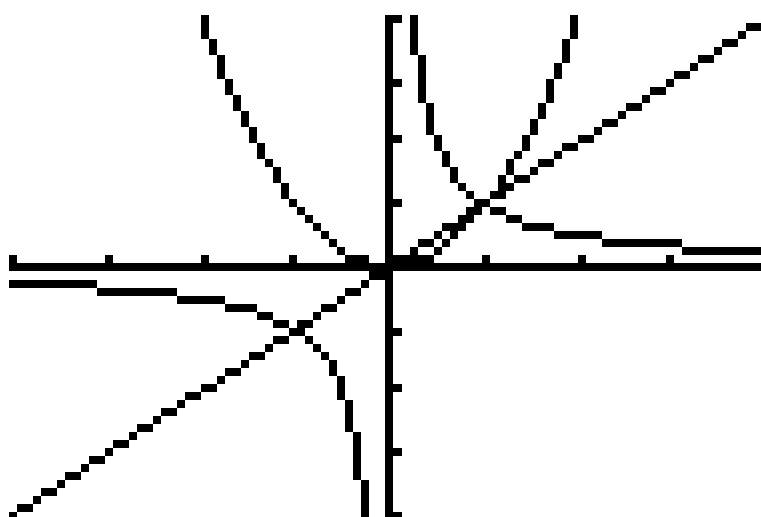


Mathematics Predicts

PreCalculus

How to add constant PerNumbers



$y = 1*x$, $y = x*x$, $y = 1/x$

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Compendium & Projects

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Mathematics Predicts

Mathematics	Mathematics contains Algebra, Geometry and Statistics
Algebra	Algebra (calculation) can predict counting processes, both the end result and the parts.
Uniting and Splitting Numbers	Geometry (earth measuring) can be used to calculate plane figures and spatial forms.
Geometry	Statistics (counting) is used for counting the actual size of different quantities.
Measuring Earth	
Statistics	
Accounting Many	

Mathematics has two main fields, Algebra and Geometry, as well as Statistics. Geometry means 'earth-measuring' in Greek. Algebra means 'reuniting' in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus together algebra and geometry give an answer to the fundamental human question: how do we divide the earth and what it produces?

Originally human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus trade was only possible with those highlanders that had silver in their mountains.

The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire. The dark Middle Ages came when the Greek silver mines were emptied and the Arabs conquered the Spanish mines. German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number system and algebra.

The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: $A+B+C = 180$ and a side law: $a^2+b^2=c^2$. Pythagoras generalized this findings by claiming: All is numbers.

This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don't know Geometry. However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation.

The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has an acceleration g ; and the distance s and the time t are connected by the formula: $s = \frac{1}{2} * g * t^2$. However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middle men. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine.

The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move?

The church said 'among the stars'. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth.

But why do things fall? The church said: everything follows the unpredictable will of our metaphysical lord only attainable through belief prayers and church attendance. Newton objected: It follows its own will, a force called gravity that can be predicted by a formula telling how a force changes the motion, which made Newton develop change-calculations, calculus. So instead of obeying the church, people should enlighten themselves by knowledge, calculations and school attendance.

Brahe used his life to write down the positions of the planets among the stars. Kepler used these data to suggest that the sun is the center of the solar system, but could not prove it without sending up new planets. Newton, however, could validate his theory by different examples of falling and swinging bodies.

Newton's discoveries laid the foundation of the Enlightenment period realizing that when an apple follows its own will and not that of a metaphysical patronizer, humans could do the same. Thus by enlightening themselves people could replace the double patronization of the church and the prince with democracy, which lead to two democracies, one in The US and one in France. Also formulas could be used to predict and therefore gain control over nature, using this knowledge to build an industrial welfare society based upon a silver-free economy emerging when the English replaced the import silk and pepper from the Far East with production of cotton in the US creating the triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton.

Calculations Predict

Calculations predict the total T. 2*4 calculation types are used to unite and split into four different types of numbers:			a \$ and n \$ total T \$:	$a+n = T$
			a \$ n times total T \$:	$n*a = T$
			r % n times total T%:	$(1+r)^n = 1+T$
			a1 kg at p1 \$/kg +	
			a2 kg at p2 \$/kg total T \$:	
			$p1*a1 + p2*a2 = T$:	$\sum p*a = T$

Algebra means re-uniting in Arabic and can be translated to predictions. Algebra thus predicts the result of uniting singles into totals or splitting totals into singles.

There are four ways of uniting numbers: addition (+), multiplication (*), power (^) and integration (\sum or \int).

Addition + predicts the result of uniting variable singles:

$$2\$ \text{ and } 3\$ \text{ and } 4\$ \text{ total T } \$: 2+3+4 = T$$

Multiplication * predicts the result of uniting constant singles:

$$2\$ + 2\$ + 2\$ + 2\$ + 2\$ = 5 \text{ times } 2\$ = T, 5*2 = T$$

Power ^ predicts the result of uniting constant percentages: 5 times 2% totals T%, $102\%^5 = 1+T$

Integration \sum or \int predicts the result of uniting constant per-numbers:

$$2\text{kg at } 7\$/\text{kg} + 3\text{kg at } 8\$/\text{kg} \text{ totals T } \$: 7*2 + 8*3 = T, \sum \$/\text{kg} * \text{kg} = T, \int p*dx = T$$

Inverse or backward calculations predicts the result of splitting a Total into singles.

$x+3 = 15$	Question: Which number added to 3 gives 15?
$x = 15-3$	Prediction: 15-3 is the number that added to 3 gives 15. Test: $3+(15-3) = 15$
Rule	Plus-numbers move across as minus-numbers, and vice versa

$x*3 = 15$	Question: Which number multiplied with 3 gives 15?
$x = \frac{15}{3}$	Prediction: $\frac{15}{3}$ is the number that added to 3 gives 15. Test: $3*\frac{15}{3} = 15$
Rule	Multiplication-numbers move across as minus-numbers, and vice versa

$x^3 = 125$	Question: Which number raised to power 3 gives 125?
$x = \sqrt[3]{125}$	Prediction: $\sqrt[3]{125}$ is the number that raised to power 3 gives 125. Test: $(\sqrt[3]{125})^3 = 125$
Rule	Exponent-numbers move across as reciprocal exponent-numbers, and vice versa

$3^x = 243$	Question: 3 raised to which power gives 243?
$x = \frac{\log 243}{\log 3}$	Prediction: 3 raised to power $\frac{\log 243}{\log 3}$ gives 243. Test: $3^{\frac{\log 243}{\log 3}} = 243$
Rule	Base-numbers move across as logarithm-numbers, and vice versa

A multi-calculation containing more calculations reduce to a single calculation by bracketing the stronger one.

$$T = 2+3*4 = 2+(3*4), T = 2+3^4 = 2+(3^4), T = 2*3^4 = 2*(3^4)$$

Priority: 1. (), 2.^, 3. *, 4. +

A formula-table can be used to document the solving of an equation.

<i>The unknown number</i>	$c = ?$	$T = a+b*c$	<i>The formula</i>
<i>The known numbers</i>	$a = 2$	$14 = 2+(3*c)$	<i>From a mixed to a single calculation by bracketing the stronger + moves across as the opposite -, and * moves across as / Bracket the calculation already present Perform the calculation</i>
	$b = 3$	$\frac{(14-2)}{3} = c$	
	$T = 14$	$4 = c$	
<i>Tests</i>	Test	$14 = 2+3*4$ $14 = 14 \quad \odot$	<i>'MATHSolver 0 = -14 + 2+3*x' gives 'x = 4'</i>

Exercises

Find the unknown number in the formula. Make more with randM (3,1)				
1. $T = a+b*c$	5. $T = a-b*c$			
2. $T = a+b/c$	6. $T = a-b/c$			
3. $T = a*b^c$	7. $T = a/b^c$			
4. $T = a+b^c$	8. $T = a-b^c$			

	T	b	a	c
I	60		12	20
II	60	1.5		20
III	60	1.5	12	

Formulas Predict

<p>A formula contains a quantity y and its calculation f, $y = f(x,z,t)$</p> <p>An equation is a formula with 1 unknown. An equation can be calculated or solved by finding the unknown.</p> <p>A function is a formula with 2 unknowns. A function can be tabled or graphed showing different scenarios: If $x = a$ then $y = f(a)$.</p>	<p>Purchase-formula: $b \\$ + x \text{ kg at a } \\$/\text{kg totals } y \\$:$ $b + x * a = y$</p> <p>Sharing-formula: $b \\$ + a \\$ shared between x persons totals y \\$:$ $b + a / x = y$</p>
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A formula contains a quantity y and a its calculation f, $y = f(x,z,t)$. Thus a formula might contain 2, 3, 4 or more variables. If the variables are replaced by fixed numbers, a formula is transformed into an equation or a function.

An equation is a formula with 1 unknown: $y = 10 + 2*3$, or $16 = b + 2*3$, or $16 = 10 + a*3$, or $16 = 10 + 2*x$

An equation can be solved manually or by a calculator using MATHSolver. After using 'solve' the solution is tested by inserting all known numbers: $16 = 10 + 2*3$ gives $16 = 16$

A function is a formula with 2 (or more) unknowns: $y = b + 2*3$, or $y = 10 + 2*x$, or $16 = b + 2*x$, or $16 = 10 + a*x$. In a function one of the unknowns is isolated and entered on the calculators y-list. Thus $x^2-y+3=0$ gives $y=x^2-3$.

Formulas are put on the y-list	Always start with Standard Zoom	Choose Graph to graph	Choose Trace to see scenarios	Calc Value gives specific values	And is used for knownx/unknowny
Knowny/unknownx y is on the y-list	The intersecting curves marked	The cursor is close to the sol.	The procedure is repeated	VARS gives access to the Y-s	The known x is put after the Y
MATHSolver is used to find y's	CLEAR old and enter new	Enter a guess	Read the solution close to guess	Enter a new guess	Read the solution close to guess
From table to formula use STAT	Enter the table as lists	Choose a formula type	Add Y1 to bring formulas to y-list	Add Plot for visual control	Adjust window before graphing

Exercises: Find the question marks in 3 different ways: manually in a formula table, using graphs and using calculation.

	1	2	3	4			
x	y=3+2*x	x	y=3-2*x	x	y=x^2-4	x	y=-x^2+5
-3.7	?	-3.7	?	-3.7	?	-3.7	?
-2.4	?	-2.4	?	-2.4	?	-2.4	?
3.1	?	3.1	?	3.1	?	3.1	?
4.5	?	4.5	?	4.5	?	4.5	?
?	-3.7	?	-3.6	?	-3.8	?	-3.2
?	-2.4	?	-2.5	?	-2.2	?	-2.6
?	3.1	?	3.2	?	3.7	?	3.3
?	4.5	?	4.6	?	4.7	?	4.3

	x	y	T
5	10	20	30
	30	50	80
6	10	15	25
	100	130	180
7	10	20	40
	100	70	10

Ans:	a	b	Formula	y	x	T
lin	2	10	$y = 10 + 2*x$	70	35	
exp	1,052	18	$y = 18 * 1,052^x$	83,33	29,2	13,6
pow	0,737	5,5	$y = 5,5 * x^{0,737}$	67,41	37,84	
lin	6	40	$y = 40 + 6*x$	190	23,33	
exp	1,054	59,17	$y = 59,17 * 1,054^x$	219,7	21,2	13,2
pow	0,647	22,54	$y = 22,54 * x^{0,647}$	180,92	24,8	
lin	-3	130	$y = 130 + -3*x$	10	40	
exp	0,965	142,86	$y = 142,86 * 0,965^x$	34,3	74,56	-19,4
pow	-0,515	327,02	$y = 327,02 * x^{-0,515}$	49	877,72	

PerNumbers and Fractions

Unit-numbers have 1 unit, per-numbers have 2 or %. Per-numbers and fractions come from double-counting.	Per-numbers must convert to unit-numbers before being added (weighted average, conditional probability).
--	--

The ReCount-formula $T = (T/B)*B$ predicts a counting result: 8 recounted in 2s gives: $8 = (8/2)*2 = 4*2$.

The ReCount-formula solves equations: $u*5 = 40 = (40/5)*5$, so $u = 40/5$ (move to opposite side with opposite sign).

Double-counting (proportionality) occurs in many places. A commodity may be double-counted as 4 kg and \$5kr thus giving the per-number 4 kg per \$5\$, or 4 kg/5 \$, or 4/5 kg/\$. Typical questions are: 60 kg = ? \$ and 60\$ = ? kg

Method 1, recounting numbers	$60 \text{ kg} = (60/4) * 4 \text{ kg} = (60/4) * 5 \$ = 75 \$$	$60\$ = (60/5)*5\$ = 60/5*4\text{kg} = 48\text{kg}$
Method 2, recounting units	$\$ = (\$/\text{kg})*\text{kg} = 5/4 * 60 = 75$	$\text{kg} = \text{kg}/\$ * \$ = 4/5 * 60 = 48$

Per-numbers as fractions and percent

With like units, per-numbers become fractions or percentages: $4\text{kg}/5\text{kg} = 4/5$, $4\text{kg}/100\text{kg} = 4/100 = 4\%$

40% of 30\$ = u\$	30\$ is u% of 75\$	30\$ is 40% of u\$
$30 = (30/100)*100$ giving $(30/100)*40 = 12$	$100 = (100/75)*75$ giving $(100/75)*30 = 40$	$100\% = (100/40)*40\%$ giving $(100/40)*30\$ = 75\$$

Adding per-numbers and fractions

Example 1. A big volumes may infer a discount: The price is 6\$/kg for the first 5kg, then it is 4\$/kg for the next 3kg.

$T1 = 5\text{kg at } 6 \text{ \$/kg} = 5*6 \$ = 30 \$$ $T2 = 3\text{kg at } 4 \text{ \$/kg} = 3*4 \$ = 12 \$$ $T = 8\text{kg at } u \text{ \$/kg} = 8*u \$ = 42 \$$ $8*u = 42 = (42/8)*8$, so $u = 42/8 = 5.25 \frac{\text{kr}}{\text{kg}}$	
---	--

Example 2. Fractions are operators needing a number to become a number: 3/5 of 5kg added to 2/3 of 3 kg totals what?

$T1 = 3/5 \text{ of } 5 \text{ kg} = 5*3/5*5 \text{ kg} = 3 \text{ kg}$ $T2 = 2/3 \text{ of } 3 \text{ kg} = 3*2/3*3 \text{ kg} = 2 \text{ kg}$ $T = u \text{ of } 8 \text{ kg} = u*8 \text{ kg} = 5 \text{ kg}$ $u*8 = 5 = (5/8)*8$, so $u = \frac{5}{8}$ <div style="border: 1px solid black; padding: 5px; display: inline-block; margin-left: 20px;"> dvs. $\frac{3}{5} + \frac{2}{3} = \frac{5}{8}$ </div>	
--	--

Notice: Per-numbers (and fractions) add as areas under the per-number graph, also called integral calculus.

Interest rate

$250\$ + 8\% = ?\$$. \$ + % doesn't work. But $100\% + 8\% = 108\%$, and 108% of $250\$ = 1.08*250\$ = (1+0.08)*250\$ = 270\$$.
Rate-formula: $T = b*(1+r)$, T: terminal number, b: beginning number, r: rent, 1+r: rate-factor = rate-multiplier.

The rate r% added n times: $T = b*(1+r)*(1+r)*... = b*(1+r)^n = b*(1+R)$, R: total rate, $1+R = (1+r)^n$

8% added 4 times gives $R = 1.08^4 - 1 = 0.360 = 36.0\% = 8\%*4 + 4.0\%$ = simple rate + compound rate (CR)

Continuous rate: 8% added continuously gives the rate-factor $e^{0.08} = 1.0833$, i.e. a rate of 8.33% including 0.33% as compound rate. $e = (1+1/n)^n$ for n very large.

Exercises

1. Density = 1.23 kg/l. 7.5 kg = ? l, ? kg = 34 l 2. Molar mass = 16 g/mol. 234 g = ? mol. ? g = 34.5 mol 3. Concentration = 2.4 mol/l. 34 mol = ? l. ? mol = 3.5 l 4. Price = 3.6 \$/kg. 346 \$ = ? kg. ? \$ = 234 kg 5. Speed = 23.4 m/s. 34 m = ? s. ? m = 56 s 6. 20 is ?% of 30. ? is 20% of 30. 20 is 30% of ?. 7: Rent = 80 \$/day. 200 \$ = ? days. ? \$ = 40 days 8: 40 kg = 60 l. 75 kg = ? l. ? kg = 200 l. 9: 40 \$ = 82 days. 75 kg = ? days. ? kg = 200 days. 10: 30 m = 20 s. 65 m = ? s. ? m = 800 s. 11: 60 J = 90 s. 55 J = ? s. ? J = 400 s.	12. Calculate the empty fields when $T = b*(1+r)^n$ <table border="1" style="width: 100%; text-align: center;"> <thead> <tr> <th></th> <th>T</th> <th>b</th> <th>r</th> <th>n</th> <th>R</th> <th>n*r</th> <th>CR</th> </tr> </thead> <tbody> <tr><td>1</td><td></td><td>10</td><td>3.2%</td><td>20</td><td></td><td></td><td></td></tr> <tr><td>2</td><td>34</td><td></td><td>3.2%</td><td>20</td><td></td><td></td><td></td></tr> <tr><td>3</td><td>34</td><td>10</td><td></td><td>20</td><td></td><td></td><td></td></tr> <tr><td>4</td><td>34</td><td>10</td><td>3.2%</td><td></td><td></td><td></td><td></td></tr> <tr><td>5</td><td>34</td><td></td><td>3.2%</td><td></td><td>72%</td><td></td><td></td></tr> <tr><td>6</td><td>34</td><td></td><td></td><td>20</td><td></td><td>50%</td><td></td></tr> <tr><td>7</td><td></td><td>10</td><td>3.2%</td><td></td><td>68%</td><td></td><td></td></tr> <tr><td>8</td><td></td><td>10</td><td></td><td>20</td><td></td><td></td><td>12%</td></tr> <tr><td>9</td><td>34</td><td></td><td>3.2%</td><td></td><td></td><td></td><td>23%</td></tr> </tbody> </table>		T	b	r	n	R	n*r	CR	1		10	3.2%	20				2	34		3.2%	20				3	34	10		20				4	34	10	3.2%					5	34		3.2%		72%			6	34			20		50%		7		10	3.2%		68%			8		10		20			12%	9	34		3.2%				23%
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Constant Change

Table

x	y
10	100
15	120
25	?
?	190

Change forms

Linear ++ change	Exponential +* change	Power ** change
$y = b+a*x$	$y = b*a^x$	$y = b*x^a$
x: +1 day y: +a \$ (change-number)	x: +1 day y: +r % (change-percent) x: + T, y: + 100% (or -50%) $T = \log 2 / \log a = \ln 2 / \ln a$	x: +1 % y: +a % (elasticity)

To model and forecast we transform a table into a formula by using regression to find the numbers a and b.

To use regression, first we select STATE EDIT. Then we enter the table. Now we select STATE CALC and e.g. ExpReg. Under the VARS Y-VARS we add Y1 to have the regression formula placed directly on the y-list. On the Y-list we add Y2 = 190 and turn on PLOT1 to see if the curve goes through the table points.

The questions asked can now be answered using TRACE 25 and CALC INTERSECTION respectively.

