THE SAME MATHEMATICS CURRICULUM FOR DIFFERENT STUDENTS

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To offer mathematics to all students, parallel tracks often occur from the middle of secondary school. The main track continues with a full curriculum, while parallel tracks might use a reduced curriculum leaving out e.g. calculus; or they might contain a different kind of mathematics meant to be more relevant to students by including more applications. Alternatively, a single curriculum may be designed for all students no matter which track they may choose if mathematics as a number-language follows the communicative turn that took place in language education in the 1970s by prioritizing its connection to the outside world higher than its inside connection to its grammar. We will consider examples of all three curricula options.

01. A NEED FOR CURRICULA FOR ALL STUDENTS

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing the Organisation for Economic Co-operation and Development (OECD, 2015) to write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change’:

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)

Other countries also experience declining PISA results; and in high performing countries not all students are doing well.

02. ADRESSING THE NEED

By saying „ All students should study mathematics in each oft the four years that they are enrolled in high school,.“ the US National Council of Teachers of Mathematics (2000, p. 18) has addressed the need for curricula for all students in their publication ‘Principles and Standards for School Mathematics’. In the overview the Council writes

> We live in a mathematical world. Whenever we decide on a purchase, choose an insurance or health plan, or use a spreadsheet, we rely on mathematical understanding (..) In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors. (..) everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics. Students need to learn a new set of mathematics basics that enable them to compute fluently and to solve problems creatively and resourcefully. (p. 1)

In this way the Council points out that it is important to master 'mathematical competence', i.e. to understand and do mathematics to solve problems creatively and to compute fluently. This will benefit the personal life, the workplace, as well as further study leading to productive futures

Consequently, the Council has included in the publication a curriculum that "is mathematically rich providing students with opportunities to learn important mathematical concepts and procedures with understanding". This in order to "provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals."

The publication includes a set of standards: "The Standards for school mathematics describe the mathematical understanding, knowledge, and skills that students should acquire from prekindergarten to grade 12.". The five standards

> present goals in the mathematical content areas of number and operations, algebra , geometry, measurement and data analysis and probability. (..) Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century" (p. 2)

In the chapter 'Number and operations', the Council writes

> Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense. Students with number sense naturally decompose numbers (..) For example, children in the lower elementary grades can learn that numbers can be decomposed an thought about in many different ways - that 24 is 2 tens and 4 ones and also two sets of 12. (p. 7)

In the chapter 'The Curriculum Principle', the Council writes

> A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades (..) for teachers at each level to know what mathematics their students have already studied and will study in future grades. (p. 3, 4)

All in all, the Council points to the necessity of designing a curriculum that is relevant in students' 'personal life, in the workplace and in further study' and that is coherent at the same time to allow teachers to know 'what mathematics their students have already studied and will study in future grades'.

03. COHERENCE AND RELEVANCE

So, in their publication, the National Council of Teachers of Mathematics stresses the importance of coherence and relevance. To allow teachers follow a prescribed curriculum effectively, and to allow students build upon what they already know, it must be 'well articulated across the grades'. And, to have importance for students a curriculum must be relevant by supplying them with 'the basic skills and understandings that students will need to function effectively in the twenty-first century'.

With 'cohere' as a verb and 'relevant' as a predicate we can ask: "to what does this curriculum cohere, and to what is it relevant?" As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.'

We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that ‘to cohere’ means ‘to hold together firmly as parts of the same mass’. As to synonyms for cohere, it lists: ‘accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.’ And as to antonyms, it lists: ‘differ (from), disagree (with).’

In the same dictionary, the word ‘relevant’ means ‘having significant and demonstrable bearing on the matter at hand’. As to synonyms for relevant, it lists: ‘applicable, apposite, apropos, germane, material, pertinent, pointed, relative.’ And as to antonyms, it lists: ‘extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.’

If we accept the verb ‘apply’ as having a meaning close to the predicate ‘relevant’, we can rephrase the above analysis question using verbs only: “to what does this curriculum cohere and apply?”

Metaphorically, we may see education as increasing skills and knowledge by bridging individual start levels to a common end level described by institutional goals. So, we may now give a first definition of an ideal curriculum: “To apply to a learning process as relevant, a curriculum coheres to the individual start levels and to the end goal, which again coheres with the need expressed by the society funding the educational institution. ”

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?
- How to make the end level cohere to goals expressed by the society?
- How to make the end level cohere to goals expressed by the learners?
- How to make the bridge cohere to previous and following bridges?
- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

04. PARALLEL TRACKS TO THE MAIN CURRULUM, EXAMPLES

In their publication chapter Grades 9 through 12, the National Council of Teachers of Mathematics discusses to the possibility to introduce parallel courses in the high school.

> In secondary school, all students should learn an ambitious common foundation of mathematical ideas and applications. This shared mathematical understanding is as important for students who will enter the workplace as it is for those who will pursue further study in mathematics and science. All students should study mathematics in each of the four years that they are enrolled in high school.
>
> Because students’ interests and inspirations may change during and after high school, their mathematics education should guarantee access to a broad spectrum of careers and educational options. They should experience the interplay of algebra, geometry, statistics, probability and discrete mathematics.
>
> High school mathematics builds on the skills and understandings developed in the lower grades. (..) High school students can study mathematics that extends beyond the material expected of all students in at least three ways. One is to include in the curriculum material that extends the foundational material in depth or sophistication. Two other approaches make use of supplementary courses. In the first students enroll in additional courses concurrent with those expected of all students. In the second, students complete a three-year version of the shared material and take other mathematics courses. In both situations, students can

> choose from such courses as computer science, technical mathematics, statistics, and calculus. Each of these approaches has the essential property that all students learn the same foundation of mathematics but some, if they wish, can study additional mathematics. (p. 18-19)

The Council thus emphasizes the importance of studying 'mathematics in each of the four years that they are enrolled in high school'. This the council sees as feasible if implementing one or more of three options allowing students to 'study mathematics that extends beyond the material expected of all students'. Some students may want to study 'material that extends the foundational material in depth or sophistication'. Others may want to take additional courses cohering to the college level, especially calculus. Others may want to take additional courses relevant to their daily life or a workplace. We will now look at two examples of that both including examples of finite mathematics, a subject that is normally outside a standard high school curriculum.

For all Practical Purposes, introduction to Contemporary mathematics

In the US, the Consortium for Mathematics and its Applications (COMAP) has worked out a material called ' For all practical purposes' (COMAP, 1988). In its preface, the material presents itself as

> (..) an introductory mathematics course for students in the liberal arts or other nontechnical curricula. The course consists of twenty-six half-hour television shows, the textbook, and this Telecourse guide. This series shows mathematics at work in todays's world. (..) For all practical purposes aims to develop conceptual understanding of the tools and language of mathematics and the ability to reason using them. We expect most students will have completed elementary algebra and some geometry in high school. We do not assume students will be pursuing additional courses in mathematics, at least none beyond the introductory level. (p. iii)

As to content, the material has five parts (p. v - vi)

Part one focuses on graph theory and linear programming illustrated with network as scheduling and planning. It includes an overview show and four additional shows called street smarts: street networks; trains, planes and critical paths; juggling machines: scheduling problems; juicy problems: linear programming.

Part two deals with statistics and probability illustrated with collecting and deducing from data. It includes an overview show and four additional shows called behind the headlines: collecting data; picture this: organizing data; place your bets: probability; confident conclusions: statistical inference.

Part three focuses on social choice, fair division and game theory illustrated by different voting systems and conflict handling. It includes an overview show and four additional shows called the impossible dream: election theory; more equal than others: weighted voting; zero-sum games: games of conflict; prisoner's dilemma: games of partial conflict.

Part four focuses on using geometry, the classical conic sections, shapes for tiling a surface, geometric growth in finance in and in population, and measurement. It includes an overview show and four additional shows called how big is too big: scale and form; it grows and grows: populations; stand up conic: conic sections; it started in Greece: measurement.

Part five focuses on computer algorithms. It includes an overview show and four additional shows called rules of the games: algorithms; counting by two's: numerical representation; creating a cde: encoding information; moving picture show: computer graphics.

The video sections are available at YouTube.

A Portuguese parallel high school curriculum

Portugal followed up on the COMAP initiative. In his paper called "Secondary mathematics for the social sciences" (Silva, 2018), Jaime Silva describes how the initiative inspired an innovative two-year curriculum for the Portuguese upper secondary school.

