The MADIF papers

2000-2020

Allan.Tarp@MATHeCADEMY.net

Content

Introduction i
Killer-Equations, Job Threats and Syntax Errors 1
Student-mathematics versus teacher-Metamatics 19
Mathematism and the Irrelevance of the Research Industry 31
The 12 Math-Blunders of Killer-Mathematics 41
Mathematics: Grounded Enlightenment - or Pastoral Salvation 49
Discourse Protection in Mathematics Education 57
Post-Constructivism 65
Golden Learning Opportunities in Preschool 73
Calculators and IconCounting and CupWriting in PreSchool and in Special Needs Education 81
Grounding Conflicting Theories 89
The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants 103
Math Competenc(i)es - Catholic or Protestant? 111
Sustainable Adaption to Quantity: From Number Sense to Many Sense 119
Per-numbers connect Fractions and Proportionality and Calculus and Equations 127
Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations 135
A Lyotardian Dissension to the Early Childhood Consensus on Numbers and Operations 143
Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis? 145

Introduction

Swedish school mathematics always fascinated me. Each second year Sweden arrange a Biennale where mathematics teachers from kindergarten to college can meet to share knowledge through exhibitions and inform themselves about new trends and ideas, and listen to foreign or local researchers having met the day before at the MADIF conference, the Swedish Mathematics Education Research Seminar arranged by the Swedish Society for Research in Mathematics Education.

Furthermore, in 1999 the Swedish government decided to establish and gracefully fund a national resource centre for mathematics education, NCM, describing its task to ‘co-ordinate, support, develop and implement the contributions which promote Swedish mathematics education from pre-school to university college’.

What a bright future for Swedish mathematics, I thought and decided to contribute with a paper at each MADIF conference and a general talk or an exhibition at each biennale.

My MADIF2 paper introduced postmodern counter research looking for hidden possible explanations for the problems in mathematics education within mathematics itself and warns against ‘killer-Equations’ and syntax errors. Furthermore, the paper suggests an alternative mathematics curriculum for the new millennium replacing the traditional Top-Down approach with a more user-friendly Bottom-Up approach. The paper was accepted for a full presentation.

However, I soon realized that it was almost impossible to establish a dialogue with the NCM and with Swedish researchers, so at the MADIF4 conference I presented a paper called ‘Mathematism and the Irrelevance of the Research Industry’ warning against supporting the irrelevance paradox in mathematics education research described by the following observation: ‘the output of mathematics education research increases together with the problems it studies - indicating that the research in mathematics education is irrelevant to mathematics education’. The paper demonstrates how to avoid mixing up mathematics with mathematism, true in the library but seldom in the laboratory.

Although accepted for a full presentation, nothing happened afterwards, so in my MADIF5 paper I decided to be much more specific by warning against twelve blunders of mathematics education. The reaction to this paper was to reduce the presentation to a short communication.

In my MADIF6 paper I draw attention to the difference between North American enlightenment schools wanting as many as possible to learn as much as possible, and European counter-Enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Bildung schools pastoral ‘metamatism’ descends from above as examples of metaphysical mystifying concepts.

The paper was rejected based upon a review process that allowed decisions to be made without specific reference to the paper reviewed.

So in my MADIF7 paper I warned against what I called ‘Discourse Protection in Mathematics Education’ and against reducing a constructive review process to what I called ‘Moo Review’ and ‘Tabloid Review’ using only one word or one sentence.

Again the paper was rejected.

One would expect the massive Swedish investment would show in the PISA scores. Here Sweden scored 502, 494, and 478 in the 2006, 2009 and 2012. Three consecutive numbers allow calculating the yearly change and the change to the change, which in the case of Sweden is -1.3 in 2006 changing yearly by -0.9 bringing the Swedish score to the zero level in 2038 if not changed.

At the same time research had demonstrated the positive effect of an early start in mathematics, so to be helpful to the Swedish research community I wrote a paper describing the golden learning opportunities in preschool accompanied by a YouTube video' Preschoolers learn Linearity & Integration by Icon-Counting & NextTo-Addition' (https://www.youtube.com/watch?v= R2PQJG3WSQY). The paper presents mathematics as natural science about the natural fact Many. To deal with Many we count and add. The school counts in tens, but preschool also allows counting in icons. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. To add next-to means adding areas also called integration. So accepting icon-counting and adding next-to offers golden learning opportunities in preschool that are lost when ordinary school begins.

And again, again the paper was rejected, this time however without using moo- or tabloid-review.

In the PISA report Denmark scored 513, 503 and 500 giving an initial yearly change of -4.5 in 2006 changing yearly by 0.8 bringing the Danish score to 629 in 2030 if not changed.

However, Denmark has not significantly increased its research activity. So the Danish success and the Swedish melt-down both indicate the correctness of the irrelevance paradox: More research creates more problems. Consequently I suggested a two year no-research pause in Sweden. It was declined because researchers had found a new research paradigm, Design Research, they hoped would change the situation in a positive way.

Design Research bases its designs on existing theory. However, in conference presentations, disagreements between conflicting theories were simply ignored or denied. And not differentiating between grounded and ungrounded theory will hardly prevent the Swedish melt-down. So, to once more offer my assistance, instead of writing yet another paper that will be rejected yet again because of discourse protection, I have decided that my contribution to the MADIF 10 conference in 2016 should be a YouTube video similar to the Paul and Allan debate on postmodern mathematics education (https://www.youtube.com/watch?v=ArKY2y_ve_U), inspired by the Chomsky-Foucault debate on human nature (www.youtube.com/watch?v=3wfNl2L0Gf8), this time called 'Grounding Conflicting Theories to avoid the Irrelevance Paradox creating the Nordic Math Melt-Down - an invitation to a dialogue on Mathematics Education and its Research'. One prominent person within the research community has declined to take part in the dialogue, but hopefully other persons will accept their responsibility and be willing to enter into a fruitful dialogue to prevent the Swedish melt-down to become reality. Money does not solve the problem, dialogue between conflicting theories does.

The MADIF papers

For the MADIF 2 conference in 2000 I wrote the paper 'Killer-Equations, Job Threats and Syntax Errors, a Postmodern Search for Hidden Contingency in Mathematics.'

The abstract says that modern mathematics is facing an exodus problem: an increasing number of students are turning away from mathematics in school, and from math-based educations within science and engineering after school. Modern research looks for explanations within human factors: students, teachers and cultures. Postmodern counter research looks for hidden possible explanations elsewhere, in this case within mathematics itself. This study identifies unnoticed syntax errors within mathematics and a problematic Top-Down practice of allowing killer-equations into the classroom. Also the study reports on a successful changing of this practice and reflects upon why a Bottom-Up approach might be more user-friendly than a Top-Down approach.

The paper contains chapters called: The Difference between Modern Research and Postmodern Counter Research, Killer-Equations in Paradise, Designing an Alternative: Rephrasing Equations, Practising the Alternative, Evaluating the Alternative, Why Might Bottom-Up Mathematics be More User-friendly?, Why Might Bottom-Up Mathematics be Unrecognised? - Rephrasing Mathematics, Mixing Different Abstraction Levels Creates Syntax Errors, Abstraction Errors, Equations Can Also be Solved by Reverse Calculations, Bottom-Up Mathematics Education Through the Social Practices that Created Mathematics, The Social Practice of Bundling and

Stacking, The Social Practices of Measuring Earth and Uniting Totals, When Will the logx Button be Included on Calculators?, The Social Practice of Building and Evaluating Models, Rephrasing Mathematical Concepts, Has Mathematics Become the God of Late Modernity?, and Fiction: “A New Curriculum for a New Millennium” - A Curriculum Architect Contest.

For the MADIF 3 conference in 2002 I wrote the paper ‘Student-mathematics versus teacher-Metamatics’.

The abstract says that the paper reports on writer’s career as an action researcher helping the students to develop their own student-mathematics, making mathematics accessible for all but being opposed by the educational system. The work took place over a 30 year-period in Danish calculus and pre-calculus classes and in Danish teacher education. As methodology a postmodern counter-research was developed accepting number-statements but being sceptical towards word-statements. Counter-research sees word-researchers as counsellors in a courtroom of correctness. The modern researcher is a counsellor for the prosecution trying to produce certainty by accusing things of being something, and the postmodern researcher is a counsellor for the defence trying to produce doubt by listening to witnesses, and by cross-examining to look for hidden differences that might make a difference. A micro-curriculum in student mathematics was developed and tested in 13 grade 11 classes showing a high degree of improvement in student performance.

The paper contains chapters called: A Confession, Methodology, the Case: Evidence and Cross-examination, and Concluding Statement.

However, I was not able to attend the conference, so instead the paper was presented at the ECER conference in 2003 and published at http://www.leeds.ac.uk/educol/documents/00003264.htm.

For the MADIF 4 conference in 2004 I wrote the paper ‘Mathematism and the Irrelevance of the Research Industry, a Postmodern LIB-free LAB-based Approach to our Language of Prediction.

The abstract says that mathematics education research increases together with the problems it studies. This irrelevance-paradox can be solved by using a postmodern sceptical LAB-research to weed out LIB-based mathematism coming from the library in order to reconstruct a LAB-based mathematics coming from the laboratory. Replacing indoctrination in modern set-based mathematism with education in Kronecker-Russell multiplicity-based mathematics turns out to be a genuine ‘Cinderella-difference’ making a difference in the classroom.

The paper contains chapters called: The Irrelevance Paradox, A Methodology: Institutional Scepticism, Sceptical LIB-free LAB-Research, Mathematics and Mathematism, Fractions and Sets - LIB-words or LAB-words?, Bringing LAB-based Mathematics to a LIB-based Academy, The MATHeCADEMY and PYRAMIDeDUCATION, and Appendix: A Kronecker-Russell Multiplicity-Based Mathematics.

For the MADIF 5 conference in 2006 I wrote the paper ‘The 12 Math-Blunders of Killer-Mathematics, Hidden Choices Hiding a Natural Mathematics.

The abstract says that mathematics itself avoids blunders by being well defined and well proven. However, mathematics education fails its goal by making blunder after blunder at all levels from grade 1 to 12. This paper uses the techniques of natural learning and natural research to separate natural mathematics from killer-mathematics. Two-digit numbers, addition, fractions, balancing equations, and calculus are examples of mathematics that has been turned upside down creating the ‘metamatism’ that killed mathematics and turned natural Enlightenment mathematics into modern missionary set-salvation.

After the initial chapter ‘Taking the Killing out of Killer-Mathematics’ the paper describes twelve, Math-Blunders: Treating both Numbers and Letters as Symbols, 2digit Numbers before Decimal Numbers, Fractions before Decimals, Forgetting the Units, Addition before Division, Fractions before PerNumbers and Integration, Proportionality before DoubleCounting, Balancing instead of

Backward Calculation, Killer Equations instead of Grounded Equations, Geometry before Trigonometry, Postponing Calculus; and the Five MetaBlunders of Mathematics Education.

For the MADIF 6 conference in 2008 I wrote the paper ‘Mathematics: Grounded Enlightenment - or Pastoral Salvation, Mathematics, a Natural Science for All - or a Humboldt Mystification for the Elite’.

The abstract says that mathematics is taught differently in Anglo-American democratic enlightenment schools wanting as many as possible to learn as much as possible; and in European pastoral Humboldt counter-Enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Humboldt Bildung schools pastoral ‘metamatism’ descends from above as examples of metaphysical mystifying concepts. To make mathematics a human right, pastoral Humboldt counter-enlightenment must be replaced with democratic grounded enlightenment.

The paper contains chapters called: Postmodern Thinking - a Short Tour, French Enlightenment and German Counter-Enlightenment, American Enlightenment and Grounded Action Theory, Deconstructing and Grounding Research, Deconstructing and Grounding the Postmodern, Deconstructing and Grounding Numbers, Deconstructing and Grounding Operations, Deconstructing and Grounding the Mathematics Curriculum, A Grounded Perspective on Pastoral Mathematics, and The Humboldt Occupation of Europe.

For the MADIF 7 conference in 2010 I wrote the paper ‘Discourse Protection in Mathematics Education’.

The abstract says that social theory describes two kinds of social systems. One uses education to enlighten its people so it can practice democracy. One uses education to force upon people open or hidden patronization. A number-language is a central part of education. Two number-languages exist. Mathematics from-below is a physical science investigating the natural fact Many in a ‘manyology’ presenting its concepts as abstractions from examples. Mathematics from-above is a meta-physical science claiming Many to be an example of ‘metamatics’ presenting its concepts as examples from abstractions. Foucault’s discourse theory explains why manyology is suppressed and why even enlightening education patronizes by presenting mathematics from-above instead of from-below.

The paper contains chapters called: Investigating the natural fact many, the absence of a manyology, Social theory, Discourse Protection and Hegemony, Moo Review and Tabloid Review, and an appendix: the case of equations.

For the MADIF 8 conference in 2012 I wrote the paper ‘Post-Constructivism’.

The abstract says that even if constructivism has been its major paradigm for several decades the relevance paradoxes in mathematics education remain; and furthermore constructivism has created a mathematics war between primary and secondary school, and between parents and teachers. Constructivism believes that numbers are meaningful and that algorithms are meaningless thus allowing students to construct their own algorithms. But maybe it is the other way around? Maybe a two-digit number is a highly abstract concept that, if not introduced slowly through cup-writing, may be meaningless to students; whereas algorithms introduced as internal trade between two neighbour cups is meaningful.

The paper contains chapters called: Constructivism, Numbers, Algorithms, Hermeneutics, Hermeneutic Research, Sceptical Cinderella Research.

However, I was not able to attend the conference, so the paper remains unpublished.

For the MADIF 9 conference in 2014 I wrote the paper ‘Golden Learning Opportunities in Preschool’.

The abstract says that preschool allows rethinking mathematics outside the tradition of ordinary school. Seeing schooling as adapting the child to the outside world containing many examples of the natural fact Many, we can ask: How will mathematics look like if built as a natural science about Many? To deal with Many we count and add. The school counts in tens, but preschool also allows counting in icons. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. To add next-to means adding areas also called integration. So accepting icon-counting and adding next-to offers golden learning opportunities in preschool that are lost when ordinary school begins.

The paper contains chapters called: Math in Preschool – a Great Idea, Postmodern Contingency Research, Building a Science about the Natural Fact Many, Comparing Manyology and the Tradition, The Traditional Preschool Mathematics, Micro-Curricula at the MATHeCADEMY.net, Five plus Two Learning Steps, Designing a Micro-Curriculum so Michael Learns to Count, Observing and Reflecting on Lesson 1.

For the MADIF 10 conference in 2016 I wrote the paper ‘Calculators and IconCounting and CupWriting in PreSchool and in Special Needs Education’.

The abstract says that to improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking preschool mathematics uncovers icon-counting in bundles less than ten implying recounting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, a calculator can predict recounting results before being carried out manually. By allowing overloads and negative numbers when recounting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary school only allowing ten-counting and on-top addition to take place.

The paper contains chapters called: Decreasing PISA Performance in spite of Increasing Research, Institutional Skepticism, Mathematics as Essence, Mathematics as Existence, Re-counting in the Same Unit and in a Different Unit, Reversing Adding On-top and Next-to, Primary Schools use Ten-counting only, Tested with a Special Needs Learner, Conclusion and Recommendation.

For the MADIF 10 conference in 2016 I also wrote the paper ‘Grounding Conflicting Theories - an invitation to a dialogue to solve the Nordic Math MeltDown Paradox, a Manuscript to a Debate on Mathematics Education and its Research. However, it was not handed in.

The abstract says with heavy funding of mathematics education research brilliant results in the PISA scores are to be expected in the Nordic countries. So it is a paradox that all Nordic counties are facing a melt-down in their PISA scores in 30 years if nothing is changed; except for Denmark that has not increased it funding significantly. This was predicted by Tarp in his MADIF papers formulating an irrelevance paradox for mathematics education: more research leads to more problems when basing research on ungrounded theories and discourse protection and moo-review.

For the MADIF 11 conference in 2018 I wrote the paper ‘The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants’.

The abstract says that educational shortages described in the OECD report ‘Improving Schools in Sweden’ challenge traditional math education offered to young male migrants wanting a more civilized education to return help develop and rebuild their own country. Research offers little help as witnessed by continuing low PISA scores despite 50 years of mathematics education research. Can this be different? Can mathematics and education and research be different allowing migrants to succeed instead of fail? A different research, difference-research finding differences making a difference, shows it can. STEM-based, mathematics becomes Many-based bottom-up Many-matics instead of Set-based top-down Meta-matics.

The paper contains chapters called: Decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Meeting Many, Children use Block-numbers to Count and Share, Meeting Many Creates a Count&Multiply&Add Curriculum, Meeting Many in a STEM Context, The Electrical circuit, an Example, Difference-research Differing from Critical and Postmodern Thinking, Conclusion and Recommendation,

For the MADIF 11 conference in 2018 I also wrote the paper 'Math Competenc(i)es - Catholic or Protestant?'

The abstract says that, introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015 OECD-report 'Improving Schools in Sweden'. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

The paper contains chapters called: Decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Defining Mathematics Competencies, Discussing Mathematics Competencies, What kind of mathematics, What kind of Mastering, Competence versus Capital, The Counter KomMod report, Quantitative Competence, Proportionality, an Example of Different Quantitative Competences, Conclusion, Recommendation: Expand the Existing Quantitative Competence,

For the MADIF 12 conference in 2020 I wrote the paper 'Sustainable Adaption to Quantity: From Number Sense to Many Sense'.

The abstract says that their biological capacity to adapt to their environment makes children develop a number-language based upon two-dimensional box- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language.

The paper contains chapters called: Decreased PISA Performance Despite Increased Research; Mathematics and its Education; Biology Looks at Education; Psychology Looks at Education; Meeting Many, Children Bundle to Count and Share; Discussing Number Sense and Number Nonsense; Conclusion and Recommendation.

For the MADIF 12 conference in 2020 I also wrote the paper 'Per-numbers connect Fractions and Proportionality and Calculus and Equations.'

The abstract says that in middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word 'percent' to also talk about other 'per-numbers' as e.g. 'per-five' thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

The paper contains chapters called: *Mathematics is Hard, or is it; The ICMT3 Conference;* Different Ways of Seeing Fractions; Different Ways of Seeing Fractions; Ratios and Rates; Per-numbers Occur when Double-counting a Total in two Units; Fractions as Per-numbers Expanding and Shortening Fractions Taking Fractions of Fractions, the Per-number Way; Direct and Inverse

Proportionality Adding Fractions, the Per-number Way; Solving Proportionality Equations by Recounting; Seven Ways to Solve the two Proportionality Questions; A Case: Peter, about to Peter Out of Teaching; Discussion and Recommendation.

For the MADIF 12 conference in 2020 I also wrote the paper ‘Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations.’

The abstract says that their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

The paper contains chapters called: A need for curricula for all students; A Traditional Precalculus Curriculum; A Different Precalculus Curriculum; Precalculus, building on or rebuilding?; Using Sociological Imagination to Create a Paradigm Shift; A Grounded Outside-Inside Fresh-start Precalculus from Scratch ; Solving Equations by Moving to Opposite Side with Opposite Sign; Recounting Grounds Proportionality; Double-counting Grounds Per-numbers and Fractions; The Change Formulas; Precalculus Deals with Uniting Constant Per-Numbers as Factors; Calculus Deals with Uniting Changing Per-Numbers as Areas; Modeling in Precalculus Exemplifies Quantitative Literature; A Literature Based Compendium; An Example of a Fresh/start Precalculus Curriculum; An Example of an Exam Question; Discussion and Conclusion.

For the MADIF 12 conference in 2020 I also wrote the workshop proposal ‘A Lyotardian Dissension to the Early Childhood Consensus on Numbers and Operations.’

The workshop proposal contains chapters called Can Sociological Imagination Improve Mathematics Education; Time Table for the Workshop; Consensus and Dissension on Early Childhood Numbers & Operations.

For the MADIF 12 conference in 2020 I also wrote the workshop proposal ‘Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis?’

The workshop proposal contains chapters called Does Mathematics Education Research have an Irrelevance Paradox; The Replication Crisis in Science; Time Table for the Workshop.

Killer-Equations, Job Threats and Syntax Errors

A Postmodern Search for Hidden Contingency in Mathematics

Abstract

Modern mathematics is facing an exodus problem: an increasing number of students are turning away from mathematics in school, and from math-based educations within science and engineering after school (Jensen et al. 1998). Modern research looks for explanations within human factors: students, teachers and cultures. Postmodern counter research looks for hidden possible explanations elsewhere, in this case within mathematics itself. This study identifies unnoticed syntax errors within mathematics and a problematic Top-Down practice of allowing killer-equations into the classroom. Also the study reports on a successful changing of this practice and reflects upon why a Bottom-Up approach might be more user-friendly than a Top-Down approach.

The Difference between Modern Research and Postmodern Counter Research

Modern research and postmodern counter research are both working in the borderland between nature and culture, between what is given and what could be different, between necessity and contingency. Out of the breakdown of premodern order, modernity saw the emergence of contingency. Scared by the idea of a contingent world modernity desperately began to reinstate order (Bauman 1992). Modern research sees contingency as hidden necessity, and tries to discover the nature of this necessity wanting to produce new convincing knowledge claims “A is B”. On the other side postmodern counter research tries to uncover hidden contingency in necessity wanting to produce new inspiring knowledge suggestions “A could also be B”.

Although some postmodern thinking might see both culture and nature as social constructions this paper recognises a borderline between nature and culture to be drawn between numbering nature and wording culture. Nature can speak through number-meters, rulers, but since no word-meter exists, the world cannot word itself, hence all phrasings are contingent, except this meta-phrasing. Phrasing is freezing, and re-phrasing is de-freezing or freeing. It is a postmodern point that a phrasing constructs what it describes and that humans are clientified by ruling phrasings and discourses (Foucault 1972). Our convictions might be not universal truths but local truths depending on the ruling phrasing, and they might change through a rephrasing. An example of a postmodern rephrasing is seen in the following case.

Killer-Equations in Paradise

Once I was invited for a two-month stay at a new four years Secondary Teacher Education College in South Africa created to solve the local 1% success problem in mathematics: 90% of the students did not enter the final exam in mathematics and 90% failed. The mathematics curriculum at the college and at the high schools followed a tradition of a Platonic Top-Down mathematics describing concepts as examples of more abstract concepts all originating from the mother concept “Set”. In the science education classes at the college the educational theory-tradition was that of curriculum 2005, Outcome Based Education (OBE) and Vygotskian constructivism.

After the first month I followed some students in their teaching practice at a high school in a local village called Paradise. The student-teachers received a textbook and a number of pages they were supposed to cover. In a grade 10 class two equations were written on the board by the student-teacher and solved by students in the following way:

Equations:	$\frac{M}{5} - \frac{M}{2} = 3$	$\frac{y+2}{4} - \frac{y-6}{3} = \frac{1}{2}$
Solutions:	$\frac{M}{10} = 3$ m = –10(3) m = –30	$\frac{6+24}{12} = 2$ y = 12·2 y = 14

After the period the student-teacher complained: "You ask them if they understand it and they say yes, but next day they have forgotten it all. They don't study at home, they have too much free time and no parent support. Their friends say mathematics is not interesting. 30 minutes lessons are too short, in private schools they have 60 minutes. The ministers take their children abroad. The new curriculum 2005 also asks us to teach these equations. Something has to be done."

Other student-teachers and teachers had similar complaints: Mathematics is difficult and can only be learned through hard work, but today's students don't like hard work. First year high school students lack fundamental mathematical knowledge from the primary school. The teaching material is outdated and in low supplies. Many secondary school teachers are not trained in mathematics. The teachers need to be workshopped in OBE. The classrooms are too crowded to practise OBE and constructivism. The instruction has to be in English, which is not the mother language.

Designing an Alternative: Rephrasing Equations

In these explanations the blame for the "bad play" is placed with external factors outside the teacher's influence: "the manager, the director and the actors". Inspired by a postmodern view looking for alternative silenced explanations I suggested looking at "the script" by rephrasing equations into two groups: Top-Down "killer-equations" and Bottom-Up "calculation stories".

Killer-equations are equations you never meet outside the classroom and which only serve one purpose, to kill off the interest of the students. Killer-equations are examples of Top-Down equations being examples of the general equation "A = B", where A and B are examples of arbitrary expressions. Calculation stories or practice-equations are questions arising from social practices: the social practice of shopping e.g. contains questions like "3 kg @ ? R/kg total 14 R including a 2 R fee" leading to the calculation story or equation "x·3+2 = 14".

Also "solving an equation by doing the same to both sides" can be rephrased as "reversing a calculation". The multiple calculation x·3+2 is reduced to a single calculation by means of a "hidden parenthesis": x·3+2 = (x·3)+2. This calculation consists of two steps: First the R/kg-number x is multiplied by the kg-number 3 to produce the R-number x·3. Then the fee 2 is added to produce the Total x·3+2, which is 14. Reversing the calculation consist of the two opposite steps: First the fee 2 is subtracted from the Total 14 to produce the R-number 12. Then the R-number 12 is divided by the kg-number 3 to produce the R/kg-number 4. The reverse calculation method is identical to the old "Move&Reverse" method: a number can be moved across the equal sign from the forward side to the backward side of an equation and vice versa by reversing its calculation sign.

Calculation direction:	*Forward*		*Backward*
Total	(x·3)+2	=	14
		+2 ↑↓ –2	
R	x·3	=	14–2 = 12
		·3 ↑↓ /3	
R/kg	x	=	12/3 = 4

the "Walk&Reverse" method

Forward		*Backward*
(x·3)+2	=	14
x·3	=	14–2 = 12
x	=	12/3 = 4

the "Move&Reverse" method

Practising the Alternative

After having discussed this rephrasing of equations with the student-teachers one of them asked me to try it out in the classroom. I accepted to take over a standard 30 minutes lesson in a grade 10 class with 50-60 students. Following the design I started to present three Bottom-Up questions:

"3 kg @ 5 R/kg total ? R"	leading to the equation	$T = 5\cdot3$
"3 kg @ 5 R/kg total ? R including a 2 R fee"	leading to the equation	$T = (5\cdot3) + 2$
"3 kg @ ? R/kg total 14 R including a 2 R fee"	leading to the equation	$14 = (x\cdot3) + 2$.

Then I introduced the reverse calculation method mentioned above. The class did a similar problem with other numbers. I then took the class to the schoolyard and asked them to line up facing me: "We start with an R-number 5 each. Now we walk forwards to steps, a "·3 step" and a "+2 step" calculating the new R-number each time". This produced the final R-number 17. "If the final number had been 14 R what did we begin with? We can guess, or we can calculate by walking backwards reversing the calculation steps." After a "–2 step" and a "/3 step" had produced 4 R we went back to the classroom and saw the resemblance between the "Walk&Reverse" method and the reverse calculation method on the board. By erasing the arrows the reverse calculation method became the "Move&Reverse" method. Some homework problems were given for the next period, where the student-teacher took over again after the students had written down their solution of the equation $4+3\cdot x=19$ on the back side of a questionnaire.

Evaluating the Alternative

The questionnaire contained a traditional quantitative opinion question and two open questions allowing for the self-phrasing of the students:

> Dear Learner. I have had the pleasure of showing you a Bottom-Up understanding of an equation 2+3x=14 seeing an equation as a story telling about the total and how it is calculated.
>
> 1. What do you think about the idea of introducing the Bottom-Up understanding of an equation in the classroom of South African secondary schools. Draw a circle around your answer (–2: Very Bad, –1: Bad, 0: Neutral, 1: Good, 2: Very Good).
>
> 2. If you have other comments to the bottom-Up understanding of an equation you can write them here.
>
> 3. You have been living with mathematics for many years now. I would be glad if you could tell me a little about your learning life with mathematics. Just write whatever falls into your mind.

I collected 50 answers. The correctness of the method and the result were graded on a (-2, -1, 0, 1, 2) scale giving the distributions (0, 3, 7, 19, 21) and (2, 9, 1, 5, 33). The answers to question 1 were (-2, -1, 0, 1, 2): (0, 0, 2, 6, 40). As to question 2, 12 answers praised the method for being easy, 25 for being understandable and 3 for being short. As to question 3 I was amazed to find among the answers 24 occurrences of a "No math - No job" myth.

So one way of motivating equations is by job threats. Another is to keep killer-equations out of the classroom only allowing practice-equations to come in.

Why Might Bottom-Up Mathematics be More User-friendly?

As other forms of life humans need to be connected to nature's flow of matter and energy (food) and information. In premodern agriculture humans add a cultural flow of food to nature's flow. In the modern industrial culture electrons are used to carry energy, and in the postmodern information culture electrons are used to carry information. The introduction of global TV into local cultures has uncovered the contingency of local traditions creating a post-traditional globalised society (Giddens in Beck et al. 1994). With the loss of external traditions to echo, identity becomes self-identity, a reflexive project, where the individuals have to create their own biographical narrative or self-story looking for authenticity and shunning meaninglessness (Giddens 1991).

By referring upwards a Top-Down sentence (“a function is an example of a relation”) can give only one answer thus creating “echo-teaching” and “echo-reluctance”. Top-Down sentences become “unknown-unknown” relations that cannot be anchored to the students' existing learning narrative. They become meaningless and only accessible as “echo-learning” (Tarp 2000).

By referring downwards a Bottom-Up sentence as e.g. “a function is a name for calculations with variable quantities” (Euler 1748) can give many examples thus becoming an “unknown-known” relation that can meaningfully be anchored to the students' existing learning narrative, thus extending this. Inspired by Ausubel (Ausubel 1963) we could say that Bottom-Up learning takes place when students get a meaningful answer to their learning-question: “Tell me something I don’t know about something I know”.

Why Might Bottom-Up Mathematics be Unrecognised? - Rephrasing Mathematics

Mathematics education is about education in mathematics - or is it? Can mathematics be rephrased and can education be rephrased? Are the actors (students and teachers) and the system clientified, caught and frozen in a “mathematics” discourse forcing them to subscribe to a Top-Down "mathematics before mathematics application" conviction?

Humans communicate about the world in two languages. A word-language assigning words to things and practices by means of sentences: “This table is high”. And a number-language assigning numbers to things and practices by means of number- or calculation-sentences called equations: “The height is forty five centimetres (h=45·cm)”, “3 kg @ 4R/kg total 3·4·R (T=3·4·R)”. And humans communicate about the languages in two meta-languages, the grammar describing the word-language, and mathematics describing the number-language. And humans communicate about the meta-languages in two meta-meta-languages, meta-grammar describing grammar, and meta-mathematics describing mathematics.

Meta-meta-language	Meta-grammar	Chomsky	Set Relation Function	Meta-mathematics
Meta-language	Grammar *of the word-language*	Subject Verb Object	Number Operation Calculation	Mathematics *Grammar of the number-language*
Language	Word-language *Applications of grammar*	Word stories Sentences	Number stories Equations	Number-language *Applications of mathematics*
WORLD		THINGS &	PRACTICES	

Mathematics as part of a language-house

The phrasing “Mathematics and applications of mathematics” creates a Top-Down conviction “Of course mathematics must be learned before it can be applied”. A rephrasing to “Grammar of the number-language and number-language” creates the opposite Bottom-Up conviction “Of course language must be learned before its grammar”. So in this case the truth is dependent upon the ruling phrasing. Frozen by the “Mathematics and applications of mathematics” phrasing modern mathematics implements a “grammar before language” practice (or even “meta-grammar before language”), which would create global illiteracy if spread from the number-language to the word-language, thus preventing a number-language from becoming a human right. Most humans are fluent in their mother language but unable to make explicit the grammatical rules they apply, grammatical competence is mostly tacit.

So mathematics education can be about education in mathematics, but it could also be about securing the human right for a number-language respecting the tacity of grammatical competence. Forcing an explication of a definite unrelatable mathematics might be blocking for this human right.

Mixing Different Abstraction Levels Creates Syntax Errors

The word-language is able to differentiate between the three language levels through the three words “language, grammar and meta-grammar”. Unwilling to use the two words “number-language” and “meta-mathematics” mathematics is unable to differentiate between the three language levels. It thus creates syntax errors violating Russell’s type-theory saying that mixing concepts from different abstraction levels creates nonsense. We can meaningfully ask “Where in France is Paris?” but not “where in Paris is France?” And self-referring sentences like “This statement is false” are meaningless. Gödel makes the same point: mathematics can prove statements, but not itself. Non the less mathematics keeps on making syntax errors by mixing different abstraction levels. Humans might accept syntax errors through “echo-learning” but computers refuse to accept syntax errors: computer programs like MathCad thus have to operate with several different equal signs.

“2+3” is a calculation and “5” is a number. A number can be counted, read and measured. A calculation can be calculated respecting priority and sometimes in reverse order. Exchanging the words “number” and “calculation” creates meaningless sentences, hence the two words are of different type. The syntax of writing “2+3 = 5” is “<calculation><identical-to><number>“, i.e. a syntax error. One way of avoiding this syntax error is to write “(2+3) = 5” meaning the result of the calculation 2+3 is identical to 5 according to the calculation “2+(3+4)” where “(3+4)” means the result of the calculation “3+4”. Another way is to write “2+3 $\rightarrow$ 5” meaning “2 and 3 gives 5”.

As with “2+3 = 5” also “x+3 = 5” is a syntax error. Writing “x+3 = 5–x” is a normal error since “x+3” and “5–x” are not identical calculations. Writing “(x+3) = (5–x)” is meaningful asking when the results of the two calculations x+3 and 5–x are identical.

Writing “f(x): x+2” meaning “let f(x) be a label for the calculation “x+2” having x as a variable number” is meaningful, but writing “f(x) = x+2” is a syntax error since x+2 is a calculation and f(x) is a label. Writing f(3) = 5 is a double error saying that 5 is a calculation with 3 as a variable number. Writing f(2x) is a syntax error since “2x” is a calculation and not a variable number. Writing 2·f(x) is a syntax error since f(x) is a label and not a number. Writing y = f(x) is a syntax error and should be written e.g. y = (x+2), or y = (<f(x)>) where <f(x)> = x+2.

Talking about ”the value of a function” is as meaningless as talking about ”the mood of a verb”. Talking about mathematics describing the world is as meaningless as talking about grammar describing the world. Mathematics and grammar describe languages, and languages describe the world. To “mathematize” the world is as meaningless as to “grammatize” the world. Mathematical models of the world are as meaningless as grammatical models of the world. The world is described by qualitative or quantitative or graphical models.

Many proofs in mathematics are based upon the power-set, the set of all subsets in a given set. A subset is meaningful, but a set of subsets cannot be a set. A set is defined by a property shared by its elements. Since no or one element cannot share anything, it is problematic to talk about an empty set and a single element set. Hence set theory and the proofs using it need a revision.

Abstraction Errors

We can say that an abstraction is true if it is true whenever you meet instances of it. An abstraction is false if there are instances where it is not true.

2 meters 3 times is always 6 meters, and 2 something 3 times is always 6 something. Hence “3·2 = 6” is a true abstraction. Although 2 meters and 3 meters are 5 meters, 2 meters and 3 centimetres are 203 centimetres, 2 days and 3 weeks are 23 days etc. Hence “2+3 = 5” is a false abstraction. Still it is taught in school as a universal truth.

In the world we always meet numbers situated in contexts carrying units, and these units have to be alike before adding. Three apples mean an apple three times: 3·apple. It is not the number “3” but the operator “3·” that is abstracted from below. Addition only has meaning if the two operators operate on the same unit, i.e. addition only has meaning within a parenthesis:

$T = 2\cdot 3 + 5\cdot 3 = (2+5)\cdot 3 = 7\cdot 3$ $\quad$ $T = 2\cdot 3+4\cdot 5 = 2\cdot 3\cdot 1+4\cdot 5\cdot 1 = 6\cdot 1+20\cdot 1 = (6+20)\cdot 1 = 26\cdot 1$

Adding fractions suffers from the same problem as adding numbers without units. According to the principle of a common denominator 2/3+4/5 = 22/15. Adding numerators and denominators 2/3+4/5 = 6/8 is considered a meaningless mistake.

However 2 cokes out of 3 cans and 4 cokes out of 5 cans total 6 cokes out of 8 cans, and not 22 cokes out of 15 cans. Now the meaningless becomes meaningful and vice versa.

Again the point is that the units should be the same before adding. 2/3 of 3 cans and 4/5 of 5 cans total 2 cans + 4 cans, i.e. 6 cans out of 8 cans, i.e. 6/8 of 8 cans.

T = 2/3 of 3 cans and 4/5 of 5 cans = 2/3·3·can+4/5·5·can = 2·can+4·can = 6·can = 6/8·8·can

In the word-language we always use full sentences to evaluate the truth of a sentence. Instead of “green” we say e.g. “This table is green”. For the same reason also the number-language should use full sentences from day one, saying “T = 3·5” instead of just “3·5” thus specifying both what is being calculated and the calculation. Standard formulations from first year mathematics as “3+5” is a third order abstraction being abstracted from reality, from the units and from the equation. Such abstractions construct mathematics as encapsulated and create serious problems to the students when they later meet wor(l)d problems.

Equations Can Also be Solved by Reverse Calculations

A Top-Down approach will phrase “2+3·x=14” as an equation only solvable after equation theory has been introduced thus showing the relevance and applicability of modern abstract algebra.

2+3·x	= 14	
(2+(3·x))-2	= 14-2	+2 has the inverse element -2
((3·x)+2)-2	= 12	+ is commutative
(3·x)+(2-2)	= 12	+ is associative
(3·x)+0	= 12	0 is the neutral element under +
3·x	= 12	by definition of the neutral element
(3·x)·1/3	= 12·1/3	·3 has the inverse element 1/3
(x·3)·1/3	= 4	· is commutative
x·(3·1/3)	= 4	· is associative
x·1	= 4	1 is the neutral element under ·
x	= 4	by definition of the neutral element
L = {x∈R | 2+3·x = 14} = {4}		

Alternatively, a Bottom-Up approach will phrase “2+(3·x) = 14” as a calculation story reporting both a calculation process (2+3·x) and a calculation product (14), thus accessible together with calculations and solvable by reversing or walking the calculations as shown above.

Bottom-Up Mathematics Education Through the Social Practices that Created Mathematics

A Platonic Top-Down understanding sees mathematics as being created by and being examples of eternal universal ideas. Alternatively, a nominalistic Bottom-Up understanding sees mathematics as being created by and abstracted from social practices. According to Giddens the competence or practical consciousness developed through exposure and participance in social practices is mainly tacit (Giddens 1984). A rephrasing of “mathematics education” could be “number-language competence” coming from bringing into the classroom the social practices of bundling, totalling and earth measuring that raises the questions creating the number language, algebra and geometry. And respecting mathematics as partly tacit knowledge. This way allows the gradual growth of tacit competencies through gradual participance in social practices (Lave 1991) in which the students are allowed to sense an authentic being or “Dasein” (Heidegger 1926). This “sociological social constructivism” is different from Vygotskian “psychological social constructivism”. The former accepts the meta-language to be tacit, the latter believes in a Platonic scientific meta-language to be made discursive.

Another option is to give the stories of these social practices the form of fairy tales, in which case we might experience automatic assessment-free learning, suggested by the long survival of fairy tales in the non-writing culture of pre-pre-modernity.

The Social Practice of Bundling and Stacking

By totalling different bundling and stacking practices are used. Thus in the case of eight apples different Total stories can be told: A 2-bundling leads to the Total story T = 4·2·apple or T=1·stack @ 4·rows per stack @ 2·apple per row. A 9-bundling leads to T = (8/9)·9·apple, a 3-bundling gives T=2·3·apple+2·apple or T= (2 2/3)·3·apple. These stories emerge from *doing* a rebundling or from *calculating* using the “rebundling-equation” T = (T/a)·a. Standardising 10-bundles leads to the decimal numbers being “Grand Totals” in disguise: T = 234 = 2·100+3·10+4·1. In Top-Down mathematics natural, integer, rational and reel numbers are existing Platonic entities. In Bottom-Up mathematics the attributes of matter, space and time might be Platonic ideas, but numbers are bundling stories abbreviated as decimal numbers able to describe these attributes with any accuracy.

In this way multiplication comes before addition and fractions before two digit numbers. Hence a Bottom-Up curriculum is different from a Top-Down curriculum from day one (Tarp 1998).

The Social Practices of Measuring Earth and Uniting Totals

Geo-metry means earth-measuring in Greek. The earth is where we live and what we live from. We divide the earth between us, and geometry grows out of questions like “How do we divide and measure earth and space?”

Algebra means reunite in Arabic. If we buy five items in a store, we don’t have to pay all the single prices, we can ask for them to be united into a total. If the total is 17 \$ we are allowed to pay e.g. 20 \$. This new total is then split into the price and the change. To check we can reunite these numbers.

So living in a money based culture means being constantly engaged in a “social practice of totalling” consisting of uniting and splitting totals, and algebra grows out of the question “How much in total?”

This question can be answered in four different ways:

Totals unite/*split into*	variable	constant
unit-numbers \$, m, s, ...	T = a+n $T-n = a$	T = a·n $\frac{T}{n} = a$
per-numbers \$/m, m/100m=%, ...	ΔT = ∫ f dx $\frac{dT}{dx} = f$	T = a^n $\sqrt[n]{T} = a$ $log_a T = n$

The operations “+” and “·” unite variable and constant unit-numbers; “∫” and “^” unite variable and constant per-numbers. The reverse operations “–” and “/” split a total into variable and constant unit-numbers; “d/dx” and “√ and log” split a total into variable and constant per-numbers

"5 \$ and 3 \$ total ? \$"	T = 5+3	or T = a+n
"5 days @ 3 \$/day total ? \$"	T = 5·3	or T = a·n
"5 days @ 3 %/day total ? %"	1+T = 1.03^5	or 1+T = a^n
"n times @ (3 %/n)/time total ? %"	1+T = (1+0.03/n)^n = (1+t)^0.03/t $= \sqrt[t]{(1+t)}$ ^0.03 ≈ e^0.03	or 1+T $= \sqrt[t]{(1+t)}$ ^r ≈ e^r where e^t = 1+t for t small e.g. e^t is locally linear
"5 sec. @ 3 m/sec increasing to 4 m/sec total ? m"	$\Delta T = \int_0^5 (3+\frac{4-3}{5}\,x)\,dx$	or $\Delta T = \int_a^b f(x)dx$

Practice based questions lead to calculation stories or equations

When Will the logx Button be Included on Calculators?

A central question as " 5%/year in ? years total 50%" leads to the equation $1.05^x = 1.50$ with the solution $x = \log_{1.05}(1.50) = 8.3$. This cannot be calculated directly on a calculator. Why not?

The Social Practice of Building and Evaluating Models

The word-language and the number-language are used to describe or model the world. Word-stories are differentiated into different genres as fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and practices. Fiddle is nonsense containing syntax errors as e.g. "this sentence is false". In the Top-Down tradition number-stories are called mathematical models or applications of mathematics. As mentioned above this phrasing is a syntax error since mathematics describes the number-language, not the world. A Bottom-Up approach can avoid this error by phrasing "number-language description" as "quantifying and calculating model" and reuse the genre distinction from the word-language by talking about fact, fiction and fiddle models (Tarp 1999).

A fact model could also be called a "since-hence" model or a "room" model. Fact models quantify and calculate deterministic quantities: "What is the area of the walls in this room?" In this case the calculated answer of the model is what is observed. Hence calculated numbers from fact models can be trusted.

A fiction model could also be called an "if-then" model or a "rate" model. A fiction model contains contingent equations that could look otherwise. Fiction models quantify and calculate non-deterministic quantities: "My debt will soon be paid off at this rate!" Fiction models are based upon contingent assumptions and produces contingent numbers that should be supplemented with calculations based upon alternative assumptions, i.e. supplemented with parallel scenarios.

A fiddle model could also be called a "risk" model. Fiddle models quantify and calculate qualities that cannot be quantified: "Is the risk of this road high enough to cost a bridge?" The basic risk model says "Risk = Consequence · Probability". In evaluating the risk of a road statistics can provide the probabilities of the different casualties, but casualties cannot be quantified. Still in some cases they are quantified by the cost to public institutions as hospitals etc. This is problematic since it is much cheaper to stay in a cemetery than in a hospital. So risk-models might be fiddle models. Fiddle models should be rejected asking for a word description instead of a number description.

Rephrasing Mathematical Concepts

In the Top-Down tradition the names of mathematical concepts come from above. A Bottom-Up approach could respect these names but supplement them with other names coming from below. "Algebra" could also be called "reuniting totals". "Geometry" could also be called "earth measuring”. “Velocity, density etc.” could also be called “per-numbers” as the opposite of “unit-numbers”. “Stochastic variables” could also be called “unpredictable numbers” as the opposite of “predictable numbers”. “Linear and exponential functions” could also be called “change by adding and multiplying”. “Differentiable” could also be called “locally linear”. “Continuos” could also be called “locally constant” as the opposite of “interval constant” resulting from interchanging the ε and δ in the ε–δ definition. “Differential equations” could also be called “change equations”.

Top down names containing syntax errors should be avoided by saying “quantify and calculate” instead of “mathematize” and “mathematical modelling”, and by saying “the value of a variable” instead of “the value of a function”.

Has Mathematics Become the God of Late Modernity?

Premodernity institutionalised the worship of God, the metaphysical creator, in the premodern story house, the church, and the rhetoric of this worship can still be heard preached in today’s churches. When Newton discovered that the nature of forces was physical and not metaphysical, and that their effects could be quantified, calculated and predicted, the basis for the industrial culture of modernity was created. This made the quantifying and calculating number-language as important as the word-language in early modernity under names as “regning” in Danish, “Rechnung” in German etc.

The metaphysical counter reformation of the mid 1900 fuelled by the technology shocks of the risk society (Beck 1986) and by the cognitive turn with constructivism (Piaget 1969, Vygotsky 1934) reintroduced a metaphysical creator in mathematics, Set, to be worshipped and teached in the story house of modernity, the school. The rhetoric of late modern Mathematics is close to that of late feudal God, e.g. “No Math-No job” and “No God-No salvation”, “Mathematics is present everywhere” and “God is present everywhere”. It is numbers and calculations that are used everywhere, not meta-stories about them. And such statements will marginalise all those who cannot see it. Dehumanised mathematics dehumanises humans. It is one of the challenges of postmodernity to revive the enlightenment dream of human empowerment: Humans become educated not by meeting metaphysical creators but by meeting the social practices that provide the daily bread.

Conclusion

Mathematics holds on to its dream of being precise and consistent in spite of its inability to fulfil it. This could be one of the hidden reasons behind today’s exodus away from mathematics and math-based educations. This paper suggests the border between necessity and contingency within mathematics is moved quite considerably leaving only decimal numbers and multiplication as necessities. Inspired by Rorty we could ask: Maybe its hidden contingency should make mathematics a little self ironic and change its solidarity from the world above to the world below, from orthodoxy to human rights (Rorty 1989). Maybe a rehumanised, Bottom-Up, meaningful, syntax error free, user-friendly mathematics will make many of today’s learning problems disappear by themselves.

Fiction: “A New Curriculum for a New Millennium” - A Curriculum Architect Contest

Last year a school in Farawaystan decided to arrange a “curriculum architect contest” in mathematics: “A new curriculum for a new millennium”. Below is a fictitious response to this contest.

Organic Bottom-Up Mathematics: A Three Level Bundling and Totalling Curriculum

The holes in the head provide humans with food for the body and knowledge for the brains: tacit knowledge for the reptile brain and discursive knowledge for the human brain. This proposal sees a school as an institutionalised knowledge house providing humans with routines and stories by making them participants in social practices and narratives, and by respecting conceptual liberty.

The chaotic learning of tacit routine knowledge can be guided by attractors (Doll 1993), in this case by social practices providing authenticity. In the case of mathematics the social practices will be those of bundling and totalling according to the Arabic meaning of the word Algebra: reunite.

In today's post-traditional society (Giddens in Beck et al. 1994) humans can no longer obtain identity by echoing traditions, they have to create their self-identity by building biographical self-stories looking for meaning and authenticity (Giddens 1991). Each individual student has his own learning story, a network of concept-relations, sentences. Resembling a widespread organic carbon structure a learning story steadily grows by adding new sentences to existing words: Tell me something I don't know about something I know (Ausubel 1968). Stories can tell about the metaphysical world above and about the physical world below. Top-Down stories from above connecting metaphysical concepts cannot be anchored to the existing learning story, they become encapsulated rote learning. Bottom-Up stories from below can, i.e. stories about the social practices providing the daily bread. The three Bottom-Up mother stories are the stories about nature, culture and humans.

