 3

MATHeCADEMY

Q10, recounting from icons to tens (multiplication) 3 7s = ? tens

"How to change unit from icons to tens?"

Asking 'T = 3 **7s** = ? **tens**', the recount-formula cannot be used since the calculator has no ten-button. However, it gives the answer directly by using multiplication alone: T = 3 **7s** = 3x7 = 21 = 2.1 **tens**, only it leaves out the unit and the decimal point.

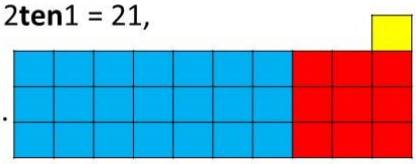
Alternatively, we may use 'less-numbers', so 7 = **ten** less 3

T = 3x7 = 3 x (**ten** less 3) = 3 x **ten** less 3x3 = 3**ten** less 9 = 2**ten**1 = 21,

or with 9 = **ten** less 1:

T = 3**ten** less (**ten** less1) = 2**ten** lessless 1 = 2**ten** & 1 = 21.

showing that 'lessless' cancel out



MATHeCADEMY

Recounting large numbers in or from tens: *same size, but new form*

Recounting 6 **47s** in **tens** Recounting 476 in **7s**

*BundleWriting seprates **INSIDE** bundles from **OUTSIDE** singles*

T = 6 x 47 = 6 x 4B7	T = 476 = **47.6 tens**
= 24B42	= 47B6
= 28B2	= 42B56
= 28.2	= 6x7B8x7
tens	= 68 x 7

MATHeCADEMY

Q11, double-counting in two units creates bridging **PerNumbers** & proportionality

"How to double-count in two units?"

DoubleCounting in kg & $, we get **4kg = 5$** or

4kg **per** 5$ = 4kg/5$ = 4/5 kg/$ = a PerNumber.

With 4kg bridged to 5$ we answer questions by recounting in the per-number.

Questions:

7kg = ?$	**8$ = ?kg**
7kg = (7/4) x 4kg = (7/4) x 5$ = 8.75$	8$ = (8/5) x 5$ = (8/5) x 4kg = 6.4kg

Answer: *Recount in the PerNumber (Proportionality)*

Q12, double-counting in the same unit creates fractions

"How to double-count in the same unit?"

Double-counted in the same unit, per-numbers are fractions, 2$ per 9$ = 2/9, or percentages, 2 per 100 = 2/100 = 2%.

To find a fraction or a percentage of a total, again we just recount in the per-number.

- **Taking 3 per 4 = taking ? per 100.** With 3 bridged to 4, we recount 100 in 4s:

100 = (100/4)*4 giving (100/4)*3 = 75, and 75 per 100 = 75%.

- **Taking 3 per 4 of 60 gives ?.** With 3 bridged to 4, we recount 60 in 4s:

60 = (60/4)*4 giving (60/4)*3 = 45.

- **Taking 20 per 100 of 60 gives ?.** With 20 bridged to 100, we recount 60 in 100s:

60 = (60/100)*100 giving (60/100)*20 = 12 .

We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number.

Q12, enlarging or shortening units

"How to enlarge or shorten units in fractions?"

Taking 2/3 of 12 means taking 2 per 3 of 12.

With 2 bridged to 3, we recount 12 in **3s**, 12 = (12/3)*3 = 4*3

So 4 times we can take 2, i.e. 8 of the 12. Thus 2 per 3 = 8 per 12.

This may be used for enlarging or shortening fractions by inserting or removing the same unit above and below the fraction line:

$\frac{2}{3} = \frac{2\ \mathbf{4s}}{3\ \mathbf{4s}} = \frac{2*4}{3*4} = \frac{8}{12}$ ● $\frac{8}{12} = \frac{2*4}{3*4} = \frac{2\ \mathbf{4s}}{3\ \mathbf{4s}} = \frac{2}{3}$ ● $\frac{12abc}{8a} = \frac{3*4*a*b}{2*4*a} = \frac{3*b\ \mathbf{4as}}{2\ \mathbf{4as}} = \frac{3b}{2}$