As to the background, Silva writes

> There are two recurring debates about the mathematics curriculum in secondary schools, especially in countries like Portugal where compulsory education goes till the 12th grade. First, should all students study mathematics (not necessarily the same) or should the curriculum leave a part of the students "happy" without the mathematics "torture"? Second, should all students study the same "classic" mathematics, around ideas from differential and integral calculus with some Geometry and some Statistics?
>
> When the 2001 revision (in great part in application today) of the Portuguese Secondary School curriculum was made (involving the 10th, 11th and 12th grades) different kinds of courses were introduced for the different tracks (but not for all of them) that traditionally existed. Mathematics A is for the Science and Technology track and for the Economics track and is a compulsory course. Mathematics B is for the Arts track and is an optional course. Mathematics Applied to the Social Sciences (MACS) is for the Social Sciences track and is an optional course. The Languages track was left without mathematics or science. Later the last two tracks were merged and the MACS course remained optional for the new merged track. The technological or professional tracks could have Mathematics B, Mathematics for the Arts or Modules of Mathematics (3 to 10 to be chosen from 16 different modules, depending on the professions). (p. 309)

As to the result of debating a reform in Portugal, Silva writes

> When, in 2001, there was a possibility to introduce a new Mathematics course for the "Social Sciences" track, for the 10th and 11th grade students, there were some discussions of what could be offered. The model chosen was inspired in the course "For All Practical Purposes" (COMAP, 2000) developed by COMAP because it "uses both contemporary and classic examples to help students appreciate the use of math in their everyday lives". As a consequence, a set of independent chapters, each one with some specific purpose, was chosen for this syllabus, that included 2 years of study, with 4.5 hours of classes per week (normally 3 classes of 90 minutes each). The topics chosen were: 10th grade Decision Methods: Election Methods, Apportionment, Fair Division; Mathematical Models: Financial models, Population models Statistics (regression); 11th grade Graph models, Probability models, Statistics (inference). (p. 310)

As to the goal of the curriculum, Silva writes

> The stated goal of this course is for the students to have *"significant mathematical experiences that allow them to appreciate adequately the importance of the mathematical approaches in their future activities"*. This means that the main goal is not to master specific mathematical concepts, but it is really to give students a new perspective on the real world with mathematics, and to change the students view of the importance that mathematical tools will have in their future life. In this course it is expected that the students study simple real situations in a form as complete as possible, and *"develop the skills to formulate and solve mathematically problems and develop the skill to communicate mathematical ideas (students should be able to write and read texts with mathematical content describing concrete situations)"*. (p. 310)

As to the reception of the curriculum, Silva writes

> This was a huge challenge for the Portuguese educational system because most of these topics had never been covered before, and most teachers did not even study Graph Theory at University. Election Methods, Apportionment and Fair Division were of course completely new to everybody. The reception was good from the part of the Portuguese Math Teacher Association APM, as it considered that *"the methodologies and activities suggested in the MACS program promote the development of the skills of social intervention, of citizenship and others"*. The reception from the scientific society SPM was rather negative because they considered the syllabus did not have enough mathematical content. (p. 310-311)

As to the present state of the curriculum, Silva writes

> After 15 years there is no thorough evaluation of how the course is run in practice in the schools, or which is the real impact on the further education or activities of the students that studied “Mathematics Applied to the Social Sciences”. In Portugal there is no institution in charge of this type of work and evaluations are done on a case by case basis. All Secondary Schools need to do selfevaluations but normally just compare internal statistics to national ones to see where they are in the national scene. In the reports consulted there was no special mention to the MACS course and so we have the impression that the MACS course entered the normal Portuguese routine in Secondary School. (p. 315)

So as to a parallel track to the traditional curriculum, the National Council of Teachers of Mathematics suggests that including a different kind of mathematics might be an option, e.g. finite mathematics. In the US this idea was taken up by the Consortium for Mathematics and its Applications (COMAP) working out a material including a textbook and a series of television shows to show ‘mathematics at work in today’s world’. Part of this material was also included in a parallel curriculum in Portugal called ‘Mathematics Applied to the Social Sciences’ (MACS) offering to Portuguese students also to study mathematics in each of their high school years, as the National Council of Teachers of Mathematics recommends.

05. PRE-CALCULUS, TYPICALLY THE LAST MANDATORY CURRICULUM

This chapter looks at the part of a mathematics curriculum called pre-calculus, typically being the first part that is described in a parallel curriculum since it contains operations as root and logarithm that is not considered part of a basic mathematics algebra curriculum. First, we look at an example of a traditional pre-calculus curriculum. The we ask what could be an ideal pre-calculus curriculum, and illustrates it with two examples. In the next chapter, we look at a special case, a Danish pre-calculus curriculum that has served both as a parallel and a serial curriculum during the last 50 years.

A traditional pre-calculus course

An example of a traditional pre-calculus course is found in the Research and Education Association book pre-calculus (Woodward, 2010). The book has ten chapters. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

An ideal pre-calculus curriculum

In their publication, the National Council of Teachers of Mathematics writes “High school mathematics builds om the skills and understandings developed in the lower grades. (p. 19)”

But why that, since in that case high school education will suffer from the lack of student skills and misunderstandings. So why not try designing a fresh-start pre-calculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability to provide and inside prediction about an outside situation as a number-language used for modeling? So, let us try to design a pre-calculus curriculum through its outside use, and not before, inspired by the communicative turn in language education that took place in the 1970s, see e.g. Halliday (1973) and Widdowson (1978).

Let students begin again with the basics of mathematics, numbers, and see that a number as 345 is really three numberings of bundles-of-bundles, bundles and unbundled as expressed in the number-formula, formally called a polynomial: T = 345 = 3*B^2 + 4*B + 5*1, with B = ten.

Let students see that a number-formula contains the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, students see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** *T – n = a*	**T = a*n** *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫a*dn** *dT/dn = a*	**T = a^n** $\sqrt[n]{T} = a \quad \log_a T = n$

Figure 01. *The 'algebra-square' shows the four ways to unite or split numbers.*

Let students see calculations as predictions, where 2+3 predicts what happens when counting on 3 times from 2; where 2*5 predicts what happens when adding 2$ 5 times; where 1.02^5 predicts what happens when adding 2% 5 times; and where adding the areas 2*3 + 4*5 predicts how to add per-numbers when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Let students see subtraction as x = 5-3 defined as the number x that added to 3 gives 5, x+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers 'to opposite sign with opposite calculation sign'; a rule that applies also to the other reversed operations:

● the division x = 5/3 is the number x that multiplied with 3 gives 5, x*3 = 5

● the root x = 3√5 is the factor x that applied 3 times gives 5, x^3 = 5, making root a 'factor-finder'

● the logarithm x = log3(5) is the number x of 3-factors that gives 5, 3^x = 5, making logarithm a 'factor-counter'.

Let students see that multiple calculations reduce to single calculations by unhiding 'hidden parentheses' where 2+3*4 = 2+(3*4) since with units, 2+3*4 = 2*1+3*4. This will prevent solving the equation 2+3*x = 14 as 5*x = 14 or x = 14/5, by allowing the hidden parentheses to be placed:

2+3*x = 14, so 2+(3*x) = 14, so 3*x = 14-2, so x = (14-2)/3, so x = 4 to be verified by testing: 2+3*x = 2+(3*x) = 2+(3*4) = 2 + 12 = 14.

Let students build or rebuild confidence by transposing formulas as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let students enjoy the power and beauty of the number-formula, containing also the formulas for constant change: T = b*x (proportional), T = b*x + c (linear), T = a * x^n (elastic), T = a * n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to ‘post-dict’ what we cannot ‘pre-dict’.

Combining linear and exponential change by n times depositing a$ to an interest rate r%, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest rate R is predicted by the formula 1+R = (1+r)^n. Such a saving may be used to neutralize a debt Do, that in the same period has changed to D = Do*(1+R).

The formula and the proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into variable per-numbers creating the operations integration and its reverse, differentiation. Likewise, we may define the core of a pre-calculus course as adding and splitting into constant per-numbers by creating the operation power and its two inverse operations, root and logarithm.

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$ changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, leading to the formula for change with a constant change-factor, also called exponential change, T = b*a^n. This formula entails two equations.

The first equation asks about an unknown change-percent. Thus, we might want to find which percentage that added ten times will give a total change-percent 70%, or, formulated with change-factors, what is the change-factor that applied ten times gives the change-factor 1.70: a^10 = 1.70. So here the job is ‘factor-finding’ which leads to defining the tenth root of 1.70, 10√1.70, as predicting the factor a that applied 10 times gives 1.70: If a^10 = 1.70 then a = 10√1.70 = 1.054 = 105.4%. This is verified by testing: 1.054^10 = 1.692. Thus, the answer is “5.4% is the percentage that added ten times will give a total change-percent 70%.”

We notice that 5.4% added ten times gives 54% only, so the 16% remaining to 70% is the effect of compound interest adding 5.4% also to the previous changes.