First the strong gravity force crunched its universe in a big bang, liberating the medium nuclear force trying to crunch the atoms of a star in small bangs liberating light. In the end the strong force crunches the star in a medium bang filling space with matter and planets and liberating the weak electromagnetic force neutralising the strong force by distant electrons. Light makes motion flow through our planet's nature creating random micro-motion and cyclic macro-motion. Molecules transfer motion through collisions and are recycled when carbon-hydrogen structures have oxygen added and removed. The weak light helps the green cells to split the weak carbon-oxygen link. The strong light, lightening, splits the strong nitrogen-nitrogen link in the air adding strength to the extended carbon-nitrogen structures from which life is build. The three life forms are black, green and grey cells. The black cells survive in oxygen free places in stomachs and on the bottom of lakes only able to take oxygen in small amounts from organic carbon-structures thus producing gas. The green cells use the weak light to remove the oxygen from the inorganic carbon dioxide structure thus producing both organic matter storing motion and the oxygen needed by the grey cells to release the motion again. Green cells form cell communities, plants, unable to move for the food and the light.

Grey cells form animals able to move for the food in form of green cells or other grey cells thus needing to collect and process information by senses and brains to decide which way to move. Animals come in three kinds. The reptiles have a reptile brain for routines. The mammals having live offspring in need of initial care have developed an additional mammal brain for feelings. Humans have developed human fingers to grasp the food, and a human brain to grasp the world in words and sentences. Thus humans can share and store not only food but also stories, e.g. stories about how to increase productivity by transforming nature to culture.

The agriculture transforms the human hand to an artificial hand, a tool, enabling humans to transform the wood to a field for growing crops. The industrial culture transforms the human muscle to an artificial muscle, a motor, integrating tools and motors to machines enabling humans to transform nature raw material to material goods. The information culture transforms the human reptile brain to an artificial brain, a computer, integrating the artificial hand, muscle and brain to an artificial human, a robot, freeing humans from routine work.

Human production and exchange of goods has developed a number-language besides the word-language to quantify the world and calculate totals. Agriculture totals crops and herds by adding. Trade totals stocks and costs by multiplying. Rich traders able to lend out money as bankers total interest percentages by raising to power. And finally industrial culture calculates the total change-effect of forces through integrating: by adding a certain amount of momentum per second and energy per meter a force changes the meter-per-second-number, which again changes the meter-number.

A Three Level Bundling, Stacking and Totalling Curriculum

This proposal presents an organic bottom-up mathematics growing out of the social practices of bundling, stacking and totalling. It is organised in three levels, level 1: 6-10 years, level 2: 10-14 years and level 3: 14-18 years. It is activity and question driven limiting the amount of written material. It is learner centred limiting the amount of in-service teacher training.

The curriculum metaphor is a tree with a trunk consisting of five fundamental social practices: bundling, stacking, totalling, coding and reporting fed by a root of basic activities. From the trunk two branches grow out, a “totals in space” branch and a “totals in time” branch reintegrating into a “totals in space and time” at three levels.

The basic activities are carried out with different piles of pellets or beads arranged and rearranged in sand or plastic boxes or frames always followed by the question “How many in total?” The pellets are bundled in different ways, illustrated graphically, reported as a Total-story, controlled on a calculator and finally coded.

One pellet only leads to one Total-story: $T = 1$

Two pellets bring the names “bundle”, “times” and “stack”. Two pellets can be bundled as a 2-bundle one time or as a 1-bundle two times. And a 2-bundle can be stacked. This produces two Total-stories:

..	. .	:
$T = 1\cdot 2$	$T = 2\cdot 1$	$T = 1\cdot 2$

Three pellets bring the names “add” and “minus” and lead to four Total-stories:

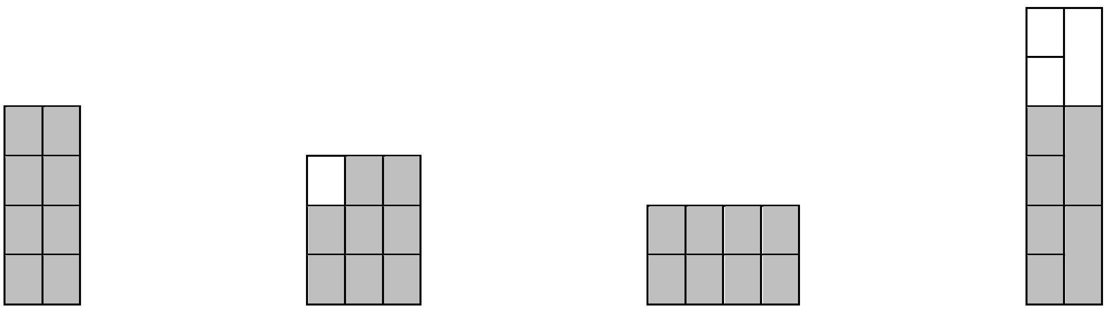
...	.. .	. ..	. . .	··0
$T = 1\cdot 3$	$T = 1\cdot 2+1\cdot 1$	$T = 1\cdot 1+1\cdot 2$	$T = 3\cdot 1$	$T = 3\cdot 1–1\cdot 1$
$T = 3$	$T = 2+1$	$T = 1+2$	$T = 3$	$T = 3–1$

(in some cases the “·1” and “1·” can be left out)

Four pellets bring the names “square”, “per” and “@” when the two 2-bundles are stacked

.. ..	::
$T = 2\cdot 2$	$T = 2\cdot 2$ = 1·stack @ 2·rows/stack @ 2·1/row

Eight tiles can lead to fractions. Some fractions can be reduced through a rebundling:



$T = 4\cdot 2$ $\quad$ $T=2\cdot 3+(2/3)\cdot 3=(2\ 2/3)\cdot 3$ $\quad$ $T = 2\cdot 4$ $\quad$ $T = 4/6 = 2/3$

Ten is used as the maximum bundle size called “X” in the beginning, $T = 3\cdot X+4\cdot 1$. Later it is abbreviated to T=34 using the sign “0” for “none”. Likewise the Roman tradition can be reused by calling hundred “C” and thousand “M”: $T = 3\cdot M + 4\cdot C + 5\cdot X + 6\cdot 1 = 3456$.

A Total-story can be coded to hide the numbers so others will have to guess:

$T = 2\cdot 5 + 1$ thus becomes $T = 2\cdot a + 1$

Coded total-stories are later called equations or functions. They can be analysed in tables and illustrated in figures on squared paper, where the ruler is introduced as a “counting stick” (fig. 1).

The numbers of the table is calculated by walking on the floor or by “finger walking” on the table:

a = 3, T = ? a = 3 —*(·2)*→ 6 —*(+1)*→ 7 = T

Walking backwards reversing the calculation signs checks the result:

a = ?, T = 7 a = 3 ←*(/2)*— 6 ←*(–1)*— 7 = T

T = 2·a+1	3	5	7	9
a	*1*	*2*	*3*	*4*

Two codings are needed to find the two numbers a and T (fig. 1):

T = 3·a–2	1	4	7	10
a	*1*	*2*	*3*	*4*

Bundling in b-bundles and d-bundles gives the Total-story the form T = a·b+c·d. Also double coding like T = 2·a + 2·b + 1 can be analysed in tables and illustrated in space using centicubes or blocks made out of paper (fig. 3). Squares with the same number can be coloured alike.

	T = 2·a + 2·b + 1:	*3*	**9**	**11**	**13**
		2	**7**	**9**	**11**
		1	**5**	**7**	**9**
		b / a	*1*	*2*	*3*

Totals in Space

This has three branches: Rebundling totals, adding totals and totalling forms and figures, geometry.

Rebundling Totals, Level 1

Rebundling or restacking questions as “T = 2·3 = ?·5” come from e.g. sharing questions. The answer can be found by a physical rebundling using pellets or beads: 2·3 = 6·1 = 1·5+1 or by a mental rebundling using a suitable calculator as e.g. Texas Instruments Math Explorer. From such activities a general “rebundle story” grows: 6 = (6/2)·2, 6 = (6/5)·5, 6 = (6/9)·9 or T = (T/a)·a. A rebundling into 2-bundles give birth to the names “even” and “odd”.

Rebundling Totals, Level 2

On this level, pellets become units, numbers become decimals, countable and measurable things become quantities and stories become equations.

Three apples become an apple three times T = 3·apple, and the counting stick now becomes a ruler counting centimetres, which can be bundled in decimetres and which has millimetres as sub-bundles: 1·dm = 10·cm and 1·cm = 10·mm. A rebundling thus can always produce a whole number giving meaning to multiplication of decimals: T = 4.3·cm = 4.3·10·mm = 43·mm.

If one of the quantities in the Total equation is a variable so is the Total: T = a·b+c·d = a·x+e. This variation can be illustrated by tables and graphs now using points instead of tiles (fig. 2 & 4).

Calculation stories now are equations now solved by reversing the calculation, i.e. moving numbers to the other side of the equal sign and reversing its calculation sign according to the rebundle story.

6 = ?·5
6 = x·5
6/5 = x

6 = ? + 5
6 = x + 5
6–5 = x

Now rebundling takes place between units thus changing e.g. kilograms to $ by a rebundling to known quantities.

T = 6·kg = 4·$

T = 9·kg = ?·$	T = 10·$ = ?·kg
T = 9·kg = (9/6)·6·kg = (9/6)·4·$ = 6·$	T = 10·$ = (10/4)·4·$ = (10/4)·6·kg = 15·kg

Another example is rebundling between meters and centimetres:

T = 100·cm = 1·m

T = 32·cm = ?·m	T = 4.1·m = ?·cm
T = 32·cm = (32/100)·100·cm = 0.32·m	T=4.1·m=(4.1/1)·1·m=4.1·100·cm=410·cm

Another example is rebundling between percent % and $:

T = 100·% = 40·$

T = 20·% = ?·$	T = 10·$ = ?·%
T=20·%=(20/100)·100·%=(20/100)·40·$=8·$	T=10·$=(10/40)·40·$=(10/40)·100·%=25·%

An alternative would be to use equation tables telling both what quantity to be calculated, what equation to use, what numbers to use in the calculation and how the calculation is done.

$=?	$=($/kg)·kg
$/kg=4/6 kg=9	$=4/6·9 $=6

m=?	m=(m/cm)·cm
m/cm=1/100 cm=32	m=1/100·32 m=0.32

$=?	$=($/%)·%
$/%=40/100 %=20	$=40/100·20 $=8

Also adding percentages can be considered an example of a rebundling, e.g. adding 5% to 40·$ two times:

To = 100·% = 40·$
T1 = 105·% = (105/100)·100·% = 1.05·40·$ which now becomes 100·%
T2 = 105·% = (105/100)·100·% = 1.05·1.05·40·$ = 1.05^2·40·$ etc. until
Tn = To·(1+r)^n

Another but slower way is to rebundle the 40·$ to 100·$ and then add 5·$ per 100·$:

40·$ = (40/100)·100·$, so we add 5·$ 40/100 times i.e. 2·$ totalling T1 = 40+2 = 42·$
42·$ = (42/100)·100·$, so we add 5·$ 42/100 times i.e. 2.1·$ totalling T1 = 42+2.1 = 44.1·$

Rebundling Totals, Level 3

On this level power calculations are reversed as logarithm and root:

6 = ?^5

6 = x^5
$\sqrt[5]{6} = x$

6 = 5^?

6 = 5^x
$\log_5 6 = x$

The quantities in the Total equation can themselves be Totals:

T = a·b+c·d = a·T2+T3·T4 = a·(mx+ny) + (px+qy)·(rx+sy), or
T = a·b+c·d = (kx+l)·(mx+n) + (px+q)·(rx+s) = A·x^2 + B·x +C

In such cases the Total is called a "polynomial" to be illustrated in a two or three dimensional co-ordinate system (fig. 5). A polynomial can be considered a mix of quantities controlling the appearance of a curve: The constant controls the initial level, the x the later direction, the x^2 the still later curvature, the x^3 the still later curvature or counter curvature etc. (fig. 6).

The change of T, ΔT can be rebundled into a change of x, Δx:

ΔT = (ΔT/Δx)· Δx	in the case of macro changes, and
dT = (dT/dx)·dx = T'·dx	in the case of micro changes

Considering $T = a \cdot b$ a stack we see that the change ΔT is

$\Delta T = \Delta a \cdot b + a \cdot \Delta b + \Delta a \cdot \Delta b$	or as per-numbers:
$\Delta T/T = \Delta a/a + \Delta b/b + \Delta a/a \cdot \Delta b/b$	in the case of macro changes, and
$dT/T = da/a + db/b$	in the case of micro changes

Thus in the case of $T = x^n$

$DT/T = n \cdot dx/x$	or
$dT/dx = n \cdot T/x = n \cdot x^{(n-1)}$	i.e. $\mathbf{d/dx\ (x^n) = n \cdot x^{(n-1)}}$

If $T = e^x$, where the Euler number e is locally linear: $e^t = 1+t$ for t a micro number, then

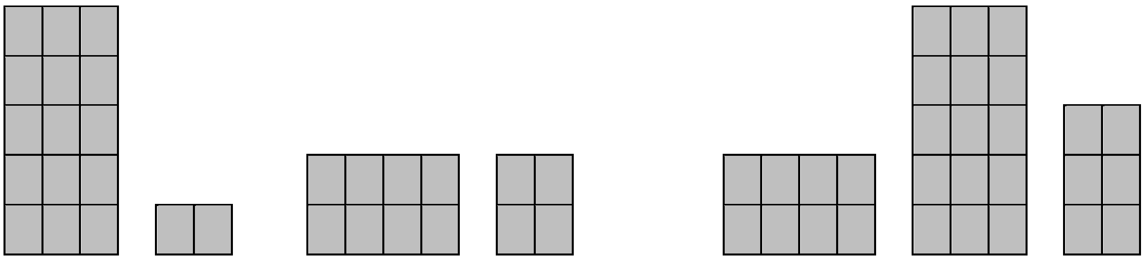
$dT = e^{(x+dx)} - e^x = e^x \cdot e^{dx} - e^x = e^x \cdot (e^{dx} - 1) = e^x \cdot (1+dx-1) = e^x \cdot dx$ or
$dT/dx = e^x$ i.e. $\mathbf{d/dx\ (e^x) = e^x}$

In the case of more variables we have e.g.

$p \cdot V = n \cdot R \cdot T$	
$dp/p + dV/V = dn/n + dT/T$	since R is a constant

Adding Totals, Level 1

Totals at different locations can be added remembering that only like bundles can be stacked



$T1 = 5 \cdot 3 + 1 \cdot 2$ $T2 = 2 \cdot 4 + 2 \cdot 2$ $\Sigma T = 2 \cdot 4 + 5 \cdot 3 + (1+2) \cdot 2$

T1 = T2 =	$5 \cdot 3 + 1 \cdot 2$ $2 \cdot 4 + 2 \cdot 2$	$= 1 \cdot 10 + 7 \cdot 1$ $= 1 \cdot 10 + 2 \cdot 1$
$T = \Sigma T =$	$2 \cdot 4 + 5 \cdot 3 + (1+2) \cdot 2$	$= (1+1) \cdot 10 + (7+2) \cdot 1$

Adding Totals, Level 2

Totals coming from different shops can be added remembering that per-numbers never add only unit-numbers do.

T1: T2:	6 kg @ 4 $/kg total 4 kg @ 7 $/kg total	24 $ 28 $
$T = \Sigma T =$	10 kg @ x $/kg total x $/kg is	52 $ 52 $/10 kg = 5.2 $/kg

Adding Totals, Level 3

Totals coming from different time intervals can be added remembering that the m/s numbers are only locally constant. In this case the question is: “5 sec at 4 m/s increasing to 6 m/s total ?m”.

dT1: dT2: dT3:	dt sec @ v1 m/sec total dt sec @ v2 m/sec total etc.	$v1 \cdot dt$ $v2 \cdot dt$

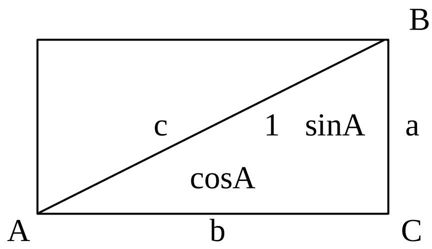
$\Delta T = \Sigma dT =$		$\int_0^5 v\cdot dt$, $v = 4 + \frac{6-4}{5}\cdot t$ e.g.

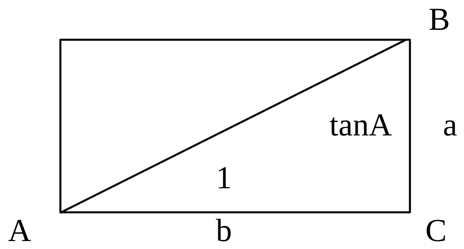
Geometry, Level 1

Geometry means "earth-measuring" in Greek. So geometry grows out of questions and activities related to dividing and measuring the earth we live on and from. A squared paper can be thought of as an island to be divided between two or more persons. Each person places a dot at a random location or starts a 6-step walk from a corner determined in some way by a dice. Then the paper has to be divided so they have equal distances to the border. Finally the question "How much did I get?" is posed. From this activity grows names as points, lines, midpoints, midlines or normals, triangles, "fourangles", rectangles, size etc. All figures can be divided into triangles, and all triangles can be wrapped into a rectangle being a stack of squares and having the double size of the triangle. A ruler becomes a square counter bundling squares into 2-bundles. Different forms as cubes and cylinders or bottles are covered with paper counting surface size. Water is poured from cubes to cubes, from cylinders to cylinders and between cubes and cylinders discussing how to count the content size of water.

Geometry, Level 2

Different figures and forms get different names. Surface and content size now becoming area and volume can be calculated by equations. Rebundling stacks become reshaping areas leading to the construction and calculation of the mean and fourth proportionals. A rectangle can be divided by the diagonal producing a right-angled triangle with an outside bundled in meters and an inside bundled in diagonals c (a = sinA·c and b = cosA·c) or in sides (a = tanA·b, b = tanB·a).





Design tasks lead to the golden section. Technical drawings can be made from front-, top- and side view and on isometric paper. All geometrical jobs are performed both on paper and in space.

Geometry, Level 3

Geometrical questions are translated to equations and vice versa by means of the co-ordinate system. Conic sections are put into equations. Technical drawing can now be made in perspective. Vectors are used to move and rotate figures in two and three dimensions.

Totals in Time, Level 1

A total T may change in time by being added a change-number ΔT. This leads to two stories, a change-story about ΔT and a Total-story about T.

Counting by 1's, 2's, 3's are examples of change stories: ΔT = 1, 2, 3 etc. Other examples are:

Constant Walk, e.g. a "+2" walk	ΔT = +2	T = 6+2+2+2+....
Walking backwards provides a "–2" walk	ΔT = –2	T = 14–2–2–....
Constant Percent Walk, e.g. a "·2" walk	ΔT = +100%	T = 6·2·2·2·....
Walking backwards provides a "/2" walk	ΔT = –50%	T = 32/2/2/....
Decreasing Walk, e.g. "a to –a" walk	ΔT = +3,..., –3	T = 10+3+2+1+0–1–2–3

Swinging Walk, e.g. "a to –a to a" walk	ΔT = +3,..., –3,..., +3	T = 10+3+2+1+0–1–2–3–2–1–0+1+2+3
Random Walk, e.g. by adding the green even dice-number and subtracting the red odd dice-numbers	ΔT = random	T = 10+4–5–1+2+...

A variation to the random walk could be "dice-number six means report-time" i.e. time for a graphical report to be made both physically with beads or pellets together with the question "rearrange so the sticks have the same length ", giving birth to the word "mean".

Another variation could be "dice-number six means tax-time" where you receive or pay 1 per 3 of your fortune depending on the next dice-number is even or odd.

Alternatively a bank could be included to receive or pay out money. If both players and bank report money transferrals the names "debit" and "credit" are introduced together with the observation that debit and credit entries always go together, thus introducing accounting at an early level.

Totals in Time, Level 2

On this level five change equations appear:

Δn = 1, ΔT = +a $	leading to linear change	T = b+a·n
Δn = 1, ΔT = +r %	leading to exponential change	T = b·a^n, a = 1+r
Δn = 1 %, ΔT = +r %	leading to potential change	T = b·n^r
Δn = 1, ΔT = + r % +a $	leading to annuities	T = a/r·R, 1+R = (1+r)^n
ΔX = random	leading to statistics	X ≈ Xmean ± 2·Xdev

The first three total equations give linear graphs on "++paper", "+·paper" and "··paper" , where the "+" means a "+scale" (0,1,2,3,...) and the "·" means a "·scale" (1,2,4,8,...).

An unpredictable number X is called a stochastic variable. A variable which is not "pre-dictable" might be "post-dictable", i.e. its previous behaviour might be described in a table from which its mean and deviance can be calculated. Based upon these numbers the variable then can be interval-predicted as a confidence interval X ≈ Xmean ± 2·Xdev. The cumulated values of a stochastic variable might give a linear graph on a normal distribution paper.

Totals in Time, Level 3

On this level the change ΔT is not constant but predictable, e.g. ΔT/Δx = x^2 or dT/dx = x^2. Such change equations are called difference and differential equations. They can all be solved by constantly adding the change: final number = initial number + change or Tf = Ti + ΔT. In the case of micro changes this means an enormous number of adding unable for a human to perform. A computer however can do it easily in no time.

Totals in Space and Time: the Quantitative Literature

Humans communicate about the world in languages. A word language with sentences assigning words to things and actions. And a number language with equations assigning numbers or calculations to things and actions. "Word-stories" are differentiated into the genres fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like "This sentence is false". "Number-stories" are often called mathematical models. Also these can be differentiated into the genres: fact, fiction and fiddle. Fact models quantify and calculate predictable quantities. Fiction models quantify and calculate non-predictable quantities. Fiddle models quantify qualities that cannot be quantified. As with word-stories also different number-stories should be treated different: Facts should be trusted, fiction should be doubted and fiddle should be rejected.

Level 1: Rebundling practices reported as Total-stories and illustrated on squared paper are examples of number- and calculation stories. Other examples are dice games of different kinds, e.g. the dice-tax-game mentioned above.

Level 2: Micro science and microeconomics. In both areas a typical question is that of rebundling one type of numbers to another kind. In physics meters are rebundled to seconds, seconds to joules, joules to degrees, volts to amperes etc. In chemistry moles are rebundled to kgs, kgs are rebundled to litres, moles to joules etc. In economics dollars are rebundled to kgs or to litres, dollars to pounds, dollars to percent etc. Statistical yearbooks are filled with tables showing quantities distributed in space and varying in time.

Level 3: Macro science and macroeconomics. In both areas the dynamics and interaction between subsystems are described and analysed, both ecological systems and economical system. Examples are Limits to Growth, Fishing Models and National Fiscal Policy Models (Tarp 1999).

References

Ausubel, D. P. (1968). *Educational Psychology*, London: Holt, Rinehart and Wilson

Bauman, Z. (1992) *Intimations of Postmodernity*, London: Routledge

Beck, U., (1986, 1992) *The Risk Society*, London: Sage Publications

Beck, U., Giddens, A. and Lash S. (1994) *Reflexive Modernization*, Cambridge: Cambridge university Press

Doll, W. (1993) *A Post-Modern Perspective on Curriculum*, New York: Teachers College Press

Euler, L. (1748, 1988): *Introduction to Analysis of the Infinite*, New York: Springer Verlag

Foucault, M. (1972) *Power/Knowledge*, New York: Pantheon Books

Giddens, A. (1984) *The Constitution of Society*, Oxford: Polity Press

Giddens, A. (1991) *Modernity and Self-identity*, Oxford: Polity Press

Heidegger, M. (1926, 62) *Being and time*, Oxford: Blackwell

Jensen J. H., Niss M. And Wedege T. (1998) *Justification and Enrolment Problems in Education Involving Mathematics or Physics*, Roskilde: Roskilde University Press

Lave J. & Wenger E. (1991) *Situated Learning*, Cambridge: Cambridge university Press

Piaget, J. (1969, 1970)*Science of Education and the Psychology of the Child*, New York: Viking Compass

Rorty, R. (1989) *Contingency, Irony and Solidarity*, Oxford: Polity Press

Tarp, A. (1998) *Postmoden Mathematics: Contextual Totalling Narratives*, Proceedings of the Norma 98 Conference, Agder College, Kistiansand, Norway

Tarp, A. (1999) *Fact, Fiction, Fiddle - Three Types of Models*, Proceedings of the ICTMA 9 Conference, Lisbon, Portugal

Tarp, A. (2000) *Why do Postmodern Students reject Modern Mathematics*, Boaler & Valero: The Social Dimensions of Mathematics Education. To appear.

Vygotsky, L. (1934, 1986) *Thought and language*, Cambridge MA, MIT press

Illustrations

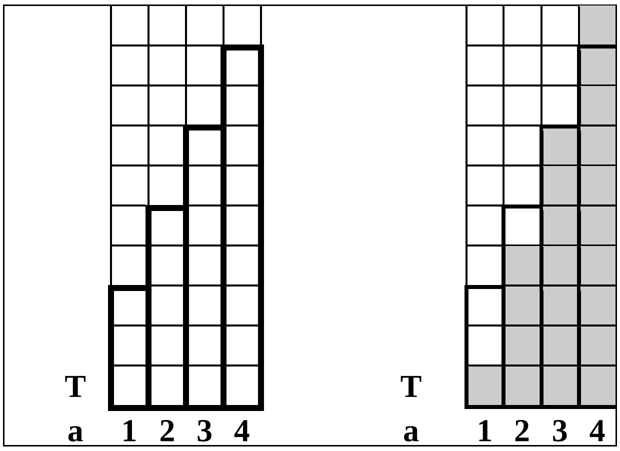


Figure 1
The coded Total-stories T = 2·a + 1 and T = 3·a – 2 illustrated on squared paper

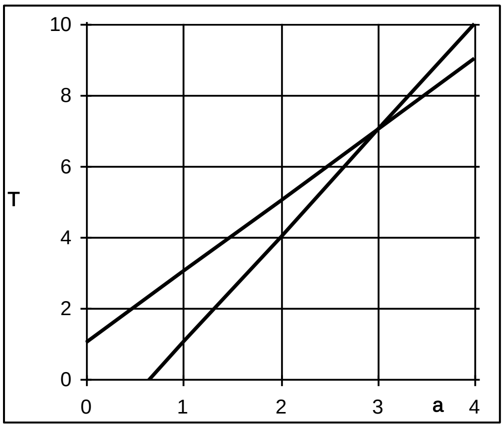


Figure 2
The equations T=2·a+1 and T=3·a–2 illustrated in a co-ordinate system

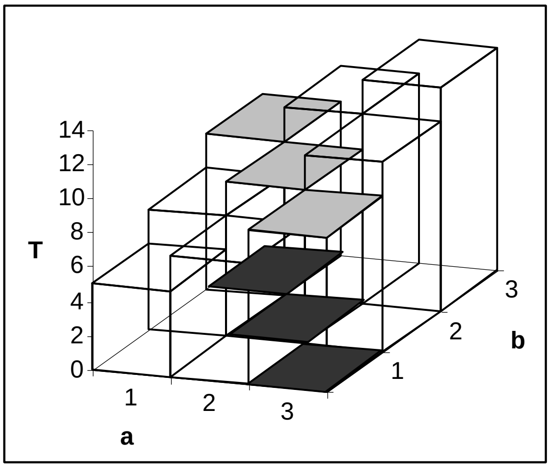


Figure 3
The coded Total-story T = 2·a+2·b+1 build on squared paper. The level-9 tiles are coloured.

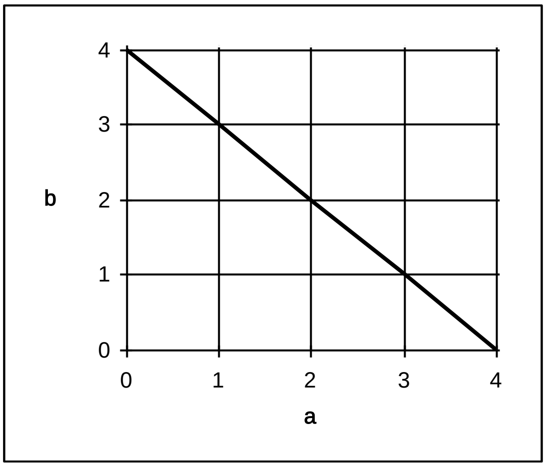


Figure 4
The level-9 line of the equation T = 2·a + 2·b+1 illustrated in a co-ordinate system

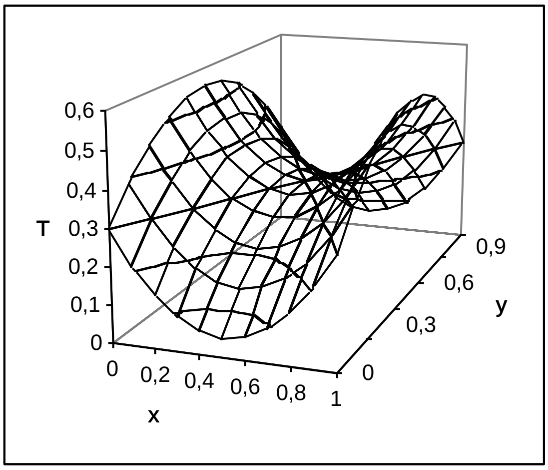


Figure 5
The equation T = x^2–y^2–x+y+0.3 illustrated in a co-ordinate system

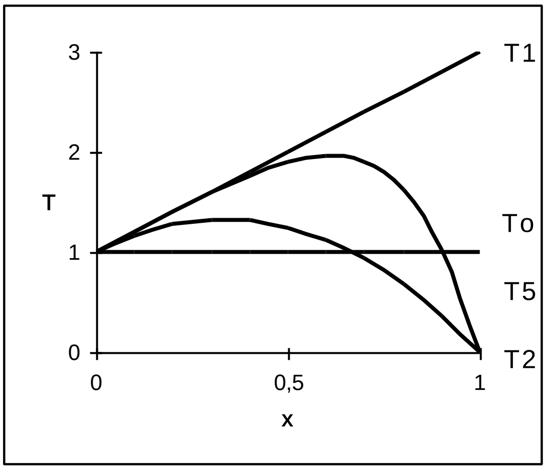


Figure 6
The equation To = 1, T1 = 1+2·x, T2 = 1+2·x –3·x^2 and T5 = 1+2·x–3·x^5 illustrated in a co-ordinate system

Student-mathematics versus teacher-Metamatics

The writer reports on his career as an action researcher helping the students to develop their own student-mathematics, making mathematics accessible for all but being opposed by the educational system. The work took place over a 30 year-period in Danish calculus and pre-calculus classes and in Danish teacher education. As methodology a postmodern counter-research was developed accepting number-statements but being sceptical towards word-statements. Counter-research sees word-researchers as counsellors in a courtroom of correctness. The modern researcher is a counsellor for the prosecution trying to produce certainty by accusing things of being something, and the postmodern researcher is a counsellor for the defence trying to produce doubt by listening to witnesses, and by cross-examining to look for hidden differences that might make a difference. A micro-curriculum in student mathematics was developed and tested in 13 grade 11 classes showing a high degree of improvement in student performance.

A Confession

I confess I have always listened to the students. 'Ich bin ein Berliner' John F. Kennedy said. We are all students, and maybe we should stay sceptical students and keep on learning; and wait to teach until we have found something that is certain, and cannot be different[i].

I was studying mathematics at the university, but during my study I learned that there was another hidden mathematics different from the one in the textbook.

According to the Danish textbooks mathematics is something above the physical world, a metaphysical subject that is studied for its own sake to obtain pure knowledge; a subject that might, but has no need to, be applied to the real world.

But looking at Anglo-Saxon textbooks and at the history of mathematics I found out that the world is not applying mathematics, the world is creating mathematics. Mathematics was born as quantitative stories about multiplicity, just as the names Geometry and Algebra clearly indicates: 'geometry' means 'earth-measuring' in Greek, and 'algebra' means 'reuniting' in Arabic. Thus geometry and algebra are answers to the two fundamental questions that arose when humans went from gathering & hunting to agriculture: How do we divide our land, and our products.

This however was not how Geometry and Algebra was presented in the Danish textbooks. Here they were presented as examples of sets. Numbers were sets, calculations were sets, and all of mathematics was examples of sets.

To name this difference between textbook-mathematics and real-world mathematics I coined the word 'meta-matics' from above as opposed to 'mathe-matics' from below.

Until then I had seen no meaning in mathematics, which I had to learn more or less by heart, and I was planning to shift away from mathematics to study architecture instead. The discovery of mathematics from below however changed this. All of a sudden I found mathematics to be a fascinating number-language that could be applied to describe the world in numbers, which can be calculated and thus predicted. And I decided to share this excitement with others, thus choosing to continue my study and become a mathematics teacher.

And the summer before starting I wrote an alternative textbook in mathematics, Calculus as mathematics from below, as opposed to the traditional textbook, Calculus as meta-matics from above.

In my Calculus textbook I showed how mathematics grows out of real-world problems. I had expected my students to be as excited as I was. Instead they said: Are we going to learn about mathematics, or are we going to learn about real-world problems? Both, I answered. But what if the real-world problems can be solved in another way, they asked.

Now I was caught in a dilemma. The students made me realise that I was trying to sell meta-matics hidden under a thin surface of applications. Thus forcing them to learn two things, meta-matics and applications.

So I had to reject my Calculus-textbook, and do as the others, follow the norm. However most students did not understand the traditional textbook. So what should I do, should I turn to architecture, or should I cross over to help the students develop their own mathematics?

I confess I became a renegade. And for the next 7 years I continually wrote new texts adjusting mathematics to the students' suggestions. Then finally we had found a text that worked so that all students were able to understand and learn calculus. I transformed this student-mathematics into a textbook, which I published, expecting that the ministry and the other teachers would welcome it as a solution to the low success rate in calculus.

But the other teachers neglected it; and the ministry told me that a textbook on calculus should cover 200 pages, and if I continued to use my 48 pages textbook I would be discharged.

So I had to reject this textbook on student-mathematics in calculus.

Instead I turned to pre-calculus, which has even bigger problems than calculus being a compulsory subject that most student find difficult to understand. Again I worked as an action researcher listening to the students' suggestions. And again the traditional 200-page textbook in meta-matics was replaced by a minor textbook in student-mathematics covering 12 pages.

But this time I did not publish the textbook. Instead I applied for a PhD scholarship in order to try out part of it with other teachers in their classes. I asked for volunteer teachers to try out a 20 lesson introductory course in student-mathematics. And I was in luck; out of approximately 1500 mathematics teacher 1.5 volunteered, a full time teacher and a temporary teacher. So other teachers I had to persuade.

For three years I followed the two volunteer teachers teaching three classes each. The teachers had big problems leaving the tradition to practise the student-mathematics. Still student-mathematics turned out to be so robust that almost all students expressed satisfaction, some even surprise to be allowed to enter the field of mathematics, which had always been closed to them[ii].

The full time teacher wanted to extend the student-mathematics to a full year programme, but the ministry turned his application down even if the ministry had called for experiments in order to save the pre-calculus mathematics, which was at risk to be terminated because around 50% of the students fail the written exam.

So I have returned to my own classroom to try out the full version of the student-mathematics myself, but again the ministry turned the application down so I had to do a little of both in the classroom. However this compromise proved to be only temporary since the external examiner complained to the ministry, that I was not following an ordinary textbook as the rest was doing. And the ministry will probably echo its standard answer 'follow the norm, or go to the dorm'.

So I also had to reject this textbook on student-mathematics in pre-calculus.

However there is a big advantage working in the research field. At the yearly teacher conference nobody wants to listen to people not following the norm[iii]. At a research conference it is different. You don't have to wait for an invitation that never comes; you can submit your own paper. After a presentation a researcher approached me. He had heard my presentation at two conferences and he was astonished that it made so much sense even if it was outside the traditional discourse on constructivist mathematics. He would like me to present student-mathematics to his students at his teacher college. So for one week we worked together presenting and translating student-mathematics to East-European students. They also were fascinated recommending that both traditional modern mathematics and student-mathematics (or postmodern mathematics as it was called) should be taught in teacher colleges[iv].

And to my luck I was asked to teach a two-year e-learning course at a Danish teacher college. So here I had the opportunity to develop a full program in student-mathematics for teacher education.

The programme was successful with the students. But halfway through the programme my temporary job was transformed into a permanent job, which I could not get since the committee called me a missionary refusing to follow the norm. And instead of finishing the programme, the new teacher ordered a cure for this programme by asking the students to read the first 60 pages of the traditional textbook for their first meeting.

So I also had to reject this textbook on student-mathematics for teacher education.

Apparently there are big problems practising student-mathematics as long as meta-matics is in power. Just like the early mammals had to survive underground when the dinosaurs ruled the world. However I am in luck. The dinosaurs are about to make themselves extinct since they cannot reproduce. Mathematics education faces an enrolment crisis all over the world since only a few students want to study for mathematics-based educations, and even fewer want to become mathematics teachers[v].

So as the bird Phoenix raises again from the fire, I plan to make student-mathematics rise again as a virtual textbook to be placed on the Internet as a self-reproducing virus.

During my short life as at the teacher college I learned, that students studying student-mathematics do not need a teacher. Meta-matics form above needs a teacher as a transmitter since it places its authority in the metaphysical world above, from which meta-matics is supposed to flow through researchers and teachers to the students.

The student-mathematics places its authority in the physical world below, in multiplicity. Multiplicity can be studied in your own living room. All the teacher needs to do is to set up an agenda for an educational meeting between the student and the multiplicity. Then learning automatically takes places, both as tacit knowledge, competence, through a 'sentence-free meeting with the sentence-subject', and as discursive knowledge, qualifications, through a 'sentence-loaded meeting with the sentence-subject'.

In this way 1 teacher can organise 16 students in 4 groups of 4 students acting by turns as instructors working together in pairs instructing the others, and being coached by one teacher over the internet.

Through pyramid-education each teacher continually produces 16 new teachers in student-mathematics, who pay for their education by each teaching a new group of 16 students. In this way student-mathematics will multiply on the Internet, until it can surface to the real world when the dinosaurs of meta-matics have died out from lack of fertility.

Now my confession ends. I am sorry that I left my tribe to join the others, those who are accused of being uneducated, uninterested, unruly, lazy, stupid, narcissistic, self-focused etc. etc. etc. to help them develop their own mathematics. But maybe this student-mathematics will survive once the 5000 years old subject mathematics has terminated its 100 years sidetrack of set-based meta-matics[vi] and returned to multiplicity-based mathematics.

Methodology

The methodology of this action research grows out of institutional scepticism, as it appeared in the enlightenment and was implemented in its two democracies, the American in the form of pragmatism and symbolic interactionism, and the French in the form of post-structuralism and post-modernism[vii]. This paper follows the postmodern scepticism towards logocentricity, i.e. towards the belief that the words represent the world[viii].

I have developed a methodology called 'postmodern counter-research' based upon a post-structuralist 'pencil-dilemma': Placed between a ruler and a dictionary a thing can show its length, but not its name - hence a thing can falsify a number-statement about its length, but not a word-statement about its kind. I.e. a thing can defend itself against a number-accusation by making a statement of difference; but against a word-accusation it can only make a statement of deference. Unless it can ask for a counsellor for the defence, a postmodern counter-researcher.

A number is an ill written icon showing the degree of multiplicity (there are 4 strokes in the number sign 4, etc.); a word is a sound made by a person and recognised in some groups and not in others. Words can be questioned and put to a vote, numbers cannot. Numbers can carry valid conclusions based upon reliable data, i.e. research. Words can carry only interpretations, that if presented as research become seduction; words can not carry truth, only hide differences to be uncovered by postmodern counter research, having quality if the difference is a genuine 'cinderella-difference', i.e. a difference that makes a difference. Thus postmodern counter-research follows in the footsteps of the ancient sophists always distinguishing between what could be different and what could not.

This difference between numbers and words is socially recognised in the two social decision institutions, the laboratory and the courtroom. A number-statement is send to the laboratory to be decided upon by asking the thing through a measurement, and the judgement of the laboratory is final and cannot be appealed. A word-statement is send to a courtroom to be decided upon by the majority of votes in a jury; but the judgement of a courtroom is not final and can always be appealed, either to a higher courtroom or to the parliament asking it to change the law.

Thus word-researchers are not researchers but counsellors in a courtroom of correctness. Modern word-researchers are counsellors for the prosecution accusing the defendant of being guilty of being something, e.g. a pencil, or unable to learn mathematics. And trying to produce certainty about its 'IS-claims'. Postmodern word-researchers are counter-researchers believing that no case can be proven. Hence counter-researchers always work as counsellors for the defence[ix] listening to the defendant through narratives, and cross-examining the witnesses of the prosecution through interviews to find a deference hiding a difference. The aim is to produce so much doubt, that the benefit of the doubt should make the defendant acquitted; e.g. by finding a hidden difference that can be shown to make a difference.

The Case: Evidence and Cross-examination

In this case the students are being accused of being unable to learn pre-calculus mathematics[x]. To prove its case the prosecution has presented the mathematics textbook, that the students cannot reproduce at the oral exam, and statistics showing that almost half of the students fail the written exam. In the pledge the prosecution asks that mathematics should be x-rated to students over 16, unless the students are able to demonstrate special talents.

As a postmodern counter-researcher I act as a counsellor for the defence. As my first witness I call a Danish high school graduate Barbie. Barbie is asked to tell about her mathematics education in her own words:

> In grade seven we were making graphs with negative and positive scales, how to draw them, and so when we asked why we made them, what purpose it kind of had, well you just had to make them, that's how it was. You didn't get any explanation as to the reality behind this mathematics. Our number two teacher, we had two different teachers that year, came in and was drunk as a lord. So we didn't learn very much.
>
> In the high school, where I had mathematics the first year, and I must say this was just what suited my head, at any case the teaching method was different, one I think should be spread out, for the teacher had a quite different way to explain, one you could understand. You really felt you learned something, even if it was difficult for you, you still learned it along the road. Even if you were a little behind, because first of all, you had a good relationship to the teacher, you felt the teacher was part of the class, not a separate part of the class thinking he has a higher authority. We really felt, the teacher was on the same level, as to authority any way, of course as to mathematics he was at a higher level. I do not know what I can explain about that method, anyway there was something about it that was incredibly attractive.

I can compare with mathematics the second year. The method, the teacher used the second year is simply one I find unsuitable and I know that many from the class agree. You felt precisely the opposite, the teacher was not so to speak a part of the class, you felt he was very authoritarian, he used his authority and taught directly from the book, and that helped us very little. When you go home and read the book and prepare your homework and then go back to school and say, that you don't understand it, the teacher explains it and mostly it helps only little for he explains it directly as it is in the book. He could have turned it, but he didn't.

Barbie describes different types of teachers. The last teacher ' taught directly from the book … he explains it directly as it is in the book. He could have turned it, but he didn't.' From this statement I will coin the words 'echo-teaching' and 'the Math-bible': The teacher enters the classroom, opens the textbook and begins to teach, but what he says is what is in the book. When he is asked to explain the book, he just repeats the book, thus practising 'echo-teaching'. And by just repeating the book even when asked to explain it, the 'teacher shows that there is only one textbook, the Textbook, the Bible of mathematics, the 'Math-bible'.

Since the teacher just echoes the textbook it helps little to cross-examine the teacher. Instead the defence will cross-examine the Math-bible.

Q: What is mathematics? A: Mathematics is what mathematicians do.

Q: Doesn't mathematics have a problem with this self-reference? A: Mathematics always uses self-reference, if not it cannot prove itself.

Q: What is the fundamental concept in mathematics? A: The set.

Q: What is a set? A: A set is a collection of well-defined elements.

Q: How can a definition be well-defined? A: A definition is well-defined through the elements of its corresponding solution set.

Q: Can a set have a set as an element? A: No problem

Q: But didn't Russell develop his type-theory to avoid the syntax-errors created by the self-reference when talking about sets of sets? A: Mathematics does not believe in type-theory.

Q: When was the concept set invented? A: The set was not invented, it was discovered around 1870.

Q: How can 5000 years of mathematics develop without its fundamental concept? A: That mathematics is not real mathematics.

Q: Why is the concept set so fundamental? A: Because all other mathematical concepts can be defined as examples of sets.

Q: What is the fundamental concept in high school mathematics? A: The function.

Q: What is a function? A: A function is an example of a set-relation with certain properties.

Q: Is it correct, that the function was invented around 1750? A: The year is correct. But again, the function was not invented, it was discovered.

Q: Is it correct, that in 1750 a function was defined differently, as a name for a calculation with a variable number, i.e. as an abstraction form examples instead of an example of an abstraction? A: Yes, but that function was not a real function, since it was not defined as an example of a set.

Q: Is it correct that all 9 mathematical operations +, -, *, /, ^, √, log, d/dx and ∫ were invented at least 50 years before the function was invented? A: Yes, but again they were not invented, they were discovered.

Q: Thank you, I have no further questions.

In this cross-examination the defence will look for deference hiding differences.

By defining a function as an example of a set, a 1700-concept is defined from a 1900-concept. This is turning the historical development upside down. A difference would be to respect the historical mathematics and the original definition of a function.

By defining a function as an example of a set, an abstract concept is defined as an example of a more abstract abstraction. A difference would be to define an abstract concept as an abstraction from less abstract examples, e.g. by using the original definition of a function, which defines a function as an abstraction from, and a name for, calculations with variable quantities.

The Math-bible does not consider historical mathematics for being real mathematics. A difference would be to consider historical mathematics for being real mathematics.

So from these observations it is possible to coin some words and formulate a counter-thesis.

To distinguish between historical mathematics and the Math-bible the defence will use the names ‘mathematics-from-below’ and ‘mathematics-from-above’. Mathematics-from-below is defining an abstract concept as an abstraction from examples. Mathematics-from-above is defining an abstract concept as an example of an abstraction. Also the defence will use the word ‘meta-matics’ for mathematics-from-above.

The defence can now formulate a counter-thesis: The prosecution accuses the students of being unable to learn mathematics. But all we can say is that the students are unable to learn ‘meta-matics’, since the Math-bible exposes them to meta-matics and not mathematics, and since the teacher practise echo-teaching. To find out if the students are unable to learn mathematics they first have to be exposed to mathematics and instead of meta-matics.

In order to expose the students to mathematics the defence has designed a micro-curriculum in mathematics-from-below covering only 20 lessons in order not to conflict with the official macro-curriculum.

The micro-curriculum recognises that the Arabic meaning of the word algebra is re-uniting, and that there are four different ways of uniting the world’s 2x2 constant and variable unit and per-numbers: The operations ‘+’ and ‘*’ unite variable and constant unit-numbers; and the operations ‘∫‘ and ‘^’ unite variable and constant per-numbers. The inverse operations ‘–’ and ‘/’ split a total into variable and constant unit-numbers; and the inverse operations ‘d/dx’ and ‘√ and log’ split a total into variable and constant per-numbers:

Calculations **Unite**/Divide Into	**Variable**	**Constant**
Unit-numbers $, m, s, ...	**T = a+n** $T-n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers $/m, m/100m = %, ...	**$\Delta T = \int f dx$** $\frac{dT}{dx} = f$	**T = a^n** $\sqrt[n]{T} = a$ $\log_a T = n$

Inspired by this perspective the traditional wording ‘linear and exponential functions’ can be reworded to ‘constant change’ emerging from questions as ‘100$ plus n days @ 5$/day total ? $’ and ‘100$ plus n days @ 5%/day total ? $’. Likewise ‘differential and integral calculus’ can be reworded to ‘variable change’ emerging from questions as ‘100$ plus n times @ (10%/n)/time total ? $’ and ‘100m plus 5 seconds @ 3m/sec increasing to 4 m/sec total ?m’.

This micro-curriculum has been tested with different teachers in 13 different classes. 5 of the classes were asked to express their satisfaction with specific parts of the curriculum on a scale with 7 degrees of satisfaction. Their overall satisfaction was 77%, 83%, 83%, 88% and 95% averaging 85%. The classes were not compared with control-groups.

All classes took a pre-test and a post-test. Performance data was obtained from 160 students showing the following distribution before and after the micro-curriculum.

Changes in performance	to High	to Medium	to Low	**Total**
from High (above 60%)	32	1	0	**33**
from Medium	52	4	1	**57**
from Low (below 40%)	50	20	0	**70**
Total	**134**	**25**	**1**	**160**

The table shows the following effects of the micro-curriculum: The category ‘High’ increased with around 100 students, half of which came from the category ‘Low’. The category ‘Medium’ was halved, and the category ‘Low’ was almost emptied.

To get a qualitative idea of the effect of the micro curriculum we can again listen to Barbie, who was exposed to mathematics-from-below in her first year at high school and mathematics-from-above the second year.

Describing her second year Barbie talks about authority. According to Barbie the teacher ‘taught directly from the book’. And when the students ask for an explanation, he ‘could have turned it, but he didn’t.’ Thus the teacher shows that neither he nor mathematics is the authority, the book is the authority, and that he intends to follow and be loyal to this authority no matter what. So the teacher is not teaching mathematics, he is preaching a bible, and teaching that is important to echo the book. Although this is a very clear message, the students instead feel that ‘the teacher was not so to speak a part of the class, you felt he was very authoritarian, he used his authority and that helped us very little’.

