DE-MODEL

Numbers & Operations & Equations

Adaptive Green Mathematics: Kids' own DoubleNumbers with Units

Bundle- & Per-Numbers in Primary & Secondary School

Understanding: From INSIDE-INSIDE to OUTSIDE-INSIDE

The Goal of Math Education, is that to

master outside many, or
master inside math (to later master outside many)

Allan.Tarp@MATHeCADEMY.net, Denmark, December 2019

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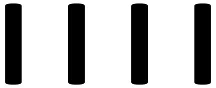
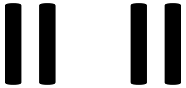
Observation 01: Is Mathematics WellDefined?

No, three Versions: MetaMatics, MatheMatism, ManyMath

This is true	Always	Never	Sometimes
2 + 3 = 5	Only with the same unit; 2weeks + 3days = 17days		**x** (MatheMatsim)
2 x 3 = 6	**x**	2x3 is 2 **3s** with 3 as unit, that recounts as 6 **1s**	(ManyMath)
$\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$		 1 of 2 apples + 2 of 3 apples gives 3 of 5 apples, and not 7 of 6	**x** (ManyMath)
$\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$	Only if taken of the same total *Fractions are not numbers, but operators, needing numbers to become numbers*		**x** (MatheMatsim)
<u>C1:</u> a **FUNCTION** is	**For example** 2+*x*, but not 2+3 i.e. a name for a calculation with an unspecified number		**(1750-1900)** (ManyMath)
<u>C2:</u>	**An example of** a SET-relation where first component identity implies second component identity		**(after 1900)** (MetaMatics)

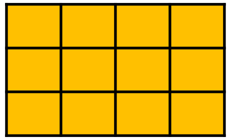
Observation 02: Adapting to Many, Children create **Flexible BundleNumbers** with Units

"How old next time?" A 3year old says "Four" showing 4 fingers: 

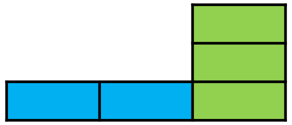
But, reacts strongly to 4 fingers held together 2 by 2: 

"That is not four, that is two **twos**"

Adapting to Many, a child uses BUNDLE-NUMBERS to describe what exists, and with units: bundles of **2s**, and 2 of them.

The block 3 **4s** has two numbers: 3 (the counting-number) & 4 (the unit-number) 

5 fingers counted in **2s**, using flexible bundle-numbers:

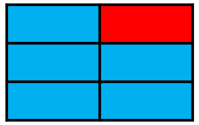


5 = 1**B**3 **2s**

Over-load

5 = 2**B**1 **2s**

Normal



5 = 3**B**-1 **2s**

Under-load

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Bundle-Numbers can Shift Unit and create a RecountFormula

$8 = (8/2)\text{x}2$

$T = (T/B)\text{x}B$

Bundle-counting 8 in 2s: 8 = ? **2s**

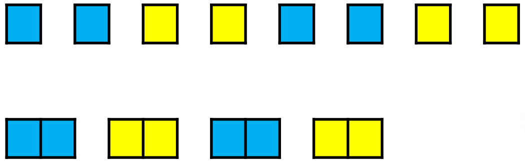
Fingers: 8 = 4 **2s**

Bricks bundle-counted: 8 = 8/2 **2s**

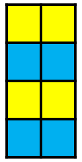
Bundles stacked: 4 **2s** = 4x2

Bundle-counting: 8 = (8/2)x2

Recount-Formula: **T = (T/B)xB**



8/2: From 8 **PUSH** away 2



4 times **LIFT** 2

*u*x2 = 8 = (8/2)x2

***u* = 8/2**

OPPOSITE side & sign

Equations ⟵ ⟶ Proportionality

Shifting unit	y = k * x
Linearity	Δy = (**Δy/Δx**) * Δx = **m** * Δx
Local linearity	dy = (**dy/dx**) * dx = **y'** * dx
Trigonometry	a = (**a/b**) * b = **tanA** * b
Trade	$ = (**$/kg**) * kg = **price** * kg
STEM	meter = (**meter/sec**) * sec = **speed** * sec

Finding: ReCounting in BundleNumbers contains Core Mathematics & STEM

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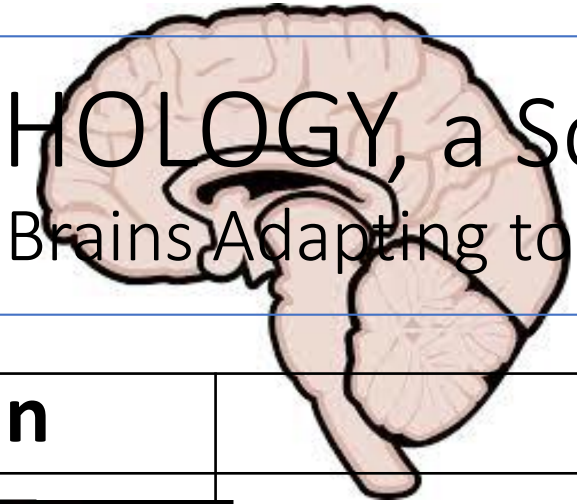
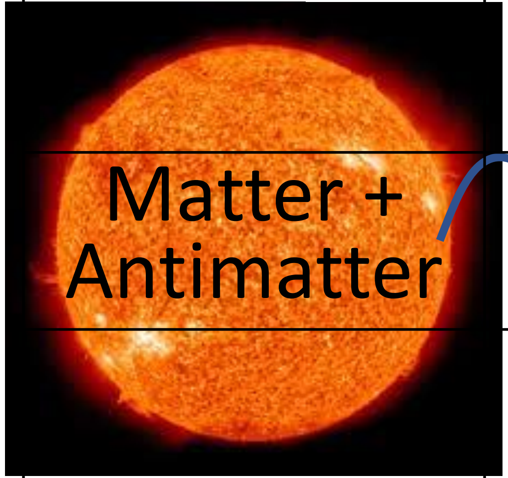
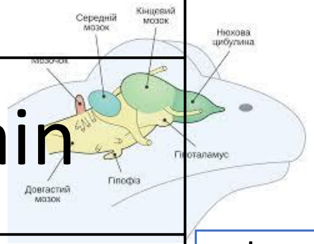
PSYCHOLOGY, a Science about Behavior & Brain

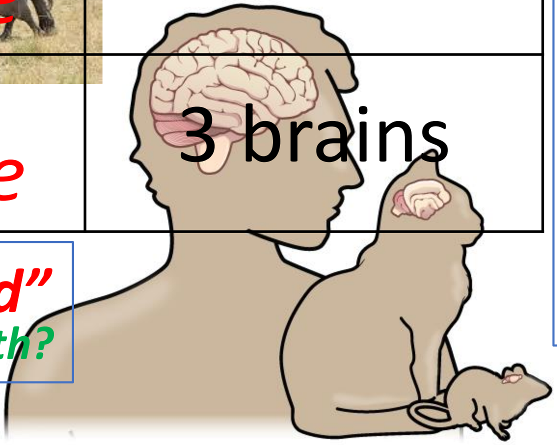
0,1,2,3 Brains Adapting to Life on the 3rd Rock from the Sun

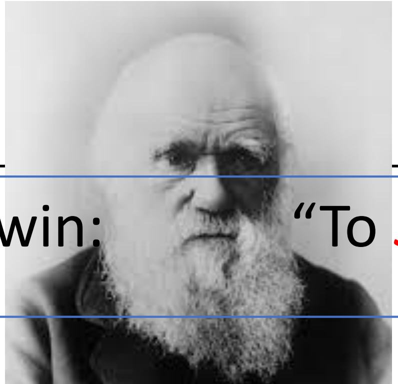


Sun		Earth	Life	Brains
		Green Cells	Plants *Quantity*	No brain
Matter + Antimatter	Energy Radiation		Reptiles *Mutation*	1 brain
		Grey Cells	Mammals *ChildCare*	2 brains
			Humans *Language*	3 brains









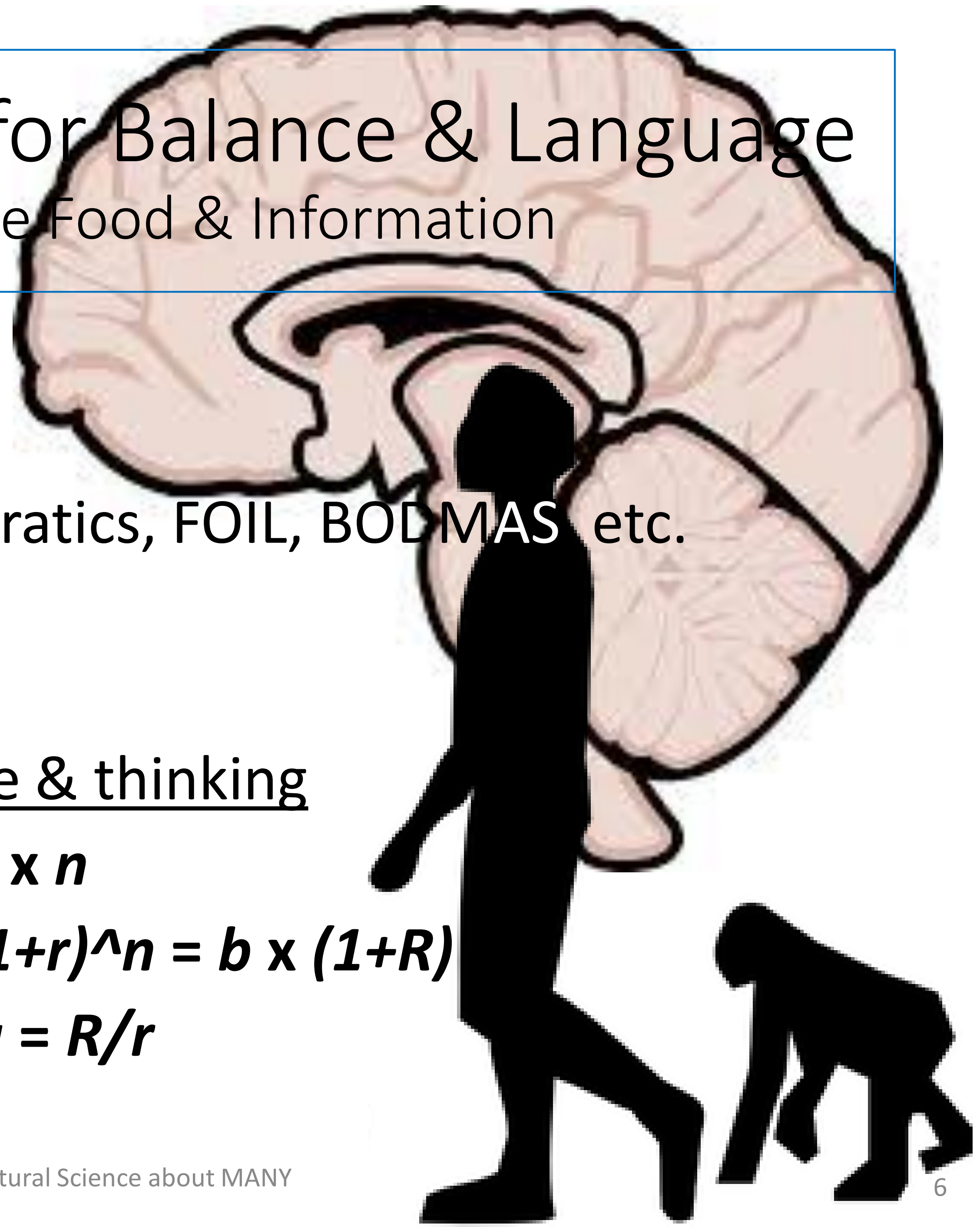
Holes in the head for food and information

Darwin: "To *SURVIVE*, you must ***ADAPT to the outside world***"

And so must math?

Standing up Created a Brain for Balance & Language

Forelegs became Hands to Grab and Share Food & Information



Humans have 3 brains:

A Reptile Brain for routines

- By heart, I know tables, formulas, quadratics, FOIL, BODMAS etc.

A Mammal Brain for feelings

- I **LIKE** math; I **DISLIKE** math

A Human Brain for information, language & thinking

- Total after *n* times adding *a*\$: $T = b + a \times n$
- Total after *n* times adding *r*% : $T = b \times (1+r)^n = b \times (1+R)$
- Total after *n* times adding *a*\$ & *r*% : $T/a = R/r$

Our two Language Houses have two Floors

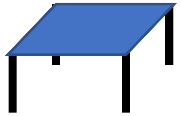
The **WORD-language** assigns words in sentences with The **NUMBER-language** assigns numbers instead with	• a subject • a verb • a predicate

Both languages have a META-language, a grammar, describing the language, that is learned before the grammar in the word-language, but in the number-language grammar is first.