<pre> 2001 CALC TESTS 1:Edit... 2:SortA(3:SortD(4:ClrList 5:SetUpEditor </pre>	<table border="1"> <thead> <tr> <th>L1</th> <th>L2</th> <th>L3</th> <th>Z</th> </tr> </thead> <tbody> <tr> <td>10</td> <td>100</td> <td></td> <td></td> </tr> <tr> <td>15</td> <td>120</td> <td></td> <td></td> </tr> <tr> <td>---</td> <td></td> <td></td> <td></td> </tr> </tbody> </table> <p>L2(C) =</p>	L1	L2	L3	Z	10	100			15	120			---				<pre> EDIT CALC TESTS 5:QuadReg 6:CubicReg 7:QuartReg 8:LinReg(a+bx) 9:LnReg 10:ExpReg 11:PwrReg </pre>
L1	L2	L3	Z															
10	100																	
15	120																	

<pre> VARS Y-VARS 1:Function... 2:Parametric... 3:Polar... 4:On/Off... </pre>	<pre> 2001 Plot2 Plot3 \Y1=69.444*1.037 ^X \Y2=190 \Y3= \Y4= \Y5= \Y6= </pre>	<p>Intersection X=27.703 Y=190</p>																

The questions asked can also be answered using formula schemes:

$\frac{y = ?}{x = 25} \quad \frac{y = 60 + 4 \cdot x}{y = 60 + 4 \cdot 25}$ $y = 160$	$\frac{y = ?}{x = 25} \quad \frac{y = 69.44 \cdot 1.037^x}{y = 69.44 \cdot 1.037^{25}}$ $y = 172.80$	$\frac{y = ?}{x = 25} \quad \frac{y = 35.48 \cdot x^{0.450}}{y = 35.48 \cdot 25^{0.450}}$ $y = 151.03$
$\frac{x = ?}{y = 180} \quad \frac{y = 60 + (4 \cdot x)}{y - 60 = 4 \cdot x}$ $\frac{y - 60}{4} = x$ $\frac{180 - 60}{4} = x$ $30 = x$	$\frac{x = ?}{y = 180} \quad \frac{y = 69.44 \cdot (1.037^x)}{\frac{y}{69.44} = 1.037^x}$ $\log\left(\frac{180}{69.44}\right) = x \cdot \log(1.037)$ $26.21 = x$	$\frac{x = ?}{y = 180} \quad \frac{y = 35.48 \cdot x^{0.450}}{\frac{y}{35.48} = x^{0.450}}$ $0.450 \sqrt[0.450]{\frac{180}{35.48}} = x$ $36.92 = x$
Test	$180 = 69.44 \cdot 1.037^{26.21}$ $180 = 179.958 \quad \odot$	$180 = 35.48 \cdot 36.92^{0.450}$ $180 = 179.991 \quad \odot$

Exercises

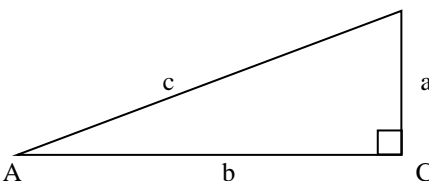
	x	y	a	b	Formula	y	x	Doubl. T
1	x	10	20	30				
	y	30	50			80		
2	x	10	15	25				
	y	100	130			180		
3	x	10	20	35				
	y	60	40			10		
4	x	10	20	40				
	y	100	70			10		

Ans:	a	b	Formula	y	x	Doubl. T
lin	2	10	$y = 10 + 2 \cdot x$	70	35	
exp	1,052	18	$y = 18 \cdot 1,052^x$	83,33	29,2	13,6
pot	0,737	5,5	$y = 5,5 \cdot x^{0,737}$	67,41	37,84	
lin	6	40	$y = 40 + 6 \cdot x$	190	23,33	
exp	1,054	59,17	$y = 59,17 \cdot 1,054^x$	219,7	21,2	13,2
pot	0,647	22,54	$y = 22,54 \cdot x^{0,647}$	180,92	24,8	
lin	-2	80	$y = 80 - 2 \cdot x$	10	35	
exp	0,96	90	$y = 90 \cdot 0,96^x$	21,77	54,19	-17,1
pot	-0,585	230,74	$y = 230,74 \cdot x^{-0,585}$	28,83	213,92	
lin	-3	130	$y = 130 - 3 \cdot x$	10	40	
exp	0,965	142,86	$y = 142,86 \cdot 0,965^x$	34,3	74,56	-19,4
pot	-0,515	327,02	$y = 327,02 \cdot x^{-0,515}$	49	877,72	

Trigonometry

Any land can be divided in triangles Any triangle can be divided into right-angled triangles	Two Greek Formula: $A+B+C = 180$ $a^2 + b^2 = c^2$ Three Arabic Formula: $\sin A = \frac{a}{c}$ $\cos A = \frac{b}{c}$ $\tan A = \frac{a}{b}$
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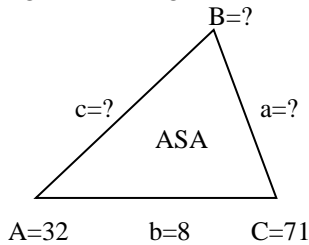
A triangle is defines by 3 pieces. The rest can be predicted by formulas. The Greeks only found two formulas, so trigonometry first was developed when the Arabs added three extra formulas.

	<p>Greek formulas $A+B+C = 180$ $a^2 + b^2 = c^2$ (Pythagoras) Arabic formulas: $\sin A = \frac{a}{c}$ (height in % of c) $\cos A = \frac{b}{c}$ (base in % of c) $\tan A = \frac{a}{b}$ A right triangle can be seen as a rectangle halved by a diagonal.</p>
---	---

In a non right-angled triangle the sine and cosine formulas have to be extended:

<p>The Sine Rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$</p>	<p>The Cosine Rule (The Extended Pythagoras) $a^2 = b^2 + c^2 - 2*b*c*\cos A$ $b^2 = a^2 + c^2 - 2*a*c*\cos B$ $c^2 = a^2 + b^2 - 2*a*b*\cos C$</p>
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Angle-Side-Angle

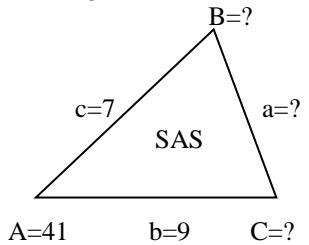


$B = ?$	$A+B+C=180$
$A=32$	$B=180-A-C$
$C=71$	$B=180-32-71$
	$B=77$

$a = ?$	$\frac{a}{\sin A} = \frac{b}{\sin B}$
$b=8$	$a = \frac{b*\sin A}{\sin B}$
$A=32$	$a = \frac{8*\sin 32}{\sin 77}$
$B=77$	$a = 4.351$

$c = ?$	$\frac{c}{\sin C} = \frac{b}{\sin B}$
$b=8$	$c = \frac{b*\sin C}{\sin B}$
$C=71$	$c = \frac{8*\sin 71}{\sin 77}$
$B=77$	$c = 7.763$

Side-Angle-Side



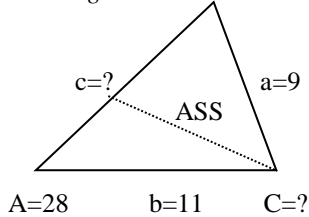
$a = ?$	$a^2 = b^2 + c^2 - 2*b*c*\cos A$
$b=9$	$a^2 = 9^2 + 7^2 - 2*9*7*\cos 41$
$c=7$	$a = \sqrt{34.907}$
$A=41$	$a = 5.908$

$B = ?$	$b^2 = a^2 + c^2 - 2*a*c*\cos B$
$b=9$	$\cos B = \frac{a^2 + c^2 - b^2}{2*a*c}$
$c=7$	$\cos B = \frac{5.9^2 + 7^2 - 9^2}{2*5.9*7}$
$a=5.9$	$B = \cos^{-1}(0.035) = 88.0$

$C = ?$	$A+B+C=180$
$A=41$	$C=180-A-B$
$B=88$	$B=180-41-88$
	$B=51$

Angle-Side-Side

The ambiguous case

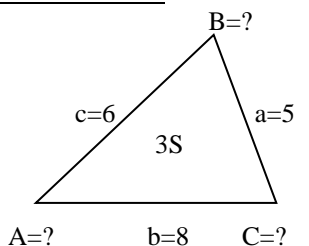


$B = ?$	$\frac{\sin B}{b} = \frac{\sin A}{a}$
$b=11$	$\sin B = \frac{b*\sin A}{a}$
$A=28$	$\sin B = \frac{11*\sin 28}{9}$
$a=9$	$B = \sin^{-1}(0.574)$
	$B = [35, 145]$

$C = ?$	$A+B+C=180$
$A=28$	$C=180-A-B$
$B=35$	$C=180-28-35$
or	$C=117$
$B=145$	or
	$C=180-28-145$
	$C=7$

$c = ?$	$\frac{c}{\sin C} = \frac{a}{\sin A}$
$a=9$	$c = \frac{a*\sin C}{\sin A}$
$A=28$	$c = \frac{9*\sin 117}{\sin 28}$
or	$c = 17.081$
$C=7$	or
	$c = 2.336$

Side-Side-Side



$a = ?$	$a^2 = b^2 + c^2 - 2*b*c*\cos A$
$a=5$	$\cos A = \frac{b^2 + c^2 - a^2}{2*b*c}$
$b=8$	$\cos A = \frac{8^2 + 6^2 - 5^2}{2*8*6}$
$c=6$	$A = \cos^{-1}(0.781)$
	$A = 38.6$

$b = ?$	$b^2 = a^2 + c^2 - 2*a*c*\cos B$
$a=5$	$\cos B = \frac{a^2 + c^2 - b^2}{2*a*c}$
$b=8$	$\cos B = \frac{5^2 + 6^2 - 8^2}{2*5*6}$
$c=6$	$B = \cos^{-1}(-0.05)$
	$B = 92.9$

$C = ?$	$A+B+C=180$
$A=38.6$	$C=180-A-B$
$B=92.9$	$C=180-38.8-92.9$
	$C=48.5$

Statistics, Stochastic Variation

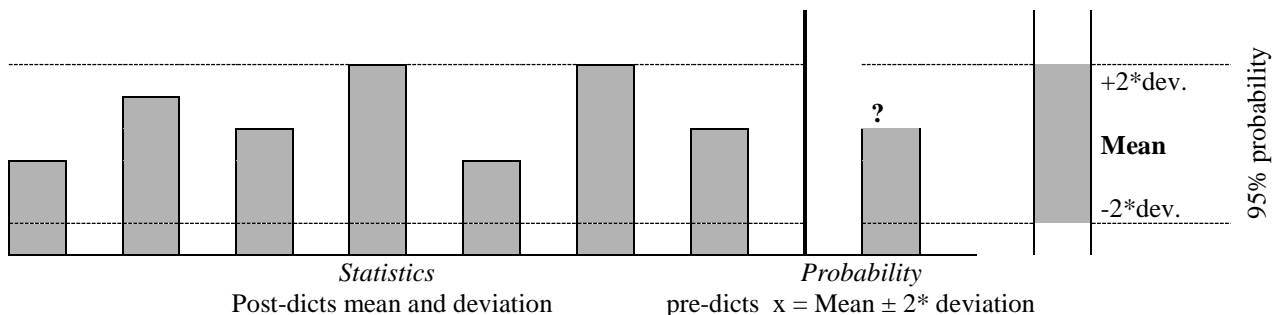
Numbers may be predictable or unpredictable. Unpredictable numbers are also called random or stochastic numbers. Numbers that cannot be pre-dicted can often be post-dicted by setting up a statistics on their former behavior. A statistical table contains two columns, one with the numbers and one with their frequencies.

If arranged in increasing order:

The median = the middle observation, 1. (3.) quartile = the middle observation in the 1. (2.) half.

A histogram shows the frequencies

An ogive shows the cumulated frequencies from which the 3 quartiles can be read and reported as a box-plot.

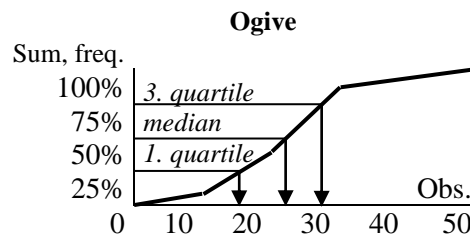


1. Observations

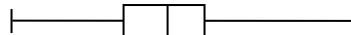
x: 10, 12, 22, 12, 15, ...

2. Grouping and counting frequencies

Observations	Frequency	Rel. Freq.	Sum. freq.
x	h	p	$\sum p$
0-10	3	$3/40=0.075$	0.075
10-20	12	0.300	0.375
20-30	18	0.450	0.825
30-50	7	0.175	1.000
Total	40	1.000	

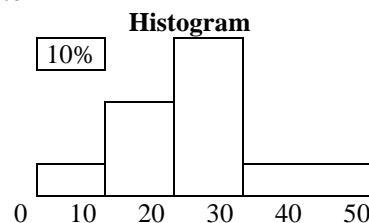


A **Boxplot** contains the median and two quartiles and the least and greatest observation



3. Mean or average: IF all the observations were the same ... however, the deviate

Observations	Frequency	Rel. Freq.	Sum. freq.	Mean
x	h	p	$\sum p$	$\mu = \sum xi * pi$
0-10	3	$3/40=0.075$	0.075	$5 * 0.075 = 0.375$
10-20	12	0.300	0.375	4.5
20-30	18	0.450	0.825	11.25
30-50	7	0.175	1.000	7
Total	40	1.000		23.1



4. Variance, deviation: IF all the deviations were the same ...

Observations	Frequency	Rel. Freq.	Sum. freq.	Mean	Distance	Variance
x	h	p	$\sum p$	$\mu = \sum xi * pi$	$ xi - \mu $	$v = \sum (xi - \mu)^2 * pi$
0-10	3	$3/40=0.075$	0.075	$5 * 0.075 = 0.375$	$ 5 - 23.1 = 18.13$	$18.13^2 * 0.075 = 24.64$
10-20	12	0.300	0.375	4.5	8.13	19.80
20-30	18	0.450	0.825	11.25	1.88	1.58
30-50	7	0.175	1.000	7	16.88	49.83
Total	40	1.000		23.1		$1 s^2 = 95.86$

Deviation $s = \sqrt{95.86} = 9.8$

5. Prediction: $x = \text{Mean} \pm 2 * \text{deviation} = \mu \pm 2 * s = 23.1 \pm 19.6$ Confidence-interval = [3.5 ; 42.7]

6. Using technology

On a GDC the interval midpoints are entered under STAT. Rel. frequency = freq/sum(freq). CumFreq = cumsum(freq).

Obs.	Freq	Rel. freq	CumFreq
0	2	.05	.050
1	5	.125	.175
2	9	.225	.400
3	12	.300	.700
4	8	.200	.900
5	4	.100	1.000

The different numbers can be calculated using 1-var statistics:

Mean $m = 2.8$

Standard deviation, $s = 1.3$

Confidence-interval = $m \pm 2 * s = [0.2; 5.4]$

1. quartile = 2

Median = 3

3. quartile = 4

Probability

A game has a 25% winning chance. Repeated 100 times, winning 25 times has the highest probability, which is less than not winning 25 times.	Probability predicts the unpredictable.
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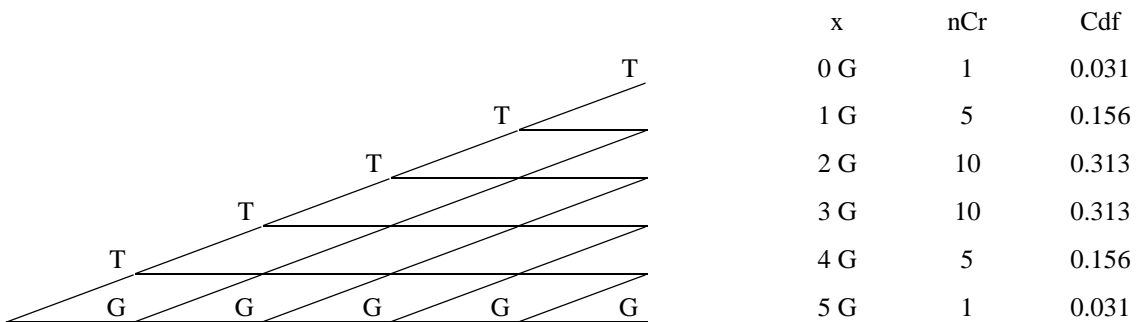
Example 1. A game with two results, Gain and Loss, is repeated 5 times. This implies 6 possible outcomes since we can win 0, 1, 2, 3, 4, 5 times. There is only 1 way to win 5 times: GGGGG, there are 5 ways to win 4 times because we can lose the 1st, 2nd, 3rd, 4th and 5th time: TGGGG, GTGGG, GGTGG, GGGTG, GGGGT. How many ways are there to win 3 times? We can count or predict the answer: the number of roads with 3 G out of 5 = $nCr(5,3) = 10$.

The 6 outcomes are equally possible, but not equally likely. The distribution obtained by repeating a 2-result experiment many times is called a **binomial distribution**. If the gain probability p is $50\% = 0.5$, then:

The probability of winning just 3 times of 5: $p(x=3) = \text{BinomCdf}(5, 0.5, 3, 3) = 0.313 = 31.3\%$.

The probability of winning 2, 3 or 4 times of 5: $p(2 \leq x \leq 4) = \text{BinomCdf}(5, 0.5, 2.4) = 0.718 = 71.8\%$

Setting up a statistics of the different outcome, we can calculate the mean and deviation. However, these numbers may be predicted by the formulas $m = n \cdot p$ and $s = \sqrt{n \cdot p \cdot (1-p)}$, where n is the number of repetitions and p is the winning chance.



Example 2. Repeating a game with a $2/3$ winning chance 30 times will give the mean value $m = n \cdot p = 30 \cdot 2/3 = 20$. The deviation is $s = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{20 \cdot 1/3} = 2.6$. The confidence interval is $20 \pm 2 \cdot 2.6 = [14.8; 25.2]$.

So there is approximately 95% chance that the next repetition will result in winning between 15 and 25 times.

If we only win 12 times, we must consider rejecting the hypothesis about a $2/3$ winning chance.

Normal distribution

With many repetitions, a binomial distribution will approach the normal distribution, which often occurs in nature where there usually will be some variation of, for example, animal height. In such cases, the random variables x may assume decimal values or negative values.

Example 3. Repeating a game with a 60% winning chance 20,000 times will give the mean value $m = n \cdot p = 20000 \cdot 0.60 = 12000$, and deviation $s = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{12000 \cdot 0.40} = 69.3$.

Binomial distribution: $P(0 < x < 12123) = \text{BinomCdf}(20000, 0.60, 0, 12122) = 0.962 = 96.2\%$

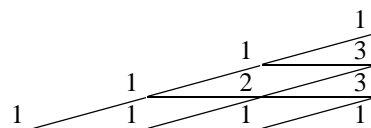
Normal distribution: $P(0 < x < 12123) = \text{NormCdf}(0, 12123, 12000, 69.3) = 0.962 = 96.2\%$

A normal distribution will have a straight sum-curve on a normal paper.

Exercises

- Describe example 1 with $p = 30\%$?
- Describe example 1 with 6 repetitions
- Describe the example 1 with 8 repetitions.
- 'when $(\text{RAND}() < 0.7, 1.0)$ ' is an game with $p = 0.7$. Combined with itself 5 times is equivalent to performing the experiment in example 1. Perform the game 32 times and make a statistics on the outcome.
- Describe example 1 with 200 repetitions and different p -values. Compare responses from the binomial and the normal distribution.
- In a population, weight is normal distributed with mean value 13.2 and deviation 2.4. What is the probability of not exceeding 12.6? at least 13.5? between 13.0 and 14.0?

- In a population height is normal distributed with mean value 132 and spread 24. What is the probability of a maximum of 126? at least 135? between 130 and 140?
- Arrange the nCr numbers systematically in the triangle below to create a 'Pascal's triangle'. Which features have this triangle? Who was Pascal?



Polynomials and Calculus

0. degree polynomial tells the (initial) point	$y = 5$
1. degree polynomial tells the (initial) gradient or steepness	$y = 5 + 2*x$
2. degree polynomial tells the (initial) bending	$y = 5 + 2*x + 0.3*x^2$
3. degree polynomial tells the (initial) counter-bending	$y = 5 + 2*x + 0.7*x^2 - 0.2*x^3$
4. degree polynomial tells the (initial) counter-counter-bending	$y = 5 + 2*x + 0.7*x^2 - 0.2*x^3 + 0.3*x^4$

Arabic numbers are polynomials: $4352 = 4*10^3 + 3*10^2 + 5*10 + 2$. General form: $y = 4*x^3 + 3*x^2 + 5*x + 2$

Polynomials with bending graphs (degree over 1) have some interesting points:

Turning points, either top-points (maximum) or bottom-points (minimum).

Intersections with the x-axis (zeros), with the y-axis (y-intercept), or with other graphs.

Intersecting other graphs (equations graphically), Intersecting vertical lines (tracing values).

Shifting bending or curvature, where the bending changes its sign.

Tangent-point. A tangent is a straight line practically coinciding with the graph around the contact point, thus showing a scenario: this is how the graph would look like if the steepness stays constant.

If the curve graphs per-numbers, the total is found as the area under the per-number graph, i.e. by integration

If the curve graphs a Total, the per-numbers are found as the steepness of the total graph, i.e. by differentiation

Differentiation twice gives the bending, being positive when bending upwards and negative when bending downwards.

Finding the steepness (gradient, slope) formula is called differentiation. Finding the area under a curve is called integration. Together differentiation and integration are inverse operations called Calculus

Example: $y = 0.5x^3 - 3x^2 + 2x + 3$

	Graphics	Formula	
Intersecting the y-axis CALC value	$y = 3$	$y1(0)$	
Intersecting the x-axis CALC zero	$x = -0.694$ $x = 1.748$ $x = 4.946$	Solve(0=Y1)	
Intersecting $y = 2$ CALC intersection	$x = -0.329$ $x = 1.181$ $x = 5.147$	Solve(0=Y1-2)	
Top CALC Maximum	$x = 0.367$ $y = 3.355$	MATH $fMax(Y1,x,0,7)$	
Bottom CALC Minimum	$x = 3.633$ $y = -5.355$	MATH $fMin(Y1,x,0,7)$	
Steepness in $x = 4$ CALC dy/dx	2	MATH $nDeriv(Y1,x,4)$	
Area from 3 to 4 CALC $\int f(x)dx$	-5.125	MATH $fnInt(Y1,x,3,4)$	
Tangent in $x = 1$ DRAW tangent $x = 1$	$y = -2.5x + 5$		

Exercises

1. Repeat as above with $y = 0.7x^3 - 4x^2 + 3x + 4$.	10. Find the cheapest pipe with double lid contain. 1 liter.
2. Repeat as above with $y = -0.4x^3 + 2x^2 - 0.5x - 3$.	11. Find the cheapest cone without lid containing 1 liter.
3. Produce your own polynomials using randM(4,1).	12. Find the cheapest cone with lid containing 1 liter.
4. Produce your own polynomials using regression	13. Find the cheapest cone with double lid contain. 1 liter.
5. Find the cheapest box without lid containing 1 liter.	14. $y1$ is a polynomial of degree 0. If $y1$ is a Total, what is its per-number? If $y1$ is a per-number, what is its total?
6. Find the cheapest pipe without lid containing 1 liter.	15. As 14 with polynomials of degree 1.
7. Find the cheapest box with lid containing 1 liter.	16. As 14 with polynomials of degree 2.
8. Find the cheapest pipe with lid containing 1 liter.	17. As 14 with polynomials of degree 3.
9. Find the cheapest box with double lid containing 1 liter.	