Talking about her first year Barbie does not even mention the textbook. So the authority is not in the textbook. Neither is it in the teacher since he is ‘part of the class, not a separate part of the class thinking he has a higher authority.’ The teacher is ‘on the same level, as to authority any way’. The teacher is recognised to be at a higher level as to mathematics, but since he is at the same level as to authority, the authority cannot be in mathematics, the authority is placed outside mathematics. Instead the teacher is helping the students by having ‘a quite different way to explain, one you could understand’.

In the middle school Barbie is missing an ‘explanation as to the reality behind this mathematics’, and ‘why we made them, what purpose it kind of had’. So here we see the authority that Barbie respects, ‘the reality behind this mathematics’.

Concluding Statement

The defence will now give its concluding statement. In this case the accused are the students of the pre-calculus mathematics class. The prosecution has presented the mathematics textbook that the students are unable to reproduce, and the prosecution has presented statistics showing that almost 50% fail the written exam. On this background the prosecution has asked the jury to vote for the decision, that the students are guilty of being unable to learn mathematics. With this decision the prosecution can advance a proposal students must pass an entrance test in order to be allowed into a pre-calculus mathematics class.

In short, the prosecution would like the jury to see the students as objects that are imperfect and resist improvement; the students are empty vessels that cannot be opened to be filled. So when the teacher teaches, only little mathematics enters into the vessels, the rest falls to the floor. And since the students are closed vessels impossible to fill, this waste of good teaching and good mathematics should be stopped.

As the jury knows, it is the task of the prosecution to prove beyond any reasonable doubt that its accusation is correct. And it is the task of the defence to produce reasonable doubt as to the correctness of this accusation. Hence the defence would like to present the jury with a counter-picture to the picture portraying the students as empty vessels resisting to be filled.

Maybe the students are not vessels; maybe the students are not passive objects, but active subjects, acting in the world to obtain a goal. Maybe there is a hidden rationality behind their apparent irrationality. If this is the case we can retell the story of the students in the form of the oldest form of tale, the fairy tale.

The structure of a fairy tale is that of a butterfly with two horizontal axes and one vertical. The vertical axis is the project axis showing that the subject has an object to achieve, a quest. The upper horizontal axis is the transport or communication axis showing that the object must be send from a sender to a receiver. The lower horizontal axis is the conflict axis showing that the subjects have both helpers and opponents in their quest.

In this case Barbie is the subject. When describing her first year at the high school, Barbie tells us 'the teacher had a quite different way to explain, one you could understand. You really felt you learned something'. Her we see, that Barbie has a rational project, she would like to learn something. And this project is fulfilled if the teacher acts as a helper providing explanations that will give an understanding to the students. However in her quest for such a communication she meets both helpers and opponents.

In the middle school Barbie meets two opponents. The first opponent is just able to perform mathematics as a ritual that the students just 'had to make'. And when the students asked why they made them and 'what purpose it kind of had' all the teacher could do was to refer to the ritual and say 'that's how it is'. This did not help the students to get what they wanted; they 'didn't get any explanation as to the reality behind this mathematics'. As a result Barbie's quest was unfulfilled since she 'didn't learn very much'.

The second opponent 'came in and was drunk as a lord'. Thus this opponent tries to escape from his obligation as a teacher by doping himself with alcohol. Again the effect on the students is negative since again 'we didn't learn very much'.

Also at the second year of high school Barbie meets an opponent, although he seems to be a helper. He has given the students a textbook, he is teaching from the textbook instead of just 'making graphs with negative and positive scales'. Also he gives the students assignments when asking them to 'go home and read the book and prepare your homework'. At home, however, the students run into problems, they 'don't understand it'. To solve this problem the students choose to take action: they 'go back to school and say, that [they] don't understand it.' Thus hoping that the teacher will be a helper by giving them a different explanation. And indeed 'the teacher explains it'; but he 'explains it directly as it is in the book. He could have turned it, but he didn't'. And the effect of such an explanation where the teacher just echoes the book is that 'mostly it helps only little'. Thus although he seems to be a helper the teacher is rather an opponent in disguise, like the wolf in the fairy tale 'Little Red Ridinghood'.

So all three opponents try to escape their teacher obligation, one by drinking, the other two by just echoing the ritual of the classroom. In one case the ritual is teaching standard techniques as 'making graphs with negative and positive scales', in the other case the ritual is to 'explain it directly as it is in the book'. In all cases Barbie's quest is unsuccessful, since 'we didn't learn very much' in the middle school, and 'that helped us very little' in the high school.

However Barbie also meets other teachers on her quest: 'In the high school, where I had mathematics the first year, and I must say this was just what suited my head.' This teacher is a helper helping Barbie to fulfil her project to learn, since he 'had a quite different way to explain, one you could understand. You really felt you learned something, even if it was difficult for you, you still learned it along the road.'

Through Barbie's evidence we learn, that the students are not passive objects, they are not empty vessels that are malfunctioning by being closed and thus impossible to fill. Instead the students are active subjects, agents who have a project to fulfil; and even if it is a difficult project because of many opponents, in the end it might be successful if they meet a helper.

But what is a helper within mathematics education?

Barbie tells us that a helper should have 'a quite different way to explain, one you could understand' when the students ask 'what purpose it kind of has' and ask for an 'explanation as to the reality behind this mathematics', or when they 'go home and read the book and prepare [their] homework and then go back to school and say, that [they] don't understand it'.

Also Barbie tells us that the question of authority is of importance: It is good when 'you felt the teacher was part of the class, not a separate part of the class thinking he has a higher authority'. And it is bad if 'You felt precisely the opposite, the teacher was not so to speak a part of the class, you felt he was very authoritarian, he used his authority and taught directly from the book.'

So where is the authentic rational authority, if the authority of the book is felt to be restrictive[xi]? To get an answer to this question we turn to the statements of the textbook, in which the teacher places so much authority that he chooses to become its echo, the Math-bible. The Math-bible admits that it defines abstract concepts, not historically correct as abstractions from examples, but as the opposite, as examples from abstractions. Thus the Math-bible has turned historical mathematics upside down and placed the authority inside mathematics' itself at the highest and youngest abstraction level with the concept of set. By building on self-reference this authority makes mathematics something that is performed for its own sake, a ritual.

The practise of defining abstractions as examples of higher abstractions is hiding a difference, which is to respect the historical development of mathematics by defining abstractions as abstractions from examples of lower abstractions. This alternative places the authority outside mathematics itself in the social practices that gave birth to geometry and algebra, i.e. in earth-measuring and in reuniting quantities. This authority is an authentic authority since it gives an authentic answer to the student asking for an 'explanation as to the reality behind this mathematics'.

To validate this fairy tale of the students' quest the defence has exposed the students to a micro-curriculum in mathematics-from-below. The effect of this counter-curriculum was very positive, changing a situation with half of the students in the low performance category, by moving these students to the medium and high performance category in the ratio 2:5.

Now the defence will rest its case. Through its evidence the defence has been able to provide, not only serious doubt, but also the conditions under which it becomes evident, that the students are not guilty of the accusation of being unable to learn mathematics. The students want to learn mathematics and they are able to do so if they are provided with teachers helping them to understand mathematics by explaining it from below from its source of authority, from the reality behind mathematics, from the social questions that created geometry and algebra 'how do we divide our land and its products?'

So the prosecution should drop the case against the students. Instead the prosecution should consider a new case against the modern institutions making it impossible to change the meta-matics of the Math-bible to student-mathematics from below.

In his famous book on 'Modernity and the Holocaust' the British sociologist Zygmunt Bauman mentions three core traits of the modern society, which can transform ordinary people into perpetrators. The three core traits are authorisation by official orders coming from legally entitled quarters; routinisation by rule-governed practices and exact specifications of roles; and dehumanisation through ideological definitions and indoctrinations[xii].

From this perspective we can see the ideological definitions of modern meta-matics indoctrinate the teachers to dehumanise the human learners of the classroom to examples of passive empty vessels resisting to be filled with knowledge; thus hiding the counter-picture of the humans as agents engaged in a rational quest for learning.

And teachers performing echo-teaching could be seen as an example of a routinisation by rule-governed practices and exact specifications of roles, where the teachers are supposed to echo the book and to persuade the students to do the same in order not to fail the exam.

And the teachers' echo-teaching and neglect of the students' asking for explanation could be seen as an example of an authorisation by official orders coming from legally entitled quarters, from the educational institution using self-reference to legitimise that the goal of mathematics education is to learn mathematics, and that mathematics is what mathematicians do.

All in all we can now see how the modern logocentricity in its most extreme form as self-reference can use the three core traits of modernity, authorisation, routinisation and dehumanisation, to transform education to 'seducation' seducing the teachers to change mathe-matics to meta-matics, thus perverting the enlightenment classroom of mathematics into a 'gas-chamber' of meta-matics, which the students naturally seek to escape from. Not because they are empty vessels irrationally avoiding the knowledge of 5000 years of accumulated quantitative knowledge, but because they are rational agents avoiding to be poisoned by the temporary infection of meta-matics.

Once closed, this case of the hidden rationality of the irrational students should be taken as a token that perhaps the time has come to transform modernity into post-modernity. This case makes it evident that a stop should be put to the era, where an institution is basing its legitimacy on logocentricity believing it is producing knowledge for the benefit of its clients, when instead it is producing clients for the benefit of its knowledge. A new era should be initiated where institutions are build upon scepticism always willing to learn from negotiating its curriculum by seeing the students as human agents engaged in a rational quest for learning and needing the curriculum to be a helper and not an opponent.

To trigger this process of transition from modern meta-matics to postmodern mathematics privately financed academies will have to be instituted to educate tomorrow's teachers in mathematics to follow today's teachers of meta-matics when they retire within the next ten years unable to reproduce themselves.

Conclusion

This paper has addressed the exodus-problem in modern mathematics education witnessing its students turning their back to mathematics and mathematical based educations. Being an example of postmodern counter-research this paper cannot report on what the solution to exodus-problem is. Instead the paper can point to hidden differences that might make and has made a difference. The paper suggests that the exodus-problem is perhaps an 'echodus'-problem making the students turn away when their asking for an explanation about the reality behind mathematics is answered by echoing a textbook, which by building on self-reference has transformed historical mathematics into ahistorical meta-matics.

References

Bauman Z. (1989) *Modernity and the Holocaust*, Cambridge UK: Polity Press

Dreyfus, H.L. & Rabinow, P. (1982) 2. ed. *Michel Foucault, beyond structuralism and hermeneutics,* Chicago: University of Chicago Press

Fromm E. (1941) *Escape from Freedom*, New York: Farrar & Rinehart

Jensen J. H., Niss M. & Wedege T. (1998) *Justification and Enrolment Problems in Education Involving Mathematics or Physics*, Roskilde: Roskilde University Press

Kline M. (1973) *Why Johnny can't add, the failure of the new math*, New York: Vintage Books

Lyotard J. (1984) *The postmodern Condition: A report on Knowledge*, Manchester: Manchester University Press

Russell B. (1945) *A History of Western Philosophy,* New York: A Touchstone Book

Tarp, A. (2001) 'Postmodern Counter-research Uncovering and Curing an 'Echodus' phenomena*, in *Proceedings of the Norma 01 Conference,* Kistianstad, Sweden
Zybartas, S. and Tarp, A. (2001) 'Postmodern Rehumanised Mathematics in Teacher Education, a Co-operation between Lithuania and Denmark', in *Proceedings of the Norma 01 Conference*, Kristianstad, Sweden
Van Maanen, J. (1988) *Tales of the field,* Chicago: Chicago University Press

---

[i] As a teacher I have been together with students for many years. Since this tale is about listening to the students I have chosen to write in an ethnographic genre. As to ethnography J. Van Maanen writes 'An ethnography is written representation of a culture' (Van Maanen 1988:1). In his book he describes three different ways of writing ethnography, the realist tale, the confessional tale and the impressionist tale carrying 'elements of both realist and confessional writing' (7). The realist tale 'provide a rather direct, matter-of-fact portrait of a studied culture, unclouded by much concern for how the fieldworker produced such a portrait' (7). The confessional tale has become 'an institutionalised and popular form of fieldwork writing' (91). Confessional work rests on 'a fundamental turning point in American social thought. No longer is the social world … to be taken for granted as merely out there as full of neutral, objective, observable facts. … Rather, social facts … are human fabrications … Fieldwork constructs now are seen by many to emerge from a hermeneutic process; fieldwork is an interpretive act, not an observational or descriptive one' (93). Since I have based my study on scepticism it has been natural to choose the sceptical genre and choose the confessional tale.

[ii] See Tarp 2001

[iii] In Denmark the high school teachers in mathematics, physics and chemistry have one yearly joint conference. Although the conference has parallel sessions, there is one session per subject. Typically researchers from the university are invited to give an update on the latest development within their subject. Thus the teachers use the conference to recreate the university instead of discussing actual educational problems in the school. At the elementary level the teachers have yearly conferences on mathematics including parallel sessions focusing on educational issues. However the two groups of teachers never have joint conferences as e.g. in Sweden. They are educated at different institutions. The high school teachers are not educated as teachers but as researchers that become teachers if they cannot of continue for a PhD degree after their master exam. The elementary school teachers get a 4-year education at a separate teacher school called a 'seminarium' allowing them to teach from grade 1-10. In other countries teachers have to have two 4-year educations to teach both in primary and in secondary school, but not in Denmark. So Danish students are educated for nine years by persons having two half teacher educations and for three years by a person without a teacher education.

[iv] See Zybartas et al 2001

[v] See e.g. Jensen et al 1998

[vi] One description of the problems with set-based mathematics can be found in Kline 1973

[vii] Institutional scepticism is part of the democratic IDC-process of information, debate and decision by making a distinction between information and debate, between natural and political correctness and authority. This distinction between 'physis' and 'nomos' was first made by the ancient Greek sophists as e.g. Antiphon saying that the command of the law is chosen, while the command of nature rests on necessity. Institutional scepticism holds that everything could be otherwise except for the few examples of natural correctness discovered by Pythagoras and Newton, showing that sounds, geometry and motion follow metaphysical number-laws. And what could be otherwise should be decided on the basis of equal authority, i.e. democracy. However in order to participate in a democracy people should be informed about what is natural correct and what can be debated. Thus democracy builds upon enlightenment. However the history of the first two democracies of the enlightenment is completely different. In the US the enlightenment developed into pragmatism and symbolic interactionism and the US still has their first republic. France now has its 5th republic, showing that the 'pastoral power' of Foucault (Dreyfus 1982: 213-215) is much stronger in France than in the US. Thus there are two kinds of institutional scepticism, an American in the form of pragmatism and symbolic interactionism leading to grounded theory; and a French in the form of post-structuralism and post-modernism leading to counter-research.

[viii] The term 'logocentricity' was coined by Derrida. Lyotard defines modern as 'any science that legitimates itself with reference to a metadiscourse' and postmodern as 'incredulity towards metanarratives' (Lyotard 1984: xxiii-xxiv). This paper includes Foucault's term 'pastoral power' (ibid.) by saying 'postmodernism means scepticism towards pastoral power'.

[ix] Russell comments on the similarity between sophists and lawyers: 'Broadly speaking, they [the sophists] were prepared, like modern lawyers, to show how to argue for or against any opinion, and were not concerned to advocate conclusions of their own' (Russell 1945: 78). However a postmodern counter-researcher is only arguing against.

[x] Pre-calculus mathematics deals with quantities showing a constant growth by having added a constant number or a constant percentage. In calculus non-constant numbers are added. Pre-calculus is introduced around grade 10.

[xi] For rational and restrictive authority se Fromm 1941

[xii] See Bauman 1989: 21

Mathematism and the Irrelevance of the Research Industry

A Postmodern LIB-free LAB-based Approach to our Language of Prediction

Mathematics education research increases together with the problems it studies. This irrelevance-paradox can be solved by using a postmodern sceptical LAB-research to weed out LIB-based mathematism coming from the library in order to reconstruct a LAB-based mathematics coming from the laboratory. Replacing indoctrination in modern set-based mathematism with education in Kronecker-Russell multiplicity-based mathematics turns out to be a genuine 'Cinderella-difference' making a difference in the classroom.

The Irrelevance Paradox

All over the world there seems to be a crisis in mathematics education:

There are strong indications of increasing justification and enrolment problems concerning mathematics and physics education, as a rather international phenomenon. During recent years, reports of a significant decline in enrolments to tertiary level education involving mathematics and physics have appeared from many parts of the world, including many countries in Europe, the US, Australia, and Japan. Also at the primary and secondary school levels mathematics and physics in many countries now seem to be receiving less interest and motivation than before amongst many categories of pupils. (Jensen et al, 1998: 15)

In Japan Yukihiko Namikawa asks 'can college mathematics in Japan survive?'

> Actually the total education system in Japan is in crisis, and so is the case of mathematics education at universities. (..) we are facing a remarkable decline of mathematical knowledge and ability of fresh students. (..) In April 1994, we established a working group in the Mathematical Society of Japan to overcome this crisis. (..) So far we made several investigations to clarify the situations. The results were much more disastrous than imagined before start and still the problems are aggravating. (Namikawa in ICME9, 2000: 94)

In Denmark proposals have been made to remove pre-calculus as a compulsory subject: In their suggestions for a reform of the Danish upper secondary Preparation High School the teacher union writes that Danish must be strengthened to improve the student's ability to write and read; that English must be compulsory and so must a second foreign language; and that all students must have a basic competence in mathematics, but not all students need to take an exam in mathematics.

Mogens Niss has formulated a 'relevance paradox'

> The discrepancy between the objective social significance of mathematics and its subjective invisibility constitutes one form of what the author often calls the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics. (Niss in Biehler et al, 1994: 371).

The 10th International Congress on Mathematical Education in 2004 shows that research in mathematics education has been going on for almost half a century. On this background I would like to supplement this 'relevance paradox' with an 'irrelevance paradox' or 'inflation paradox': 'the output of mathematics education research increases together with the problems it studies - indicating that the research in mathematics education is irrelevant to mathematics education'.

A Methodology: Institutional Scepticism, Sceptical LIB-free LAB-Research

To get an answer to the 'irrelevance paradox' we obviously have to use a counter-methodology. Historically research originated as bottom-up 'LAB-LIB research' where the LIB-statements of the library are induced from and validated by reliable LAB-data from the laboratory. However the word-based 'LIB-research' has created a 'LIB-LAB war' or 'science-war' exemplified by 'Sokal's bluff' or by the 'number&word-paradox': Placed between a ruler and a dictionary a thing can point to a number but not to a word, so a thing can falsify a number-statement in the laboratory but not a word-statement in the library; thus numbers are reliable LAB-data able to carry research, whereas words carry interpretations, which presented as research becomes seduction - to be lifted by the counter-seduction of sceptical LIB-free LAB-research replacing LIB-words with LAB-words being validated by being, not ‘truth’, but ‘Cinderella-differences’ making a difference. (Tarp 2003)

The inflation in today’s LIB-research comes from library cells inhabited by persons with little or no practical classroom experience, which reminds of the production of scholastic scriptures in medieval monasteries. So a proper counter-methodology could be inspired by counter-scholasticism as e.g. the institutional scepticism of the enlightenment as it was implemented in its two democracies, the American in the form of pragmatism, symbolic interactionism and grounded theory, and the French in the form of post-structuralism and post-modernism.

In America Blumer talks about practical experience, symbolic interactionism and research:

> I merely wish to reassert here that current designs of ‘proper’ research procedure do not encourage or provide for the development of firsthand acquaintance with the sphere of life under study. Moreover, the scholar who lacks that firsthand familiarity is highly unlikely to recognize that he is missing anything. Not being aware of the knowledge that would come from firsthand acquaintance, he does not know that he is missing that knowledge. (..) Respect the nature of the empirical world and organize a methodological stance to reflect that respect. This is what I think symbolic interactionism strives to do. (..) Sociological thought rarely recognizes or treats human societies as composed of individuals who have selves. Instead they assume human beings to be merely organisms with some kind of organization, responding to forces which play upon them. (Blumer, 1998: 37-38, 60, 83)

America still has its first republic whereas France now has its fifth republic. The American settlers emigrated to avoid the feudal institutions of Europe and to install ‘freedom under God’. So what Foucault calls ‘pastoral power’ was not present in America; but very much present both inside France and around it, and several revolutions had to be fought forcing the French republic to organise the state as a military camp where French philosophers has developed a special sensitivity towards any attempt to overthrow the democracy of ‘la Republique’.

Thus the French institutional scepticism is quite different from the American by turning the question of representation upside down and focusing upon, not how outside structure installs internal concepts, but how internal concepts install outside structure; and how words can be used as counter-enlightenment to patronise and ‘clientify’ people by installing pastoral power.

Derrida calls the belief that words represent the world for ‘logocentrism’. Lyotard defines modern as ‘any science that legitimates itself with reference to a metadiscourse’; and postmodern as ‘incredulity towards metanarratives’ (Lyotard, 1984: xxiii-xxiv). Foucault describes pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1982: 213, 215)

In this way Foucault opens our eyes to the salvation promise of the generalised church: ‘You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction centre, the hospital, and do exactly what we tell you’.

So according to Foucault pastoral power comes from words installing an abnormality and a normalizing institution to cure this abnormality through new words installing a new abnormality etc. (Foucault 1995). Thus the pastoral word ‘educate’ installs the ‘un-educated’ to be ‘cured’ by the institution ‘education’; failing its ‘cure’ it is ‘cured’ by the institution ‘research’ installing new ‘scientific’ words as ‘competence’ installing the ‘in-competent’ to be ‘cured’ by the institution ‘competence development’; failing its ‘cure’ it is again being ‘cured’ by new ‘research’ installing new ‘scientific’ words etc.

Thus pastoral power is installed by a self-supporting top-down LIB-LAB-industry of research and education using self-created LAB-problems to invent new ‘scientific’ LIB-words that are exported to the LAB through master educated inspectors creating new problems funding new research etc.

To increase its productivity the LIB has replaced verb-based words as ‘educate’ with words that are not verb-based such as ‘competence’. So where the ‘clients’ themselves knew when they were ‘educating’ themselves or others, they do not know when they are ‘competencing’ themselves or others, only the pastors know – in full accordance with the view of the inventor of pastoral power, Plato, arguing that the democracy of the sophists should be replaced by the autocracy of the ‘philo-sopists’ educated at Plato’s academy.

By its distinction between words and numbers sceptical LIB-free LAB-research is inspired by the French postmodern scepticism by saying that ‘postmodernism means institutional scepticism towards the pastoral power of words’; and by the ancient Greek sophists always distinguishing between necessity and choice, between natural and political correctness, between logos and nomos, according to the two prerequisites of democratic decisions: information and debate. Thus Plato’s half brother the sophist Antifon writes:

> Correctness means not breaking any law in your own country. So the most advantageous way to be correct is to follow the correct laws in the presence of witnesses, and to follow nature’s laws when alone. For the command of the law follows from arbitrariness, and the command of nature follows from necessity. The command of the law is only a decision without roots in nature, whereas the command of nature has grown from nature itself not depending on any decisions. (Antifon in Haastrup et al 1984: 82, my tranlation).

By transforming seduction back into interpretation scepticism transforms the library from a hall of fact to a hall of fiction to draw inspiration from, especially from the tales that have been validated by surviving through countless generations, the fairy tales. Hence the preferred interpretation genre in institutional scepticism is the fairytale. Grounded theory uses categorised LAB-data for axial ‘fairytale-coding’. Sceptical LIB-free LAB-research looks into institutional LAB-texts to replace opponent LIB-words with proponent LAB-words found by discovering forgotten or unnoticed alternatives at different times and places inspired by the genealogy and archaeology of Foucault; and by inventing alternatives using sociological imagination inspired by Mills (Mills 1959).

The aim of sceptical LIB-free LAB-research is not to extend the existing seduction of the library, so no systematic reference to the existing ‘research’ literature takes place. The aim is to solve LAB-problems by searching for hidden Cinderella-differences in the LAB, i.e. by 1) identifying the pastoral LIB-word installing the problem 2) renaming the LIB-word to a LAB-word through discovery and imagination, 3) testing the LAB-word to see if it is a Cinderella-differences making a difference, and 4) publish the alternative so the problem can be decreased instead of increased.

Mathematics and Mathematism

Mathematics education is an institution instituted to cure ‘mathematical uneducated-ness’. Not being verb-based ‘mathematics’ is a LIB-word to be translated into a verb-based LAB-word by observing what goes on in the laboratory of mathematics education, the classroom. The first day of secondary school we witness a ‘fraction test’ as e.g.:

The teacher	The students
Welcome to secondary School! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

Apparently we have a ‘fraction-paradox’ in the mathematics classroom:

Inside the classroom	20/100 = 20%	+ 10/100 = + 10%	= 30/100 = = 30%
Outside the classroom e.g. in the laboratory	20%	+ 10% or	= 32% in the case of compound interest = b% (10<b<20) in the case of the total average

20% of 300 + 10% of 300 = (20%+10%) of 300 = 30% of 300 since the common total 300 can be put outside a parenthesis. But the fraction-paradox shows that this is not always the case.

So 20/100 = 20%, but no general rule says that 20%+10% = 30% or 20/100+10/100 = 30/100.

Since a part of mathematics cannot be validated outside the classroom we can distinguish between ‘mathematics’, which is a science that can be validated in the laboratory, and ‘mathematism’, which is a doctrine, an ideology, a scholasticism, that cannot be validated in the laboratory.

This gives a possible answer to the irrelevance paradox: What is disguised as ‘education in mathematics’ is really indoctrination in ‘mathematism’ teaching ‘killer-mathematics’ only existing inside classrooms, where it kills the relevance of mathematics.

As validation a killer-free LIB-free LAB-mathematics must be uncovered through a combination of concept archaeology and imagination and tested in the laboratory of learning i.e. the classroom.

Fractions and Sets - LIB-words or LAB-words?

In the laboratory we talk about ‘fractions of’ as e.g. 2/3 of 6. The textbook however talks about plain ‘fractions’ as e.g. 2/3. To see if this is a LIB-word or a LAB-word we look at its definition:

The set Q of rational numbers is defined as a set of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [sets of equivalence sets in a product set of two sets of [Peano-numbers]]]; such that the number (a,b) is equivalent to the number (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same rational number ½. (See any textbook in modern mathematics, e.g. Griffith et al 1970)

Since fractions are defined as examples of ‘sets’ the question is whether ‘set’ is a LIB-word or a LAB-word. To separate LIB-math from LAB-math we travel back in time in the mathematics laboratory. As to the prospects for the enlightenment eighteenth century, Morris Kline writes:

> The enormous seventeenth-century advances in algebra, analytic geometry, and the calculus; the heavy involvement of mathematics in science, which provided deep and intriguing problems; the excitement generated by Newton's astonishing successes in celestial mechanics; and the improvement in communications provided by the academies and journals all pointed to additional major developments and served to create immense exuberance about the future of mathematics. (..) The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 398-99)

So the enormous creativity in seventeenth-century mathematics was a result of neglecting the LIB-restrictions of classical Greek geometry by practising ‘a complete disregard of logical scruples’ and instead being inspired by the laboratory’s ‘physical insight’ and ‘confidence of intuition’.

> If the seventeenth century has correctly been called the century of genius, then the eighteenth may be called the century of the ingenious. Though both centuries were prolific, the eighteenth-century men, without introducing any concept as original and as fundamental as the calculus, but by exercising virtuosity in technique, exploited and advanced the power of the calculus to produce what are now

> major branches (..) Far more than in any other century the mathematical work of the eighteenth was directly inspired by physical problems. In fact one could say that the goal of the work was not mathematics, but rather the solution of physical problems. (..) The physical meaning of the mathematics guided the mathematical steps and often supplied partial arguments to fill in nonmathematical steps. The reasoning was in essence no different from a proof of a theorem of geometry, wherein some facts entirely obvious in the figure are used even though no axiom or theorem supports them. Finally, the physical correctness of the conclusions gave assurance that the mathematics must be correct. (..) Lagrange wrote to d'Alembert on September 21, 1781, 'It appears to me also that the mine [of mathematics] is already very deep and that unless one discovers new veins it will be necessary sooner or later to abandon it. Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities'. (..)This fear was expressed even as early as 1754 by Diderot in Thoughts on the Interpretation of Nature: ' I dare say that in less than a century we shall not have three great geometers [mathematicians] left in Europe. This science will very soon come to a standstill (..) We shall not go beyond this point.' (614, 616, 617, 623)

The seventeenth century saw the arrival of the last form of calculations, calculus, and the eighteenth century developed the many LAB-applications of calculus within physics. Only little new mathematics was added; and around 1800 mathematicians felt that there was no more mathematics to develop as expresses by Diderot. However LIB-mathematics soon came back. In spite of the fact that calculus and its applications had been developed without it logical scruples now were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. These LIB-scruples lead to the introduction of 'set'. So as numbers were introduced to distinguish between different degrees of multiplicity having 1 as its unit, sets were introduced to distinguish between different degrees of infinity having the natural numbers as a unit. However changing infinity from a quality to a quantity involves the question of actual and potential infinity:

> The central difficulty in the theory of sets is the very concept of an infinite set. Such sets had naturally come to the attention of mathematicians and philosophers from Greek times onward, and their very nature and seemingly contradictory properties had thwarted any progress in understanding them. Zeno's paradoxes are perhaps the first indication of the difficulties. Neither the infinite divisibility of the straight line nor the line as an infinite set of discrete points seemed to permit rational conclusions about motion. Aristotle considered infinite sets, such as the set of whole numbers, and denied the existence of an infinite set of objects as a fixed entity. For him, sets could be only potentially infinite. (..) Cauchy, like others before him, denied the existence of infinite sets because the fact that a part can be put into one-to-one correspondence with the whole seemed contradictory to him. The polemics on the various problems involving sets were endless (992-993)

Kronecker objected to set theory and Russell objected to talking about sets of sets:

> A radically different approach to mathematics has been undertaken by a group of mathematicians called intuitionists. As in the case of logicism, the intuitionist philosophy was inaugurated during the late nineteenth century when the rigorization of the number system and geometry was a major activity. The discovery of the paradoxes stimulated its further development. The first intuitionist was Kronecker, who expressed his views in the 1870s and 80s. To Kronecker, Weierstrass's rigor involved unacceptable concepts, and Cantor's work on transfinite numbers and set theory was not mathematics but mysticism. Kronecker was willing to accept the whole numbers because these are clear to the intuition. These 'were the work of God.' All else was the work of man and suspect. (..) after the paradoxes were discovered, intuitionism were revived and became a widespread and serious movement. The next strong advocate was Poincaré. (..) He agreed with Russell that the source of the paradoxes was the definition of collections of sets that included the object itself. Thus the set A of all set contains A. But A cannot be defined until each member of A is defined, and if A is one member the definition is circular. (..) This idea that the whole numbers derive from the intuition of time has been maintained by Kant, William R. Hamilton in his 'algebra as a Science of Time,' and the philosopher Arthur Schopenhauer. (1197-1200).

As to the paradoxes in set-theory even Cantor saw problems asking Dedekind in 1899 whether the set of all cardinal numbers is itself a set; because if it is, it would have a cardinal number larger than any other cardinal (1003). Another paradox is the Russell paradox showing that self-reference leads to contradiction, as in the classical liar-paradox ‘this sentence is false’, when talking about sets of sets as e.g. the set M of all sets that are not a member of themselves:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

Russell solves this paradox by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Since fractions are defined as sets of sets of numbers they cannot be considered numbers themselves making the addition ‘2+3/4’ meaningless. Not wanting a fraction-problem modern LIB-mathematics has chosen to neglect Russell’s type-theory until computer language, needing to avoid syntax errors, has brought a renaissance to Russell’s type-theory.

To avoid the type-theory Zermelo and Fraenkel invented an axiom system making self-reference legal by not distinguishing between an element of a set and the set itself, which removes the distinction between examples and abstractions and between different abstraction levels thus hiding that mathematics historically developed through layers of abstractions; and hiding the difference between an object and its predicate or interpretation means subscribing to the logocentrism criticised by poststructuralist thinking and by the number&word-paradox.

So ‘set’ is a LIB-word derived from axioms and not abstracted from the LAB. Since the definitions of modern mathematics are based upon the concepts set, this ‘LIB-virus’ makes all definitions LIB-words different from the LAB-words of the historical LAB-definitions. Thus we can name modern LIB-based mathematics ‘meta-matics’ to distinguish it from historical LAB-based ‘mathe-matics’.

The difference between LIB-based meta-matics, LIB-MATH, and LAB-based mathe-matics, LAB-MATH, can be seen in the word ‘function’ defined by modern meta-matics as ‘an example of a set of ordered pairs where first-component identity implies second-component identity’; and defined by Euler in 1748 as a common name for calculations with a variable quantity:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1988:3)

Bringing LAB-based Mathematics to a LIB-based Academy

A LAB-based mathematics should respect two fundamental principles: A Kronecker-principle saying that only the natural numbers can be taken for granted. And a Russell-principle saying that we cannot use self-reference and talk about sets of sets. The appendix shows an example of a Kronecker-Russell mathematics based on the LAB-words ‘repetition in time’ and ‘multiplicity in space’ creating a LIB-free, set-free, fraction-free and function-free ‘Count&Add-laboratory’ where addition predicts counting-results making mathematics our language of prediction (.#1)

This multiplicity-based mathematics makes a difference in the Danish pre-calculus classroom (Tarp 2003), in teacher education in Eastern Europe (Zybartas et al 2001) and in Africa (Tarp 2002). Thus the irrelevance paradox can be solved if set-based mathematism is replaced by multiplicity-based mathematics. But as a pastoral power LIB-based research is interested in, not solving, but guarding the fundraising irrelevance paradox by continuing to research the indoctrination of mathematism instead of researching the education of mathematics.

To test this hypothesis I applied for a job at a LIB-based academy. The verdict of the committee (#2) shows that challenging LIB-based meta-matics with LAB-based mathematics is not considered an asset; you are only admitted to a LIB-based academy if you are loyal to its interpretation and willing and able to expand it even if it is seduction and irrelevant to the field it studies. Hence to solve the irrelevance paradox an alternative sceptical LAB-based academy has to be installed.

The MATHeCADEMY and PYRAMIDeDUCATION

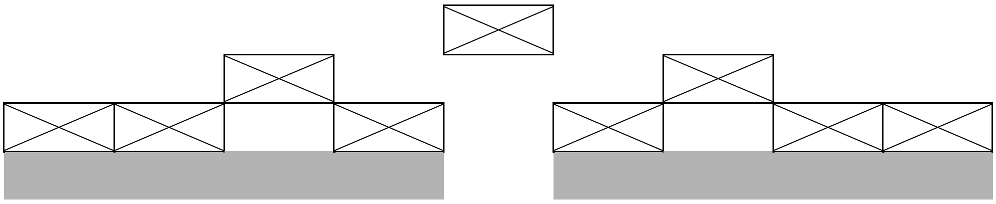
MATH**e**CADEMY.net is an example of an alternative sceptical LAB-based academy building on the sophist distinction between choice and necessity; and solving the irrelevance paradox by introducing a count&add laboratory posing the educational questions: ‘How to count and predict multiplicity in bundles and stacks? How to unite stacks and per-numbers?’; thus respecting that ‘reuniting’ is the original meaning of the Arabic word ‘algebra’.

At the MATH**e**CADEMY Primary school mathematics is learned through educational sentence-free meetings with the sentence-subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic ‘grasp-to-grasp’ learning.

Secondary school mathematics is learned through educational sentence-loaded fairy tales abstracted from and validated in the laboratory, i.e. through automatic ‘gossip-learning’.

In PYRAMID**e**DUCATION 8 student teachers are organised in 2 teams of 4 students choosing 3 pairs and 2 instructors by turn. The coach coaches the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In each pair each student corrects the other student’s routine-assignment. Each pair is the opponent on the essay of another pair. Each student pays for the education by coaching a new group of 8 students.

1 coach
2 instructors
3 pairs
8 students in 2 teams



In this way multiplicity-based mathematics will multiply as a self-reproducing virus on the Internet, until it can surface in ten years when half of the mathematics teachers have retired unable to reproduce by failing to make set-based mathematism relevant to the mathematics students.

References

Biehler R Scholz R W Strässer R & Winkelmann B (1994) *Didactics of Mathematics as a Scientific Discipline*, Dordrecht: Kluwer Academic Press

Blumer H (1998) *Symbolic Interactionism,* Berkely, Ca.: University of California Press

Dreyfus H L & Rabinow P (1982) 2. ed. *Michel Foucault, beyond structuralism and hermeneutics,* Chicago: University of Chicago Press

Euler L (1988): *Introduction to Analysis of the Infinite*, New York: Springer Verlag

Foucault M (1995) *Discipline & Punish*, New York: Vintage Books

Griffiths H B & Hilton J P (1970) *A Comprehensive Textbook of Classical Mathematics,* London: Van Nostrand Reinhold Company

Gymnasieskolernes Lærerforening (2001) *GLs hf-politik*, 2. sektion af Gymnasieskolen nr. 11

Haastrup G & Simonsen A (1984) *Sofistikken*, København: Akademisk forlag

ICME9 (2000) *Abstracts of Plenary Lectures and Regular Lectures*, Tokyo/Makuhari, Japan

Jensen J H, Niss M & Wedege T (1998) *Justification and Enrolment Problems in Education Involving Mathematics or Physics*, Roskilde: Roskilde University Press

Kline M (1972) *Mathematical Thoughts from Ancient to Modern Times,* New York: Oxford University Press

Lyotard J (1984) *The postmodern Condition: A report on Knowledge,* Manchester: Manchester University Press

Mills C W (1959) *The Sociological Imagination,* Oxford: Oxford University Press

Tarp A (2002) *Killer-Equations, Job Threats and Syntax Errors*, in C. Bergsten, G. Dahland & B. Grevholm Research and Action in the Mathematics Classroom, Linkoping, SMDF

Tarp A (2003) *Student-Mathematics versus Teacher-Metamatics*, ECER 2003, Hamburg, http://www.leeds.ac.uk/ educol/documents/00003264.htm.

Zybartas S, Tarp A (2001) *Postmodern Rehumanised Mathematics in Teacher Education, a Co-operation between Lithuania and Denmark*, Proceedings of the Norma 01 Conference, Kristianstad College, Kristianstad, Sweden

Notes

#1)Through his successor-principle Peano is forcing an additive structure upon the natural numbers seducing us to believe that 2+2 = 4. However this is an example of killer-mathematics, since outside the classroom we meet many examples where 2+2 is not 4: 2*meter + 2*cm = 202*cm, 2*week + 2*day = 14*day, 2*ten + 2*one = 22*one etc.

As we can see the numbers here are per-numbers and should be added accordingly, as the integration formula 'T2 = T1 + ∫a*dx' tells us. I.e. they have to be transformed to totals first; then they can be added, but only inside a parenthesis securing that the units are the same: T= 2 3s + 4 5s= 2*3 + 4*5= 6*1 + 20*1= (6 + 20)*1= 26*1= 26/3*3= 8 2/3*3= 26/5*5= 5 1/5*5. So in this case 2+4 can give both 26, 8 2/3 and 5 1/5. Thus 2 3s + 4 5s is not 6 8s; whereas 2 3rds + 4 5ths = 6 8ths in the case of e.g. 3 and 5 bottles: 2/3*3+4/5*5 = 2+4 = 6 = 6/8*8.

Hence there is a need for a 'Peano II' giving the natural numbers a multiplicative structure so they will represent directly what they describe, i.e. stacks. And so that mathematical knowledge can grow out of the count&add-laboratory, where rules are generalised through induction and validated by counting the deduced predictions. This leads to a new kind of natural numbers, stack-numbers always having the form T = a*b = (a,b). A relation can be set up identifying stacks with identical totals by saying that the stacks (a,b) and (c,d) are identical if a*b*1 = c*d*1 as e.g. (2,6) and (3,4).

Thus a natural number becomes an equivalence class in the set of stacks where n = (a,b) if n = a*b*1 as e.g. 8 = (2,4) since 8 = 2*4*1. The natural numbers then becomes the total 'area' of a stack; identical numbers occur though a re-bundling of their stacks; and prime numbers are stacks that cannot be rebundled. This stack-representation of the natural numbers is what Kuhn calls a new paradigm. It remains to be seen if number theory will look different within this stack-paradigm, and whether special problems as Fermat's last theorem will be easier to solve within this stack-paradigm.

Reformulated as stacks the Fermat theorem a^n + b^n = c^n becomes a^n = c^n – b^n. Here a^n is an n-dimensional stack, an n-stack. And c^n – b^n is a binomial that, to become an n-stack, has to factorised as a combination of n basic binomials of the form (c-b) or (c+b). For n=2 the 2 basic polynomials can contain different signs, making it possible to reduce the product of two binomials, normally having four terms, to two terms: (c+b)*(c-b) = c^2 – b^2. But with three binomials, or more, one of the signs is repeated thus creating a trinomial, which then has to be reduced to a binomial by being multiplied with a binomial.

#2) 'The applicant presents, on a normative basis referring only to sociology, an original new formulation of the specific mathematically content. However the distance is far too big to the reality and the problems that on a practical level can be connected to the teaching of mathematics. No publications show direct signs of cooperation with other research with a deviating and a more general accepted starting point, which will be a central part of the work of the applicant. On this basis the committee does not find the applicant qualified for the job'.

Appendix I. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ‘from T take away 4’ can by iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4’, a ‘per-number’. So the count&stack calculation T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking.

5. A calculation T=3*4 = 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 6 litres = 100 % = 5 $ then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal’s length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by re-counting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:

C = diameter * n * sin(180/n) -> diameter * π.

9. Stacks can be added by removing overloads:

T = 38 + 29 = 3ten 8 + 2ten 9 = 5ten 17 = 5ten 1ten 7 = (5+1)ten 7 = 6ten 7 = 67

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number ‘a’ is multiplied with the day-number ‘b’ before being added to the total $-number T: T2 = T1 + a*b.

2days @ 6$/day + 3days @ 8$/day = 5days @ (2*6+3*8)/(2+3)$/day = 5days @ 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration	Reversed addition of per-numbers -> differentiation
T2 = T1 + a*b T2 - T1 = + a*b $\Delta T = \sum a*b$ $\Delta T = \int a*db$	T2 = T1 + a*b (T2-T1)/b = a $\Delta T/\Delta b = a$ dT/db = a

Only in the case of adding constant per-numbers, as a constant interest of e.g. 5%, the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years @ 5 % /year = 21.6% , since 105%*105%*105%*105% = 105%^4 = 121,6%

Conclusion. A Kronecker-Russel multiplicity-based mathematics can be summarised as a ‘count&add-laboratory’ adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word ‘algebra’, reuniting:

	Constant	Variable
Totals m, s, kg, $	T = a*b T/b = a	T2 = T1 + a*b T2-T1 = a*b
Per-numbers m/s, $/kg, $/100$ = %	T = a^b $\sqrt[b]{T} = a$ $\quad \log_a T = b$	T2 = T1 + ∫a*db $dT/db = a$

The Count&Add-Laboratory

The 12 Math-Blunders of Killer-Mathematics

Hidden Choices Hiding a Natural Mathematics

Mathematics itself avoids blunders by being well defined and well proven. However, mathematics education fails its goal by making blunder after blunder at all levels from grade 1 to 12. This paper uses the techniques of natural learning and natural research to separate natural mathematics from killer-mathematics. Two digit numbers, addition, fractions, balancing equations, and calculus are examples of mathematics that has been turned upside down creating the 'metamatism' that killed mathematics and turned natural Enlightenment mathematics into modern missionary set-salvation.

Taking the Killing out of Killer-Mathematics

Killer-mathematics is the authorized routines (Baumann, 1989) that threatens to kill the enrolment to mathematics-based education by creating 'strong indications of increasing justification and enrolment problems concerning mathematics and physics education, as a rather international phenomenon. (..) Also at the primary and secondary school levels mathematics and physics in many countries now seem to be receiving less interest and motivation than before amongst many categories of pupils' (Jensen et al, 1998: 15); and that threatens to kill the relevance of mathematics by creating a 'discrepancy between the objective social significance of mathematics and its subjective invisibility' (Niss' 'relevance paradox' in Biehler et al, 1994: 371).

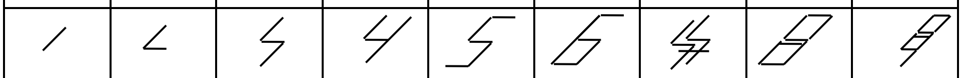
Based upon the oldest research method, the Greek sophist distinction between nature and choice, a hypothesis can be made saying that mathematics education has turned into killer-mathematics because some of the choices made has been faux pas. To identify these math-blunders we must first be able to locate the hidden choices in mathematics education. To tell choice from nature we use the principles of natural learning and natural research to recreate a natural mathematics.

Mammal offspring adapts to the environment through natural learning. Piaget says: 'In other words, intelligence is adaptation in its highest form, the balance between a continuous assimilation of things to activity proper and the accommodation of those assimilative schemata to things themselves' (Piaget, 1969: 158). In this Piagetean 'natural constructivism' natural learning takes place when the individual constructs or accommodate schemata to be able to assimilate stimuli. Adopting the principles of natural learning, Grounded theory becomes a natural research method by using observations to induce schemata, who are validated or adapted through deducing predications that are assimilated to, or leads to the accommodation of existing schemata (G. Tarp, 2005).

So using the principles of natural learning and natural research means basing concepts and theory upon laboratory observations and validations. In this way a natural mathematics can be recreated from which we can see the hidden choices to be reconsidered to avoid creating math-blunders.

Math-Blunder1, Treating both Numbers and Letters as Symbols

In primary school both numbers and letters are treated as symbols. However, numbers are not symbols, but icons representing different degrees of multiplicity. If written in a less sloppy way it becomes clear that there are four strokes in the icon 4, five in the icon 5 etc. (Zybartas et al, 2004)

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

A letter is not representing a distinct sound in nature. On the contrary, a letter constructs and installs a sound to be distinct. Treating letters and numbers alike makes it difficult later to distinguish between the truth-values of number-statements and word-statements i.e. between nature and choice.

Math-Blunder2, 2digit Numbers before Decimal Numbers

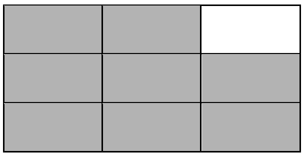
Mathematics can be introduced using 1digitnumbers alone (Zybartas et al, 2004). However, the traditional mathematics curriculum introduces two digit numbers from the beginning thus creating problems to many students:

> Richardson: 27 now. A 2 and a 7. (Chester writes it). Richardson: 29 then 30. (..) Richardson: 32 now. Chester writes it as 23 – a common mistake for him. (Brown, 1997: 112)

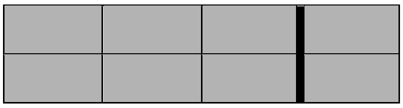
The traditional way of making sense of 2digit numbers is 32 = 3*10 + 2*1. But then we cannot make sense of the number 10 since defining ten as 10 = 1*10 + 0*1 is a meaningless circular self-reference only becoming meaningful through constructing a meaning. The problem is that ten is the only number having a name but not a symbol unless we use the Roman symbol: 10 = 1*X + 0*1, which is problematic since X is not a number symbol.

In the laboratory 2digit numbers accounts for leftovers when counting a total T in b-bundles. Often the bundle-size is 1 making us count in 1s, but it may as well be 2s making us count in 2s. We count in 2s by taking away 2s. The manual process ‘from 8 take away 2s’ can be symbolically represented as ‘8/2’, which is the symbol for division. While the manual process ‘from 8 take away 2’ can be symbolically represented as ‘8-2’, which is the symbol for subtraction, making division repeated subtraction. So the calculation 8/2 can be interpreted in two ways: as an instruction to an action ‘from 8 take away 2s how many times’, and as a prediction of the result, 8/2 = 4 since 4 times we can take away 2s from 8. Thus the result of counting T in bs can be predicted by the recount-equation T = (T/b) *b (Zybartas et al, 2004).

Recounting the total 8 in 3s produces 2 leftovers T = (8/3) *3 = 2*3 + 2*1. When stacking we have to choose between two options. We can count the 2 leftovers in 3s (2 = 2/3 * 3) and put them on top of the existing ‘single-stack’ of 3-bundles, or we can place the 2 leftovers as a separate stack next to the existing stack of 3-bundles, thus producing a ‘multi-stack’ of 3s and 1s.



T = 2 2/3 3s = 2 2/3 *3

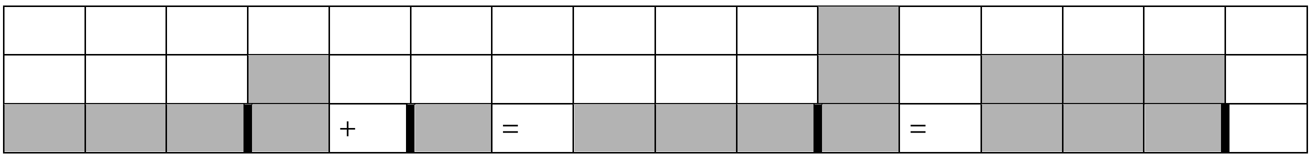


T = 2 3s + 2 1s = 2*3 + 2*1 = 2.2 3s = 2.2*3 = 2)2) = 22

Here 2digit-numbers occur as decimal numbers 2.2 or ‘cup-numbers’ 2)2) or pure numbers 22 avoiding the meaningless self-reference using the number ten: 22 = 2)2) = 2.2*3 = 2*3 + 2*1.