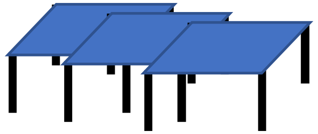
	WORD language	NUMBER language
language, grammar	*'is' is a verb*	*'x' is an operation*
Language	This is a table	T = 3x4

Mathematics

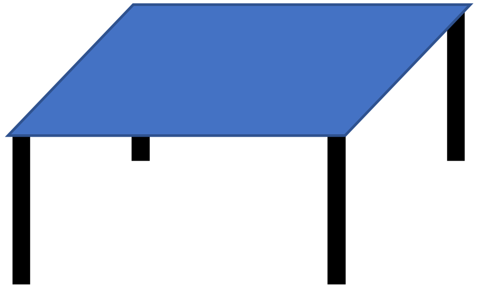
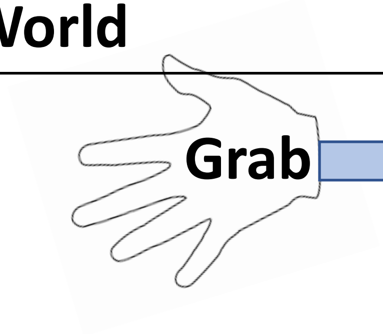
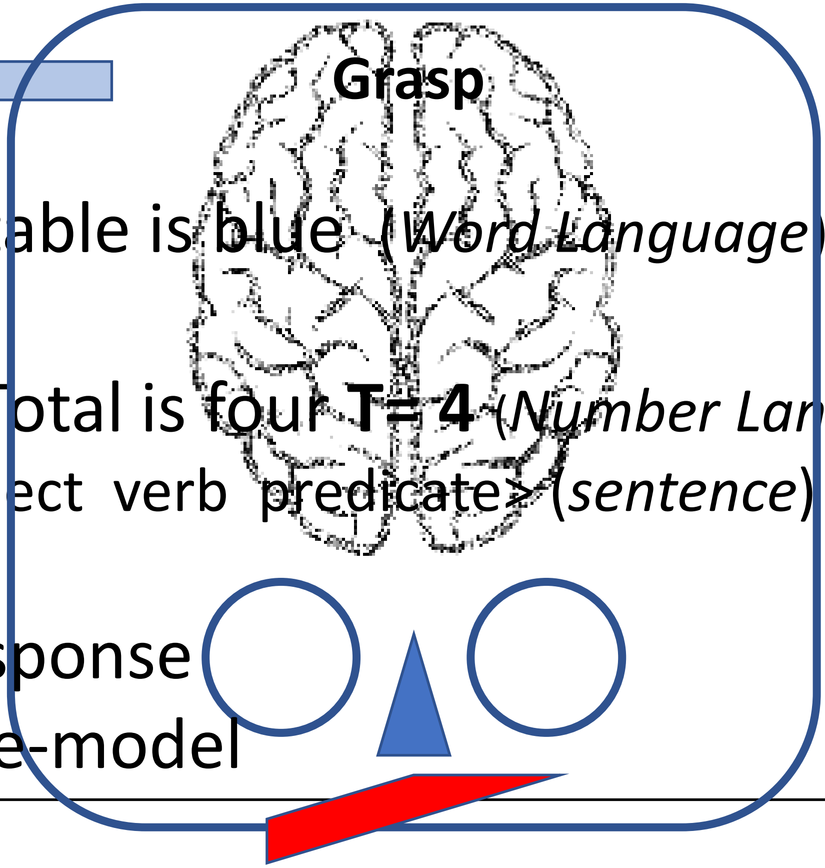
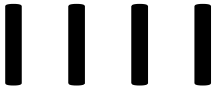
Mother tongue



WORLD

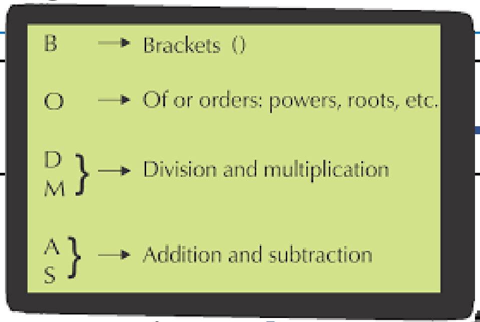


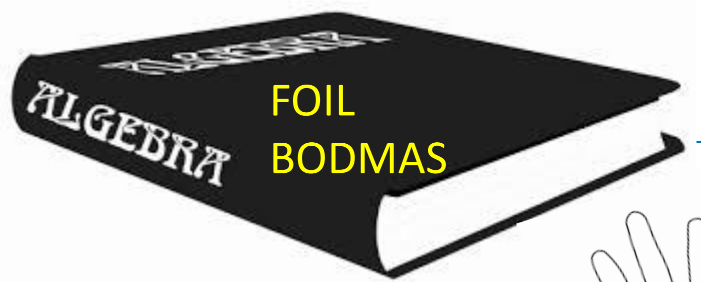
A Brain adapts through Stimuli & Response
OUTSIDE Stimulus ⏩ ⏪ INSIDE Response

Outside World	**Inside Brain**
  **Grab**	to **Grasp** 
	The table is blue (*Word Language*)
	The Total is four **T = 4** (*Number Language*)
	<Subject verb predicate> (*sentence*)
Stimulus ⏩	⏪ Response
Model ⏩	⏪ De-model

Inside-Outside Skinner-learning (I)

Reptile & Mammal Brain Learning





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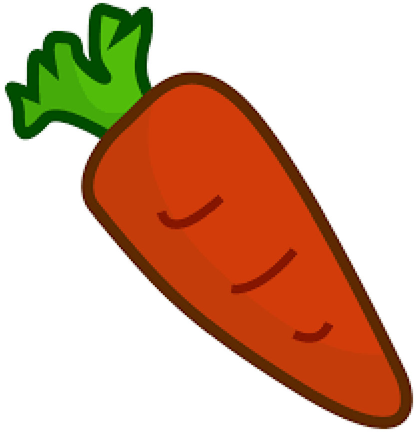
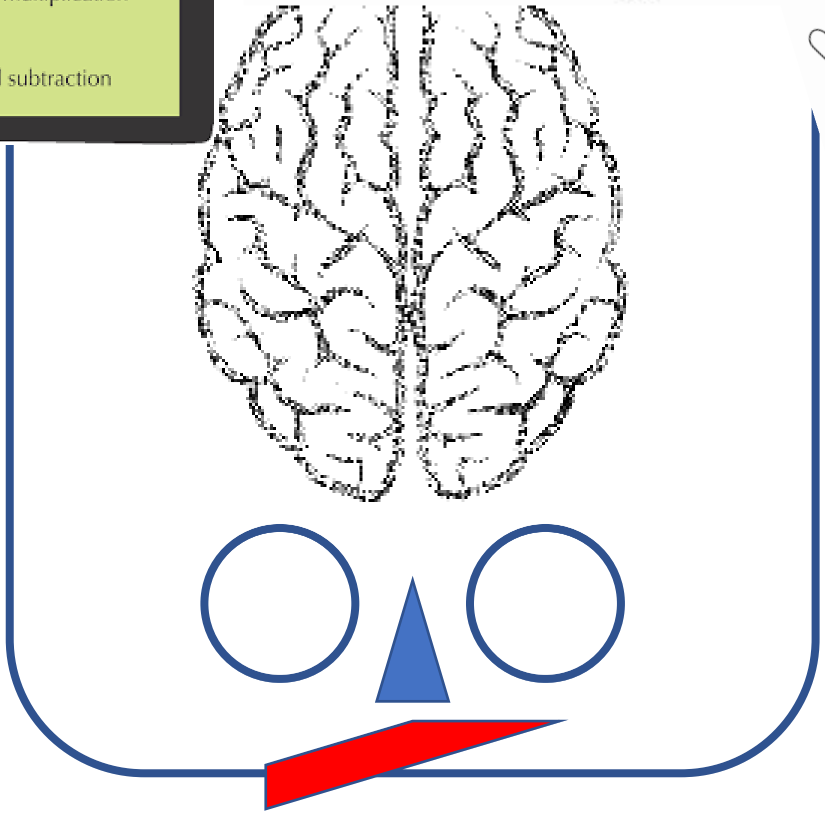
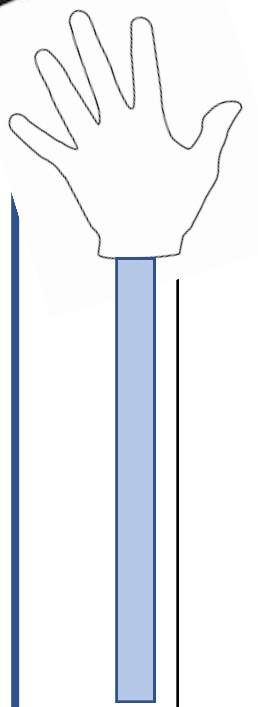
Outside World

Reward or Punishment







Objection: *Skills OK, but no understanding*

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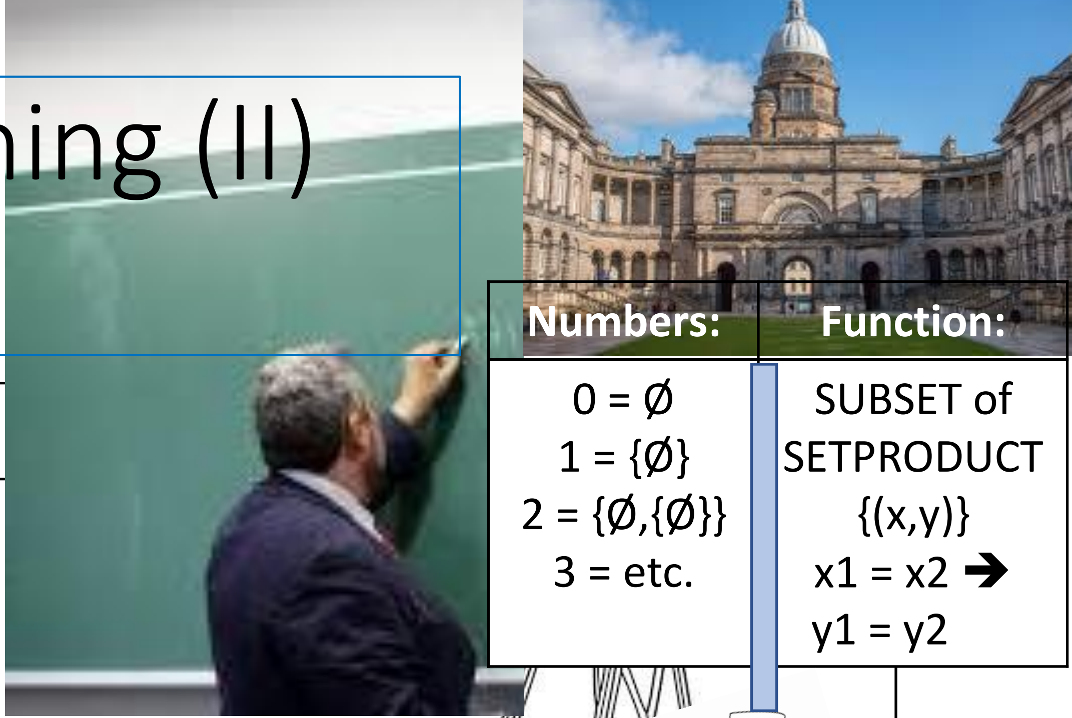
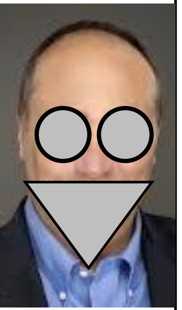
Inside-Inside Vygotsky-learning (II)

A major Brain teaches (colonizes?) minor Brains abstract TopDown Understanding & Enculturation



Numbers:	Function:
0 = Ø 1 = {Ø} 2 = {Ø,{Ø}} 3 = etc.	SUBSET of SETPRODUCT {(x,y)} x1 = x2 ➔ y1 = y2

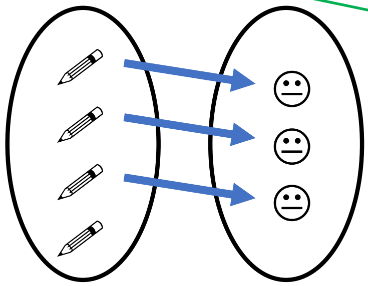


Bruner & Skemp & Vygotsky

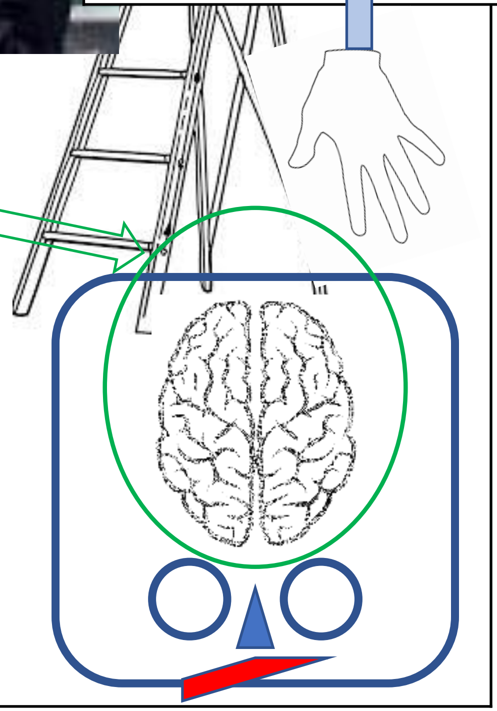
School subjects must mirror university subjects to structure a good teaching, providing a scaffolding as the ladder down to the learners ‘Zone of Proximal Development’ (ZPD, Vygotsky).

To understand numbers, first you must understand understanding; then cardinality as equivalence classes in the set of sets. So children first draw arrows between sets to learn number-names.

“What a child can do today with assistance, she will be able to do by herself tomorrow”. So, good teaching by a more knowledgeable other matters. So does good teacher education and good Professional Development.