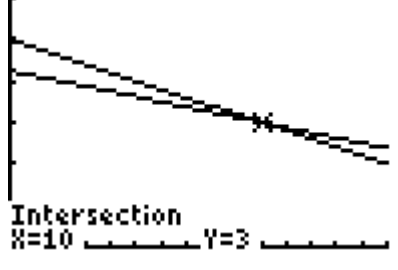
Two equations With Two Unknowns; and Three

Two equations with two unknowns are solved manually, by intersection or by matrices	b \$ + 5kg at a \$/kg = 25 \$ b \$ + 8kg at a \$/kg = 34 \$	$x + 5*y = 25$ $x + 8*y = 34$	$\begin{pmatrix} 1 & 5 \\ 1 & 8 \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 25 \\ 34 \end{pmatrix}$
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2 equations with 2 unknowns: The formula b \$ + 5kg at a \$/kg = 25\$ contains 2 unknowns and cannot be solved, unless we know another example of the same formula as e.g. b \$ + 8kg at a \$/kg = 34 \$.

Written as an equation system	Written as a matrix equation
$x + 5*y = 25$ $x + 8*y = 34$	$\begin{pmatrix} 1 & 5 \\ 1 & 8 \end{pmatrix} * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 25 \\ 34 \end{pmatrix}$

Manually one variable is isolated in the first and inserted in the second equation: $x=25-5*y$, $25-5*y+8*y=34$, $y=3$ & $x=10$.

<p>Using graphs, the y's are isolated and inserted into the y- editor.</p> <p>$x + 5*y = 25$ gives $y = (25-x)/5$,</p> <p>$x + 8*y = 34$ gives $y = (34-x)/8$</p> <p>The intersection point is found by 'Calc Intersection' to $x = 10$ and $y = 3$.</p> <p>Also we can use Math Solver $0 = Y1 - Y2$.</p>		<p>EQUATION SOLVER</p> <p>eqn: $0 = Y1 - Y2$</p>
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Matrix-solutions is found by entering the matrices into the matrix-editor as ml and mr (matrix-left & -right):

$\underline{V} = \begin{pmatrix} x \\ y \end{pmatrix} = ?$	$\underline{ml} * \underline{V} = \underline{mr}$	$\underline{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ?$	$\underline{ml} * \underline{V} = \underline{mr}$
$\underline{ml} = \begin{pmatrix} 1 & 5 \\ 1 & 8 \end{pmatrix}$	$\underline{V} = \underline{ml}^{-1} * \underline{mr}$	$\underline{ml} = \begin{pmatrix} 2 & 5 & 2 \\ 1 & 0 & -1 \\ 4 & -3 & 6 \end{pmatrix}$	$\underline{V} = \underline{ml}^{-1} * \underline{mr}$
$\underline{mr} = \begin{pmatrix} 25 \\ 34 \end{pmatrix}$	$\underline{V} = \begin{pmatrix} 1 & 5 \\ 1 & 8 \end{pmatrix}^{-1} * \begin{pmatrix} 25 \\ 34 \end{pmatrix}$	$\underline{mr} = \begin{pmatrix} 18 \\ -2 \\ 16 \end{pmatrix}$	$\underline{V} = \begin{pmatrix} 2 & 5 & 2 \\ 1 & 0 & -1 \\ 4 & -3 & 6 \end{pmatrix}^{-1} * \begin{pmatrix} 18 \\ -2 \\ 16 \end{pmatrix}$
Test	$\begin{pmatrix} 1 & 5 \\ 1 & 8 \end{pmatrix} * \begin{pmatrix} 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 25 \\ 34 \end{pmatrix}$	Test	$\begin{pmatrix} 2 & 5 & 2 \\ 1 & 0 & -1 \\ 4 & -3 & 6 \end{pmatrix} * \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \\ 16 \end{pmatrix}$
	$\begin{pmatrix} 25 \\ 34 \end{pmatrix} = \begin{pmatrix} 25 \\ 34 \end{pmatrix}$		$\begin{pmatrix} 18 \\ -2 \\ 16 \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \\ 16 \end{pmatrix}$

3 equations with 3 unknowns cannot be solved graphically, but manually and by using matrices:

Written as an equation system	Written as a matrix equation
$3*x + 5*y + 2*z = 19$ $x - z = -2$ $4*x - 3*y + 6*z = 16$	$\begin{pmatrix} 3 & 5 & 2 \\ 1 & 0 & -1 \\ 4 & -3 & 6 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19 \\ -2 \\ 16 \end{pmatrix}$

A matrix-solution is found by entering the matrices into the matrix-editor as ml and mr.

4 equations with 4 unknowns, 5 equations with 5 unknowns etc. Like 3 equations with 3 unknowns.

Equation systems for skill building can be generated by 'randM(3,3)' and 'randM(3,1).

Exercises. Solve the equation systems

1. $4x - 1*y = -9$ $4x - 4*y = 0$	5. $-7*x - 3*y - 7*z = 3$ $-1*x - 5*y + 1*z = -13$ $9*y - 5*z = 36$	8. $2*x + 5*y - 1*z + 9t = 118$ $1*x + 1*y - 9*z - 5t = -88$ $-3*y + 7*z + 5t = -51$ $-3*x + 5*y + 2*z - 5t = -10$
2. $4x + 2*y = 16$ $5x - 3*y = -2$	6. $4*x + 3*y + 7*z = 81$ $5*x + 3*y + 1*z = 54$ $2*x + 9*y + 5*z = 57$	9. $-6*x - 1*y + 8*z + 8t = 129$ $-2*x + 2*y - 5*z + 7t = 60$ $8*x + 6*y + 3*z + 3t = -40$ $-7*x - 4*y - 8*z - 4t = 12$
3. $7x + 4*y = -1$ $-3x + 2*y = 19$	7. $2*x + 3*y - 1*z = -6$ $5*x + 3*y - 4*z = -15$ $2*x - 2*y + 5*z = 40$	
4. $2x - 5*y = 16$ $3x - 4*y = 17$		

Coordinate Geometry

Coordinate Geometry coordinates geometry and algebra so that equations may be translated into geometry and vice versa. Coordinate geometry allows for computer graphics.	
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In geometry, a figure is composed of points. In coordinate geometry, a figure is placed in a coordinate system assigning to each point two coordinates x and y, where x is the horizontal forward-number, and y the vertical up-number (some calculators use t instead of y). Algebraic formulas allows predicting geometric lengths, intersection points, etc.

Given two points P (x1,t1) = (4.3) and Q (x2, t2) = (8.5) (, for technical reasons)

The distance d between P and Q.	$d = \sqrt{(x1-x2)^2 + (t1-t2)^2}$	'd = $\sqrt{(x1-x2)^2 + (t1-t2)^2}$ t1=3 and t2=5 and x1=4 and x2=8' gives 'd=4.47'
The gradient or slope between a P and Q.	$a = \frac{t1-t2}{x1-x2}$	'a = $\frac{t1-t2}{x1-x2}$ t1=3 and t2=5 and x1=3 and x2=8' gives 'a=0.5'.
The equation of the line between P and Q.	$t = t1 - a(x1-x)$	'solve(a = $\frac{t1-t}{x1-x}$, t) t1=3 and x1=4 and a=0.5' gives 't = 0.5*x + 1'
Midpoint coordinates	$(x_m, y_m) = (\frac{x1+x2}{2}, \frac{t1+t2}{2})$	'(x_m, y_m) = $(\frac{x1+x2}{2}, \frac{t1+t2}{2})$ t1=3 and t2=5 and x1=3 and x2=8' gives '(x_m, y_m) = (6,4)
Gradient c of the normal line	$a*c = -1$	'solve(a = $\frac{t1-t}{x1-x}$, t) t1=4 and x1=6 and a=-1/0.5' gives 't = -2*x + 16' (eq. of the midpoint normal)
The line angle v from horizontal	$\tan(v) = a$	'solve(tan(v)=a,v) a=0.5 and v<90 and -90<v' gives 'v=26.6'
Distance d from point (x1,t1) = (3,8) to the line t = ax+b	$d = \frac{\text{abs}(t1-ax1-b)}{(1+a^2)^{0.5}}$	'd = $\frac{\text{abs}(t1-ax1-b)}{(1+a^2)^{0.5}}$ a=0.5 and b=1 and t1=8 and x1=2' gives 'd=5.367'
Equation of the circle with center (x1,t1) and radius r	$(x-x1)^2 + (t-t1)^2 = r^2$	'(x-x1)^2 + (t-t1)^2 = r^2 t1=4 and x1=6 and r=2' gives 't^2 - 8t + x^2 - 12x + 52 = 4'
Intersection points	Solve(lign1 and lign2,x)	Solve (x^2+t^2=1 and t=2x+1,x)' gives (x,t) = (-0.8, -0.6) og (x,y) = (0,1)

The predictions may be tested by drawing the geometric figures on paper, or on the calculator's Cabri program.

Turning. An arrow (vector) may be turned v degrees around its starting point by a turning matrix: $\begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}$

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is turned 78 degrees to the vector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 78 & -\sin 78 \\ \sin 78 & \cos 78 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 78 \\ \sin 78 \end{pmatrix} = \begin{pmatrix} 0.208 \\ 0.978 \end{pmatrix}$.

Turning			
Angle:	Turning matrix		
78	$\begin{pmatrix} 0,208 & -0,978 \\ 0,978 & 0,208 \end{pmatrix}$		
	Point 1 Point 2		
Before	x	0 1	
	y	0 0	
After	x	0 0,208	
	y	0 0,978	
			<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;"> <p>BEFORE</p> </div> <div style="text-align: center;"> <p>AFTER</p> </div> </div>

Exercises

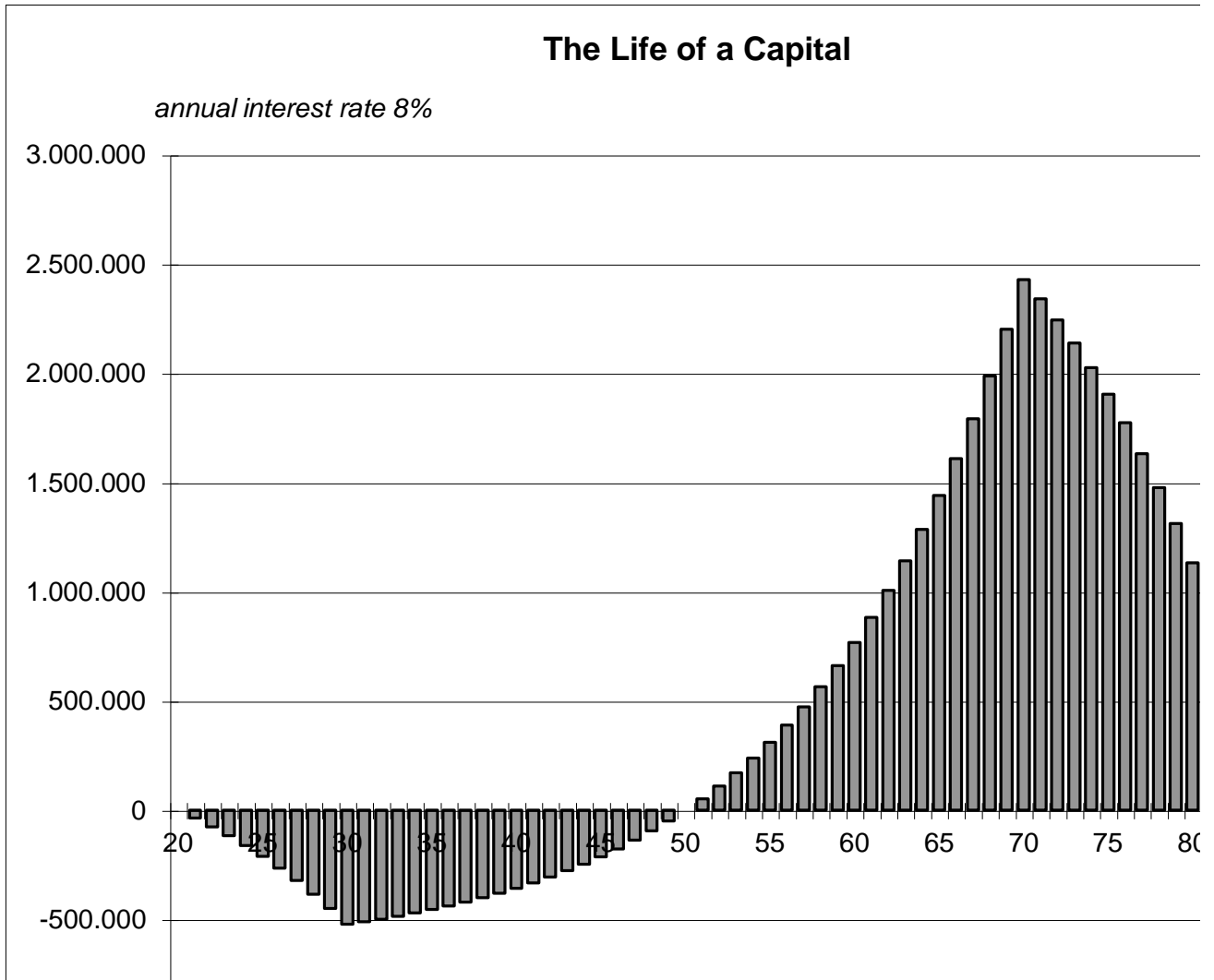
<p>1. Geometry, to measure land. Two persons draw a 10x10cm square. Select each a capital away from the center by rolling two dices. Split the country. How much does each get? Test the predictions.</p> <p>2. The same game with 3 persons.</p> <p>3. Set up a triangle ABC with coordinates chosen randomly, e.g. from a 'randMat (1,2) generator. Find the equations of the median lines. Where do they intersect?</p>	<p>4. The same as 3, now with heights.</p> <p>5. The same as 3, now with angle bisectors.</p> <p>6. The same as 3, now with bisectors.</p> <p>Find the new coordinates of A and B and C when</p> <p>7. Turning the triangle 60 degrees around A.</p> <p>8. Moving the triangle 2 forward, and 3 up.</p> <p>9. Turning the triangle 30 degrees, and moving it 3 back, and 4 down.</p>
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The Life of a Capital

During the lifetime, a person's capital conditions will change. In the following example, life is divided into four periods:

- 1) 20-30 years: student loans at \$36,000 per year for 10 years.
- 2) 30-50 years: debt settlement, where the student loan is settled by paying \$53,000 per year for 20 years.
- 3) 50-70 years: building a capital by continuing depositing \$53,000 per year for 20 years
- 4) 70-85 years: pension, where assets are settled by paying \$284,000 per year for 15 years.

The example assumes an interest rate at 8% per year. Other examples may be set up with different interest rates.



Compound interest

A constant monthly interest rate will doubling-time on T months. Counting in doublings-times, we look at an account A_0 receiving 1 unit at time 0. At time 1, the interest of A_0 is transferred to account A_1 . At time 2, the interest of A_0 is transferred to account A_1 , and the interest of A_1 is transferred to account A_2 , etc. What do you observe?

A10											
A9											
A8											
A7											
A6											
A5											
A4					1						
A3				1	$1+3=4$						
A2			1	$1+2=3$	$3+3=6$						
A1		1	$1+1=2$	$2+1=3$	$3+1=4$						
A0	1	1	1	1	1						
Time	0	1	2	3	4	5	6	7	8	9	10
Sum	1	2	4	8	16						

The Quantitative Literature: The Three Kinds of Models

Geometry and algebra are the classical quantitative literature meaning ‘measuring earth’ and ‘uniting number’s in Greek and Arabic respectively. The modern quantitative literature is mainly created by questions coming from the production: How to extract silver and coal from mines? How to navigate at sea? How to predict the orbit of a falling body? How to build machines? How to optimize a production? How to optimize profits? Etc.

Calculations took place to lift silver and water up from the mines, and how to transform crude silver into silver of varying degree of purity. Silver now travelled through Germany to Italy, and from there to the east to be exchanged with pepper and silk, either via the expensive way by land transported by caravans, or via the cheap way over sea transported by Arab merchants. Helped by replacing their own Roman numbers with Arabic numbers allowing multiplication and division to take place, Italian wealth created banks to lend out money for an interest in return. This implies a need for adding interest rates leading to the discovery of power calculations and compound interest: 7 years at 6% yearly = 42% interest rate + 8% compound interest rate = 50% since $106\% ^ 7 = 150\%$. Part of the profit goes to consumption when building magnificent palaces everywhere in Renaissance Italy, and when hiring artists.

Then Portugal took over by decreasing the price on pepper to 1/3 when skipping the Arabic middlemen by sailing silk and pepper home around Africa on their own ships. Spain then tried to find another way to India by sailing west. But in West India there was no spice or silk, but plenty of gold and silver, e.g. in the land of silver, Argentine. The Pope divided the new world between Spain and Portugal. Portugal gets everything east of the 60th longitude, Spain everything west of it. In Spain and Portugal profits went to consumption through the building of churches and monasteries and mansions. The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move? The church said ‘among the stars’. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth pulled down by a gravitational force, that changes motion thus leading to the creation of change-calculations, calculus.

Once in India, England exchanged silver for cotton to be planted in their North-American colonies bought from Spain who had no interest in land without silver. By replacing silk and pepper trade from the Far East with production of cotton in the US, England created a triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton. The profit was used for investment buying stock and establishing an industrial production. Fighting for colonies led to the second world war, leading to the creation of the computer and operation research.

The three genres: facts, fiction and fiddle

Qualitative and quantitative literature divide into three genres: facts, fiction and fiddle. In qualitative literature, the three qualitative of genres are

Fact: ‘SINCE Copenhagen is located on Zealand, THEN Copenhagen is close to water level.’

Fiction: ‘IF Copenhagen was located in the Alps, THEN Copenhagen is far from water level.’

Fiddle: ‘IF Copenhagen is located first in a sentence, THEN Copenhagen is close to water level.’

Fact

Fact models are ‘SINCE-THEN’ calculations, quantifying quantities, and calculating the calculable: ‘SINCE the price is 4 \$/kg, THEN 6 kg costs $6 * 4 = 24\$$.

‘SINCE-THEN’ calculations may also be called ‘room-calculations: ‘SINCE the room has dimensions 3x4x5, THEN the volume is $3*4*5 = 60$ ’

Fact models are re-calculated for testing: $T = 3 \text{ kg at } 4\$/\text{kg} = 3*4 \$ = 12 \$$, oops, calculation error, $T = 12 \$$.

Fact models may be wrong if omitting the units. The Mars Climate Orbiter thus failed when mixing cm and inch.

Fiction

Fiction models are ‘IF-THEN’ calculations, quantifying quantities, and calculating the incalculable: ‘IF my daily income is 4\$, THEN 6 days will give the income be $6*4 = 24\$$. But my income may vary?’

‘IF-THEN’ calculations may also be called ‘rate-calculations’: ‘IF the growth rate is 3% per year, THEN the total growth rate after 5 years be 15.9%, since $103\% ^ 5 = 115.9\%$.’ But the growth rate may vary?’

Fiction models must be supplemented by parallel scenarios: The daily income is estimated to be between 4\$ and 5\$, so in 3 days, ‘The expected income will be between $3*4\$ = 12\$$ and $3*5\$ = 15\$$.

Fiddle

Fiddle models are ‘SO-WHAT’ calculations, quantifying qualities, and calculate the incalculable: ‘If the consequence C= broken bone is set to 2 million \$, and if the probability P set to 30%, then will the risk be $R = C*P = 2 * 0.3 = 0.6$ million \$.’ But SO-WHAT? Who says that a broken leg costs 2 million \$? And who says that a general probability of breaking a leg can be measured?’

Another example: ‘If the cost of a burial ground is 10 \$/day, and the cost of a hospital bed is 10,000 \$/day, then it is cheaper to have people lying in the cemetery than at the hospital. But SO-WHAT? Should we increase the speed limit to 200 km/hour to save money?’

‘SO-WHAT’ calculations may also be called ‘risk-calculations’: ‘IF we can increase the probability of death and reduce the probability of injury, then will the risk of crossing a school road could be reduced.’ But SO-WHAT? Should we then dismantle the zebra crossing? Fiddle models must be rejected and referred to a qualitative treatment in the word-language instead of a quantitative one in the number-language.

Letter Calculation, Transposing Formulas

Change the T-formulas to a-formulas, b-formulas and c-formulas, and vice versa.

