New Curricula for Primary & Middle & High School

Sustainable Adaption to Quantity:
From Number Sense to Many Sense

Per-numbers connect Fractions and Proportionality and Calculus and Equations

Sustainable Adaption to Double-Quantity:
From Pre-calculus to Per-number Calculations

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Their biological capacity to adapt to their environment make children develop a number-language based upon two-dimensional block- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language. And math education must accept that its goal is not to mediate the truth regime of setcentric university math, but to develop the child's already existing adaption to Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden.

However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

> The highest performing education systems across OECD countries are those that combine excellence with equity. A thriving education system will allow every student to attain high level skills and knowledge that depend on their ability and drive, rather than on their social background. Sweden is committed to a school system that promotes the development and learning of all its students, and nurtures within them a desire for lifelong learning. PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively

> participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD 2015, p. 3).

Widespread innumeracy also resides in Denmark, where the use of multi-year office-directed lines with fixed classes from secondary school has lowered the exam passing limit at the end of lower and upper secondary school to about 15% and 20% compared to the North-American limit at 70%, using instead self-chosen half-year blocks to uncover and develop the student's individual talent.

Furthermore, two different forms of mathematics are taught, one accepting and one rejecting the 'New Math' occurring around 1960.

Mathematics and its Education

The Pythagoreans used the word 'mathematics' as a common label for their knowledge about Many by itself and in space and time: arithmetic, geometry, music and astronomy. (Freudenthal, 1973)

Without the two latter, mathematics later became a label for arithmetic, algebra and geometry, which may be called pre-setcentric (Derrida, 1991) math, replaced by the present setcentric 'New Math' in 1960 despite it never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we adapt to the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon. Consequently, continental Europe taught reckoning, called 'Rechnung' in German, until the arrival of the New Math. And, when opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only teach set-centric mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. But also here precalculus is seen as a very difficult class to teach, discouraging many students from taking calculus classes.

However, in their 'Learning framework 2030', OECD (2018) points to the necessity of a solid background for all in literacy and numeracy, which raises the 'Cinderella question': with pre-setcentric and setcentric mathematics unable to 'make the prince dance', is there a third hidden post-setcentric alternative, that may prove sustainable so it will last?

The nature of education has been studied by different sciences. To discuss how to find a sustainable solution we should begin with biology, specializing in sustainability through adaption.

Biology Looks at Education

As a life science, biology sees life as built from green, grey and black cells.

Green cells form plants able to perform photosynthesis that store the energy form solar photons in carbon hydrate molecules by replacing oxygen with water in carbon dioxide molecules. To survive, plants must access light and water where they are situated since they are unable to move.

Grey cells form animals able to release the energy from plants or other animals by the replacing hydrogen with oxygen when inhaling oxygen and exhaling carbon dioxide through breathing. To survive, animals must move using muscles and limbs, as well as a brain to decide which way to move. Also, according to ethology (Darwin, 2003) they must adapt to the environment.

Black cells exist individually in oxygen depleted areas on the bottom of lakes or in the stomach of animals, surviving by removing oxygen from carbon hydrate molecules, thus being transformed to carbon or carbon hydrogen or oil allowing energy to be used by machines.

The holes in their head allow animals to satisfy their two basic needs for information and food. Animals come in three forms.

Reptiles have one brain allowing it to transform outside information into a choice between alternative actions.

Mammals also have a second brain for feelings binding them to a mate and to the offspring to allow it to gradually adapt to the environment through childhood before having offspring themselves.

Finally, humans also have a third brain to store and share information, made possible by transforming forelegs to arms with hands that can grasp food and things that are named by sounds, thus developing a language for mutual sharing information about what they observe and know about the six core ingredients of their life: I, you1, it, we, you2, and they; or in German: ich, du, es, wir, ihr, sie.

The combination of individual and collective adaption is so effective that to reproduce, humans only need two to three offspring in a lifetime, where other mammals need it per year.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured e.g. by a social institution called a school.

With life existing in space and time, institutional education has to answer two core questions: what things and events in the environment is important to address in education? And will learning take place through a meeting allowing individual representations to be created, or will it need to be mediated through the teaching of socially constructed representations.

To answer this, we now turn to three other sciences: philosophy, psychology and sociology.

Philosophy Looks at Education

Philosophy looks at the relation between outside existence, ontology, and inside representation, epistemology, or, in other words, the 'it-we' relation. Within philosophy, precedence is given to outside phenomena by empiricism, to inside rationality by rationalism, and to questioning ruling knowledge claims by skepticism.

A controversy within philosophy began in ancient Greece where the sophists pointed out that to practice democracy, a population must be enlightened, especially about the difference between nature and choice to avoid being patronized by political choice masked as unpolitical nature. In opposition to this, Plato saw choice as an illusion since all physical is but examples of metaphysical forms only visible to philosophers educated at the Plato academy. (Russell, 1945)

Later, the Christian church changed the academies into monasteries, where some changed back into universities after the reformation; and after Newton's discovery of physical laws controlling nature without being physical or metaphysical patronized. This inspired the Enlightenment Century installing two republics, one in North America taking over empiricism when developing 'it is right if it works' pragmatism, symbolic interactionism, grounded theory, and action research; and one in France taking over skepticism after seeing its republic turned over several times because of resistance from its German speaking neighbors.

Today, opposition against rationalism is seen within existentialism, claiming with Sartre (2007) that existence precedes essence. And explicated by Heidegger (1962) arguing that in defining verdict-sentences "subject is predicate", the subjects should be respected for naming what exist outside, whereas the predicates should be questioned and appealed since they represent an inside choice between alternatives in risk of being masked as nature.

In contrast to this, continental rationalism gives precedence to inside essence developed by rational universities through peer-reviewed research.

Psychology Looks at Education

Psychology looks at cognitive aspects of learning, or, in other words, the 'it-I' relation. Here, the philosophical controversy between outside existence and inside essence becomes a controversy between different forms of inside constructivism.

Supporting the philosophical existence stance, Piaget (1971) sees learning as a biological process of adapting inside to the outside environment through outside assimilation and inside accommodation, where assimilation makes the outside conform to inside schemata, whereas accommodation makes inside schemata conform to the outside resistance against assimilation.

Thus, to Piaget, learning takes place in the meeting between outside existence and inside schemata that accommodate through outside operations and inside peer communication. Here, teaching socially constructed schemata should be kept to a minimum to not influence the construction of individual schemata.

Siding with Piaget, Ausubel says that "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1978, p. vi).

Supporting the philosophical essence stance, Vygotsky (1986) sees learning as adapting to the socially institutionalized knowledge mediated through good teaching respecting that the knowledge taught must be attachable to what the learners already know in their zone of approximate development. Consequently, high quality must be given to teacher education and textbooks to provide good teaching. And teaching should be structured and well-organized aiming at students being able to reproduce what teachers teach.

Sociology Looks at Education

Sociology looks at the social aspects of human interaction, or, in other words, the 'they-I' relation. Here, the philosophical controversy between outside existence and inside essence carries on as a controversy between different forms of social theory emphasizing individual agency or social structure.

Individual agency is emphasized in the first Enlightenment republic in North America showing strong resistance against institutional answers since they may lead to a goal displacement (Baumann, 1990) becoming an inside goal itself instead of staying as an inside means to an outside goal, thus suppressing the 'sociological imagination' (Mills, 1959) that might keep the answer fluid instead of fossilizing into what Weber (1930) calls an 'disenchanting Iron Cage'.

Structuralism is preeminent in continental Europe seeing established science as being a true inside representation of the outside environment if accepted by the society's knowledge institutions, the universities.