MATHeCADEMY

Q13, recounting the sides in a block

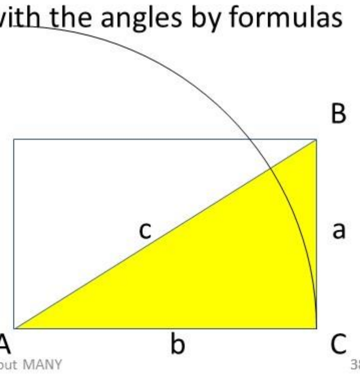
Geometry means to measure earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula. And connected with the angles by formulas recounting a side in the other side or in the diagonal:

A+B+C = 180

a*a + b*b = c*c (the Pythagoras formula)

sinA = a/c; cosA = b/c; tanA = a/b = Δy/Δx = gradient

Circle: circum./diam. = π **= n*tan(180/n)** for n large



MATHeCADEMY

Q14, double-counting gives per-numbers in STEM multiplication formulas I

STEM (Science, Technology, Engineering, Math) typically contains multiplication formulas with per-numbers coming from double-counting.

Examples:

- kg = (**kg/cubic-meter**) x cubic-meter = **density** x cubic-meter
- force = (**force/square-meter**) x square-meter = **pressure** x square-meter
- meter = (**meter/sec**) x sec = **velocity** x sec
- energy = (**energy/sec**) x sec = **Watt** x sec
- energy = (**energy/kg**) x kg = **heat** x kg

MATHeCADEMY

$\mathbb{Q}$14, double-counting gives per-numbers in STEM multiplication formulas II

Extra STEM examples:

- gram = (**gram/mole**) x mole = **molar mass** x mole;
- Δ momentum = (Δ **momentum/sec**) x sec = **force** x sec;
- Δ energy = (Δ **energy/ meter**) x meter = **force** x meter = work;
- **energy/sec** = **(energy/charge)** x **(charge/sec)** or **Watt** = **Volt** x **Amp**;
- dollar = (**dollar/hour**) x hour = **wage** x hour;
- dollar = (**dollar/meter**) x meter = **rate** x meter
- dollar = (**dollar/kg**) x kg = **price** x kg.

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 40

MATHeCADEMY

$\mathbb{Q}$15, navigating on a squared paper

First steps into coordinate geometry, to always keep algebra and geometry together.

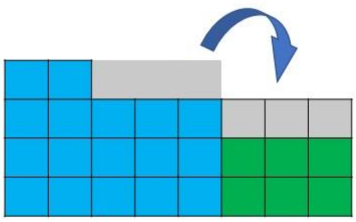
"Collect treasures on the rocks "	**"Plan a trip to treasure island"**
Three rocks are placed on a squared paper.	Departure point: 3cm out & 2cm up
The rocks have the values -1, 1, and 2.	Destination point: 7cm out & 4cm up.
A journey begins in the midpoint.	Plan a voyage with 1 out per day.
Two dices tell the out- and up- change,	How many days before reaching the island?
where odd numbers are negative.	What is your position after 2 days?
How many points before reaching the edge?	What is your position after n days?
Predict and measure angles on the journey.	What is the angle traveled?

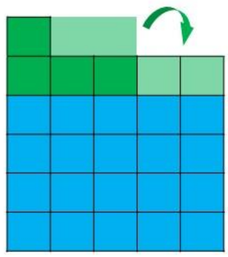
MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 41

MATHeCADEMY

Counted & recounted, Totals can be added

BUT: NextTo →	or OnTop ↑
4 **5s** + 2 **3s** = 3**B**2 **8s**	4 **5s** + 2 **3s** = 4 **5s** + 1**B**1 **5s** = 5**B**1 **5s**
The areas are integrated *Adding areas = Integration*	The units are changed to be the same *Change unit = Proportionality*





MATHeCADEMY

Four ways to unite into a Total

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2 + 4***B** + 5 (a polynomial) shows the four ways to add: +, *, ^, next-to block-addition (integration).
Addition and multiplication add changing and constant unit-numbers.
Integration and power add changing and constant per-numbers.
We might call this beautiful simplicity the ‘Algebra Square’.

Operations unite	changing	constant
Unit-numbers *m, s, $, kg*	**T = a + n**	**T = a * n**
Per-numbers *m/s, $/kg, m/(100m) = %*	**T = ∫ a dn**	**T = a^n**

See the rest of the PPP on

http://mathecademy.net/the-childs-own-twin-curriculum/