	T	a	b	c
1	$T = a + b \cdot c$	$a = T - b \cdot c$	$b = \frac{T-a}{c}$	$c = \frac{T-a}{b}$
2	$T = a - b \cdot c$	$a = T + b \cdot c$	$b = \frac{a-T}{c}$	$c = \frac{a-T}{b}$
3	$T = a + \frac{b}{c}$	$a = T - \frac{b}{c}$	$b = (T-a) \cdot c$	$c = \frac{b}{T-a}$
4	$T = a - \frac{b}{c}$	$a = T + \frac{b}{c}$	$b = (a-T) \cdot c$	$c = \frac{b}{a-T}$
5	$T = (a + b) \cdot c$	$a = \frac{T}{c} - b$	$b = \frac{T}{c} - a$	$c = \frac{T}{a+b}$
6	$T = (a - b) \cdot c$	$a = \frac{T}{c} + b$	$b = a - \frac{T}{c}$	$c = \frac{T}{a-b}$
7	$T = \frac{a+b}{c}$	$a = T \cdot c - b$	$b = T \cdot c - a$	$c = \frac{a+b}{T}$
8	$T = \frac{a-b}{c}$	$a = T \cdot c + b$	$b = a - T \cdot c$	$c = \frac{a-b}{T}$
9	$T = \frac{a}{b+c}$	$a = T \cdot (b+c)$	$b = \frac{a}{T} - c$	$c = \frac{a}{T} - b$
10	$T = \frac{a}{b-c}$	$a = T \cdot (b-c)$	$b = \frac{a}{T} + c$	$c = b - \frac{a}{T}$
11	$T = \frac{a}{b} + c$	$a = (T-c) \cdot b$	$b = \frac{a}{T-c}$	$c = T - \frac{a}{b}$
12	$T = \frac{a}{b} - c$	$a = (T+c) \cdot b$	$b = \frac{a}{T+c}$	$c = \frac{a}{b} - T$
13	$T = a \cdot b^c$	$a = \frac{T}{b^c}$	$b = \sqrt[c]{\frac{T}{a}}$	$c = \frac{\log(\frac{T}{a})}{\log b}$
14	$T = \frac{a}{b^c}$	$a = T \cdot b^c$	$b = \sqrt[c]{\frac{a}{T}}$	$c = \frac{\log(\frac{a}{T})}{\log b}$
15	$T = (a \cdot b)^c$	$a = \frac{\sqrt[c]{T}}{b}$	$b = \frac{\sqrt[c]{T}}{a}$	$c = \frac{\log T}{\log(a \cdot b)}$
16	$T = (\frac{a}{b})^c$	$a = \sqrt[c]{T} \cdot b$	$b = \frac{a}{\sqrt[c]{T}}$	$c = \frac{\log T}{\log(\frac{a}{b})}$
17	$T = (a + b)^c$	$a = \sqrt[c]{T} - b$	$b = \sqrt[c]{T} - a$	$c = \frac{\log T}{\log(a+b)}$
18	$T = (a - b)^c$	$a = \sqrt[c]{T} + b$	$b = a - \sqrt[c]{T}$	$c = \frac{\log T}{\log(a-b)}$
19	$T = a + b^c$	$a = T - b^c$	$b = \sqrt[c]{T-a}$	$c = \frac{\log(T-a)}{\log b}$
20	$T = a - b^c$	$a = T + b^c$	$b = \sqrt[c]{a-T}$	$c = \frac{\log(a-T)}{\log b}$
21	$T = a^{(b+c)}$	$a = (b+c) \sqrt{T}$	$b = \frac{\log T}{\log a} - c$	$c = \frac{\log T}{\log a} - b$
22	$T = a^{(b-c)}$	$a = (b-c) \sqrt{T}$	$b = \frac{\log T}{\log a} + c$	$c = b - \frac{\log T}{\log a}$

Homework

- In the triangle ABC, C is 90, A = 42, c = 5. Find the rest.
- In the triangle ABC, C is 90, A = 34, a = 6. Find the rest.
- In the triangle ABC, C is 90, A = 28, b = 7. Find the rest.
- In the triangle ABC, C is 90, a = 5, c = 7. Find the rest.
- In the triangle ABC, C is 90, b = 4, c = 7. Find the rest.
- In the triangle ABC, C is 90, a = 4, b = 5. Find the rest.
- In the triangle ABC, A is 32.6, b = 4.6, c = 5.2. Find the rest.
- In the triangle ABC, A is 34.8, b = 5.6, a = 7.2. Find the rest.
- In the triangle ABC, A is 42.6, B = 74.6, c = 6.2. Find the rest.
- In the triangle ABC, A is 34.8, C = 54.6, a = 5.2. Find the rest.

11. (all lin, exp & pow)		12		13		14		15		16	
x	y	x	y	x	y	x	y	x	y	x	y
2	10	3	8	1	20	10	80	12	64	3	50
7	15	7	12	5	30	20	62	18	42	12	28
9	?	9	?	9	?	30	?	25	?	20	?
?	30	?	28	?	80	?	30	?	24	?	10

- In 1993 there was 420 \$. In 1998 there was 630 \$. In 2005 there was ? \$. In ? there was 950 \$. Linear and exponential and power change.
- In 1994 there was 520 \$. In 1998 there was 630 \$. In 2004 there was ? \$. In ? there was 1250 \$. Linear and exponential and power change.
- In 1992 there was 920 \$. In 1996 there was 730 \$. In 2005 there was ? \$. In ? there was 450 \$. Linear and exponential and power change.
- In 1994 there was 720 \$. In 1998 there was 630 \$. In 2004 there was ? \$. In ? there was 250 \$. Linear and exponential and power change.
- A capital had 753 \$. increased with 20% 4 times and became ? \$. What is the doubling-time?
- A capital had 956 \$. decreased with 25% 5 times and became ? \$. What is the half-time?
- A capital had 486 \$. increased with 30% ? times and became 2345.83 \$. What is the doubling-time?
- A capital had 324 \$. decreased with 35% ? times and became 25.88 \$. What is the half-time?
- A capital had 743 \$. increased with ?% 4 times and became 2854.32 \$. What is the doubling-time?
- A capital had 896 \$. decreased with ?% 5 times and became 45.09 \$. What is the half-time?
- A capital had ? \$. increased with 50% 6 times and became 2423.83 \$. What is the doubling-time?
- A capital had ? \$. decreased with 55% 7 times and became 2.45 \$. What is the half-time?

31. (Polynomial regr.)		32		33		34		35		36	
x	y	x	y	x	y	x	y	x	y	x	y
2	10	3	8	1	20	10	60	12	74	3	9
7	30	7	5	5	30	20	120	18	22	12	28
9	35	11	12	7	35	30	30	20	43	15	8
12	?	9	?	9	?	40	70	25	41	17	14
?	30	?	28	?	10	50	?	30	?	20	?
?	turn	?	turn	?	turn	?	80	?	34	?	10
						?	turn	?	turn	?	turn

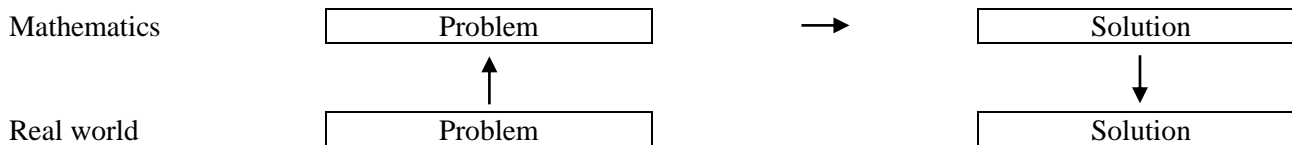
41. (Mean, ogive. & boxplot)		42		43		44		45		46	
Obs	Freq	Obs	Freq	Obs	Freq	Obs	Freq	Obs	Freq	Obs	Freq
0-10	6	0-10	50	0-10	16	0-10	16	0-10	12	0-10	23
10-20	9	10-20	20	10-20	29	10-20	29	10-20	56	10-20	45
20-30	12	20-30	10	20-30	52	20-30	32	20-30	42	20-30	25
30-40	15	30-40	20	30-40	25	30-40	45	30-40	13	30-40	12
40-50	6	40-50	30	40-50	16	40-50	56	40-50	73	40-50	86
						50-60	66	50-60	25	50-60	23
										60-70	45

- Solve the equation $2+3*(1+x)^4 = 20$
- Solve the equation $4+5*(1+x)^6 = 30$
- Solve the equation $40-3*(1-x)^4 = 20$
- Solve the equation $50-4*(1-x)^5 = 10$
- Transpose the equations $T = d - e$, $T = d - \frac{e}{f}$, $T = d - \frac{e-f}{g}$
- Transpose the equations $T = \frac{d}{e}$, $T = \frac{d}{e} - f$, $T = \frac{d-e}{f} - g$

01. Project Forecasting

Problem: How to set up a forecast assuming constant growth?

A mathematical model



1. The Real World Problem

A capital is assumed to grow constantly. From two data sets we would like to establish a forecast predicting the capital at a certain time and when a certain level will be reached.

2. The Mathematical Problem

We set up a table showing the capital to two different times. x are years, y is 1000 \$

<table border="1"> <tr><td>x</td><td>y = ?</td></tr> <tr><td>2</td><td>10</td></tr> <tr><td>5</td><td>30</td></tr> <tr><td>8</td><td>?</td></tr> <tr><td>?</td><td>60</td></tr> </table>	x	y = ?	2	10	5	30	8	?	?	60	<p>1. Linear ++ growth $y = a*x + b$</p> <p>2. Exponential +* growth $y = a*b^x = a*(1+r)^x$</p> <p>3. Power ** growth $y = a*x^b$</p>	<p>x: +1, y: +a (gradient, slope)</p> <p>x: +1, y: + r% (interest rate, $b=1+r$)</p> <p>x: +1%, y: + r% (elasticity)</p>
x	y = ?											
2	10											
5	30											
8	?											
?	60											

3. Solution to the Mathematical Problem

First we find the y-formulas using regression. We enter the table as lists L1 and L2 und STAT.

'LinReg Y1' produces a linear model transferred to the y-list as Y1

'ExpReg Y1' produces an exponential model transferred to the y-list as Y1

'PowerReg Y1' produces a power model transferred to the y-list as Y1

Linear growth	Exponential growth	Power growth																														
<table border="1"> <tr><td>y = ?</td><td>$y = 6.667*x - 3.333$</td></tr> <tr><td>Test</td><td>x = 2 and 5 gives y = 10 and 30</td></tr> <tr><td>Trace</td><td>30</td></tr> <tr><td>x = 8</td><td>$y = 6.667*8 - 3.333 = 50$</td></tr> <tr><td>Test</td><td>Trace x = 8 gives y = 50</td></tr> </table>	y = ?	$y = 6.667*x - 3.333$	Test	x = 2 and 5 gives y = 10 and 30	Trace	30	x = 8	$y = 6.667*8 - 3.333 = 50$	Test	Trace x = 8 gives y = 50	<table border="1"> <tr><td>y = ?</td><td>$y = 4.807 * 1.442^x$</td></tr> <tr><td>Test</td><td>x = 2 and 5 gives y = 10 and 30</td></tr> <tr><td>Trace</td><td>30</td></tr> <tr><td>x = 8</td><td>$y = 4.807 * 1.442^8 = 89.9$</td></tr> <tr><td>Test</td><td>Trace x = 8 gives y = 89.9</td></tr> </table>	y = ?	$y = 4.807 * 1.442^x$	Test	x = 2 and 5 gives y = 10 and 30	Trace	30	x = 8	$y = 4.807 * 1.442^8 = 89.9$	Test	Trace x = 8 gives y = 89.9	<table border="1"> <tr><td>y = ?</td><td>$y = 4.356*x^{1.199}$</td></tr> <tr><td>Test</td><td>x = 2 and 5 gives y = 10 and 30</td></tr> <tr><td>Trace</td><td>30</td></tr> <tr><td>x = 8</td><td>$y = 4.356*8^{1.199} = 52.7$</td></tr> <tr><td>Test</td><td>Trace x = 8 gives y = 52.7</td></tr> </table>	y = ?	$y = 4.356*x^{1.199}$	Test	x = 2 and 5 gives y = 10 and 30	Trace	30	x = 8	$y = 4.356*8^{1.199} = 52.7$	Test	Trace x = 8 gives y = 52.7
y = ?	$y = 6.667*x - 3.333$																															
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<table border="1"> <tr><td>x = ?</td><td>$y = 6.667*x - 3.333$</td></tr> <tr><td>y = 60</td><td>$60 = (6.667*x) - 3.333$ $60 + 3.333 = 6.667*x$ $63.333/6.667 = x$ $9.5 = x$</td></tr> <tr><td>Test1</td><td>$60 = 6.667*9.5 - 3.333$ 60 = 60</td></tr> <tr><td>Test2</td><td>MathSolver 0 = Y1-60 Gives x = 9.5</td></tr> <tr><td>Test3</td><td>CALC Intersection with y2=60 gives x = 9.5</td></tr> </table>	x = ?	$y = 6.667*x - 3.333$	y = 60	$60 = (6.667*x) - 3.333$ $60 + 3.333 = 6.667*x$ $63.333/6.667 = x$ $9.5 = x$	Test1	$60 = 6.667*9.5 - 3.333$ 60 = 60	Test2	MathSolver 0 = Y1-60 Gives x = 9.5	Test3	CALC Intersection with y2=60 gives x = 9.5	<table border="1"> <tr><td>x = ?</td><td>$y = 4.807 * 1.442^x$</td></tr> <tr><td>y = 60</td><td>$60 = 4.807 * (1.442^x)$ $60/4.807 = 1.442^x$ $\log(60/4.807)/\log 1.442 = x$ $6.89 = x$</td></tr> <tr><td>Test1</td><td>$60 = 4.807 * 1.442^{6.89}$ 60 = 60</td></tr> <tr><td>Test2</td><td>MathSolver 0 = Y1-60 Gives x = 6.89</td></tr> <tr><td>Test3</td><td>CALC Intersection with y2=60 gives x = 6.89</td></tr> </table>	x = ?	$y = 4.807 * 1.442^x$	y = 60	$60 = 4.807 * (1.442^x)$ $60/4.807 = 1.442^x$ $\log(60/4.807)/\log 1.442 = x$ $6.89 = x$	Test1	$60 = 4.807 * 1.442^{6.89}$ 60 = 60	Test2	MathSolver 0 = Y1-60 Gives x = 6.89	Test3	CALC Intersection with y2=60 gives x = 6.89	<table border="1"> <tr><td>x = ?</td><td>$y = 4.356*x^{1.199}$</td></tr> <tr><td>y = 60</td><td>$60 = 4.356*(x^{1.199})$ $60/4.356 = x^{1.199}$ $1.199\sqrt[1.199]{(60/4.356)} = x$ $8.91 = x$</td></tr> <tr><td>Test1</td><td>$60 = 4.356*8.91^{1.199}$ 60 = 60</td></tr> <tr><td>Test2</td><td>MathSolver 0 = Y1-60 Gives x = 8.91</td></tr> <tr><td>Test3</td><td>CALC Intersection with y2=60 gives x = 8.91</td></tr> </table>	x = ?	$y = 4.356*x^{1.199}$	y = 60	$60 = 4.356*(x^{1.199})$ $60/4.356 = x^{1.199}$ $1.199\sqrt[1.199]{(60/4.356)} = x$ $8.91 = x$	Test1	$60 = 4.356*8.91^{1.199}$ 60 = 60	Test2	MathSolver 0 = Y1-60 Gives x = 8.91	Test3	CALC Intersection with y2=60 gives x = 8.91
x = ?	$y = 6.667*x - 3.333$																															
y = 60	$60 = (6.667*x) - 3.333$ $60 + 3.333 = 6.667*x$ $63.333/6.667 = x$ $9.5 = x$																															
Test1	$60 = 6.667*9.5 - 3.333$ 60 = 60																															
Test2	MathSolver 0 = Y1-60 Gives x = 9.5																															
Test3	CALC Intersection with y2=60 gives x = 9.5																															
x = ?	$y = 4.807 * 1.442^x$																															
y = 60	$60 = 4.807 * (1.442^x)$ $60/4.807 = 1.442^x$ $\log(60/4.807)/\log 1.442 = x$ $6.89 = x$																															
Test1	$60 = 4.807 * 1.442^{6.89}$ 60 = 60																															
Test2	MathSolver 0 = Y1-60 Gives x = 6.89																															
Test3	CALC Intersection with y2=60 gives x = 6.89																															
x = ?	$y = 4.356*x^{1.199}$																															
y = 60	$60 = 4.356*(x^{1.199})$ $60/4.356 = x^{1.199}$ $1.199\sqrt[1.199]{(60/4.356)} = x$ $8.91 = x$																															
Test1	$60 = 4.356*8.91^{1.199}$ 60 = 60																															
Test2	MathSolver 0 = Y1-60 Gives x = 8.91																															
Test3	CALC Intersection with y2=60 gives x = 8.91																															
<p>Intersection X=9.4999 Y=60</p>	<p>Intersection X=6.8934 Y=60</p>	<p>Intersection X=8.9131 Y=60</p>																														

4. Solution to the Real World Problem

We see that forecast can be made by using technology's regression lines.

The forecasts give different answers to the same questions since different forms of growth is assumed.

Linear growth assumes that the gradient is constant.

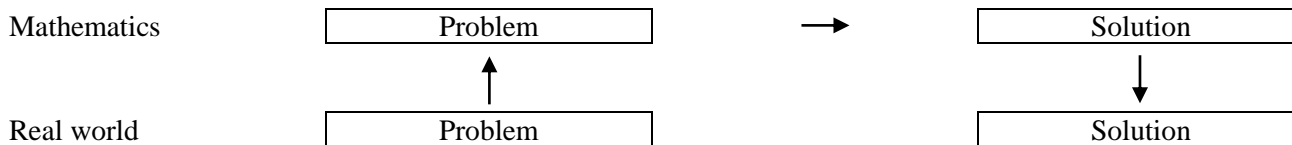
Exponential growth assumes that the interest rate is constant.

Power growth assumes that the elasticity is constant.

02. Project Population and Food Growth

Problem: When will the population exceed food supply?

A mathematical model



1. The Real World Problem

Around the year 1800, the English economist Malthus (1766-1834) predicted a future food crisis, "since the world's population grows exponentially and food supply linearly, the population will one day overtake the food supply with famine to follow" (Malthus ' principle of population). Is Malthus right?

2. The Mathematical Problem

We set up a table of time x as the number of years after 1850; and the world's population, which is assumed to be 1.59 billion in 1900 and 5.3 billion in 1990; and world food production, which is assumed to be 1,800 billion daily rations in 1900 and 4.5 billion daily rations in 1990. The population is assumed to grow exponentially, and food quantity is assumed to grow linearly. The table scope is assumed to be $0 < x < 250$.

x Years after 1850	Y1 World population in mio.	Y2 World food supply in mio. in daily rations
(1900) 50	1590	1800
(1990) 140	5300	4500

3. Solving the mathematical problem

On a TI-82, the x -numbers are on list L1, and the y -numbers are on lists L2 and L3. The formula for the population are determined by L1, L2, ExpReg Y1. The result is $y_1 = 815 * 1.013^x$. So, when x is 0 in 1850, the population is $y = 815$; and when x increases by 1, the population increases by 1.3%. The formula for the volume of food is determined by LinReg L1, L3, Y2. The result is $y_1 = 300 + 30x$. So, when x is 0 in the 1850, the food amount is $y = 300$; and when x increases by 1, the food increases by 30.

Famine occurs where Y1 is greater than Y2. Y1 and Y2 are the intersection found with 'Calc Intersection' to around $x = 30$ and 123. So, as to the model there was famine from 1850 to 1880, and again after 1973.

Assume instead that the world population is growing by 1% a year, and the amount of food with 40 per year. Then the formulas become $Y_3 = 815 * 1.01^x$, and $Y_4 = 300 + 40x$. They will intersect at approximately $x = 16$ and $x = 258$. I.e. in this case, there was famine from 1850 to 1866, and again after 2108.

Intersections points can be controlled using MathSolver with a guess.

<table border="1"> <thead> <tr> <th>L1</th> <th>L2</th> <th>L3</th> <th>3</th> </tr> </thead> <tbody> <tr> <td>50 140</td> <td>1590 5300</td> <td>1800 4500</td> <td></td> </tr> <tr> <td colspan="4">L3(3) =</td> </tr> </tbody> </table>	L1	L2	L3	3	50 140	1590 5300	1800 4500		L3(3) =				<pre> EDIT [] TESTS 4: LinReg(ax+b) 5: QuadReg 6: CubicReg 7: QuartReg 8: LinReg(a+bx) 9: LnReg 0: ExpReg </pre>	<pre> Plot1 Plot2 Plot3 \Y1 814.533*1.01 3^X \Y2 30X+300 \Y3 = \Y4 = \Y5 = \Y6 = </pre>
L1	L2	L3	3											
50 140	1590 5300	1800 4500												
L3(3) =														
<pre> WINDOW Xmin=0 Xmax=250 Xscl=1 Ymin=0 Ymax=7000 Yscl=1 Xres= </pre>														

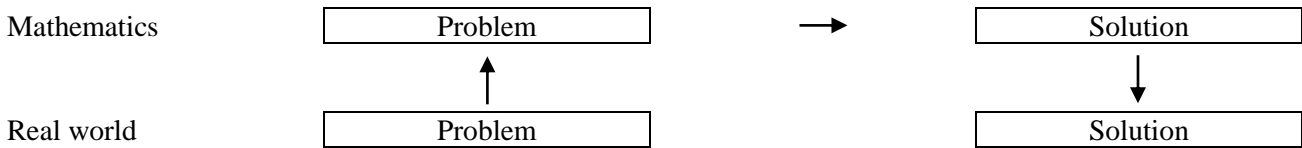
4. Solving the real world problem

Malthus is right in saying that there will be famine if the world's population continues to grow exponentially, and the quantity of food continues to grow linearly, for a curved path will always outpace a straight. If food grows by 30 million daily rations per year, then famine will occur years 1973 if the world's population is growing by 1.3% per year, and in the year 2108 if the world's population is growing by 1% per year and food production grows by 40/year. However, it will be earlier if a part of the food is used for fueling cars instead.

03. Project Saving and Pension

Problem: How much pension will a saving provide?

A mathematical model



1. The Real World Problem

A saving comes from sending in a fixed amount each month to a bank. In the end, a saving can be used to drawing out a fixed amount each month. What is the relationship between the monthly saving input and the pension output?

2. The Mathematical Problem

If paying \$1000 monthly for 30 years, what will the monthly pension be for 10 years? The interest rate is 0.4% monthly. By saving two formulas apply, the first applies to a single deposit, the second for many monthly deposits:

1) $K = K_0(1+R)$, $1+R = (1+r)^n$, K/K_0 : terminal/initial capital, r : monthly rate, R : total interest, n number of months.

2) $K/a = R/r$, K : terminal capital, a : monthly deposit, r : monthly rate, R : total interest,

3. Solving the mathematical problem

First we find the total interest rate per year R : $1+R = (1+r)^n = (1+0.004)^{12}$, so $R = 1.049 - 1 = 0.049 = 4.9\%$ per year. Then we find the total interest rate for 30 years R : $1+R = (1+r)^n = (1+0.004)^{(30*12)} = 4.209$. So. $R = 4.209 - 1 = 3.209 = 321\%$.