In the second Enlightenment republic in France, institutional skepticism inspired by Heidegger led to French post-structuralism where Derrida, Lyotard, Foucault and Bourdieu warn against hidden patronization in our most basic institutions: words, correctness, diagnosing discourses, curing institutions, and education especially that might make mathematics so difficult it may serve as symbolic violence to establish a new knowledge nobility.

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006: 41)

As to education, Foucault (1995) sees schools as 'pris-pitals' mixing social power techniques from a prison and a hospital. Here students are 'pati-mates' forced to return to the same class, hour after hour, day after day, month after month for several years. Furthermore, self-reference is used to create diagnose ‘ignorant’ without specifying of what: ‘you don’t know math, so we must teach you math’. Consequently, humans are placed in a Kafkaesque (2015) situation where they, unable to have their diagnose defined, finally accept it and therefore also accept to be retained and treated to be cured.

In Germany, Habermas (1981) theorizes the possibility of creating a third Enlightenment republic in post-war Germany. Inspired by Weber’s warning that rationalization carried too far might lead to an iron cage dis-enchanting the world, Habermas warns against system-worlds tending to colonize the life-world and recommends a power free communication rationality to prevent this from happening where “peers exchange views”.

Likewise, Arendt (1963) points out that by definition, institutions lack competition, forcing employees to follow order, which might lead to the banality of evil, where ordinary citizens must act evilly to keep the job.

Agency-based education sees knowledge construction as best taking place in symbolic interaction between peers exchanging views about the sentence subject in order to negotiate a common view. This resonates with Piagetian constructivism, and with philosophical existentialism giving precedence to existence; and with the construction of social knowledge by using Grounded Theory (Glaser and Strauss, 1967). Secondary education should be block-organized to support the student’s identity work through self-chosen half-year blocks with teachers teaching only one subject.

Structuralism sees democratically controlled educational institutions as the best way to mediate the university knowledge heritage. This resonates with Vygotskyan constructivism. Secondary education should be line-organized to supply the state with skilled academic and non-academic workers, thus forcing students to choose career line early, and to start all over if changing career.

Meeting Many, Children Bundle to Count and Share

How children adapt to Many can be observed from preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, ‘That is not four, that is two twos’, thus describing what exists: bundles of 2s, and 2 of them. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units.

Likewise, children use bundle-numbers when talking about Lego bricks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3 more’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, placing them next-to each other, they typically say ‘2 7s and 3 more’.

Children love placing four cars or dolls in patterns; and they smile when the items form a 4-icon. Likewise, they like to form number-icons with footprints in the sand, with body-parts etc.

Children love counting their fingers in 4s using a rubber band to hold the bundles together. They smile when seeing that the fingers can be counted in 4s as 1Bundle6, 2B2 or 3B less2. Or, if counting in 3s, as 1B7, 2B4, 3B1, or 4Bless2. Some even see that 3 bundles is the same as one bundle of bundles, 3B = 1BB.

Likewise, children love bundle-counting the fingers in e.g. 4s as 0Bundle1, 0B2, 0B3, 0B4 no 1B0, 1B1, 1B2, 1B3, 1B4 no 2B0, 2B1, 2B2.

A special case is counting in pairs or 2s. Here the fingers can be counted as 1B8, 2B6, 3B4, 4B2, 5B0. A different color for the rubber band used for the bundle of bundles will allow the fingers to be counted as 1BundleBundle6, 2BB2, 3BBless2. Some might suggest a new color for the bundles of bundles of bundles, thus counting the fingers as 1BBB2 or 1BBB1B0; or even 1BBB0BB1B0.

And children don't mind writing using 'bundle-writing' with a full sentence containing a subject, a verb and a predicate as in the word-language: T = 8 = 1B5 = 2B2 = 3B-1 3s. Some might even write T = 8 = 3B-1 = 1BB-1 3s.

Also, children smile when they see that, counting in hands, T = 5 = 1B0 5s, thus realizing that ten is written as 10 because ten becomes 1B0 if we count in tens.

Sharing 8 cakes, 2 children take away 2 to have one each; and smile when they see that entering '8/2', a calculator predicts they can have 4 each; thus seeing the division sign as an icon for a broom pushing away 2s. This motivates rooting division by 2 as counting in 2s.

Likewise, when counting 9 cubes in 2s they may stack the 2s on-top as a block of 4 2s, smiling when they see that entering '4x2', a calculator predicts they have a total of 8; thus seeing the multiplication sign as an icon for a lift pushing up 2s.

And again, they smile they see that entering '8 – 4*2', a calculator predicts that 1 is left when pulling away a stack of 4 2s from 8; thus seeing the subtraction sign as an icon for a rope pulling away the 4 2s.

Children thus see that counting involves three processes: pushing away, pushing up and pulling away, that can be performed by a broom, a lift and a rope; and that can be predicted on a calculator by using division, multiplication and subtraction. Some may even accept that the counting prescription 'From the total 8, 8/2 times, 2s can be pushed away' may be shortened to the calculation formula '8 = 8/2x2', later with unspecified numbers becoming a core formula expressing proportionality, the recount-formula 'T = (T/B)*B'.

Exposed to counting, children adapt in a natural way to the three basic operations division, multiplication and subtraction; and typically enjoys using a calculator, or even the recount-formula, to predict the counting result.

A Contemporary Mathematics Curriculum

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5.

This offers a ‘natural’ curriculum with multidigit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about the results of operating on such numbers, e.g. that (a+b) * (a-b) = a^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles and areas. Later, geometry and algebra are coordinated in coordinate geometry.

To be followed by halving a rectangle by its diagonal to create a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

In a calculation, changing the input x will change the output y, making y a function of x, y = f(x), using f for an unspecified calculation. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about the results of operating on calculations, e.g. that (f*g)’/(f*g) = f’/f + g’/g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6*x^2dx = 2*x^3 + c, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

The Difference to a Typical Contemporary Mathematics Curriculum

Thus, typically the core of a curriculum is about how to operate on specified and unspecified numbers and calculations.

Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent.

Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles, bundles, and bundles of bundles.

Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting ‘(ten and) 1 left’, ‘(ten and) 2 left’. Seldom children may say ‘bundle’ instead of ten, or to say ‘ten-ten’ or ‘bundle-bundle’ instead of hundred.

Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit.

Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit.

Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed.

Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction.

In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children’s existing mastery and quantitative competence.

Mathematics as a Number-Language

Wanting to respect the child’s own number-language, Tarp (2018, p. 103) talks about word- and number-languages with word- and number-Sentences:

> Observing the quantitative competence children bring to school, (..) we discover a different ‘Many-matics’. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

Discussing Number Sense and Number Nonsense

The basic question in grade one mathematics is: shall education be about numbering or about numbers? Shall education guide and support the development of the children’s already existing adaption to quantity, or shall education teach numbers? Shall the ‘I’ keep on adapting to the ‘it’ directly, or indirectly by having the adaption replaced by what is mediated by the ‘they’?

Choosing numbers over numbering, the US National Council of Teachers of Mathematics, NCTM, in their publication ‘Principles and Standards for School Mathematics’ (2000) says: “Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number.

Central to the number and operation standard is the development of number sense (p. 7).”

Likewise choosing numbers over numbering, ICMI study 23 creates a WNA-discourse (Whole Number Arithmetic) asking:

> To what extent is basic number sense inborn and to what extent is it affected by socio-cultural and educational influences? How is the relationship between these precursors/foundations of WNA, on the one hand, and children’s whole number arithmetic development?” (Bussi and Sun, 2018, pp 500-501)

Thus, both to the NCTM and in the WHA discourse, the concept ‘number sense’ is central, although not being that well defined (Griffin, 2004). In the ICMI study there are several references to Sayers and Andrews (2015) that based upon reviewing research in the WHA domain create a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

However, several questions may be raised to this FoNS framework.