Here we solve the equation a^10 = 1.70 by moving the exponent to the opposite side with the opposite calculation sign, the tenth root, a = 10√1.70. This resonates with the ‘to opposite side with opposite calculation sign’ method that also solved the equations a+3 = 7 by a = 7-3, and a*3=20 by a = 20/3.

The second equation asks about a time-period. Thus, we might want to find how many times 7% must be added to give 70%, 1.07^n = 1.70. So here the job is factor-counting which leads to defining the logarithm log1.07(1.70) as the number of factors 1.07 that will give a total factor at 1.70: If 1.07^n = 1.70 then n = log1.07(1.70) = 7,84 verified by testing: 1.07^7.84 = 1.700.

We notice that simple addition of 7% added ten times gives 70%, but with compound interest it gives a total change-factor 1.07^10 = 1.967, i.e. an additional change at 96.7%-70% = 26.7%, explaining why only 7.84 periods are needed instead of ten.

Here we solve the equation 1.07^n = 1.70 by moving the base to the opposite side with the opposite calculation sign, the base logarithm, n = log1.07(1.70). Again, this resonates with the 'to opposite side with opposite calculation sign' method.

An ideal pre-calculus curriculum could de-model the constant percent change exponential formula T = b*a^n to outside real-world problems as a capital in a bank, or as a population increasing or decreasing by a constant change-percent per year.

De-modeling may also lead to situations where the change-elasticity is constant as in science multiplication formulas wanting to relate a percent change in T with a percent change in n.

An example is the area of a square T = s^2 where a 1% change in the side s will give a 2% change in the square, approximately: With To = s^2, T1 = (s*1.10)^2 = s^2*1.01^2 = s^2* 1.0201 = To*1.0201.

Once mastery of constant change-percentage is established, it is possible to look at time series in statistical tables asking e.g. "How has the unemployment changed over a ten-year period?" Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b+a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n. These average numbers allow setting up a forecast predicting the yearly numbers in ten-year period, if the numbers were predictable. However, they are not, so instead of predicting the number with a formula, we might 'post-dict' the numbers using statistics dealing with unpredictable numbers, but still trying to predict a plausible interval by calculating a mean and a deviation after having described the variation by a median and quartiles.

Likewise, real-world phenomena as unemployment occur in both time and space, so unemployment may also change with space, e.g. from one region to another. This leads to double tables sorting the workforce according to two categories, region and employment status. The numbers lead to percentages within each of the categories. To find the total employment percent, the single percentages do not add, they must be multiplied back to numbers to find the total percentage. However, once you multiply you create an area, and adding per-numbers by areas is what calculus is about, thus here introduced in a natural way through double-tables from statistical materials.

An example: in one region 10 persons have 50% unemployment, in another, 90 persons have 5% unemployment. To find the total, the unit-numbers can be added directly to 100 persons, but the percentages must be multiplied back to numbers: 10 persons have 10*0,5 = 5 unemployed; and 90 persons have 90*0,05 = 4.5 unemployed, a total of 5+4.5 unemployed = 9.5 unemployed among 100 persons, i.e. a total of 9.5% unemployment, also called the weighted average.

With per-numbers as the core of a pre-calculus curriculum, it is only natural to include also per-numbers in geometry introduced by trigonometry. Geometry means to measure earth in Greek. The earth can be divided into triangles that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula. The sides connect with the angles by formulas recounting one side in the other side or in the diagonal: a = (a/c)*c = sinA * c; b = (b/c)*c = cosA * c; a = (a/b)*b = tanA * b. With a circle filled from the inside by right triangles, this also allows phi to be found from a formula: circumference/diameter = $\pi \approx$ n*tan(180/n) for n large.

Furthermore, the occurrence of graphic display calculators allows authentic modeling to be included in a pre-calculus curriculum thus giving a natural introduction to the following calculus curriculum.

Regression thus allows a table to be translated into a formula. Here a two data-set table allows modeling with a degree1 polynomial with an algebraic parameter representing the geometrical steepness. And a three data-set table allows modeling with a degree2 polynomial with algebraic parameters representing the geometrical initial steepness and bending. And a four data-set table allows modeling with a degree3 polynomial with algebraic parameters representing the geometrical initial steepness, initial bending and counter-bending. Here the bending means that the steepness changes from a number to a formula, and disappears in top- and bottom points, easily located by a graphical display calculator, that also finds the area under a graph to add piecewise or locally constant per-numbers.

Regression thus allows giving a practical introduction to calculus by analysing a road trip where the per-number speed was measured in five second intervals to respectively 10 m/s, 30 m/s, 20 m/s 40 m/s and 15 m/s. With a five data-set table we can choose to model with a degree4 polynomial found by regression. Within this model we can predict when the driving began and ended, what the speed and the acceleration was after 12 seconds, when the speed was 25m/s, when acceleration and braking took place, what was the maximum speed, and what distance was covered in total and in the different intervals.

Another example is the project ‘Population versus food’ looking at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2- line table regression can produce two formulas: with x counting years after 1850, the population is modeled by Y1= 815*1.013^x and the food by Y2= 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

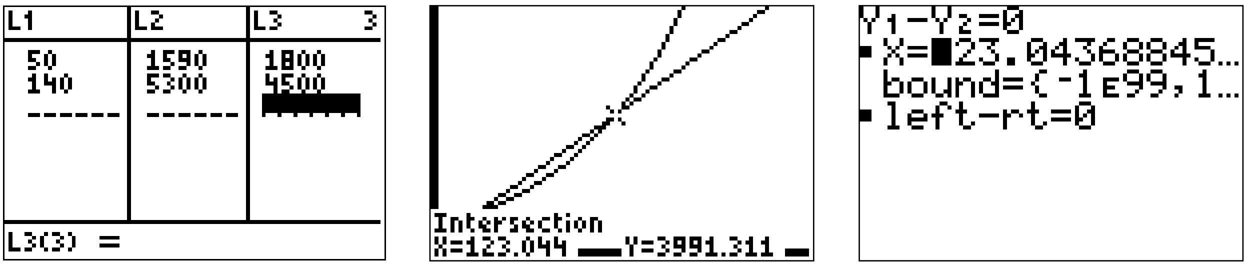

Figure 02. A Malthusian model of population and food levels

An example of an ideal pre-calculus curriculum is described in a paper called ‘Saving Dropout Ryan With a Ti-82’ (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' (Tarp, 2009) replaced the textbook. A formula's left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

The compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

With the increased educational interest on STEM (Science, Technology, Engineering and Mathematics) modeling also allows including science-problems as e.g. the transfer of heat taking place when placing an ice cube in water or in a mixture of water and alcohol, or the transfer of energy taking place when connecting an energy source with energy consuming bulbs in series or parallel; as well as technology problems as how to send of a golf ball to hit a desired hole, or when to jump from a swing to maximize the jumping length; as well as engineering problems as how to build a road inclining 5% on a hillside inclining 10%.

Furthermore, pre-calculus allows students to play with change-equations, later called differential equations since change is calculated as a difference, $\Delta T = T2 - T1$. Using a spreadsheet, it is fun to see the behavior of a total changing with a constant number or a constant percentage, as well as with a decreasing number or a decreasing percentage, as well as with half the distance to a maximum value or with a percent decreasing until disappearing at a maximum value. And to see the behavior of a total accelerating with a constant number as in the case of gravity, or with a number proportional to its distance to an equilibrium point as in the case of a spring.

So by focusing on uniting and splitting into constant per-numbers, the ideal pre-calculus curriculum having the constant change-percent as its core formula will cohere with a previous curriculum on constant change-number or linearity, as well to the next curriculum in calculus created to unite and split into locally constant per-numbers deals with local linearity. Likewise, such a pre-calculus curriculum is relevant to the workplace where forecasts based upon assumptions of a constant change-number or change-percent are frequent. This curriculum is also relevant to the students' daily life as participants in civil society where understanding tables presented in the media is frequent.

Two curriculum examples inspired by an ideal pre-calculus curriculum

An example of a curriculum inspired by the above outline was tested in a Danish high school around 1980. Its goal was that the students know how to deal with quantities in other school subjects and in their daily life. Its means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

Later, around year 2000, another version was designed but not tested.

Its goal was that students develop their number-language so they can participate in social practices involving quantitative descriptions of change and shape. It means included

1. Numbers and calculations. Quantities and qualities. Number-language, word-language, meta-language. Unit-numbers and per-numbers, and how to calculate their totals. Equations as predicting statements. Forwards and reverse calculations.

2. Change calculations. Change measuring change with change-number and change-percent and index-number. Calculation rules for the change of a sum, a product and a ratio.

3. Constant change. Change with a constant change-number. Change with a constant change-percent. Change with both.

4. Unpredictable change. Fractals, mean and deviation, 95% confidence interval. Binomial distribution approximated by a normal distribution.

Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

5. Trigonometry. Dividing and measuring earth. Calculation the sides and angles in a triangle.

06. PRECALCULUS IN THE DANISH PARALLEL HIGH SCHOOL, A CASE STUDY

In the post-war era, the Organization for Economic Co-operation and Development (OECD) called for increasing the population knowledge level, e.g. by offering a second chance to take a high school degree giving entrance to tertiary education. In Denmark in 1966, this resulted in creating a two-year education called 'Higher preparation exam' as a parallel to the traditional high school. Two levels of two-years mathematics courses were included, a basic pre-calculus course for those who did not choose the calculus course.

The 1966 curriculum

The pre-calculus curriculum came from leaving out small parts of the calculus curriculum, thus being an example of a reduced curriculum.

The goal of the calculus course stated it should 'supply students with knowledge about basic mathematical thinking and about applications in other subject areas, thus providing them with prerequisites for carrying through tertiary education needing mathematics.'

The goal of the pre-calculus course was reduced to ‘supplying students with an impression of mathematical thinking and method and to mediate mathematical knowledge useful also to other subject areas.’

So, where the calculus curriculum has to cohere and be relevant to tertiary education needing mathematics, the pre-calculus course is a parallel curriculum meant to be relevant to the students themselves and to other high school subjects.

The content of the pre-calculus curriculum had five sections.

The first section contained basic concepts from set theory as sets, subsets, complementary set, union, intersection, product, difference. The function concept. Mapping into an on a different set, one-to one mapping, inverse mapping (inverse function), composite mappings. The calculus curriculum added nothing here.

Section two contained concepts from abstract algebra: Composition rules. The associative law. The commutative law. Neutral element. Inverse element. The group concept with examples. Rules for operations on real numbers. Numeric value. Here the calculus curriculum added the distributive law, the concept of a ring and a field, the ring of whole numbers as well as quotient classes. The calculus curriculum added nothing here.

Section three contained equations and inequalities. Examples on open statements in one or two variables. Equations and inequalities of degree one and two with one unknown. Equations and inequalities with the unknown placed inside a square root or a numeric sign. Simple examples of Equations and inequalities of degree one and two with two unknowns. Graphical illustration. The calculus curriculum added nothing here.

Section four contained basic functions. The linear function in one variable. A piecewise linear function. The second-degree polynomial. The logarithm function with base ten, the logarithmic scale, the calculator stick, the use of logarithm tables. Trigonometric functions, tables with functions values. Calculations on a right-angled triangle using trigonometric functions. Here the calculus curriculum added rational functions in one variable, exponential functions, and the addition formulas and logarithmic formulas in trigonometry.

Section five contained combinatorics. The multiplication principle. Permutations and combinations. Here the calculus curriculum added probability theory, probability field, and examples of probability based upon combinatorics.

Finally, the calculus curriculum added a section about calculus.

The new set-based mathematics coming into education around 1960 inspired the 1966 pre-calculus curriculum thus cohering with the university mathematics at that time, but it was not especially relevant to the students. Many had difficulties understanding it and they often complained about seeing no reason for learning it or why it was taught.

In my own class, I presented it as a legal game where we were educating us to become lawyers that could convince a jury that we were using lawful methods to solving equations in one of two different methods by referring to the relevant paragraphs in the law. The first method was the traditional one used at that time way by moving numbers to the opposite side with opposite calculation sign, now legitimized by the theorem that in a group the equation a*u = b has as a solution a^-1*b. The second method was a new way with many small steps where, for each step, you have to refer to laws for associativity, and commutativity etc.; and, where a group contained exactly the paragraphs needed to use this method. Once seen that way, the students found it easy but boring. However they accepted since they needed the exam to go on, and we typically finished the course in half time allowing time for writing a script for a movie to be presented at the annual gala party.

So all in all, the 1963 curriculum was coherent with the next step, calculus, and with the university math view at that time, set-based; but it was mostly irrelevant to the students.

The 1974 curriculum

The student rebellion in 1968 asked for relevance in education, which led to a second pre-calculus in 1974 revision. Here the goal was stated as ‘giving the students a mathematical knowledge that could be useful to other subjects and to their daily life, as well as an impression of mathematical methods thinking’. Now the curriculum structure was changed from a parallel one to a serial one where all students took the pre-calculus course and some chose to continue with the calculus course afterwards just specifying in its curriculum what was needed to be added.

The 1974 pre-calculus curriculum now had four sections.

The first section contained concepts from set theory and logic and combinatorics. Set, subset; solution set to an open statement, examples on solving simple equations and inequalities in one variable; the multiplication principle, combinations.

Section two contained the function concepts: Domain, function value, range; injective function; monotony intervals; inverse function, composite function.

Section three contained special functions; graphical illustration. A linear function, a piecewise linear function, an exponential function; examples of functions defined by tables; coordinate system, logarithmic paper.

Section four contained descriptive statistics. Observations described by numbers; frequency and their distribution and cumulated distribution; graphical illustration; statistical descriptors.

Section five described probability and statistics. A random experiment, outcome space, probability function, probability field; sampling; binomial distribution; binomial testing with zero hypothesis, critical set, significance level, single and double-sided test, failure of first degree.

Section six was called ‘Free lessons’. 20m lessons are to be used for studying details in one of the above sections, or together with one or more other school subjects to work with an area applying mathematics.

The second 1974 curriculum thus maintains a basis of set-theory but leaves out the abstract algebra. As to functions, it replaces the second-degree polynomial with the exponential function. Here trigonometry is excluded to be included in the calculus curriculum.

The combinatorics section is to great extent replaced by descriptive statics.

Finally, the section has been added with quite detailed probability theory and testing theory within statistics.

All in all, the coherence with the university set-based mathematics has been softened by leaving out abstract algebra and second-degree polynomial. Instead of introducing a first-degree polynomial together with a second-degree polynomial, the former now is introduced as a linear function together with the exponential function allowing modelling outside change with both a constant change-number and a constant change-percent. This makes the curriculum more relevant to the students individually as well as to other high school subjects as required by the goal statement.

The quite detailed section on testing theory was supposed to make the curriculum more relevant to students but the degree of detail make it fail to do so by drowning in quite abstract concepts.

The 1990 curriculum

As the years passed on it was observed that the free hours were used on trigonometry, and on savings and instalments, the first cohering with the following calculus course, the latter highly relevant to many students, and at the same time combining linear and exponential change, the core of the curriculum. This let to designing an alternative curriculum around 1990 to choose instead of the standard curriculum if wanted.

The 1990 curriculum did not change the goal but included the following subjects

1) Numbers, integers, rational and real numbers together with their calculation rules. Number sets. Calculations with power and root.

2) Calculations including percentages and interest rates: Average percentage, index number, weighed average. Simple and compound interest, saving and installments.

3) Geometry and trigonometry. Similar triangles. Right triangles. Calculations on sides and angles.

4) Functions. The function concept, domain, functional values, range, monotony. Various ways to define a function. Elementary functions as linear, piecewise linear and exponential growth and decay. Coordinate system. Examples of simple equations and inequalities including the functions mentioned above.

5) Probability and statistics. A stochastic experiment. Discrete stochastic variables, probability distribution, mean value, binomial distribution, observation sets described graphically, representation by statistical descriptors, examples of a normal distribution, normal distribution paper.

6) Calculation aids. Pocket calculator, formulas, tables, semi logarithmic paper, normal distribution paper.

The 2005 curriculum

Then a major reform of the Danish upper secondary high school was planned for 2005. As to pre-calculus, it was inspired the occurrence of graphical display calculators and computer assisted systems allowing regression to transform tables into formulas, thus allowing realistic modeling to be included.

Now the goal defined the competences students should acquire:

The students can

- handle simple formulas and translate between symbolic and natural language and use symbolic language to solve simple problems with a mathematical content.

- apply simple statistical models for describing a given data set, pose questions based upon the model and sense what kind of answers are to be expected and knows how to formulate conclusion in a clear language.
- apply relations between variables to model a given data set, can make forecasts, and can reflect on them and their domain of relevance
- describe geometrical models and solve geometrical problems
- produce simple mathematical reasoning
- demonstrate knowledge about mathematical methods, applications of mathematics, and examples of cooperation between mathematics and other sciences, as well as its cultural and historical development
- apply information technology for solving mathematical problems

The means include

- The hierarchy of operations, solving equations graphically and with simple analytical methods, calculating percentages and interest rates, absolute and relative change
- Formulas describing direct and inverse proportionality as well as linear, exponential ad power relations between variables
- Simple statistical methods for handling data sets, graphical representation of statistical materials, simple statistical descriptors
- Ratios in similar triangles and trigonometry used for calculations in arbitrary triangles.
- xy-plot of data sets together with characteristics of linear, exponential and power relations, the use of regression.
- Additional activities for 25 lessons are examples of mathematical reasoning and proofs, modeling authentic data sets, examples of historical mathematics.