~~Numbers~~



Outside-Inside Piaget-learning (III)

Peer-Brains teach eaxh other Bottom-up Understanding
through concrete Examples provided by Guiding Teachers

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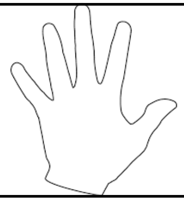
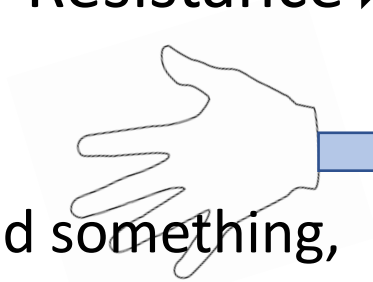
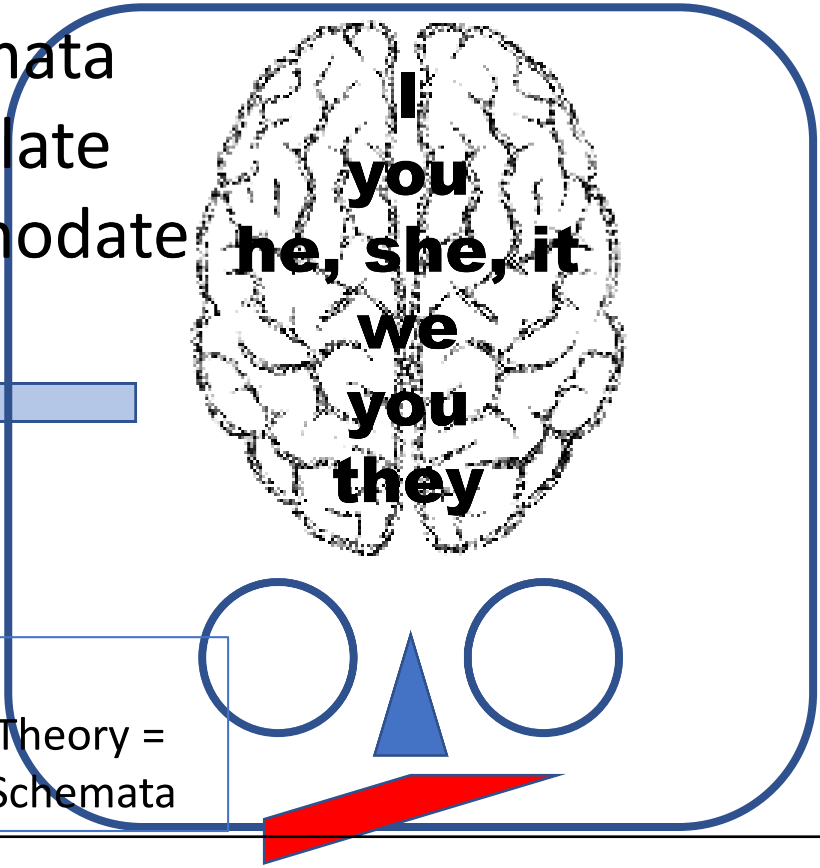
Outside World	Inside Brain

Numbers, Operations *manual*	**Functions are Sentences *verbal***
	T = 2+3 No T = 2+? Yes

Adaption ⏩ Schemata
Validate ⏪ Assimilate
Resistance ⏩ Accomodate





"Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover for himself will remain with him visible for the rest of his life"

Research:
Grounded Theory =
Collective Schemata

Research Question

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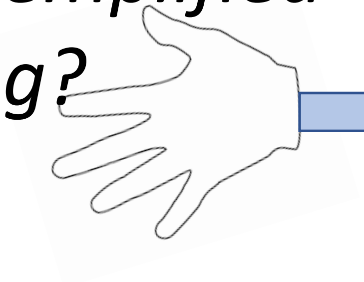
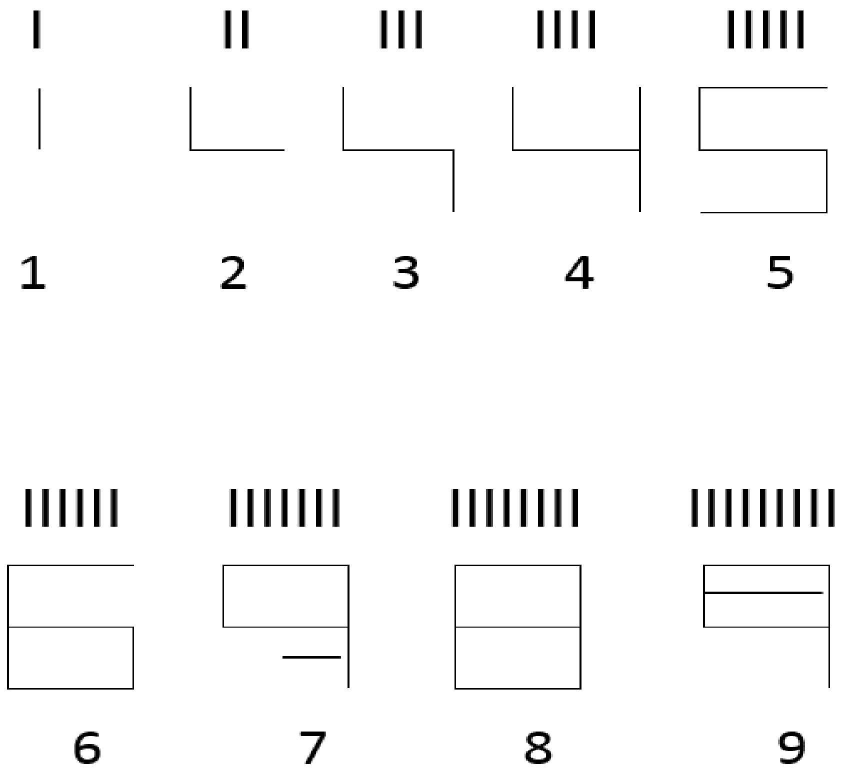
Outside World	Inside Brain
Can children discover/invent mathematics themselves to obtain a concrete exemplified understanding?  Method: De-model or re-ify numbers and operations etc.	

DeModel Digits as Icons

with as Many Sticks as they Represent: 4 Sticks in the 4-Icon etc.

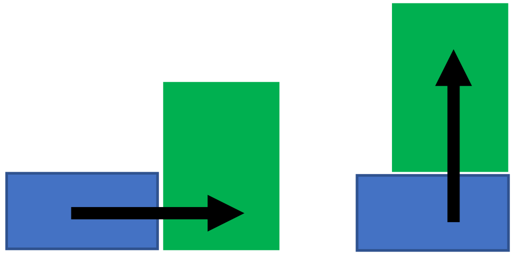
Outside World	Inside Brain
	What? Digits are icons with as many sticks as they represent! *But, why does ten not have an icon?* Oh, it is a **Bundle**, so ten is 1**Bundle**0. So eleven is 1**B**1, and twelve is 1**B**2. Ah, the Vikings liked to shorten: 1**B**1 became 1-left (one-leven), 1**B**2 became 2-left (twe-leven).

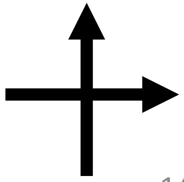
DeModel Division & Multiplication & Subtraction & Addition as Icons also

- From 9 **PUSH** away **4s** we write 9/4 iconized by a broom, called *division*.

- 2 times **LIFTING** the **4s** to a stack we write 2x4 iconized by a lift called *multiplication*.
- From 9 **PULL** away 2 **4s'** to find un-bundled we write 9 – 2x4 iconized by a rope, called *subtraction*.

- **UNITING** next-to or on-top we write **A**+**C** iconized by two directions, called *addition.*





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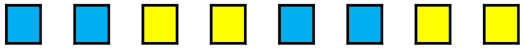
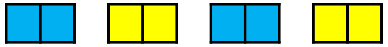
A BundleNumber carries a Unit that is changed by a RecountFormula

$8 = (8/2)x2$

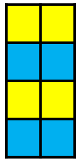
$T = (T/B)xB$



8/2: from 8 **PUSH** away 2



4x2: 4 times **LIFT** 2



Bundle-counting 8 in 2s: 8 = 8/2 **2s**

Bundles stacked: 4 **2s** = 4x2

Bundle-counting prediction: 8 = (8/2)x2

Recount-Formula: **T = (T/B)xB**

- Solves equations

OPPOSITE side & sign:

*u*x2 = 8 = (8/2)x2

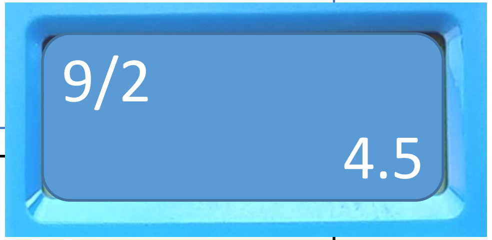
u = 8/2

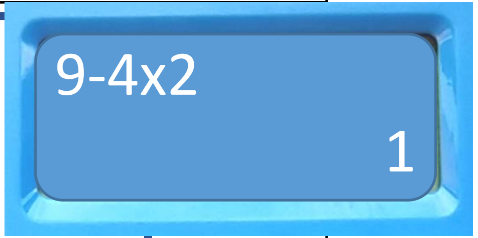
- Is all over **S**cience, **T**echnology, **E**ngineering, **M**ath:

Unit shift	y = k * x
Linearity	Δy = (**Δy/Δx**) * Δx = **m** * Δx
Local linearity	dy = (**dy/dx**) * dx = **y'** * dx
Trigonometry	a = (**a/b**) * b = **tanA** * b
Trade	$ = (**$/kg**) * kg = **price** * kg
STEM	kg = (**kg/m^3**) * m^3 = **density** * m^3 meter = (**meter/sec**) * sec = **speed** * sec Joule = (**Joule/sec**) * sec = **watt** * sec Joule = (**Joule/kg**) * kg = **heat** * kg

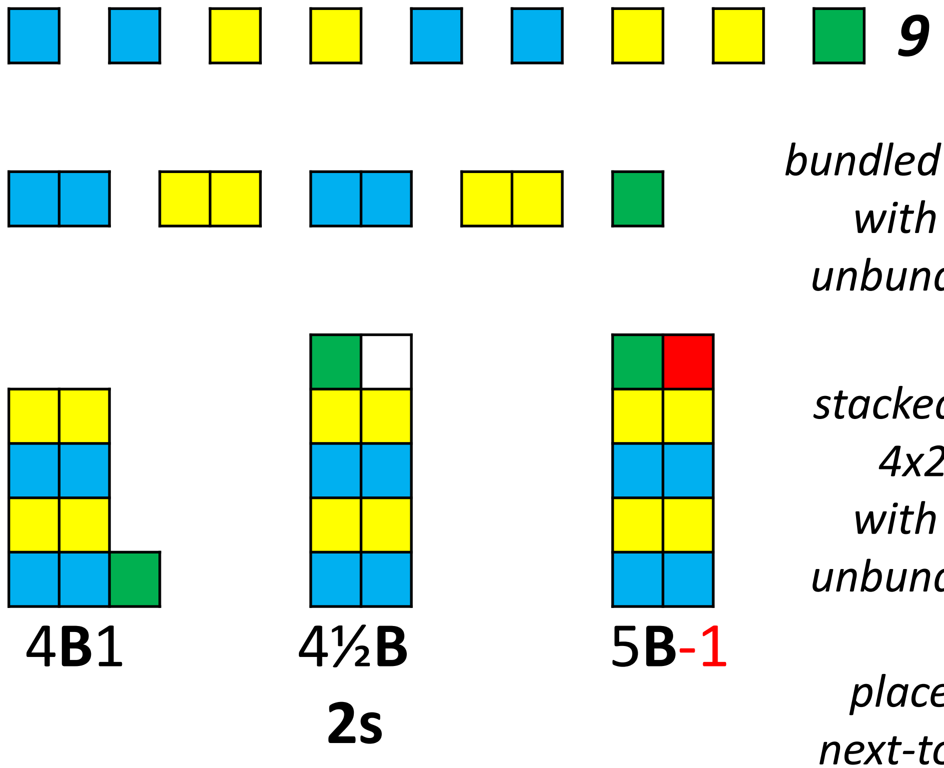
BundleCounting a Total of 9 in **2s**

9/2 **4.some**
9 – 4x2 **1**





Outside World



*bundled in **2s** with 1 unbundled*

stacked as 4x2 with 1 unbundled

placed next-to or on-top

Inside Brain

From 9, 9/2 times push away 2.
From 9, pull away 4 **2s**, leaving 1.
Prediction by the recount-formula:

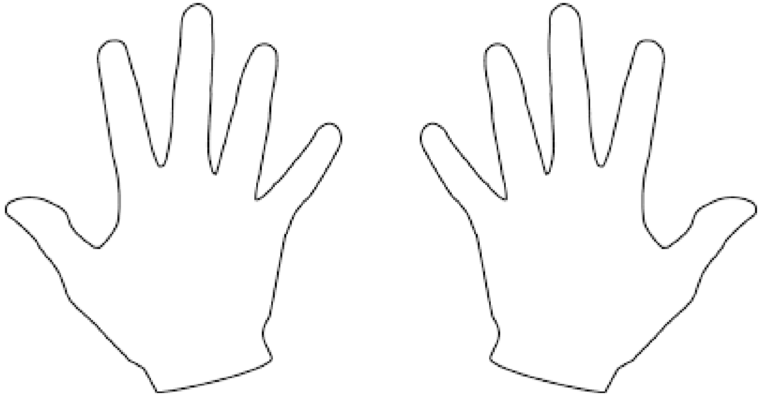
T = 9 = **4B1 2s**

The unbundled can be placed

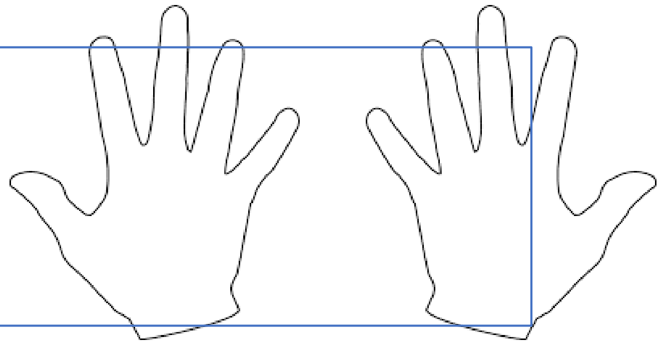
- next-to the stack iconized by a dot named a decimal point; 4.1 **2s**; or on-top of the stack
- counted in bundles as 1 = (1/2) x 2 giving 4½**B 2s**,
- counting what is missing in a full bundle, 5**B**-1 **2s.**

This de-models decimals, fractions & negatives.