In his book, Dantzig (2007) uses the term ‘number sense’ for a natural property shared by humans and animals. However, from a biological view it is sensing the environment that is fundamental to all grey cells. And as human constructs, numbers are not part of the environment, in contrast to what they number and what is embedded in human language as the singular in plural forms, the physical fact many or ‘more-ness’. Using the term ‘cardinality’ just adds a religious power aspect demanding respect for the Cardinal.

Thus, the term ‘many sense’ is more precise than the term ‘number sense’. Especially since, with its reference to numbers, ‘number sense’ becomes a self-reference that removes meaning from four of the eight categories.

Furthermore, using the word ‘understanding’ makes three categories dubious since there are many different understandings of the word understanding.

What is left is category seven, simple arithmetic competence, which is about adding and subtraction, thus neglecting that division and multiplication come first when counting in bundles.

Thus, it seems difficult to define number-sense without self-reference and without referring to a tradition giving priority to addition and subtraction.

A grounded definition of number-sense or many-sense should come from how numbers emerge in the numbering process counting and recounting a total in bundles, to allow seeing the link between the number and what it numbers by including the ‘missing link’, the bundle and the unit, absent in everyday use: T = 6B7 tens = 67. Therefore, a short definition could be: Having number-sense or many-sense means including the word ‘bundle’ as a unit for the numbers. That is:

To bridge the outside total with an inside numbering by bundling creating flexible bundle-numbers expressed in a full number-language sentence with an outside subject, a verb and an inside predicate, e.g. T = 2 3s.

To count 5 fingers in fives as 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0; and as 1Bundle less 4, 1B-3, 1B-2, 1B-1, 1B0; and to recount five fingers with ‘flexible bundle-numbers’ with overload, underload or fraction, i.e. as 1B3 2s, 2B1 2s or 3B-1 2s or 2 ½B 2s, and later as 1BB 0B1 2s or 1BB1B-1 2s. And to recount ten fingers in 3s as 1B7, 2B4, 3B1, 4B-2, 31/3, 1BB0B1, or 1BB1B-2. And to let 67 = 6B7 = 5B17 = 7B-3 = 6.7 tens = 7.-3 tens. And 678 = 67B8 = 6BB7B8. (Tarp, 2019)

To see the digits as icons with as many sticks or strokes as they represent if written less sloppy; and with ten needing no icon when used as bundle-size.

To see the operations as icons coming from the counting process, where division iconizes a broom pushing away bundles, where multiplication iconizes a lift pushing up bundles into a block, where subtraction iconizes a rope pulling away the block to find unbundles singles, and where addition iconizes placing blocks next-to or on-top.

To see the counting process predicted by the recount-formula T = (T/B)*B, saying ‘From the total T, T/B times, B-bundles can be pushed way’; and to use a calculator to enter ‘9/4’ giving ‘2’, and ‘9-2*4’ giving ‘1’ to predict that from 9, 4s can be pushed away 2 times, and that pulling away the 2 4s from 9 leaves 1, thus predicting that 9 may be recounted as 2B1 4s.

To see totals as double described both as outside blocks and as inside bundles.

To see 678 as a numbering containing four numbers counting unbundled, bundles, bundles of bundles and specifying the bundle-size.

To see a multiplication task as recounting from icons to tens, facilitated by using flexible block&bundle numbers so that 6*8 = 1B-4 * 1B-2 = 1BB – 4B – 2B + 4*2 = 4B8 = 48, thus realizing that -*- is + since the corner was pulled away twice. And to see that 4*67 may be calculated as 4*6B7 giving 24B28, which may be recounted without an overload as 26B8 or 268.

To see a multiplication equation 4*x = 20 as recounting from tens to icons, solved by the recount-formula.

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**10~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	10**B**10
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~

1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

Figure 1. A counting table that includes the bundles in the number names

The WHA discourse defines numbers by internal reference as a set of whole numbers included in the set of integers, included in etc. All created to describe what is called cardinality which is claimed to be linear and represented by a number-line.

The WHA discourse thus presents 678 as one number, or if asked to be more precise, as 6 numbers: 6, 7, 8, ones, tens and hundreds, even if the correct answer is fou4r numbers: 6, 7, 8 and bundles, which typically is ten where it is twenty when the French and the Danes count four twenties instead of eight tens.

Furthermore, 67 is not even a whole number but decimal number that might include a negative number as well:

67 = 6ten7 = 6B7 tens = 7B-3 tens, or 6ten7 = 6.7 tens = 7.-3 tens.

The WNA discourse subscribes to setcentric mathematics. Even if Russell proved that self-reference leads to the nonsense of the classical liar paradox, ‘this sentence is false’, since the set of sets not belonging to itself will belong if and only if it will not.

Russell’s point is that it is OK to talk about elements and sets since that is how a language is organized, but when you talk about sets of sets you talk from a meta-level that should not be mixed with the language level, even if this was precisely what Zermelo and Fraenkel did when trying to save the set theory by disregarding the difference between a set and its elements, thus disregarding the difference between examples and abstractions that is the basis in any language.

Grounded in outside observations, the numbers zero, one and two are rooted in fingers on a hand. Defined inside the WNA discourse, zero is defined as the empty set Ø = {}. With 0 = Ø, 1 is defined as the set containing the set Ø, 1 = {Ø}, but as a set of sets, this places 1 on a different language level where it cannot be added to 0. Then 2 is defined as the set that contains a set, and a set of sets, Ø and 1, 2 = {Ø, {Ø}} thus placing 0, 1 and 2 on three different language levels. Which is nonsense according to Russell.

As to the sociological effect of creating an educational concept ‘number-sense’ we should remember that sociologically, a school is a pris-pital. So, the moment you introduce a new construct you may also introduce a new diagnose: this child lacks number sense, so it must be treated. Especially since it is claimed that children who start with a poorly developed understanding of numbers remain low achievers throughout school (Geary, 2013). And with eight diagnose components, you need eight cures. This might be good news for universities selling teacher education courses, but bad news for the curers, the teachers, now having three times eight additional tasks forced upon them: How to understand the diagnoses, how to find material to use in the cure, and how to evaluate if the cure works.

Introducing diagnoses is an example of what Foucault calls pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1982: 213, 215)

In this way Foucault describes the salvation promise of the generalized church: 'You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and do exactly what we tell you. If not, you only have yourself to than for your decline'.

Introducing diagnoses may also be seen as an example of 'symbolic violence' used as an exclusion technique to keep today's knowledge nobility in power (Bourdieu 1977).

To master Many, humans invented numbers as a means, typically rooted in the hands as the Roman numbers bundling fingers in hands and double hands (Dantzig, 2007). But numbers may lose their outside link and become examples of inside abstractions instead of abstractions from the outside. Likewise, outside quantity may become an example of inside cardinality. In that moment numbers undertake what Baumann calls a goal displacement, where inside derived setcentric numbers become the goal instead with outside quantity as a means thus leaving Many as what Weber calls disenchanted.

The situation with eight components in number sense reminds of the claimed eight 'mathematical competencies' (Niss, 2003) also made meaningless by self-reference, but meaningfully reduced to two competences, count and add (Tarp, 2002).

Likewise, both situations remind of the eight sacraments in the catholic church, challenged by the two sacraments of the protestant church.

To look for meaningful diagnoses in a sustainable mathematics education adapted to quantity we must ask: What is it in the outside world that the children are not adapted to? Will bringing this inside the classroom allow children to extend their existing adaption?