Math-Blunder3, Fractions before Decimals

The traditional mathematics curriculum introduces decimal numbers as examples of fractions, thus having to postpone decimals until fractions are taught around grade 4. In a natural approach both fractions and decimals occur together in grade 1 as different ways of accounting for leftovers as shown above. After that fraction should be allowed to rest until they reoccur as per-numbers in double-counting (se below). Whereas decimal numbers and multi-stacks leads directly to the idea of carrying when adding two multi-stacks produces an overload:



1.2*3 + 0.1*3 = 1.3*3 = 2.0*3

Teaching addition of fractions leads to the following ‘welcome to secondary school ceremony’:

The teacher:	The students:
Welcome! What is $\frac{1}{2}+\frac{2}{3}$?	$\frac{1}{2}+\frac{2}{3}=\frac{1+2}{2+3}=\frac{3}{5}$
No, $\frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$	But $\frac{1}{2}$ of 2 cokes + $\frac{2}{3}$ of 3 cokes is $\frac{3}{5}$ of 5 cokes! How can it be 7 cokes out of 6 cokes?
In mathematics $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$!	

Apparently adding fractions without their 'units' creates 'mathematism', i.e. mathematics that is true in the library, but not in the laboratory (A. Tarp, 2004b).

Math-Blunder4, Forgetting the Units

The traditional mathematics curriculum treats numbers without units, and considers fractions, roots, π and e as numbers. In a natural approach fractions, roots, π and e are calculations, where p = n*sin(180/n) and e = (1+1/n)^n for n very big. A number is a decimal-polynomial: 345.6 = 3*n^2 + 4*n^1 + 5*n^0 + 6*n^-1, where normally n is ten. This shows that numbers can occur as a numerator 3* or as a unit *7. Adding numbers without their units leads to mathematism as shown above. Thus 2+3=5 is seldom true while 2*3=6 is always true: 2weeks+3days=17days, 2m+3cm = 203cm etc., while 2 3s always can be recounted as 6 1s. Also the integration formula tells directly that the per-number f must be multiplied with its unit 'dx' before being added: $\Delta F = \int f\, dx$.

Math-Blunder5, Addition before Division

The traditional mathematics curriculum introduces addition as the first operation. This however leads directly to the need of using 2digit numbers that are ten-based, and thus directly to Math-Blunder2. In a natural approach the first thing we do when meeting multiplicity is to count it predicting the result by the recount-equation T = (T/b)*b. Thus we count be dividing.

Multiplication specifies the height and the bundle size or unit of a stack T = 3*6 = 3 6s. Of course a stack of 3 6s can always be recounted in tens as T = (3*6/10) *10 = 1.8*10 = 1*10 + 8*1 = 18.

Since ten is our standard-bundle it is convenient that recounting in tens can be shown directly by a multiplication: T = 3*6 = 18. However, producing 2digit numbers based on ten, multiplication leads directly to MathBlunder2. Instead subtraction should be introduced after division leading directly to the idea of carrying when internal trade is needed to be able to sell 0.3 4s from a stock of 2.1 4s:

T= 2.1 4s= 2)1) = 2-1)4+1)= 1)5)= 1)5-3) &)3)= 1)2) &)3)= 1.2 4s & 0.3 4s (Zybartas et al, 2004)

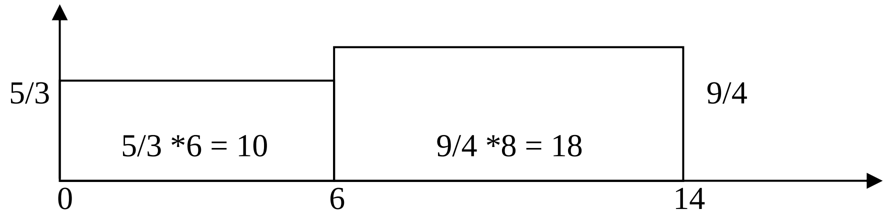
Later, when the students have grown accustomed to decimal numbers through recounting and internal trade and cup-writing, it is time to recount in tens and introduce addition and multiplication.

Math-Blunder6, Fractions before PerNumbers and Integration

The traditional mathematics curriculum only talks about per-numbers in connection with percentages, and percentages are taught as examples of fractions, thus having to wait until fractions are taught around grade 4. In a natural approach fractions first occur as 'proto-fractions' when recounting in number-units: 2 = (2/3) *3. Later fractions occur as 'per-numbers' when double-counting in two different units creates a 'guide-equation' 4kg = 5$, which is re-described as 'per-numbers': 4kg per 5$ = 4kg/5$ = 4/5 kg/$, or 5$ per 4kg = 5$/4kg = 5/4 $/kg.

Here again it makes no sense to add fractions without units. Instead adding per-numbers, as when blending tea, leads directly to integration where the total is the area under the per-number curve:

6 kg @ 5/3 $/kg +8 kg @ 9/4 $/kg = 5/3 *6+9/4 *8 = 10+18 = 28 = 28/14*14 = 14 kg @ 28/14 $/kg

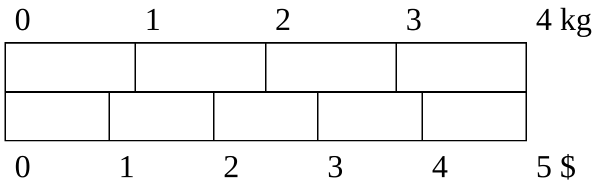


And to differentiation in the case of backward calculation:

$$10\$ + 8\text{ kg @ } ?\ \frac{\$}{\text{kg}} = 28\$,\quad ? = \frac{28-10}{14-6} = \frac{T2-T1}{x2-x1} = \frac{\Delta T}{\Delta x}$$

Math-Blunder7, Proportionality efore DoubleCounting

The traditional mathematics curriculum sees proportionality as an example of a homomorphism thus having to wait until functions are taught around grade 8. In a natural approach proportionality is just another name for 'double-counting' occurring when a quantity can be counted in two different units. Double-counting takes place already in grade 1 where a total of squares can be counted both in 2s and in 3s raising questions as T = 5 2s = ? 3s. Later, when double-counting in kgs and $ we get a 'guide-equation' like 4kg = 5$



To answer questions as 10 kg = ?$ we recount the number: T= 10 kg= $\frac{10}{4}$ * 4 kg= $\frac{10}{4}$ * 5 $= 12.5 $

Or we can choose to recount the unit: $ = $\frac{\$}{\text{kg}}$ * kg = $\frac{5}{4}$ * 10 = 12.5

Double-counting is the most important example of applied mathematics. As just a special kind of recounting it can be introduced together with counting in grade 1 even before addition is introduced.

Math-Blunder8, Balancing instead of Backward Calculation

The traditional mathematics curriculum sees an equation as an example of a statement having a truth set. By performing identical operation on both sides of the equation sign the statement is changed without changing its truth set. In this way equations are solved by balancing, using double arrows to indicate that the truth set is maintained.

In a natural approach an equation is just another name for backward calculation. In the beginning the understanding is helped by adding double arrows showing the forward calculation on the left side (first *3, then +2) and the backward calculation on the right side (first –2, then /3). Later we leave out the arrows and just use the rule 'move across to opposite calculation sign'

2+3*x	= 14
3*x	= 14 - 2
x	= 12 / 3

Or with the 'hidden double-arrows':

Forward		Backward
2+3*x	=	14
	+2 ↑↓ -2	
3*x	=	14-2
	*3 ↑↓ /3	
x	=	12/3

The balancing (or neutralising) method builds upon the abstract algebra of set-based mathematics not seeing 3*4 as a calculation predicting the result of uniting 4 3 times, but as a number-name equivalent to other number-names as '10+2', '24/2' etc. (see e.g. Griffith et al, 1970).

2+3*x = 14	’2+3*x’ and ’14’ are equivalent number-names that are connected by the equivalence relation ‘=’ in the set of number names.
↕ (2+3*x) + (-2) = 14 + (-2) ^ 14 + (-2) = 12	Both number-names are changed by adding –2, the inverse number to 2 under addition. Not changing the truth set, the two statements are equivalent and can be connected by the equivalence relation’ ↕’ in the set of open statements. ’14 + (-2)’ and ’12’ are equivalent number-names. ‘^’ is the conjunction between two statements.
↕ (3*x + 2) + (-2) = 12	Since addition is commutative 2 and 3*x can commute. Since an equivalence relation is transitive we write 12 in stead of 14 + (-2).
↕ 3*x + (2 + (-2)) = 12	Since addition is associative the parenthesis can be moved.
↕ 3*x + 0 = 12	Since 2 and –2 are mutual inverse 2+(-2) becomes the neutral number under addition, i.e. 0.
↕ 3*x = 12	According to the definition of the neutral element.
↕ 3*x = 12 … ↕ x = 4	Over again: ’3*x’ and ’12’ are equivalent number-names that are connected by the equivalence relation ‘=’ in the set of number names. Both number-names are changed by multiplying etc. etc. etc.

Following the double arrows we see that since an equivalence relation is transitive the statements ’2+3*x = 14’ and ’x = 4’ are equivalent therefore having the same truth set.

Math-Blunder9, Killer Equations instead of Grounded Equations

The traditional mathematics curriculum doesn’t mind ‘killer-equations’, i.e. equations we only meet inside the classroom where they only serve one purpose, to kill the interest of the students.

In a natural approach an equation is grounded as an abstraction form a real life situation, typically a word problem as e.g. ‘2$ plus 3kg @ ?$/kg total 14$’ leading to the equation ‘2+3*x=14’.

In Africa I witnessed a student teacher’s fruitless attempt to be loyal to a textbook not respecting the difference between grounded equations and killer equations:

Equations:	$\frac{M}{5} - \frac{M}{2} = 3$	$\frac{y+2}{4} - \frac{y-6}{3} = \frac{1}{2}$
Solutions proposed by the students at the board:	$\frac{M}{10} = 3$ m = –10(3) m = –30	$\frac{6+24}{12} = 2$ y = 12*2 y = 14

After the period the student-teacher complained: ‘You ask them if they understand it and they say yes, but next day they have forgotten it all. They don’t study at home, they have too much free time and no parent support. Their friends say mathematics is not interesting. 30 minutes lessons are too short, in private schools they have 60 minutes. The ministers take their children abroad. The new curriculum also asks us to teach these equations. Something has to be done.’

In these explanations the blame for the ‘bad play’ is placed with external factors outside the teacher’s influence: ‘the manager, the director and the actors’. Inspired by the sophists looking for hidden choices I suggested looking at ‘the script’ by rephrasing equations into two groups: Top-Down ‘killer-equations’ and Bottom-Up ‘calculation stories’ (A. Tarp, 2002, 2005).

Math-Blunder10, Geometry before Trigonometry

Geometry means ‘measuring earth’ in Greek. Areas can be divided into triangles, which again can be divided into right-angled triangles. However, the Greeks only had two equations to predict the three unknowns in a triangle, so instead they developed the axiomatic geometry of Euclid that led to the Plato Academy, cloisters and universities, but that also froze mathematical thinking for 2 thousands years until it was finally neglected in the flourishing period of Enlightenment mathematics:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline, 1972: 398-99)

Later the Arabs provided two equations to replace the Pythagorean theorem by simply recounting the two sides of a right-angled triangle in the long side: a = a/c*c = sinA*c and b = b/c*c = cosA*c.

Tested in the classroom of an African teacher college the trigonometry approach to geometry turned out to be very successful (A. Tarp, 2005).

Math-Blunder11, Postponing Calculus

The traditional mathematics curriculum sees calculus as dealing with two examples of limits, the gradient and the integral. Thus calculus cannot be introduced before the real numbers and the concepts of functions, limits and continuity are introduced late in secondary school. Furthermore calculus is considered so difficult that only very few students are advised to take calculus classes.

In a natural approach calculus is an abstraction from examples of uniting variable per-numbers. This takes place from grade 1, where 4 3s and 2 5s can be united as 1s, as 3s, as 5s or as 8s. Asking '2*5 + 4*3 = ?*8' is integration since the total is the sum of the stacks 2*5 and 4*3, i.e. the area under the stacks' height-curve. And asking '2*5 + ?*3 = 4*8' is differentiation (A. Tarp, 2004c).

In lower secondary school variable per-numbers are united when blending tea as shown above. In upper secondary school the per-numbers are not piecewise constant anymore, but locally constant. Before the CAS-calculators special uniting techniques had to be learned lading to the limit concept, but after the CAS-calculators we just have to enter the formula y= and ask for the gradient formula if y is a Total-formula, and ask for the area formula if y is a per-number formula.

Math-Blunder12, the Five MetaBlunders of Mathematics Education

Besides the numerous concrete blunders mathematics education has also made several meta-blunders at the curriculum's meta-level.

1. The Preclusion of Prediction. In Greek the word 'mathematics' means 'what we know', i.e., what we can use to predict with. In the 1600s the predicting ability of mathematics was used to replace political correctness with natural correctness by showing that the Pope was wrong claiming that a falling object obeys a metaphysical will that is unpredictable, so that all humans should do is believe, go to church and learn to prey. Instead Brahe, Kepler and Newton used knowledge validated by its predicting ability, to prove that physical things move according to a physical will, a force, that is predictable since it can be described in numbers and formulas; so from now on humans should enlighten themselves by going to school and learn how to calculate. The predicting ability of mathematics thus laid the foundation of the Enlightenment and its two democracies developing two different forms of natural research: American grounded theory discovering the nature of things, and French post-structuralism discovering hidden choices presented as nature.

The fascination by seeing physical structures as examples of geometry, again being an example of meta-physical axioms, led to Plato's Academy for the study of how physical thing could be understood as examples of metaphysical structures. Algebra became like geometry when set-theory created its library–mathematics where all mathematics is defined and proved through self-reference within set-theory. However, Russell and Gödel shoved that self-reference leads to paradoxes. Also, by accepting the mathematism mentioned above mathematics looses its ability to predict. Still the traditional curriculum follows the old Plato dream by defining its concepts as examples of abstractions (meta-matics) instead of as abstractions from examples (natural mathematics).

It is easy to revive the predicting ability of mathematics. Uniting 23 and 45 stones, the total may be found by counting on: 24, 25, … , 67, 68. This result is predicted by a calculation T = 23+45 = 68.

Likewise, uniting 6 stones 8 times can be done by practising the 6 table: 6, 12, 18, … , 42, 48. Again, this result is predicted by a calculation T = 8*6 = 48.

In the same way power can predict that adding 6% 8 times gives 59.4%: 1+T = 1.06^8 = 1.594

And integration can predict the result of adding per-numbers. Thus 5 seconds @ 6m/s increasing to 8m/s gives 35 m, a result that is predicted by the calculation $\int_0^5 (6 + (8-6)/5 *x)\, dx = 35$.

The inverse operations predict the result of backward calculation. Thus the question '?+3' = 9 can be solved by trial and error: 2+3=5, no. 3+3=6, no. …, 6+3=9, yes! So ? = 6. This result is predicted by a backward-calculation T = 9-3 = 6.

I a similar way the other inverse calculations, division, root, log, d/dx can be used to predict the answer to the backward calculations '?*3=15','?^3=125', '3^?=81', '∫?dx = x^3'.

In this way we see that the operations are means to predict the result of uniting or separating four different kinds of numbers according to the fact that in Arabic the word 'algebra' means reuniting.

Uniting (separation) is predicted by	unlike	like
+numbers	+ (-)	* (/)
*numbers	∫ (d/dx)	^ (√, log)

2. Interchanging Product and Process. Through thousands of years mathematics has been constructed through a collective learning process abstracting concepts and theory grounded in laboratory observations, thus following the principle of natural learning of Piaget. Mathematics is not like biology teaching about factual biological objects and processes present on the earth before mankind came along. Mankind constructs mathematics, and as such the students can reconstruct it.

Freudenthal calls this 'guided reinvention' (Freudenthal, 1973). However, by using the undifferentiated word 'mankind' Freudenthal does not pay respect to the fact that knowledge is situated and local. The library fact that the ancient Egyptians added fractions doesn't necessarily mean that children should learn to add fractions in the early grades. Instead the students should be allowed to reinvent their own sentences through natural learning in sentence-free educational laboratory meetings with the subject of mathematics, multiplicity. Thus a distinction should be made between laboratory-guided reinvention and library-guided reinvention.

3. Interchanging Goal and Means. Mathematics should be a means to an outside goal, a number-language enabling us to predict the world by numbers and calculations. However, this relationship is turned upside down, so mathematics has become the goal and the world a means. This mathematical somersault is enforced by set-based mathematics defining its concepts from above as examples of abstractions, and validating its theorems through deduction from axioms, thus replacing natural Enlightenment mathematics that define its concept as abstractions from examples and validate its theorems in the laboratory instead of in the library. And enforced by accepting the phrasing 'the world applies mathematics' instead of the phrasing 'the world creates mathematics'. To apply mathematics we must know mathematics, hence education is set up having mathematics as its goal. The natural thing is to formulate the goal in outside terms as algebra and geometry, meaning reuniting numbers and measuring earth. So a natural goal for mathematics education would be: mathematics is a means to develop a number language for predicting quantities.

4. Funding Library Research Instead of Laboratory Research. Mathematics education research has an 'irrelevance paradox' since the number of research articles increase with the number of problems they try to solve (A. Tarp, 2004b). Examples of 'irrelevant' research are 'lackey-research', 'ghost-research' and 'mirror-research'. Lackey-research accepts the hidden choices of mathematics education and search for understandings of the problems these choices cause instead of searching for hidden alternatives. Ghost-research or 'master+ research' sets up hypotheses based upon library concepts that cannot be operationalised and therefore have to be installed as 'ghosts' in order to be studied. Mirror research is research in mathematics education research instead of in mathematics education itself. To solve the irrelevance paradox, funding must be given to natural research uncovering nature; and to counter-research uncovering hidden choices presented as nature.

5. Turning Natural Mathematics into Metamatism. However, turning natural multiplicity-based mathematics upside down so it becomes set-based meta-matics not able to tell predicting mathematics from 'mathematism', is the mother of all meta-blunders. To change this 'metamatism' (A. Tarp, 2004a) back to natural mathematics, the laboratory has to replace the library as the authority so we can be re-enlightened and learn how to tell nature from choice.

Conclusion

The paper has identified some of the hidden choices of mathematics education leading to math-blunders. To make mathematics education blunder-free and killer-free, killer-mathematics deduced from the library must be replaced with mathematics induced from the laboratory as done e.g. in the 'MATHeCADEMY.net' (A. Tarp, 2004b). In this way a natural mathematics created through natural research performed on the root of mathematics, multiplicity, will lead to natural learning avoiding the math-blunders that turned natural mathematics for all into killer-mathematics for few.

References

Bauman, Z. (1989). *Modernity and the Holocaust*. Oxford: Polity Press

Biehler, R., Scholz, R. W., Strässer, R. & Winkelmann, B. (1994). *Didactics of Mathematics as a Scientific Discipline*. Dordrecht: Kluwer Academic Press.

Brown, T. (1997). *Mathematics Education and Language.* Dordrecht: Kluwer Academic Publishers.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: D. Reidel Publishing Company.

Griffiths, H. B. & Hilton, J. P. (1970). *A Comprehensive Textbook of Classical Mathematics.* London: Van Nostrand Reinhold Company.

Jensen, J. H., Niss, M. & Wedege, T. (1998). *Justification and Enrolment Problems in Education Involving Mathematics or Physics*. Roskilde: Roskilde University Press.

Kline, M. (1972). *Mathematical Thoughts from Ancient to Modern Times*. New York: Oxford University Press.

Piaget, J. (1969). *Science of Education of the Psychology of the Child*. New York: Viking Compass.

Tarp, A. (2002). Killer-Equations, Job Threats and Syntax Errors. In C. Bergsten, G. Dahland & B. Grevholm (Eds.) *Research and Action in the Mathematics Classroom*. Linkoping, Sweden: SMDF No. 1.

Tarp, A. (2004a). *Applying Mathe-Matics, Mathe-Matism or Meta-Matics.* Paper accepted for presentation at the Topic Study Group 20. The 10th International Conference on Mathematics Education. Copenhagen, Denmark.

Tarp, A. (2004b). Mathematism and the Irrelevance of the Research Industry, A Postmodern LIB-free LAB-based Approach to our Language of Prediction. In C. Bergsten & B. Grevholm (Eds.) *Mathematics and Language*. Proceedings of the 4th Swedish Mathematics Education Research Seminar, MADIF 4 (pp. 229-241). Linkoping, Sweden: SMDF No. 3.

Tarp, A. (2004c). *Per-Number Calculus.* Paper accepted for distribution at the Topic Study Group 12. The 10th International Conference on Mathematics Education. Copenhagen, Denmark.

Tarp, A. (2005). *Killer Equations in Africa.* A PhD thesis in progress. To be handed in 2005.

Tarp, G. (2005). *Grounded Theory as a Dialectical Constructivist Learning Method for Ethnographers*. Paper to be published.

Zybartas, S. & Tarp, A. (2004). *One Digit Mathematics*. Paper accepted for presentation at the Topic Study Group 1. The 10th International Conference on Mathematics Education. Copenhagen, Denmark.

Mathematics: Grounded Enlightenment - or Pastoral Salvation

Mathematics, a Natural Science for All - or a Humboldt Mystification for the Elite

Mathematics is taught differently in Anglo-American democratic enlightenment schools wanting as many as possible to learn as much as possible; and in European pastoral Humboldt counter-Enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Humboldt Bildung schools pastoral 'metamatism' descends from above as examples of metaphysical mystifying concepts. To make mathematics a human right, pastoral Humboldt counter-enlightenment must be replaced with democratic grounded enlightenment.

Introduction

This paper is written for a conference theme 'perspectives on mathematical knowledge' translating into 'perspectives on knowledge knowledge' since in Greek 'mathematics' means knowledge. So to give it meaning, this paper interprets the theme as 'perspectives on the contemporary university discourse called mathematics.' This theme is an example of a more general theme called 'perspectives on the contemporary university discourse called knowledge production'. Thus a natural approach to such a theme is to identify perspectives in the general discourse and exemplify them in the mathematics discourse. At the general discourse level during the last three decades a fierce debate has taking place between modern and postmodern perspectives on knowledge. So it seems natural to import this discussion in to the discussion about mathematical knowledge.

Postmodern Thinking, a Short Tour

As to defining the word 'postmodern', the literature often refers to Lyotard's scepticism towards modern science legitimising its truths as examples of a truth above, a meta-truth.

I will use the term *modern* to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..) Simplifying to the extreme, I define *postmodern* as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

As to legitimising postmodern research, Lyotard says that postmodern research should produce paralogy in the sense of parallel knowledge that invents not truth, but differences and dissension. In other words, postmodern research means searching for hidden differences, contingency:

> Where, after the metanarratives, can legitimacy reside? The operativity criterion is technological; it has no relevance for judging what is true or just. Is legitimacy to be found in consensus obtained through discussion, as Jürgen Habermas thinks? Such consensus does violence to the heterogeneity of language games. And invention is always born of dissension. Postmodern knowledge is not simply a tool of the authorities; it refines our sensitivity to differences and reinforces our ability to tolerate the incommensurable. Its principle is not the expert's homology, but the inventor's paralogy. (xxiv-xxv)

Lyotard writes inside the French post-structural Enlightenment tradition, also including Derrida and Foucault. Inspired by Heidegger, Derrida has inaugurated

> a project of deconstructing Western metaphysics or 'logocentrism' with its characteristic hierarchizing oppositions (..) Derrida's claim is that these conceptual orderings are not in the nature of things, but reflect strategies of exclusion and repression that philosophical systems have been able to maintain only at the cost of internal contradictions and suppressed paradoxes. The task of 'deconstruction' is to bring these contradictions and paradoxes to light, to undo, rather that to reverse, these hierarchies, and thereby to call into question the notions of Being as presence that give rise to them (Baynes 1987: 119)

Later Derrida demystifies the term 'deconstruction' by saying in an interview '(..) in order to demystify or, if you prefer, to deconstruct (..) (Derrida in Royle 2003: 35). So Derrida expresses scepticism towards excessive trust in words, logocentrism. Some words might enlighten what they describe, others instead mystify; and thus needs to be demystified or deconstructed to be enlightening.

Inspired by Nietzsche, Foucault writes about knowledge-power, or pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1983: 213, 215)

Foucault thus sees modern institutions as generalised churches using pastoral discourses to offer salvation promises: ‘You are un-saved, un-healthy, un-social, un-educated. But do not fear! For we, the saved, healthy, social, educated, will save you. All you have to do is to repent, and go to our salvation institution, the church, hospital, correction centre, school, and become a loyal lackey’.

Common to Derrida, Lyotard and Foucault is a revival of scepticism towards hidden patronisation. Together they describe the compulsion techniques of modern pastoral knowledge: compulsive pastoral mystifying words installing what they describe as nature instead of choice; compulsive pastoral statements installing their claims as nature instead of choice; and compulsive pastoral salvation institutions mediating discursive servility instead of enlightenment.

The first generation of sceptical thinkers were the ancient Greek sophists claiming that in order to practise democracy people must be enlightened to tell the difference between nature and choice; if not, patronisation in disguise would arise presenting its choice as nature. Thus Plato’s half-brother, the sophist Antifon, writes:

> Correctness means not breaking any law in your own country. So the most advantageous way to be correct is to follow the correct laws in the presence of witnesses, and to follow nature’s laws when alone. For the command of the law follows from arbitrariness, and the command of nature follows from necessity. The command of the law is only a decision without roots in nature, whereas the command of nature has grown from nature itself not depending on any decisions. (Antifon in Haastrup et al 1984: 82, my translation).

Plato claimed that choice is an illusion; all is nature since all physical phenomena are examples of metaphysical forms only visible to the philosophers who therefore are the only ones to name them. Hence people should abandon democracy and accept the pastoral patronisation of philosophers educated at Plato’s academy.

In Greece democracy disappeared with the silver mines financing import of silk and spice from the Far East. The academy, however, survived and was later renamed to monasteries by the Christian church sympathising strongly with the academy’s pastoral salvation techniques. Later some monasteries developed into universities, as visible in Cambridge and Oxford; and at universities in general still organised like a monastery with long corridors of cells where people sit and produce writings extending and referring to the ruling pastoral discourse.

Robbing Spanish silver on the Atlantic was no problem for the British. But sailing to the Far East only following the moon to avoid Portuguese fortification of Africa was. Newton rejected the official knowledge saying that the moon moves among the stars following the unpredictable will of a metaphysical Lord. Instead he claimed that the moon falls towards the earth as does the apple, both following an internal physical will that can be predicated through calculations and later tested.

Newton’s scepticism led to the Enlightenment: when an apple only obeys its own will, why shouldn’t people do the same and replace patronisation with democracy?

French Enlightenment and German Counter-Enlightenment

The Enlightenment established two democracies, in America and in France. America still has its first republic, France its fifth. In France the Germen autocracy send in the army to stop the French democracy. However, they send in an army of mercenaries that was no match to the French army of conscripts only to aware of the feudal alternative to democracy. So not only was the German armies rolled back to the border, the French occupied Germany itself.

Napoleon was shocked to see the many different measurers in the many principalities of Germany and Italy created to guard the silver on its journey from the Harz to Venice where it financed the import of spice and silk that financed the Italian Renaissance. So he cancelled the Second Reich, the Holy Roman Empire, having lasted almost 1000 years; and installed the meter system by force.

Being unable to use the army, the German autocracy turned to education to stop the spreading of democracy from France. So they asked the father of new-humanism, Humboldt, to develop counter-enlightenment and reinstall pastoral schools that could stop the democratic enlightenment schools.

Mixing Hegel philosophy with romanticism, Humboldt developed 'Bildung' to reinstall a metaphysical Spirit present all over nature, in minerals, plants, animals and humans, and expressing itself in art. To understand art, people need the Bildung of the Humboldt school system. However, Bildung is only accessible to the chosen few, so not everybody is allowed enter into the Humboldt schools. Thus today's Humboldt university refuses to receive the students directly from the democracy's secondary schools, first they must pass an entrance exam at a Humboldt-gymnasium. However, only the most gifted half of the students is allowed to enter the Humboldt gymnasium, and again only the best half is allowed to enter the Humboldt University, where a half is failed so that only 13% finally gets a university degree (OECD 2004: 6).

The elitism of the Humboldt schools was enthusiastically accepted by the other European autocracies. When later turning into democracies they kept the Humboldt Bildung system.

American Enlightenment and Grounded Action Theory

In America, Enlightenment developed into pragmatism showing scepticism towards traditional philosophy by developing 'symbolic interactionism' with its own methodology called 'grounded theory'. Grounded Theory respects agents as independent actors:

> Actors are seen as having, though not always utilizing, the means of controlling their destinies by their responses to conditions. They are able to make choices according to their perceptions, which are often accurate, about the options they encounter. Both Pragmatism and Symbolic Interactionism share this stance. Thus, grounded theory seeks not only to uncover relevant conditions, but also to determine how the actors respond to changing conditions and to the consequences of their actions. It is the researcher's responsibility to catch this interplay. (Corbin & Strauss 1990: 5)

As to the question about being guided by existing theory, Grounded Theory gives the advice to ignore the literature and theory on the area under study in order to assure that the emergence of categories will not be contaminated by concepts more suited to different areas:

> Although categories can be borrowed from existing theory, provided that the data are continually studied to make certain that the categories fit, generating theory does put a premium on emergent conceptualizations. (..) In short, our focus on the emergence of categories solves the problems of fit, relevance, forcing, and richness. An effective strategy is, at first, literally to ignore the literature of theory and fact on the area under study, in order to assure that the emergence of categories will not be contaminated by concepts more suited to different areas. Similarities and convergences with the literature can be established after the analytic core of categories has emerged. (Glaser et al 1967: 36-37)

So instead of going to the library, Grounded Theory listens to the agent's own accounts and narratives from which categories and relations are discover and constantly checked or accommodated through new data. In this way grounded research could be named ‘systematic natural learning’ reminding very much of the ‘individual natural learning’ described by Piaget:

> Is childhood capable of this activity, characteristic of the highest forms of adult behaviour: diligent and continuous research, springing from a spontaneous need? – that is the central problem of the new education. (..) But all these psychologists agree in accepting that intelligence begins by being practical, or sensorimotor, in nature before gradually interiorising itself to become thought in the strict sense, and in recognizing that its activity is a continuous process of construction. (..) In other words, intelligence is adaptation in its highest form, the balance between a continuous assimilation of things to activity proper and the accommodation of those assimilative schemata to things themselves. (Piaget 1969: 152, 158)

Piaget thus is the father of constructivist learning theories believing that learning takes place through a ‘grasping before grasping’ or ‘greifen vor begreifen’ process. With physical grasping always preceding mental grasping, the mental concepts will automatically enlighten the physically grasped. Contrary to this the Vygotsky social constructivism tries to adapt the learner to a pre-existing pastoral mystifying vocabulary calling itself ‘scientific’. Likewise, the importance of physical grasping is absent in Luhmann’s pragmatic constructivism seeing the individual embedded in two systems, a reflective and a communicational system, both being self-referential. Luhmann’s theory of self-generating and self-referring systems seems to be created to support and legitimise the self-reference taking place at the pastoral Humboldt counter-enlightenment universities.

To avoid the self-reference of the Humboldt University and instead make research usable to the public, some American enlightenment universities recommend action research.

> Our universities have a monastic origin, and they have specialized in being centers of higher learning, functions originally given by the Church to monasteries. (..) The form of the university most familiar to us today is mainly a Prussian invention whose architect and champion was Wilhelm von Humboldt (..) The collegial system and its related peer review structures centered on an effort to gain intellectual freedom from the constraints of theological doctrine and political manipulation. Although addressing this problem was obviously important, the solution adopted has subsequently done much to weaken the social articulation of the university to all groups other than powerful elites. (..) Not surprisingly, society at large occasionally thinks it should be getting a more useful return for its investment and the freedom it gives to the professoriate. This situation is predictable because the autopoetic research process provides important supports for intellectual freedom but simultaneously opens the door to useless research and academic careerism divorced from attention to important public social issues. (..) While we advocate action research as a promising way of moving the academic social sciences to socially meaningful missions, we do not base our claims for action research only on its putative moral superiority. Central to our argument is the claim that action research creates the valid knowledge, theoretical development, and social improvements that the conventional social sciences have promised. Action research does better what academic social science claims to do. (Greenwood & Levin in Denzin & Lincoln. 2000: 85-89)

Deconstructing and Grounding Research

Lyotard’s postmodern paralogy research creating dissension to the ruling consensus by searching for hidden differences, contingency, resonates with the ancient sophist advice: know the difference between nature and choice to avoid hidden patronisation presenting choice as nature. Also including the American enlightenment sociology advocating theory being grounded by assigning names to things that can be observed, it is now possible to design a postmodern research paradigm that could be called ‘anti-pastoral enlightenment research’: To avoid hidden patronisation, uncover pastoral choices presented as nature by replacing self-referring mystification with grounded enlightenment.

Thus linear and exponential functions are pastoral terms since they describe a Renaissance calculation formula using a word from around 1750. These terms can be demystified by terms grounded in and enlightening their nature as e.g. ‘change by adding and by multiplying’.

Re-grounding mathematics in its historical roots, the nature of many, the names 'metamatics' and 'mathematism' can be given to ungrounded self-referring mathematics (Tarp 2004).

The roots of mathematics are revealed by its two sub-discourses, algebra and geometry. In Greek Geometry means 'earth measuring'; and in Arabic Algebra means 'reuniting'. Together they answer two fundamental questions 'How to divide the earth and its products?' Or simpler 'How to divide and unite many?' So mathematics is created as a grounded theory about many; and as such it was very successful in the Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 398-99)

Later with the set-concept, all concepts seemed to be examples of sets. This re-installed pastoral mathematics until Russell and Gödel showed that a self-referring mathematics can be neither well-defined nor well-proven. Russell's set-paradox 'if $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$' shows that concepts can't be self-referring. And Gödel proved that any axiom system would contain true statements that cannot be proven. So mathematics is still a natural grounded science.

Deconstructing and Grounding the Postmodern

To demystify and deconstruct the word 'postmodern' we can ask if the words 'postmodernity' and 'postmodernism' can be grounded in 'laboratory' observations.

As other biological animals also humans need a constant supply of matter, energy and information. Knowledge about supply techniques, technology, has developed through human history.

First matter technology using iron invented artificial hands, tools, enabling a transition from gather/hunter culture to agriculture. Then energy technology using electrons to carry energy invented artificial muscles, motors, combining with tools to machines, enabling a transition from agriculture to industrial culture. Then information technology using electrons to carry also information invented artificial brains, computers, combining with tools and motors to robots, enabling a transition from industrial culture to information culture, postmodernity. And since the robot is the end of the line, the term post-postmodernity has no meaning.

To control machines, the modern industrial culture needed the brain to be educated creating well-defined jobs. Modern thinking then means choosing between a set of well-defined identities.

In the postmodern information culture the human brain is not needed for routine jobs, making most traditional training redundant. Furthermore, by informing also about alternatives that were before hidden, information technology un-hides hidden contingency. Thus the individual now sees the world full of choices in areas where before was only nature to obey. Most identities now are liquid (Baumann 2000). To get an identity, the individual now has to build its own identity as a biographical narrative shunning meaninglessness and looking for authenticity (Giddens 1991).

Postmodernism means presenting choice as choice creating more personal and social choices. Post-postmodernism means presenting choice as nature resulting in a return of pastoral patronisation.

Deconstructing and Grounding Numbers

The different degrees of many are enlightened by names and icons. Counting a given total T by bundling and stacking can be predicted by the recount-equations as $T = (T/5)*5$. Many different icons have been used. Today the most frequent icon systems are the Roman and the Arabic.

The Arabic system rearranges the given number of strokes into an icon so there are four strokes in the icon 4 etc. Ten is chosen as the standard bundle-size in which to bundle singles, bundles, bundles-of-bundles etc. Thus a total T can be iconised as e.g. T = 3BBB,5BB,7B,1, or leaving out the bundles, T = 3571; or T = 3501 if all bundles can be re-bundled into bundles-of-bundles.

The Roman system uses strokes for unbundled, and the letters V, X, L, D for certain bundle-sizes. However, bundling is not systematic: V means a 5-bundle, X means 2 V-bundles etc.

Arabic numbers are introduced from grade 1 in all pastoral mathematics curricula. In grounded mathematics 2digit numbers are banned from grade 1 since they refer to the number ten. As the only number with its own name but without its own icon the number ten becomes a cognitive bomb if presented too early: a 2digit number as 23 is explained as 2 10s and 3 1s, thus referring to ten. And the 2digit number 10 is explained as 1 10 and no 1s, i.e. through circular self-reference to ten. Instead 2digit numbers should be introduced slowly through bundling, stacking and cup-writing: A total of sixteen sticks can be counted in 5-bundles and stacked as 3 5-bundles and 1 unbundled: T = 3*5 + 1*1. Counted in 8-bundles produces 2 8-bundles that can be stacked as T = 2*8.

The bundles and the unbundled are put in a left and a right cup. Later a stone and later again a stick is used as a symbol of a full bundle, knowing that a stick in the left cup symbolises a full bundle.

The manual activity of cup-filling leads to the mental activity of ‘cup-writing’ T = 3)1) worded as 3 bundles and 1 unbundled in the case of 5-bundling; and T = 2)) worded as 2 bundles and no unbundled in the case of 8-bundling. Later the cups can be left out and a 0 introduced as an icon for an empty cup: T = 2)) = 20 worded as 2 bundles and no unbundled. Now 10 means 1)) thus being defined by a two-cup physical reality, which makes the circular self-reference disappear.

Likewise in a grounded approach, fractions and decimal numbers are introduced simultaneously in grade 1 as ways of dealing with the unbundled, where e.g. 2 can be counted in 5s as 2 = (2/5)*5 and put on top of the 5-stack and written as T = 3 2/5 *5; or the unbundled can be put next to the 5 stack as a separate stack of 1s written as T = 3.2 *5. In fact, all of mathematic can be introduced using 1digit numbers alone, including equations and calculus since equations is just another word for backward calculation (3 + ? = 8); and calculus is just another word for horizontal addition instead of vertical: 3 5s + 2 3s = ? 8 s instead of 3 5s + 2 3s = ? 5s, or 3 5s + 2 3s = ? 3s. (Zybartas 2005)

Deconstructing and Grounding Operations

In Greek, mathematics means knowledge, and knowledge can be used for prediction. Thus ‘number-prediction’ is one possible demystification or deconstruction of mathematics, which grounds operations as number-prediction techniques. Without addition, wanting to unite 32 and 64 becomes a very time-consuming task involving a high risk of making errors, since we have to count-on from 32 64 times: ‘33, 34,…, 96, 97, I think; or maybe it is 98?’ To be sure, one has to make an accounting by writing down one stroke per count. It would be nice to be able to predict counting-results. Addition does this: T = 32 + 64 = 96. Likewise multiplication predicts adding many like numbers, and power predicts multiplying many like numbers.

To avoid trying out many numbers, it would be nice also to predict the answer to the questions 3+? = 8, 3*? = 15, 3^? = 81 and ?^5 = 32. This grounds inverse operations as the answers 8-3, 15/3, log3(81) and 5√(32); and offers an simple technique of solving equations: just move a number to the other side by changing its calculation sign:

$3 + x = 8$	$3 * x = 15$	$3 \wedge x = 81$	$x \wedge 5 = 32$
$x = 8 - 3$	$x = \frac{15}{3}$	$x = \log_3(81)$	$x = \sqrt[5]{32}$

Pastoral mathematics needs all the concepts of abstract algebra to solve the equation: neutral and inverse elements, commutative and associative laws:

2+3*x = 14 , (2+3*x) + -2 = 14 + -2 = 12 , (3*x + 2) + -2 = 12 , 3*x + (2 + -2) = 12, 3*x+0 = 12

$$3*x = 12 , (3*x)*\frac{1}{3} = 12*\frac{1}{3} = 4 , (x*3)*\frac{1}{3} = 4 , x*(3*\frac{1}{3}) = 4 , x*1 = 4 , x = 4$$

However, this is impossible to bring to the classroom. Instead a lever is introduced to teach the method of doing the same to both sides, cheating students by reducing an understanding to a ritual.

Deconstructing and Grounding the Mathematics Curriculum

In a grounded mathematics curriculum mathematics is learned as a natural science exploring many. This means that both teachers and students re-discover mathematics through the CATS-approach: Count&Add in Time&Space as presented by the MATHeCADEMY.net. Thus in the lower primary school a grounded mathematics curriculum introduces the whole of mathematics working with 1digit cup-numbers alone (Zybartas 2005). Addition and subtraction of cup-numbers is learned through re-bundling and internal trade between neighbour cups: Thus, in the case of 5-bundles

T = 3)4) + 4)2) = 7)6) = 7+1)6-5) = 8)1) = 0+1)8-5)1) = 1)3)1)

In upper primary school this curriculum is repeated, now using multi-digit numbers. And per-numbers are introduced now using the recount-equation T = (T/b)*b to describe recounting in different units by recounting a given total in the given base unit, e.g. recounting 8 in 3s: If 3kg = 5$ then 8kg = (8/3)*3kg = (8/3)*5$ = 13.3 $. Geometry is introduced as trigonometry considering sin, cos and tan as percent-numbers and tan as an easy protractor.

Secondary school algebra deals with change equations: constant change, i.e. linear change ($\Delta y = a$) and exponential change ($\Delta y = r\%$); variable predictable change (dy/dx = formula); and unpredictable change, i.e. statistics and probability. Geometry is extended to include non-linear forms, and later geometry becomes coordinate geometry and vector geometry.

A Grounded Perspective on Pastoral Mathematics

The pastoral approach to mathematics makes many learning-blunders (Tarp 2006) transforming it into metamatism only accessible to the elite. This is precisely what the Humboldt university wants: It witnessed how the Enlightenment was created by mathematics' ability to predict numbers, so a counter-enlightenment must reinstall mathematics as a pastoral knowledge descended from above. Thus in Germany teaching fractions as metamatism, e.g. 1/2 + 2/3 = 7/6 instead of 3/5 enables the Humboldt system to split the students into three groups: Realschule, Hauptschule und Gymnasium.

Still acting as a province governed from Holstein, Denmark has taken the Humboldt counter-enlightenment to an extreme. In school, most marks are oral being unreliable since they are based upon the personal subjective judgement of the person who has also given the education, and not on written performances. Being unable to prove the absent learning with written tests, the teachers are forced to give most students middle marks making it possible to sit off both school and teacher education since a teacher can function by just handing out middle marks. Sitting off of course means disaster at written exams. Thus the international standard of 60% correctness as passing limit is lowered to 40% in the Danish Gymnasium and to 20% in the secondary school. Likewise the Danish Humboldt university refuses to include other tertiary educations as e.g. teacher education.

The Humboldt Occupation of Europe

The Humboldt University's 200-years occupation of Europe created no problems in the industrial culture needing less than 10% to attend university. But in a postmodern information culture needing more that 50%, it presents an unmatched disaster since the Humboldt University will wipe out the population in 200 years by holding on to its youth in its Humboldt maze of uncoordinated non-modularized educations, that keep the youth from producing and keeps the reproduction rate at 1.5 child per couple. However, the European population is unaware of this since the counter-enlightenment of the Humboldt Bildung has kept the majority of the population including its politicians unenlightened while sorting out the elite for its own reproduction. Likewise as lackey-research supporting metamatism education, mathematics education research has turned into a research industry producing huge amounts of irrelevant research only useful for personal careerism.

Conclusion

A postmodern perspective on mathematical knowledge enlightens what is nature and what is choice within mathematical knowledge; and what is pastoral choice presented as nature. This again makes plain to Europe's democracies the choice they face: will they continue to support the occupation of Europe by the Humboldt counter-enlightenment Bildung system; that will wipe out the European population in 200 years by holding its youth caught in its pastoral salvation institutions in the crucial years where elsewhere they get their university degree, a job, and a family; that instead of teaching mathematics preach metamatism in order to sort out the elite; and that allows its universities to be self-referring and to produce useless research only usable for careerism. Or will they finally introduce democracy also into education; by changing the Humboldt counter-enlightenment system to the Anglo-American enlightenment system that has been adopted as international standard outside Europe; by changing pastoral metamatism salvation to mathematics enlightenment; and by only funding action research forcing research to ground its theories in society's needs and concerns. As a first step to this decision, the European democracies should privatise its Humboldt universities and Humboldt gymnasia in order to enable free competition with Anglo-American enlightenment education.

References

Bauman Z (2000) *Liquid Modernity*, Oxford: Polity Press

Baynes K, Bohman J & McCarthy T (1987) *After Philosophy*, Cambridge Ma: the MIT press

Corbin J & Strauss A (1990) Grounded Theory Research: Procedures, Canons and Evaluative Criteria In *Qualitative Sociology*, Vol. 13, No. 1

Denzin N K & Lincoln Y S (2000) *Handbook of Qualitative Research 2nd ed.*, London: Sage

Dreyfus H L & Rabinow P (1983) *Michel Foucault*, Chicago: University of Chicago Press

Giddens A (1991) *Modernity and Self-identity*, Oxford: Polity Press

Glaser B G & Strauss A L (1967) *The Discovery of Grounded Theory*, New York: Aldine de Gruyter

Haastrup G & Simonsen A (1984) *Sofistikken*, København: Akademisk forlag

Kline M (1972) *Mathematical Thoughts from Ancient to Modern Times*, New York: Oxford University Press

Lyotard J (1986) *The postmodern Condition*, Manchester: Manchester University Press

OECD (2004) *University Education in Denmark, Examiner's Report,* http://www.videnskabsministeriet.dk

Piaget J (1969) *Science of Education of the Psychology of the Child*, New York: Viking Compass

Royle N (2003) *Jaques Derrida*, London: Routledge

Tarp, A. (2004a). *Applying Mathe-Matics, Mathe-Matism or Meta-Matics.* Paper accepted for presentation at the Topic Study Group 20. The 10th International Conference on Mathematics Education. Copenhagen, Denmark.

Tarp, A. (2006). *The 12 Math-Blunders of Killer-Mathematics*. Paper rejected at the Madif 5 conference in Malmoe Sweden 2006. www.MATHeCADEMY.net

Zybartas S & Tarp A (2005). One Digit Mathematics. *Pedagogika (78/2005)*, Vilnius, Lithuania.

Discourse Protection in Mathematics Education

Social theory describes two kinds of social systems. One uses education to enlighten its people so it can practice democracy. One uses education to force upon people open or hidden patronization. A number-language is a central part of education. Two number-languages exist. Mathematics from-below is a physical science investigating the natural fact Many in a 'manyology' presenting its concepts as abstractions from examples. Mathematics from-above is a meta-physical science claiming Many to be an example of 'metamatics' presenting its concepts as examples from abstractions. Foucault's discourse theory explains why manyology is suppressed and why even enlightening education patronizes by presenting mathematics from-above instead of from-below.

Investigating the natural fact many

To survive in space and time humans must deal with Many being present all over: a trip has many steps, a period has many sunsets, a tree has many fruits, a pig has many offspring many times etc.

Surviving by agriculture it becomes necessary to distinguish between different degrees of Many by counting, assigning numbers to Many by bundling & stacking resulting in e.g. 'five tens three' sticks, or more precisely five bundles and three unbundled sticks. Since also bundles can be bundled we might also meet 'three bundles-of-bundles four bundles five unbundled' also called three hundred forty five, or more precisely three tens-of-tens and four tens and five, which can be written with symbols as 3*X*X + 4*X + 5 or 3*X^2 + 4*X + 5 using the Roman symbol X for ten, alternatively written as 10 to show that ten ones is the same as 1 ten-bundle and 0 unbundled.

So a given degree of Many will always be split into a union of unbundled, bundles, bundles-of-bundles, bundles-of-bundles-of-bundles etc. This creates the root of splitting and uniting Many, i.e. of reuniting Many, called algebra in Arabic. And uniting sticks is the root of operations as addition, repeated addition called multiplication, and repeated multiplication called power. Once created, operations can take place not only with bundles but also with numbers creating calculations for prediction: 5+3 predicts the result when counting on from 5 3 times, 5*3 predicts the result when adding 5 3times, 5^3 predicts the result when multiplying with 5 3 times.

Adding two numbers might create an overload to be rebundled: 4ten7 + 3ten 8 = 7ten15 = 8ten5. And multiplying two numbers can be illustrated by a 2x2 square splitting the numbers into its bundles and unbundled. Thus multiplying 47 and 38 leads to overloads to be rebundled:

	4ten	7	
	12tenten	21ten	3ten
	32ten	56	8
12tenten	53ten	56	

4ten7*3ten8 = 12tenten + 32ten + 21ten + 56 = 12tenten + 53ten + 56 = 12tenten + 58ten + 6 = 17tenten + 8ten + 6 = 1tententen + 7tenten + 8ten + 6 = 1 thousand 7 hundreds 8 tens 6.