BundleCounting Fingers in **5s**

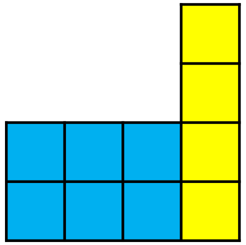
Outside World	Inside Brain
	One hand counted in **5s**, using **B** for Bundle: • ~~1, 2, 3, 4, 5~~ ; • 0**B**1, 0**B**2, 0**B**3, 0**B**4, ~~0**B**5~~ **no** 1**B**0; or • 1 **Bundle** less 4, 1**B**-3, 1**B**-2, 1**B**-1, 1**B**0. Two hands counted in **5s**: • 0**B**1, 0**B**2, 0**B**3, 0**B**4, ~~0**B**5~~ **no** 1**B**0, 1**B**1, …, 1**B**4, ~~1**B**5~~ **no** 2**B**0 • 1**B**-4, 1**B**-3, 1**B**-2, 1**B**-1, 1**B**0, 2**B**-4, 2**B**-3, 2**B**-2, 2**B**-1, 2**B**0

BundleCounting Fingers in **3s**

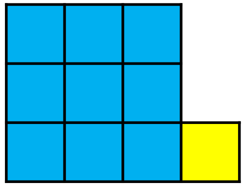


Over-load, Normal, Under-load

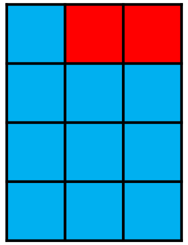
Two hands bundle-counted in **3s**:



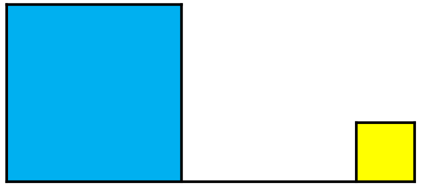
T = 2**B**4
over-load



T = 3**B**1
normal



T = 4**B**-2
under-load



T = 1**BB** 0**B** 1 = 101 **3s**

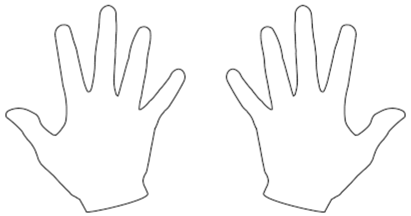
Singles, Bundles, Bundle-Bundles

Counting-sequence bundle-counting in **3s**:

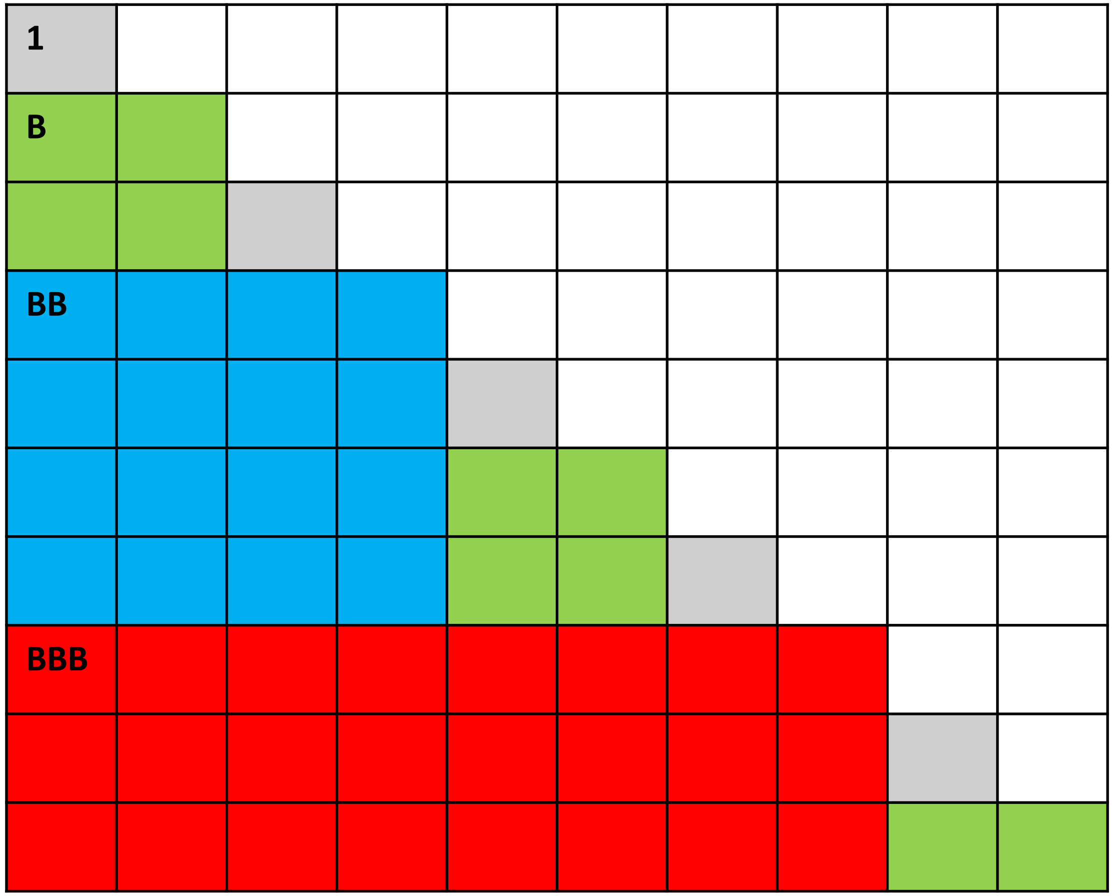
0**B**1, 0**B**2, ~~0**B**3~~ **no** 1**B**0,
1**B**1, 1**B**2, ~~1**B**3~~ **no** 2**B**0,
2**B**1, 2**B**2, ~~2**B**3~~ **no** 3**B**0,
3**B**1 or ten

But 3 **Bundles**, is 1 **Bundle-of-Bundles.**
So T = 9 = 1**BB 3s** or
T = ten = 3**B**1 **3s** = 1**BB**1 **3s** or
T = ten = 1**BB**0**B**1 **3s** or 1**BB**1**B**-2 **3s** or
T = ten = 101 **3s**

BundleCounting Fingers in **2s**



1	1	1
2	1**B**0	10
3	1**B**1	11
4	1**BB**00	100
5	1**BB**01	101
6	1**BB**1**B**0	110
7	1**BB**1**B**1	111
8	1**BBB**000	1000
9	1**BBB**001	1001
Ten	1**BBB**01**B**0	1010



This can be shown with Lego bricks having different colors:

a green 2-brick is **B**

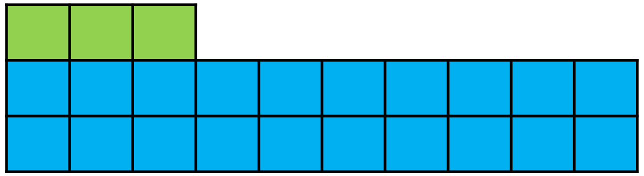
a blue 4-brick is **BB**

a red 8-brick is **BBB**

Counting in Tens with UnderLoad & OverLoad

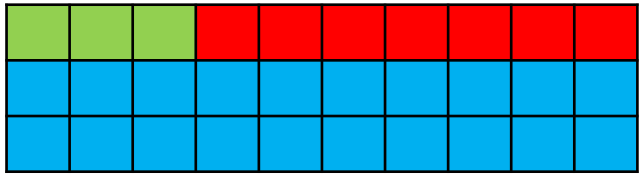
Q: Where to put the unbundled singles with tens?

Counting in tens, an outside Total of 2 **tens** + 3 can be inside described as T = 2.3 **tens**, or as 23 if leaving out the unit; or as



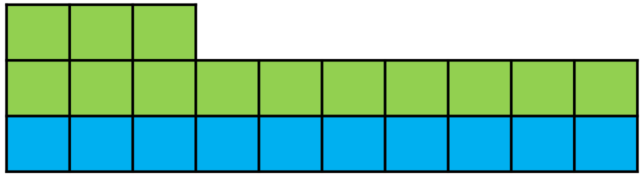
T = 2**B**3 **tens**

normal



T = 3**B**-7 **tens**

under-load



T = 1**B**13 **tens**

over-load

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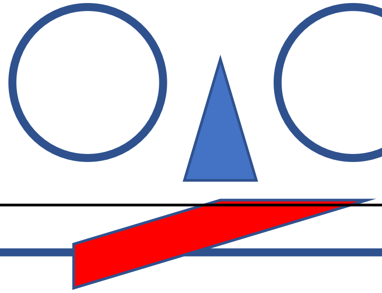
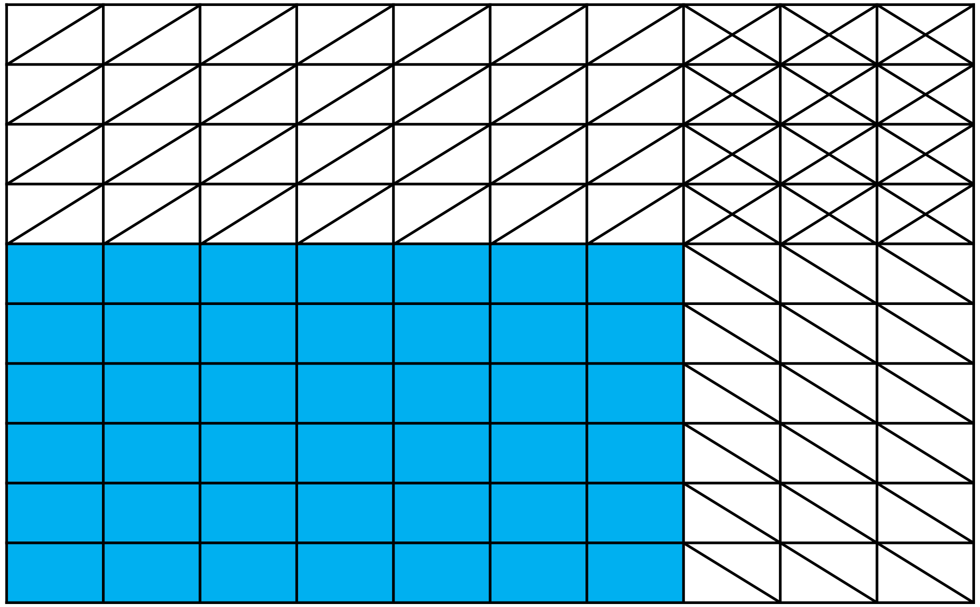
Recounting into Tens: Multiplication Tables
The Small Table from 2 to 5

Outside World	Inside Brain
see table below	The small table comes from using hands and fingers. With **2s**, a finger counts 1. With **3s**, a finger counts 3. With **4s**, 4 is a **H**and less 1, 4 = **H**-1. With **5s**, 5 is a **H**and with 2**H** = 1**B**.

Count in	1	2	3	4	5
2s	2	4	6	8	1B0
3s	3	6	9	1B2	1B5
4s	**H**-1 4	**B**-2 8	**BH**-3 1B2	**BB**-4 1B6	**BB** 2B0
5s	**H** 5	**B** 1B0	**BH** 1B5	**BB** 2B0	**BBH** 2B5



Recounting into Tens: Multiplication Tables
The Big Table until Ten

Outside World	**Inside Brain**
6 7s = ? tens  This roots the algebraic formula *(a – b) x (c – d) = a x c – a x d – b x c + b x d*	On a bead pegboard, 2 rubber bands make a ten by ten square; and 2 bands make a 6 by 7 block. So 6 **7s** is ten bundles, less 4 bundles, less 3 bundles, plus the 3 **4s** that are removed twice: T = 6 **7s** = 6 x 7 = (**B**-4) x (**B**-3) = 10**B** – 4**B** – 3**B** + 4 **3s** removed twice = 3**B**12 = 4**B**2 = 42 *Interesting, negative x negative = positive!*

6 x7 gives the less-numbers 4 & 3. So from tens we subtract the sum, and add the product.

Recounting BundleBundles in Tens (Squares: ..., 4 **4s** = ? **tens**, 5 **5s** = ? **tens**, ...)

Using the multiplication table, we recount the different bundle-bundles (called squares) in **tens**:

S4 = 4 **4s** = 4x4 = 16

S5 = 5 **5s** = 5x5 = 25, etc.

We see that to get to the next square we add the sides twice, + 1:

5*5 = 4*4 + 2*4 + 1, or with 4 = n:

(n+1)*(n+1) = n*n + 2*n + 1, or

$(n+1)^2 = n^2 + 2*n + 1$

	1	2	3	4	5	6	7	8	9	10
1	1									
2		4								
3			9							
4				16						
5					25					
6						36				
7							49			
8								64		
9									81	
10										100

Recounting Large Numbers into Tens: Block Schemes or **FOIL**-lines

First
Outside
Inside
Last

Outside World	Inside Brain

27 **36s** = ? **tens**

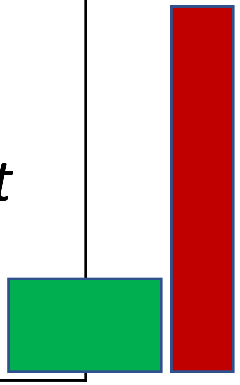
	2**B**	7	
	6**BB**	21**B**	3**B**
	12**B**	42	6
6**BB**	33**B**	42	
6**BB**	37**B**	2	
9**BB**	7**B**	2	
9	7	2	

This roots the algebraic formula
(a + b) x (c + d) = a x c + a x d + b x c + b x d

Question: T = 27 **36s** = ? **tens,** or 27 x 36 = ?
We use a block scheme top-down, or FOIL-lines.