So, instead of using the eight number sense components as diagnoses, we may use the alternative definition of number sense given above as diagnoses to be cured by guiding questions to outside subjects brought inside to receive common predicates, thus reifying the subject in the number language sentences.

Conclusion and Recommendation

This paper asked if there is a third hidden post-setcentric alternative, that may prove sustainable so it will last? The answer is yes, and maybe, since testing for sustainability has to be carried out on what may be called post-setcentric mathematics respecting instead of colonizing the way children adapt to quantity by using two-dimensional bundle-numbers with units instead of the one-dimensional line-numbers forced upon them by setcentric education. Thus, mathematics education should see itself as a language education allowing children develop their quantitative number-language like their qualitative word-language, both using sentences typically with a subject, a verb and a predicate.

A core question in language education is the following: should education develop further the children's own language, or should education colonize it by replacing their native language with a foreign language. And should language be taught before, together with or after its grammar?

Word-language education chose to respect the children's native language and to develop it before introducing a grammar. Likewise, with foreign language after the language revolution in the 1970s made language be taught before grammar (Widdowson, 1978; and Halliday, 1973).

Number-language education chose to disrespect the children's native language. Furthermore, its revolution in the 1970s made language be taught after its grammar, that was introduced not through bottom-up reference to examples, but as top-down examples of the abstraction Set.

So, to establish as sustainable tradition that will allow all to learn and practice a number-language, mathematics education must stop using a setcentric grammar-based foreign language to colonize the children's own native language.

The consequences of not decolonizing is seen in the OECD-report on the Swedish school system as well as in the widespread innumeracy documented by various PISA studies. The time has come for a paradigm shift (Kuhn, 1962) in early childhood education in adaption to quantity by developing the children's already existing many-sense.

Therefore, if the goal is a sustainable mathematics education it might be a good idea to respect and develop the natives' own natural number-language; and to say: 'only cure the diagnosed'.

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Per-numbers connect Fractions and Proportionality and Calculus and Equations

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In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word 'percent' to also talk about other 'per-numbers' as e.g. 'per-five' thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

Mathematics is Hard, or is it

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature.

That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969.

Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Witnessing poor PISA performance, Denmark has lowered the passing limit at the final exam is to around 15% and 20 % in lower and upper secondary school.

Other countries also witness poor PISA performance. And high-ranking countries admit they have a high percentage of low scoring students.

As to finding the cause, Kilpatrick, Swafford, and Findell (2001, p. 36) points out that "what is actually taught in classrooms is strongly influenced by the available textbooks". Personally, working ethnographically in schools in Denmark and abroad, listening to teachers and students confirms the picture that textbooks are followed quite strictly.

So, it seems only natural to look at what is currently being discussed in textbook research e.g. by looking at the Third International Conference on Mathematics Textbook Research and Development, ICMT3, in Germany.

The ICMT3 Conference

The September 2019 ICMT3 conference consisted of 4 keynote addresses, 15 symposium papers, 2 workshops, 40 oral presentations and 13 posters.

The name ‘fraction’ occurred 212 times in the proceedings, and one of the keynotes addressed the problems students have when asked to find 3/5 of 2/4.

As to fractions, Ripoll and Garcia de Souza writes that “The integer numbers structure and the idea of equivalence are elementary in the mathematical construction of the ordered field of the rational numbers. Hence, the concept of equivalence should not be absent in the Elementary School’s classrooms and textbooks.” (Rezat et al, 2019, p. 131). Looking at 13 Brazilian textbooks from 4th to 7th grade they conclude that

> The conclusion, with respect to equivalence, was that no (complete) characterization of equivalent fractions is present in the moment the content fractions is carried on in the 6th grade Brazilian textbooks, like “Two given fractions a/b and c/d are equivalent if and only if ad = bc.” In most cases only a partial equivalence criterion is presented, like “Two fractions are equivalent if one can transform one into the other by multiplying (or dividing) the numerator and the denominator by the same natural number.”

The authors thus take it that fractions should obey the New Math ‘setcentrism’ (Derrida, 1991) by saying: in a set-product of integers, a fraction is an equivalence class created by the equivalence relation statin that a/b ~c/d if a*d = b*c; and thus neglect the pre-setcentric version mentioned above where a fraction keeps its value by being expanded or shortened; as well as the post-setcentric version seeing a fractions as an example of a per-number, described later in this paper.

Confirming in the afterwards discussion that fractions are introduced by the part-whole model, an argument was made that if a fraction is defined as a part of a whole then a fraction must always be a fraction of something; thus being an operator needing a number to become a number, and not a number in itself.

Of course, in a 30 minutes presentation there is little time to discuss the nature of fractions thoroughly, so this question needs to be addressed in more details.

Also addressing middle school problems, Watanabe writes that “Ratio, rate and proportional relationships are arguably the most important topics in middle grades mathematics curriculum before algebra. However, many teachers find these topics challenging to teach while students find them difficult to learn.” (p. 353)

And, talking about proportionality, Memis and Yanik writes that “Proportional reasoning is an important skill that requires a long process of development and is a cornerstone at middle school level. One of the reasons why students cannot demonstrate this skill at the desired level is the learning opportunities provided by textbooks.“ (p. 245)

Textbooks must follow curricula, and middle school problems were also mentioned at the International Commission on Mathematical Instruction Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, in Japan November 2018. Here in his plenary talk, McCallum after noting that “a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships” challenged the audience by asking “What is the difference between 5/3 and 5÷3?” (ICMI, 2018, p. 4).

So, this paper will focus on these challenges by asking: “Is there a hidden different way to see and teach core middle school concepts as fractions, quotients, ratios, rates and proportionality?” A question that might be answered answer by Difference-research (Tarp, 2018) using sociological imagination (Mills, 1959) to search for differences making a difference by asking two questions: ‘Can this be different – and will the difference make a difference?’

Different Ways of Seeing Fractions

In a typical curriculum using a ‘part-whole’ approach, fractions are introduced after division has been taught as sharing a whole in equal parts: 8/4 is 8 split in 4 parts or 8 split by 4.

Representing the whole geometrically as a bar or a circle, dividing in 4 parts creates 4 pieces each called ¼ of the total. Assigning numbers to the whole allows finding ¼ of e.g. 8 by the division 8/2. Then the fraction ¾ means taking ¼ three times, so that taking ¾ of 8 involves two calculations, first 8/4 as 2, then 3*2 as 6, so that ¾ of 8 is 8/4*3, later reformulated to one calculation, 8*¾, multiplying the integer 8 with the rational number ¾.

However, in the ‘part-whole’ approach a fraction is a fraction of something, thus introducing a fraction as an operator needing a number to become a number.

This becomes problematic when the fraction later is claimed to be a point on a number line, i.e. a number in its own right, a rational number, defined by setcentrism as an equivalence class in a set-product as described above.

Furthermore, setcentrism is problematic in itself by making mathematics a self-referring ‘Meta-matics’, defined from above as examples from abstractions instead of from below as abstractions from examples.

And, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

To avoid self-reference Russell introduced a type theory allowing reference only to lower degree types. Consequently, fractions could not be numbers since they refer to numbers in their setcentric definition.

Neglecting the Russell paradox by defining fractions as rational numbers leads to additional educational questions: When are two fractions equal? How to shorten or expand a fraction? What is a fraction of a fraction? Which of two fractions is the bigger? How to add fractions? Etc.

Fraction later leads on to percentages, the special fractions having 100 as the denominator; which leads to the three percentage questions coming from the part-whole formula defining a fraction, fraction = part/whole.