The simple interest rate is $30*12*0.4\% = 144\%$. So the effect of compound interest is $321\% - 144\% = 177\%$.

With $a = 1000$, $r = 0.4\%$ the saving after x deposits will be $K = a*R/r = 1000*(1.004^x - 1) / 0.004$.

We find the saving after 10, 20 and 30 years:

Months	120	240	360
Saving	153632	401675	802147

The total deposit after 30 years is $1000*360 = 360000$. The effect of compound interest is $802147 - 360000 = 442147$.

We observe that the saving is 500000 after 275 deposits.

$x = ?$ $\frac{1000*(1.004^x - 1)}{0.004} = 500000$ $1.004^x - 1 = \frac{500000*0.004}{1000}$ $1.004^x = 2+1$ $x = \frac{\ln(3)}{\ln(1.004)} = 275.2$	<p>Test 2</p>	<p>$x = 120, K = ?$</p>
<p>Test1</p> $\frac{1000*(1.004^{275.2} - 1)}{0.004} = 500000$ $499994 = 500000$		

To use the saving for a 10 years pension we use two accounts.

On the first, the saving grows from 10 years of interest to $K = K_0(1 + R) = 802147 * (1 + 0.004)^{120} = 1295089$.

The second is used for a 'negative saving' with a monthly redraw, a , that makes the two accounts balance after 10 years:

$K = a*R/r = K_0(1+R)$, giving the equation $a*(1,004^{120} - 1)/0.004 = 1295089$ that is solved by $a = 8430$.

Thus the relationship between output and input is $(10*12*8430)/(30*12*1000) = 2.8$.

Repeating the calculations with a monthly interest rate of 0.3% and 0.5% gives a different relationship:

Monthly %	Yearly %	Saving	Monthly pension	Relationship between output and input
0.3%	3.7%	646640	6425	$(10*12*6425)/(30*12*1000) = 2.1$
0.4%	4.9%	802147	8430	$(10*12*8430)/(30*12*1000) = 2.8$
0.5%	6.2%	1004515	11152	$(10*12*11152)/(30*12*1000) = 3.7$

4. Solving the real world problem

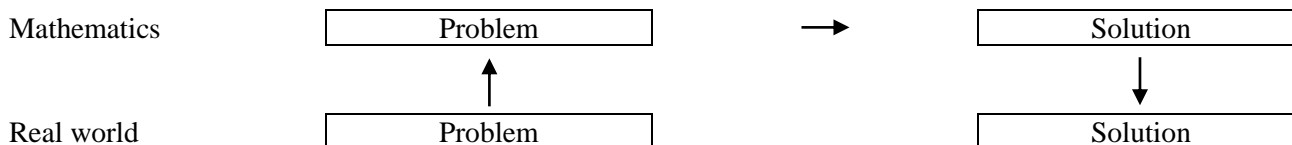
When saving, an account grows from three monthly sources: a deposit, and an interest rate of the total deposit as well as of the total interest amount. When terminated, a saving continues to grow, but, used as a pension fund, the monthly input is replaced by a monthly output, the pension. Depositing \$1000 monthly for 30 years allows taking out monthly \$8430 in 10 years. With a monthly interest rate of respectively 0.3%, 0.4% and 0.5% the output-input ratio is respectively. 2.1, 2.8 and 3.7. However, it should be remembered that 40 years of inflation will reduce this factor.

Proof of the saving-formula: Account 1 contains the amount a/r . Each month we transfer the interest amount, $r*a/r = a$, to account 2, also receiving the monthly interest of its own amount. Account 2 thus will contain a saving K , but at the same time it contains total interest amount R of account 1, i.e. $R*a/r$. Therefore $K = R*a/r$, or $K/a = R/r$.

04. Project Supply, Demand and Market Price

Problem: How does supply and demand determine the market price?

A mathematical model



1. The Real World Problem

We assume we know supply curve and the demand for a given commodity, e.g. apples. That is, we know the market price determines supply and demand. If supply is larger than demand, a decrease in price should increase demand and decrease supply. If supply is less than demand, an increase in price should decrease demand and increase supply. The equilibrium price therefore should occur where supply equals demand.

2. The Mathematical Problem

We set up a table showing the relationship between price and demand and supply. The table is supposed to be valid for prices between 0 and 10, $0 < x < 10$. Regression allows finding the two formulas, becoming an equation when set to be equal, thus finding the point of intersection that determines the equilibrium price.

Linear graphs			Bending graphs		
Price x	Supply S	Demand D	Price x	Supply S	Demand D
2	40	80	2	40	80
4	60	50	4	60	50
			6	75	30

3. Solving the mathematical problem

Entering a table to the data/matrix-editor of a graphic display calculator allows finding regression formulas.

With 2 data-sets we choose LinREg, giving a degree 1 polynomial without bending.

With 3 data-sets we choose QuadREg, giving a degree 2 polynomial with bending.

The intersection point is found graphically, or algebraically by solving two equations with two unknowns.

Degree one polynomial		Degree two polynomial	
x = ?	Supply = Demand	x = ?	Supply = Demand
S = 10x+20	10x+20 = -15x+110	S = -0.625x ² +13.75x+15	-0.625x ² +13.75x+15 = 1.25x ² -22.5x+120
D = -15x+110	10x + 15x = 110-20	D = 1.25x ² -22.5x+120	-1.875x ² +36.25x-105 = 0
	25x = 90	Faktorising	-1.875*(x-15.79)*(x-3.55) = 0
	x = 90/25 = 3.6	Zero rule	x = 15.79 og x = 3.55
Test1	y1(x) x=3.6 gives y=56		15.79 is outside the validity area
	y2(x) x=3.6 gives y=56	Test1	y1(x) x=3.55 gives y=55.91
Test2	Solve(y1(x)=y2(x),x) gives x=3.6		y2(x) x=3.55 gives y=55.91
Test3	Graphical reading gives (x,y)=(3.6,56) (intersection)	Test2	Solve(y1(x)=y2(x),x) gives x=3.55
		Test3	Graphical reading gives (x,y)=(3.55,55.91) (intersection)

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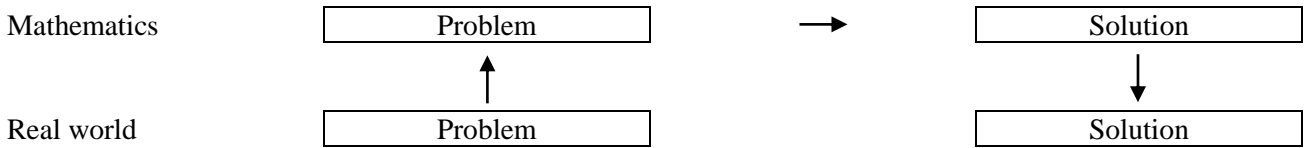
4. Solving the real world problem

We see that with linear supply and demand curves, the equilibrium price is \$3.6 resulting in an equilibrium level at 56 units. And we see that with bending supply and demand curves, the equilibrium price is \$3.55 resulting in an equilibrium level at 55.9 units. The solution assumes that the tables are unchanged. If changed, the regression formulas will change accordingly, and so will the solutions.

05. Project Collection, Laffer Curve

Problem: How Which ticket price will optimize a collection income?

A mathematical model



1. The Real World Problem

We want to collect a charity fund among the school's 500 students by selling tickets at a fixed price. Which of the following three collection models will provide the highest contribution?

A) No marketing. B) Marketing. C) Marketing and lottery.

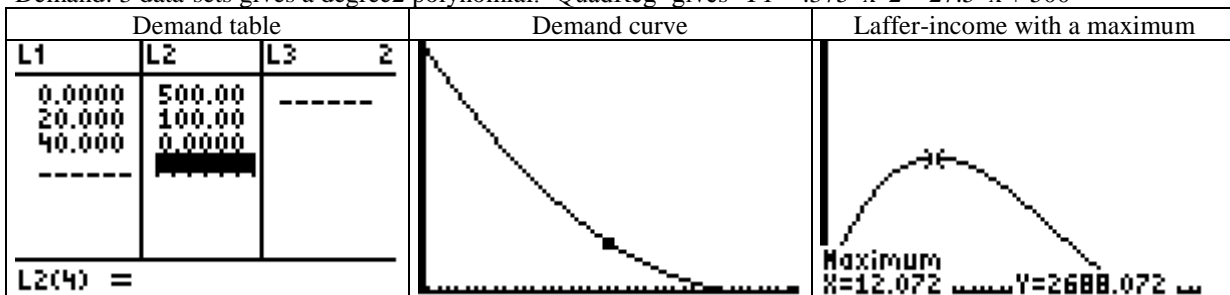
2. The Mathematical Problem

The demand Y_1 will depend on the price x . The collected fund then will be $Y_2 = Y_1 * x$.

3. Solution to the Mathematical Problem

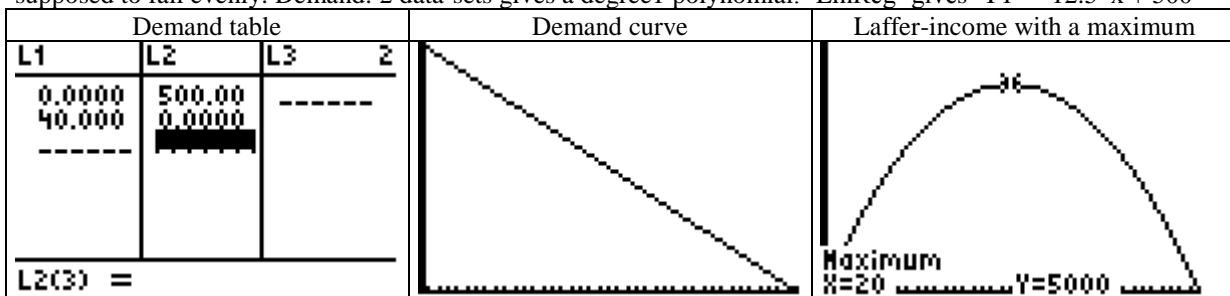
Model A. We assume that all 500 customers will buy a ticket at the price \$0, that no one will provide over \$40; and that demand is falling rapidly as only 100 customers will give \$20.

Demand: 3 data-sets gives a degree2 polynomial. 'QuadReg' gives $Y_1 = .375 * x^2 - 27.5 * x + 500$



Test: Calc $dy/dx \approx 0$ in $x = 12.07$. $dy/dx =$ gradient, slope

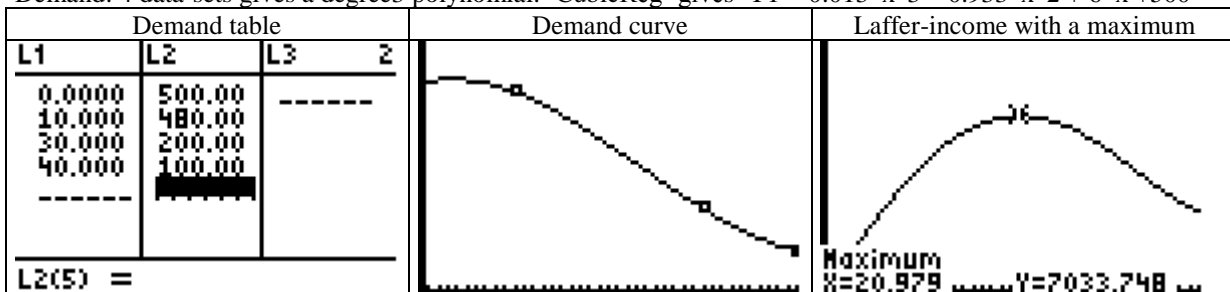
Model B. We assume that with marketing, 500 will buy a ticket at \$0, no one will give more than \$40, demand is supposed to fall evenly. Demand: 2 data-sets gives a degree1 polynomial. 'LinReg' gives $Y_1 = -12.5 * x + 500$



Test: Calc $dy/dx \approx 0$ in $x = 20$. $dy/dx =$ gradient, slope

Model C. Here marketing includes a lottery with a Grand Prize \$500 and 3 extra prizes of \$200. We assume this will result in all 500 customers will buy a ticket at the price \$0, 480 customers will give \$10, 400 customers will give \$20, 200 customers will give \$30 and 100 customers will give \$40.

Demand: 4 data-sets gives a degree3 polynomial. 'CubicReg' gives $Y_1 = 0.013 * x^3 - 0.933 * x^2 + 6 * x + 500$



Test: Calc $dy/dx \approx 0$ in $x = 21.28$. $dy/dx =$ gradient, slope

4. Solution to the Real World Problem

Collection without marketing will provide an income of \$2688 at a ticket price of \$12.

Marketing without lottery will give an income of \$5000 at a price at of \$20.

Marketing with lottery will provide an income of $7034 - (500 + 3 * 200) = 5934$ at a price at \$21.

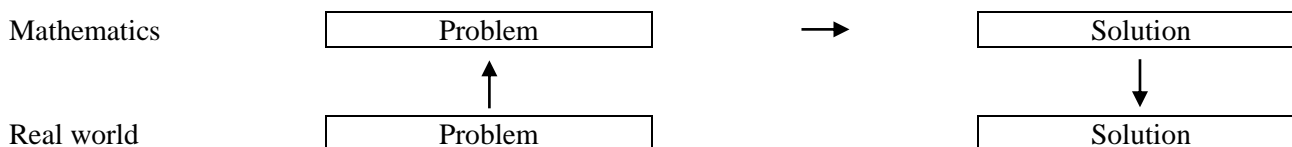
The Laffer-curve is an argument for increasing income tax together with decreased tax percentage.

The demand curve tells then that with growing tax rate, undeclared work will also grow.

06. Project Statistics

Problem: How to give a short description of the results from a questionnaire?

A mathematical model



1. The Real World Problem

A questionnaire produces numbers that change unpredictably and can therefore not be described by a formula. How can we provide a comprehensive summary of unpredictable numbers changing in a seemingly random way?

2. The Mathematical Problem

In a questionnaire two questions were: what distance do you have to school? How many children has your mother given birth to? Although non pre-dictable, the answers are post-dictable when listed in a table containing observations and frequencies. The distance-numbers are grouped, the child-number are not. In the group 15-20, 15 is not included, 20 is.

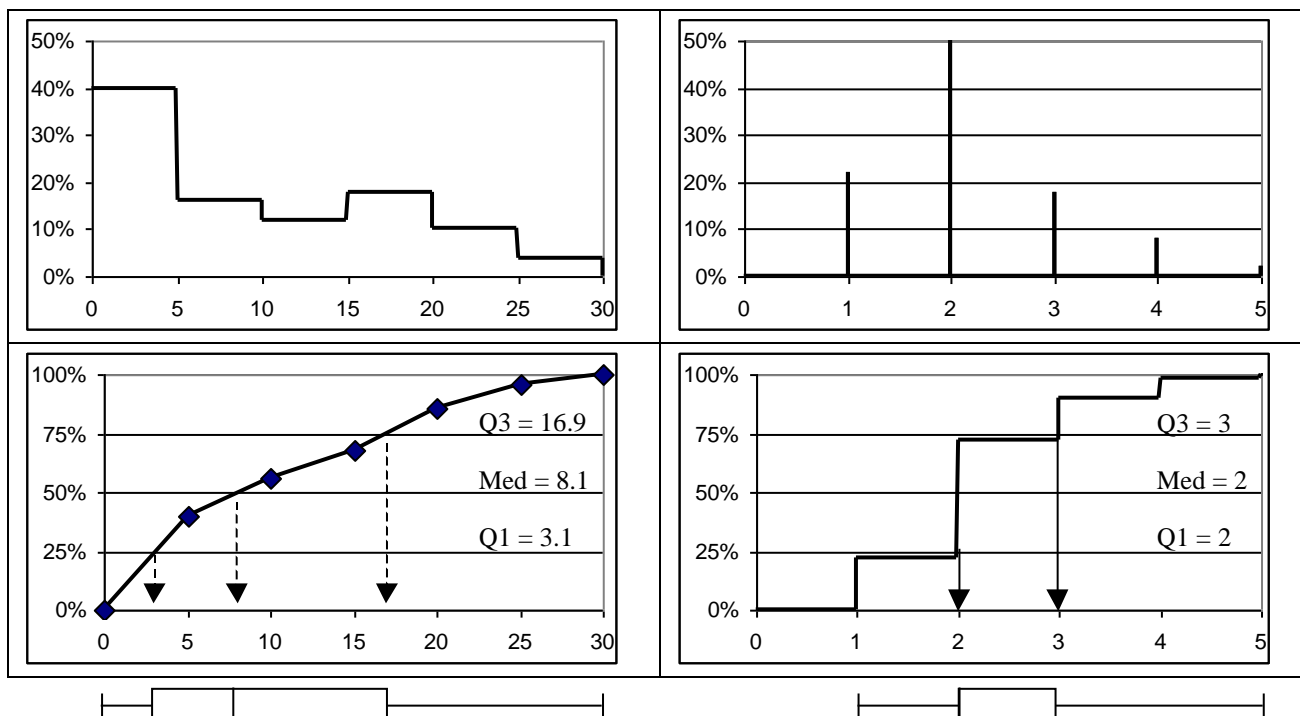
Obs.	Freq.	Rel. Freq.	Cumulated	Mean, average
00-05	20	$20/50 = 0.40$	0.40	$2.5 * 0.40 = 1.00$
05-10	8	0.16	0.56	$7.5 * 0.16 = 1.20$
10-15	6	0.12	0.68	1.50
15-20	9	0.18	0.86	3.15
20-25	5	0.10	0.96	2.25
25-30	2	0.04	1.00	1.00
50		1.00		M = 10.2

Obs.	Freq.	Rel. Freq.	Cumulated	Mean, average
1	11	$11/50 = 0.22$	0.22	$1 * 0.22 = 0.22$
2	25	0.50	0.72	$2 * 0.50 = 1.00$
3	9	0.18	0.90	0.54
4	4	0.08	0.98	0.32
5	1	0.02	1.00	0.10
50		1.00		M = 2.2

3. Solution to the Mathematical Problem

The relative frequency indicates the frequency as a percentage of all responses. The 10% in the group 20-25 means that 10% of the respondents have from 20 km up to and including 25 km to school. For grouped observations, the frequency is illustrated graphically with a histogram, and ungrouped observations use a stick diagram. The cumulative frequency adds up the relative frequencies, thus 86% of the respondents have at most 20 km to school. The cumulative frequency is illustrated by an ogive, from which we can read the first, the second (the median) and the third quartile at the 25%, 50% and 75% level. A box plot shows the minimum and maximum observation as well as the three quartiles.

The mean or average value indicate the answer given, if all answers were the same. Which they are not, they deviate from the average, motivating computing a standard deviation measuring the average deviation, roughly. When grouping, the middle observation is used when calculating averages. The values may also be found by entering the observations and frequencies as lists in a graphical display calculator, and then use Calc 1-Var Stats L1, L2.

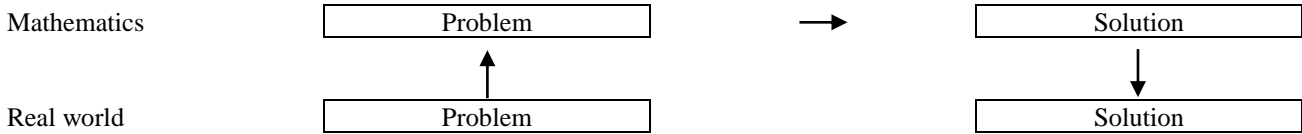


4. Solution to the Real World Problem. Although non pre-dictable, the answers in a questionnaire are post-dictable when listed in a table containing observations and frequencies, and graphed as a histogram or a stick diagram. This allows many answers to be reduced to two values: one for the center, the mean or the median; and one or two for the spread, the deviation or the quartiles described in a box-plot.

07. Project Hypothesis Test

Problem: How to test the hypothesis This tetraether is symmetrical?

A mathematical model



1. The Real World Problem

Rolling a symmetrical tetrahedron, each outcome has the Win- and Loose probability $\frac{1}{4}$ and $\frac{3}{4}$. Rolling it 4 times the Win-result '1' can occur 0, 1, 2, 3 or 4 times with 31.6% chance for occurring 0 times. So with 40 repetitions we expect it to occur $40 \cdot 31.6\% \cdot 40 = 12.6$ times. But we observed the data 16, 14, 7, 2, 1 for winning 4, 3, 2, 1 or 0 times, making us compare our observations with the theoretical expectations in order to test whether the tetrahedron is symmetrical.

2. The Mathematical Problem

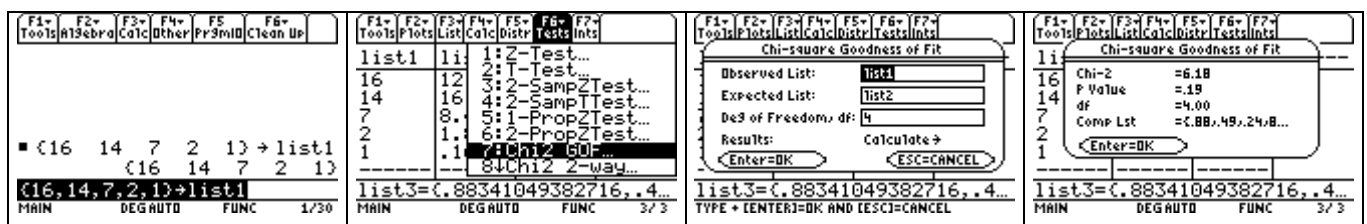
1/4 W	Win	Result	Runs	Probability	LIStat.Binompdf(4,0.25)
W	4	4 W og 0 L	1	$1 \cdot (1/4)^4 \cdot (3/4)^0$	0,004
3/4 L W	3	3 W og 1 L	4	$4 \cdot (1/4)^3 \cdot (3/4)^1$	0,047
W W	2	2 W og 2 L	6	$6 \cdot (1/4)^2 \cdot (3/4)^2$	0,211
W L	1	1 W og 3 L	4	$4 \cdot (1/4)^1 \cdot (3/4)^3$	0,422
L L L L	0	0 W og 4 L	1	$1 \cdot (1/4)^0 \cdot (3/4)^4$	0.316

In an experiment with two outcomes, with a 25% winning probability, the result of four repetitions ($n = 4$) is predicted by the binomial numbers, $nCr(4,2) = 6$ saying that there are 6 ways to win 2 times of 4; and the binomial distribution $TIStat.Binompdf(4, 0.25)$, saying e.g. that the probability of winning 2 times of 4, $P(2,4)$, is 21.1%. We observe the occurrence of the Pascall triangle, and that the probability of winning r times of n is $P(n, r) = nCr \cdot p^r \cdot (1-p)^{(n-r)}$. We want to test the null hypothesis 'the tetrahedron is symmetrical with probability $\frac{1}{4}$ for all four outcomes.