The 2017 curriculum

Then in 2017 e new reform was made to inspire more students to continue with the calculus level by moving some subjects to the pre-calculus level:

- interpreting the slope of a tangent as a growth rate in a mathematical model
- combinatorics, basic probability theory and symmetrical probability space
- the function concept and characteristics of linear, exponential and power functions and their graphs
- graphical handling of a quadratic function, and the logarithm functions and their characteristics
- graphical determination of a tangent, and monotony intervals, as well as finding extrema values in a closed interval
- prime characteristics at mathematical models and simple modelling using the functions above alone or in combination.

Relevance and coherence

The 1966 had internal coherence with the previous and following curriculum, but with the emphasis on abstract algebra, there was little external coherence. It was indirectly relevant to students wanting later to take a calculus course but only little relevant to the daily life of students

The 1972 curriculum took the consequence and changed from a parallel curriculum to a serial curriculum so that it had internal coherence to the calculus curriculum, and by replacing quadratics with exponential functions, it obtained an external relevance to change calculations with a constant change-number or a constant change-percent. Also by including considerable amount of probability gave coherence to eternal testing situations, however these were not part of student daily life, so they didn't add to the relevance for students. However, including the free lessons allowed the students to choose areas that they found relevant, in this case interest rates and saving and installment calculations as well as trigonometry.

The 1990 curriculum was inspired by this and re-included trigonometry and interest rates while at the same time reducing probability a little.

The 2005 reform was informed by the occurrence of competence concept as well as the advances in calculation technology. Her the function concept was replaced by variables to make it cohere more with external applications in science and economics and daily life. Now the probability was gone, so this curriculum showed coherence and relevance to external appliers and to the student's daily life as well for other school subjects. It was close to the ideal pre-calculus curriculum.

The 2017 reform was inspired by the wish to motive more to continue with a calculus course, so part of this was moved down to the pre-calculus level, making the two levels cohere better, however the things imported had little relevance to the students' daily life.

07. A REFUGEE CAMP CURRICULUM

The name 'refugee camp curriculum' is a metaphor for a situation where mathematics is taught from the beginning and with simple manipulatives. Thus, it is also a proposal for a curriculum for early childhood education, for adult education, for educating immigrants and for learning mathematics outside institutionalized education. It considers mathematics a number-language parallel to our word-language, both describing the outside world in full sentences, typically containing a subject and a verb and a predicate. The task of the number-language is to describe the natural fact Many in space and time, first by counting and recounting and double-counting to transform outside examples of Many to inside sentences about the total; then by adding to unite (or split) inside totals in different ways depending on their units and on them being constant or changing. This allows designing a curriculum for all students inspired by Tarp (2018) that focuses om proportionality, solving equations and calculus from the beginning, since proportionality occurs when recounting in a different unit, equations occur when recounting from tens to icons, and calculus occurs when adding block-numbers next-to and when adding per-numbers coming from double-counting in two units.

Talking about 'refugee camp mathematics' thus allows locating a setting where children do not have access to normal education, thus raising the question 'What kind and how much mathematics can children learn outside normal education especially when residing outside normal housing conditions and without access to traditional leaning materials?'. This motivates another question 'How much mathematics can be learned as ‘finger-math’ using the examples of Many coming from the body as fingers, arms, toes and legs?’

So the goal of ‘refugee camp mathematics’ is to learn core mathematics through ‘Finger-math’ disclosing how much math comes from counting the fingers.

Focus 01: Digits as icons with as many outside sticks and inside strokes as they present

Activity 01. With outside things (sticks, cars, dolls, animals), many ones are rearranged into one many-icon with as many things as it represents. Inside, we write the icon with as many strokes as it represents. Observe that the actual digits from 1 to 9 are icons with as many strokes as they represent if written less sloppy. A discovery glass showing nothing is an icon for zero. When counting by bundling in tens, ten become ‘1 Bundle, 0 unbundled’ or 1B0 or just 10, thus needing no icon since after nine, a double-counting takes place of bundles and unbundled.

Focus 02. Counting ten fingers in various ways

Activity 01. Double-count ten fingers in bundles of 5s and in singles

● Outside, lift the finger to be counted; inside say “0 bundle 1, 0B2, 0B3, 0B4, 0B5 or 1B0. Then continue with saying “1B1, …, 1B5 or 2B”. ● Outside, look at the fingers not yet counted; inside say “1 bundle less4, 1B-3, 1B-2, 1B-1, 1B or 1B0. Then continue with saying “2B-4, …, 2B or 2B0”. ● Outside, show the fingers as ten ones. ● Outside, show ten fingers as 1 5s and 5 1s; inside say “The total is 1Bundle5 5s” and write ‘T = 1B5 5s’. ● Outside, show ten fingers as 2 5s; inside say “The total is 2Bundle0 5s” and write ‘T = 2B0 5s’.

Activity 02. Double-count ten fingers in bundles of tens and in singles

● Outside, lift the finger to be counted; inside say “0 bundle 1, 0B2, 0B3, …, 0B9, 0Bten, or 1B0”. ● Outside, look at the fingers not yet counted; inside say “1 bundle less9, 1B-8, …, 1B-2, 1B-1, 1B or 1B0.

Activity 03. Counting ten fingers in bundles of 4s using ‘flexible bundle-numbers’.

● Outside, show the fingers as ten ones, then as one tens. ● Outside, show ten fingers as 1 4s and 6 1s; inside say “The total is 1Bundle6 4s, an overload” and write ‘T = 1B6 4s’ ● Outside, show ten fingers as 2 4s and 2 1s; inside say “The total is 2Bundle2 4s, a standard form” and write ‘T = 2B2 4s’. ● Outside, show ten fingers as 3 4s less 2; inside say “The total is 3Bundle, less2, 4s, an underload” and write ‘T = 3B-2 4s’.

Activity 04. Counting ten fingers in bundles of 3s using ‘flexible bundle-numbers’.

● Outside, show ten fingers as 1 3s and 7 1s; inside say “The total is 1Bundle7 3s, an overload” and write ‘T = 1B7 3s’. ● Outside, show ten fingers as 2 3s and 4 1s; inside say “The total is 2Bundle4 3s, an overload” and write ‘T = 2B4 3s’. ● Outside, show ten fingers as 3 3s and 1 1s; inside say “The total is 3Bundle1 3s, a standard form” and write ‘T = 3B1 3s’. ● Outside, show ten fingers as 4 3s less 2; inside say “The total is 4Bundle, less2, 3s, an underload” and write ‘T = 4B-2 3s’.

Activity 05. Counting ten fingers in bundles of 3s, now also using bundles of bundles.

● Outside, show ten fingers as 3 3s (a bundle of bundles) and 1 1s; inside say “The total is 1BundleBundle1 3s” and write ‘T = 1BB1 3s’. Now, inside say “The total is 1BundleBundle 0 Bundle

1 3s" and write 'T = 1BB 0B 1 3s'. Now, inside say "The total is 1BundleBundle 1 Bundle, less2, 3s" and write 'T = 1BB 1B -2 3s'.

Focus 03. Counting ten sticks in various ways

The same as Focus 02, but now with sticks instead of fingers.

Focus 04. Counting ten cubes in various ways

The same as Focus 02, but now with cubes, e.g. centi-cubes or Lego Bricks, instead of fingers. When possible, transform multiple bundles into 1 block, e.g. 2 4s = 1 2x4 block; inside say "The total is 1 2x4 block" and write 'T = 2B0 4s.'

Focus 05. Counting a dozen of finger-parts in various ways

Except for the thumps, our fingers all have three parts. So four fingers have three parts four times, i.e. a total of T = 4 3s = 1 dozen finger-parts.

Focus 05 is the same as focus 02, but now with a dozen finger-parts instead of ten fingers.

Focus 06. Counting a dozen sticks in various ways

Focus 06 is the same as focus 03, but now with a dozen sticks instead of ten.

Focus 07. Counting a dozen cubes in various ways

Focus 07 is the same as focus 04, but now with a dozen cubes instead of ten.

Focus 08. Counting numbers with underloads and overloads.

Activity 01. Totals counted in tens may also be recounted in under- or overloads.

● Inside, rewrite T = 23 as T = 2B3 tens, then as 1B13 tens, then as 3B-7tens. ● Try other two-digit numbers as well. ● Inside, rewrite T = 234 as T = 2BB3B4 tens, then as T = 2BB 2B14, then as T = 2BB 4B-6. Now rewrite T = 234 as T = 23B4, then as 22B14, then as 24B-6. Now rewrite T = 234 as T = 3BB-7B4, then as 3BB-6B-6. ● Try other three3-digit numbers as well.