Seeing fractions as, not numbers, but operators still raises the first three questions whereas the two latter are meaningless since the answer depends on what whole they are taken of as seen by 'the fraction paradox' where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes, and never 7 cokes of 6 bottles.

Adding numbers without units may be called 'mathe-matism', true inside but seldom outside the classroom. And strangely enough the two latter questions are only asked with fractions and seldom with percentages.

Ratios and Rates

When introduced, ratios are often connected to fractions by saying that splitting a total in the ratio 2:3 means splitting it in 2/5 and 3/5.

Where fractions and ratios typically are introduced without units, rates include units when talking e.g. about speed as the ratio between the meter-number and the second-number, speed = 2m/3s.

Per-numbers Occur when Double-counting a Total in two Units

The question "What is 2/3 of 12?" is typically rephrased as "What is 2 of 3 taken from 12?" Seldom it is rephrased as "What is 2 per 3 of 12?". Even if the word 'per' occurs in many connections, meter per second, per hundred, etc.

When we rephrase "taking 30% of 400" as "taking 30 per 100 of 400", why don't we rephrase "taking 3/5 of 400" as "taking 3 per 5 of 400" ?

In short, why don't we rephrase 3/5 both as '3 of 5' and as '3 per 5'?

In his conference paper, Tarp (p. 332) introduces per-numbers and recounting:

> An additional learning opportunity is to write and use the 'recount-formula' T = (T/B)*B, saying "From T,T/B times B can be taken away", to predict counting and recounting examples. (..) Another learning opportunity is to observe how double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Of course, you might argue that we cannot write '6$ = 9kg' since the units are not the same. But then again, we write '2 meter = 200 centimeter' even if the units are different, and we are allowed to do so since the bridge between the two units is the per-number 1m/100cm. Likewise, we should be allowed to write '6$ = 9kg' since the bridge between the two units for now is the per-number 2$/3kg.

The difference is that the per-number between meter and centimeter is globally valid, whereas the per-number between kilogram and dollar is only locally valid. Still, it has validity as long as you are talking about the same outside total.

The interesting thing is that by including units, per-numbers connects fractions and proportionality. And that by including units, the recount-formula gives an introduction to fractions saying that 1/3 is ‘1 counted in 3s’: 1 = (1/3)*3 = 1/3 3s.

Fractions as Per-numbers

With per-numbers coming from double-counting the same total in two units, we see that when double-counting in the same unit, the unit cancels out and we get a ratio between two numbers without units, a fraction as e.g. 3$/8$ = 3/8.

Reversely, inside fractions without units may be ‘de-modeled’ outside by adding new units, e.g. ‘good’ and ‘total’ transforming 3/8 to 3g/8t. This allows per-numbers and recounting to be used when solving the three fraction questions:

“What is 3/4 of 60?”, and “20 is what of 60?”, and “20 is 2/3 of what?”

Asking “What is ¾ of 60” means asking “What is 3 per 4 of 60”, or de-modeled with units, “What is 3g per 4t of 60t”,

Of course, 60t is not 4t, but 60 can be recounted in 4s by the recount-formula, 60t = (60/4)*4t = (60/4)*3g = 45g, giving the inside answer “¾ of 60 is 45”.

Asking “20 is which fraction of 60” means asking “What fraction is 20 per 60”, or with units, “Which per-number is 20g per 60t”, giving the answer directly as 20g/60t or 20/60 g/t. Here we might look for a common unit in 20 and 60 to cancel out, e.g. 20, giving 20/ 60 = 1 20s/3 20s = 1/3. This allows transforming the outside answer “20 per 60 is 1 per 3” to the inside answer “20 is 1/3 of 60”.

Asking “20 is 2/3 of what” means asking “20 is 2 per 3 of what”, or with units, “20g is 2g per 3t of which total”. Of course, 20g is not 2g, but 20 can be recounted in 2s by the recount-formula, 20g = (20/2)*2g = (20/2)*3t = 30t. This allows transforming the outside answer “20 is 2 per 3 of 30” to “20 is 2/3 of 30.”

Expanding and Shortening Fractions

With fractions as per-numbers coming from double counting in the same unit that has cancelled out, we are always free to add a common unit to both numbers.

Using numbers as units will expand the fraction:

2/3 = 2 7s/3 7s = 2*7/3*7 = 14/21

Reversely, if both numbers contain a common unit, this will cancel out:

14/21= 2*7/3*7 =2 7s/ 3 7s = 2/3

Taking Fractions of Fractions, the Per-number Way

One of the keynotes pointed out that to understand that 6/20 is the answer to the question “What is 3/5 of 2/4?” we must draw a rectangle with 4 columns if which 2 are yellow, and with 5 rows of which 3 are blue. Then 6 double-colored squares out of a total of 20 squares gives an understanding that 3/5 of 2/4 is 6/20, which also comes from multiplying the numerators and the denominators.

Seeing fractions as per-numbers the question "What is 3/5 of 2/4?" translates into "What is 3 per 5 of 2 per 4". Knowing that using per-numbers to bridge two units involves recounting them in the per-number which again involves division, we might begin with a number that is easily recounted in 4s and 5s, e.g. 4*5 = 20, and reformulate the question to "3 per 5 of 2 per 4 is what per 20?".

To find 2 per 4 of 20 means finding 2g per 4t of 20t, so we recount 20 in 4s:

20t = (20/4)*4t = (20/4)*2g = 10g, so 2 per 4 of 20 is 10.

To find 3 per 5 of 10 means finding 3g per 5t of 10t, so we recount 10 in 5s:

10t = (10/5)*5t = (10/5)*3g = 6g, so 3 per 5 of 10 is 6

Thus, we can conclude that 3 per 5 of 2 per 4 is the same as 6 per 20, or, with fractions, that 3/5 of 2/4 is 6/20, again coming from multiplying the numerators and the denominators.

Of course, we could discuss, which method gives a better understanding, but we might never reach an answer, given the many different understandings of the word 'understanding'

Adding Fractions, the Per-number Way

Adding per-numbers occurs in mixture problems asking e.g. "What is 2kg at 3$/kg plus 4kg at 5$/kg?". We see that the unit-numbers 2 and 4 add directly, whereas the per-numbers cannot add before multiplication changes them to unit-numbers. However, multiplication creates the areas 2*3 and 4*5, which gives the answer: 2kg at 3$/kg + 4kg at 5$/kg gives (2+4)kg at (2*3+4*5)/(2+4)$/kg.

So we see that per-numbers add by the areas under the per-number graph in a coordinate system with the kg-numbers and the per-numbers on the axes.

But adding area under a graph is what integral calculus is all about. Only here, the per-number graph is piecewise constant, where the velocity graph in a free fall, is not piecewise, but locally constant, which means that the total area comes from adding up very many small area-strips.

This may be done by observing that the total area always changes with the last area-strip thus creating a change equation ΔA = p*Δx, which motivates differential calculus to answer questions as dA/dx = p, thus finding the area formula that differentiated gives the give per-number formula p, e.g. d/dx (x^2) = 2*x.

Interchanging epsilon and delta to change piecewise constancy to local may be postponed to high school, that would benefit considerably by a middle school introduction of integral calculus as adding locally constant per-numbers by the area under the per-number graph, using differential calculus to find the area in a quicker way than asking a computer to add numerous small area-strips.

Directly and Inversely Proportionality

Using a coordinate system with decimal numbers comes natural if bundle-writing totals in tens so e.g. T = 26 becomes T = 2.6 tens. This allows fixing a 3x5 box in the corner with the base and the height on the x- and y-axes. The recount-formula T = (T/B)*B then shows a total T as a box with base x = B and height y = T/B.