3. Solution to the Mathematical Problem

We create columns.	Win.	Obs.	Exp.	Dev.	Dev.^2	Ki^2
Column 1 contains the observed frequencies To.	4	1	0,16	0,84	0,71	4,56
Column 2 contains the expected frequencies with $Te = 40 \cdot TIStat.Binompdf(4, 0.25)$.	3	2	1,88	0,13	0,02	0,01
Column 3-5 contains the ki^2 deviation-numbers calculated as $ki^2 = \sum (To-Te)^2/Te$. This gives 6.18.	2	7	8,44	-1,44	2,07	0,24
	1	14	16,88	-2,88	8,27	0,49
	0	16	12,66	3,34	11,18	0,88
						6,18

On a CAS calculator we can import the two lists to the Stats/List editor. Then we perform a ki^2 GOF (Goodness of Fit) test, which shows each ki^2 number, as well as the total ki^2 number 6.18.



The ki^2 is zero if the observed and expected numbers are the same. So here there is a difference between the two numbers. The question is whether the difference is so great that it is critical. The critical limit can be found in a ki^2 -table (5% significance). First we find the degrees of freedom. With 5 categories, 4 can vary freely, while the fifth will always be the rest. Likewise horizontal: with 2 categories, 1 can vary freely, the other is the rest. Now the degree of freedom is calculated by the formula, degrees of freedom = (rows - 1) * (columns - 1) = (5 - 1) * (2 - 1) = 4. A table shows that at 4 degrees of freedom the critical limit is 9.49, if we want 5% significance (error-probability).

Degree of freedom	1	2	3	4	5	6	7	8
Critical limit	3.841	5.991	7.815	9.488	11.07	12.592	14.067	15.507

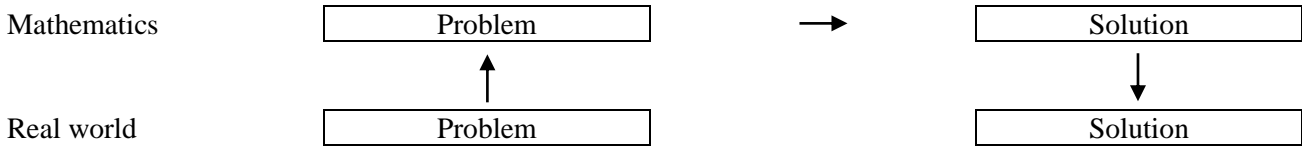
4. Solution to the Real World Problem.

The hypothesis 'this tetrahedron is symmetrical' was tested with 40 repetitions of rolling 4 times. We expect to get the numbers close to the binomial form, but our observed numbers differ from the expected numbers. The deviation is measured with a ki^2 GOF test at a 5% significance level shows a difference at 6.18. This is less than the critical difference at 9.49 at 4 degrees of freedom. We can therefore conclude that the difference is not critical, so our zero-hypothesis is accepted at a 5% significance level, IE. that there is a 95% chance that the study's answer is correct. The same distribution with 80 repetitions would give twice that ki^2 number, and this would mean rejecting the hypothesis.

08. Project Linear Programming

Problem: Which product will maximize my outcome under given restrictions?

A mathematical model



1. The Real World Problem

A market booth sells water (max 15 boxes) and beer (max 10 boxes). The cost for water and beer is per box \$25 and \$100 respectively. A maximum of \$1200 DKK can be invested. At most 21 boxes can be sold during opening hours. Income is \$80/120 per box beer/water. How to optimize the income?

2. The Mathematical Problem

Outside world	Inside equations	Graphics
Boxes of water Boxes of beer	x y	
Restriction on goods: The booth has place for a maximum of 15 boxes of water 10 boxes of beer	$0 \leq x \leq 15$ $0 \leq y \leq 10$	
Restriction on capital \$1200, given the cost: \$25 per box of water \$100 per box of beer	$25*x + 100*y \leq 1200$ $(100*y \leq -25*x + 1200$ $y \leq -\frac{1}{4}*x + 12)$	
Restriction on labor: At most 21 boxes can be sold during opening hours.	$x + y \leq 21$ $(y \leq -x + 21)$	
Total income T is \$80/120 per box water/beer.	$T = 80*x + 120*y$ $y = -\frac{2}{3}*x + \frac{T}{120}$ N0: T = 0: y = -2/3*x N600: T = 600: y = -2/3*x + 5	
Solution Buying 12 boxes of water and 9 boxes of beer will maximize the total income to \$2040.	'Solve(-1/4*x+12 = -x+21,x)' gives x = 12 'y = -x + 21 x=12' gives y = 9 $T = 80*x + 120*y x=12 \text{ and } y = 9$ gives T = 2040	

3. Solution to the Mathematical Problem

Graphically, the restrictions gives a polygon. The level-lines are since the income T only influences the intersection with the y-axis. A parallel translation of the level-lines across the polygon thus will increase or decrease T. So, he optimal value comes where a level-line leaves or is a tangent to the polygon, which will always happen in a corner. We can therefore predict the optimal situation by calculating all corner points (n equations with n unknowns), as well as T's values in these points (simplex method). This method is used when the number of variable is greater than 2.

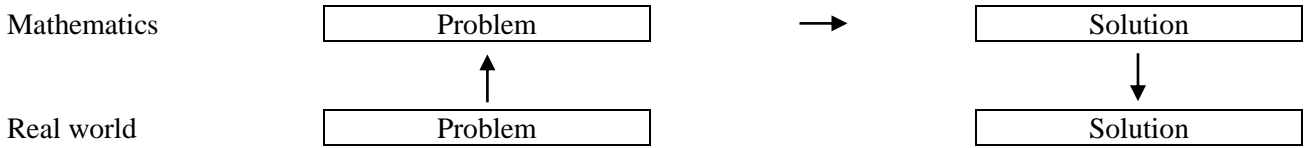
4. Solution to the Real World Problem

We see that the maximum income will be \$2040 when 12 boxes of water and 9 boxes of beer are sold. Furthermore, we see that the effective restrictions are the opening hours and the invested capital. Linear programming is used to optimize a particular quantity (to maximize profit, to minimize cost, etc.) within a series of restrictions on other quantities.

09. Project Game Theory

Problem: Which strategy will maximize my outcome?

A mathematical model



1. The Real World Problem

Two players A and B choose between different strategies. The outcome table shows what B must pay to A. The 2person game is called a ZeroSum game, since what A wins, B loses, and vice versa.

2. The Mathematical Problems

We look at two different games:

I

		B	
		b1	b2
A	a1	5	0
	a2	15	10

II

		y	B	1-y
		b1	b2	
A	x	5	0	
	1-x	-5	10	

3. Solution to the Mathematical Problems

The two players analyse the game:

A: With a1 is I risk outcome 0, with a2 I risk 10. In order to maximize my minimum-number I should choose a2. And, if I was B, I would choose b2, so I choose a2.

B: With b1 is I risk to pay 15, with b2 I risk 10. In order to minimize my maximum-pay I should choose a2. And, if I was A, I would choose a2, so I choose b2

The pair (a2, b2) is called the game **equilibrium** with the game value 10. No player will gain from not choosing the equilibrium strategy: A risks to get 0 instead of 10, so A sticks with the **maximin**-strategy. B risks paying 15 instead of 10, so B sticks with the **minimax**-strategy.

The two players analyse the game:

A: With a1 is I risk outcome 0, with a2 I risk -5. In order to maximize my minimum-number I should choose a1.

B: With b1 is I risk to pay 5, with b2 I risk 10. In order to minimize my maximum-pay I should choose b1.

But the maximin and minimax strategies a1 and b1 are not in equilibrium, since B gain by choosing b2, making A choose a2, making B choose b1, making A choose a1, etc.

Thus, with no equilibrium point, A should mix a1 and a2 in the ratio x to 1-x, randomly, and likewise B.

If B chooses b1 or b2, the outcomes will be respectively

$$P1 = 5*x - 5*(1-x) = 10*x - 5.$$

$$P2 = 0*x + 10*(1-x) = -10*x + 10.$$

The two outcomes are like with $10*x - 5 = -10*x + 10$, or $20*x = 15$, or $x = 15/20 = 3/4$ giving $P = 10*3/4 - 5 = 2.5$.

So by mixing the strategies a1 and a2 in the ration 3 to 1, player A will secure an average outcome 2.5 no matter what B chooses. So this game has the value 2.5.

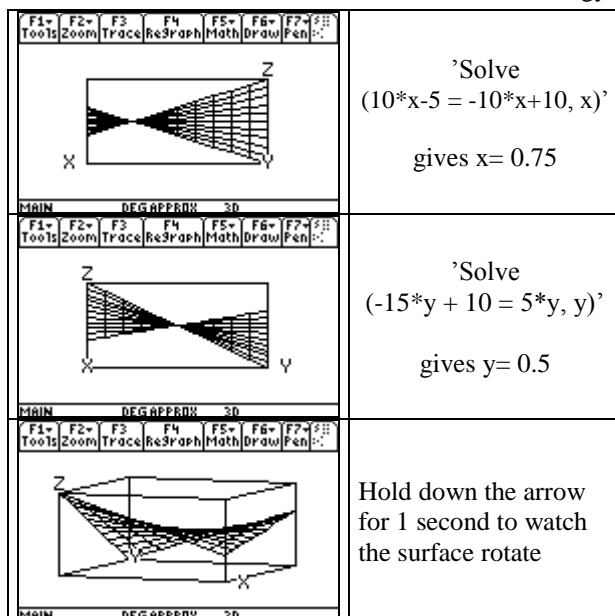
Likewise we find that by mixing the strategies b1 and b2 in the ration 1 to 1, player B will secure an average loss 2.5 no matter what A chooses.

With A and B randomly mixing their strategies in the ratios x to 1-x, and y to 1-y respectively, the outcome will be

$$P = 5*x*y + 0*x*(1-y) - 5*(1-x)*y + 10*(1-x)*(1-y)$$

$$P = -10*x - 15*y + 20*x*y + 10$$

On a graphical display calculator P becomes a surface called a **saddle point**, going down one way, and up the other way.



4. Solution to the Real World Problem

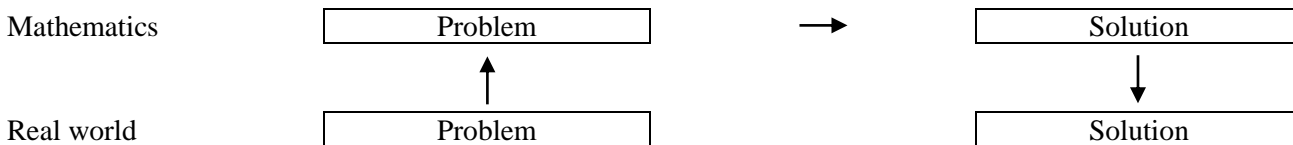
In game I, A and B should choose strategy a2 and b2 respectively resulting B losing 10 per game. In game II, the solution $x = .75$, $y = .5$, $P = 2.5$ means that with a deck of cards, A should choose a2 when dragging clubs, otherwise a1. And that B should choose b2 when dragging black, otherwise b1. B will have an average loss at 2.5 per game.

We see that a 2person ZeroSum-game always has an equilibrium point in a saddle point going up to one side and down to the other. With the saddle point in a corner, this corner is the solution to the problem. With an inside saddle point, the strategies must be mixed randomly in a ration found by looking at the saddle point from the side.

10. Project Distance to a Far-away Point

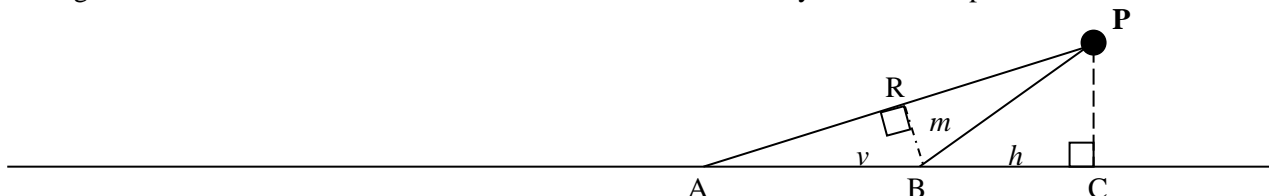
Problem: How to determine the distance to an inaccessible distant point?

A mathematical model



1. The Real World Problem

From a given baseline we want to determine the distance to a far-away inaccessible point P.



2. The Mathematical Problem

From a known baseline AB we measure the angles A and B to the inaccessible point P.

From the three right angled triangles ABR, BRP and BCP we calculate RB, BP as well as the distance PC.

Measurements: AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees

<p>$90-34=56$ $c = 366$ $a = ?$ $A(A) = 34$ $C(R) = 90$</p>	<p>$180-55-56=69$ $c = ?$ $a = 205$ $A(P) = 90-69=21$ $C(R) = 90$</p>	<p>$c = 572$ $a = ?$ $A(B) = 55$ $C(C) = 90$</p>
--	--	---

3. Solution to the Mathematical Problem

We set up three formula tables

Triangle ABR		Triangle PBR		Triangle PBC	
$a = ?$	$\sin A = \frac{a}{c}$	$c = ?$	$\sin A = \frac{a}{c}$	$a = ?$	$\sin A = \frac{a}{c}$
$A = 34$ $c = 366$	$\sin 34 = \frac{a}{366}$ $\sin 34 * 366 = a$ $205 = a$	$A = 21$ $a = 205$	$\sin 21 = \frac{205}{c}$ $c * \sin 21 = 205$ $c = \frac{205}{\sin 21}$ $c = 572$	$A = 55$ $c = 572$	$\sin 55 = \frac{a}{572}$ $\sin 55 * 572 = a$ $469 = a$
Test1 ☺	$\sin 34 = \frac{205}{366}$ $0.559 = 0.560$	Test1 ☺	$\sin 21 = \frac{205}{572}$ $0.358 = 0.358$	Test1 ☺	$\sin 55 = \frac{469}{572}$ $0.819 = 0.820$
Test2 ☺	Math Solver $0 = \frac{x}{366} - \sin 34$ gives $x = 205$	Test2 ☺	Math Solver $0 = \frac{205}{x} - \sin 21$ gives $x = 572$	Test2 ☺	Math Solver $0 = \frac{x}{572} - \sin 55$ gives $x = 469$

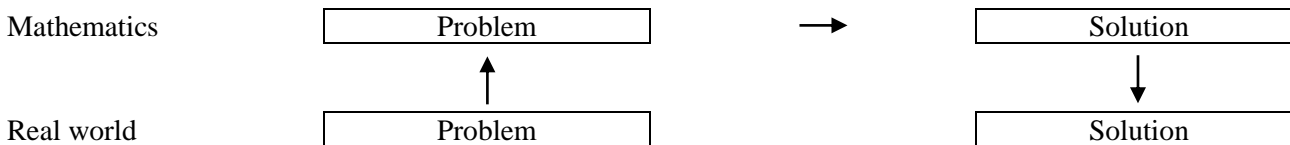
4. Solution to the Real World Problem

By using trigonometry we are able to determine the distance to the inaccessible point P to 469 cm.

11. Project Bridge

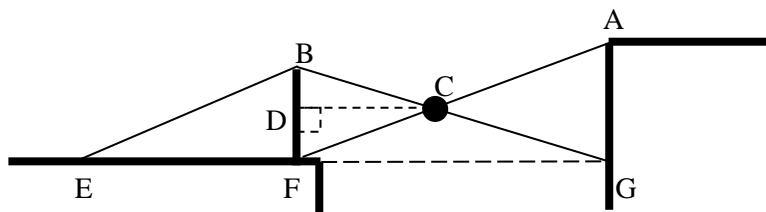
Problem: How to determine the dimensions of a bridge?

A mathematical model



1. The Real World Problem

Over a canyon a suspension bridge made of steel is fastened to the cliff and to a vertical upright. We want to determine the length of the 3 beams as well as the welding point. The left fixing angle must be 30 degrees.



2. The Mathematical Problem

From the right angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between the lines BG and FA.

Measurements: angle FEB = 30 degrees, FB = 3.5m, FG = 8m + 1m = 9m and AG = 5m.

<p>A(E) = 30 b C(F) = 90</p>	<p>A(F) b = 8+1=9 C(G) = 90</p>	<p>I a coordinate system with F as zero the following coordinates emerge: F: (0,0) and A: (9,5), as well as B: (0,3.5) and G: (9,0). Using linear regression we determine the equations for the lines FA and BG.</p>
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3. Solution to the Mathematical Problem

We set up formula tables

Triangle EFB	Triangle FGA og GFB	Lines BG and FA
$c = ?$ $\sin A = \frac{a}{c}$ <hr/> $A = 30$ $a = 3.5$ $\sin 30 = \frac{3.5}{c}$ $\sin 30 * c = 3.5$ $c = 3.5 / \sin 30$ $c = 7.0$	$c = ?$ $a^2 + b^2 = c^2$ <hr/> $a = 5$ $b = 9$ $5^2 + 9^2 = c^2$ $\sqrt{106} = c$ $10.30 = c$	$BG: ?$ $y = ax + b$ <hr/> $y = -0.389x + 3.5$ Found by LinReg L1, L2, Y1 Test Trace x=0 gives 3.5 Trace x=9 gives 0 StatPlot fits
Test1 ☉ $\sin 30 = \frac{3.5}{7}$ $0.5 = 0.5$	Test1 & Test2 $c = ?$ $a = 3.5$ $b = 9$ $a^2 + b^2 = c^2$ $3.5^2 + 9^2 = c^2$ $\sqrt{93.25} = c$ $9.66 = c$	Likewise we find FA: ? $y = 0.556x$ Calc Intersection gives $x = 3.71$ and $y = 2.06$ In the triangle FDC, DC = 3.71 and FD = 2.06 Pythagoras gives: $FC = \sqrt{3.71^2 + 2.06^2} = 4.24$ In the triangle BDC, DC = 3.71 and $FD = 3.6 - 2.06 = 1.54$ Pythagoras gives: $BC = \sqrt{3.71^2 + 1.54^2} = 4.02$

4. Solution to the Real World Problem

Using trigonometry we have found the lengths of the three steel beams as EB = 7.00 m, FA = 10.30 m and BG = 9.66

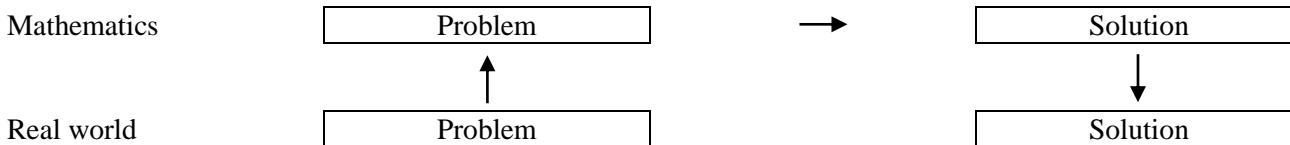
The welding point is determined by FC = 4.24 m and BC = 4.02 m.

As an extra control the bridge can be drawn and build by pipe cleaners in the ration 1:100.

12. Project Driving

Problem: How far and how did Peter drive?

A mathematical model



1. The Real World Problem

When driving, the velocity 100 km/t is $100 \cdot 1000 / (60 \cdot 60) = 27.8$ m/s. A camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s and 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

Time x sec	Velocity y m/s	Accel. dy/dx
5	10	?
10	30	?
15	20	?
20	40	?
25	15	?

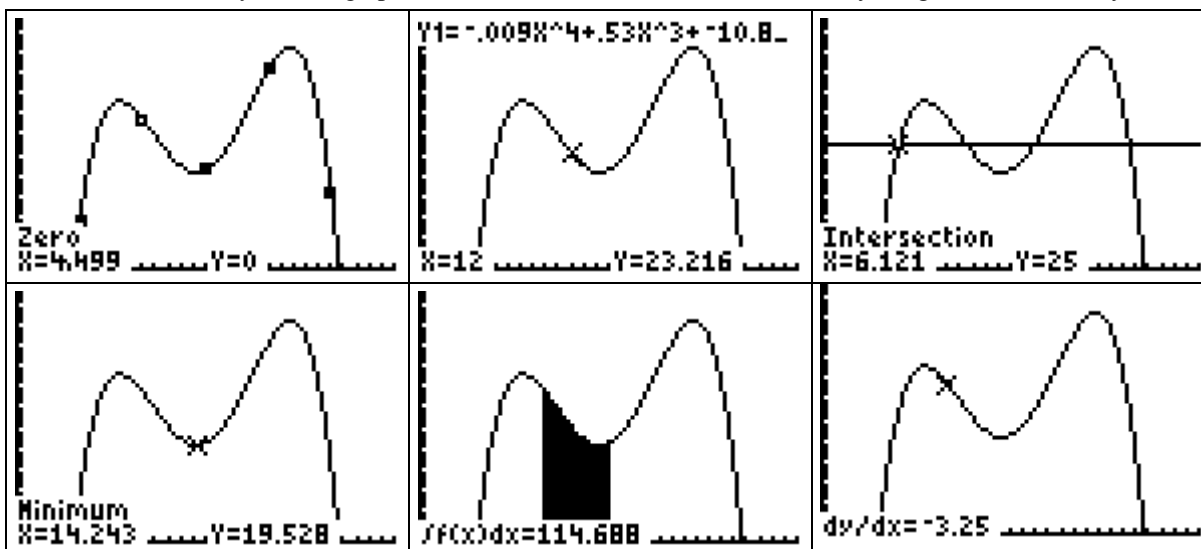
2. The Mathematical Problem

We set up a table showing time x and velocity y. The domain of the table is taken to be $0 < x < 30$.