Answer: T = 2**B**7 x 3**B**6
= (2**B** + 7) x (3**B** + 6)
= 6**BB** + 12**B** + 21**B** + 42
= 6**BB** + 33**B** + 4**B**2
= 6**BB** + 37**B** + 2
= 9**BB**7**B**2 = 972 = 97.2 tens.

It makes sense that decreasing the unit base from 36 to 10 will increase the height of the stack from 27 to 97.2.



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Recounting Large Numbers into Icons: Reversing Block Schemes or FOIL-lines

Outside World	Inside Brain
16.8 tens = ? 7s	Question: 16.8 **tens** = ? **7s**; or 168 /7 = ? We use a block scheme bottom-up, or reverse the FOIL-lines. Answer:

Outside World

16.8 tens = ? 7s

		7	
		14**B**	? = 2**B**
		28	? = 4
	14**B**	28	
	16**B**	8	
1	6	8	

Inside Brain

Question: 16.8 **tens** = ? **7s**; or 168 /7 = ?
We use a block scheme bottom-up, or reverse the FOIL-lines. Answer:

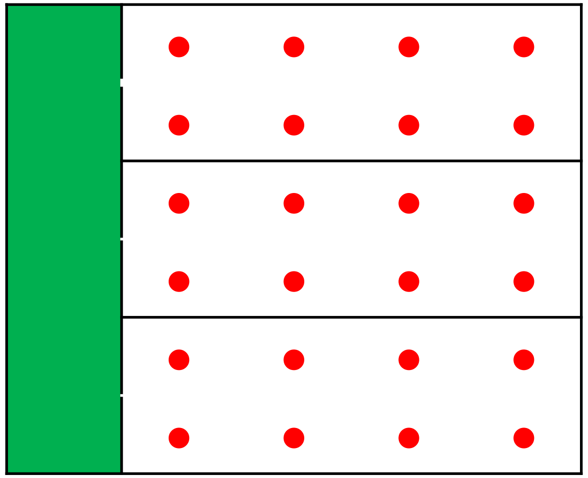
168 = 16**B**8 = 14**B**28 = 7 x 2**B**4
= 7 x 24 = 24 **7s**

Question: 16.9 **tens** = ? **7s**; or 169 /7 = ?

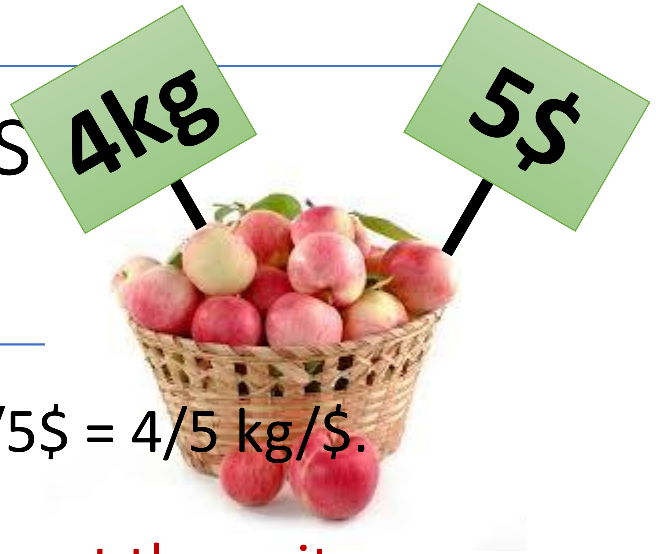
169 = 16**B**9 = 14**B**29 = 14**B**28 + 1
= 7 x 2**B**4 + (1/7) x 7
= 7 x (24+1/7) = 24 1/7 **7s**

It makes sense that decreasing the unit base from 10 to 7 will increase the height of the stack from 16.8 to 24.

DoubleCounting gives PerNumbers Bridging Units

Outside World	Inside Brain
On a 2x4 brick, **double-counting** rows and dots gives the **per-number** 2 rows per 8 dots, or 2r/8d or 2/8 r/d. 	Q: **6** rows = ? dots. A: recount 6 in the per-number 2. T = 6r = (6/2) x 2r = (6/2) x 8d = 24dots Q: ? rows = 56 dots? A: recount 56 in the per-number 8. T = 56d = (56/8) x 8d = (56/8) x 2r = 14rows

DoubleCounting in two Units creates **PerNumbers** & Proportionality



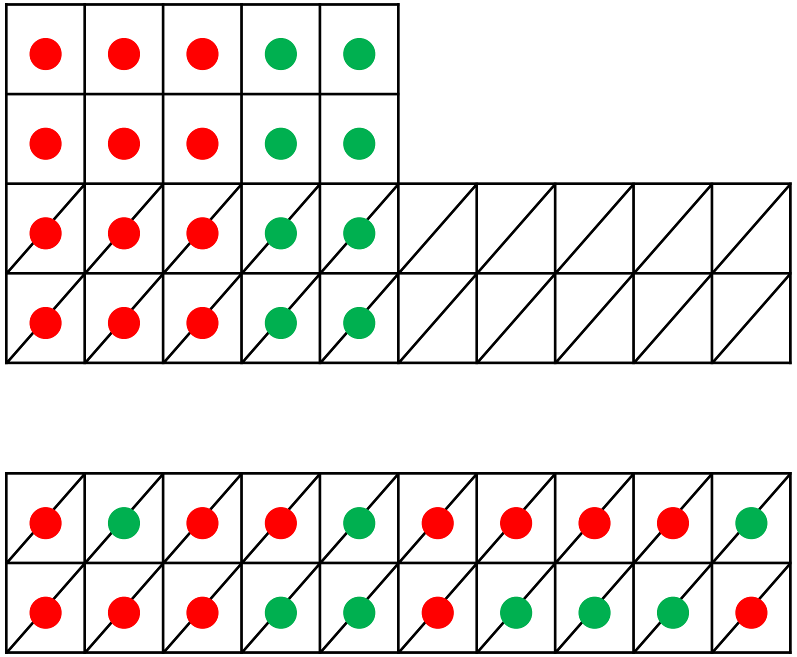
DoubleCounting in kg & \$, we get a **PerNumber** 4kg **per** 5\$ = 4kg/5\$ = 4/5 kg/\$.

With 4kg bridged to 5\$, we recount in the per-number. Or we recount the units directly. Or we equate the per-numbers. Or we use the before 1900 ‘Rule of 3’ (regula de tri) alternating the units, and, from behind, first multiply, then divide.

Questions:

12kg = ?\$	**20\$ = ?kg**
12kg = (12/4) x 4kg = (12/4) x 5\$ = 15\$	20\$ = (20/5) x 5\$ = (20/5) x 4kg = 16kg
\$ = (\$/kg) x kg = 5/4 x 12 = 15	kg = (kg/\$) x \$ = 4/5 x 20 = 16
u/12 = 5/4, so *u* = 5/4 x 12 = 15	*u*/20 = 4/5, so *u* = 4/5 x 20 = 16
If 4kg is 5\$, then 12kg is ?\$; answer: 12x5/4 = 15	If 5\$ is 4kg, then 20\$ is ?kg; answer: 20x4/5 = 16

With like Units, PerNumbers become Fractions, both Operators Needing Numbers to Become Numbers

Outside World	Inside Brain
In a box filled with 3 red per 5 apples, double-counting reds and apples gives the FRACTION **3/5** reds/apples. 	Q: ? red in 20 apples. A: Recount 20 in **5s** (the per-number) T = 20a = (20/5) x 5a gives (20/5) x 3r = 12 red apples Or, we equal the per-numbers: *u*/20 = 3/5; so *u* = 3/5 x 20 = 12. *Moving 20 to opposite side with opposite sign*

DoubleCounting the Sides in a Block

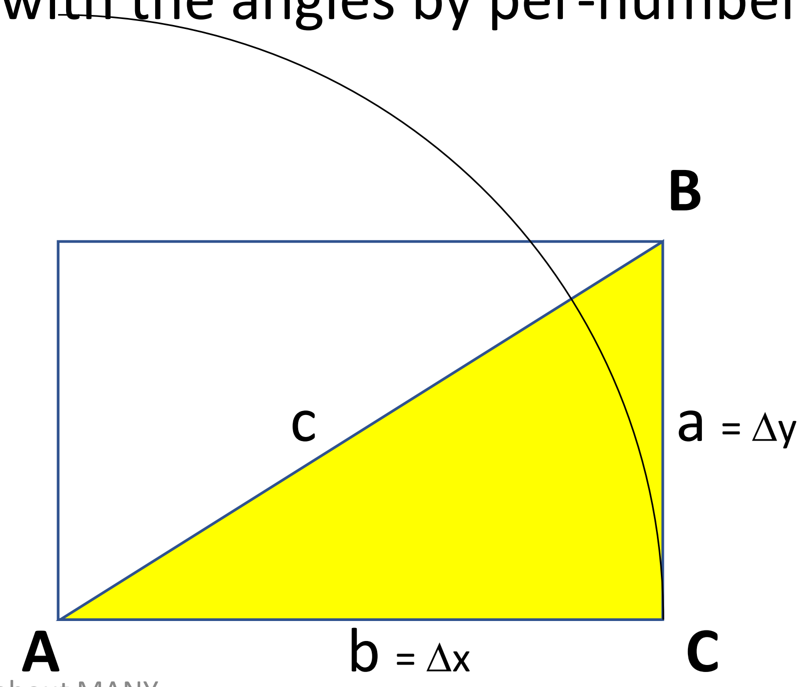
Geometry means to measure earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula. And connected with the angles by per-number formulas double-counting the sides pairwise.

A + **B** + **C** = 180

a*a + b*b = c*c (the Pythagoras formula)

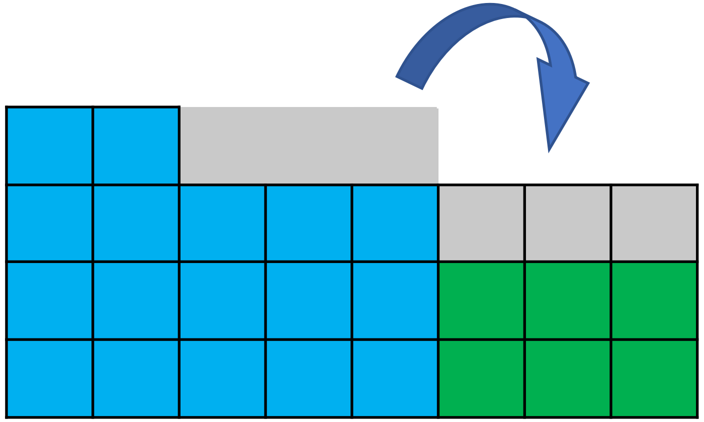
a = (a/c) x c = sin**A** x c; tan**A** = a/b = $\Delta y/\Delta x$ = gradient

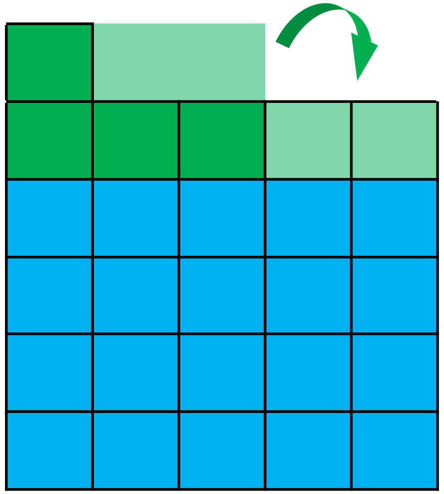
Circle: circum./diam. = **π = n*sin(180/n)** for n large



Counted & Recounted, Totals may be Added

BUT: NextTo	**or OnTop**
4 5s + 2 3s = 3**B**2 8s	4 5s + 2 3s = 4 5s + 1**B**1 5s = 5**B**1 5s
The areas are integrated *Adding areas = Integration*	The units are changed to be the same *Change unit = Proportionality*





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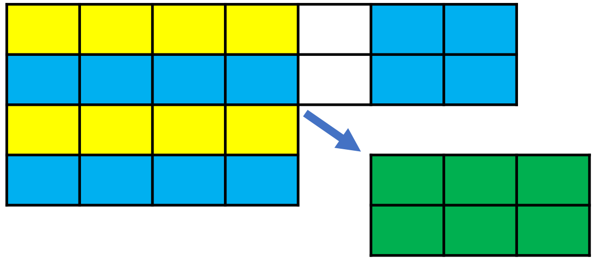
Reversed NextTo Addition



"If T1 = 2 **3s** and T2 add next-to as 4 **7s**, what is T2?"

<u>Outside</u>, we remove the initial block T1 and recount the rest in **4s**.

Thus reversed next-to addition geometrically means subtracting areas.

Reversed next-to addition is also called differential calculus.

<u>Inside</u>, the recount formula algebraically predicts the result.

Here subtraction precedes division; which is natural as reversed integration.