The need to reverse calculations is the root of inverse operations predicting the answers. Thus x=20–5 predicts the answer to the reversed calculation x+5=20, x = 20/5 predicts the answer to x*5=20, x=5√20 predicts the answer to x^5=20, and x=log5(20) predicts the answer to 5^x=20.

Present as physical quantities, Many always carries units as e.g. $ or kg. Double-counting the same quantity in two different units create per-numbers as 4$ per 5 kg or 4$/5kg or 4/5 $/kg.

Per-numbers must be transformed to unit-numbers before being added:

10kg at 4$/5kg + 24kg at 7$/8kg = 8$ + 21$ = 29$ = 34kg at 29/34 $/kg.

Finally adding stacks with different bundle-sizes as 2 3s and 4 5s can take place on-top or next-to. In the case of on-top addition the units must be the same, which is the root of changing units. In the case of next-to addition the units are added, thus creating the root of adding by integration.

Agriculture implies measuring land, which becomes the root of land-measuring called geometry in Greek. Any form can be seen as a union of triangles, themselves seen as a union of two right-angled triangles. A triangle consists of three copies of a line turned three times. Two intersecting lines form a double-angle adding up to half a full-turn. So three double-angles add up to three times half a full-turn, but since the outside angles add up to one full-turn, the inside angles must add up to half a full-turn. Also it is straightforward to see that in a square both the base and the height and the diagonal can create squares where the diagonal squared is the sum of the squares of the height and the base, a relation that holds when tested on rectangles. Likewise, in a rectangle the base and the height can be counted in diagonals instead of in meters, thus creating sine and cosine.

So as to knowledge about Many, 'manyology', it is natural to adopt the Arabic and Greek names algebra and geometry for its two main parts, reuniting numbers and measuring land.

The absence of a manyology

And indeed, what is called mathematics does contain geometry and algebra as its two main ingredients. But in geometry calculating triangles is isolated in a separate field called trigonometry, reserving geometry itself to be deducing theorems from axioms. And algebra doesn't recognize its Arabic meaning as the task of reuniting numbers. Instead algebra presents itself as the art of searching for patterns. Almost no manyology is present in what is called mathematics (NCTM, 2000) that never introduces e.g. 1digit mathematics (Zybartas, 2005).

In manyology a natural number is a decimal number caring a unit and using the decimal to separate the bundles from the unbundled, as e.g. 2.1 3s. In mathematics only ten-bundling is allowed and 2.1 tens is written as 21 leaving out the unit ten and misplacing the decimal point.

In manyology 10 is a sloppy way of writing 1 bundle and no unbundled, e.g. 1.0 3s or 1.0 8s or 1.0 tens. Thus 10 might be the follower of 2, 7 or 9 depending on the bundle size. In mathematics 10 can only mean 1.0 tens and here 10 IS the follower of nine.

In manyology the natural operation order is: first division used in the counting process to take away bundles of e.g. 5s, then multiplication when 3 5s are stacked as 3*5, then subtraction when the overload 7 5s is rebundled by removing 5 1s as 1 5s to 1.2 5s. Finally addition has two meanings, adding next-to or adding on-top. In mathematics the order is the opposite and addition IS on-top.

In manyology adding on-top and next-to is introduced in grade 1 thus becoming the roots of changing units and integration. In mathematics changing units is called proportionality, which is postponed to middle school and presented as an example of linearity. And adding next-to is called integration and postponed to late secondary school and presented as an example of a limit process.

In manyology per-numbers must be transformed into unit-numbers before adding thus being a middle school generalization of primary school's adding next-to. In mathematics per-numbers are renamed to rational numbers presented as examples of equivalence classes in a set-product; and added as fractions without respect to their units; and only including the unit in integral calculus where the per-number f(x) is transformed into a unit-number f(x)dx before added as integration.

In manyology a calculation is a number-prediction. In mathematics a calculation is called a number-name and presented as an example of an element in a set organized with a binary operation.

In manyology moving a number to the other side of the equal sign with a reversed calculation sign solves a reversed calculation. In mathematics a reversed calculation is called an equation presented as an example of an equivalence relation with a truth set determined by performing identical operation to both sides of the equal sign.

In manyology formulas describe how a total T is created by uniting numbers, thus T = b+a*x describes how T is an initial number b united x times with a constant number a. In mathematics a formula as T = b+a*x is called a linear function being presented as an example of a general function again being presented as an example of a set relation having the property that first component identity implies second component identity.

From these observations we see, that in manyology concepts are rooted in examples and presented ‘from-below’ as abstractions from examples; and that in mathematics concepts are rooted in abstractions and presented ‘from-above’ as examples of abstractions. So basically mathematics is manyology turned upside down. Social theory might be able to explain this ‘upside-down paradox’ turning the natural science about Many, manyology, upside down to ‘metamatism’ combining ‘metamatics’, presenting concepts as examples of abstractions instead of as abstractions from examples, with ‘mathematism’ true in the library but often not in the laboratory (Tarp, 2009).

Social theory

Social theory has human interaction as its main focus. As to communication, the most basic interaction, Berne has developed a transactional analysis describing three different ego-states:

> In a given individual, a certain set of behaviour patterns corresponds to one state of minds, while another set is related to a different psychic attitude, often inconsistent with the first. These changes and differences give rise to the idea of *ego states*. (..) Colloquially their exhibitions are called Parent, Adult and Child (..) The unit of social intercourse is called a transaction (..) Simple transactional analysis is concerned with diagnosing which ego state implemented the transactional stimulus, and which one executed the transactional response (Berne, 1964: 23, 29)

Berne’s concepts reflect the social fact that interaction between human beings can be patronized and non-democratic, or it can be non-patronized and democratic. In a family the interaction between children and parents will typically be one of patronization. In a society adult interaction typically will be non-patronized, unless the society is a non-democratic autocracy where patronization is carried on into adulthood. In this way Berne describes the main problem in human interaction, the choice between patronization and self-determination or ‘Mündigkeit’. The fact that the German word ‘Mündigkeit’ does not have en English equivalent indicates that social interaction is quite different outside the EU and inside where the presence of and resistance against patronization createted the label ‘Mündigkeit’; whereas the absence of patronization doesn’t call for labeling resistance against patronization.

In his theory Berne points out is that in order to be successful, transactions must be parallel: Both parts must agree as to whether patronization is needed or not in the given situation. If the transaction is crossing, the interaction is unbalanced and no information can be exchanged.

The debate on patronization runs all the way though the history of social theory (Russell, 1945; Ritzer, 1996). In ancient Greece the sophists warned against hidden patronization coming from choices presented as nature. Hence to protect democracy, people should be enlightened to tell choice from nature. To the philosophers choice was an illusion since according to their view everything physical is examples of meta-physical forms only visible to people educated at the Plato academy. Consequently patronization was a natural order with the philosophers as protectors.

In the middle age the patronization question reappeared in the controversy on universals between the realists and the nominalists. Here the realist took the Plato standpoint by renaming his metaphysical forms to universals claimed to have independent existence and to be exemplified in the physical world, and consequently waiting to be discovered by philosophers. In contrast to this the nominalist saw universals as invented names facilitating human interaction.

The Renaissance period saw a protestant uprising against the patronization of the Roman Catholic Church resulting in the bloody 30year war from 1618. To avoid the chaos of war Hobbes in his book ‘Leviathan’ argues that to protect themselves against their natural egoistic state, humans would have a much better life if accepting the patronization of an autocratic monarch.

In natural science Newton discovered that the moon doesn’t move among the stars, instead it falls towards the earth, as does the apple, both following their own physical will and not the will of a metaphysical patronizor. This discovery inspired Locke to argue against patronization: ‘John Locke is the apostle of the Revolution of 1688, the most moderate and the most successful of all revolutions. Its aims were modest, but they were exactly achieved, and no subsequent revolution has hitherto been found necessary in England’ (Russell, 1945). Locke’s chief work, the Essay Concerning Human Understanding, was highly inspirational in the Enlightenment 1700-century, which resulted in two democracies being installed, one in the US and one in France.

American sociology describes human interaction based upon enlightenment and freed from patronization. Its ‘it is true if it works’-pragmatism expressed by Peirce and James leads on to symbolic interactionism and to the natural empery-rooted research paradigm Grounded Theory resonating with the principles of natural learning expressed by Piaget. In harmony with this the US enlightenment school, being organized in blocks and aiming at enlightening as many as possible as much as possible, has set the international standard followed worldwide outside Europe.

Inside Europe reaction against the Enlightenment came from Germany where Hegel reinstalled metaphysical patronization in the form of a Spirit expressing itself through the history of the people.

Marx develops Hegel thinking into Marxism claiming that until a socialist utopia has been established a socialist party serving the interest of the working people should patronize people. In contrast to this Nietzsche argued that only by freeing itself from meta-physical philosophical hegemony would the western individuals be able to realize their full potentials. Marxist thinking developed into critical theory in the Frankfurt school infiltrating the 1968 student revolt so that EU’s Bildung universities could carry on protecting its Hegel-based patronizing discourses.

Wanted to protect its republic against patronization, France developed post-structuralism inspired by Nietzsche’s opposition against Hegel and by Heidegger’s question ‘what is IS?’

Derrida introduces ‘logocentrism’ to warn against patronizing words installing what they label and recommends that such categories be deconstructed. Lyotard introduces ‘postmodern’ to warn against sentences taking the form of ‘meta-narratives’ claiming to be truths and recommends paralogy as research inventing dissensus to the ruling consensus. Foucault uses the word ‘pastoral power’ to warn against patronizing institutions promising to cure human abnormalities installed by discourses claiming to be disciplines producing truths about humans. He shows how disciplines discipline itself and its object, in contrast to a natural discipline disciplining itself by its objects. Foucault also describes doctrines and other techniques used for discourse protection.

Bauman points out that by following authorized routines modernity can create both gas turbines and gas chambers (Baumann, 1989). Analyzing the latter, Ahrendt (Ahrendt, 1968) shows how in industrialized societies patronization might become totalitarian thus reintroducing evil actions this time rooted not in inspiration from a devil but in the sheer banality of just following orders.

Discourse Protection and Hegemony

Mathematics can be rooted in examples ‘from-below’ as well as in abstractions ‘from-above’, but only the latter presentation exists in mathematics education. Can social theory explain this? If the question of patronization is the key issue in social theory, this question can be reformulated to ‘does mathematics education contain elements of hidden patronization?’

From the perspective of the ancient Greek sophists, mathematics from-above is an example of hidden patronization installed by a choice presented as nature; a choice made by their opponents, the philosophers, seeing geometry as demonstrating how physical forms are examples of meta-physical structures only visible to them, consequently needed for patronizing through education.

Ancient Greek thus created two different forms of schooling: an enlightening school wanting to inform the people about the difference between choice and nature to prevent hidden patronization by choices presented as nature; and a patronizing school wanting to demonstrate how philosophical knowledge is exemplified in everyday life thus in the need of openly philosophical patronization.

The Enlightenment period installed two democracies, one in the US and one in France. The US democracy created an enlightening school organized in blocks to be chosen freely. Today this is the international school standard outside the EU. Inside the EU its Bildung schools are still organized in lines forcing students to follow predetermined block combinations and forcing them to wait for years for an exam that cannot be retaken; in contrast to the block-organized schools having half-year exams that can always be retaken. At enlightening schools the outside world determines the curriculum and the exams. To determine the content of Bildung, EU needs to be patronized by strong central administrations and by a special educational discourse called didactics. Historically, the Bildung schools were invented in Prussia just after 1800 using Hegel based romanticism to obtain three goals: to keep the people unenlightened so it will not ask for democracy as in France; to install a feeling of nationalism into the people so that it could protect itself against the French and their democracy; and to sort out the population elite for central administration offices.

From the perspective of the contemporary sophists, the French poststructuralists, presentations can be seen as examples of discourses fighting each other to win the monopoly of representing truth and thus to establish what Foucault calls pastoral power and discourse protection.

At universities the mathematics from-above discourse took over power with the introduction of set-theory just before 1900. And it has managed to stay in power despite of its internal problems as demonstrated by Russell showing a set-based definition will never be well defined (If M = $\{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$'); and by Gödel showing that truths is not always provable.

At schools the mathematics from-above discourse took over power as ‘modern mathematics’. The traditional ‘Rechnung’ discourse disappeared since it was no longer seen as the root of but only as a simple application of mathematics that ‘of course’ must be learned before it can be applied.

It seems natural that Bildung schools with its patronizing goal and wish to sort out the elite for the central administration has chosen the mathematics from-above discourse as its curriculum But it seems odd that also enlightening schools does the same since it keeps many students unenlightened by using its defining IS-statements to forces false identities upon the natural fact Many.

Thus 2 ten-bundles and 3 unbundled is sentenced to be an example of a position system description 23 instead of enjoying its nature as the double stack consisting of 2.3 tens. 3*6 is sentenced to be an example of the category ‘number-name’ instead of enjoying its nature as a calculation predicting that 3 6s can be recounted as 1.8 tens. 3*x = 18 is sentenced to be an example of an equation and is forced to be solved by performing identical operation to both sides of the equation sign, instead of enjoying its nature as a reversed calculation that that can be re-reversed by moving numbers to the other side and reversing its calculation sign. 1/2 and 2/3 are sentenced to be examples of rational numbers and are forced to be added without respect to their units instead of enjoying their nature as per-numbers needing their units to be added. Shifting units as 2$ = ?£ is sentenced to be an example of proportionality instead of enjoying its nature as a recounting problem. The question 2 3s + 4 5s is sentenced to deportation from the discourse instead of enjoying its nature as two stacks being added either on-top or next-to thus constituting the root of proportionality and integration. Adding repeatedly 3$ or 3% to 200$ is sentenced to be examples of linear and exponential functions, instead of enjoying their nature as growth by adding or by multiplying. A function is sentenced to be an example of a set-product where first component identity implies second component identity instead of enjoying its nature as a formula containing only two unknowns. The question ‘5 seconds at 4m/s increasing to 6m/s gives ?m’ is sentenced to be an example of integral calculus, again being sentenced to be an example of a limit process, instead of enjoying its natures as uniting per-numbers by the area under the per-number graph.

With false identities forced upon it by the ruling discourse, students are not allowed to meet the root of mathematics, Many, in its materiality but only as examples of false identities. Thus the ruling from-above discourse becomes a clear example of hidden patronization becoming pastoral by hiding its natural alternative, manyology, rooting mathematics from-below in the natural fact Many.

The fact that also enlightening schools chooses the patronizing mathematics from-above discourse shows that this discourse has developed into a totalitarian discourse penetrating all levels of education thus forcing the teachers to perform 'Eichmann-teaching' just following the orders. To allow teachers to instead become enlighteners, they should be exposed to both mathematics discourses. However, discourse protection makes this very difficult. This author has observed many different examples of discourse protection preventing his work on manyology to be publicly know: applications for university jobs were refused on the ground that the work contains too few references to the ruling consensus; application for a professorship was refused by a professor having himself written neither a phd or a dissertation on the ground that the work did not contain articles published in the ruling journals; application for defense of a paralogy thesis on postmodern mathematics (Tarp, 2007) was refused by a person having not written a dissertation on the ground that the work falls outside the ruling discourse, which is precisely the point of paralogy research.

This leaves only conferences on mathematics education as breathing holes. The first MADIF conferences allowed presentation of off-discourse papers (Tarp, 2001). This however has changed.

Moo Review and Tabloid Review

At the MADIF 6 conference reviewers were asked to answer the following questions:

> Note that a paper can be philosophic/ theoretical, without presenting new empirical data, or an empirical research report. In both cases the headings below apply, though possibly to different aspects and degrees.
>
> 1 Does the paper state clear research question(s)? Yes/No, Comments:
>
> 2 Does the paper present a relevant theoretical framework? Yes/No, Comments:
>
> 3 Does the paper relate to relevant literature in the area? Yes/No, Comments:
>
> 4 Does the paper show methods used in a transparent way? Comments:
>
> 5 Does the paper expose results and discuss them linking the theory to the data, and discussing the answers to the research questions? Yes/No, Comments:
>
> 6 How do you judge the scientific quality of the paper? High/Intermediate/Low, Comments:
>
> 7 Is the paper interesting /relevant to the mathematics education research community? Yes/No, Comments
>
> 8 Do you recommend accepting the paper for presentation at the conference? Yes/No, Comments:
>
> 9 Which are your suggestions to the author in order to improve the paper:

The questions 1, 2, 4 and 5 relate to the research-genre characterized as a text generated by a research question and using a theoretical framework and a method to reach a result. The rest of the questions contain verdict-adjectives as 'relevant, scientific, acceptable, interesting, and improvable' without specifying what qualities must be present as grounds for such judgments.

The absence of grounding questions as 'Does the paper follow traditional paradigms and correctness, or does it provide new perspectives and new paradigms?' suggests the existence of a hidden doctrine wanting to restrict papers to those respecting the ruling discourse. This suspicion is confirmed by the fact that the conference allows 'moo-review' containing only a single sound as nay or aye, and 'tabloid review' containing only a single sentence; both being in conflict with the research genre demanding referenced arguments.

Thus at the conference the paper 'Mathematics: Grounded Enlightenment - or Pastoral Salvation' (Tarp, 2008) had three reviewers. The reviews contained 13 examples of moo-review, and 12 examples of tabloid-review. Only 2 statements contained 2 sentences. One answered question 3 as: 'No, since there are no research questions, it is impossible to say. It seems as if the authors try to prove a political statement'. This case shows how Foucault discourse protection is carried out at conferences by mixing genre-related and discourse-related questions in the review task, and by not rejecting reviews using moo-review and tabloid-review.

This paper will probably be rejected by the same technique.

Conclusion

The natural fact Many can be presented as the root of or as examples of mathematical abstractions. However, in mathematics and its education only the latter case exist. Seeing presentations as discourses, social theory explains how discourses fight for domination and how the victorious discourse becomes a discipline by claiming to represent truth. Once in power, a discipline uses discourse protection to discipline itself, as well as its subjects by forcing false identities upon them so that the natural fact Many IS an example of the ruling discourse. Thus the ruling discourse becomes totalitarian penetrating all levels of education including textbooks and teaching transforming the individual teacher from an enlightener to a patronizor just following orders. Unable to defend itself against opponents, the ruling discourse protects itself by bureaucratic reference-counting and closes the potential breathing holes for dialogues, conferences, by allowing paper refusal to be ungrounded. Maybe the time has come to replace from-above patronization with from-below enlightenment - and to replace gas with jazz? To do so, conferences should respect its scientific purpose; and should stop practicing discourse protection by allowing moo-review.

References

Ahrendt, H. (1968). *The Origins of Totalitarianism.* Orlando: Hartcourt.

Bauman, Z. (1989). *Modernity and the Holocaust.* Oxford: Polity Press.

Berne, E. (1964). *Games People Play.* New York: Ballantine Books.

NCTM (2000). *Principles and Standards for School Mathematics.* National Council of Teachers of Mathematics. Reston VA.

Ritzer, G. (1996). *Modern Sociological Theory.* New York: The McGraw Hill Companies.

Russell B. (1945). *A History of Western Philosophy.* New York: A Touchstone Book.

Tarp, A. (2002). Killer-Equations, Job Threats and Syntax Errors, A Postmodern Search for Hidden Contingency in Mathematics. In C. Bergsten, G. Dahland & B. Grevholm (Eds.), *Research and Action in the Mathematics Classroom.* Proceedings of the 2nd Swedish Mathematics Education Research Seminar, MADIF 2 (pp. 138-161). Linkoping, Sweden: SMDF No. 1.

Tarp, A. (2007). *PostModern Mathematics, a Hidden Difference to Modern MetaMatism*. Thesis. http://mathecademy.net/papers/thesis/.

Tarp, A. (2008). *Mathematics: Grounded Enlightenment - or Pastoral Salvation*. Paper rejected at the Madif 6 conference in Stockholm Sweden 08. http://mathecademy.net/papers/other-papers/.

Tarp, A. (2009). Applying Pastoral Metamatism or re-applying Grounded Mathematics. In Blomhøj M. & Carreira S. (eds.) *Mathematical applications and modelling in the teaching and learning of mathematics.* Proceedings from Topic Study Group 21 at ICME 11 2008. Roskilde, Denmark.

Zybartas, S. & Tarp, A. (2005). One Digit Mathematics. *Pedagogika* (78/2005), Vilnius, Lithuania.

Appendix: the case of equations

Equations are among the most important concepts in mathematics, appearing when a formula has only one unknown left after entering the known numbers. A typical introduction to equations at the basic algebra level is the following:

An example of an equation is the following open statement 3*x+14 = 5*x+2. By entering x = 2 we get 20 = 12 so x = 2 is not a solution. By entering x = 3 we get 23 = 17 so x = 3 is not a solution. By entering x = 6 we get 32 = 32 so x = 6 is a solution. To find the solution directly we are allowed to perform identical operations to both sides of the equation:

3*x + 14 = 5*x + 2 3*x + 14 –2 = 5*x + 2 –2 3*x + 12 = 5*x	3*x + 12 –3*x = 5*x –3*x 12 = 2*x 12/2 = 2*x/2	6 = x so x = 6 is the solution

At first sight this seems to be a good introduction to equations. After all, equations exist out there and of course good education will prepare the students to meet what exists, so nothing seems to be wrong here. However, a closer look will uncover this approach to be a pastoral approach that by presenting its choice as nature hides its natural alternative.

It is correct that equations exist, but they exist inside a discourse. The important question to ask is: what outside the discourse created the discourse? What is the root of the discourse? What is the root of equations?

In Arabic, algebra means reuniting. The uniting-question '3 and 5 unite to what?' can be written as '3+5 = x'; and its opposite question '3 and what unite to 8?' as '3+x = 8'. The latter can be seen as a splitting question asking '8 can be split into 3 and what?' Also the latter can be seen as an example of backward or reversed calculation: 'What should I do to 8 to find the number that added to 3 gives 8?' In both cases the answer can be found by guessing, but the root of mathematics is number-prediction, so an operation is invented that predicts the answer directly, in this case subtraction, the opposite of addition. Thus the answer to the question 3+x = 8 is predicted by the calculation x = 8–3.

Repetition is the root of forward operations: 3+5 predicts the answer to the question: what happens when counting on from 3 5 times? 3*5 predicts the answer to the question: what happens when adding 3 5 times? And 3^5 predicts the answer to the question: what happens when 3 is a factor 5 times. And since any calculation can be reversed, reverse operations are invented to give the answers to the following reversed calculations also called equations:

3 + x = 7 x = 7 – 3	3*x = 12 $x = \frac{12}{3}$	3^x = 243 $x = \frac{\log 243}{\log 3}$	x^5 =243 $x = \sqrt[5]{243}$

Thus the definition of the reverse operations gives a very simple way of solving equations: moving numbers to the other side of the equal sign reversing their calculation sign solves equations. So the roots of equations are splitting jobs and reversed calculations that at the same time give a method for solving equations.

However, both are hidden by the ruling discourse becoming pastoral by hiding its alternatives and by presenting its unnatural choices as nature.

With a graphical display calculator the left hand and a right hand side are called respectively Y1 and Y2. They can be entered on the Y-list and graphed providing the geometry solution as the intersection point. And as to algebra, the Math-solver 0 = Y1-Y2 gives the same solution. So all that is needed is to enter the two sides of the equation on the Y-list. Using technology to solve the equation, human brainpower can be used to set up equations, typically by choosing between different regressions formulas, since a calculator can use regression to transform tables to formulas.

The case of equations illustrates the difference between pastoral discourse-protecting research and anti-pastoral contingency research. The discourse-protecting research accepts as nature the choices of mathematics 'from-above' that becomes patronizing by being presented as nature thus hiding its alternatives. Contingency-research or paralogy research uncovers the pastoral nature of these choices by discovering hidden alternatives (Tarp, 2007).

So confronted with student learning problems within equations, discourse protecting research can only describe problems, it cannot solve the problems by suggesting and trying out alternatives to a pastoral tradition. And neglecting the hidden alternatives has big advantages since it protects not only the discourse itself, but also it protects the learning problems that finance the discourse.

And in Bildung line organized schools one further advantage is that protecting learning problems is an effective way to sort out the elite for the central administration and to keep the general population unenlightened. In contrast to this, contingency research is able to suggest alternatives, one of which might be the root of the actual concept; and is able to test alternatives to see if they make a positive difference to the purpose of schooling, learning. So contingency research might be able to solve learning problems so that all learns all.

Post-Constructivism

Even if constructivism has been its major paradigm for several decades the relevance paradoxes in mathematics education remain; and furthermore constructivism has created a mathematics war between primary and secondary school, and between parents and teachers. Constructivism believes that numbers are meaningful and that algorithms are meaningless thus allowing students to construct their own algorithms. But maybe it is the other way around? Maybe a two-digit number is a highly abstract concept that, if not introduced slowly through cup-writing, may be meaningless to students; whereas algorithms introduced as internal trade between two neighbour cups is meaningful.

The background of this paper is the worldwide crisis in mathematics education indicated by enrolment problems in mathematical based educations (Jensen et al, 1998); by 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371); and by an 'irrelevance paradox' created by the fact that the volume of the mathematics education research increases together with the volume of problems it studies and aims to solve, thus being unable to be validated by solving the problems of the mathematics classroom (Tarp, 2004).

In a plenary address to the ICME10 conference Anna Sfard mentioned the second focus turn in mathematics education research, first the constructivist-turn from the curriculum to the learner and now the participant-turn from the learner to the teacher. This turn away from constructivism seems to indicate that constructivism will not be able to solve the crisis in mathematics education.

Constructivism

To see where constructivism went wrong we return to Piaget:

> To educate means to adapt the individual to the surrounding social environment. ... The traditional school imposes his work on the student: it "makes him work". ... The new school, on the contrary, appeals to real activity, to spontaneous work based upon personal need and interest. ... Is childhood capable of this activity, characteristic of the highest forms of adult behaviour: diligent and continuous research, springing from a spontaneous need? – that is the central problem of the new education. ... But all these psychologists agree in accepting that intelligence begins by being practical, or sensorimotor, in nature before gradually interiorising itself to become thought in the strict sense, and in recognizing that its activity is a continuous process of construction. ... In other words, intelligence is adaptation in its highest form, the balance between a continuous assimilation of things to activity proper and the accommodation of those assimilative schemata to things themselves. (Piaget, 1969, pp. 151, 152, 158)

So according to Piaget the individual constructs internal schemata to create meaning in outside practise through assimilation, and when meeting something meaningless existing schemata are accommodated or new schemata are constructed.

In mathematics education constructivism grew out of observing that instead of using the algorithms they were taught students invented their own algorithms. Traditionally this has been interpreted that numbers made sense, but algorithms did not, therefore students should be allowed to construct their own algorithms, and teaching should focus on developing students' number sense.

This constructivist no-forced-algorithms principle however created 'mathematics-wars', one between primary and secondary school, and one between primary school and the parents. Secondary school complained that many students did not have the knowledge necessary to begin secondary education. And parents rebelled against the schools unwillingness to teach algorithms by teaching them at home.

Sceptical Cinderella research, to be described later, sees a typical Cinderella situation here. Maybe the labels good and bad should be turned around, maybe numbers are meaningless and algorithms meaningful?

Numbers

This question can be illustrated by a classroom example described by Brown, also sceptical towards constructivism and advocating a hermeneutic approach to be discussed later.

> A group of six year old students were working on some problems set by their teacher. These involve using "base ten strips in tackling double digit addition. The teacher's speech was very brief and sparse, consisting, almost entirely, of requests such as "Make 34", "Now make 21" and "Now put them together". The students, it seemed, were expected to make the appropriate arrangements with the strips and then write the sums in their books. It was quite noticeable that few of the teacher's requests were carried out immediately, but rather, arrangements of strips were made after much deliberation. (Brown, 1997, 104-105)

In this lesson the students should think and communicate about two-digit addition while solving the problems in a practical way by means of 'base ten strips'. Apparently this follows the constructivist principle: addition of two-digit numbers is a central part of mathematics, and it should be learned through working with and communicating about practical materials. But still the students had problems.

The subjects of the lesson are the two words 'addition' and 'two-digit number'. Clearly addition has the practical root of bringing things together. Also 1digit numbers have a practical root in the degree of many they describe.

With two-digit numbers, however, it is different. The traditional way of making sense of two-digit numbers is 23 = 2*10 + 3*1. But then we cannot make sense of the number 10: 10 = 1*10 + 0*1, which is a meaningless circular self-reference only becoming meaningful through constructivism. The problem is that ten is the only number having a name but not a symbol unless we use the Roman symbol: 10 = 1*X + 0*1 which is problematic since X is not a number symbol.

So the Cinderella question arises: is there a hidden Cinderella-difference that makes a difference by having a practical root? Can we find a way to two digit-numbers that avoid the problems with the number ten, and that follows the Piaget 'from practice to thought' advice?

As an exemplary case we take a total of nine matches. First we count in ones, and the one taken away is added to the icon already built from the four strokes in the number icon 4, etc. thus observing that a number icon is just a rearrangement of the degree of many it describes if written in a less sloppy way. This observation holds until ten.

Then we count the total in bundles of 2s, 3s, 4s, 5s etc. to be stacked as a single stack (T = 3*3) or as a stock consisting of two stacks (T = 2*4 + 1*1). Then we symbolise or code the bundle with e.g. a plastic C making T = 2*C + 1*1. Then we introduce a cup for the Cs and a cup for the 1s. Then we begin to use matches in both cups knowing that the matches in the left cup count Cs and in the right cup counts 1s. Then we change to cup-writing T = 2*C + 1*1 = 2)1). Later we become lazy and just writes 21 knowing that the 21 = 2)1) = 2*C + 1*1 = 2*4 +1*1 = 2* IIII + I, i.e. that 2 belongs to the left cup, thus counting Cs, which are 4s, where 4 is an icon for the multiplicity IIII.

Finally we count in stacks, e.g. 2*2 stacks, symbolizing a 2bundle as a plastic C and a 2*2 stack as a plastic S: T = 7 = 1*S + 1*C + 1*1 = 1)1)1) = 111, a 3digit number.

Here a path is discovered leading from practice to though where the schemata are gradually accommodated to assimilate and make sense of 2digit numbers.

Now we can practise changing: 'Counted in 3s, 25 = 32 since 1 3s can be changed to 3 1s: T = 25 = 2)5) = 2+1)-3+5) = 3)2) = 32'

Also we can practise recounting: 'Counted in 3s I have 24, what do I get counting in 5s?' Or in written form: T= 2*3+4*1= ?*5. The numbers 3 and 5 can be chosen by throwing a dice where 1 means e.g. 7.

After having learned how to count and recount the rest of mathematics can be learned with one digit numbers only (see Zybartas et al., 2004, for details). Or, if the parents are impatient, 24 can also be interpreted as 2 tens and 4 1s giving no problem since its definition go back to a practice: 24 = 2)4) = 2*X + 4*1 = 2*IIIIIIIIII + 4*I. Now the definition of ten is not circular any more: 10 = 1)0) = 1*X + 0*1 = 1*IIIIIIIIII.

During these counting and recounting practices the students have learned to divide since counting in 2s is division by 2 which can be predicted by a calculator able to show integer division T = (9/2)*2 = 4*2 + 1*1 (the ‘recount-equation’).

Through recounting, multiplication becomes division depending on the bundle-size: T = 3*4 = 2*5+2*1 (i.e. 3*4 = 22) since 3 4s can be recounted as 2 5s + 2 1s. T = 3*4 means that the total is counted as 3 4s; and 3*4 is only 12 if recounted in tens. But since ten has been chosen as our standard-bundle we accept that the calculator recounts in tens.

In this way calculation and calculators becomes predictions and predictors introducing the scientific method of prediction and verification at a very early stage.

Algorithms

Once 2digit numbers make sense it is time for addition, although the problems with overloads and carrying might be eased by introducing subtraction before addition. When practising addition or subtraction a 2digit number is a store with two cup-managers, mister C and mister 1. They can do internal trade by changing 1 C to e.g. 5 1s and visa versa. They also do bookkeeping to account for what goes in and out. Thus selling 3) from a stock of 4)2) involves internal trade where 1 C is traded for 5 1s:

T= 4)2)= 4-1)+5+2)=3)7)=3)7-3+3)=3)4) & 3) using the ‘restack-equation’: T=T-3+3.

Addition as 2+3 = 5 is unreliable having countless counter-examples, e.g. 2*weeks + 3*days = 17*days. Addition only holds if the units are alike, so abstract numbers cannot be added before the units are included. Whereas 2*3 = 6 is reliable saying that 2 3s can be recounted as 6 1s. Here the unit is already present as 3s: 2*3 = 2 3s.

Using cup-writing the multiplication algorithm is:

7*23 = 7* 2)3) = 14)21) = 14+2)-20+21) = 16)1 = 161

17*23 = 17* 2)3) = 34)51) = 34+5)-50+51) = 39)1 = 391

Using cup-writing the opposite division algorithm is using the recount-equation:

85= 8)5)= 8/6*6)5)= 1*6+2)5)= 1*6)25)= 1*6)25/6*6)= 1*6)4*6+1)= 1*6)4*6) & 1

So 85 = 6*14 + 1 = 6*14 + 1/6*6 = 14 1/6 *6, so 85/6 = 14 1/6.

These algorithms constitute a direct link from the internal trade practise to the traditional algorithms thus making these meaningful. This however is only possible if 2digit numbers have been introduced as a lazy variant of cup-writing.

Thus a Cinderella point has been made: it is the numbers and not the algorithms that are meaningless. So the whole starting point of two decades of constructivism was wrong making constructivism fail its task. Should it be replaced by hermeneutics or should it return to its starting point and this time listen carefully to Piaget’s advice?

Hermeneutics

In his book about Mathematics Education and Language Brown says:

> Recognising that the perspective of participants is becoming more central within analyses of social situations, this book offers a theoretical approach to discussing the world as understood through the eyes of the participants ... Broadly this book concerns the way in which language and interpretation

> underpin the teaching and learning of mathematics. ... In particular, issues of language, understanding, communication and social evolution, all of which are tackled by recent mathematics education research under the banner of constructivism and related areas, are central themes in post-war western thinking on philosophy and the social sciences, yet research in mathematics education seems to under-utilise the resource of work done in the broader context. ... In developing my theoretical framework I will be calling on certain key-writers such as: Gadamer and Ricoeur on hermeneutics, Habermas on critical social theory, Saussure on linguistics, Derrida, Foucault and Barthes in post-structuralism and Schütz on social phenomenology. I seek to show how language is instrumental in developing mathematical understanding (Brown, 1997, p.3)

Arguing that mathematics education research should be informed by theoretical thinkers working within other areas Brown gives an overview:

> How do we reconcile the social and individual dimensions of developing mathematical understanding? Habermas recent work in social theory has sought to combine two traditions that dominated theoretical thinking during the nineteenth and early twentieth century, namely, positivism and hermeneutics. This can be seen as an attempt to reconcile scientific overviews of social situations with the experience of the people living within these situations. It displays a growing recognition of a need to integrate a fuller account of the participant's understandings within analyses of social situations ... Habermas argues that neither of these two traditions enable us in bringing into question the current status quo. ... Habermas promotes an approach to social understanding which transcends both positivism and hermeneutics. ... He sees the task of post-positivist methodology within social inquiry as being to combine the philosophical and practical with the methodological rigour of positivism, "the irreversible achievement of modern science". ... For mathematics education research, I suggest this means examining how mathematics is embedded in the performance of it. (Brown, 1997, pp.7-8)

I would like to supplement the thorough work of Brown with a different set of theoretical thinkers. Not only because they are different but because I would like my choice of theoretical thinkers to be guided by Piaget's 'from practice to thought' advice by uncovering the practical situations that lead to the work of these thinkers.

In doing so I begin with the basic interaction between people, the conversation. Here Berne has developed what he calls a transactional analysis:

> In a given individual, a certain set of behaviour patterns corresponds to one state of minds, while another set is related to a different psychic attitude, often inconsistent with the first. These changes and differences give rise to the idea of ego states. ... Colloquially their exhibitions are called Parent, Adult and Child ... The unit of social intercourse is called a transaction ... Simple transactional analysis is concerned with diagnosing which ego state implemented the transactional stimulus, and which one executed the transactional response (Berne, 1964, pp. 23, 29)

Berne's concepts reflect the social fact that interaction between human beings can be democratic or non-democratic. Growing up in a family and as a society the parent-child non-democratic interaction will often precede the democratic interaction, which may never occur. Hence it is interesting to see what kind of thought the practice of democratic interaction could provoke. The first sources of democratic practise are from ancient Greece, where it was acknowledged that in order to practise democracy you need knowledge, sofia. And in the Greek democracy two kinds of knowledge-men were competing, the sophists and the filo-sophists or philosophers. As to the sophists Russell writes:

> The great pre-Socratic systems ... were confronted in the latter half of the fifth century by a sceptical movement ... The word "Sophist" had originally no bad connotation; it meant, as nearly as may be, what we mean by "professor." A Sophist was a man who was living by teaching young men things that, it was thought, would be useful to them in practical life. ... Plato devoted himself to caricaturing and vilifying them, but they must not be judged by his polemics ... To some extent ... the odium which the Sophists incurred, not only with the general public, but with Plato and subsequent philosophers, was due to their intellectual merit. ... The Sophists were prepared to follow an argument wherever it might lead them. Often it led to scepticism. (Russell, 1945, pp. 73-75, 78)

Democracy bases its choices on information and debate. To practise a democracy the Sophists, as e.g. Plato's half brother Antifon, taught the importance of distinguishing between information and debate, between necessity and decision:

> Correctness means not breaking any law in your own country. So the most advantageous way to be correct is to follow the correct laws in the presence of witnesses, and to follow nature's laws when alone. For the command of the law follows from arbitrariness, and the command of nature follows from necessity. The command of the law is only a decision without roots in nature, whereas the command of nature has grown from nature itself not depending on any decisions. (Antifon in Haastrup et al, 1984, p. 82, my translation).

However, the sophists were demonised by Plato arguing in his cave-story that debate was lack of enlightenment since the physical world had to be understood as a shadow of metaphysical structures. This thinking was later copied by the Christian church substituting the metaphysical structures with the metaphysical will of the Lord.

In the late Renaissance, however, scepticism reoccurred when Brahe, Kepler and Newton, by introducing the laboratory as the courtroom of correctness, were able to show that a physical will was ruling the physical world. Thus we did not need the patronisation by the Lord any more if we became enlightened instead of saved.

This scepticism created the Enlightenment and its two democracies, the French and the American. The French democracy however had a difficult time now seeing its 5th republic, which made French philosophers very sensitive toward any attack on 'la Republique', especially from words. Thus Derrida warns us against words, they are not representing but installing what they describe. Lyotard warns us against sentences, they are not representing but installing knowledge.

Foucault uses the word 'pastoral power' to warn us against believing that institutions become 'rational' by building upon the words and sentences of 'human science'; rather they use 'scientific' words as a means to install a new non-democratic patronisation:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. ... It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being ... And this implies that power of pastoral type, which over centuries ... had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions ... those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1983, pp. 213, 215)

So pastoral power comes from words installing an abnormality, and a normalizing institution to cure this abnormality with its salvation promise: 'You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction centre, the hospital, and do what we tell you'.

Thus the pastoral word 'educate' installs the 'un-educated' to be 'cured' by the institution 'education'; failing its 'cure' it is 'cured' by the institution 'research' installing new 'scientific' words as 'competence' installing the 'in-competent' to be 'cured' by 'competence development'; failing its 'cure' it is again etc.etc.

The American scepticism developed into pragmatism and symbolic interactionism with a research methodology called grounded theory sceptical to words not created as schemata from observations, in accordance with Piaget's theory of adaptation; and in accordance with Bauman's warning us against authorized routines being able to produce both a welfare society and a Holocaust (Baumann, 1989, p. 21).

Sceptical Cinderella research combines American and French institutional scepticism with a number&word paradox called the pencil-dilemma: Placed between a ruler and a dictionary a pencil can point to numbers but not to words. Hence numbers belong to necessity and can be used to produce valid conclusion based on reliable data, i.e. research. Words are not reliable but chosen interpretations, that if presented as research becomes seduction; to be lifted by sceptical Cinderella research using words, not for research but for counter-research uncovering hidden Cinderella differences making a difference. So sceptical Cinderella research could also be called postmodern counter-research inspired by Lyotard's description of postmodern science as the search for instabilities:

> Simplifying to the extreme, I define postmodern as incredulity towards metanarratives ... there are two different kinds of 'progress' in knowledge: One corresponds to a new move (a new argument) within the established rules; the other, to the invention of new rules, in other words, a change to a new game. ... We no longer have recourse to the grand narratives – we can resort neither to the dialectic of Spirit nor even to the emancipation of humanity as a validation for postmodern scientific discourse. But as we have just seen, the little narrative remains the quintessential form of imaginative invention, most particular in science. (Lyotard, 1984, pp. xxiv, 60)

Hermeneutic Research

The difference between hermeneutical research and sceptical Cinderella research can be illustrated by the classroom example from Brown's book. Brown examines

> how the students reads the situation they are in and how the significance of the teacher's input shows itself in their activity. ... My intention is to trace out some of the facets of the filter which translates teacher intention into responses by students. … Here, the same students are counting up in twos, having started with 2, 4, 6, 8,…. Their progress has not been without controversy. The following sequence occurs several minutes into the activity. Richardson: 27 now. A 2 and a 7. (Chester writes it). Richardson: 29 the 30. ... Richardson: 32 now. Chester writes it as 23 – a common mistake for him. ... Richardson: 70 now. Chester: How 70 going? Clifford: A 4 and a 6. ... Here it seems clear that the boys are affecting the contexts for each other's actions. ... They seem, however, to have completely lost touch with the direction anticipated by the teacher and their progress now has a life of its own (Brown, 1997, pp. 103-104, 112-113)

Apparently a hermeneutic approach sees the task of working with 2digit numbers as a relevant task to give to six year old students, and focuses upon developing an understanding of why the students seem to have problems, instead of discussing if the problems could be avoided by changing the task. It is as if this approach takes for granted not only the mathematical task but also mathematics itself and the fact that mathematics is difficult and by necessity will create problems to many students.

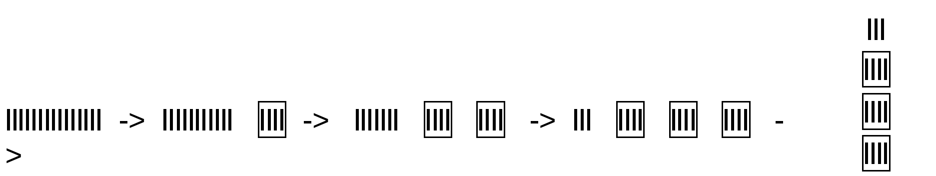
Sceptical Cinderella Research

Contrary to this, Cinderella researchers would be sceptical towards the task. Constantly needing to learn from observations Cinderella researchers give priority to the laboratory over the library by working halftime in classrooms and halftime at the university. They focus on the concerns of typical classrooms as expressed by students and teachers in their 'stories of complaints' as in the case above where the teacher complains about how difficult is to be a teacher having to accept that 'few of the teacher's requests were carried out immediately' (Brown, 1997, p.105). And the students complain about having problems understanding what the teacher is talking about.

A sceptical Cinderella researcher begins to look closely at the words in use. Are the words abstractions from observable laboratory examples, or are they examples of abstractions from the library? In short, are they LAB-words or LIB-words? This approach leads to the analysis of addition of 2digit numbers mentioned above showing that 2digit-numbers is a very abstract concept that should be introduced very carefully and slowly, and not at all be taken for granted. So the Cinderella-question arise: are there other neglected options in the house that might make the prince dance? Will it be possible to introduce some interesting mathematics dealing with one-digit numbers alone while waiting for the two digit numbers to gradually develop? So we go to the library to look up sceptical Cinderella research. Here we find a paper called 'One Digit

Mathematics’ (Zybartas et al, 2004). And at the MATHeCADEMY.net we find the agenda for ‘A Multiplicity-Based Mathematics: The Count&Add Laboratory’ included at the end of the paper.

The Zybartas paper suggests activities where multiplicity is counted as a stack by bundling and stacking: First the total is lined up, then it is bundled and equal bundles are stacked and finally the height is counted as e.g. T = 3 4s = 3*4. Also leftovers can be counted as 4s: 3 = ¾*4:

	T= 3*4 + ¾ *4 = 3 ¾ *4

Another option is to work with ‘One Digit Equations’ asking: “ How can we reverse addition? One answer is reversed calculations, also called solving equations. The recount- and the restack-equation show that equations are solved when moving a number to the other side of the equation sign reversing its calculation sign:

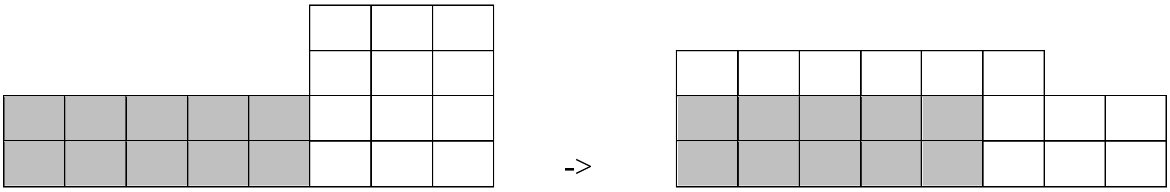
Recounting:	T	= (T/4) * 4	Restacking:	T	= (T-4) + 4
Equation Solution	T T/4	= x * 4 = x	Equation Solution	T T-4	= x + 4 = x

Still another option is to do ‘One Digit Calculus’ asking: ‘How can stocks be added differently?’ The stacks 2 5s and 4 3s can be ‘added in time’ as 3s or as 5s, or ‘added in space’ as 8s, which is called integration or calculus.

Added as 3s: T= 2 5s + 4 3s= 2*5 + 4*3= (2*5)/3*3 + 4*3= 3 1/3 * 3 + 4*3= 7 1/3 * 3

Added as 5s: T= 2 5s + 4 3s= 2*5 + 4*3= 2*5 + (4*3/5)*5= 2*5 + 2 2/5 * 5= 4 2/5 * 5

Added as 8s: T = 2 5s + 4 3s = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8

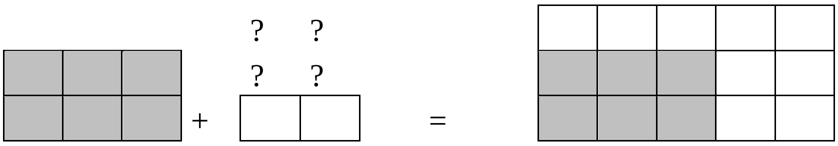


T = 2*5 + 4*3 = 2 6/8 *8 = (2*5+4*3)/8*8

Thus integration adds the per-numbers 2 and 4 as heights in stacks: 2 + 4 = 2 6/8. So 2 + 4 can give many different results, unless the units are the same:

T= 2*3 + 4*3 = 6*3 if added in time; and T= 2*3 + 4*3 = (2*3 + 4*3)/6*6 = 3*6 if added in space.

The addition process can be reversed by asking 2 3s + ? 2s = 3 5s:



The answer can be obtained by removing the 2 3s from the 3 5s and then recounting the remaining 9 in 2s as (9/2)*2= 4 1/2 * 2. Thus ?= 4 ½. This process is called differentiation.

Conclusion

Both a hermeneutic approach and a sceptical Cinderella approach are sceptical towards the contemporary constructivist tradition. But where a hermeneutic approach wants to interpret the student failure when trying to cope with the authorized routines (Bauman, 1989) of constructivism, a sceptical Cinderella approach wants to reinstall the original constructivism by replacing the authorized routines of the mathematics library with the authentic routines of a count&add laboratory.

From a hermeneutic perspective the important thing is to interpret what is happening when the students are carrying out a task sat by the teacher. From a sceptical Cinderella perspective educational tasks should be set, not by a teacher, but by a practical situation, e.g. where 19 matches are given to the group together with the task to count up in 2s, and afterwards in 3s after having predicted the result.

References

Bauman, Z. (1989). *Modernity and the Holocaust.* Oxford: Polity Press.

Berne, E. (1964). *Games People Play.* New York: Ballantine Books.

Biehler, R., Scholz, R. W., Strässer, R. & Winkelmann, B. (1994). *Didactics of Mathematics as a Scientific Discipline.* Dordrecht: Kluwer Academic Press.

Brown, T. (1997). *Mathematics Education and Language.* Dordrecht: Kluwer Academic Publishers.

Dreyfus, H. L. & Rabinow, P. (1983). *Michel Foucault.* Chicago: University of Chicago Press.

Haastrup, G. & Simonsen, A. (1984). *Sofistikken.* København: Akademisk forlag.