3. Solution to the Mathematical Problem

On TI-84 the table is entered as the lists L1 and L2. 5 data sets allow quartic regression (a 4. degree polynomial with a 3-fold parabola) providing the formula $y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235$ placed as Y1. No the question asked can be answered using formula tables, or using technology, i.e. graphical readings or calculations.

Starting and ending points are found using 'CALC Zero'. Y-numbers are found using 'TRACE'. X-numbers are found using 'CALC Intersection'. Maximum and minimum are found with 'CALC Maximum/Minimum'. The total meter-number is obtained by summing up the $m/s \cdot s = \int Y1 dx$. Acceleration is found by the gradient 'CALC dy/dx'.



y = ?	y = y1
x=12	y = y1(12) = 3.667
Test	TRACE x = 12 gives y = 23.216

x = ?	y = y1
y = 25	MATH Solver 0 = y1 - 25 gives x = 6.12 and ...
Test1	y1(3) = 6, y1(8) = 6
Test2	CALC intersection gives x = 6.12, 11.44, 16.86 and 24.47

y _{max} = ?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx = 0 at x = 5.5

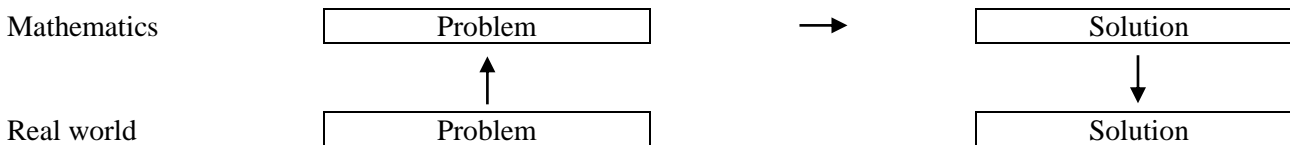
4. Solution to the Real World Problem

The driving began after 4.50 sec. and ended after 25.62 sec. After 12 sec the velocity was 23.2 m/s. And it was 25m/s after 6.12 sec, 11.44 sec, 16.86 sec and 24.47 sec. Acceleration took place in the time-intervals (4.50; 8.19) and (14.24; 21.74). Deceleration in the intervals (8.19; 14.24) and (21.74; 25.62). Max-velocity was 44.28 m/s = 159 km/t. after 21.7 sec. In the time-intervals (5; 10), (10; 15), (15; 20) and (20; 25) the distance traveled was 142.8 m, 114.7 m, 142.8 m and 189.7 m. The acceleration in the beginning of these time-intervals were 17.75, -3.25, 1.25, 4.25, -21.25 m/s². The total distance traveled was 597.4 m.

13. Project Vine Box

Problem: What are the dimensions of a 3 liters vine bag with the least surface area?

A mathematical model



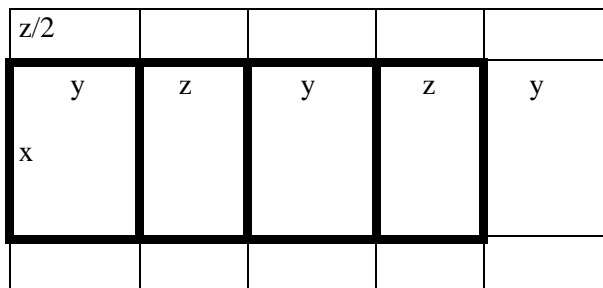
1. The Real World Problem

Vine is sold in bottles or in boxes. A 3 liter bag will be constructed by cutting out a piece of cardboard.

2. The Mathematical Problem

The cardboard dimensions are called x , y & z all in dm. We express the volume V and the Surface S as formulas:

$$V = x \cdot y \cdot z = 3, \quad S = x \cdot (3y + 2z) + 2 \cdot z / 2 \cdot (3y + 2z)$$



3. Solution to the Mathematical Problem

We expand the S -formula: $S = x \cdot (3y + 2z) + 2 \cdot z / 2 \cdot (3y + 2z) = 3xy + 2xz + 3yz + 2z^2$

We now insert $z = 3/(x \cdot y)$ so that S only depends on two variables x and y :

$$S = 3xy + 2xz + 3yz + 2z^2 \text{ and } z = 3/(x \cdot y) \text{ gives } S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 \cdot y^2}$$

Scenario A. We assume that y should be half the length of x : $y = 0.5 \cdot x$. This restriction is inserted:

$$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 \cdot y^2} = 1.5x^2 + \frac{21}{x} + \frac{72}{x^4}, \text{ which gives } \frac{dS}{dx} = 3x - \frac{21}{x^2} - \frac{288}{x^5} = 0 \text{ for } x = 2.4$$

Graphing this S -formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point $x = 2.4$ and $S = 19.56$, so $y = 0.5 \cdot x = 0.5 \cdot 2.4 = 1.2$, and $z = 3/(2.4 \cdot 1.2) = 1.0$

Scenario B. We assume that y should be the same length of x : $y = x$. This restriction is inserted:

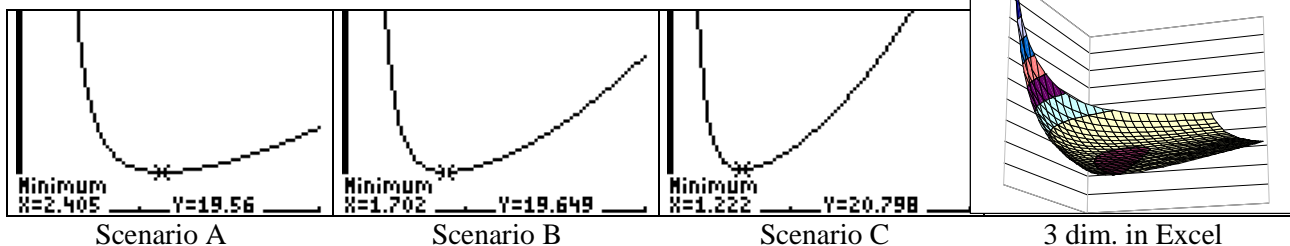
$$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 \cdot y^2} = 3x^2 + \frac{15}{x} + \frac{18}{x^4}, \text{ which gives } \frac{dS}{dx} = 6x - \frac{15}{x^2} - \frac{72}{x^5} = 0 \text{ for } x = 1.7$$

Graphing this S -formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point $x = 1.7$ and $S = 19.65$, so $y = x = 1.7$, and $z = 3/(1.7 \cdot 1.7) = 1.0$

Scenario C. We assume that y should be double the length of x : $y = 2 \cdot x$. This restriction is inserted:

$$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 \cdot y^2} = 6x^2 + \frac{12}{x} + \frac{4.5}{x^4}, \text{ which gives } \frac{dS}{dx} = 12x - \frac{12}{x^2} - \frac{18}{x^5} = 0 \text{ for } x = 1.2$$

Graphing this S -formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point $x = 1.2$ and $S = 20.80$, so $y = 2x = 2 \cdot 1.2 = 2.4$, and $z = 3/(1.2 \cdot 2.4) = 1.0$



4. Solution to the Real World Problem

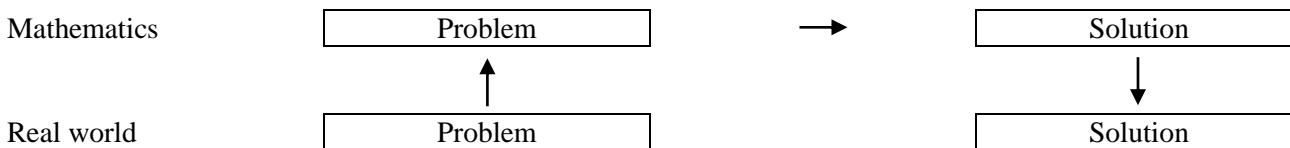
We see that the minimum surface area is a little above 19 dm^2 . Using an Excel-spreadsheet we can find the optimal solution to be $x = 2.1$ and $y = 1.4$ and $z = 1.0$, giving a minimum surface area at 19.47 dm^2 .

(Graphing $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 \cdot y^2}$ does not give a curve but a surface as shown on the Excel-pict.)

14. Project Golf

Problem: How to hit a golf hole hidden behind a hedge?

A mathematical model



1. The Real World Problem

From a position on a 2 meter high flat hill we want to send a golf ball over a 3 meter hedge 2 meter away on the hill to hit a hole situated 12 meters away at level zero.

What is the orbit of the ball? How high is the ball at the distance 10 meters? When does the ball have a height of 6 meters? How high does the ball go? What is the direction of the ball in the beginning, at 10 meters distance and at the impact?

2. The Mathematical Problem

We set up a table with the length x and the height y having the domain $0 < x < 12$.

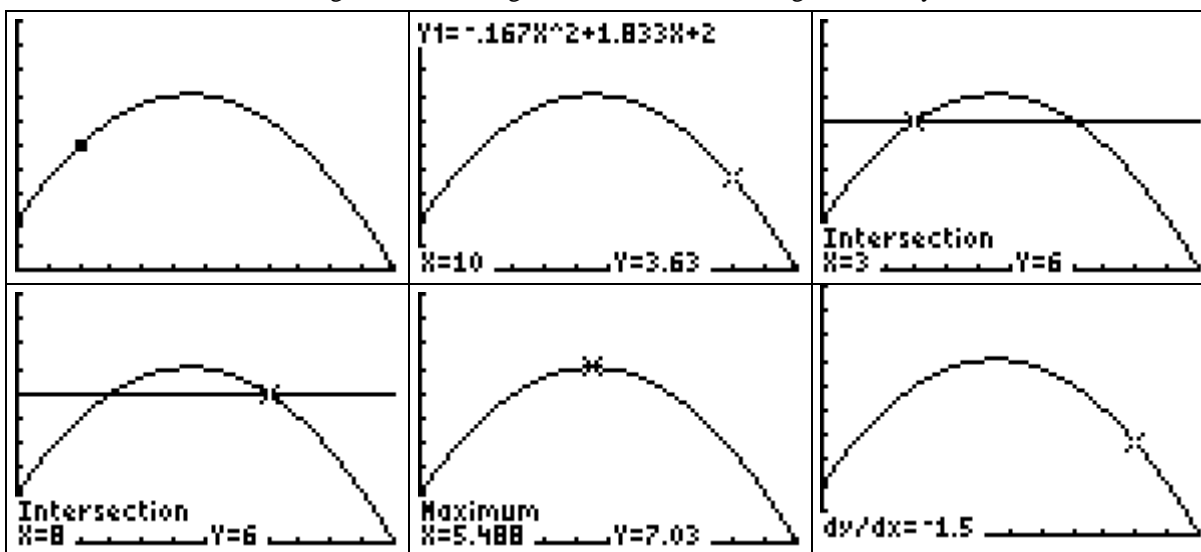
Length x	Height y	Direction v
0	2	?
2	5	
12	0	?
10	?	?
?	6	

3. Solution to the Mathematical Problem

We insert the table as lists L1 and L2. Three data-sets allows a 2nd degree polynomial, quadratic regression, which produces the formula $y = -0.167x^2 + 1.833x + 2$, which is transferred to Y1.

Now the questions asked can be answered using formula tables and a calculator for graphing or calculating. The Y-number can be found by CALC Value, the x-number by CALC Intersection, the maximum by CALC Maximum, and the gradient by CALC dy/dx.

To determine the direction angle v we use the gradient formula: $\tan v = \text{gradient} = dy/dx$.



$y = ?$	$y = y1$
$x = 10$	$y = y1(10) = 3.667$
Test	Trace $x=10$ gives $y=3.67$

$x = ?$	$y = y1$
$y = 6$	Math solver $0 = y1 - 6$ gives $x = 3$ & $x = 8$
Test1	$y1(3) = 6, y1(8) = 6$
Test2	CALC Intersection gives $x = 3$ and 8

$y_{\max} = ?$	$y = y1$
	Calc maximum gives $y = 7.042$ at $x = 5.5$
Test	$dy/dx \approx 0$ at $x = 5.5$

$v = ?$	$\tan v = dy/dx$
$x = 12$	$\tan v = -2.167$ $v = \tan^{-1}(-2.167)$ $v = -65.2$

$v = ?$	$\tan v = dy/dx$
$x = 0$	$\tan v = 1.833$ $v = \tan^{-1}(1.833)$ $v = 61.4$

$v = ?$	$\tan v = dy/dx$
$x = 10$	$\tan v = -1.5$ $v = \tan^{-1}(-1.5)$ $v = -56.3$

4. Solution to the Real World Problem

The orbit of the ball is a parabola. The height of the ball at the distance 10 meters is 3.67 meters? At the distances 3 meters and 8 meters the ball has a height of 6 meters. The ball goes to the maximum height 7.04 meters? The direction of the ball in the beginning, at 10 meters distance and at the impact are 61.4 grader, -65.2 grader and -56.3 grader.

15. Project Family Firm

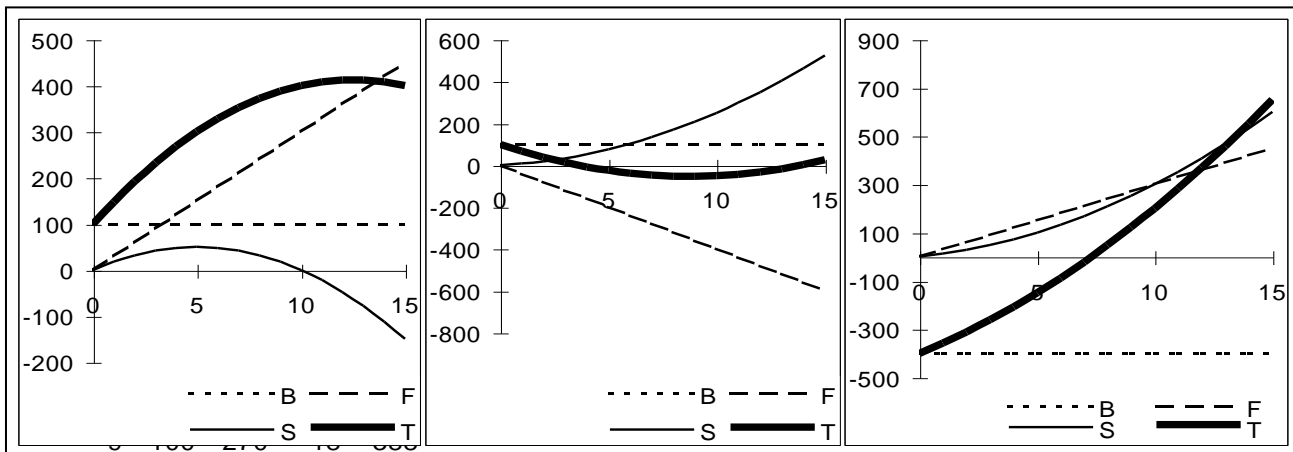
Problem: How to predict the development of a three generation family firm?

A family firm builds its total capital T through income from three generations. The grandfather has retired and has left the capital T_0 \$. The father has established a routine which earns b \$/day. The son has just come back from the university, where he has learned a new technology enabling him to slowly raise the daily income d \$/day to $s = s_0 + d \cdot n$. The total capital after n days, since the family is calculated as a sum of polynomials:

Grandfather B	T_0	degree 0 polynomial, a constant
Father F	$b \cdot n$	degree 1 polynomial, a line
Son S	$s \cdot n = (s_0 + \frac{1}{2} \cdot d \cdot n) \cdot n = s_0 \cdot n + \frac{1}{2} \cdot d \cdot n^2$	degree 2 polynomial, a bended line
Total T	$T = T_0 + (b+s_0) \cdot n + d \cdot n^2$	degree 2 polynomial, a parabola

Gatherers and Spreaders

Some families have both a spreader and a gatherer. The spreader may be the son, the father or the grandfather



Spreader: Son

Father

Grandfather

Pricing the tea

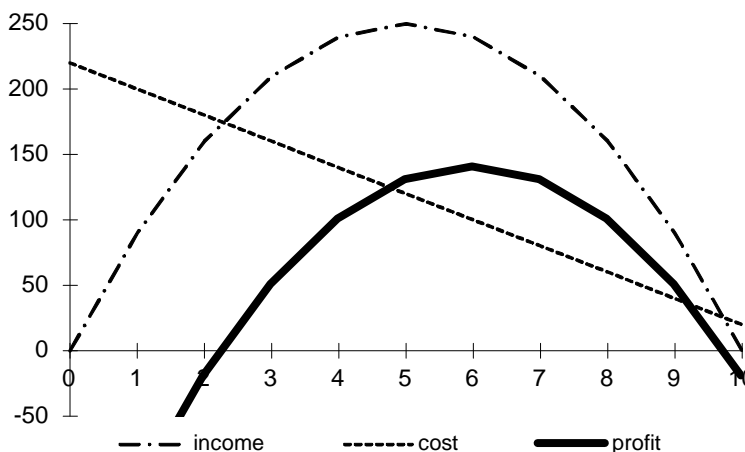
In the family firm they discuss if increasing the unit price by necessity will make the sale decrease.

Grandfather: The sale will decrease with an increasing unit price. I believe in a linear relation $y = a + b \cdot x$ found from the table	Father: The sale will decrease slower at high unit prices. I believe in a degree 2 polynomial $y = a + b \cdot x + c \cdot x^2$ found from the table	Son: The sale will decrease more at low and high unit prices. I believe in a degree 3 polynomial $y = a + b \cdot x + c \cdot x^2 + d \cdot x^3$ found from the table																								
<table border="1"> <thead> <tr> <th>price x</th> <th>sale y</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>100</td> </tr> <tr> <td>10</td> <td>0</td> </tr> </tbody> </table>	price x	sale y	0	100	10	0	<table border="1"> <thead> <tr> <th>price x</th> <th>sale y</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>100</td> </tr> <tr> <td>5</td> <td>80</td> </tr> <tr> <td>10</td> <td>0</td> </tr> </tbody> </table>	price x	sale y	0	100	5	80	10	0	<table border="1"> <thead> <tr> <th>price x</th> <th>sale y</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>100</td> </tr> <tr> <td>2</td> <td>60</td> </tr> <tr> <td>8</td> <td>40</td> </tr> <tr> <td>10</td> <td>0</td> </tr> </tbody> </table>	price x	sale y	0	100	2	60	8	40	10	0
price x	sale y																									
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price x	sale y																									
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10	0																									
price x	sale y																									
0	100																									
2	60																									
8	40																									
10	0																									

The grandfather scenario

The sale will be $y = 100 - 10 \cdot x$ found by regression. The total income T is $T = \text{unit price} \cdot \text{sale} = x \cdot y = x \cdot (100 - 10 \cdot x) = 100 \cdot x - 10 \cdot x^2$, a degree 2 polynomial. The cost C to produce y units consists of a fixed cost $c_0 = 20$ and a variable unit-cost $m = 2$. So, $C = c_0 + m \cdot y = 20 + 2 \cdot y = 20 + 2 \cdot (100 - 10 \cdot x) = 20 + 200 - 20 \cdot x = 220 - 20 \cdot x$. The profit P will be when $P = T - C = (100 \cdot x - 10 \cdot x^2) - (220 - 20 \cdot x) = -220 + 80x - 10 \cdot x^2$, i.e. again a degree 2 polynomial.

price	sale	income	cost	profit
0	100	0	220	-220
1	90	90	200	-110
2	80	160	180	-20
3	70	210	160	50
4	60	240	140	100
5	50	250	120	130
6	40	240	100	140
7	30	210	80	130
8	20	160	60	100
9	10	90	40	50
10	0	0	20	-20