To keep the total unchanged, increasing the base will decrease the height (and vice versa) making the upper right corner create a curve called a hyperbola with the formula height = T/base, or y = T/x, showing inverse proportionality.

In a 3x5 box, the raise of the diagonal is the per-number 3/5. Expanding or shortening the per-number by adding or removing extra units will make the diagonal longer or shorter without changing direction. This will make the upper right corner move along a line with the formula 3/5 = height/base = y/x, or y = 3/5*x, showing direct proportionality.

Solving Proportionality Equations by Recounting

Reformulating the recount-formula from T = (T/B)*B to T = c*B shows that with an unknown number u it may turn into an equation as 8 = u*2 asking how to recount 8 in 2s, which of course is found by the recount-formula, u*2 = 8 = (8/2)*2, thus providing the equation u*2 = 8 with the solution u = 8/2 obtained by isolating the unknown by moving a number to the opposite side with the opposite sign.

This resonates with the formal definition of division saying that 8/2 is the number u that multiplied by 2 gives 8: if u*2 = 8 then u = 8/2.

Setcentrism of course prefers applying and legitimizing all concepts from abstract algebra's group theory (commutativity, associativity, neutral element and inverse element) to perform a series of reformulations of the original equation: 2*u = 8, so (2*u)*½ = 8*½, so (u*2)*½ = 4, so u*(2*½) = 4, so u*1 = 4, so u = 4.

Seven Ways to Solve the two Proportionality Questions

The need to change units has mad the two proportionality questions the most frequently asked questions in the outside world, thus calling for multiple solutions.

As an example, we look at a uniform motion where the distance 2meter needs 5second. The two questions then go from meter to second and the other way, e.g.

Q1: "7 meters need how many seconds?"

Q2: "How many meters is covered in 12 seconds?"

● Europe used 'Regula-de-tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.

● Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter.

● Equating the rates. The velocity rate is constantly 2meter/5second. So we can set up an equation equating the rates. Q1: 2/5 = 7/x, where cross-multiplication gives 2*x = 7*5, which gives x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, where cross-multiplication gives 5*x = 12*2, which gives x = (12*2)/5 = 4.8.

● Recount in the per-number. Double-counting produces the per-number 2m/5s used to recount the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.

● Multiply with the per-number. Using the fact that T = 2m, and T = 5s, division gives T/T = 2m/5s = 1, and T/T = 5s/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.

● Modeling. The set-concept introduce modeling with linear functions to show the relevance of abstract algebra's group theory: Let us define a linear function f(x) = c*x from the set of meter-numbers to the set of second-numbers, having as domain DM = {x∈R | x>0}. Knowing that f(2) = 5, we set up the equation f(2) = c*2 = 5 to be solved by multiplying with the inverse element to 2 on both sides, and by applying the associative law: c*2 = 5, so (c*2)*½ = 5*½, so c*(2*½) = 5/2, so c*1 = 5/2, so c = 5/2. With f(x) = 5/2*x, the inverse function is f-1(x) = 2/5*x. So with 7meter, f(7) = 5/2*7 = 17.5second; and with 12second, f-1(12) = 2/5*12 = 4.8meter.

The two proportionality questions thus consist of multiplication and division. Multiplication precedes division when using the regula-de-tri, equating the rates, and multiplying with the per-number. And, division precedes multiplication when using the unit rate, recounting in the per-number, and recounting the units.

Modeling seems more suited to have a discussion of how abstract algebra's group theory may be applied to the outside world having really little need for it.

A Case: Peter, about to Peter Out of Teaching

As a new middle school teacher, Peter is looking forward to introducing fractions to his first-year class coming directly from primary school where the four basic operations have been taught so that Peter can build upon division when introducing fractions in the traditional way.

However, Peter is shocked when seeing many students with low division performance, and some even showing dislike when division is mentioned. So, Peter soon is faced with a class divided in two, a part that follows his introduction of fractions, and a part that transfers their low performance or dislike from divisions to fractions.

The following year seeing his new class behaving in the same way, Peter is about to give up teaching when a colleague introduces him to a different approach where division is used for bundle-counting instead of sharing called ‘Recounting fingers with flexible bundle-numbers’. The colleague also recommends some YouTube videos to watch and some material to download from the MATHeCADEMY.net to try it yourself.

Inspired by this, Peter designs a micro-curriculum for his class aiming at introducing the class to bundle-counting leading to the recount-formula leading to double-counting in two units leading to per-numbers having fractions as the special case with like units.

“Welcome class, this week we will not talk about fractions!” “?? Well, thank you Mr. teacher, then what will we do?” “We will count our five fingers.” “Ah, Mr. teacher we did that in preschool!” “Correct, in preschool we counted our fingers in ones, now we will bundle-count them in 2s and 3s and 4s using bundle-writing. In this way we will see that a total can be counted in three different ways: overload, standard and underload. Look here:

Outside we have IIIII = ⧺III = ⧺⧺I = ⧺⧺⧺

Inside we write: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s

We will call this to recount 5 with flexible bundle-numbers. Now count the five fingers in 3s and 4s in the same way. Later, we will count all ten fingers.”

The following class, Peter began by rehearsing.

“Welcome class. Yesterday we saw that an outside total can be recounted in different units, and that the result inside can be bundle-written in three ways, with overload, standard and underload. Today we will begin by recounting twenty in hands, in six-packs and in weeks. Why twenty? Because counting in twenties was used by the Vikings who also gave us the words eleven and twelve, meaning one-left and two-left in Viking language.”

Later, Peter introduced the recount-formula:

“Here we have 6 cubes that we will count in 2s. We do that by pushing away 2-bundles, and write the result as T = 6 = 3B 2s. We see that the inside division stroke looks like an outside broom pushing away the bundles. And asking the calculator, 6/2, and we get the answer 3 predicting it can be done 3 times. We can illustrate this prediction with a recount formula ‘T = (T/B)xB’ saying that ‘from the total T, T/B times, B can be pushed away’. So, from now on, 6/2 means 6 recounted in 2s; and 3x2 means 3 bundles of 2s. And since it is counted in tens, 42 is seen as 4B2 or 3B12 or 5B-8 using flexible bundle-numbers.

Now let us read 42/3 as 4bundle2 tens recounted in 3s; and let us use flexible bundle-numbers to rewrite 4B2 with an overload as 3B12. Then we have T = 42 /3

= 4B2 /3 = 3B12 / 3 = 1B4 = 14. We notice that squeezing a box from base 10 to base 3 will increase the height, here from 4.2 to 14.

And, by the way, flexible bundle-numbers also come in handy when multiplying: Here 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 bundles and 56 unbundled singles, which can be recounted to remove the overload:

T = 7 x 4B8 = 28B56 = 33B6 = 336.

The third day Peter repeated the lesson with 7 cubes counted in 3s to show that where the unbundled single was placed would decide if the total should be written using a decimal number when placed next-to as separate box of ones, T = 2B1 3s = 2.1 3s. Placed on-top means missing 2 to form a bundle, thus written as T = 3B-2 3s = 3.-2 3s. Or it means recounting 1 in 3s as 1 = (1/3)x3 = 1/3 3s, thus becoming a fraction, here seen as 1 counted in 3s, where 2 would be 2 counted in 3s; and where 4 would be 4 counted in 3s giving 1 bundle and 1 counted in 3s, 4/3 = 1 1/3.

Later, Peter introduced per-numbers and fractions as described above, which allowed Peter to work with fractions and ratios and proportionality at the same time; and later to introduce calculus as adding fractions and per-numbers by areas.