T2 = (T2/**B**) x **B**

= ((4x7 – 2x3)/4) x 4 = 5.2 **4s**

(4x7 – 2x3)/4	**5.some**
(4x7 – 2x3) – 5x4	**2**

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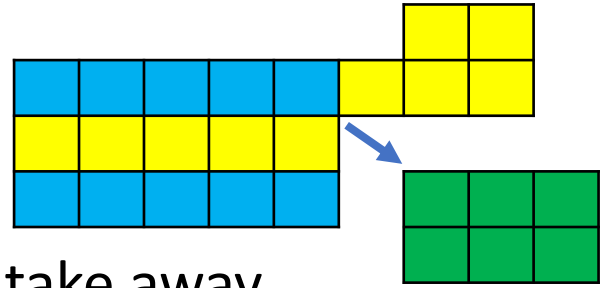
Reversed Addition = Solving Equations

OppoSite Side with OppoSite Sign		**NextTo**
2 x ? = 8 = (8/2) x 2	**2 + ? = 8** = (8-2) + 2	2 **3s** + ? **5s** = 3.2 **8s**
? = 8/2 *Solved by ReCounting*	? = 8-2 *Solved by ReStacking*	? = (3.2 **8s** – 2 **3s**)/5 *Solved by differentiation: (T-T1)/5 = ΔT/5*



<u>Hymn to Equations</u>

Equations are the best we know,
they are solved by isolation.
But first, the bracket must be placed
around multiplication.

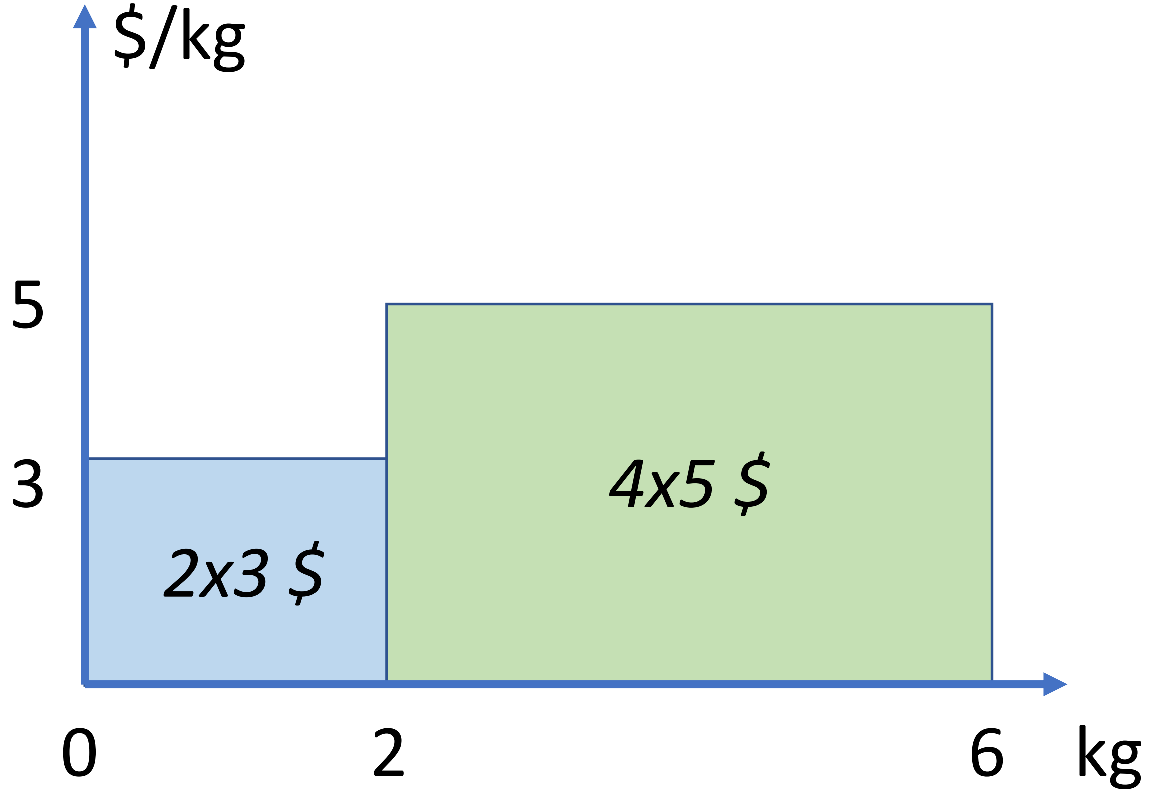
We change the sign and take away
and only x itself will stay.
We just keep on moving, we never give up.
So feed us equations, we don’t want to stop!

Adding PerNumbers as Areas (Integration)

"2kg at **3$/kg** + 4kg at **5$/kg** = 6kg at **?** **$/kg**?"

2 kg	at	**3 $/kg**
+ 4 kg	at	**5 $/kg**
(2+4) kg	at	**? $/kg**

- Unit-numbers add on-top.
- Per-numbers must be multiplied to unit-numbers, thus adding as **areas** under the per-number graph.
- Here, multiplication before addition



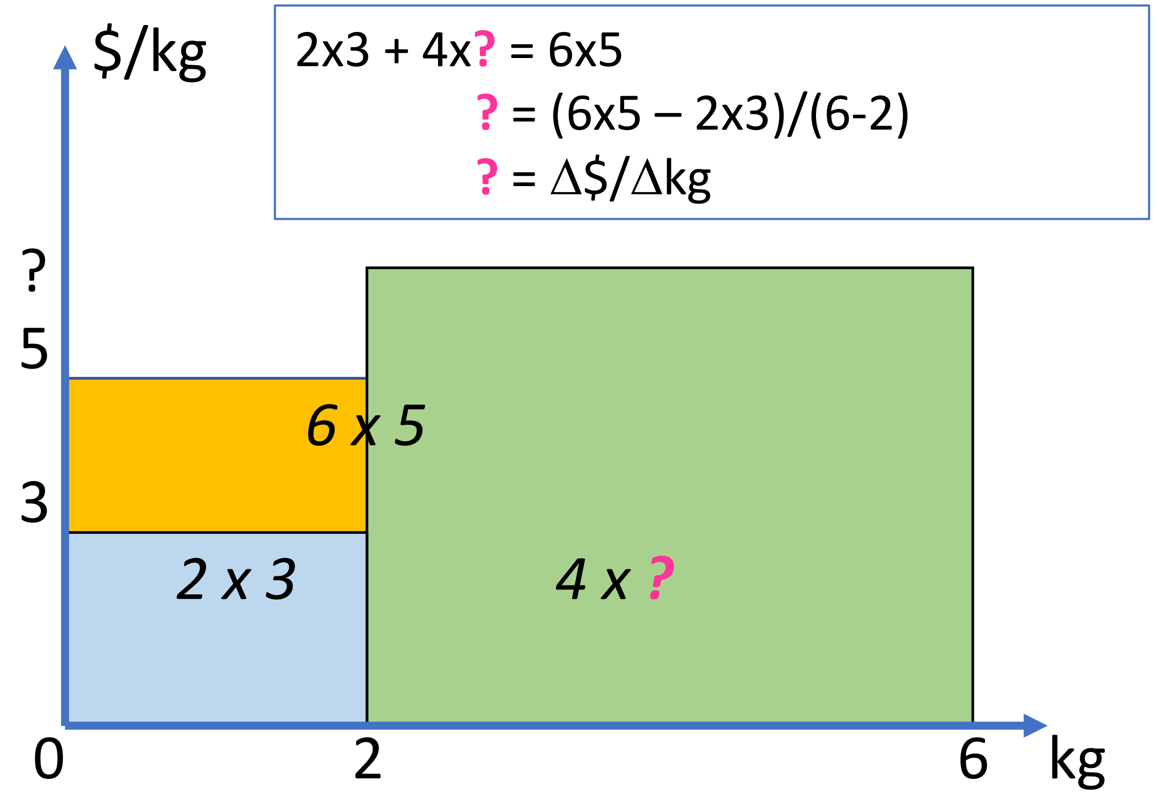
Subtracting PerNumbers (Differentiation)

“2kg at 3$/kg + 4kg at **what** = 6kg at 5$/kg?”

	2 kg	at	3 $/kg
+	4 kg	at	? $/kg
	6 kg	at	5 $/kg

Outside, we remove the initial 2x3 block and recount the rest in 4s. Geometrically, reversed per-number addition means subtracting areas to be reshaped, called differential calculus.

Inside, the recount-formula algebraically predicts the result. Here subtraction (giving a change, Δ) comes before division.



Flexible BundleNumbers Ease Operations

Over-load and under-load come in handy:

T = 65 + 27 = 6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92

T = 65 - 27 = 6**B**5 - 2**B**7 = 4**B**-2 = 3**B**8 = 38

T = 7 x 48 = 7 x 4**B**8 = 28**B** 56 = 33**B**6 = 336

T = 336 /7 = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

T = 336 /7 = 33**B**6 /7 = 35**B**-14 /7 = 5**B**-2 = 48

Adding or Subtracting Unspecified Numbers

"Only add like units, so how to add $T = 4ab^2 + 6abc$?"

Here units come from folding (factoring):

a factor-filter

$T = 4ab^2 + 6abc = T1 + T2$

$= 2 \times 2 \times a \times b \times b \quad + \quad 2 \times 3 \times a \times b \times c$

$= 2 \times b \times (2 \times a \times b) \quad + \quad 3 \times c \times (2 \times a \times b)$

$= 2b\ \mathbf{2ab}\text{s} + 3c\ \mathbf{2ab}\text{s}$

$= 2b+3c\ \mathbf{2ab}\text{s}$

$= (2b+3c) \times \mathbf{2ab}$

T1	2	2	*a*	*b*	*b*
T2	2	3	*a*	*b*	*c*
unit	**2**		***a***	***b***	
T1 left		2			*b*
T2 left		3			*c*

Conclusion

We ask: Can children discover/invent mathematics themselves to obtain a concrete exemplified understanding?

The answer is YES, if we

- de-model digits as icons with as many sticks as they represent
- use the flexible bundle-numbers with units that children develop when adapting to Many
- de-model operations as means for bundle-counting 8 as 8/2 **2s**, leading directly to the recount-formula $T = (T/B) \times B$, used to change units, and to
- solve equations as 'How many 2s in 8?' by recounting 8 in 2s

ux2 = 8 = 8/2x2
so u = 8/2

- use double-counting to construct per-numbers, fractions and trigonometry
- add both next-to and on-top, so calculus becomes addition of per-numbers

Core Findings

- Digits are icons with as many sticks as they represent
- Operations are icons created by counting: Division is a broom pushing away bundles; multiplication is a lift stacking the bundles; subtraction is a rope pulling away the stack to find unbundled; and addition is a choice between uniting stacks on-top or next-to
- Recounting predicted by the recount-formula **T = (T/B)xB** makes factors units
- Equations are solved by moving to opposite site with opposite sign
- Double-counting creates per-numbers or fractions adding with units as calculus
- Using also in the number-language full sentences with a subject, a verb and a predicate allows modeling to take place from grade 1, so totals carry units always

Discussion: What is the Difference?

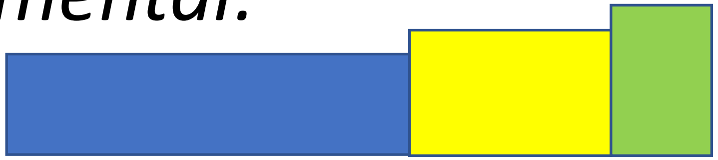
		Traditional Math	**Adaptive Green Math**
Digits	4	Symbol	Icon with four strokes
Numbers	456	One number	Three numberings, 4**BB**5**B**6
Division	8/2	8 split in 2	8 counted in **2s**
Multiplication	6 x 7	42	6 **7s with** 7 as a **unit** (or 4**B**2 **tens**)
Addition	2+3	2+3 = 5	2 **4s** + 3 **5s** = 2**B**3 **9s** 2 **4s** + 3 **5s** = 4**B**1 **5s**
Equations	3 x *u* = 12	Neutralize (3 x *u*) x 1/3 = 12 x 1/3 (*u* x 3) x 1/3 = 4 *u* x (3 x 1/3) = 4 *u* x 1 = 4 *u* = 4	Opposite side & sign *u* x 3 = 12 = (12/3) x 3 *u* = 12/3 = 4
Fractions	2/3	Numbers 1/2 + 2/3 **IS** 7/6	Per-numbers, i.e. operators, needing numbers to become numbers: 1/2 of 2 + 2/3 of 3 **IS** 3/5 of 5

What Mastery of Many does the child have already?

Children typically see Many as blocks with a number of bundles, and use flexible bundle-numbers with units, and with over- or underloads

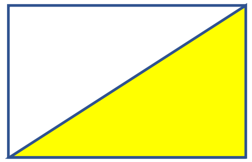
So, in green ManyMath, BLOCKS are fundamental:

- in numbers: 456 = 4**BB**5**B**6 = three blocks

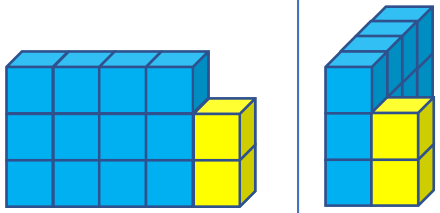


- in algebra: adding blocks next-to or on-top

- in geometry: recounting half-blocks



The Child’s own Twin Math Curriculum



1) Digits are (sloppy) icons, with as many sticks as they represent.

2) Totals are counted by bundling, giving <u>outside</u> geometrical multi-blocks, & (when turned to hide the units behind) <u>inside</u> algebraic bundle-numbers.

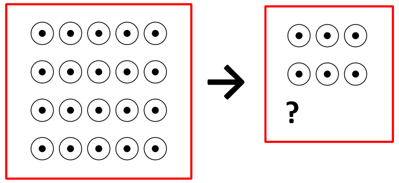
3) Operations are icons, showing the 3 counting steps: Pushing & stacking bundles & pulling stacks to find the unbundled.

4) The operation order is division first, then multiplication, then subtraction. Addition next-to & on-top comes later after totals are counted & re-counted.