Exercises

1. Set up the father's scenario.

2. Set up the son's scenario.

Revision Problems Using TI-84

1.	<table border="1"> <thead> <tr> <th>x</th> <th>y = ?</th> </tr> </thead> <tbody> <tr> <td>3</td> <td>12</td> </tr> <tr> <td>7</td> <td>16</td> </tr> <tr> <td>10</td> <td>?</td> </tr> <tr> <td>?</td> <td>40</td> </tr> </tbody> </table>	x	y = ?	3	12	7	16	10	?	?	40	<p>Answer the question marks in case of a linear model.</p> <p>Answer the question marks in case of an exponential model. What is the doubling time?</p> <p>Answer the question marks in case of a power model.</p>				
x	y = ?															
3	12															
7	16															
10	?															
?	40															
2.	<table border="1"> <thead> <tr> <th>x</th> <th>y = ?</th> </tr> </thead> <tbody> <tr> <td>3</td> <td>12</td> </tr> <tr> <td>7</td> <td>16</td> </tr> <tr> <td>10</td> <td>18</td> </tr> <tr> <td>15</td> <td>?</td> </tr> <tr> <td>?</td> <td>10</td> </tr> </tbody> </table>	x	y = ?	3	12	7	16	10	18	15	?	?	10	<p>Answer the question marks in case of a quadratic model.</p> <p>Find maxima or minima.</p> <p>Find the equation for the tangent line in $x=2$.</p> <p>Find the gradient formula.</p> <p>Find the gradient number in $x = 5$</p> <p>Find the area formula</p> <p>Find the area number from $x= 1$ to $x = 6$</p> <p>Find the intersection points with the line $y = 3 + 2x$</p>		
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x	y = ?															
3	12															
7	16															
10	14															
12	18															
15	?															
?	30															
4.	$3x + 4y = 15$ & $5x - 6y = 12$	Solve the simultaneous equations														
5.	Given two points in a coordinate system P(2,4) and Q(6,10)	<p>Find the midpoint of the line PQ.</p> <p>Find the equation for the line through P and Q</p> <p>Find the equation for the normal line to PQ passing through P</p> <p>Find the angle between PQ and the x-axis.</p> <p>Find the distance between P and Q</p> <p>Find the distance from the line PQ to the point S(8,1)</p> <p>Find the equation for the circle through P and Q and with the midpoint of PQ as center.</p> <p>Find the intersection point between the circle and the line $y = 12-2x$</p>														
6.	Let X be a normal random variable with mean $m = 100$ and standard deviation $d = 12$	<p>$P(X < 89) = ?$</p> <p>$P(X > 108) = ?$</p> <p>$P(93 < X < 109) = ?$</p>														
7.	X counts the numbers of wins in 100 repetitions of a game with 65% winning chance.	<p>$P(X < 70) = ?$</p> <p>$P(X \leq 60) = ?$</p> <p>$P(X \geq 58) = ?$</p> <p>$P(63 < X \leq 72) = ?$</p>														
8.	$\sin(3x) = 0.4, \quad 0 \leq x \leq 2\pi$ $\cos(\frac{1}{2}x) = -0.3, \quad 0 \leq x \leq 2\pi$ $\tan(2x) = 0.7, \quad 0 \leq x \leq 2\pi$	<p>Find the solutions: <i>Remember to adjust the window</i></p> <p>Find the solutions:</p> <p>Find the solutions:</p>														
9.	$A = 40, b = 7, C = 90$	Find a, B and c.														
10.	$a = 4, c = 7, C = 90$	Find A, B and b.														
11.	$A = 40, b = 7, C = 68$	Find a, B and c.														
12.	$A = 40, b = 7, c = 6.8$	Find a, B and C.														
13.	$A = 40, b = 7, a = 6.2$	Find c, B and C.														
14.	$a = 4, b = 7, c = 6.8$	Find A, B and C.														
15.	$T = \frac{d}{e-f} + g$	Transpose the T-formula to a d-, e-, f-, and g-formula														
16.	The capital 785 increased with 2.7% 5 times and became ?	<p>Find the answer</p> <p>Find the corresponding doubling time.</p>														
17.	The capital 785 increased with 2.7% ? times and became 980	<p>Find the answer</p> <p>Find the corresponding doubling time.</p>														
18.	The capital 785 increased with ?% 5 times and became 980	<p>Find the answer</p> <p>Find the corresponding doubling time.</p>														
19.	-21	As 16-18, but with \$ instead of %														

Problem 1. Linear model

Equation:	$y=ax+b$ $y=x+9$, found by Stat, Calc, LinReg
Test	$y1(3) = 12$ ☺

$y=?$	$y=x+9$
$x=10$	$y=19$ found by $y1(10)$
Test	$y=19$ found by CalcValue ☺

$x=?$	$y=x+9$
$y=40$	$x=31$, found by Math, Solver $0=y1-40$
Test	$y1(31) = 40$ ☺

Exponential model

Equation:	$y=a*b^x$ $y=9.671*1.075^x$, found by Stat, Calc, ExpReg
Test	$y1(3) = 12$ ☺

$y=?$	$y=9.671*1.075^x$
$x=10$	$y=19.853$ found by $y1(10)$
Test	$y=19.853$ found by CalcValue ☺

$x=?$	$y=9.671*1.075^x$
$y=40$	$x=19.740$, found by Math, Solver $0=y1-40$
Test	$y1(19.740) = 40$ ☺

Doubling time $T = \log 2 / \log b = \log 2 / \log 1.075 = 9.6$

Power model

Equation:	$y=a*x^b$ $y=8.264*x^{0.340}$ found by Stat, Calc, PwrReg
Test	$y1(3) = 12$ ☺

$y=?$	$y=8.264*x^{0.340}$
$x=10$	$y=18.060$ found by $y1(10)$
Test	$y=18.060$ found by CalcValue ☺

$x=?$	$y=8.264*x^{0.340}$
$y=40$	$x=104.024$ found by Math, Solver $0=y1-40$
Test	$y1(104.024) = 40$ ☺

Problem 2. Quadratic model

Equation:	$y=a*x^2+b*x+c$ $y=-0.048x^2+1.476x+8$ found by Stat, Calc, QuadReg
Test	$y1(3) = 12$ ☺

$y=?$	$y=-0.048x^2+1.476x+8$
$x=15$	$y=19.429$ found by $y1(15)$
Test	$y=19.429$ found by Graph, Calc, Value ☺

$x=?$	$y=-0.048x^2+1.476x+8$
$y=10$	$x=1.420$ or 29.580 found by Math, Solver $0=y1-10$
Test	$y1(1.420) = 10$ $y1(29.580) = 10$ ☺

Maximum:	$y=-0.048x^2+1.476x+8$ $(x,y) = (15.500,19.140)$ found by Graph, Calc, Maximum
Test	$dy/dx = 0$ for $x = 15.5$ $y1(15.5) = 19.14$ ☺

Tangent in $x=2$	$y=-0.048x^2+1.476x+8$ $x=2$ $y=1.286x + 8.190$ found by Graph, Draw, Tangent
------------------	--

Gradient formula	$y=-0.048x^2+1.476x+8$ $y' = -0.095*x + 1.476$, found by TI89
Test	$\int y' dx = -0.048x^2+1.476x$ found by TI89 ☺

Gradient number:	$y=-0.048x^2+1.476x+8$ $x=5$ $dy/dx = 1$ for $x=5$ found by Graph, Calc, dy/dx
Test	1, found by Math, nDeriv ☺

Area formula:	$y=-0.048x^2+1.476x+8$ $x=2$ $\int y dx = -0.016*x^3 + 0.738*x^2 + 8.000*x$ found by TI89
Test	$d(\int y dx)/dx = -0.048x^2+1.476x+8$ found by TI89 ☺

Area number:	$y=-0.048x^2+1.476x+8$ 6 $\int_1^6 y dx = 62.421$, found by Graph, Calc, $\int f(x) dx$
Test	62.421, found by Math, fnInt ☺

Intersection points	$y = -0.048x^2+1.476x+8$ and $y = 3+2x$ ($y1 = y3$) $(x,y) = (-17.130,-31.260)$ and $(x,y) = (6.130, 15.260)$, found by Math, Solver $0=y1-y3$ and $y1(-17.130) = -31.260$ etc.
Test	tested by Graph, Calc, Intersect ☺

Problem 3. Cubic model

Equation:	$y=a*x^3+b*x^2+c*x+d$ $y=0.086x^3-1.952x^2+13.752x-14$, found by Stat, Calc, CubicReg
Test	$y1(3) = 12$ ☺

$y=?$	$y=0.086x^3-1.952x^2+13.752x-14$
$x=15$	$y=42.286$ found by $y1(15)$
Test	$y=42.286$ found by Graph, Calc, Value ☺

$x=?$	$y=0.086x^3-1.952x^2+13.752x-14$
$y=30$	$x=13.885$ found by Math, Solver $0=y1-30$
Test	$y1(13.885) = 30$ ☺

Maximum Minimum:	$y=0.086x^3-1.952x^2+13.752x-14$ Max: $(x,y) = (5.552, 16.841)$ found by Graph, Calc, Maximum Min: $(x,y) = (9.634, 13.925)$ found by Graph, Calc, Minimum
Test	$dy/dx = 0$ for $x = 5.552$ $y1(5.552) = 16.841$ $dy/dx = 0$ for $x = 9.643$ $y1(9.643) = 13.925$ ☺

Tangent in $x=2$	$y=0.086x^3-1.952x^2+13.752x-14$ $x=2$ $y = 6.971x - 7.562$ found by Graph, Draw, Tangent
------------------	--

Gradient formula	$y=0.086x^3-1.952x^2+13.752x-14$ $y' = 0.257*x^2 - 3.905*x + 13.752$, found by TI89
Test	$\int y' dx = 0.086x^3-1.952x^2+13.752x$ found by TI89 ☺

Gradient number:	$y=0.086x^3-1.952x^2+13.752x-14$
$x=5$	$y'(5) = 0.657$ found by Graph, Calc, dy/dx
Test	0.657, 1 found by Math, nDeriv ☺

Area formula:	$y=0.086x^3-1.952x^2+13.752x-14$
$x=2$	$\int ydx = 0.021*x^4 - 0.651*x^3+6.876*x^2+14*x$ found by TI89
Test	$d(\int ydx)/dx = 0.086x^3-1.952x^2+13.752x-14$ found by TI89 ☺

Area number:	$y=0.086x^3-1.952x^2+13.752x-14$
	6 $\int_1^6 ydx = 58.496$, found by Graph, Calc, $\int f(x)dx$
Test	58.496, 62.421 found by Math, fnInt ☺

Intersection points with $y=3+2x$: $(x,y) = (2.129, -7.259)$ and $(x,y) = (6.657, 16.315)$ and $(x,y) = (13.991, 30.981)$ found by Math, Solver $0=y1-y3$, tested by Graph, Calc, Intersect.

Problem 4

Solutions: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix}$, found by $A*B=C$, $B=A^{-1}*C$, where $A = \begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix}$ and $B = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$

Tested by $A*B=C$: $A*B = \begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix} * \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix} = C$ ☺

Problem 5

Midpoint:	$(x,y) = (\frac{x1+x2}{2}, \frac{y1+y2}{2})$
$x1=2$ $x2=6$ $y1=4$ $y2=10$	$(x,y) = (\frac{2+6}{2}, \frac{4+10}{2})$ $(x,y) = (4,7)$
Test	Tested geometrically ☺

Gradient PQ:	$a = \frac{y2-y1}{x2-x1}$
$x1=2$ $x2=6$ $y1=4$ $y2=10$	$a = \frac{10-4}{6-2}$ $a = 3/2$ $a = 1.5$
Test	Tested geometrically ☺

Line PQ:	$y = y1 + a*(x - x1)$
$a=1.5$ $x1=2$ $y1=4$	$y = 4 + 1.5*(x - 2)$ $y = 1.5*x + 1$
Test	Tested geometrically ☺

Gradient perpend.:	$c*a = -1$
$a=3/2$	$c = -2/3$ found by Math, Solver $0 = c*3/2+1$
Test	Tested geometrically ☺

Normal:	$y = y1 + a*(x - x1)$
$a=-2/3$ $x1=2$ $y1=4$	$y = 4 + -2/3*(x - 2)$ $y = -2/3*x + 5.333$
Test	Tested geometrically ☺

Distance PQ	$d = \sqrt{(x2-x1)^2 + (y2-y1)^2}$
$x1=2$ $x2=6$ $y1=4$ $y2=10$	$d = \sqrt{(6-2)^2 + (10-4)^2}$ $d = 7.21$
Test	Tested geometrically ☺

Distance point-line	$d = \frac{ y1 - a*x1 - b }{\sqrt{1 + a^2}}$
$a=1.5$ $b=1$ $x1=8$ $y1=1$	$d = \frac{ 1 - 1.5*8 - 1 }{\sqrt{1 + 1.5^2}}$ $d = 6.66$
Test	Tested geometrically ☺

Circle equation	$(x-c1)^2 + (y - c2)^2 = r^2$
$r=1/2*7.21$ $r=3.61$ $c1=4$ $c2=7$	$(x - 4)^2 + (y - 7)^2 = 3.61^2$ $(x - 4)^2 + (y - 7)^2 = 13.03$
Test	Tested geometrically ☺

Intersection	$(x-c1)^2 + (y - c2)^2 = r^2$ and $y = 12-2x$
$r = 1/2*7.21$ $=3.61$ $c1 = 4$ $c2 = 7$	$(x,y) = (1.30, 9.40)$ and $(4.30, 3.40)$ found by Math, Solver $0 = (x-4)^2 + (12-2x-7)^2 - 3.61^2$
Test	Tested geometrically ☺

Angle: $\tan(v) = a$, $a=3/2$; $v = 56.31$ found by Math, Solver $0 = \tan v - 3/2$, $v > 0$ and $v < 90$. Tested geometrically ☺

Problem 6

$p(X < 115) = 0.894$, found by normalCdf(1EE-99,115,100,12)
$p(X < 89) = 0.180$, found by normalCdf(1EE-99,89,100,12)
$p(X > 108) = 0.253$, found by normalCdf(108,1EE99,100,12)
$p(93 < X < 109) = 0.494$, found by normalCdf(93,109,100,12)

Problem 7

$p(X < 70) = 0.827$, found by binomCdf(100,0.65,0,69)
$p(X \leq 60) = 0.172$, found by binomCdf(100,0.65,0,60)
$p(X \geq 58) = 0.941$, found by binomCdf(100,0.65,58,100)
$p(63 < X < 72) = 0.571$, found by binomCdf(100,0.65,64,72)

Problem 8

$x=?$	$\sin(3x) = 0.4$
	$x = 0.137$, or 0.910 , or 2.232 or 3.004 or 4.326 or 5.099 found by Math, Solver $0=y1-0.4$
Test	tested by Graph, Calc, Intersect ☺

$x=?$	$\cos(1/2x) = -0.3$
	$x = 3.745$ found by Math, Solver $0=y1+0.3$
Test	tested by Graph, Calc, Intersect ☺

$x=?$	$\tan(2x) = 0.7$
	$x = 0.305$, or 1.876 , or 3.447 or 5.018 found by Math, Solver $0=y1-0.7$
Test	tested by Graph, Calc, Intersect ☺

Problem 9

$a=?$	$\tan A = a/b$
$A=40$ $b=7$	$a = 5.874$ found by Math, Solver $0=a/7-\tan 40$
Test	$\tan 40 = 5.874/7$ $0.839 = 0.839$ ☺

$c=?$	$\cos A = b/c$
$A=40$ $b=7$	$c = 9.138$ found by Math, Solver $0=7/c-\cos 40$
Test	$\cos 40 = 7/9.138$ $0.766 = 0.766$ ☺

$B=?$	$A + B = 90$
$A=40$	$B = 50$ found by Math, Solver $0=40+B-90$
Test	$50+40 = 90$ $90 = 90$ ☺

Problem 10

$b = ?$	$a^2 + b^2 = c^2$
$a = 4$	$b = 5.745$
$c = 7$	found by Math, Solver $0 = 4^2 + b^2 - 7^2$
Test	$4^2 + 5.745^2 = 7^2$ $49 = 49$ ☺

$A = ?$	$\sin A = a/c$
$a = 4$	$A = 34.85$
$c = 7$	found by Math, Solver $0 = 4^2 + b^2 - 7^2$
Test	$\sin 34.85 = 4/7$ $0.571 = 0.571$ ☺

$B = ?$	$A + B = 90$
$A = 34.85$	$B = 55.15$
	found by Math, Solver $0 = 34.85 + B - 90$
Test	$34.85 + 55.15 = 90$ $90 = 90$ ☺

Problem 11

$B = ?$	$A + B + C = 180$
$A = 40$	$B = 72$
$C = 68$	found by Math, Solver $0 = 40 + B + 68 - 180$
Test	$40 + 72 + 68 = 180$ $180 = 180$ ☺

$a = ?$	$a/\sin A = b/\sin B$
$A = 40$	$a = 4.731$
$B = 72$	found by Math, Solver $0 = a/\sin 40 - 7/\sin 72$
Test	$4.731/\sin 40 = 7/\sin 72$ $7.360 = 7.360$ ☺

$c = ?$	$c/\sin C = b/\sin B$
$C = 68$	$c = 6.824$
$B = 72$	Math, Solver $0 = c/\sin 68 - 7/\sin 72$
Test	$6.824/\sin 68 = 7/\sin 72$ $7.360 = 7.360$ ☺

Problem 12

$a = ?$	$a^2 = c^2 + b^2 - 2*c*b*\cos A$
$A = 40$	$a = 4.724$
$c = 6.8$	found by Math, Solver $0 = a^2 - 6.8^2 - 7^2 + 2*6.8*7*\cos 40$
Test	$4.724^2 = 6.8^2 + 7^2 - 2*6.8*7*\cos 40$ $22.316 = 22.316$ ☺

$B = ?$	$a/\sin A = b/\sin B$
$A = 40$	$B = 72.3$
$b = 7$	found by Math, Solver $0 = 4.724/\sin 40 - 7/\sin B$
Test	$4.724/\sin 40 = 7/\sin 72.3$ $7.348 = 7.348$ ☺

$C = ?$	$A + B + C = 180$
$A = 40$	$C = 67.7$
$B = 72.3$	found by Math, Solver $0 = 40 + 72.3 + C - 180$
Test	$40 + 72.3 + 67.7 = 180$ $180 = 180$ ☺

Problem 13

$B = ?$	$a/\sin A = b/\sin B$
$A = 40$	$B = 46.53$ or $B = 133.47$
$a = 6.2$	found by Math, Solver or $0 = 6.2/\sin 40 - 7/\sin B$
Test	$6.2/\sin 40 = 7/\sin 46.53 = 7/\sin 133.47$ $9.645 = 9.645 = 9.645$ ☺

$C = ?$	$A + B + C = 180$
$A = 40$	$C = 93.47$ or $C = 6.53$
$B = 46.53$	found by Math, Solver or $0 = 40 + B + C - 180$
Test	$40 + 46.53 + 93.47 = 180$ $180 = 180$ ☺

$c = ?$	$a/\sin A = c/\sin C$
$A = 40$	$c = 9.628$ or $C = 1.097$
$a = 6.2$	found by Math, Solver $0 = 6.2/\sin 40 - c/\sin C$
Test	$6.2/\sin 40 = 9.628/\sin 93.47 = 9.628/\sin 6.53$ $9.645 = 9.645 = 9.645$ ☺

Problem 14

$A = ?$	$a^2 = c^2 + b^2 - 2*c*b*\cos A$
$a = 4$	$A = 33.66$
$c = 6.8$	found by Math, Solver $0 = 4^2 - 6.8^2 - 7^2 + 2*6.8*7*\cos A$
Test	$4^2 = 6.8^2 + 7^2 - 2*6.8*7*\cos 33.66$ $16 = 16$ ☺

$B = ?$	$b^2 = a^2 + c^2 - 2*a*c*\cos B$
$a = 4$	$B = 75.91$
$c = 6.8$	found by Math, Solver $0 = 7^2 - 4^2 - 6.8^2 + 2*6.8*4*\cos B$
Test	$7^2 = 4^2 + 6.8^2 - 2*6.8*4*\cos 75.91$ $49 = 49$ ☺

$C = ?$	$A + B + C = 180$
$A = 33.66$	$C = 70.43$
$B = 75.91$	found by Math, Solver $0 = 33.66 + 75.91 + C - 180$
Test	$33.66 + 75.91 + 70.43 = 180$ $180 = 180$ ☺

Problem 15

$d = ?$	$T = \frac{d}{e-f} + g$
	$T = \left(\frac{d}{e-f}\right) + g$ $d = (e-f)*(T-g)$
Test	$T = \frac{(e-f)*(T-g)}{e-f} + g = T$

$e = ?$	$T = \frac{d}{e-f} + g$
	$T = \left(\frac{d}{e-f}\right) + g$ $(T-g)(e-f) = d$ $e = \frac{d}{T-g} + f$
Test	$T = \frac{d}{\frac{d}{T-g} + f - f} + g = T$

$f = ?$	$T = \left(\frac{d}{e-f}\right) + g$
	$(T-g)(e-f) = d$ $e = \frac{d}{T-g} + f$ $e - \frac{d}{T-g} = f$
Test	$T = \frac{d}{e - e - \frac{d}{T-g}} + g = T$

$g = ?$	$T = \frac{d}{e-f} + g$
	$T = \left(\frac{d}{e-f}\right) + g$ $T - \left(\frac{d}{e-f}\right) = g$
Test	$T = \frac{d}{e-f} + T - \left(\frac{d}{e-f}\right) = T$

Problems 16-18

$y = ?$	$y = a*b^x$
$a = 785$	$y = 785*1.027^5$
$b = 1.027$	$y = 896.85$
$x = 5$	

$x = ?$	$y = a*b^x$
$a = 785$	$x = 8.3$
$b = 1.027$	found by Math, Solver $0 = 785*1.027^x - 980$
Test	$980 = 785*1.027^{8.3}$ $980 = 980$ ☺

$b = ?$	$y = a*b^x$
$a = 785$	$b = 1.045 = 1 + 4.5\%$
$y = 980$	found by Math, Solver $0 = 785*b^5 - 980$
Test	$980 = 785*1.045^5$ $980 = 980$ ☺

$T = \log(2)/\log(1.027) = 26.0$

$T = \log(2)/\log(1.027) = 26.0$

$T = \log(2)/\log(1.045) = 15.7$

Problems 19-21

$y = ?$	$y = a*x + b$
$b = 785$	$y = 2.7*5 + 785$
$a = 2.7$	$y = 798.5$
$x = 5$	

$x = ?$	$y = a*x + b$
$b = 785$	$x = 72.2$
$a = 2.7$	found by Math, Solver $0 = 2.7*x + 785 - 980$
Test	$980 = 2.7*72.2 + 785 = 980$ ☺

$a = ?$	$y = a*x + b$
$b = 785$	$a = 39$
$y = 980$	found by Math, Solver $0 = a*5 + 785 - 980$
Test	$980 = 39*5 + 785 = 980$ ☺