Observing the increase of performance and the disappearance of dislike, the headmaster suggested to the headmaster of the nearby primary school that Peter be used as a facilitator for in-service teacher training. This would introduce flexible bundle-numbers with units, recounting and double-counting, thus allowing primary school children to meet fractions and negative numbers and proportionality when recounting a total in a new bundle-unit.

Discussion and Recommendation

This paper asked "Is there a hidden different way to see and teach core middle school concepts as fractions, quotients ratios, rates and proportionality?" The answer is yes: per-numbers includes them all as examples, as well as integral calculus and equations.

Accepting the double-numbers with units as 2 3s that children bring to school leads to recounting in another unit predicted by a recount-formula T = (T/B)*B expressing proportionality occurring when double-counting in two units leads to per-numbers as 2$/3kg, becoming fractions with like units as 2$ per 3$ = 2/3.

In this way questions about fractions, quotients, ratios, rates and proportionality can be reformulated to questions about per-numbers answered by recounting, which also allows solving proportional equations by moving numbers to opposite side with opposite calculation sign.

Questions as comparing or adding fractions, quotients ratios, rates are rejected since they are meaningless when not including the units. Instead, including the units leads to adding fractions by areas thus exemplifying integral calculus.

So introducing per-numbers through double-counting the same total in two units makes a difference by allowing fractions, quotients, rates and ratios to be seen and taught as examples of per-numbers, and by allowing integral calculus to be introduced in middle school, and by allowing a more natural way to solve multiplication equations, and by allowing STEM examples in the classroom since most STEM formulas are proportional formulas.

Furthermore, introducing recounting with flexible bundle-numbers allows math dislike to be cured by taking the hardness out of division, seen traditionally as the basis for fractions but becoming a tumbling stone instead if not learned well.

Consequently, it is recommended that primary school accepts and develops the double-numbers children bring to school. And that middle school introduces students to recounting in flexible bundle-numbers from the start to provide a strong division foundation for fractions that becomes connected with quotients, ratios, rates, proportionality, equations and calculus if introduced as per-numbers coming from double-counting in two units that may be the same.

So yes, mathematics is hard, not by nature, but by a choice replacing it with a mixture of top-down meta-matics and mathe-matism seldom true outside the class.

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Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations

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Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

A need for curricula for all students

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

Traditionally, a school system is divided into a primary school for children and a secondary school for adolescents, typically divided into a compulsory lower part, and an elective upper part having precalculus as its only compulsory math course. So, looking for a change we ask: how can precalculus be sustainably changed?

A Traditional Precalculus Curriculum

Typically, basic math is seen as dealing with numbers and shapes; and with operations transforming numbers into new numbers through calculations or functions. Later, calculus introduces two additional operations now transforming functions into new functions through differentiation and integration as described e.g. in the ICME-13 Topical Survey aiming to "give a view of some of the main evolutions of the research in the field of learning and teaching Calculus, with a particular focus on established research topics associated to limit, derivative and integral." (Bressoud et al, 2016)

Consequently, precalculus focuses on introducing the different functions: polynomials, exponential functions, power functions, logarithmic functions, trigonometric functions, as well as the algebra of functions with sum, difference, product, quotient, inverse and composite functions.

Woodward (2010) is an example of a traditional precalculus course. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

A Different Precalculus Curriculum

Inspired by difference research (Tarp, 2018) we can ask: Can this be different; is it so by nature or by choice?

In their 'Principles and Standards for School Mathematics' (2000), the US National Council of Teachers of Mathematics, NCTM, identifies five standards: number and operations, algebra, geometry, measurement and data analysis and probability, saying that "Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century (p. 2)." In the chapter 'Number and operations', the Council writes: "Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number (p. 7)."

Their biological capacity to adapt to their environment make children develop a number-language allowing them to describe quantity with two-dimensional block- and bundle-numbers. Education could profit from this to teach children primary school calculus that adds blocks (Tarp, 2018). Instead, it imposes upon children one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must be learned before it can be applied to operate on the functions introduced at the precalculus level.

However, listening to the Ausubel (1968) advice "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly (p. vi).", we might want to return to the two-dimensional block-numbers that children brought to school.

So, let us face a number as 456 as what it really is, not a one-dimensional linear sequence of three digits obeying a place-value principle, but three two-dimensional blocks numbering unbundled singles, bundles, bundles-of-bundles, etc., as expressed in the number-formula, formally called a polynomial:

T = 456 = 4*B^2 + 5*B + 6*1, with ten as the international bundle-size, B.

This number-formula contains the four different ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant 'double-numbers' or 'per-numbers'. We might call this beautiful simplicity 'the algebra square' inspired by the Arabic meaning of the word algebra, to re-unite.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Figure 01. The 'algebra-square' has 4 ways to unite, and 5 to split totals

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers, creating the operations integration and its reverse operation, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power, and its two reverse operations, root and logarithm.

Precalculus, building on or rebuilding?

In their publication, the NCTM writes "High school mathematics builds on the skills and understandings developed in the lower grades (p. 19)."

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they may have from the lower grades?

Furthermore, what kind of mathematics has been taught? Was it 'grounded mathematics' abstracted 'bottom-up' from its outside roots as reflected by the original meaning of ‘geometry’ and ‘algebra’ meaning ‘earth-measuring’ in Greek and ‘re-uniting’ in Arabic? Or was it ‘ungrounded mathematics’ or ‘meta-matics’ exemplified ‘top-down’ from inside abstractions, and becoming ‘meta-matism’ if mixed with ‘mathe-matism’ (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept ‘function’, Euler saw it as a bottom-up name abstracted from ‘standby calculations’ containing specified and unspecified numbers. Later meta-matics defined a function as an inside-inside top-down example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may also be seen as an outside-inside bottom-up number-language sentence containing a subject, a verb and a predicate allowing a value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group operation applying the associative and commutative law as well as the neutral element and inverse elements, thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2:

2*u =6, so (2*u)*½ =6*½, so (u*2)*½ =3, so u*(2*½) =3, so u*1 =3, so u =3.

However, the equation 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula ‘T = (T/B)*B’ (Tarp, 2018), present all over mathematics as the proportionality formula, thus solved in one step:

2*u = 6 = (6/2)*2, giving u = 6/2 = 3.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology.

Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as a number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar.

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction for an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

A Grounded Outside-Inside Fresh-start Precalculus from Scratch

Let students see that both the word-language and the number-language provide ’inside’ descriptions of ‘outside’ things and actions by using full sentences with a subject, a verb, and an object or predicate, where a number-language sentence is called a formula connecting an outside total with an inside number or calculation, shortening ‘the total is 2 3s’ to ‘T = 2*3’;

Let students see how an outside degree of Many at first is iconized by an inside digit with as many strokes as it represents, five strokes in the 5-icon etc. Later the icons are reused when counting by bundling, which creates icons for the bundling operations as well. Here division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation. Writing ‘y = f(x)’ means that the y-number is found by specifying the x-number and the f-calculation. Thus, with x = 3, and with f(x) = 2+x, we get y = 2+3 = 5.

Let students see how calculations predict: how 2+3 predicts what happens when counting on 3 times from 2; how 2*5 predicts what happens when adding 2$ 5 times; how 1.02^5 predicts what happens when adding 2% 5 times; and how adding the areas 2*3 + 4*5 predicts adding the ‘per-numbers’ when asking ‘2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?’

Solving Equations by Moving to Opposite Side with Opposite Sign

Let students see the subtraction ‘u = 5-3’ as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers ‘to opposite side with opposite calculation sign’; a rule that applies also to the other reversed operations:

- the division u = 5/3 is the number u that multiplied with 3 gives 5, thus solving the equation u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, thus solving the equation u^3 = 5, and making root a ‘factor-finder’
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, thus solving the equation 3^u = 5, and making logarithm a ‘factor-counter’.