Focus 09. Operations as icons showing pushing, lifting and pulling

Activity 01. Transform the three outside counting operations (push, lift and pull) into three inside operation-icons: division, multiplication and subtraction.

● Outside, place five sticks as 5 1s. ● Outside, push away 2s with a hand or a sheet; inside say "The total 5 is counted in 2s by pushing away 2s with a broom iconized as an uphill stroke" and write 'T = 5 = 5/2 2s'. ● Outside, rearrange the 2 2s into 1 2x2 block by lifting up the bundles into a stack; inside say "The bundles are stacked into a 2x2 block by lifting up bundles iconized as a lift" and write 'T = 2 2s = 2x2'. ● Outside, pull away the 2x2 block to locate unbundled 1s; inside say "The 2x2 block is pulled away, iconized as a rope" and write 'T = 5 – 2x2 = 1'.

Five counted in 2s:

I I I I I (push away 2s) II II I (lift to stack) II/II I (pull to find unbundles ones) II/II I

Focus 10. The inside recount-formula T = (T/B)xB predicts outside BundleCounting results

Activity 01. Use a calculator to predict a bundle-counting result by a recount-formula T = (T/B)xB, saying "from T, T/B times, B is pushed away", thus using a full number-language sentence with a subject, a verb and a predicate.

● Outside, place five cubes as 5 1s. ● Outside, push away 2s with a 'broom'; inside say "Asked '5/2', a calculator answers '2.some', meaning that 2 times we can push ways bundles of 2s. ● Outside , stack the 2s into one 2x2 stack by lifting; inside say "We lift the 2 bundles into one 2x2 stack, and we write T = 2 2s = 2x2 ● Outside, we locate the unbundled by, from 5 pulling away the 2x2 block; inside we say "Asked '5-2x2', a calculator answers '1'. We write T = 2B1 2s and say "The recount-formula predicts that 5 recounts in 2s as T = 2B1 2s, which is tested by recounting five sticks manually outside."

Activity 02. The same as activity 01, but now with 4 3s counted in 5s, 4s and 3s.

Focus 11. Discovering decimals, fractions and negative numbers.

Activity 01. When bundle-counting a total, the unbundled can be placed next-to or on-top.

● Outside, chose seven cubes to be counted in 3s. ● Outside, push away 3s to be lifted into a 2x3 stack to be pulled away to locate one unbundled single. Inside use the recount-formula to predict the result, and say "seven ones recounts as 2B1 3s" and write T = 2B1 3s. ● Outside, place the single next-to the stack. Inside say "Placed next-to the stack the single becomes a decimal-fraction '.1' so now seven recounts as 2.1 3s" and write T = 2.1 3s. ● Outside, place the single on-top of the stack. Inside say "Placed on-top of the stack the single becomes a fraction-part 1 of 3, so now seven recounts as 2 1/3 3s" and write T = 2 1/3 3s. Now, inside say "Placed on-top of the stack the single becomes a full bundle less 2, so now seven recounts as 3.-2 3s" and write T = 3.-2 3s. Finally, inside say "With 3 3s as 1 bundle-bundle of 3s, seven recounts as 1BB-2 3s."

Activity 02. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 3s.

Activity 03. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 4s.

Activity 04. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 5s.

Focus 12. Recount in a new unit to change units, predicted by the recount-formula

Activity 01. When bundle-counting, all numbers have units that may be changed into a new unit by recounting predicted by the recount-formula.

● Outside, chose 3 4s to be recounted in 5s. ● Outside, rearrange the block in 5s to find the answer T = 3 4s = 2B2 5s. Inside use the recount-formula to predict the result, and say "three fours recounts as 2B2 5s" and write T = 3 4s = 2B2 5s = 3B-3 5s = 2 2/5 5s. Repeat with other examples as e.g. 4 5s recounted in 6s.

Focus 13. Recount from tens to icons

Activity 01. A total counted in tens may be recounted in icons, traditionally called division.

● Outside, chose 29 or 2B9 tens to be recounted in 8s. ● Outside, rearrange the block in 8s to find the answer T = 29 = 3B5 8s and notice that a block that decreases its base must increase its height to keep the total the same. Inside use the recount-formula to predict the result, and say "With the recount-formula, a calculator predicts that 2 bundle 9 tens recounts as 3B5 8s" and write T = 29 = 2B9 tens = 3B 5 8s = 4B-3 8s = 3 5/8 8s. Repeat with other examples as e.g. 27 recounted in 6s.

* Now, inside reformulate the outside question 'T = 29 = ? 8s' as an equation using the letter u for the unknown number, u*8 = 24, to be solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign. Use an outside ten-by-ten abacus to see that when a block decreases its base from ten to 8, it must increase its height from 2.4 to 3. Repeat with other examples as e.g. 17 = ? 3s.

Focus 14. Recount from icons to tens

Activity 01. Oops, without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. Use an outside ten-by-ten abacus to see that when a block increases its base from 7 to ten, it must decrease its height from 3 to 2.1.

Activity 02. Use 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently 'less less 1' means adding 1.

Focus 15. Double-counting in two physical units

Activity 01. We observe that double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$. Repeat with other examples as e.g. 4$ per 5days.

Focus 16. Double-counting in the same unit creates fractions

Activity 01. Double-counting in the same unit creates fractions and percentages as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%. We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Repeat with other examples as e.g. 2$/5$.

Focus 17. Double-count mutually the sides in a block halved by its diagonal

Activity 01. Recount sides in a block halved by its diagonal? Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. Use these formulas to predict the sides in a half-block with base 6 and angle 30 degrees. Use these formulas to predict the angles and side in a half-block with base 6 and height 4.

Focus 18. Adding NextTo

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Since 3*5 is an area, adding next-to in 8s means adding areas, called integral calculus. Asking a calculator, the two answers, '2.some' and '5', predict the result as 2B5 8s.

Focus 19. Reversed adding NextTo

Activity 01. With T1 = 2 3s and T2 adding next-to as T = 4 7s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Asking '3 5s and how many 3s total 2B6 8s?', using sticks will give the answer 2B1 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula

y' = Δy/t = (y2 – y1)/t.

Focus 20. Adding OnTop

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Adding on-top in 5s, '3 5s + 2 3s = ? 5s?', re-counting must make the units the same. Asking a calculator, the two answers, '4.some' and '1', predict the result as 4B1 5s.

Focus 21. Reversed adding OnTop

Activity 01. With T1 = 2 3s and T2 as some 5s adding to T = 4 5s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting. Reversed addition is called backward calculation or solving equations.

Focus 22. Adding tens

Activity 01. With T1 = 23 and T2 = 48, what is T1+T2 id added as tens?" Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Focus 23. Subtracting tens

Activity 01. "If T1 = 23 and T2 add to T = 71, what is T2?" Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Focus 24. Multiplying tens

Activity 01. "What is 7 43s recounted in tens?" Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Focus 25. Dividing tens

Activity 01. "What is 348 recounted in 6s?" Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Focus 26. Adding per-numbers

Activity 01. “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining local constancy by epsilon and delta.

Activity 02. Two groups of voters have a different positive attitude to a proposal. How to find the total positive attitude?

● Asking “20 voters with 30% positive + 60 voters with 10% positive = 80 voters with ? positive.” Here we see that the unit-numbers 20 and 40 add directly whereas the per-numbers 30% and 10% add by areas since they must first transform to unit-numbers by multiplication, creating the areas.

Focus 27. Subtracting per-numbers

Activity 01. “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Focus 28. Adding differences

Activity 01. Adding many numbers is time-consuming, but not if the numbers are changes, then the sum is simply calculated as the change from the start to the end-number.

● Write down ten numbers vertically. The first number must be 3 and the last 5, the rest can be any numbers between 1 and 9. In the next column write down the individual changes ‘end-start’. In the third column add up the individual changes along the way. Try to explain why the result must be 5-3 regardless of the in-between numbers.

● Draw a square with side n. Let n have a small positive change t. Show that the square will change with two next blocks when disregarding the small txt square. This shows that the change in an n*n square is 2*n*t, so if we want to add arears under a y = 2*n curve we must add very many small areas y*t = 2*n*t. However, since each may be written as a change in a square, we just have to find the change of the square from the start-point to the end-point. That is how integral calculus works.

Focus 29. Finding common units

Activity 01. “Only add with like units, so how add T = 4ab^2 + 6abc?”. Here units come from factorizing: T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b).