Jensen, J. H., Niss, M. & Wedege, T. (1998). *Justification and Enrolment Problems in Education Involving Mathematics or Physics.* Roskilde: Roskilde University Press.

Lyotard, J. (1984). *The postmodern Condition: A report on Knowledge.* Manchester: Manchester University Press.

Piaget, J. (1969). *Science of Education of the Psychology of the Child.* New York: Viking Compass.

Russell, B. (1945). *A History of Western Philosophy.* New York: A Touchstone Book.

Tarp, A. (2004). 'Mathematism and the Irrelevance of the Research Industry, A Postmodern LIB-free LAB-based Approach to our Language of Prediction'. In C. Bergsten & B. Grevholm (eds.), *Mathematics and Language.* Linkoping: SMDF.

Zybartas, S. & Tarp, A. (2004). *One Digit Mathematics.* Paper for Topic Study Group 1, ICME 10. Accessible at www.MATHeCADEMY.net.

Golden Learning Opportunities in Preschool

Preschool allows rethinking mathematics outside the tradition of ordinary school. Seeing schooling as adapting the child to the outside world containing many examples of the natural fact Many, we can ask: How will mathematics look like if built as a natural science about Many? To deal with Many we count and add. The school counts in tens, but preschool also allows counting in icons. Once counted, totals can be added. To add on-top the units are made the same through recounting, also called proportionality. To add next-to means adding areas also called integration. So accepting icon-counting and adding next-to offers golden learning opportunities in preschool that are lost when ordinary school begins.

Math in Preschool – a Great Idea

Mathematics is considered one of the school's most important subjects. So it seems to be a good idea to introduce mathematics in preschool - provided we can agree upon what we mean by mathematics.

As to its etymology Wikipedia writes that the word mathematics comes from the Greek máthēma, which, in the ancient Greek language, means "that which is learnt". Later Wikipedia writes:

In Latin, and in English until around 1700, the term mathematics more commonly meant "astrology" (or sometimes "astronomy") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. (http://en.wikipedia.org/wiki/Mathematics)

This meaning resonates with Freudenthal writing:

> Among Pythagoras' adepts there was a group that called themselves mathematicians, since they cultivated the four "mathemata", that is geometry, arithmetic, musical theory and astronomy. (Freudenthal 1973: 7)

Thus originally mathematics was a common word for knowledge present as separate disciplines as astronomy, music, geometry and arithmetic. This again resonates with the educational system in the North American republics offering courses, not in mathematics, but in its separate disciplines algebra, geometry, etc.

In contrast to this, in Europe with its autocratic past the separate disciplines called Rechnung, Arithmetik und Geomtrie in German were integrated to mathematics from grade one with the arrival of 'modern mathematics' wanting to revive the rigor of Greek geometry by defining mathematics as a collection of well-proven statements about well-defined concepts all defined as examples of the mother concept set.

Kline sees two golden periods, the Renaissance and the Enlightenment that both created and applied new mathematics by disregarding Greek geometry:

> Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

Furthermore, Gödel has proven that the concept of being well-proven is but a dream. And Russell's set-paradox questions the set-based definitions of modern mathematics by showing that talking about sets of sets leads to self-reference and contradiction as in the classical liar-paradox 'this sentence is false' being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

With no general agreement as to what mathematics is and with the negative effects of imposing rigor, preschool mathematics should disintegrate into its main ingredients, algebra meaning reuniting numbers in Arabic, and geometry meaning measuring earth in Greek; and both should be grounded in their common root, the natural fact Many. To see how, we turn to sceptical research.

Postmodern Contingency Research

Ancient Greece saw a controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that in a republic people must be enlightened about choice and nature to prevent being patronized by choices presented as nature. In contrast to this philosophers saw everything physical as examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become patronizors.

Enlightenment later had its own century that created two republics, an American and a French. Today the sophist warning is kept alive in the French republic in the postmodern sceptical thinking of Derrida, Lyotard, Foucault and Bourdieu warning against when categories, discourses, institutions and education become patronising by presenting their choices as nature (Tarp 2004).

Thus postmodern sceptical research discovers contingency, i.e. hidden alternatives to choices presented as nature. To make categories, discourses and institutions non patronizing they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget's principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

With only little agreement as to what mathematics is we ask: How will mathematics look like if built as a natural science about Many?

Building a Science about the Natural Fact Many

To deal with the natural fact Many we iconize and bundle. What could be called 'first order counting' bundles sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way icons are created for numbers until ten, the only number with a name, but without an icon.

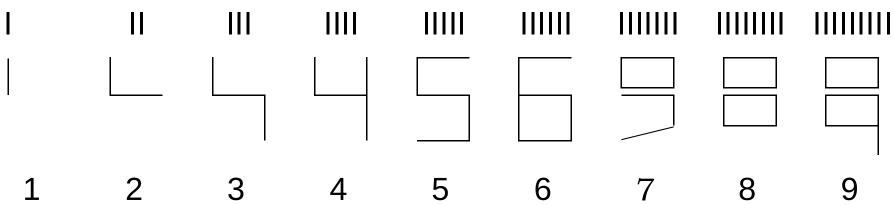


Figure 1: Icons contain as many sticks as they represent

What could be called 'second order counting' bundles in icon-bundles. So a total T of 7 1s can be bundled in 3s as T = 2 3s and 1, and placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Then the cup-contents is described by icons, first using cup-writing 2)1), then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III I -> III III) I) ->) I) -> II) I) -> 2)1) -> 2.1 3s

Using squares or LEGO blocks or an abacus, the two 3-bundles can be stacked on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a decimal number.

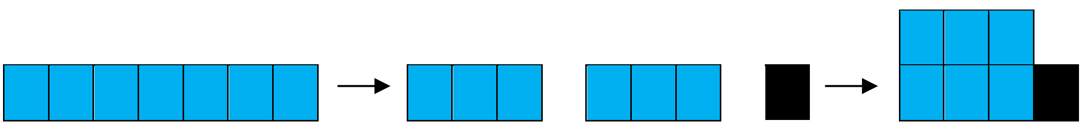


Figure 2: Seven 1s first becomes 2 3s & 1, and then 2x3 + 1 or 2.1 3s

With overloads also bundles can be bundled and placed in a new cup to the left. Thus in 6.2 3s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s:

III III) II) -> II)) II), or 6)2) = 2)0)2, or 6.2 3s = 20.2 3s.

Adding an extra cup to the right shows that multiplying with the bundle-size just moves the decimal point:

T = 2.1 3s = 2)1) -> 2)1)) = 21.0 3s

Operations iconize the bundling and stacking processes. Taking away 4 is iconized as – 4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 showing a 3 times lifting of the 4s. Placing a stack of 2 singles next to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. as 1 3^2-bundle.

Numbers and operations can be combined to calculations in formulas predicting the counting results. Counting a total T in bs can be predicted by a 'recount-formula' T = (T/b)*b telling that 'From a total T, T/b times, b can be taken away'. Thus recounting a total T = 3 5s in 6s, the prediction says T = (3x5)/6 6s.

Using a calculator we get the result '2.some' where the some is found by dragging away the 2 6s, predicted by the 'restack-formula' T = (T–b) + b telling that 'From a total T, T–b is left, when b is taken away and placed next-to'.

3 x 5 / 6	2.some
3 x 5 – 2 x 6	3

Figure 3: A calculator predicts that recounting 3 5s in 6s is 2.3 6s

The combined prediction T = 3 5s = 2 6s + 3 1s = 2.3 6 holds when tested:

IIIII IIIII IIIII -> IIIIII IIIIII III

Once counted, totals can be added on-top or next-to. To add on-top, the units must be the same, so one total must be recounted in the other total's unit. Adding stacks with the same unit might create an overload forcing the sum to be recounted in the same unit. Adding totals next-to means adding the areas, which is also called integration. Again, a next-to addition of e.g. 4 3s and 1 5s can be predicted by a calculator using the recount- and restack-formulas.

(4 x 3 + 1 x 5) / 8	2.some
(4 x 3 + 1 x 5) – 2 x 8	1

Figure 4: A calculator predicts that adding 4 3s and 1 5s as 8s is 2.1 8s

Addition can be reversed by taking away what was added. If on-top addition created an overload that was removed it must be recreated in order to take away what was added. In next-to addition what is left, when what was added is taken away, must be recounted in the original unit. Reversed addition on-top is called subtraction and reversed addition next-to is called differentiation.

The tradition counts in tens only, which can be called third order counting.

Written in its full form, 354 = 3*10^2 + 5*10 + 4*1 becomes a sum of areas placed next-to each other, thus showing the four ways to unite numbers: Addition unites variable unit numbers, multiplication unites constant unit numbers, integration unites variable per-numbers, and power unites constant per-numbers.

De-uniting a total is predicted by the inverse operations that are named subtraction, division, root and logarithm, and differentiation. Thus it makes good sense that algebra means reuniting in Arabic.

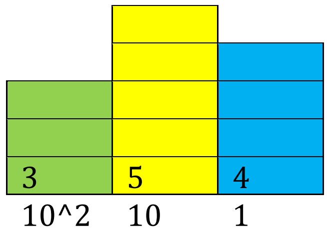


*Figure 5: The number 354 = 3*10^2 + 5*10 + 4*1 shown as stacks*

Comparing Manyology and the Tradition

Using postmodern contingency research we have discovered a natural science about Many that can be called Manyology and that allows us to deal with Many by counting and adding: First we count in icons, then in icon-bundles allowing a total to be written in a natural way as a decimal number with a unit where the decimal point separates the bundles from the unbundled. To add on-top and next-to we change the unit by recounting, predicted by a recount- and a restack-formula. Written out fully as stacked bundles, numbers show the four ways to unite: on-top and next-to addition, multiplication, and power. And to reverse addition we need inverse operations (Zybartas et al 2005), (YouTube), (Tarp 2014).

Counting Many by cup-writing and as stacked bundles contains the core of the mathematical sub-disciplines algebra and geometry. However there are fundamental differences between Manyology and traditional mathematics.

In the first an icon contains as many sticks or strokes as it represents, in the second an icon is just a symbol. In the first a natural number is a decimal number with a unit using the decimal point to separate bundles and unbundled; in the second a natural number hides the unit and misplaces the decimal point one place to the right.

The first presents operations as icons with the natural order division, multiplication, subtraction and two kinds of addition, on-top and next-to; the second presents operations as symbols; the order is the opposite; and next-to addition is neglected.

The first uses a calculator for number prediction. The second neglects it. The first allows counting in icons, the second only allows counting in tens.

With ten as THE bundle-size, recounting becomes irrelevant and impossible to predict by a calculator since asking '3 8s = ? tens' leads to T = (3x8/ten) tens that cannot be entered. Now the answer is given by multiplication, 3x8 = 24 = 2 tens + 4 1s, thus transforming multiplication into division. Likewise adding next-to is neglected and adding on-top becomes THE way to add.

Furthermore the tradition changes mathematics into 'metamatism', a combination of 'meta-matics' and 'mathema-tism' where metamatics turns mathematics upside down by presenting concepts as examples of abstractions instead of as abstractions from examples, thus insisting that numbers are examples of sets in one-to-one correspondence; and where mathematism allows addition without units, thus presenting '1+2=3' as a natural fact in spite of its many counterexamples as 1 week + 2 days = 9 days, 1 m + 2 cm = 102 cm etc.

Thus the goal of a preschool curriculum should be the golden learning opportunities coming from icon-counting and next-to addition since they both disappear when traditional metamatism suppresses Manyology from day one in school. So Manyology is an example of postmodern paralogy described by Lyotard to be a dissension to the ruling consensus (Lyotard 1984, 61).

The Traditional Preschool Mathematics

At the twelfth International Congress on Mathematical Education, ICME 12, the topic study group on Mathematics education at preschool level contains two interesting contributions from Sweden (http://www.icme12.org/sub/tsg/ tsg_last_ view.asp?tsg_param=1). The second discusses the content knowledge needed for preschool teachers to guide mathematical learning; and the first discusses the difficulties trying to categorize children behaviour according to the revised preschool curriculum in Sweden from 2011, inspired by five categories claimed by Bishop to constitute mathematics (Bishop 1988).

The five categories are counting, i.e. the use of a systematic way to compare and order discrete phenomena; locating, i.e. exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means; measuring, i.e. quantifying qualities for the purposes of comparison and ordering; designing, i.e. creating a shape or design for an object or for any part of one's spatial environment; and playing, i.e. devising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by.

Bishop's five activities reminds of Niss' eight competencies: thinking mathematically; posing and solving mathematical problem; modelling mathematically ; reasoning mathematically; representing mathematical entities; handling mathematical symbols and formalisms; communicating in, with, and about mathematics; and making use of aids and tools (Niss 2003). Both define mathematics with action words. Bishop uses general words whereas Niss is caught in self-reference by including the term mathematics in its own definition.

However, both exceed in numbers vastly the two activities of Manyology, counting and adding, so sceptical thinking could ask: Since the numbers of activities alone makes it almost impossible for teachers and children to learn, is there a hidden patronizing agenda in these longs lists since just two activities or competences are needed to deal with the natural fact Many? And is it mathematics or metamatism these lists define?

To illustrate the issue we now look at the web-based training of in-service teachers at the MATHeCADEMY.net using 'pyramid-education'.

Micro-Curricula at the MATHeCADEMY.net

The MATHeCADEMY.net sees mathematics as Manyology, the natural science about the natural fact Many. It teaches teachers to teach this natural science about Many to learners by allowing both teachers and learners to learn mathematics through investigations guided by educational questions and answers.

Seeing counting and adding as the two basic competences needed to deal with Many, it uses a CATS method, Count & Add in Time & Space, in a Count&Add laboratory where addition predicts counting-results, thus making mathematics a language for number-prediction. The website contains 2x4 study units with CATS1 for primary school and CATS2 for secondary school.

In pyramid-education 8 in-service teachers are organized in 2 teams of 4 teachers, choosing 3 pairs and 2 instructors by turn. The Academy coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach helps the instructors to correct the count&add problems. In each pair each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair. Having finished the course, each in-service teacher will 'pay' by coaching a new group of 8 in-service teachers.

Five plus Two Learning Steps

The in-service teachers learn in the same way as their students by carrying out five learning steps: to do, to name, to write, to reflect and to communicate. For a teacher two additional steps are added: to design and to carry out a learning experiment, while looking for examples of cognition, both existing recognition and new cognition. To give an example, wanting children to learn that 5 is an icon with five sticks, the steps could be:

Do: take 5 sticks and arrange them next to each other, then as the icon 5.

Say: a total of five sticks is rearranged as the number icon 5, written as T=5.

Reflect. That five sticks is called five is old cognition. It is new cognition that five sticks can be rearranged as a 5-icon and that this contains the number of sticks it represents.

Communicate. Write a postcard: ‘Dear Paul. Today I was asked to take out five sticks and rearrange them as a 5-icon. All of a sudden I realized the difference between the icon 5 and the word five, the first representing what it describes and the second representing just a sound. Best wishes’.

Design an experiment: I will help Michael, who has problems understanding 2digit numbers. Once he tries to build a number symbol for ten, eleven and twelve, he will realize how smart it is to stop inventing new symbols and instead begin to double-count bundles and unbundled. So I design an experiment asking the children to build the first twelve number-icons by rearranging sticks.

Carry out the experiment: It is my impression that constructing the number icon for ten was what broke the ice for Michael. It seems as if it enabled Michael to separate number-names from number-icons, since it made him later ask ‘Why don’t we say one-ten-seven instead of seventeen? It would make things much easier.’ This resonates with what Piaget writes:

> Intellectual adaptation is thus a process of achieving a state of balance between the assimilation of experience into the deductive structures and the accommodation of those structures to the data of experience (Piaget 1970: 153-154).

Designing a Micro-Curriculum so Michael Learns to Count

This 5-lesson micro-curriculum uses activities with concrete material to obtain its learning goals. In lesson 1 Michael learns to use sticks to build the number icons up to twelve, and to use strokes to draw them, thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy.

In lesson 2 Michael learns to count a given total in 1s and in 4s; and to count up a given total containing a specified number of 1s or of 4s.

Lesson 3 repeats lesson 2, now counting in 3s.

Lesson 4 combines lesson 2 and 3, now counting in 1s, 3s and 4s.

In lesson 5 Michael learns to recount in 4s a total already counted in 3s, both manually and by using a calculator; and vice versa.

As concrete materials anything goes in lesson 1. The other lessons will use fingers, sticks, pegs on a pegboard, beads on an abacus, and LEGO blocks.

Another 5-lesson micro-curriculum could make Michael learn to add on-top and next-to to be able to answer questions like 2 3s + 4 5s = ? 3s = ?5s = ?8s. This will not be discussed further here.

Lesson 1, Building and Drawing Number Icons

On the floor the children place six hula hoop rings next to each other as six different lands: empty-land, 1-land, 2-land, 3-land, 4-land and 5-land shown by the corresponding number of chopsticks on a piece of paper outside the ring.

Each child is asked to find a thing to place in 1-land, and to explain why. Then they are asked to turn their thing so it has the same direction as the chopstick. Finally the group walks around the room and points out examples of ‘one thing’ always including the unit, e.g. 1 chair, 1 ball, etc.

In the same way each child is asked to find a thing to place in 2-land. The instructor shows how the two chopsticks can be rearranged to form one 2-icon. The children are asked to pick up two sticks and do the same; and to draw many examples of the 2-icon on a paper discussing with the instructor why the 2-icon on the wall is slightly different from the ones they draw. Now the children are asked to rearrange their 2s in 2-land so they have the same form as the 2-icon. And again the group walks around the room and points out examples of ‘two things’ that is also called ‘one pair of things’.

This is now repeated with 3-land where three things are called one triplet.

Before going on to 4-land the instructor asks the children to do the same with empty-land. Since the empty-icon cannot be made by chopsticks the instructor ask for proposals for an empty-icon hoping that one or more will suggest the form of the ring, i.e. a circle. And again the group walks around the room to try to locate examples of ‘no things’ or zero things.

Now the activity is repeated with 4-land. Here the instructor asks the children to suggest an icon for four made by four sticks. When summing up the teacher explains that the adults have rejected the square since it reminds too much of a zero, so the top stick is turned and placed below the square to the right. Here the children are asked to rearrange their 4s in 4-land so they have the same form as a square, and as the 4-icon. And again the group walks around the room and points out examples of ‘four things’ that is also called ‘a double pair’.

Now the activity is repeated with 5-land. Here the instructor asks the children to suggest an icon for five made by 5 sticks. When summing up the teacher explains that the adults have decided to place the five stick in an s-form. When walking around the room to point out examples a discussion is initiated if ‘five things’ is the same as a pair plus a triplet, and as a double pair plus one.

This activity can carry on to design icons for the numbers from six to twelve realizing that the existing icons can be recycled if bundling in tens.

Observing and Reflecting on Lesson 1

Having designed a micro-curriculum, the in-service teacher now carries it out in a classroom looking for examples of recognition and new cognition.

One teacher noticed the confusion created by asking the children to bring things to empty-land. It disappeared when one child was asked what he had just put into the ring and answered no elephant. Now all of the children were eager to put no cars, no planes etc. into the ring.

Later the teacher witnessed children discussing why the 3-icon was not a triangle, and later used the word four-angle for the square. Also this teacher noticed that some children began to use their fingers instead of the chopsticks.

Under the walk around the room a fierce discussion about cheating broke out when a child suggested that clapping his hand three times was also an example of three things. Its not, another child responded. It is. No its not! Why not? Because you cannot bring it to 3-land! Let’s ask the teacher!

After telling about space and time, children produced other examples as three knocks, three steps, three rounds around a table, three notes. Other children began to look at examples of threes at their own body soon finding three fingers, three parts on a finger, and three hands twice when three children stood side by side and the middle one lent out his two hands to his neighbours.

Conclusion

To find which mathematics can be treated in preschool, postmodern contingency research uncovered Manyology as a hidden alternative to the ruling tradition. Dealing with the natural fact Many means counting in icons, and recounting when adding on-top or next-to thus introducing linearity and calculus. However, these golden learning opportunities are lost when entering grade one, where the monopoly of ten-counting prevents both from happening; and furthermore grounded mathematics is replaced with metamatism when introducing one-to-one corresponding sets and when teaching that 1+2 IS 3. So maybe someone should tell the governments that in a republic the educational system must not present choice as nature. Instead governments should accept the historic fact that long, long ago the antique collective name mathematics was split up into independent disciplines. So instead of teaching mathematics, schools should prepare for the outside world by teaching the two competences needed to deal with the natural fact Many, to count and to add. Consequently, the golden learning opportunities in preschool mathematics should enter ordinary school instead of being suppressed by it.

References

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education.* Dordrect: Klüwer.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht: D. Reidel Publishing Company

Glaser B. G. & Strauss A. L. (1967). *The Discovery of Grounded Theory.* New York: Aldine de Gruyter.

Kline, M. (1972). *Mathematical Thoughts from Ancient to Modern Times*. New York: Oxford University Press.

Lyotard, J. (1984). *The postmodern Condition: A report on Knowledge*. Manchester: Manchester University Press.

Niss, M. A. ´(2003). Mathematical competencies and the learning of mathematics : the Danish KOM project. *3rd Mediterranean Conference on Mathematical Education - Athens, Hellas 3-4-5 January 2003.* Editors A. Gagatsis; S. Papastavridis. Athen : Hellenic Mathematical Society, 2003. p. 116-124.

Piaget, J. (1970). *Science of Education of the Psychology of the Child.* New York: Viking Compass.

Tarp, A. (2004). *Pastoral Power in Mathematics Education.* An ICME Triology. http://dev.mathecademy.net/wp-content/uploads/2013/05/An-ICME-Trilogy-online.pdf. p.59-70.

Tarp, A. (2014 expected). *ManyMath - MyMath.* Publisher to be decided.

YouTube video (2013). http://youtu.be/qgCwVZnALXA.

Zybartas S. & Tarp A. (2005). One Digit Mathematics. *Pedagogika (78/2005).* Vilnius, Lithuania.

Calculators and IconCounting and CupWriting in PreSchool and in Special Needs Education

To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking preschool mathematics uncovers icon-counting in bundles less than ten implying recounting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, a calculator can predict recounting results before being carried out manually. By allowing overloads and negative numbers when recounting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary school only allowing ten-counting and on-top addition to take place.

Decreasing PISA Performance in spite of Increasing Research

Being highly useful to the outside world, math is a core part of education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise funding has increased witnessed by e.g. the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA results in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This got OECD to write the report ‘Improving Schools in Sweden’ describing the Swedish school system as being ‘in need of urgent change’ (OECD 2015).

Created to help students cope with the outside world, schools are divided into subjects that are described by goals and means with the outside world as the goal and the subjects as means. However, goal/means confusions might occur where the subject become the goal and the outside world a means.

A goal/means confusion is problematic since while there is one goal there are many means to be replaced if not leading to the goal, unless an ineffective means becomes a goal itself, leading to a new discussing about which means will best lead to this false goal; thus preventing looking for alternative means that would more effectively lead to the original goal. So we can ask: Does mathematics education build on a goal-means confusion seeing mathematics as the goal and the outside world as a means? Institutional skepticism might offer an answer.

Institutional Skepticism

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget’s principles of natural learning (Piaget 1970). In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting patronizing choices as nature (Lyotard 1984).

Building on Kierkegaard, Nietzsche and Heidegger, Sartre defines existentiallism by saying that to existentialist thinkers ‘existence precedes essence, or (..) that subjectivity must be the starting point’ (Marino 2004: 344). Kierkegaard was skeptical to institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone ‘may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.’ (Marino 2004: 186-187). Inspired by Heidegger, Arendt divided human activity into labor and work both focusing on the private sphere, and action focusing on the political, creating institutions to be treated with care to avoid the banality of evil by turning totalitarian (Arendt 1963).

Since one existence gives rise to many essence-claims, the existentialist distinction offers a perspective to distinguish between one goal and many means.

Mathematics as Essence

In ancient Greece the Pythagoreans used the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music now as independent knowledge areas, today mathematics is a common label for the two remaining activities, Geometry and Algebra replacing Greek Arithmetic (Freudenthal 1973).

Textbooks see mathematics as a collection of well-proven statements about well-defined concepts, defined ‘from above’ as examples from abstractions instead of ‘from below’ as abstractions from examples. The invention of the set-concept allowed mathematics to be self-referring. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by its inability to separate concrete examples from abstract essence.

And, as expected, teaching meaningless self-reference creates learning problems.

Mathematics as Existence

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry.

Meaning to reunite numbers in Arabic, Algebra contains four ways to unite as shown when writing out fully the total T = 354 = 3*B^2 + 5*B + 4*1 = 3 bundles of bundles and 5 bundles and 4 unbundled. Here we see that we reunite by using on-top addition, multiplication, power and next-to addition, called integration. So, with a human need to describe the physical fact Many, algebra was create as a natural science about Many.

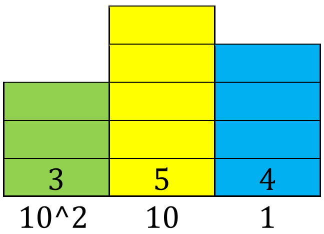


Figure 1: 354 = 3*10^2 + 5*10 + 4*1 shown as stacked bundles

To deal with Many, first we iconize, then we count by bundling and stacking. With ‘first order counting’ we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written in a less sloppy way. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in e.g. fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc. (Zybartas et al, 2005).

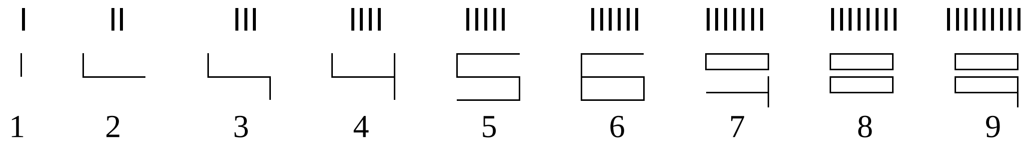
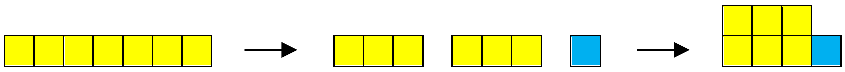


Figure 2: Digits as icons containing as many sticks as they represent

With ‘second order counting’ we bundle a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. The unbundled can be placed in a right single-cup; and in a left bundle-cup we trade the bundles, first with a thick stick representing a bundle glued together, then with a normal stick representing the bundle. The cup-contents is described by icons, first using ‘cup-writing’ 2)1), then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. In addition, we can also use plastic letters as B, C or D for the bundles.

IIIIIII → III III I → **II**) I) → II) I) → 2)1) → 2.1 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a decimal number, 2 3s & 1 or 2.1 3s.



We live in space and in time. To include both when counting, we can introduce two different ways of counting: counting in space, geometry-counting, and counting in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.

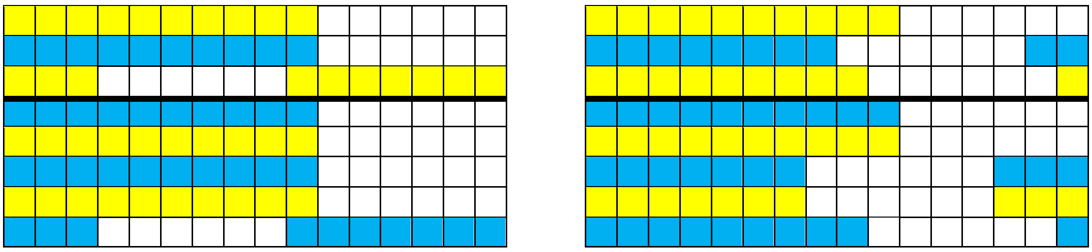


Figure 3: 7 counted in 3s on an abacus in geometry and algebra mode

To predict the counting result we can use a calculator. Building a stack of 2 3s is iconized as 2x3 showing a jack used 2 times to lift the 3s. As for the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once. Thus by entering ‘7/3’ we ask the calculator ‘from 7 we can take away 3s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 3s by asking ‘7 – 2x3’. From the answer ‘1’ we conclude that 7 = 2.1 3s. Showing ‘7 – 2x3 = 1’, a display indirectly predicts that 7 can be recounted as 2 3s and 1, or as 2.1 3s.

7 / 3	2.some
7 – 2 * 3	1

Re-counting in the Same Unit and in a Different Unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3.2 2s or as 2.4 2s. Likewise 4.2s can be recounted as 5 2s less 2; or as 6 2s less 4 thus leading to negative numbers:

Letters	Sticks	Calculator		T =
B B B B	II II II II			4.0 2s
B B B I I	II II II I I	4x2 – 3*2	2	3.2 2s
B B I I I I	II II I I I I	4*2 – 2*2	4	2.4 2s
B B B B B	II II II II II	4*2 – 5*2	-2	5.-2 2s
B B B B B B	II II II II II II	4*2 – 6*2	-4	6.-4 2s

Figure 4: Recounting 4 2s in the same unit creates overloads or deficits

To recount in a different unit means changing unit, called proportionality or linearity also. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2.2 5s.

IIII IIII IIII → IIIII IIIII I I → 2) 2) 5s → 2.2 5s

With letters, C = BI so that BBB → BB IIII → CC II

Using a calculator to predict the result we enter '3*4/5' to ask 'from 3 4s we take away 5s how many times?' The calculator gives the answer '2.some'. To find the leftovers we take away the 2 5s and ask '3*4 – 2*5'. Receiving the answer '2' we conclude that 3 4s can be recounted as 2 5s and 2, or as 2.2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Once counted, totals can be added on-top or next-to. Asking '3 5s and 2 3s total how many 5s?' we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s giving 1.1 s that added to the 3 5s gives a grand total of 4.1 5s. With letters: 3B + 2C = 3B III III = 4BI. With sticks:

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4) 1) 5s → 4.1 5s,

Using a calculator to predict the result we use a bracket before counting in 5s: Asking '(3*5 + 2*3)/5', the answer is 4.some. Taking away 4 5s leaves 1.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas called integration. Asking '3 5s and 2 3s total how many 8s?' we use sticks to get the answer 2.5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2) 5) 8s → 2.5 8s

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking '(3*5 + 2*3)/8', the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2.5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing Adding On-top and Next-to

To reverse addition is also called backward calculation or solving equations. To reverse next-to addition is called reversed integration or differentiation. Asking '3 5s and how many 3s total 2.5 8s?', using sticks will get the answer 2 3s:

IIIII IIIII IIIII III III ← IIIII III) IIIII III) IIIII ← 2) 5) 8s ← 2.5 8s

Using a calculator to predict the result the remaining is bracketed before counted in 3s. Adding the two stacks 2 3s and 3 5s next-to each other means multiplying before adding. Reversing integration means subtracting before dividing, as in the gradient formula y' = dy/t = (y2 – y1)/t.

(2 * 8 + 5 – 3 * 5)/ 3	2
(2 * 8 + 5 – 3 * 5) – 2 * 3	0

Primary Schools use Ten-counting only

In primary school textbooks, numbers are counted in tens to be added, subtracted, multiplied and divided. This leads to questions as '3 4s = ? tens'. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount- and restack-formula is impossible since the calculator has no ten buttons. Instead it is programmed to give the answer directly in a special form that leaves out the unit and misplaces the decimal point one place to the right.

3 * 4	12

Recounting icon-numbers in tens is called doing times tables to be learned by heart. So from grade 1, 3*4 is not 3 4s any more but has to be recounted in tens as 1.2 tens, or 12 in the abbreviated form.

Recounting tens in icons by asking ‘38 = ? 7s’, is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens 0.3 7s must be written in fraction form as 3/7 and calculated as 0.428…, shown directly by the calculator, 38/7 = 5.428…

38 / 7	5.some
38 – 5 * 7	3

Without icon-counting, primary school labels the problem ‘38 = ? 7s’ as an example of an equation ’38 = x*7’ to be postponed to secondary school.

Where icon-counting involves division, multiplication, subtraction and later next-to and on-top addition, primary school turns this order around and only allows on-top addition using carrying instead of overloads. Using cup-writing with overloads or deficits instead of carrying, the order of operations can be turned around to respect, that totals must be counted before being added.

<table>
<tr><th></th><th>Carry</th><th>Cup-writing</th><th>Words</th></tr>
<tr><td>Add</td><td>1
4 5
1 7
6 2</td><td>4) 5)
1)7)
5)12)
6)2) = 62</td><td>4 ten 5
1 ten 7
5 ten 12
5 ten 1 ten 2
6 ten 2 = 62</td></tr>
<tr><td>Subtract</td><td>1
4 5
1 7
2 8</td><td>4) 5)
1)7)
3)-2)
2)10-2)
2)8) = 28</td><td>4 ten 5
1 ten 7
3 ten less2
2 ten 8 = 28</td></tr>
<tr><td>Multiply</td><td>4
2 6 * 7
1 8 2</td><td>7 * 2) 6)
14)42)
18) 2) = 182</td><td>7 times 2 ten 6
14 ten 42
14 ten 4 ten 2
18 ten 2 = 182</td></tr>
<tr><td>Divide</td><td>2 4 rest 1
3| 7 3
6
1 3
1 2
1</td><td>7)3) counted in 3s
6)13)
6)12) + 1
2 3s)4 3s) + 1
24 3s + 1
73 = 24*3 + 1</td><td>7ten3
6ten 13
6ten12 + 1
3 times 2ten4 + 1
3 times 24 + 1</td></tr>
</table>

Figure 5: Cup-writing with overloads and deficits instead of carrying

As to addition, subtraction and multiplication, carrying occurs indirectly as an overload to be removed or created by recounting in the same unit. As to division the recounting is guided by 3-tables showing which numbers should occur in the cups and how much to move to the next cup or outside.

Tested with a Special Needs Learner

A special needs learner taken out of her normal grade six class agreed to test the effects of using icon-counting, cup-writing, next-to addition and a calculator for number-prediction. As to the learner’s initial level, when asked to add 5 to 3 she used the fingers to count on five times from three. To avoid previous frustrations from blocking the learning process, the word ‘mathe-matics’ was replaced by ‘many-matics’. The material was 8 micro-curricula for preschool using activities with concrete material to obtain its learning goals in accordance with Piaget’s principle ‘greifen vor begrifen’ (grasp to grasp) (MATHeCADEMY.net/ preschool).

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using two cups for the bundled and the unbundled reported with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and deficits. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eights, the learner reverses next-to addition. The micro-curricula M2-M8 used the recount- and restack formula on a calculator to predict the result:

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2) 1) 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	'2.7 5s is also how many 5s?' IIIII IIIII IIIIIII = V V V II = V V V V III 2)7) = 2+1)7-5) = 3)2) = 3+1)2-5) = 4)-3) So 2.7 5s = 3.2 5s = 4.-3 5s,	(2*5+7)/5 (2*5+7) -3*5 (2*5+7) -4*5	3.some 2 -3
M4	2 5s is how many 4s?' IIIII IIIII = IIII I IIII I = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2 * 4	2.some 2
M5	'2 5s and 4 3s total how many 5s?' IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2
M6	'2 5s and 4 3s total how many 8s?' IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 (2*5+4*3) – 2*8	2.some 6
M7	'2 5s and ? 3s total 4 5s?' IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1
M8	'2 5s and ? 3s total how 2.1 8s?' IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1

Figure 6: A calculator predicts counting and adding results

One curriculum used silent education where the teacher demonstrates and guides through actions only, not using words; and one curriculum was carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator quickly took over the communication. Examples of statements are given below.

Activity	Examples of statements
Icon-creation with a folding rule	Oh that's where the digits come from.
Icon-counting	So that means that 3*5 is 3 5s and not a tables-question?
Recounting in the same unit	That is the same as changing coins or getting back change.
Recounting in a different unit	Wow, a calculator can predict the result before I carry it out. Can I please keep this calculator?
Adding on-top	Oh, I see, balconies are not allowed
Adding next-to	This is like building with Lego blocks
Reversed adding on-top	Well, you just take away what was added and then count in 3s
Reversed adding next-to	Take away and count, again.

Recounting icon-numbers in tens	Hey, you just have to enter 3*4 to recount in tens.
Recounting tens in icon-numbers	So recounting in icons is just another word for solving equations?
Removing overloads with addition and multiplication 35+47 = 7)12) = 8)2) = 82 3 * 58 = 15)24) = 17)4) = 174	Hey, its fun to trade bundles for singles and vice versa.
Creating overloads with subtraction 35-17 = 2)15)-1)7) = 1)8) = 18, or 35-17 = 3)5)-1)7) = 2)-2) = 1)8) = 18	Why didn't the teacher teach me this method the first time?
Creating overloads with division 86/3 = ?; 86 = 28*3 + 2 since 8)6)= 6)26)= 6)24)+2= 2 3s)8 3s)+2	Now I see why tables are useful. They find the contents of the cups.
Creating per-numbers as bridges when double-counting in 2 different physical units With 3$/4kg, 5kg = (5/4)*4kg = (5/4)*3$ = 3.75$ 5$ = (5/3)*3$ = (5/3)*4kg = 6.67kg	OK, so recounting dollars in kgs is just like recounting 3s in 5s, isn't it? And again, we just use the calculator to predict the answer.

Figure 7: Examples of comments

At the end the learner went back to her normal class where proportionality lessons created learning problems. The learner suggested renaming it to double-counting but the teacher insisted in following the textbook. However, observing that the class gradually took over the double-counting method, he finally gave in and allowed proportionality to be renamed and treated as double-counting.

When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integration.

Thus icon-counting and a calculator for predicting recounting results allowed the learner to get to the goal, mastery of Many, by following an alternative to the institutionalized means that had become a stumbling block to her.

In the beginning the learner solved adding and subtraction problems by using the counting sequence forwards and backwards and she had given up with tables and division. With icon-counting, the order is turned around and the operations take on meanings rooted in activities: 7/3 now means 7 counted in 3s. 4*5 now means 4 5s. 7 – 2*3 now means to drag away a stack of 2 3-bundles from 7 to look for unbundled leftovers. Addition now comes in two versions, first next-to addition then on-top addition. In all cases a calculator predicts the result.

Finally,double-counting in two physical units and recounting tens in icons allowed her to master proportionality and equations without following the traditionally road of institutionalized education. And performing and reversing next-to addition gave her an introduction to calculus way before this is included in the tradition.

Conclusion and Recommendation

Institutionalized education sees mathematics, not as a means to an outside goal but as a goal in itself to be reached by hindering learners in learning to count; by insisting that only ten-counting is allowed; by using the word natural for numbers with misplaced decimal point and the unit left out; by reversing the natural order of the basic operations division, multiplication, subtraction and addition; and by neglecting activities as creating or removing overloads and double-counting.

To find how mathematics looks like if built as a natural science about its root, the physical fact Many, institutional skepticism has used the existentialist distinction between existence and essence to uncover 'ManyMatics' as a hidden alternative to the ruling tradition. Dealing with Many means bundling and counting in icons, and recounting when adding on-top or next-to thus introducing proportionality and calculus. Likewise reversing on-top or next-to addition leads to solving equations and differentiation. That totals must be counted before being added means introducing the operations division, multiplication, subtraction before addition.

Consequently, mathematics education suffers from a goal-means confusion to be removed to improve PISA-results. To respect its outside goal, mathematics education must develop mastery of Many by teaching mathematics as grounded ManyMatics, and not as self-referring 'MetaMatism', a mixture of 'MetaMatics' turning mathematics upside down by presenting concepts as examples of abstractions instead of as abstractions from examples, and 'MatheMatism' true inside a classroom but not outside where claims as '1+2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

In short: Don't preach essence, teach existence.

References

Arendt, H. (1963). *Eichmann in Jerusalem, a Report on the Banality of Evil.* London: Penguin Books.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht-Holland: D. Reidel Publishing Company.

Glaser B. G. & Strauss A. L. (1967). *The Discovery of Grounded Theory.* New York: Aldine de Gruyter.

Lyotard, J. (1984). *The postmodern Condition: A report on Knowledge.* Manchester: Manchester University Press.

Marino, G. (2004). *Basic Writings of Existentialism.* New York: Modern Library.

OECD. (2015). *Improving Schools in Sweden: An OECD Perspective.* http://www.oecd.org/edu/school/improving-schools-in-sweden-an-oecd-perspective.htm.

Piaget, J. (1970). *Science of Education of the Psychology of the Child.* New York: Viking Compass.

Zybartas, S. & Tarp, A. (2005). One Digit Mathematics. *Pedagogika (78/2005).* Vilnius, Lithuania.

Grounding Conflicting Theories

An invitation to a dialogue to solve the Nordic Math MeltDown Paradox

With heavy funding of mathematics education research brilliant results in the PISA scores are to be expected in the Nordic countries. So it is a paradox that all Nordic counties are facing a melt-down in their PISA scores in 30 years if nothing is changed; except for Denmark that has not increased it funding significantly. This was predicted by Tarp in his MADIF papers formulating an irrelevance paradox for mathematics education: more research leads to more problems when basing research on ungrounded theories and discourse protection and moo-review.

Background

The meltdown in Nordic mathematics as illustrated by the Pisa results since 2006

	2006	2009	2012		2030
Finland	548	541	519		72
Denmark	513	503	500		629
Iceland	506	507	493		94
Norway	490	498	489		78
Sweden	502	494	478		214
OECD	498	499	494		338

Using the PISA scores from 2006, 2009 and 2012 it is possible to create a quadratic model that can predict future values by a bending curve. The curves are all declining and bending downwards i.e. the yearly increase becomes more and more negative. The only exception is Denmark with an initial negative yearly decrease at -4.5 but with an upwards bending curve adding 0.8 points yearly to the increase. In the case of Finland, Island, Norway, Sweden and the OECD, the yearly increase in 2006 was at 0.2, 2.8, 5.5, -1.3 and 1.3 increasing yearly with -1.7, -1.7, -1.9, -0.9 and -0.7, thus reaching the zero-level in 2032, 2032, 2032, 2038 and 2047 if the trend continues.

The paradox comes the fact that countries as Sweden, Norway has invested huge funding in mathematics research and created centres for math education research as well as special institutions for the development of mathematics education such as e.g. Only Denmark has been reluctant to increase funding.

Thus, In 1999 the Swedish government decided to establish and gracefully fund a national resource centre for mathematics education, NCM, describing its task to ‘co-ordinate, support, develop and implement the contributions which promote Swedish mathematics education from pre-school to university college’.

However, I soon realized that it was almost impossible to establish e dialogue with the NCM and with Swedish researchers, so at the MADIF4 conference I presented a paper called ‘Mathematism and the Irrelevance of the Research Industry’ warning against supporting the irrelevance paradox in mathematics education research described by the following observation: ‘the output of mathematics education research increases together with the problems it studies - indicating that the research in mathematics education is irrelevant to mathematics education’. The paper demonstrates how to avoid mixing up mathematics with mathematism, true in the library but seldom in the laboratory.

Although accepted for a full presentation, nothing happened afterwards, so in my MADIF5 paper I decided to be much more specific by warning against twelve blunders of mathematics education. The reaction to this paper was to reduce the presentation to a short communication.

In my MADIF6 paper I draw attention to the difference between North American enlightenment schools wanting as many as possible to learn as much as possible, and European counter-

Enlightenment Bildung schools only wanting the elite to be educated. In the enlightenment school enlightenment mathematics is grounded from below as a natural science enlightening the physical fact many. In the Bildung schools pastoral 'metamatism' descends from above as examples of metaphysical mystifying concepts.

The paper was rejected based upon a review process that allowed decisions to be made without specific reference to the paper reviewed.

So in my MADIF7 paper I warned against what I called 'Discourse Protection in Mathematics Education' and against reducing a constructive review process to what I called 'Moo Review' and 'Tabloid Review' using only one word or one sentence. Again the paper was rejected.

As said, one would expect the massive Swedish investment would show in the PISA scores. Here Sweden scored 502, 494, and 478 in the 2006, 2009 and 2012. Three consecutive numbers allow calculating the yearly change and the change to the change, which in the case of Sweden is -1.3 in 2006 changing yearly by -0.9 bringing the Swedish score to the zero level in 2038 if not changed.

In the PISA report Denmark scored 513, 503 and 500 giving an initial yearly change of -4.5 in 2006 changing yearly by 0.8 bringing the Danish score to 629 in 2030 if not changed.

However, Denmark has not significantly increased its research activity. So the Danish success and the Swedish melt-down both indicate the correctness of the irrelevance paradox: More research creates more problems. Consequently I suggested a two year no-research pause in Sweden. It was declined because researchers had found a new research paradigm, Design Research, they hoped would change the situation in a positive way.

Design Research bases its designs on existing theory. However, in conference presentations, disagreements between conflicting theories were simply ignored or denied. And not differentiating between grounded and ungrounded theory will hardly prevent the Swedish melt-down. So, to once more offer my assistance, instead of writing yet another paper that will be rejected yet again because of discourse protection, I have decided that my contribution to the MADIF 10 conference in 2016 should be a YouTube video similar to the Paul and Allan debate on postmodern mathematics education (https://www.youtube.com/watch?v=ArKY2y_ve_U), inspired by the Chomsky-Foucault debate on human nature, this time called 'Grounding Conflicting Theories to avoid the Irrelevance Paradox creating the Nordic Math Melt-Down - an invitation to a dialogue on Mathematics Education and its Research'. One prominent person within the research community has declined to take part in the dialogue, but hopefully other persons will accept their responsibility and be willing to enter into a fruitful dialogue to prevent the Swedish melt-down to become reality. Money does not solve the problem, dialogue between conflicting theories does.

Manuscript to a Debate on Mathematics Education and its Research

Bo: Welcome to the MATHeCADEMY.net channel. My name is Bo. Today we discuss Mathematics education and its research. Humans communicate in languages, a word-language and a number-language. In the family, we learn to speak the word language, and we are taught to read and write in institutionalized education, also taking care of the number-language under the name Mathematics, thus emphasizing the three r's: Reading, Writing and Arithmetic. Today governments control education, guided by a growing research community. Still international tests show that the learning of the number language is deteriorating in many countries. This raises the question: If research cannot improve Mathematics education, then what can? I hope our two guests will provide some answers. I hope you will give both a statement and a comment to the other's statement.

Welcome to Allan. Allan has been working as an ethnographer in different parts of education from secondary school to teacher education. Allan has created the web-based MATHeCADEMY.net teaching teachers to teach Mathematics as a natural science about Many. In addition, Allan has written a book about this approach called 'ManyMath – MyMath'.

Allan: Thank you Bo

B0: And welcome to John. John has …

John: Thank you Bo

1. Mathematics Itself

Bo: We begin with Mathematics. The ancient Greeks Pythagoreans used this word as a common label for what we know, which at that time was Arithmetic, Geometry, Astronomy and Music. Later Astronomy and Music left, and Algebra and Statistics came in. So today, Mathematics is a common label for Arithmetic, Algebra, Geometry and Statistics, or is it? And what about the so-called 'New Math' appearing in the 1960s, is it still around, or has it been replaced by a post New-Math, that might be the same as pre New-Math? In other words, has pre-modern Math replaced modern Math as post-modern Math? So, I would like to ask: 'What is Mathematics, and how is it connected to our number-language?'

Allan: To me, it is the need to communicate about the natural fact 'Many' that created the number-language. In space, we constantly see many examples of Many; and in time Many is present as repetition. So, if Mathematics means what we know, we might want to add about Many, and use the word 'Manyology' as a parallel word for Mathematics.

To deal with Many we perform two actions, we count and we add to answer the basic question 'how many'. This resonates with the action-words algebra and geometry meaning to reunite numbers in Arabic and to measure land in Greek. We count a given total in singles, bundles, bundles of bundles, etc. as shown by a number as five hundred and forty three, consisting of 3 singles, 4 ten-bundles and 5 ten-bundles of ten-bundles. We see that all numbers carry units as ones, tens, ten-tens etc. Having the same unit, the 4 ten-bundles are added on-top of each other; and having different units, the 5 tens-tens and the 4 tens are added next-to each other as areas, also called integration, where shifting unit is called linearity. So, a three digit number shows the core of Mathematics, which is linearity and integration. The number also shows the four different ways to unite numbers: by multiplication as in 4 tens, by power as in ten-tens, by vertical on-top addition as in 3 ones, and by horizontal next-to addition as in the juxtaposition of the three blocks with different units. Showing its bundle-size ten when written as 54.3 tens, the total also shows that singles can be written as decimals or as fractions where the 3 singles become 0.3 tens or 3 counted in tens, 3/10. With unspecified bundle-number, a three-digit number becomes a formula, where the bundle-number can be found by reversing addition, also called solving equations.

So, Mathematics is very easy; and also very easy to make hard. You just replace Mathematics with 'Metamatism', a mixture of 'Meta-matics' and 'Mathema-tism'.

Mathematism is true in a library but not in a laboratory. Thus statements as '2 + 3 is 5' are found in any textbook even if it is falsified by countless outside examples, as e.g. 2 weeks and 3 days total 17 days.