5) Counting & re-counting & double-counting is big fun, when predicted by a calculator with the recount formula: **T = (T/B)xB** (from T, T/B times, Bs are pushed away)

*Question: T = 4 **5s** = ? **3s** ● Answer: T = 4 **5s** = 6B2 **3s** ● Prediction:*

4x5/3	6.some
4x5 – 6x3	2



Solving Equations BottomUp or TopDown

ManyMath

2 + u = 5 = (5-2) + 2	Solved by re-stacking 5
u = 5-2 = 3	Test: 2 + 3 = 5 OK

2 x u = 5 = (5/2) x 2	Solved by re-bundling 5
u = 5/2 = 2½	Test: 2 x 3 = 6 OK

MatheMatics

2 + u = 5	Addition has 0 as its neutral element, and 2 has -2 as its inverse element
(2 + u) + (-2) = 5 + (-2)	Adding 2’s inverse element to both number-names
(u + 2) + (-2) = 3	Applying the commutative law to u + 2, 3 is the short number-name for 5+(-2)
u + (2 + (-2)) = 3	Applying the associative law
u + 0 = 3	Applying the definition of an inverse element
u = 3	Applying the definition of a neutral element. *With arrows, a test is not needed*

Four Ways to Unite and Split a Total

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2 + 4***B** + 5 (a polynomial) shows the 4 ways to add: +, *, ^, next-to block-addition (integration). Addition and multiplication add changing and constant unit-numbers. Integration and power add changing and constant per-numbers.
We might call this beautiful simplicity the 'Algebra Square' since in Arabic, algebra means to reunite.

Operations unite / *split into*	**changing**	**constant**
Unit-numbers *m, s, $, kg*	**T = a + n** *T – a = n*	**T = a * n** *T/n = a*
Per-numbers *m/s, $/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $\log_a T = n$, $\sqrt[n]{T} = a$

The 4 uniting operations each has a reverse splitting operation: Addition has subtraction (–), and multiplication has division (/). Power has factor-finding (root, √) and factor-counting (logarithm, log). Integration has per-number finding (differentiation dT/dn = T').
Reversing operations is solving equations, done by moving to **opposite side** with **opposite sign**.

Defining Mathematics BottomUp

Pythagoras: mathematics, meaning knowledge, is a common label for 4 areas describing Many by itself and in space & time.

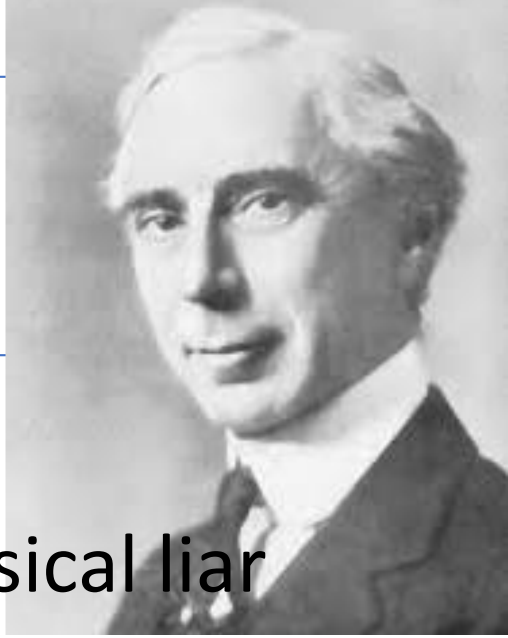
Together they formed the '**quadrivium**' recommended by Plato as a general curriculum after the '**trivium**' consisting of grammar & logic & rhetoric.

Grounded in Many as shown by names:

Geometry means to measure earth in Greek

Algebra means to reunite numbers in Arabic

Defining Mathematics TopDown



Around 1900, **SET** made mathematics self-referring.

However, Russell said: Self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true & opposite.

Let M be the set of sets not belonging to itself, $M = \{A \mid A \notin A\}$.

Then $M \in M \Leftrightarrow M \notin M$. Forget about sets. Use type theory instead.

So, by self-reference, fractions cannot be numbers.

Mathematics: Forget about Russell, he is not a mathematician.

Of course fractions are numbers, they are rational numbers.

Three Kinds of Mathematics

- Mathematics becomes **mathe-matism** when adding without units, thus only valid inside, but seldom outside classrooms where 2+3 = 5 meets counter-examples as 2weeks + 3days = 17days, in contrast to 2x3 = 6 stating that 2 **3s** may be recounted as 6 **1s**.
- Mathematics becomes **meta-matics** when defining concepts from above as examples of abstractions, instead of from below as abstractions from examples.
- Mathematics becomes adaptive **many-math** when accepting the double-numbers children develop when adapting to Many: bundle-numbers in primary and per-numbers in secondary school

Three Kinds of Mathematics Education

- Pre set-centric **mathe-matism** rejects set-centrism as too abstract and goes back to basic traditions resting heavily on root learning, thus using Skinner psychology
- Present set-centric **meta-matics** defining concepts as examples of set is used more or less all over the world, except in the US. Its definitions become abstract self-reference without meaning forcing learners to construct their own meaning by Vygotskian social constructivism
- Post set-centric adaptive **many-math** defines concepts as abstractions from concrete examples, thus becoming a natural science about Many and using Piaget psychology

Recommendation: Learners should be Researchers, Extending their already existing Adaption to Many

- To survive, also math must adapt to the outside world . So it should adopt the double-numbers children develop before school; and accept fractions as per-numbers, both operators needing numbers to become numbers.
- Hence to survive math must learn from children, not the other way around.
- Designing a micro- or macro-curriculum we should always ask: What is it out there that the learners need to adapt to?
- When adapting, learners should use grounded theory to answer the guiding learning questions listed in the curriculum.
- Teaching should be minimized to supplying concrete material and extra guiding questions, and to be opponents on the learners' findings.

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Question Guided Teacher Education

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Teaches Teachers to Teach MatheMatics as ManyMath, a Natural Science about MANY.

To learn Math, Count & Add MANY, using the CATS method:

Count & Add in Time & Space

- Primary: C1 & A1 & T1 & S1
- Secondary: C2 & A2 & T2 & S2

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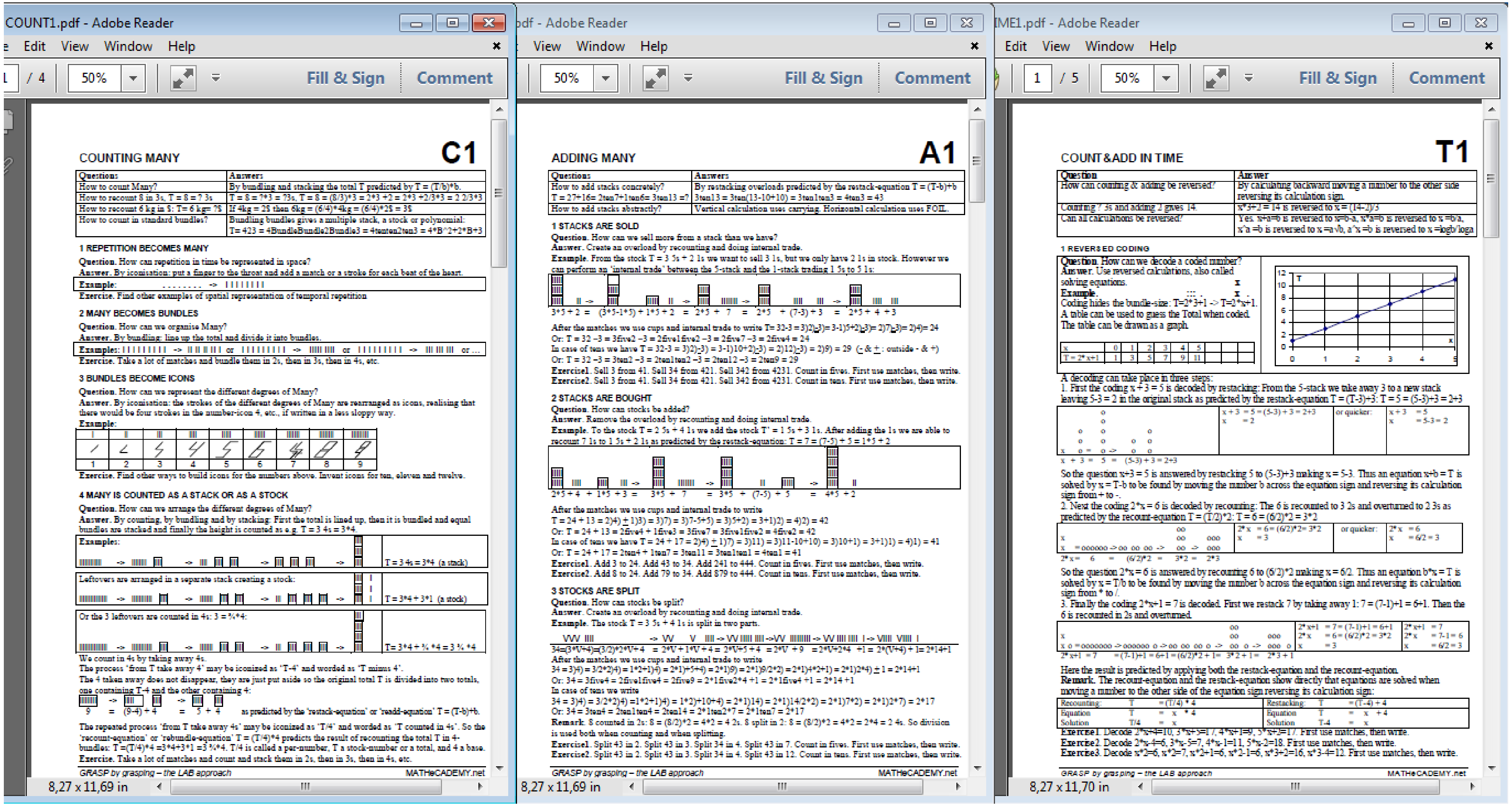
a VIRUSeCADEMY:
ask Many, not the Instructor

SUMMARY

	QUESTIONS	ANSWERS
C1 COUNT	How to count Many? How to recount 8 in 3s: T= 8 = ? 3s How to recount 6kg in $: T=6kg=?$ How to count in standard bundles?	By bundling and stacking the total T predicted by T = (T/b)*b T= 8= ?*3= ?3s, T= 8=(8/3)*3 = 2*3 +2 = 2*3 +2/3*3 = 2 2/3*3 If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$ Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle+2Bundle+3 = 4tenten2ten3 = 4*B^2+2*B+3
C2 COUNT	How can we count possibilities? How can we predict unpredictable numbers?	By using the numbers in Pascal's triangle We 'post-dict' that the average number is 8.2 with the deviation 2.3. We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation)
A1 ADD	How to add stacks concretely? T=27+16= 2ten7+1ten6= 3ten13=? How to add stacks abstractly?	By restacking overloads predicted by the restack-equation T= (T-b)+b T = 27+16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43 Vertical calculation uses carrying. Horizontal calculation uses FOIL
A2 ADD	What is a prime number? What is a per-number? How to add per-numbers?	Fold-numbers can be folded: 10=2fold5. Prime-numbers cannot: 5=1fold5 Per-numbers occur when counting, when pricing and when splitting. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b
T1 TIME	How can counting & adding be reversed ? Counting ? 3s and adding 2 gave 14. Can all calculations be reversed?	By calculating backward, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign. x*3+2=14 is reversed to x = (14-2)/3 Yes. x+a=b is reversed to x=b-a, x*a=b is reversed to x=b/a, x^a=b is reversed to x=a√b, a^x=b is reversed to x=logb/loga
T2 TIME	How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable?	By using constant change-equations: If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44 If Ko = 30 and ΔK/K = r = 2%, then K7= Ko*(1+r)^n= 30*1.02^7= 34.46 By solving a variable change-equation: If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx
S1 SPACE	How to count plane and spatial properties of stacks and boxes and round objects?	By using a ruler, a protractor and a triangular shape. By the 3 Greek Pythagoras', mini, midi & maxi By the 3 Arabic recount-equations: sinA=a/c, cosA=b/c, tanA=a/b
S2 SPACE	How to predict the position of points and lines? How to use the new calculation technology?	By using a coordinate-system: If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14) Computers can calculate a set of numbers (vectors) and a set of vectors (matrices)
QL	What is quantitative literature? Does quantitative literature also have the 3 different genres: fact, fiction and fiddle?	Quantitative literature tells about Many in time and space The word and the number language share genres: Fact is a since-so calculation or a room-calculation Fiction is an if-then calculation or a rate-calculation Fiddle is a so-what calculation or a risk-calculation

Teacher Training in CATS ManyMath
Count & Add in Time & Space



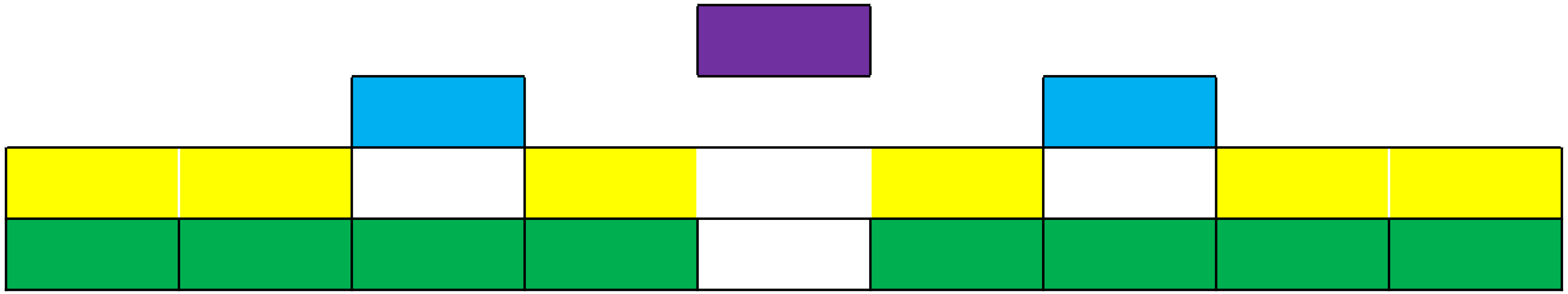
PYRAMIDeDUCATION

To learn MATH: Count&Add MANY
Always ask Many, not the Instructor
MATHeCADEMY.net - a VIRUSeCADEMY