Let students see multiple calculations reduce to a single calculation by unhiding ‘hidden brackets’ where 2+3*4 = 2+(3*4) since, with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s.

This prevents solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5. Allowing to unhide the hidden bracket we get:

2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4

This solution is verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy a 'Hymn to Equations': "Equations are the best we know, they're solved by isolation. But first the bracket must be placed, around multiplication. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don't want to stop!"

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let students place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, three terms disappear if adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = b^2/4 – c.

Moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c, so u + b/2 = $\pm\sqrt{b^2/4 - c}$, so u = -b/2 $\pm\sqrt{b^2/4 - c}$

Recounting Grounds Proportionality

Let students see how recounting in another unit may be predicted by a recount-formula 'T = (T/B)*B' saying "From the total T, T/B times, B may be pushed away" (Tarp, 2018). In primary school this formula recounts 6 in 2s as 6 = (6/2)*2 = 3*. In secondary school the task is formulated as an equation u*2 = 6 solved by recounting 6 in 2s as u*2 = 6 = (6/2)*2 giving u = 6/2, thus again moving 2 'to opposite side with opposite calculation sign'.

Thus an inside equation u*b = c can be 'demodeled' to the outside question 'recount c from ten to bs', and solved inside by the recount-formula: u*b = c = (c/b)*b giving u = c/b.

Let students see how recounting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. And see how filling a circle with right triangles from the inside allows phi to be found from a formula: circumference/diameter = π ≈ n*tan(180/n) for n large.

Double-counting Grounds Per-numbers and Fractions

Let students see how double-counting in two units create 'double-numbers' or 'per-numbers' as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number:

Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg.

Asking '9kg = ?$' we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Let students see how double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 40%.

To find 40% of 20$ means finding 40$ per 100$, so we re-count 20 in 100s:

T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$:

T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

Let students see how double-counting physical units create per-numbers all over STEM (Science, Technology, Engineering and mathematics):

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter;
meter = (meter/second) * second = velocity * second;
joule = (joule/second) * second = watt * second

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, T = 456 = 4*B^2 + 5*B + 6*1, containing the formulas for constant change:

T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change. So where constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics to ‘post-dict’ what we can’t ‘pre-dict’; and using confidence for predicting intervals.

Combining linear and exponential change by n times depositing a$ to an interest percent rate r, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest percent rate R is predicted by the formula 1+R = (1+r)^n. This saving may be used to neutralize a debt Do, that in the same period changes to D = Do*(1+R).

This formula and its proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Uniting Constant Per-Numbers as Factors

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$, changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, the formula for change with a constant change-factor, also called exponential change.

Reversing the question, this formula entails two equations. Asking 600 = 300*a^5, we look for an unknown change-factor. So here the job is ‘factor-finding’ which leads to defining the fifth root of 2, i.e. 5√2, found by moving the exponent 5 to opposite side with opposite calculation sign, root.

Asking instead 600 = 300*1.2^n, we now look for an unknown time period. So here the job is ‘factor-counting’ which leads to defining the 1.2 logarithm of 2, i.e. log1.2(2), found by moving the base 1.2 to opposite side with opposite calculation sign, logarithm.

Calculus Deals with Uniting Changing Per-Numbers as Areas

In mixture problems we ask e.g. ‘2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?’ Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before added, thus adding by areas. So here multiplication precedes addition.

Asking inversely ‘2kg at 3$/kg + 4kg at how many $/kg gives 6kg at 5 $/kg?’, we first subtract the areas 6*5 – 2*3 before dividing by 4, a combination called differentiation, $\Delta T/4$, thus meaningfully postponed to after integration.

Statistics Deals with Unpredictable Change

Once mastery of constant change is established, it is possible to look at time series in statistical tables asking e.g. “How has the unemployment changed over a ten-year period?” Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b+a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n.

The average numbers allow calculating all totals in the period, assuming the numbers are predictable. However, they are not, so instead of predicting the number with a formula, we might ‘post-dict’ the numbers using statistics dealing with unpredictable numbers. This, in turn, offers a likely prediction interval by describing the unpredictable random change with nonfictional numbers, median and quartiles, or with fictional numbers, mean and standard deviation.

Calculus as adding per-numbers by their areas may also be introduced through cross-tables showing real-world phenomena as unemployment changing in time and space, e.g. from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add. First, they must multiply back to unit-numbers to find the total percent. However, multiplying creates areas, so per-numbers add by areas, which is what calculus is about.

Modeling in Precalculus Exemplifies Quantitative Literature

Furthermore, graphing calculators allows authentic modeling to be included in a precalculus curriculum thus giving a natural introduction to the following calculus curriculum, as well as introducing ‘quantitative literature’ having the same genres as qualitative literature: fact, fiction and fiddle (Tarp, 2001).

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator thus using a natural science proof, valid until finding counterexamples.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A' = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A' = x' = 1; and that if A = ½*x^2, then y = A' = (½*x^2)' = x; and that if A = 1/3*x^3, then y = A' = (1/3*x^3)' = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to 3, instead of adding small strips we just calculate the change in the area over this distance, later called the fundamental theorem of calculus.

A Literature Based Compendium

An example of an ideal precalculus curriculum is described in 'Saving Dropout Ryan With a Ti-82' (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' (Tarp, 2019) replaced the textbook. A formula's left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

An Example of a Fresh/start Precalculus Curriculum

This example was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the

domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

An Example of an Exam Question

Authentic material was used both at the written and oral exam. The first had specific, the second had open questions as the following asking ‘What does the table tell?

Agriculture: Number of agricultural farms allocated over agricultural area

	1968	1969	1970	1971	197?	1973	1974	1975	1976	1977
Farms in total	**161142**	**154 694**	**148 512**	**144 070**	**143093**	**141 137**	**137712**	**13424S**	**130 7S3**	**127117**
0,0- 4,9 ha	25 285	23 493	21 533	21623	22123	21872	21093	19915	18 852	17 833
5.0- 9.9-	34 644	32129	30 235	28 404	27693	26 926	26109	25072	24066	23152
10,0-19.9-	48 997	46482	43 971	41910	40850	39501	38261	36 702	35 301	34 343
20.0-29.9-	25670	25 402	25181	24 472	24 195	23 759	23 506	23134	22737	22376
30,0-49.9-	18 505	18 779	18 923	18 705	18 968	18 330	19 095	19 304	10 305	19 408
50,0-99.9-	6 552	6 852	7 076	7 275	7 549	7956	7 847	8247	8 556	8723
100.0 ha and over	1489	1 557	1611	1681	1 715	1791	1801	1871	1934	1882

Figure 02. A table found in a statistical survey used at an oral exam.

Discussion and Conclusion

Asking “how can precalculus be sustainably changed?” an inside answer would be: “By its nature, precalculus must prepare the ground for calculus by making all function types available to operate on. How can this be different?”

An outside answer could be to see precalculus, not as a goal but as a means, an extension to the number-language allowing us to talk about how to unite and split into changing and constant per-numbers. This could motivate renaming precalculus to per-numbers calculations.

In this way, precalculus becomes sustainable by dealing with adding, finding and counting change-factors using power, roots and logarithm. Furthermore, by including adding piecewise constant per-numbers by their areas, precalculus gives a natural introduction to calculus by letting integral calculus precede and motivate differential calculus since an area changes with the last strip, thus connecting the unit number, the area, with the per-number, the height.

Finally, graphing calculators allows authentic modeling to take place so that precalculus may be learned through its use, and through its outside literature.

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