Focus 30. Finding square roots

Activity 01. A 7x7 square can be recounted in tens as 4.9 tens. The inverse question is how to transform a 6x7 block into a square, or in other words, to find the square root of 4.2 tens. A quick way to approach a relevant number is to first find two consecutive numbers, p and p+1, that squared are too low and too high. Then the an approximate value for the square root may be calculated by using that if p^2 < N < (p+1)^2, then √N ≈ (N+p^2)/2p

Conclusion

A curriculum for a refugee camp assumes that the learners have only the knowledge they acquire outside traditional education. The same is the case for street children living outside traditional homes; and for nomadic people always moving around.

However, a refugee camp curriculum might also be applied in a traditional school setting allowing the children to keep on to the two dimensional block numbers they bring to school allowing them to learn core mathematics as proportionality, equations, functions and calculus in the first grade, thus not needing parallel curricula later on.

So, the need for parallel curricula after grade 9 is not there by nature, but by choice. It is the result of disrespecting the mastery of many children bring to school and force them to adopts numbers as names along a number line, and force them to add numbers that are given to them without allowing them to find them themselves by counting, recounting and double-counting.

08. DO WE REALLY NEED PARALLEL CURRICULA

Why do we need different curricula for different groups of students? Why can't all students have the same curriculum? After all, the word-language does not need different curricula for different groups, so why does the number-language?

Both languages have two levels, a language level describing the 'outside' world, and a grammar level describing the 'inside' language. In the word-language, the language level is for all students and includes many examples of real-world descriptions, both fact and fiction. Whereas grammar level details are reserved for special students. Could it be the same with the number-language, teaching the language level to all students including many examples of fact and fiction? And reserving grammar level details to special students?

Before 1970, schools taught language as an example of its grammar (Chomsky, 1965). Then a reaction emerged in the so-called 'communicative turn' in language education. In his book 'Explorations in the function of language' Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a "communicative approach to the teaching of language (p. ix)" allowing more students to learn a less correct language to be used for communication about outside things and actions.

Thus, in language teaching the communicative turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language. So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use

- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

After all, the word language seems more voluminous with its many letters, words and sentence rules. In contrast, a pocket calculator shows that the number language contains ten digits together with a minor number of operations and an equal sign. And, where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. (Tarp, 2018)

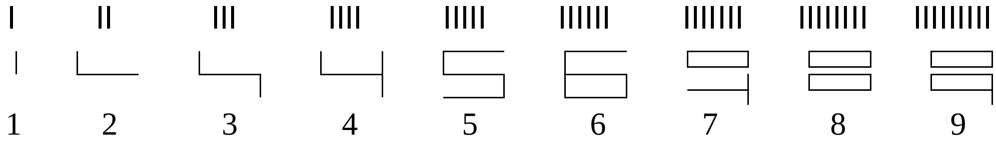

Figure 03. Digits as icons with as many sticks as they represent.

Furthermore, also the operations are icons describing how we total by counting unbundled, bundles, bundles of bundles etc. Here division iconizes pushing away bundles to be stacked, iconized by a multiplication lift, again to be pulled away, iconized by a subtraction rope, to identify unbundled singles that are placed next-to the stack iconized by an addition cross, or by a decimal point; or on-top iconized by a fraction or a negative number.

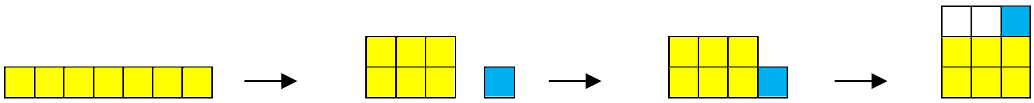

Figure 04. Seven counted as 2 3s & 1 or 2*B*1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

The operations allow predicting counting by a recount-sentence or formula ‘T = (T/B)*B’ saying that ‘from T, T/B times, B can be taken away’, making natural numbers as bundle- or block numbers as e.g. T = 3B2 4s or T = 3*4+2. And, using proportionality to change the unit when two blocks need the same unit to be added on-top, or next-to in a combined unit called integral calculus.

So, it seems as if early childhood education may introduce core mathematics as proportionality and integral calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction. Thus, there is indeed an opportunity to design a core curriculum in mathematics for all students without splitting it up in tracks. But, only if the word- and the number-language are taught and learned in the same way by describing outside things and actions in words and in numbers coming from counting and adding.

However, instead of teaching children how to number, the tradition teaches children about numbers and operation to be explained and learned before they can be applied to the outside world. Thus, where learning word-language takes place in the space between the language and the outside world, learning the number-language is expected to take place in the space between the language and its meta-language or grammar, which makes the number-language more abstract, which leaves many educational challenges unsolved despite close to half a century of mathematics education research.

So maybe we should go back to the mother Humboldt university in Berlin and reflect on Karl Marx thesis 11 written on the staircase: ” Die Philosophen haben die Welt nur verschieden interpretiert; es kömmt drauf an, sie zu verändern.” (The philosophers have only interpreted the world, in various ways. The point, however, is to change it.)

09. CONCLUSION

Let us return to the dream of the National Council of Teachers of Mathematics, to “provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.” Consequently, “everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics.” Furthermore, let us also accept what the council write about numbers: “Number pervades all areas of mathematics.”

So let us look for a curriculum that allows the students to understand and use and numbers, and see how far such a curriculum can carry all students without splitting into parallel tracks.

Now, what does it mean to understand a number like 456?

Is the ability to say that the three digits obey a place-value system where, from right to left, the first digits is ones, then tens, then hundred, then thousands, then, oops no-name unless we use the Chinese name wan, then no-name, then million, then no-name, then no-name, then billions or milliards, etc. Names and lack of names that give little meaning to children where only few understand why ten has its own name but not its own icon but has two digits as 10.

On the other hand, is it the ability to understand that of course ten becomes 10 since it is short for ‘1 bundle and no singles’? And, that it would have been 20 had we counted in bundles of 5s instead as they do on an eastern abacus, where the two digits 10 then would be used for the bundle size 5.

And that ten is just another word for bundle, and hundred for bundle-bundle, i.e. 2 times bundling; and thousand for bundle-bundle-bundle, i.e. or bundling 3 times, etc. where we never end in a situation with no name. Isn’t it both power and beauty to transform an unorganized total into a repeated bundling with the ability that only the decimal point moves if you change the number of bundling, T = 32.1 tens = 3.21 tentens, which is not the case with romans bundling where 3 tens is 6 fives. The romans didn’t stick to bundling bundles since they bundled in both fives and tens and fifties but not in 5 5s, i.e. in 25s. Power and beauty comes from bundle bundles only.

Consequently, to understand the number 456 is to see it, not as one number, but as three numberings of a total that has been bundled 0 times, bundled 1 times, bundled 2 times, etc. And to read the total as 4 bundled 2 times and 5 bundled once and 6 not bundled, or as 4 bundle-bundles and 5 bundles and 6 unbundles singles. And to write the total as T = 4BB 5B 6. And to allow the same total to be recounted with an underload as T = 4BB 6B -4, or with an overload as T = 45B 6 = 4BB 56; or as T = 45B -4 if combining overload and underload.

This understanding allows an existing unorganized total become a number-language sentence connecting the outside subject T to an inside calculation, T = 4*B^2 + 5*B^1 + 6*B^0. Which again is an example, or specification, of an unspecified number-formula or polynomial T = a*x^2 + 5*x + 6.

The power and beauty of the number-formula is manifold. It shows four ways to unite: power, multiplication, addition and next-to block addition also called integration. By including the units, we realize that there are only four types of numbers in the world as shown in the algebra-square above, constant and changing unit-numbers and per-numbers, united by precisely these four ways: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers.

Furthermore, we observe that splitting a total into parts will reverse uniting parts into a total, meaning that all uniting operations have reverse operations: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers. This makes root a factor-finder, and logarithm a factor-counter, and differentiation a finder of per-numbers.

And, if we use the word 'equation' for the need to split instead of unite, we observe that solving an equation means isolating the unknown by moving numbers to the opposite side with opposite calculation sign. Furthermore, using variables instead of digits we observe that the number-formula contains the different formulas for constant change as shown above.

As to a non-constant change, there are two kinds. Predictable change roots calculus as shown by the algebra-square; and unpredictable change roots statistics to instead 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for unpredictable numbers.

Thus the 'power and beauty' of mathematics resides in the number-formula, as does the ability 'to use mathematics in students' personal life, in the workplace and in further study'. So, designing a curriculum based upon the number-formula will 'provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.'

Furthermore, a number-formula based curriculum need not split into parallel curricula until after calculus, i.e. until after secondary education.

So, one number-language curriculum for all is possible, as it is for the word-language. Thus, it is possible to allow all students to learn about the four ways to unite and the five ways to split a total.