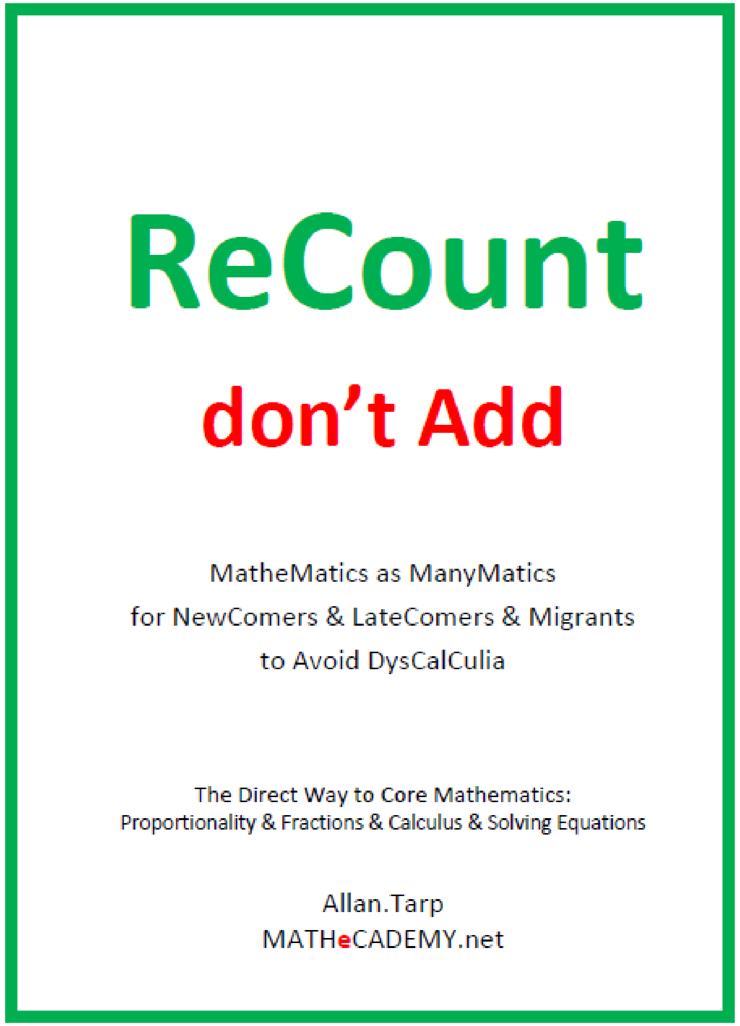
In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

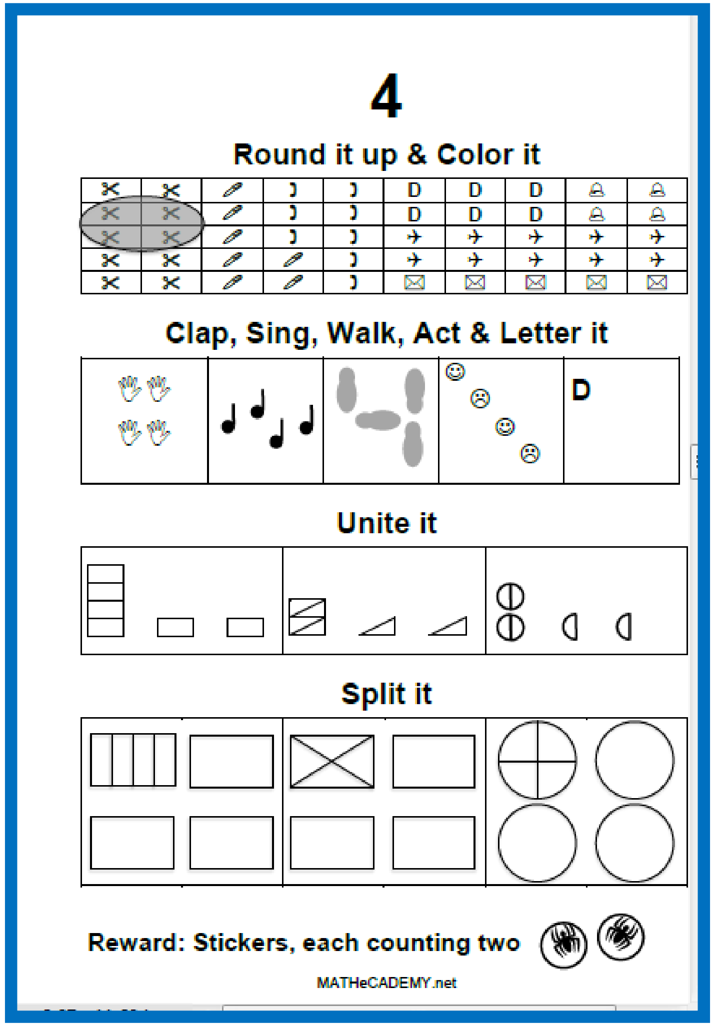
- Each pair works together to solve Count&Add problems.
- The coach assists the instructors when instructing their team and when correcting the Count&Add assignments.
- Each teacher pays by coaching a new group of 8 teachers.

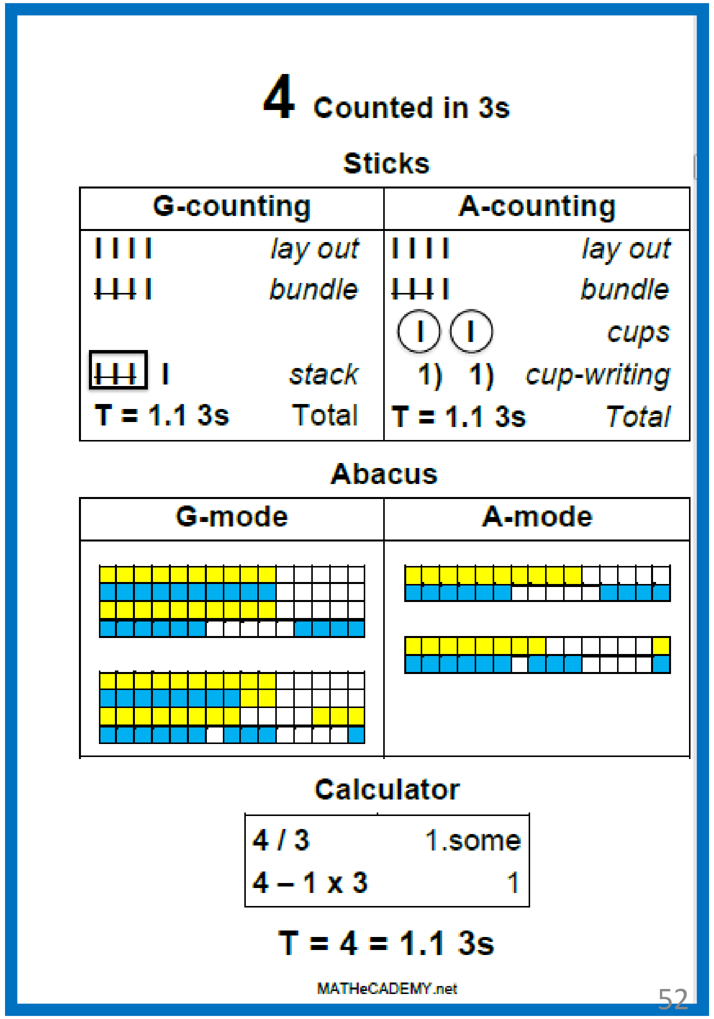
1 Coach
2 Instructors
3 Pairs
2 Teams



Action Learning and Research Materiel at mathecademy.net/ & mathecademy.net/preschool/

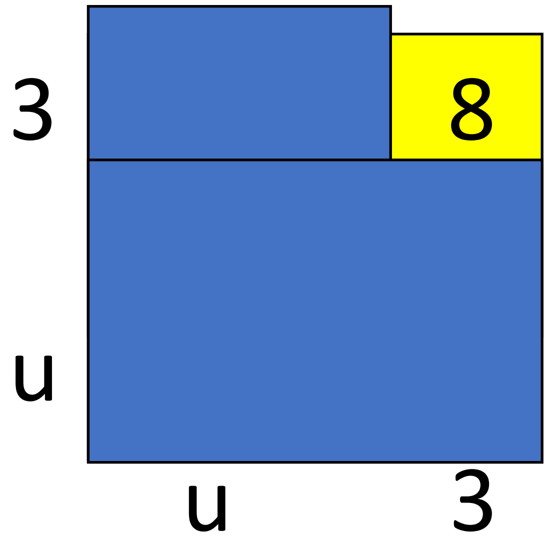






Quadratic Equations with 3 Cards

Solve the quadratic equation $\color{red}{\mathbf{u^2 + 6u + 8 = 0}}$



$$(u+3)^2 = u^2 + 6u + 8 + 1$$

$$(u+3)^2 = 0 + 1$$

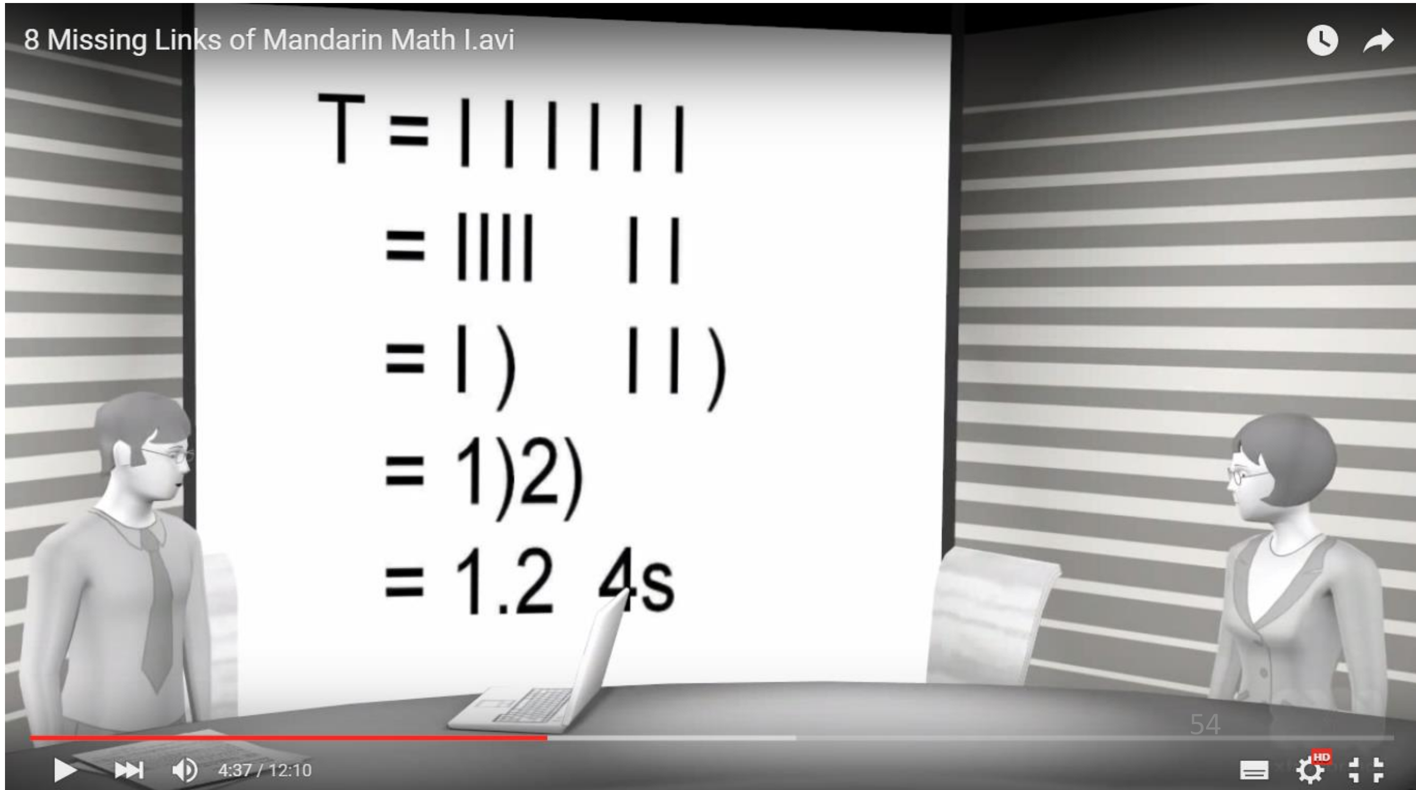
$$u+3 = \pm 1$$

$$u = -3 \pm 1$$

Solution: $\color{red}{\mathbf{u = -4, u = -2}}$

MrAlTarp YouTube Videos

- Postmodern Mathematics Debate
- CupCounting removes Math Dislike
- IconCounting & NextTo-Addition
- PreSchool Mathematics
- Fractions
- PreCalculus
- Calculus
- Mandarin Mathematics
- World History



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Theoretical Background

Tarp, A. (2018). Mastering Many by counting, recounting and double-counting before adding on-top and next-to. *Journal of Math Education, March 2018, 11*(1), 103-117.