Metamatics defines its concepts as examples of abstractions instead of as abstractions from examples, i.e. top-down and from above instead of bottom-up and from below. Thus, Metamatics defines a formula as an example of a set-product where first-component identity implies second-component identity, instead of, as Euler did, as a name for a calculation containing both numbers and letters. Defining concepts as examples of the ultimate abstraction, a set, makes Metamatics self-referring, and thus meaningless according to Russell's set-paradox saying that the set of sets not belonging to itself will belong to itself if it does not belong, and vice versa. To avoid this paradox, Russell proposed a type-theory to distinguish between examples and abstractions, meaning e.g. that a fraction is not a number. Unwilling to accept this, modern set theory removes the difference between an element and a set, i.e. between an example and an abstraction, which still makes Metamatics meaningless since you can survive on examples of food but not on the label food; they enter different holes in the head.

Summing up, Mathematics can be a grounded natural science about the natural fact Many, thus becoming a number-language showing how numbers are built by using four different ways to unite: multiplication, power, on-top and next-to addition, that can all be reversed. However, Mathematics can also be an ungrounded self-referring Metamatism with set-derived definition and with statements that are claimed to be true even when confronted by counter-examples. In other words, Mathematics can be easy and accessible to all, or it can be made hard and accessible to an elite only.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

2. Education in General

Bo: Thank you, John and Allan. Now let us talk about education in general. On our planet, life takes the form of single black cells, or green or grey cells combined as plants or animals. To survive, plants need minerals, pumped in water from the ground through their leaves by the sun. Animals instead use their heart to pump the blood around, and use the holes in the head to supply the stomach with food and the brain with information. Adapted through genes, reptiles reproduce in high numbers to survive. Feeding their offspring while it adapts to the environment through experiencing, mammals reproduce with a few children per year. Humans only need a few children in their lifetime, since transforming the forelegs to hands and fingers allows humans to grasp the food, and to share information through communication and education by developing a language when associating sounds to what they grasp. Where food must be split in portions, information can be shared. Education takes place in the family and in the workplace; and in institutions with primary, secondary and tertiary education for children, for teenagers and for the workplace. Continental Europe uses words for education that do not exist in the English language such as Bildung, unterricht, erziehung, didactics, etc. Likewise, Europe still holds on to the line-organized office preparing education that was created by the German autocracy shortly after 1800 to mobilize the population against the French democracy, whereas the North American republics have block-organized talent developing education from secondary school. As to testing, some countries use centralized test where others use local testing. And some use written tests and others oral tests. So, my next question is 'what is education?'

Allan: We adapt to the outside world through experience and advice, i.e. we are educated by the outside world and by other human beings. Children like to feel the outside world; teenagers like to gossip about it and about themselves; and adults must exchange actions with money to support a family. Thus, it makes sense to institute both primary, secondary and tertiary education to serve the needs of children, teenagers and adults. As an institution, education contains an element of force. Our language came from naming what we can grasp or point to, i.e. through a from-the-hand-to-the-head principle, called greifen-begreifen in German. So guiding children with concrete material to grasp, and teenagers with gossip to listen to makes education successful as described in Psychology by Piaget and Ausubel. On the other hand, forcing abstractions upon children and teenagers before introducing concrete materials or gossip excludes many children and teenagers from learning, thus creating a monopoly of knowledge as described in Sociology by e.g. Foucault and Bourdieu.

As to the space-and-time structure of education, primary education for children should be line-organized with yearly age-group-nannies as guides bringing the outside world to the classroom to develop concepts about nature described by a number-language, and concepts about society described by a word-language. In late primary school, this double-nanny becomes two different nannies. Daily, the children also express themselves through music, art, or motion. The priority of to-do-subjects over to-be-subjects changes from primary to secondary school.

Transformed from children to teenagers able to have children of their own, the curiosity changes from the outside to the inside world, from things to persons. Being biologically programmed to remember gossip is useful if information about nature and society takes the form of gossip, i.e. statements with known subjects. Experimenting now is with what is inside oneself, e.g. as to talents. Consequently, secondary school should offer daily lessons in self-chosen half-year blocks to allow the individual teenager to test personal talents. If successful, the school says ‘good job, you need more of this’. If not, the school says ‘good try, you need to try something else’ to express admiration for the courage it takes to try out something new. This is how the North American republics organize a bottom-up secondary and tertiary education.

Being highly institutionalized, Europe hangs on to its line-organized school system preparing for public, created by Humboldt in Berlin shortly after 1800. Furthermore, the word ‘education’ is replaced by words as ‘unterricht’ and ‘erziehung’ and ‘Bildung’. Unterricht means handing down to those below you, and erziehung means dragging them up. These top-down words come from the Platonic patronizing view that the goal of education is to transmit and exemplify abstract knowledge.

The success of the French Enlightenment republic came from enlightening its population. To protect autocracy, the Prussian king asked Humboldt to create a school that could replace the blood-nobility unable to stop the French with a knowledge-nobility to occupy a strong public administration and to receive Bildung so it could go to court. This Bildung school should have two more goals: to prevent democracy, the population must not be enlightened; instead, the population must be transformed into a people proud of its history and willing to protect it against other people, especially the people from the French republic. To hide its anti-enlightening agenda, teacher education is based upon a special subject called didactics, confusing the teachers by claiming to determine the content of Bildung.

So to sum up, education can be bottom-up enlightenment allowing children to experiment with the outside world brought to the classroom, and allowing teenager to experiment with their inside talents through daily lessons in self-chosen half-year blocks that inform about the outside world in the form of gossip. Or, education can be top-down Bildung trying to make the students accept patronization by abstract knowledge created at a distant university, where the best of them might be accepted later.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

3. Mathematics Education

Bo: Thank you, John and Allan. Now let us talk about education in Mathematics, seen as one of the core subjects in schools together with reading and writing. However, there seems to be a difference here. If we deal with the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. However, you cannot Math, you can reckon. At the European continent reckoning, called ‘Rechnung’ in German, was an independent subject until the arrival of the so-called new Mathematics around 1960. When opened up, Mathematics still contains subjects as fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc. Today, Europe only offers classes in Mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. Therefore, I ask, ‘what is Mathematics education?’

Allan: The outside world contains many examples of Many: many persons, many houses, many days, etc. So, to adapt to the outside world, humans need to be deal with the natural fact Many, and this should be the goal of Mathematics education since the main contents of Mathematics was created as precisely that: statistics to count Many, algebra to reunite Many and geometry to count spatial forms. To deal with Many, we count and add. Counting takes place in the family and therefore integrates into preschool in a natural way. Since primary school only allows counting in tens, preschool can profit from the golden learning opportunities coming from icon-counting in numbers less than ten. Here first-order counting allows five ones to be bundled as one fives, transformed into one five-icon containing five strokes if written in a less sloppy way. Now second-order counting can count in icons so that seven sticks can be recounted in 1 five-bundle and two unbundled singles, written as 1 and 2 5s, or as 1.2 5s using the decimal point to separate bundles and unbundled. Which again can be recounted as 2.1 3s where changing units later is called proportionality and linearity. Once counted, totals can be added. To add on-top the units must be the same, so one of the totals must be recounted in the other's unit. Added next-to each other, the totals are added as areas which is called integration. And reversing addition means creating opposite operations to predict the result. Here the operations occur in their natural order, which is the opposite of what the school presents: to count in 5s we take away 5s many times, which is division. Then the bundles are stacked, which is multiplication. We might want to recount a stack by taking away one bundle to change it into singles, which is subtraction. Finally stacks can be added on-top or next-to. By meeting concrete examples of Many, children learn to count and recount by bundling and stacking; and to add on-top and next-to. Later physical units introduce children to per-numbers when double-counting in two different units as e.g. 5 $ per 3 kg, or 5/3 $/kg.

Telling Mathematics as gossip makes learning easy for teenagers, biologically programmed to remember statements about known subjects. The formula for a number as 543, i.e. 5 tens-tens and 4 tens and 3 ones show the four ways to unite numbers: Multiplication, power, on-top addition and next-to addition, also called integration. With an unknown bundle-number, the number-formula becomes a polynomial containing basic relations between variable numbers as proportional, linear, exponential, power and quadratic formulas that tabled and graphed show the different forms of constant changing unit-numbers in pre-calculus. As to calculus, per-numbers can be constant in three different ways: globally, piecewise and locally also called continuous; all added to totals by the area under the per-number graph i.e. by combining multiplication and addition. Reversed, the combination of subtraction and division, called differentiation, allows the per-number to be determined from the area. Many teenagers enjoy the beauty of uniting geometry and algebra in coordinate-geometry allowing a geometrical prediction of algebraic solutions and vice versa; as well as the fascinating post-diction by statistics of unpredictable numbers in probability.

To sum up: Mathematics education can be easy if grounded in the roots of Mathematics, the natural fact Many, to be dealt with by counting and adding making a natural number a decimal number with a unit. Counting and recounting in icons before counting in tens brings the core of Mathematics, linearity and integration, to preschool; and allows solving equations and fractions to be introduced in the beginning of primary school as reversed addition and double-counting in different physical units. Or Mathematics can be hard by allowing only counting in tens, by presenting a natural number without a decimal point and a unit, and by transforming Mathematics to Mathematism by adding numbers without units, claiming e.g. that 2 plus 3 is 5 in spite of many counterexamples; and by postponing proportionality and integration to the beginning and end of secondary school.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

4. The Learner

Bo: Thank you, John and Allan. Now let us talk about at the humans involved in Mathematics education: Governments choose curricula, build schools, buy textbooks and hire teachers to help learners learn. We begin with the learners. The tradition sees learning taking place when learners follow external instructions from the teacher in class and from the textbook at home. Then constructivism came along suggesting that instead learning takes place through internal construction. Therefore, I ask ‘what is a learner?’

Allan: Again, let us assume that we adapt to the outside world through actions, physical and verbal. So learning means acquiring proper actions, some verbalized and some tacit. Repetition makes learning effective. Repetition takes place in the family and in the workplace, and can take place in school through daily lessons both for children and for teenagers. Also, allowing learners to carry out most of the homework at school will minimize the effect of the learners’ different social backgrounds.

Again we must distinguish between a child, a teenager and an adult. Its biology programs a child to learn by grasping as described by Piaget, and a teenager to learn by gossip as described by Ausubel stressing the importance of connecting new knowledge to what the learner already knows. An adult is motivated to learn something from its use in the workplace.

Piaget describes individual learning as creating schemata that can assimilate new examples, or be accommodated to assimilate divergent examples. In contrast, Vygotsky describes learning as being able to connect the learner’s individual knowledge zone with the abstract concepts of the actual knowledge regime.

The four answers to the question: “Where do concepts come from? From above or from below? Form the outside or from the inside?” create four learning rooms. The two traditional rooms, the transmitter room and the constructivist room, say “above and outside” and “above and inside”. The two hidden alternatives, the “fairy-tale room” and the apprentice room, say “below and outside” and “below and inside”. The traditional rooms take Mathematics for granted and see the world as applying Mathematics. The hidden rooms have the opposite view seeing Many as granted and as a creator of Mathematics through the principle ‘grasping by grasping’. The transmission room and the fairy-tale room facilitate learning through sentences with abstract and concrete subjects. The constructivist room and apprentice room facilitate learning through sentence-free meetings with abstract or concrete subjects.

A block-organized education allows the learners to change classes twice a year with a “good job” greeting if successful and a “good try” greeting if less successful aiming at keeping alive the curiosity of the teenager as to which talent is hidden inside. In Europe, its line-organized education forces the learner to stay in the class even if being less successful, or to be removed from class to special education, of to be to leave education and find a job as an unskilled worker.

To summarize: As to children, learning can be concept-building through daily contact with concrete materials. Or, learning can prevent concept-building by excluding concrete materials and by sporadic lessons. As to teenagers, learning can be expanding their personal narrative with authorized gossip enforced by daily lessons in self-chosen half-year blocks. Or learning can be preventing their narratives from growing by teaching unknown fact about unknown subjects, again enforced by sporadic lessons. Finally, to adults learning can be grounded in workplace examples, or learning can be ungrounded encapsulated knowledge claimed to become maybe useful later.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

5. The Teacher

Bo: Thank you, John and Allan. Now let us talk about the teacher. It seems straightforward to say that the job of a teacher is to teach learners so that learning takes place, checked by written tests. However, continental Europe calls a teacher a ‘Lehrer’ thus using the same word as for learning. In addition, a Lehrer is supposed to facilitate ‘unterrichtung and erziehung and to develop qualifications and competences. In teacher education, the subject didactics, meant to determine the content of Bildung, is unknown outside the continent. And until lately, educating lehrers took place outside the university in special lehrer-schools. Thus, being a teacher does not seem to be that well-defined. Therefore, my next question is 'what is a teacher?'

Allan: As with learning, we must differentiate between teaching children in primary school, teaching teenagers in secondary school and teaching adults in tertiary schools.

A parent is an adult helping the child to supply its stomach with food and its brain with information, based upon a relationship of trust. Removed from the home in an institution, a child will look for a substitute parent, a nanny. To prevent them from becoming competing parents, a nanny only teaches one year-group and has only one class. The first year of primary school the nanny slowly splits up the outside world in things that we count and humans that we communicate with or about, thus laying the foundation to the two basic knowledge areas: nature with a number-language and society with a word language. At the end of primary school a class has two nannies specialized in each of the two basic knowledge areas.

In secondary school, the teacher role changes from a nanny to an expert with special training in one or two subjects. Now teachers have their own classroom where they teach the different daily half-year groups in their subject in the form of gossip. Half-year classes allow the teachers and the learners to maintain a good relationship, since at the end of the half year all learners leave the class thanked with a "good job" if successful and a "good try" if less successful.

In tertiary education, the degree of specialization is higher demanding a master degree in a theoretical subject or a license in a trade or in a craft.

At a block-organized university taking additional blocks allows a teacher to change career from primary to secondary or tertiary education, or to business, engineering or other crafts, and vice versa. And the final choice between teaching preschool or primary or secondary school can be postponed to later in teacher education. In contrast, Europe's line-organized education forces a choice between the different level to be made before tertiary school, and forces teachers to stay in their public office for the rest of their working life.

To summarize, a teacher have different roles at block- and line-organized schools. At the former, a teacher for children is a nanny splitting up the world in two subject areas: nature with a number-language and society with a word-language. And for teenagers teachers are experts telling about their specific knowledge area in the form of gossip. Both are educated at a university and able to change career by taking additional blocks. In line-organized education, a teacher specializes in several subjects, have several classes each day, and follows a class for several years. And once a teacher, always a teacher, since line-organized universities typically force students to start all over if wanting to change form one line to another.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

6. The Political System

Bo: Thank you, John and Allan. Now let us talk about governments. Humans live together in societies with different degrees of patronization. In the debate on patronization, the ancient Greek sophists argued that humans must be enlightened about the difference between nature and choice to prevent patronization by choices presented as nature. In contrast, the philosophers saw choice as an illusion since physical phenomena are but examples of metaphysical forms only visible to philosophers educated at Plato's Academy who consequently should be accepted as patronizors. Still today, democracies come in two forms with a low and high degree of institutionalized patronization using block-organized education for individual talent developing or using line-organized education for office preparation. As to exams, some governments prefer them centralized and some prefer them decentralized. As to curricula, the arrival of new Mathematics in the 1960s integrated its subfields under the common label Mathematics. Likewise, constructivism meant a change from lists of concepts to lists of competences. However, these changes came from Mathematics and education itself. So my question is: ‘Should governments interfere in Mathematics education?’

Allan: A government must create an educational institution forcing children and teenagers to spend so much of their life in it that some Greenland teenagers even talk about being condemned to school. Thus, a government must decide how much force it will allow the educational institution to exercise. Likewise, a government should know the root and agenda of their present educational institution as well as alternatives practiced elsewhere in the world.

As to curricula, a government must decide if schools present concepts as exemplified from above or abstracted from below. As to structure, a government must choose between the block-organized enlightenment education of the North American Democracies aiming at developing individual talents; and the line-organized Bildung education in Europe created in Berlin around 1800 to prevent democracy from spreading from France and aiming at preparing for public offices.

Besides politicians, a government also includes public servants, called mandarins in the ancient Chinese empire. In Europe the French sociologist Bourdieu has pointed out that the mandarin class forms a new knowledge-nobility using the educational system to exercise symbolic violence so that their children inherit the parents’ lucrative public offices; and that Mathematics is especially well suited for this purpose. Some countries, as e.g. Denmark, even hold on to oral exams, thus giving additional advantages to mandarin children.

In Europe, spreading out economical capital by creating a welfare state made socialist parties strong. However, they seem to neglect to spread out knowledge capital as well. After all, where economical capital is split up in a ‘what I win, you lose’ game, knowledge capital can be enjoyed by all in an all-win game. To me this paradox shows the strength of the mandarin class in Europe.

So to sum up. Yes, governments must create educational institutions, but should minimize its force as much as possible. Consequently, education should be block-organized from secondary school, and school subjects should be teaching grounded categories and knowledge. That is, Mathematics education must meet the human need to deal with the natural fact Many by counting and adding, i.e. by recounting in different units to root proportionality, by adding also next-to to root integration, and to reverse addition to root solving equations. And no, Europe should not hold on the its Humboldt line-organized Bildung preparing the mandarin children to inherit their parents’ public offices, created 200 years ago by the German nobility to induce nationalism into the population to keep democracy from spreading from France.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

7. Research

Bo: Thank you, John and Allan. Now let us talk about research. Tradition often sees research as a search for laws built upon reliable data and validated by unfalsified predictions. The ancient Greek Pythagoreans found three metaphysical laws obeyed by physical examples. In a triangle, two angles and two sides can vary freely, but the third ones must obey a law. In addition, shortening a string must obey a simple ratio-law to create musical harmony. Their findings inspired Plato to create an academy where knowledge meant explaining physical phenomena as examples of metaphysical forms only visible to philosophers educated at his academy by scholasticism as ‘late opponents’ defending their comments on an already defended comment against three opponents. However, this method discovered no new metaphysical laws before Newton by discovering the gravitational law brought the priority back to the physical level, thus reinventing natural science using a laboratory to create reliable data and test library predictions. This natural science inspired the 18th century Enlightenment period, which again created counter-enlightenment, so today research outside the natural sciences still uses Plato scholastics. Except for the two Enlightenment republics where American Pragmatism used natural science as an inspiration for its Grounded Theory, and where French post-structuralism has revived the ancient Greek sophist skepticism towards hidden patronization in categories, correctness and institutions that are ungrounded. Using classrooms to gather data and test predictions, Mathematics education research could be a natural science, but it seems to prefer scholastics by researching, not Math education, but the research on Math education instead. To discuss this paradox I therefore ask, 'what is research in general, and within Mathematics education specifically?'

Allan: A 'pencil-paradox' illustrates the trust-problem in research. Placed between a ruler and a dictionary, a pencil can itself falsify a number by pointing to a different number, but it cannot falsify a word by pointing to a different word, so where number-statements may express natural correctness, word-statements express a political correctness valid inside a ruling truth regime. In other words, using numbers, natural science produces universal truth, and using words, human and social sciences produce local and temporary truths always threatened by competing truth regimes or paradigms as Kuhn called them. Psychology has a paradigm war between behaviorists and constructivists, and within constructivism between Vygotsky and Piaget disagreeing as to whether the learner shall adapt to the ruling paradigm or the other way around. Sociology has a paradigm war called the actor-structure controversy, where the North American republics see social life as created by the symbolic interaction between independent actors, while the institutionalized Europe traditionally sees social life as determined by structures similar to the gravitational laws of natural science. But accepting word-statements as being not nature but choice has created a research genre studying the social construction of different word-paradigms.

The two Enlightenment republics have found ways around the pencil-paradox. North American reaction against traditional philosophy has created American Pragmatism and its symbolic interactionism insisting that categories and theory be grounded in observations. Thus, you must not enter a field with preconceived categories, and generated categories must accommodate to field resistance, thus paralleling the generation of collective and individual knowledge as described by Piaget both accepting the priority of observations as in natural science. Here counter-examples do not reject a category but splits it into sub-categories. In other words, both the courtroom and Grounded Theory base their categories upon action-statements and reject is-statements as prejudice, reserved for the judge and the researcher.

In the second Enlightenment republic, the French, patronization hidden in ungrounded words, sentences and institutions has developed the post-structural thinking of Derrida, Lyotard and Foucault. Derrida recommends deconstructing patronizing categories. Lyotard recommends challenging political correctness by inventing paralogy as dissension to the ruling consensus. Foucault recommends using concept archeology to uncover the pastoral power of the so-called human sciences, instead being disciplines disciplining themselves and their subject, thus silencing competing disciplines and forcing ungrounded identities upon humans as diagnoses to be cured by normalizing institutions applying these human sciences.

Inspired by this French skeptical thinking, postmodern contingency research has found another solution to the pencil paradox. Often postmodern thinking is seen as meaningless since its skepticism also must apply to itself. However, postmodern skepticism is a meta-statement about statements about the world and therefore not one of the statements about the world, against which it directs its skepticism. Of course, the liar paradox saying 'this sentence is false' and being false if true and vice versa makes self-reference problematic, but postmodern thinking avoids self-reference by its meta-statement 'Everything can be different, except the fact that everything can be different'. Thus the ancient sophist warning against mixing up nature and choice makes it possible for postmodern contingency research to discover false nature by finding hidden alternatives to choices presented as nature. Within Mathematics education research, contingency research has successfully pointed out hidden alternatives to unquestioned traditions within numbers, operations, equations, teacher education, etc. as seen on the MATHeCADEMY.net website.

To sum up, research can be a bottom-up activity using outside world observations to generate categories and theories to test predictions, especially successful with the number-statements of natural sciences. Or research can be a top-down activity forcing the outside world to assimilate to operationalized categories from the ruling paradigm, and using scholasticism to produce new researchers as late opponents defending comments on already defended comments against three opponents.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

8. Conflicting Theories

Bo: Thank you, John and Allan. Of course, Mathematics education research builds upon and finds inspiration in external theories. However, some theories are conflicting. Within Psychology, constructivism has a controversy between Vygotsky and Piaget. Vygotsky sees education as building ladders from the present theory regime to the learners' learning zones. Piaget replaces this top-down view with a bottom-up view inspired by American Grounded Theory allowing categories to grow out of concrete experiences and observations. Within Sociology, disagreement about the nature of knowledge began in ancient Greece where the sophists wanted it spread out as enlightenment to enable humans to practice democracy instead of allowing patronizing philosophers to monopolize it. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. As counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe's line-organized office preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the post-structuralism of the French Enlightenment republic. Likewise, the two extreme examples of forced institutionalization in 20th century Europe, both terminated by the low institutionalized American Enlightenment republics, made thinkers as Baumann and Arendt point out that what made termination camps work was the authorized routines of modernity and the banality of evil. Reluctant to follow an order, you can find another job in the private sector, but not in an institution. Here the necessity of keeping a job forces you to carry out both good and evil orders. As an example of a forced institution, this also becomes an issue in Mathematics Education. So I ask: What role do conflicting theories play in Mathematics education and its research?

Allan: To me, Sociology is the basic theory when discussing Mathematics education and its research. Sociology asks the basic question: in the social space, do we need patronization or can we find mutual solutions using the threefold information-debate-choice method of a democracy? As pointed out, the debate on patronization began in ancient Greece between the philosophers and the sophists; and the debate is still with us today between socialist top-down critical theorists and skeptical bottom-up postmodern theorists. As a social institution, education contains an element of force, that can be patronizing or emancipating providing what is called 'Mündigkeit' in German. Europe maximizes the force-component by using line-organized office preparing education to force humans to stay in the line as long as possible, and to accept that their difficulties are caused by their inferiority to the children of the public office holders helping their children inherit their offices created to patronize the population. Whereas North America from secondary school minimizes the force-component by using daily lessons in self-chosen half-year blocks to uncover and develop the individual talent of the learner.

Likewise, Mathematics can serve both purposes. Presented from above as top-down falsified Metamatism, it becomes so hard to learn that it forces many learners to stop learning it. This is a minor problem with half-year blocks since leaving Mathematics does not force you to leave school, but it is a big problem at line-organized schools where leaving the line means leaving school for good. Presented bottom-up from below grounded in the natural fact Many, Mathematics becomes easy to learn; and the learner can keep on choosing more blocks until the interest may disappear, or in Europe the ordinary learner can stay longer on the line to the dislike of the public office holders, the mandarins.

Likewise, the controversy within Psychology between Vygotsky and Piaget as to how learning takes place also serves both sociological purposes. Presented top-down from above, concepts become hard to learn and force many learners to stop learning the concepts and to accept patronization by those who succeed learning them. In contrast, bottom-up concepts grounded from below in the outside world are easy to learn for children through the concrete material that roots the concepts; and for teenagers since knowing the subject of the sentence gives a Grounded Theory the form of gossip.

The need to keep their job forces teacher to follow the orders of their specific institution. When trained, teachers should as potential change agents be informed about the many choices of an educational institution and within Mathematics, so the individual teacher knows the difference between choice and nature, i.e. what can be changed and what cannot, in order to prevent being a victim of the banality of evil.

To sum up, a civilized teacher education should inform about the many examples of conflicting theories in Mathematics, in education and in research and should put more emphasis on the sociological consequences of unnecessary force in these three institutions.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

9. Me and Mathematics Education and Research

Bo: Thank you, John and Allan. Now let us talk about your own experiences with Mathematics education and its research. In addition, I would like to ask you who are the most important theorists in Math education research in your opinion?

Allan: I met Mathematics before the arrival of the so-called new Math. In elementary school we had reckoning, and in middle school we had written and oral reckoning together with arithmetic and geometry, and finally about 5% of us went on to the European high school called a 'gymnasium' where we met the word Mathematics for the first time; finally, at the university, Mathematics was to new Math from day one. Repetition and its roots to the outside world made reckoning easy to learn, likewise with geometry where we learned to construct different figures and met formal definitions and proofs. Introduced as letter-reckoning made arithmetic strange and difficult, especially when reducing letter fractions came along. At the gymnasium, the epsilon-delta definition of real numbers from day one killed the interest of most students; and likewise during the first year at the university when geometry was replaced by n-dimensional linear algebra. Here Mathematics changed to Metamatics with top-down set-derived definitions and general proofs without examples to sort out the elite for graduate studies. Most students dropped out or failed the exam. I passed, but to get a meaningful job I decided to shift to architecture. However, at a Belgian library I met American textbooks presenting algebraic topology bottom-up as abstractions from examples instead of the other way around and I decided to become a Math teacher teaching bottom-up meaningful

Mathematics instead of the top-down meaningless Metamatics, that made the textbooks so hard to access for the students in the gymnasium.

As a teacher I learned, that using words derived from its roots made concepts much more understandable. Thus, most students had problems with the traditional textbook definitions and theorems of exponential functions introduced after the set-derived definition of a function. In contrast, telling that when growing by a constant multiplier, the end value y is the initial value b multiplied with the multiplier c x times, written as y equal b multiplied with c to the power of x made one student remark: 'Hey mr. Teacher, this we already know, when do you teach us something we don't know?' So I began to look for root-based names for the Mathematical concepts and was surprised to find the root of calculus as adding variable per-numbers, and to find that when epsilon and delta changes places we define a piecewise instead of a locally constant formula. Likewise, introducing integral calculus before differential calculus took the hardness out of calculus.

The discovery that hidden alternatives can change Mathematics from hard to easy brought me to Mathematics education research. Here the beauty and simplicity of the ancient Greek sophist warning against false nature by saying that unenlightened about the difference between nature and choice we risk being patronized by choices presented as nature made me develop contingency research aiming at discovering hidden alternatives to choices presented as nature. Likewise, I admired the beauty and simplicity of American Sociology where Berne talks about the three states of communication, parent, child and adult. These three states create two effective ways of communicating, child-parent where both accept the presence of authority, and adult-adult where both accept its absence; and several ineffective ways not agreeing upon the role of authority. In addition, I was fascinated about the resemblance between Piaget in Psychology and American Grounded Theory both inspired by natural science and describing how individual and collective learning means adapting knowledge to the outside world by assimilation and accommodation. And finally I was caught by postmodern or post-structural skeptical thinking developed in the threatened French Enlightenment republic warning against patronization in our most basic institutions: our words, beliefs, cures and schools. Here I saw the patronizing techniques of the school: hiding understandable alternatives forces children and teenagers to accept the ruling choices as nature.

Searching for contingency, I found hidden words as icon-counting, next-to addition, reversed addition, and per-numbers. In addition, I found that Mathematics was created as a natural science about the natural fact Many. By teaching in the US I found that teenagers can be allowed to develop their personal talent if Europe's line-organized office preparing education with forced classes are replaced with North American block-organized talent developing education with daily lessons in self-chosen half-year blocks. Furthermore, I found that Bourdieu might be right when warning against a knowledge nobility that use their public offices to protect the line-organized education to ensure that their children inherit their offices. And finally, Baumann's and Arendt's work on the extreme institutionalization in 20th century Europe made me realize that the problems in Mathematics education and its research might be caused by an exaggerated institutionalization that by forcing teachers to follow authorized routines makes them subjects to the banality of evil without knowing it and without wanting to be so.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

10. How to Improve Mathematics Education

Bo: Thank you, John and Allan. Let us finish by looking at what this is all about, Mathematics education. The first International Congress on Mathematics Education, ICME 1, took place in 1968, so we can say that Mathematics education research has about the same age as the new Mathematics emerging in the 1960s. With half a century of research, we should expect the problems in Mathematics education to have disappeared or at least decreased considerably. However, the decreasing results of international tests indicates that the opposite is the case. The paradox that researching Mathematics education seems to create more problems than solutions motivates my last question 'how can Mathematics education be improved?'

Allan: Indeed, we have a paradox when the problems in Math education increase with its research. To solve it we can ask how well defined Mathematics and education and research is? Or, as in the fairy tale Cinderella we can look for hidden alternatives that might please the Prince and make the paradox disappear? The ruling tradition presents Mathematics as ungrounded Metamatism with meaningless self-referring concepts, and with statements falsified by the outside world. The hidden alternative presents Mathematics as grounded science about the natural fact Many. These two alternatives entail two different forms of teaching. One presents concepts as created from above as examples from abstractions as shown in the textbooks; the other show how concepts are created from below as abstractions from examples, facilitated by concrete material for children and relevant gossip for teenagers.

Theorists also come in two forms. One uses the Platonic tradition to present physical phenomena as examples of metaphysical forms discovered by and investigated by philosophers. The other sees theory as grounded in and adapting to its underlying reality that generates the theory's concepts and validates its statements.

Research also comes in two forms. One is self-referring scholasticism commenting on comments already defended against three opponents. The other is Grounded Theory seeing individual and collective knowledge creation as parallel processes, creating schemata that adapt to the outside world. Finally, education also comes in two forms, as line-organized office-preparation or as block-organized talent-developing.

So to me, the choice within four factors determines the success of Mathematics education. Problems occur if Mathematics presents itself as Metamatism, if only top-down theorists are used, if research is scholastic, or if education uses force by choosing line-organized office preparation. When chosen simultaneously as in Europe, Mathematics education is in deep trouble, which of course suits the knowledge nobility well. To be successful, Mathematics must grows from its roots in the natural fact Many, only grounded bottom-up theorists must be used, research must be a natural science using the classrooms to generate categories and test predictions; and education must minimize its force by choosing block-organized talent development from secondary school. Having implemented the three latter, the North American republics only need to change Metamatism to grounded Mathematics to make their Mathematics education successful.

John: Sentence. Sentence. Sentence. …

Allan: I would like to comment on what John said. Sentence. Sentence. Sentence.

John: I would like to comment on what Allan said. Sentence. Sentence. Sentence.

Bo: Thank you, John and Allan. I began by expressing the hope that you could provide some answers to the question 'If research cannot improve Mathematics education then what can?' I now see that this debate has resulted in a several suggestions that I am sure practitioners and politicians will be eager to work with and be inspired by. Thank you, John and Allan, for your time and for sharing your views with us.

Allan: You are welcome, Bo. I enjoyed very much to take part in this debate.

John: So did I, Bo.

The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants

Educational shortages described in the OECD report 'Improving Schools in Sweden' challenge traditional math education offered to young male migrants wanting a more civilized education to return help develop and rebuild their own country. Research offers little help as witnessed by continuing low PISA scores despite 50 years of mathematics education research. Can this be different? Can mathematics and education and research be different allowing migrants to succeed instead of fail? A different research, difference-research finding differences making a difference, shows it can. STEM-based, mathematics becomes Many-based bottom-up Many-matics instead of Set-based top-down Meta-matics.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Math Education. But, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

To find an unorthodox solution let us pretend that a university in southern Sweden arranges a curriculum architect competition: 'Theorize the low success of 50 years of mathematics education research, and derive from this theory a STEM-based core mathematics curriculum for young male migrants.'

Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is a rational organization, 'in which the *end* is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Such a goal displacement occurs if saying 'The goal of mathematics education is to teach and learn mathematics'. Furthermore, by its self-reference such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts. In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2x3=6' stating that 2 3s can be re-counted as 6 1s.

So, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about shapes and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement, according to what Bauman (1992, p. ix) calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'?

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. (Heidegger, 1962, pp. 126, 169)

Heidegger has inspired existentialist thinking, described by Sartre (2007, p. 22) as holding that 'existence precedes essence'. In France, Heidegger inspired Derrida, Lyotard, Foucault and Bourdieu in poststructuralist thinking pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and to your world (Lyotard, 1984; Bourdieu, 1970; Foucault, 1995).

As to the political aspects of research, Foucault says:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006, p. 41; also on YouTube)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics and education and research are examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as an inside goal instead, thus marginalizing or forgetting its original outside goal.

To avoid this, difference-research is based upon the Greek sophists, saying 'Know nature form choice to unmask choice masked as nature.'; and Heidegger saying 'In sentences, trust the subject but question the rest.'; and Sartre saying 'Existence precedes essence'; and Foucault, seeing a school as a 'pris-pital' mixing power techniques of a prison and a hospital by keeping children and adolescents locked up daily to be cured without being properly diagnosed. For it is differences that unmask false nature, and unmask prejudice in predicates, and uncover alternative essence, and cure an institution from a goal displacement.

Furthermore, difference-research knows the difference between what can be different and what cannot. From a Heidegger view an is-sentence contains two things: a subject that exists and cannot be different, and a predicate that can and that may be gossip masked as essence, provoking 'the banality of Evil' (Arendt, 2006) if institutionalized. So, to discover its true nature, we need to meet the subject, the total, outside its predicate-prison of traditional mathematics. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help develop or rebuild their original countries.

So, to restore its authenticity, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many, Children use Block-numbers to Count and Share

How to deal with Many can be learned from preschool children. Asked 'How old next time?', a 3year old will say 'Four' and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not 4, that is 2 2s. Children also use block-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked 'How many 3s when united?' they typically say '5 3s and 3 extra'; and when asked 'How many 4s?' they say '5 4s less 2'; and, placing them next-to each other, they say '2 7s and 3 extra'.

You don't need research to observe how children love digital counting by bundling, replacing a bundle of 2 1s with 1 Lego Brick with 2 knobs to be placed in a cup for the bundles; and they don't mind exchanging 2 2s with 1 Lego brick with 4 knobs to be placed in a cup for 4s. And they have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 1 4s &3, etc. And children don't mind writing a total of 7 using 'cup-writing' as T = 7 = 1]5 = 2]3 = 3]1 = 1]0]3 = 1]1]1. And with 1 plastic S for 1 borrowed, some children even writes T = 7 = 3]1 = 4]S = 5]SSS. Also, children love to count in 3s and 4s. Recounting in 5s is unfortunately not possible since Lego refuses to produce bricks with 5 knobs.

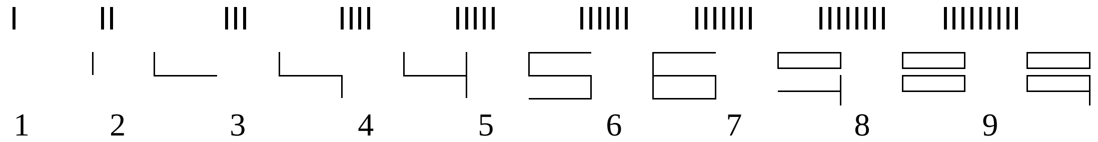
Sharing 9 cakes, 4 children takes one by turn as long as possible; with 4s taken out they say ‘I take 1 of each 4’, and with 1 left they say ‘let’s count it as 4’. And they smile when seeing that sharing 4 5s by 3 is predicted by asking a calculator 4*5/3. Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. So children master sharing, taking parts and splitting into parts before having learned about division and counting- and splitting-fractions, which they would like to learn before being forced to add.

Children thus show core mastery of Many before coming to school, allowing school to build upon this knowledge instead of rejecting it. So, school could ask research to design a curriculum, that counts totals in two-dimensional block-numbers instead of one-dimensional line-numbers; that counts and re-counts and double-counts totals before they are added, and then both on-top and next-to; that teaches 8/4 as 8 counted in 4s giving 2 4s instead of as 8 split between 4 giving 4 2s; and that root counting-fractions and splitting-fractions in per-numbers and re-counting. Difference-research gladly takes on such a curriculum design.

Meeting Many Creates a Count&Multiply&Add Curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding to create a number-language sentence, T = 2 3s, containing a subject, a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents, icons can be used as units when counting: four strokes in the 4-con, five in the 5-icon, etc.



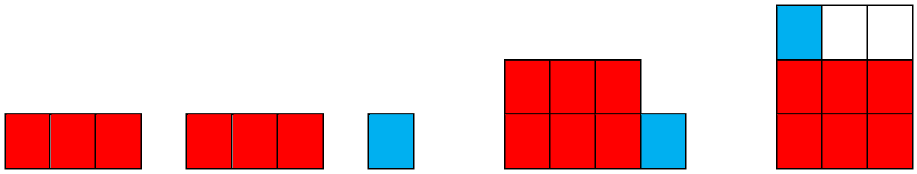
We count in bundles to be stacked as block-numbers to be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count we take away bundles (thus rooting division as a broom wiping away the bundles) to be stacked (thus rooting multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting subtraction as the trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)xB saying that ‘from T, T/B times, B can be taken away’:

7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

Finally, bundle- or cup-writing double-counts the bundles inside the bundle-cup and the singles outside, where an overload or underload is removed or created by re-counting in the same unit, T = 7 = 2B1 3s = 2]1 3s = 1]4 3s = 3]-2 3s.

Likewise, placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations to be solved by re-counting, resulting in moving numbers to the opposite side with the opposite sign: u x 7 = 42 = (42/7)x7 gives u = 42/7.

Double-counting in physical units creates per-numbers bridging the units, thus rooting proportionality. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. Then both on-top and next-to addition can be reversed, thus rooting equations and differential calculus.

In a rectangle split by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Meeting Many in a STEM Context

Having met Many by itself, now we meet Many in time and space in the present culture based upon STEM, described by OECD as follows:

> In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature’s physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature’s three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground since motion transfers through collisions, now present as increased motion in molecules; so the motion has lost its order and can no longer work.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making it combine to stars and planets and galaxies. Some planets have a size and a distance from its star that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles and mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves.

Technology is knowledge about ways to satisfy human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for motion and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language for predicting Many by calculation sentences, formulas, expressing counting and adding processes. First Many is double-counted in bundles and singles to create a total T that might be re-counted in the same or in a new unit or into or from tens; or double-counted in two physical units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. Finally, triangulation predicts spatial forms.

So, a core STEM curriculum could be about cycling water. Heating pumps in motion transforming water from solid to liquid to gas, i.e. from ice to water to steam; and cooling pumps motion out. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform moving water to moving electrons, electricity. To get to the dam, we build roads on hillsides.

The Electrical circuit, an Example

To work properly, a 2000Watt water kettle needs 2000Joule per second. The socket delivers 220Volts, a per-number double-counting Joules per charge-unit.

Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle has a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per charge-unit) *(charge-unit per second) we also have a second formula Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So surprisingly, the 120Watt bulb only receives half of the Joules of the 60Watt bulb.

Difference-research Differing from Critical and Postmodern Thinking

Together with difference-research, also critical thinking and postmodernism show skepticism towards knowledge claims, so how does difference-research differ?

As to critical thinking, Skovsmose & Borba (2000) describes a Brazilian research group that, focusing on issues related to new technologies and mathematics education, has developed software and work with students at different levels and with teachers. The group was approached by a teacher from a nearby school where she had some tough problems to face and hoped that the computers would be able to help her. She was teaching rational numbers to a class of 5th graders, with a mixture of 11year old students and 15year old repeaters having given up rational numbers and turning to violence.

> The teacher was enthusiastic about a software, which deals with rational numbers. (..) Both researchers and teacher had the ‘intuition’ that the computer might have a positive effect in this class and maybe could avoid that the students had to repeat this grade again. (p. 7)

By recommending computers, the researchers showed criticism, not towards fractions as such, but towards how they are taught. Critical thinking thus sees mathematics as an unquestionable goal, only how it is taught can be questioned.

Contrary to this, difference-research sees fractions as a means rooted in double-counting, and recommends fractions introduced as per-numbers via the 'fraction-paradox': 1 red of 2apples and 2red of 3apples total 3red of 5apples and not 7red of 6apples as says the textbook. Fractions thus add by their areas as integral calculus. Adding fractions of the same total can be treated later. Introducing fractions via per-numbers and separating core-mathematics from 'footnote-mathematics' will side the teacher with the learner against the textbook.

As to postmodern thinking, the book 'Mathematics Education within the Postmodern' (Walshaw, 2004) contains 12 chapters divided into three parts: thinking otherwise for mathematics education, postmodernism within classroom practices, and within the structures of mathematics education. The preface says:

> It is a groundbreaking volume in which each of the chapters develops for mathematics education the importance of insights from mainly French intellectuals of the post: Foucault, Lacan, Lyotard, Deluze. (p. vii)

Although the book wants to be skeptical towards both mathematics and its education, it is only the educational part that is scrutinized; and most authors describes how what is labeled postmodern thinking can be exemplified in educational contexts, they don't see mathematics itself as a social construction that could be questioned also. A central thinker as Derrida is mentioned only in the two survey chapters, and the core concept of deconstruction is not mentioned at all despite its fundamental importance to a postmodern perspective to mathematics education (Tarp, 2012).

By going behind French thinking to its root in Heidegger existentialism, difference-research is the only skeptical thinking raising the basic sociological question about a possible goal displacement in mathematics itself.

Conclusion and Recommendation

The task of the curriculum architect competition was 'Theorize the low success of 50 years of mathematics education research, and derive from this theory a STEM-based core mathematics curriculum for young male migrants.'

One explanation sees the situation caused by mathematics itself as very hard to teach and learn. This paper, however, sees it caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many. The two views lead to different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying 'To master Many, count to produce block-numbers and per-numbers that might be constant or variable, to be united by adding or multiplying or powering or integrating.

Thus, this simplicity of mathematics as expressed in a Count&Multiply&Add curriculum allows learners to keep their own block-numbers, and to acquire core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young male migrants learn core STEM subjects at the same time, thus allowing them to become pre-teachers or pre-engineers after two years to return help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

References

Arendt, H. (2006). *Eichmann in Jerusalem.* London: Penguin Books.
Bauman, Z. (1990). *Thinking Sociologically*. Oxford, UK: Blackwell.
Bauman, Z. (1992). *Intimations of Postmodernity.* London, UK: Routledge.
Bourdieu, P. (1970). *Reproduction in Education, Society and Culture*, London: Sage.
Chomsky, N. & Foucault, M. (2006). *The Chomsky-Foucault Debate on Human Nature.*

Foucault, M. (1995). *Discipline & Punish.* New York: Vintage Books.
Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht-Holland: D. Reidel Publishing Company.
Glaser, B. G. & Strauss, A. L. (1967). *The Discovery of Grounded Theory.* New York: Aldine de Gruyter
Han, S., Capraro, R. & Capraro MM. (2014). How science, technology, engineering, and mathematics (STEM) project-based learning (PBL) affects high, middle, and low achievers differently: The impact of student factors on achievement. *International Journal of Science and Mathematics Education. 13* (5), 1089-1113.
Heidegger, M. (1962). *Being and Time.* Oxford, UK: Blackwell.
Lyotard, J. (1984). *The postmodern Condition: A report on Knowledge.* UK: Manchester University Press.
Mills, C. W. (1959). *The Sociological Imagination.* UK: Oxford University Press
Negt, O. (2016). *Soziologische Phantasie und exemplarisches Lernen: Zur Theorie und Praxis der Arbeiterbildung.* Germany: Steidl.
OECD. (2015a). *Improving Schools in Sweden: An OECD Perspective.* Retrieved from: www.oecd.org/edu/school/improving-schools-in-sweden-an-oecd-perspective.htm.
OECD. (2015b). *OECD Forum 2015.* Retrieved from http://www.oecd.org/forum/ oecdyearbook/we-must-teach-tomorrow-skills-today.htm.
Piaget, J. (1969). *Science of Education of the Psychology of the Child.* New York: Viking Compass
Sartre, J.P. (2007). *Existentialism is a Humanism.* CT. Yale University Press
Skovsmose, O. & Borba, M. (2000). *Research Methodology and Critical Mathematics Education.* Roskilde University: Research in Learning Mathematics n. 17.
Tarp, A. (2012). *An ICME Trilogy. Papers, Posters and other Material from ICME 10, 11 and 12*. Retrieved from http://mathecademy.net/papers/icme-trilogy/
Tarp, A. (2017). *Math Ed & Research 2017.* Retrieved from http://mathecademy.net/ 2017-math-articles/.
Walshaw, M. (2004). *Mathematics Education within the Postmodern.* CT, Greenwich: Information Age Publishing.

Math Competenc(i)es - Catholic or Protestant?

Introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015 OECD-report 'Improving Schools in Sweden'. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other Scandinavian countries also have experienced declining PISA results. Which came as a surprise since they all adopted the idea of the eight mathematics competencies introduced by Niss (2003) as a means to solve poor mathematics performance. Of course, new ideas cannot work overnight, but after close to two decades it is time to ask: Why does math competencies not work?

Since education and textbooks are social constructions meant to solve important problems by common social institutions, maybe sociology can provide an answer to the lacking success of the eight mathematics competencies.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959), and by Bauman (1990) saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16). As to rationality as the base for social organizations, Bauman says (pp. 79, 84):

> **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right.

As an institution, mathematics education is a public organization with a rational action 'in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas, arithmetic and geometry and music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time, i.e. as a ‘Many-matics’. And together forming the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught ‘reckoning’ (Rechnung in German) in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here the invention of the concept SET created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming a meaningless language by mixing concrete examples and abstract concepts. In this way, SET transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘1 + 2 IS 3’ meet counter-examples as e.g. 1week + 2days is 9 days. So, mathematics has meant different things during its long history.

Defining Mathematics Competencies

In the paper ‘Mathematical Competencies and the Learning of Mathematics: The Danish Kom Project’ Niss writes (2003, p. 1):

> The fundamental idea of the project is to base the description of mathematics curricula primarily on the notion of a “mathematical competency”, rather than on syllabi in the traditional sense of lists of topics, concepts, and results. This allows for an overarching conceptual framework which captures the perspectives of mathematics teaching and learning at whichever educational level.

Niss writes (pp. 4-5) that the project was initiated in 2000 by the Danish Ministry of Education asking the following questions:

- To what extent is there a need for innovation of the prevalent forms of mathematics education?
- Which mathematical competencies need to be developed with students at different stages of the education system?
- How do we ensure progression and coherence in mathematics teaching and learning throughout the education system?
- How do we measure mathematical competence?
- What should be the content of up-to-date mathematics curricula?
- How do we ensure the ongoing development of mathematics as an education subject as well as of its teaching?
- What does society demand and expect of mathematics teaching and learning?

- What will mathematical teaching materials look like in the future?
- How can we, in Denmark, make use of international experiences with mathematics teaching?
- How should mathematics teaching be organised in the future?

Next, Niss defines what it means to master mathematics (pp. 5-6, 8):

> The Committee based its work on an attempt to answer the following question: *What does it mean to master mathematics?*' (..) To master mathematics means to possess mathematical competence. (..) To possess a competence (to be competent) in some domain of personal, professional or social life is to master (to a fair degree, modulo the conditions and circumstances) essential aspects of life in that domain. *Mathematical competence* then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. (..) *A mathematical competency* is a clearly recognisable and distinct, major constituent of mathematical competence. (..) There are eight competencies which can be said to form two groups. The first group of competencies are to do with the ability to *ask and answer questions in and with mathematics*. (..) The other group of competencies are to do with the ability to deal with and *manage mathematical language and tools*:

Before writing that 'Possessing a mathematical competency (to some degree) consists in being prepared and able to act mathematically on the basis of knowledge and insight (p. 10)' Niss lists (pp. 7-9) and specify the two groups of four mathematical competencies

1. Thinking mathematically (mastering mathematical modes of thought)
2. Posing and solving mathematical problems
3. Modelling mathematically (i.e. analysing and building models)
4. Reasoning mathematically
5. Representing mathematical entities (objects and situations)
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools (IT included)

Discussing Mathematics Competencies

As to the definition of mathematics competencies, Niss is very clear: Mathematics competencies are the eight constituents of mathematics competence, defined as the ability to master mathematics. What is not so clear is what Niss means with these two words, mathematics and master.