The most effective way to design a curriculum for all students is to adopt the curriculum designed refugee camp from the beginning since it accepts and develops the number-language children bring to school. Presenting figures and operations as icons, it bridges outside existence with inside essence. All four uniting methods occur in grade one when counting and recounting in different units, and when adding totals next-to and on-top. It respects the natural order of operations by letting division precede multiplication and subtraction, thus postponing addition until after counting, recounting and double-counting have taken place. It introduces the core recounting-formula expressing proportionality when changing units from the beginning, which allows a calculator to predict inside an outside recounting result. By connecting outside blocks with inside bundle-writing, geometry and algebra are introduced as Siamese twins never to part. Using flexible bundle-numbers connects inside decimals, fractions and negative numbers to unbundled leftovers placed next-to or on-top the outside block. It introduces solving equations when recounting from tens to icons. It introduces per-numbers and fractions when double counting in units that may be the same or different. And, it introduces trigonometry before geometry when double-counting sides in a block halved by its diagonal.

Another option is to integrate calculus in a pre-calculus course by presenting integral calculus before differential calculus, which makes sense since until now inverse operations are always taught after the operation, subtraction after addition etc. Consequently, differential calculus should wait until after it has been motivated by integral calculus that is motivated by adding changing per-numbers in trade and physics, and by adding percentages in statistical double-tables.

In their publication, the National Council of Teachers of Mathematics writes “High school mathematics builds om the skills and understandings developed in the lower grades. (p. 19)” If this has to be like that then high school education will suffer from lack of student skills and misunderstandings; and often teachers say that pre-calculus is the hardest course to teach because of a poor student knowledge background.

So, we have to ask: Can we design a fresh-start curriculum for high school that integrates pre-calculus and calculus? And indeed, it is possible to go back to the power and beauty of the number-formula as described above, and build a curriculum based upon the algebra-square. It gives an overview of the four kinds of numbers that exist in the outside world, and how to unite or split them. It shows a direct way to solve equations based upon the definitions of the reverse operations: move to opposite side with opposite calculation sign.

Furthermore, it provides 2x2 guiding questions: how to unite or split into constant per-numbers, as needed outside when facing change with a constant change-factor? And how to unite or split into changing per-numbers that are piecewise or locally constant, as needed outside when describing e.g. the motion with a changing velocity of a falling object.

As a reverse operation, differential calculus is a quick way to deliver the change-formula that solve the integration problem of adding the many area-strips coming from transforming locally constant per-numbers to unit-numbers by multiplication. Also, by providing change-formulas, differential calculus can extend the formulas for constant change coming from the number-formula. An additional extension comes from combining constant change-number and change-percent to one of the most beautiful formulas in mathematics that is too often ignored, the saving-formula, $A/a = R/r$, a formula that is highly applicable in individual and social financial decisions.

Working with constant and changing change also raises the question what to do about unpredictable change, which leads directly into statistics and probability.

So designing and implementing a fresh-start integrated pre-calculus and calculus curriculum will allow the National Council of Teachers of Mathematics to have their dream come through, so that in the future high schools can provide all students “with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.”

As a number-language, mathematics is placed between its outside roots and its inside meta-language or grammar. So, institutionalized education must make a choice: should the number-language be learned through its grammar before being applied to outside descriptions; or should it as the word-language be learned through its use to describe the outside world? I short, shall mathematics education tach about numbers and operations and postpone applications till after this has been taught? Or shall mathematics education teach how to number and how to use operations to predict a numbering result thus teaching rooting instead of applications?

Choosing the first ‘inside-inside’ option means connecting mathematics to its grammar as a ‘meta-matics’ defining concepts ‘from above’ as top-down examples from abstractions instead of ‘from below’ as bottom-up abstractions from examples. This is illustrated by the function concept that can be defined from above as an example of a set-product relation where first component identity implies second-component identity, or from below as a common name for ‘stand-by’ calculations containing unspecified numbers.

Choosing the inside-inside 'mathematics-as-metamatics' option means teaching about numbers and operations before applying them. Here numbers never carry units but become names on a number-line; here numbers are added by counting on; and the other operations are presented as inside means to inside tasks: multiplication as repeated addition, power as repeated multiplication, subtraction as inverse addition, and division as inverse multiplication. Here fractions are numbers instead of operators needing numbers to become numbers. Here adding numbers and fractions without units leads to 'mathe-matism', true inside classrooms where 2+3 is 5 unconditionally, but seldom outside classrooms where counterexamples exist as e.g. 2weeks + 3days is 17days or 2 3/7 weeks. Here geometry and algebra occur independently and before trigonometry. Here primary and lower secondary school focus on addition, subtraction, multiplication and division with power and root present as squaring and square roots, thus leaving general roots and logarithm and trigonometry to the different tracks in upper secondary school where differential calculus is introduced before integral calculus, if at all.

Choosing the inside-outside 'mathematics-as-manymath' option means to teach digits as icons with as many strokes as they represent. And to also teach operations as icons, rooted in the counting process where division wipes away bundles to be stacked by multiplication, again to be removed by subtraction to identify unbundled singles. This will allow giving a final description of the total using a full sentence with a subject, a verb and a predicate predicted by the recount-formula T = (T/B)*B, e.g. T = 2Bundle 1 3s = 2.1 3s = 2 1/3 3s thus including decimal numbers and fractions in a natural number. Here a double description of Many as an outside block and an inside bundle-number allows outside geometry and inside algebra to be united from the start. Once counted, totals can be recounted. First in the same unit to create overloads and underloads introducing negative numbers. Then between icon- and ten-bundles introducing the multiplication table and solving equations. Then double-counting in two units creates per-numbers becoming fractions with like units. Finally, recounting the sides in a block halved by its diagonal will root trigonometry before geometry, that integrated with algebra can predict intersection points. Then follows addition and reversed addition in its two versions, on-top or next-to. On-top addition calls for recounting the totals in the same unit, thus rooting proportionality. And next-to addition means adding blocks as areas, thus rooting integral calculus. Reversed addition roots equations and differential calculus. Per-numbers are added as operators including the units, thus rooting integral calculus, later defined as adding locally constant per-numbers. Thus, this option means that the core of mathematics is learned in primary school allowing ample of time in secondary school to enjoy the number-language literature by examining existing models or producing models yourself. And it means that only one curriculum is needed for all students as in the word-language.

Furthermore, the root and use of calculus to add changing per-numbers is easily introduced at the pre-calculus level when adding ingredients with different per-numbers and when adding categories in statistics with different percentages.

And, the fact that the difficulty by adding many numbers disappears when the numbers can be written as change-numbers since adding up any number of small changes total just one change from the start- to the end-number. Which of course motivates differential calculus.

Consequently, there is no need for a parallel curriculum to the traditional, since everybody can learn calculus in a communicative way. Of course, one additional optional course may be given to look at all the theoretical footnotes.

To offer a completely different kind of mathematics as graph theory and game theory and voting theory risks depriving the students of the understanding that mathematics is put in the world as a number-language that use operations to predict the result of counting, recounting and double-counting. A language that only needs four operations to unite parts into a total, and only five operations to split a total into parts.

Without calculus in the final high school curriculum, students may not understand how to add per-numbers and might add them as unit-numbers instead of as areas; and this will close many 'doors to productive futures' as the US National Council of Teachers of Mathematics talks about.

References

Chomsky, N. (1965). *Aspects of the Theory of Syntax.* Cambridge, MA: MIT press.

COMAP (1988, 2000). *For All Practical Purposes: Mathematical Literacy in Today's World.* New York, NY: W.H. Freeman.

Halliday, M. A. K. (1973). *Explorations in the function of language.* London, UK: Edward Arnold.

OECD. (2015). *Improving schools in Sweden: an OECD perspective.* Retrieved 07/01/19 from: www.oecd.org/edu/ school/improving-schools-in-sweden-an-oecd-perspective.htm.

Silva, J. (2018). Secondary mathematics for the social sciences. In ICMI study 24. *School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities.* Pre-conference proceedings. Editors: Yoshinori Shimizu and Renuka Vithal, pp. 309-316.

Tarp, A. (2009). *Mathematics Predicts, PreCalculus, Compendium & Projects.* Retrieved 07/01/19 from http://mathecademy.net/various/us-compendia/.

Tarp, A. (2012). *Saving Dropout Ryan With a Ti-82.* Paper presented in ICME 12 at Topic Study Group 18: Analysis of uses of technology in the teaching of mathematics. Retrieved 07/01/19 from http://mathecademy.net/papers/icme-trilogy/, pp. 229-237.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

The National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics an overview.* Reston, VA.

Widdowson, H. G. (1978). *Teaching language as communication.* Oxford, UK: Oxford University Press.

Woodward, E. (2010). *Pre-calculus.* New Jersey, US: Research & Education Association.