What kind of mathematics

As to mathematics, at least two kinds of mathematics exits as shown above, a bottom-up and a top-down version, the original Greek grounded Many-matics and the modern self-referring meta-matism. Likewise, on the background of the science wars and mathematics wars in the previous decades, it would be relevant to clarify what kind of mathematics Niss is talking about: the original Greek version, the 'back to basics' pre-NewMath version, the set-based NewMath version, or a post-NewMath version in its constructivist or postmodern forms (Tarp, 1998, 2000).

Instead Niss refers to the fact that in Denmark, as one of the few countries if not the only, teacher education is not allowed to take place at universities where only research directed set-based mathematics is taught forcing students to include a master degree before being allowed to teach in upper secondary school.

Niss describes this difference in teacher background by saying that before upper secondary school, teachers 'are ambassadors of the student to the subject', whereas 'the university graduates who end up teaching mathematics see themselves as ambassadors of mathematics to the student' (pp. 2-3).

> A further aspect of the cultural and institutional differences that exist in Danish mathematics education is that mathematics is perceived and treated so differently at the different levels that one can hardly speak of the same subject, even if it carries the same name throughout the system. (..) The main problem is that the different educational levels tend to see themselves as competitors rather than as agents - acting at different sections of the education system - of the same overall endeavour and a common project, namely to increase and strengthen the mathematical competence of all students who receive some form of mathematics education.

On this background it seems clear that what Niss means with mathematics is the set-based university mathematics introduced with the NewMath. So what Niss points out is which competences are needed to master inside set-based university mathematics, not which are needed to master its outside root, Many. Thus, the question about what could be called quantitative competence is left unanswered.

What kind of Mastering

In the final report Niss left out two of the original Ministry questions, 'How can education take into account the new student type?' and 'What impact will a modified education have for teacher training?'. And in two questions, 'Which competences and qualifications can be acquired at the various stages of the education' and 'How can competences and qualifications be measured?', the word qualification is left out and the word mathematics is added. Likewise, the original term competence has replaced by his own term, competency (Tarp, 2002).

The difference between qualifications and competence might be illustrated by the fact that learning is a process shared by all three kinds of animals, reptiles and mammals and humans, all producing offspring to reproduce, but in different numbers since the chances of survival are different because of different learning abilities. Darwin's 'survival of the fittest' principle points to the fact that to survive you must fit to the surrounding outside world. Reptiles survive by their genes that might change over generations through mutations. Mammals feed their offspring until sexual maturity so they can adapt to the outside surroundings by guidance from their parents in an informal learning setting that could be called apprenticeship or learning from the master, providing the learner with tacit knowledge, also called abilities or know-how or competences. Likewise, humans learn basic living skills and the mother language as competences through apprenticeship guided by caring parents and adults. However, humans benefit from an additional learning possibility occurring when expanding the brain to keep the balance when standing up freed the forelegs to become graspers. Now the brain was also able to store sounds to mentally grasp what was grasped physically (in German: 'greifen & begreifen'), thus developing a word-language and a number-language for outside qualities and quantities allowing for life-long learning.

Language allowing information to be transferred between brains thus creates more competences quicker and more effective. And creates a formal learning setting called education or schooling using rational goal-means descriptions to qualify the learners to obtain the goal by following the means.

Thus, where animals develop competences from 'ex-ducational' informal learning outside school, humans learn additional qualification from 'in-ducational' formal learning inside schools. So human knowledge comes from two channels, from inside school as qualifications and from outside school as competences.

Inside teaching can take place through mediation to qualify or through guidance to develop competences. This discussion takes place between traditional teaching and constructivism; and within constructivism, between a social and a radical version where Vygotsky points to teaching, and Piaget to guidance.

Competence versus Capital

Niss uses no theoretical reference to mathematics or education, but points out that the report is supposed to be a response to question posed by the Ministry (p. 6).

Thus, there is no discussion of parallel and more developed or used concepts describing the same reality as does competences. As an example, Bourdieu (1977) has developed a theory on habitus and capital describing how in a social filed, your social or knowledge capital depends on your habitus within the field. Thus, it seems as if competence is a parallel concept to capital. If that is the case then, according to Bourdieu, capital is only obtainable by informal learning processes.

The Counter KomMod report

The KomMod report (Tarp, 2002) shows the original 12 Ministry questions and how they can be answered in a different way. In the end it compares the two reports by talking about a catholic and a protestant version of mathematics with eight and two competences respectively (p. 3):

> Defining competence as insight-based, the report assumes that mathematics is already learned, after which the rest of the time can be used to apply mathematics, not on the outside world, but on mathematics itself through eight internal competencies leading to exercising mathematical professionalism. This makes it a report on 'catholic mathematics' with eight sacraments, through which the encounter with science can take place. In contrast to this, the counter-report portrays a 'protestant mathematics' that emphasizes the importance of a direct meeting between the individual and the knowledge root, Many, through two sacraments, count and add.

Quantitative Competence

In the outside world, Many often occurs in time and space. To master Many, you must have quantitative competence from informal learning or quantitative qualifications from formal learning.

Meeting Many, we ask 'How many in total?' To answer, we count and add to get a number for a number-language sentence telling that the total is e.g. T = 456, thus containing a subject and a verb and a predicate as in the word-language. By counting and adding you build different know-how as to how to master Many:

- A digit has as many strokes as it represents, e.g. four strokes in the 4-icon, etc.
- Counting the fingers on a hand, the total cannot be different, but how to count it can be different, e.g. T = 5 1s = 2 2s & 1 = 1 3s & 2 1s = 1 3s & 1 2s etc.
- The sentence T = 456 is a short way of writing T = 4*BB + 5*B + 6*1, describing what exists, three blocks with 6 1s and 5 bundles and 4 bundles-of- bundles, typically using ten as the bundle-size and therefore needing no icon since ten then is 1*B. This shows that a number is the result of several countings: of unbundled ones, of bundles, of bundles-of-bundles etc.; and shows that all numbers have units: ones, bundles, bundles-of-bundles, etc.
- Writing out fully, T = 456 also shows the four ways to unite totals: on-top addition creating a block described by multiplication as repeated addition, power describing repeated multiplication when forming bundles-of-bundles, and finally integration as next-to addition when juxtaposing blocks.
- Operations are icons also: division is iconized as a broom wiping away the bundles; multiplication as a lift stacking the bundles into a block; subtraction as a trace left when dragging away the blocks to look for unbundled singles; and addition as a cross since blocks may be added both on-top or next-to.
- To deal with leftover singles when bundling we introduce a decimal point to separate the bundles from the singles, e.g. T = 7 = 2B1 3s = 2.1 3s, or we count the singles in bundles also even if a part only, T = 7 = 2B1 3s = 2 1/3 3s.

- A total can be recounted to change unit. Recounting in the same unit creates overload or underload e.g. T = 42 = 4B2 = 3B12 = 5B-8. This is useful when performing standard operations as e.g. T = 5*43 = 5*4B3 = 20B15 = 21B5 = 215. Or, we just move the decimal point separating the bundle from the unbundled, e.g. T = 4.3 hundreds = 43 tens = 0.43 thousands.
- To recount in another bundle size we use a 'recount formula' T = (T/B)*B saying that 'from T, T/B times B can be taken away' as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the 'restack formula' T = (T–B)+B saying that 'from T, T–B is left when B is taken away and placed next-to', as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. The recount formula turns out to be a very basic formula turning up repeatedly: In proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right-angled triangles, and in calculus as dy = (dy/dx)*dx = y'*dx when counting steepness on a curve.
- To recount icons in tens we use the multiplication table, e.g. T = 6 7s = 6*7 = 42. To recount tens in icons we solve equations, e.g. T = 42 = ? 7s = x*7 solved by x = 42/7, i.e. by moving numbers to opposite side with opposite sign.
- Double-counting a quantity in physical units creates per-numbers as e.g. 4$/5kg or 4/5 $/kg allowing the two units to be bridges by recounting in the per-number: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$, etc. With like units we get fractions, or percentages.
- Adding means uniting unit- and per-numbers, that can be constant or variable. So to predict, we need four uniting operations: addition and multiplication uniting variable and constant unit-numbers; and integration and power uniting variable and constant per-numbers. As well as four splitting operations: subtraction and division splitting into variable and constant unit-numbers; and differentiation and root/logarithm splitting into variable and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite.
- Blocks can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

Proportionality, an Example of Different Quantitative Competences

A question asks 'If 5kg costs 30$ what does 8kg cost; and what does 54$ buy?

A 1867 reguladetri 'long way-method' says: 'Make the outer units like, then multiply and divide, but from behind'. So, after reformulating the second question to '30$ buys 5kg, what does 54$ buy?' the first answer is 8*30/5$ = 48$; and the second answer is 54*5/30kg = 9kg.

A 1917 unit-method says: 1kg costs 30/5 = 6$, so 8 kg costs 6*8 = 48$.

A 1967 function-method says: With f(5) = 30, the linear function f(x) = c*x becomes f(x) = 6*x. So f(8) = 6*8 = 48. And 54 = 6*x is an equation. To neutralize 6, both sides are multiplied with its inverse element, 1/6, giving x = 54*1/6 = 9.

A 2017 back-to-basics method says 'cross-multiply' the price equation: 30/5 = x/8 gives 5*x = 8*30, so x = 48. And 30/5 = 54/x gives 30*x = 5*54, so x = 9.

A 2067 double-counting method recounts in the per-number 5kg/30$. So 8kg = (8/5)*5kg = (8/5)*30$ = 48$. And 54$ = (54/30)*30$ = (54/30)*5kg = 9kg.

Conclusion

Invented to improve mathematics education, the eight mathematics competencies inspired Scandinavian educational reforms that failed as witnessed by low PISA results decreasing until 2015. This paper asked why the competencies failed.

Formal education can use mediation to qualify or constructivism to create competences by guided meetings with the outside subjects for which education is supposed to prepare the learner. With Niss we can discuss which competences to create and how, but only in a constructivist setting that accepts the original Greek meaning of mathematics as knowledge about Many in time and space.

Niss may be right that his eight mathematical competences are needed to survive at a university that holds on to the original set-based version of mathematics introduced with the NewMath and recommended by Bruner to also be mediated in schools. But to master the outside goal Many, two competences will do, count & add, since they allow answering the standard question 'How many in total' by producing a number created by counting and adding as shown when writing out fully a number as a combination of blocks.

So the eight mathematics competences failed because university mathematics and school mathematics have different goals. At the university, education prepares you for the inside goal of staying at the university as a researcher; and in school, education prepares you for the task of mastering Many as it appears outside school in time and space.

Recommendation: Expand the Existing Quantitative Competence

By distinguishing between 4 and 2 2s at the 4th birthday, a child shows that before formal learning begins in school, the informal learning of growing up makes the child develop the two core quantitative competences, counting and adding. By counting in 2dimensional block-numbers supplied with some leftovers, children show a basic competence in double-counting a total in bundles and unbundled. And, when adding blocks, they answer by using one of the units or by uniting the units, thus showing a basic competence in proportionality and calculus.

Seeing expanding the learner's quantitative competence as the goal of mathematics education, school may choose to use guiding 'footnote-teaching':

- Show that digits are icons with as many strokes as they represent by inviting the child to build up a 5-icon with five dolls or cars or animals, etc.
- Ask the child to use cups for the bundles when re-counting a total in icons thus emphasizing that counting means double-counting, first bundles to be placed in a bundle-cup, then unbundled singles to be left outside, allowing a total to be counted in three ways: normal, and with outside overload or underload.
- Show that the four operations are icons as well, created to allow a calculator to predict the result when recounting a total in another unit; especially from icons to tens predicted directly by the multiplication table; or from tens icons, becoming equations solved by recounting in the icon, and technically by moving numbers to opposite side with opposite sign.
- Accept overload or underload, quickly created or removed by recounting, with standard operations as adding, subtracting, multiplying or dividing.
- Show that totals can be added both on-top after recounting them in the same unit thus rooting proportionality, and next-to recounting them in the united unit thus rooting integral calculus.
- Show that reversed on-top addition roots equations, again solved by recounting, i.e. by moving to the opposite side with opposite sign; and that reversed next-to addition roots differential calculus by using subtraction to remove the initial block, and division to recount the rest.

Once school has allowed the child to use and develop its own quantitative competence, it will be possible to expand this by introducing double-counting in physical units to create per-numbers, becoming fractions if using the same physical unit. Adding per-numbers and fractions by their areas then becomes just another example of adding blocks next-to each other, also by their areas. (Tarp, 2017)

So, formal school mathematics education can choose to expand the child's existing two quantitative competences, to count and to add. Or it can choose to discard them and force upon the child eight mathematics competencies about one-dimensional number-names arranged in a place-value system, and about more or less obscure algorithms when adding, subtracting, multiplying and dividing, and about fractions as numbers that can be added without considering the units.

In short, the school can choose to strengthen or weaken the mastery of Many that the child brings to school. Wanting to improve mathematics education, maybe it would be a good idea to choose the former and stop practising the latter.

So, we can celebrate the 500year Luther anniversary by saying: The subject of mathematics education, Many, we can meet directly without being mediated by its 'latinized' version in the form of a self-referring meta-matism.

References

Bauman, Z. (1990). *Thinking Sociologically*. Oxford, UK: Blackwell.

Bourdieu, P. (1977). *Outline of a Theory of Practice.* UK: Cambridge University Press.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht-Holland: D. Reidel Publishing Company.

Mills, C. W. (1959). *The Sociological Imagination.* UK: Oxford University Press.

Niss, M. (2003). *Mathematical Competencies and the Learning of Mathematics: The Danish Kom Project.* Retrieved at http://www.math.chalmers.se/Math/Grundutb/ CTH/mve375/1112/docs/KOMkompetenser.pdf.

OECD. (2015). *Improving Schools in Sweden: An OECD Perspective.* Retrieved from: www.oecd.org/edu/school/improving-schools-in-sweden-an-oecd-perspective.htm.

Tarp, A. (1998). *Postmodern Mathematics: Contextual Totalling Narratives.* In Breiteig, T. & Brekke, G. (Eds.), Theory into practise in Mathematics Education. Proceedings of the Second Nordic Conference on Mathematics Education (pp. 244-252). Kistiansand, Norway: Agder College, Research Series No. 13.

Tarp, A. (2000). *Postmodern Enlightenment, Schools and Learning.* Copenhagen, Denmark: Danish University of Education, Skolefag, Læring & Dannelse, Arbejdspapir 32.

Tarp, A. (2002, 2017). The 'KomMod Report', a Counter Report to the Ministry's Competence Report. In Tarp, A. *Math Ed & Research 2017.* Retrieved from http://mathecademy.net/ 2017-math-articles/.

Sustainable Adaption to Quantity: From Number Sense to Many Sense

Their biological capacity to adapt to their environment makes children develop a number-language based upon two-dimensional box- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden.

However, despite increased research and funding, this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

Widespread innumeracy also resides in Denmark, where the use of multi-year office-directed lines with fixed classes from secondary school has lowered the exam passing limit at the end of lower and upper secondary school to about 15% and 20% compared to the North-American limit at 70%, using instead self-chosen half-year blocks to uncover and develop the student's individual talent.

Furthermore, two different forms of mathematics are taught, one accepting and one rejecting the 'New Math' occurring around 1960.

Mathematics and its Education

The Pythagoreans used the word 'mathematics' as a common label for their knowledge about Many by itself and in space and time: arithmetic, geometry, music and astronomy. Without the two latter, mathematics later became a label for arithmetic, algebra and geometry, which may be called pre-setcentric (Derrida, 1991) math, replaced by the present setcentric 'New Math' in 1960 despite it never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we adapt to the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon. Consequently, continental Europe taught reckoning, called 'Rechnung' in German, until the arrival of the New Math. And, when opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only teach set-centric mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. But also here precalculus is seen as a very difficult class to teach, discouraging many students from taking calculus classes.

However, in their 'Learning framework 2030', OECD (2018) points to the necessity of a solid background for all in literacy and numeracy, which raises the 'Cinderella question': with pre-setcentric and setcentric mathematics unable to 'make the prince dance', is there a third hidden post-setcentric alternative, that may prove sustainable so it will last?

The nature of education has been studied by different sciences. To discuss how to find a sustainable solution we should begin with biology, specializing in sustainability through adaption.

Biology Looks at Education

As a life science, biology sees life as built from green, grey and black cells.

Grey cells form animals able to release the energy from plants or other animals by the replacing hydrogen with oxygen when inhaling oxygen and exhaling carbon dioxide through breathing. To survive, animals must move using muscles and limbs, as well as a brain to decide which way to move. Also, according to ethology (Darwin, 2003) they must adapt to the environment.

The holes in their head allow animals to satisfy their two basic needs for information and food. Animals come in three forms.

Reptiles have one brain allowing it to transform outside information into a choice between alternative actions. Mammals also have a second brain for feelings binding them to a mate and to the offspring to allow it to gradually adapt to the environment through childhood before having offspring themselves.

Finally, humans also have a third brain to store and share information, made possible by transforming forelegs to arms with hands that can grasp food and things that are named by sounds, thus developing a language for mutual sharing information about what they observe and know about the six core ingredients of their life: I, you1, it, we, you2, and they; or in German: ich, du, es, wir, ihr, sie.

The combination of individual and collective adaption is so effective that to reproduce, humans only need two to three offspring in a lifetime, where other mammals need it per year.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured e.g. by a social institution called a school.

With life existing in space and time, institutional education has to answer two core questions: what things and events in the environment is important to address in education? And will learning take place through a meeting allowing individual representations to be created, or will it need to be mediated through the teaching of socially constructed representations. To answer this, we now turn to psychology.

Psychology Looks at Education

Psychology looks at cognitive aspects of learning, or, in other words, the 'it-I' relation. Here, the philosophical controversy between outside existence and inside essence becomes a controversy between different forms of inside constructivism.

Supporting the philosophical existence stance, Piaget (1971) sees learning as a biological process of adapting inside to the outside environment through outside assimilation and inside accommodation, where assimilation makes the outside conform to inside schemata, whereas accommodation makes inside schemata conform to the outside resistance against assimilation.

Thus, to Piaget, learning takes place in the meeting between outside existence and inside schemata that accommodate through outside operations and inside peer communication. Here, teaching socially constructed schemata should be kept to a minimum to not influence the construction of individual schemata.

Siding with Piaget, Ausubel says that "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1978, p. vi).

Supporting the philosophical essence stance, Vygotsky (1986) sees learning as adapting to the socially institutionalized knowledge mediated through good teaching respecting that the knowledge taught must be attachable to what the learners already know in their zone of approximate development.

Consequently, high quality must be given to teacher education and textbooks to provide good teaching. And teaching should be structured and well-organized aiming at students being able to reproduce what teachers teach.

Meeting Many, Children Bundle to Count and Share

How children adapt to Many can be observed from preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, ‘That is not four, that is two twos’, thus describing what exists: bundles of 2s, and 2 of them. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units.

Likewise, children use bundle-numbers when talking about Lego bricks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3 more’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, placing them next-to each other, they typically say ‘2 7s and 3 more’.

Children love placing four cars or dolls in patterns; and they smile when the items form a 4-icon. Likewise, they like to form number-icons with footprints in the sand, with body-parts etc.

Children love counting their fingers in 4s using a rubber band to hold the bundles together. They smile when seeing that the fingers can be counted in 4s as 1Bundle6, 2B2 or 3B less2. Or, if counting in 3s, as 1B7, 2B4, 3B1, or 4Bless2. Some even see that 3 bundles is the same as one bundle of bundles, 3B = 1BB.

Likewise, children love bundle-counting the fingers in e.g. 4s as 0Bundle1, 0B2, 0B3, 0B4 no 1B0, 1B1, 1B2, 1B3, 1B4 no 2B0, 2B1, 2B2.

A special case is counting in pairs or 2s. Here the fingers can be counted as 1B8, 2B6, 3B4, 4B2, 5B0. A different color for the rubber band used for the bundle of bundles will allow the fingers to be counted as 1BundleBundle6, 2BB2, 3BBless2. Some might suggest a new color for the bundles of bundles of bundles, thus counting the fingers as 1BBB2 or 1BBB1B0; or even 1BBB0BB1B0.

And children don’t mind writing using ‘bundle-writing’ with a full sentence containing a subject, a verb and a predicate as in the word-language: T = 8 = 1B5 = 2B2 = 3B-1 3s. Some might even write T = 8 = 3B-1 = 1BB-1 3s.

Also, children smile when they see that, counting in hands, T = 5 = 1B0 5s, thus realizing that ten is written as 10 because ten becomes 1B0 if we count in tens.

Sharing 8 cakes, 2 children take away 2 to have one each; and smile when they see that entering ‘8/2’, a calculator predicts they can have 4 each; thus seeing the division sign as an icon for a broom pushing away 2s. This motivates rooting division by 2 as counting in 2s.

Likewise, when counting 9 cubes in 2s they may stack the 2s on-top as a box of 4 2s, smiling when they see that entering ‘4x2’, a calculator predicts they have a total of 8; thus seeing the multiplication sign as an icon for a lift pushing up 2s.

And again, they smile they see that entering ‘8 – 4*2’, a calculator predicts that 1 is left when pulling away a stack of 4 2s from 8; thus seeing the subtraction sign as an icon for a rope pulling away the 4 2s.

Children thus see that counting involves three processes: pushing away, pushing up and pulling away, that can be performed by a broom, a lift and a rope; and that can be predicted on a calculator by using division, multiplication and subtraction. Some may even accept that the counting prescription ‘From the total 8, 8/2 times, 2s can be pushed away’ may be shortened to the calculation formula ‘8 = 8/2x2’, later with unspecified numbers becoming a core formula expressing proportionality, the recount-formula ‘T = (T/B)*B’.

Exposed to counting, children adapt in a natural way to the three basic operations division, multiplication and subtraction; and typically enjoys using a calculator, or even the recount-formula, to predict the counting result.

Discussing Number Sense and Number Nonsense

The basic question in grade one mathematics is: shall education be about numbering or about numbers? Shall education guide and support the development of the children’s already existing adaption to quantity, or shall education teach numbers? Shall the ‘I’ keep on adapting to the ‘it’ directly, or indirectly by having the adaption replaced by what is mediated by the ‘they’?

Choosing numbers over numbering, the US National Council of Teachers of Mathematics, NCTM, in their publication ‘Principles and Standards for School Mathematics’ (2000) says: “Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense (p. 7).”

Likewise choosing numbers over numbering, ICMI study 23 creates a WNA-discourse (Whole Number Arithmetic) asking:

> To what extent is basic number sense inborn and to what extent is it affected by socio-cultural and educational influences? How is the relationship between these precursors/foundations of WNA, on the one hand, and children’s whole number arithmetic development?” (Bussi and Sun, 2018, pp 500-501)

Thus, both to the NCTM and in the WHA discourse, the concept ‘number sense’ is central, although not being that well defined (Griffin, 2004). In the ICMI study there are several references to Sayers and Andrews (2015) that based upon reviewing research in the WHA domain create a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

However, several questions may be raised to this FoNS framework.

In his book, Dantzig (2007) uses the term ‘number sense’ for a natural property shared by humans and animals. However, from a biological view it is sensing the environment that is fundamental to all grey cells. And as human constructs, numbers are not part of the environment, in contrast to what they number and what is embedded in human language as the singular in plural forms, the physical fact many or ‘more-ness’. Using the term ‘cardinality’ just adds a religious power aspect demanding respect for the Cardinal.

Thus, the term ‘many sense’ is more precise than the term ‘number sense’. Especially since, with its reference to numbers, ‘number sense’ becomes a self-reference that removes meaning from four of the eight categories.

Furthermore, using the word ‘understanding’ makes three categories dubious since there are many different understandings of the word understanding.

What is left is category seven, simple arithmetic competence, which is about adding and subtraction, thus neglecting that division and multiplication come first when counting in bundles.

Thus, it seems difficult to define number-sense without self-reference and without referring to a tradition giving priority to addition and subtraction.

A grounded definition of number-sense or many-sense should come from how numbers emerge in the numbering process counting and recounting a total in bundles, to allow seeing the link between the number and what it numbers by including the ‘missing link’, the bundle and the unit, absent in everyday use: T = 6B7 tens = 67. Therefore, a short definition could be: Having number-sense or many-sense means including the word ‘bundle’ as a unit for the numbers. That is:

To bridge the outside total with an inside numbering by bundling creating flexible bundle-numbers expressed in a full number-language sentence with an outside subject, a verb and an inside predicate, e.g. T = 2 3s.

To count 5 fingers in fives as 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0; and as 1Bundle less 4, 1B-3, 1B-2, 1B-1, 1B0; and to recount five fingers with ‘flexible bundle-numbers’ with overload, underload or fraction, i.e. as 1B3 2s, 2B1 2s or 3B-1 2s or 2 ½B 2s, and later as 1BB 0B1 2s or 1BB1B-1 2s. And to recount ten fingers in 3s as 1B7, 2B4, 3B1, 4B-2, 31/3, 1BB0B1, or 1BB1B-2. And to let 67 = 6B7 = 5B17 = 7B-3 = 6.7 tens = 7.-3 tens. And 678 = 67B8 = 6BB7B8. (Tarp, 2018)

To see the digits as icons with as many sticks or strokes as they represent if written less sloppy; and with ten needing no icon when used as bundle-size.

To see the operations as icons coming from the counting process, where division iconizes a broom pushing away bundles, where multiplication iconizes a lift pushing up bundles into a box, where subtraction iconizes a rope pulling away the box to find unbundles singles, and where addition iconizes placing boxes next-to or on-top.

To see the counting process predicted by the recount-formula T = (T/B)*B, saying ‘From the total T, T/B times, B-bundles can be pushed way’; and to use a calculator to enter ‘9/4’ giving ‘2’, and ‘9-2*4’ giving ‘1’ to predict that from 9, 4s can be pushed away 2 times, and that pulling away the 2 4s from 9 leaves 1, thus predicting that 9 may be recounted as 2B1 4s.

To see totals as double described both as outside boxes and as inside bundles.

To see 678 as a numbering containing four numbers counting unbundled, bundles, bundles of bundles and specifying the bundle-size.

To see a multiplication task as recounting from icons to tens, facilitated by using flexible box&bundle numbers so that 6*8 = 1B-4 * 1B-2 = 1BB – 4B – 2B + 4*2 = 4B8 = 48, thus realizing that -*- is + since the corner was pulled away twice. And to see that 4*67 may be calculated as 4*6B7 giving 24B28, which may be recounted without an overload as 26B8 or 268.

To see a multiplication equation 4*x = 20 as recounting from tens to icons, solved by the recount-formula.

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**10~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	~~10**B**10~~
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

Figure 1. A counting table that includes the bundles in the number names

The WHA discourse defines numbers by internal reference as a set of whole numbers included in the set of integers, included in etc. All created to describe what is called cardinality which is claimed to be linear and represented by a number-line.

The WHA discourse thus presents 678 as one number, or if asked to be more precise, as 6 numbers: 6, 7, 8, ones, tens and hundreds, even if the correct answer is fou4r numbers: 6, 7, 8 and bundles, which typically is ten where it is twenty when the French and the Danes count four twenties instead of eight tens.

Furthermore, 67 is not even a whole number but decimal number that might include a negative number as well:

67 = 6ten7 = 6B7 tens = 7B-3 tens, or 6ten7 = 6.7 tens = 7.-3 tens.

The WNA discourse subscribes to setcentric mathematics. Even if Russell proved that self-reference leads to the nonsense of the classical liar paradox, 'this sentence is false', since the set of sets not belonging to itself will belong if and only if it will not.

Russell's point is that it is OK to talk about elements and sets since that is how a language is organized, but when you talk about sets of sets you talk from a meta-level that should not be mixed with the language level, even if this was precisely what Zermelo and Fraenkel did when trying to save the set theory by disregarding the difference between a set and its elements, thus disregarding the difference between examples and abstractions that is the basis in any language.

Grounded in outside observations, the numbers zero, one and two are rooted in fingers on a hand. Defined inside the WNA discourse, zero is defined as the empty set Ø = {}. With 0 = Ø, 1 is defined as the set containing the set Ø, 1 = {Ø}, but as a set of sets, this places 1 on a different language level where it cannot be added to 0. Then 2 is defined as the set that contains a set, and a set of sets, Ø and 1, 2 = {Ø, {Ø}} thus placing 0, 1 and 2 on three different language levels. Which is nonsense according to Russell.

As to the sociological effect of creating an educational concept 'number-sense' we should remember that sociologically, a school is a pris-pital (Foucault, 1995). So, the moment you introduce a new construct you may also introduce a new diagnose: this child lacks number sense, so it must be treated. Especially since it is claimed that children who start with a poorly developed understanding of numbers remain low achievers throughout school (Geary, 2013). And with eight diagnose components, you need eight cures. This might be good news for universities selling teacher education courses, but bad news for the curers, the teachers, now having three times eight additional tasks forced upon them: How to understand the diagnoses, how to find material to use in the cure, and how to evaluate if the cure works.

Introducing diagnoses may be seen as an example of 'symbolic violence' used as an exclusion technique to keep today's knowledge nobility in power (Bourdieu 1977).

To master Many, humans invented numbers as a means, typically rooted in the hands as the Roman numbers bundling fingers in hands and double hands (Dantzig, 2007). But numbers may lose their outside link and become examples of inside abstractions instead of abstractions from the outside. Likewise, outside quantity may become an example of inside cardinality. In that moment numbers undertake what Baumann calls a goal displacement, where inside derived setcentric numbers become the goal instead with outside quantity as a means thus leaving Many as what Weber calls disenchanted.

The situation with eight components in number sense reminds of the claimed eight 'mathematical competencies' (Niss, 2003) also made meaningless by self-reference, but meaningfully reduced to two competences, count and add (Tarp, 2002).

Likewise, both situations remind of the eight sacraments in the catholic church, challenged by the two sacraments of the protestant church.

To look for meaningful diagnoses in a sustainable mathematics education adapted to quantity we must ask: What is it in the outside world that the children are not adapted to? Will bringing this inside the classroom allow children to extend their existing adaption?

So, instead of using the eight number sense components as diagnoses, we may use the alternative definition of number sense given above as diagnoses to be cured by guiding questions to outside subjects brought inside to receive common predicates, thus reifying the subject in the number language sentences.

Conclusion and Recommendation

This paper asked if there is a third hidden post-setcentric alternative, that may prove sustainable so it will last? The answer is yes, and maybe, since testing for sustainability has to be carried out on what may be called post-setcentric mathematics respecting instead of colonizing the way children adapt to quantity by using two-dimensional bundle-numbers with units instead of the one-dimensional line-numbers forced upon them by setcentric education. Thus, mathematics education should see itself as a language education allowing children develop their quantitative number-language like their qualitative word-language, both using sentences typically with a subject, a verb and a predicate.

A core question in language education is the following: should education develop further the children's own language, or should education colonize it by replacing their native language with a foreign language. And should language be taught before, together with or after its grammar?

Word-language education chose to respect the children's native language and to develop it before introducing a grammar. Likewise, with foreign language after the language revolution in the 1970s made language be taught before grammar (Widdowson, 1978; and Halliday, 1973).

Number-language education chose to disrespect the children's native language. Furthermore, its revolution in the 1970s made language be taught after its grammar, that was introduced not through bottom-up reference to examples, but as top-down examples of the abstraction Set.

So, to establish as sustainable tradition that will allow all to learn and practice a number-language, mathematics education must stop using a setcentric grammar-based foreign language to colonize the children's own native language.

The consequences of not decolonizing is seen in the OECD-report on the Swedish school system as well as in the widespread innumeracy documented by various PISA studies. The time has come for a paradigm shift (Kuhn, 1962) in early childhood education in adaption to quantity by developing the children's already existing many-sense.

Therefore, if the goal is a sustainable mathematics education it might be a good idea to respect and develop the natives' own natural number-language; and to say: 'only cure the diagnosed'.

References

Ausubel, D., Novak, J., & Hanesian, H. (1978). *Educational Psychology: A Cognitive View (2nd Ed.).* New York: Holt, Rinehart & Winston.

Bourdieu, P. (1977). *Reproduction in Education, Society and Culture.* London: Sage.

Bussi, M. G. B. B. & Sun, X. H. (Editors) (2018). *Building the Foundation: Whole Numbers in the Primary Grades, The 23rd ICMI Study.* Springer Open.

Dantzig, T. (2007). *Number: the language of science.* London, UK: Penguin Plume Book.

Darwin, C. (2003). *The origin of species.* New York, US: Penguin Putnam Inc.

Derrida, J. (1991). *A Derrida Reader: Between the Blinds.* P. Kamuf (ed.). New York: Columbia University Press.

Foucault, M. (1995). *Discipline & punish.* New York: Vintage Books.

Geary, D. (2013). Early foundations for mathematics learning and their relations to learning disabilities. *Current Directions in Psychological Science, 22*(1), 23-27.

Griffin, S. (2004). Building number sense with Number Worlds: A mathematics program for young children. *Early Childhood Research Quarterly, 19*(1), 173-180.

Halliday, M. A. K. (1973). *Explorations in the function of language.* London, UK: Edward Arnold.
Kuhn T.S. (1962). *The Structure of Scientific Revolutions.* Chicago: University of Chicago Press.
Niss, M. (2003). *Mathematical competencies and the learning of mathematics: the Danish KOM project.* Retrieved from http://www.math.chalmers.se/Math/Grundutb/CTH/mve375/1112/docs/KOMkompetenser.pdf.
OECD. (2015). *Improving schools in Sweden: an OECD perspective.* Retrieved from: www.oecd.org/education/school/improving-schools-in-sweden-an-oecd-perspective.htm.
OECD. (2018). *The future of education and skills, education 2030.* https:// www.oecd.org/education/2030/E2030%20Position%20Paper%20(05.04.2018).pdf.
Piaget, J. (1971). *Science of education of the psychology of the child.* New York: Viking Compass.
Sayers, J., & Andrews, P. (2015). Foundational number sense: The basis of whole number arithmetic competence. In Sun, X., Kaur, B., & Novotna, J. (Eds.). *Conference proceedings of the ICMI study 23: Primary mathematics study on whole numbers.* (pp. 124–131).
Tarp, A. (2002). *The 'KomMod report', a counter report to the ministry's competence report.* In Tarp, A. Math ed & research 2017. Retrieved from http://mathecademy.net/2017-math-articles/.
Tarp, A. (2018). Mastering Many by Counting, Recounting and Double-counting before Adding On-top and Next-to. *Journal of Mathematics Education, March 2018, 11*(1), pp. 103-117.
The National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics an overview.* Reston, VA.
Vygotsky, L. (1986). *Thought and language.* Cambridge MA: MIT press.
Widdowson, H. G. (1978). *Teaching language as communication.* Oxford, UK: Oxford University Press.

Per-numbers connect Fractions and Proportionality and Calculus and Equations

In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word 'percent' to also talk about other 'per-numbers' as e.g. 'per-five' thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

Mathematics is Hard, or is it

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature.

That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969.

Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Witnessing poor PISA performance, Denmark has lowered the passing limit at the final exam is to around 15% and 20 % in lower and upper secondary school.

Other countries also witness poor PISA performance. And high-ranking countries admit they have a high percentage of low scoring students.

As to finding the cause, Kilpatrick, Swafford, and Findell (2001, p. 36) points out that "what is actually taught in classrooms is strongly influenced by the available textbooks". Personally, working ethnographically in schools in Denmark and abroad, listening to teachers and students confirms the picture that textbooks are followed quite strictly.

So, it seems only natural to look at what is currently being discussed in textbook research e.g. by looking at the Third International Conference on Mathematics Textbook Research and Development, ICMT3, in Germany.

The ICMT3 Conference

The September 2019 ICMT3 conference consisted of 4 keynote addresses, 15 symposium papers, 2 workshops, 40 oral presentations and 13 posters.

The name 'fraction' occurred 212 times in the proceedings, and one of the keynotes addressed the problems students have when asked to find 3/5 of 2/4.

As to fractions, Ripoll and Garcia de Souza writes that "The integer numbers structure and the idea of equivalence are elementary in the mathematical construction of the ordered field of the rational numbers. Hence, the concept of equivalence should not be absent in the Elementary School's classrooms and textbooks." (Rezat et al, 2019, p. 131). Looking at 13 Brazilian textbooks from 4th to 7th grade they conclude that

> The conclusion, with respect to equivalence, was that no (complete) characterization of equivalent fractions is present in the moment the content fractions is carried on in the 6th grade Brazilian textbooks, like "Two given fractions a/b and c/d are equivalent if and only if ad = bc." In most cases only a partial equivalence criterion is presented, like "Two

> fractions are equivalent if one can transform one into the other by multiplying (or dividing) the numerator and the denominator by the same natural number."

The authors thus take it that fractions should obey the New Math 'set-centrism' (Derrida, 1991) by saying: in a set-product of integers, a fraction is an equivalence class created by the equivalence relation statin that a/b ~c/d if a*d = b*c; and thus neglect the pre-setcentric version mentioned above where a fraction keeps its value by being expanded or shortened; as well as the post-setcentric version seeing a fractions as an example of a per-number, described later in this paper.

Confirming in the afterwards discussion that fractions are introduced by the part-whole model, an argument was made that if a fraction is defined as a part of a whole then a fraction must always be a fraction of something; thus being an operator needing a number to become a number, and not a number in itself.

Of course, in a 30 minutes presentation there is little time to discuss the nature of fractions thoroughly, so this question needs to be addressed in more details.

Also addressing middle school problems, Watanabe writes that "Ratio, rate and proportional relationships are arguably the most important topics in middle grades mathematics curriculum before algebra. However, many teachers find these topics challenging to teach while students find them difficult to learn." (p. 353)

And, talking about proportionality, Memis and Yanik writes that "Proportional reasoning is an important skill that requires a long process of development and is a cornerstone at middle school level. One of the reasons why students cannot demonstrate this skill at the desired level is the learning opportunities provided by textbooks." (p. 245)

Textbooks must follow curricula, and middle school problems were also mentioned at the International Commission on Mathematical Instruction Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, in Japan November 2018. Here in his plenary talk, McCallum after noting that "a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships" challenged the audience by asking "What is the difference between 5/3 and 5÷3?" (ICMI, 2018, p. 4).

So, this paper will focus on these challenges by asking: "Is there a hidden different way to see and teach core middle school concepts as fractions, quotients, ratios, rates and proportionality?" A question that might be answered answer by Difference-research (Tarp, 2018) using sociological imagination (Mills, 1959) to search for differences making a difference by asking two questions: 'Can this be different – and will the difference make a difference?'

Different Ways of Seeing Fractions

In a typical curriculum using a 'part-whole' approach, fractions are introduced after division has been taught as sharing a whole in equal parts: 8/4 is 8 split in 4 parts or 8 split by 4.

Representing the whole geometrically as a bar or a circle, dividing in 4 parts creates 4 pieces each called ¼ of the total. Assigning numbers to the whole allows finding ¼ of e.g. 8 by the division 8/2. Then the fraction ¾ means taking ¼ three times, so that taking ¾ of 8 involves two calculations, first 8/4 as 2, then 3*2 as 6, so that ¾ of 8 is 8/4*3, later reformulated to one calculation, 8*¾, multiplying the integer 8 with the rational number ¾.

However, in the 'part-whole' approach a fraction is a fraction of something, thus introducing a fraction as an operator needing a number to become a number.

This becomes problematic when the fraction later is claimed to be a point on a number line, i.e. a number in its own right, a rational number, defined by set-centrism as an equivalence class in a set-product as described above.

Furthermore, set-centrism is problematic in itself by making mathematics a self-referring 'Meta-matics', defined from above as examples from abstractions instead of from below as abstractions from examples.

And, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If M = $\{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

To avoid self-reference Russell introduced a type theory allowing reference only to lower degree types. Consequently, fractions could not be numbers since they refer to numbers in their setcentric definition.

Neglecting the Russell paradox by defining fractions as rational numbers leads to additional educational questions: When are two fractions equal? How to shorten or expand a fraction? What is a fraction of a fraction? Which of two fractions is the bigger? How to add fractions? Etc.

Fraction later leads on to percentages, the special fractions having 100 as the denominator; which leads to the three percentage questions coming from the part-whole formula defining a fraction, fraction = part/whole.

Seeing fractions as, not numbers, but operators still raises the first three questions whereas the two latter are meaningless since the answer depends on what whole they are taken of as seen by 'the fraction paradox' where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes, and never 7 cokes of 6 bottles.

Adding numbers without units may be called 'mathe-matism', true inside but seldom outside the classroom. And strangely enough the two latter questions are only asked with fractions and seldom with percentages.

Ratios and Rates

When introduced, ratios are often connected to fractions by saying that splitting a total in the ratio 2:3 means splitting it in 2/5 and 3/5.

Where fractions and ratios typically are introduced without units, rates include units when talking e.g. about speed as the ratio between the meter-number and the second-number, speed = 2m/3s.

Per-numbers Occur when Double-counting a Total in two Units

The question "What is 2/3 of 12?" is typically rephrased as "What is 2 of 3 taken from 12?" Seldom it is rephrased as "What is 2 per 3 of 12?". Even if the word 'per' occurs in many connections, meter per second, per hundred, etc.

When we rephrase "taking 30% of 400" as "taking 30 per 100 of 400", why don't we rephrase "taking 3/5 of 400" as "taking 3 per 5 of 400" ?

In short, why don't we rephrase 3/5 both as '3 of 5' and as '3 per 5'?

In his conference paper, Tarp (p. 332) introduces per-numbers and recounting:

> An additional learning opportunity is to write and use the 'recount-formula' T = (T/B)*B, saying "From T,T/B times B can be taken away", to predict counting and recounting examples. (..) Another learning opportunity is to observe how double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Of course, you might argue that we cannot write '6$ = 9kg' since the units are not the same. But then again, we write '2 meter = 200 centimeter' even if the units are different, and we are allowed to do so since the bridge between the two units is the per-number 1m/100cm. Likewise, we should be allowed to write ‘6$ = 9kg’ since the bridge between the two units for now is the per-number 2$/3kg.

The difference is that the per-number between meter and centimeter is globally valid, whereas the per-number between kilogram and dollar is only locally valid. Still, it has validity as long as you are talking about the same outside total.

The interesting thing is that by including units, per-numbers connects fractions and proportionality. And that by including units, the recount-formula gives an introduction to fractions saying that 1/3 is ‘1 counted in 3s’: 1 = (1/3)*3 = 1/3 3s.

Fractions as Per-numbers

With per-numbers coming from double-counting the same total in two units, we see that when double-counting in the same unit, the unit cancels out and we get a ratio between two numbers without units, a fraction as e.g. 3$/8$ = 3/8.

Reversely, inside fractions without units may be ‘de-modeled’ outside by adding new units, e.g. ‘good’ and ‘total’ transforming 3/8 to 3g/8t. This allows per-numbers and recounting to be used when solving the three fraction questions:

“What is 3/4 of 60?”, and “20 is what of 60?”, and “20 is 2/3 of what?”

Asking “What is ¾ of 60” means asking “What is 3 per 4 of 60”, or de-modeled with units, “What is 3g per 4t of 60t”,

Of course, 60t is not 4t, but 60 can be recounted in 4s by the recount-formula, 60t = (60/4)*4t = (60/4)*3g = 45g, giving the inside answer “¾ of 60 is 45”.

Asking “20 is which fraction of 60” means asking “What fraction is 20 per 60”, or with units, “Which per-number is 20g per 60t”, giving the answer directly as 20g/60t or 20/60 g/t. Here we might look for a common unit in 20 and 60 to cancel out, e.g. 20, giving 20/ 60 = 1 20s/3 20s = 1/3. This allows transforming the outside answer “20 per 60 is 1 per 3” to the inside answer “20 is 1/3 of 60”.

Asking “20 is 2/3 of what” means asking “20 is 2 per 3 of what”, or with units, “20g is 2g per 3t of which total”. Of course, 20g is not 2g, but 20 can be recounted in 2s by the recount-formula, 20g = (20/2)*2g = (20/2)*3t = 30t. This allows transforming the outside answer “20 is 2 per 3 of 30” to “20 is 2/3 of 30.”

Expanding and Shortening Fractions

With fractions as per-numbers coming from double counting in the same unit that has cancelled out, we are always free to add a common unit to both numbers.

Using numbers as units will expand the fraction:

2/3 = 2 7s/3 7s = 2*7/3*7 = 14/21

Reversely, if both numbers contain a common unit, this will cancel out:

14/21= 2*7/3*7 =2 7s/ 3 7s = 2/3

Taking Fractions of Fractions, the Per-number Way

One of the keynotes pointed out that to understand that 6/20 is the answer to the question “What is 3/5 of 2/4?” we must draw a rectangle with 4 columns if which 2 are yellow, and with 5 rows of which 3 are blue. Then 6 double-colored squares out of a total of 20 squares gives an understanding that 3/5 of 2/4 is 6/20, which also comes from multiplying the numerators and the denominators.

Seeing fractions as per-numbers the question “What is 3/5 of 2/4?” translates into “What is 3 per 5 of 2 per 4”. Knowing that using per-numbers to bridge two units involves recounting them in the per-number which again involves division, we might begin with a number that is easily recounted in 4s and 5s, e.g. 4*5 = 20, and reformulate the question to "3 per 5 of 2 per 4 is what per 20?".

To find 2 per 4 of 20 means finding 2g per 4t of 20t, so we recount 20 in 4s:

20t = (20/4)*4t = (20/4)*2g = 10g, so 2 per 4 of 20 is 10.

To find 3 per 5 of 10 means finding 3g per 5t of 10t, so we recount 10 in 5s:

10t = (10/5)*5t = (10/5)*3g = 6g, so 3 per 5 of 10 is 6

Thus, we can conclude that 3 per 5 of 2 per 4 is the same as 6 per 20, or, with fractions, that 3/5 of 2/4 is 6/20, again coming from multiplying the numerators and the denominators.

Of course, we could discuss, which method gives a better understanding, but we might never reach an answer, given the many different understandings of the word 'understanding'

Direct and Inverse Proportionality

Using a coordinate system with decimal numbers comes natural if bundle-writing totals in tens so e.g. T = 26 becomes T = 2.6 tens. This allows fixing a 3x5 box in the corner with the base and the height on the x- and y-axes. The recount-formula T = (T/B)*B then shows a total T as a box with base x = B and height y = T/B.

To keep the total unchanged, increasing the base will decrease the height (and vice versa) making the upper right corner create a curve called a hyperbola with the formula height = T/base, or y = T/x, showing inverse proportionality.

In a 3x5 box, the raise of the diagonal is the per-number 3/5. Expanding or shortening the per-number by adding or removing extra units will make the diagonal longer or shorter without changing direction. This will make the upper right corner move along a line with the formula 3/5 = height/base = y/x, or y = 3/5*x, showing direct proportionality.

Adding Fractions, the Per-number Way

Adding per-numbers occurs in mixture problems asking e.g. "What is 2kg at 3$/kg plus 4kg at 5$/kg?". We see that the unit-numbers 2 and 4 add directly, whereas the per-numbers cannot add before multiplication changes them to unit-numbers. However, multiplication creates the areas 2*3 and 4*5, which gives the answer: 2kg at 3$/kg + 4kg at 5$/kg gives (2+4)kg at (2*3+4*5)/(2+4)$/kg.

So we see that per-numbers add by the areas under the per-number graph in a coordinate system with the kg-numbers and the per-numbers on the axes.

But adding area under a graph is what integral calculus is all about. Only here, the per-number graph is piecewise constant, where the velocity graph in a free fall, is not piecewise, but locally constant, which means that the total area comes from adding up very many small area-strips.

This may be done by observing that the total area always changes with the last area-strip thus creating a change equation $\Delta A = p*\Delta x$, which motivates differential calculus to answer questions as dA/dx = p, thus finding the area formula that differentiated gives the give per-number formula p, e.g. d/dx (x^2) = 2*x.

Interchanging epsilon and delta to change piecewise constancy to local may be postponed to high school, that would benefit considerably by a middle school introduction of integral calculus as adding locally constant per-numbers by the area under the per-number graph, using differential calculus to find the area in a quicker way than asking a computer to add numerous small area-strips.

Solving Proportionality Equations by Recounting

Reformulating the recount-formula from T = (T/B)*B to T = c*B shows that with an unknown number u it may turn into an equation as 8 = u*2 asking how to recount 8 in 2s, which of course is found by the recount-formula, u*2 = 8 = (8/2)*2, thus providing the equation u*2 = 8 with the solution u = 8/2 obtained by isolating the unknown by moving a number to the opposite side with the opposite sign.

This resonates with the formal definition of division saying that 8/2 is the number u that multiplied by 2 gives 8: if u*2 = 8 then u = 8/2.

Set-centrism of course prefers applying and legitimizing all concepts from abstract algebra's group theory (commutativity, associativity, neutral element and inverse element) to perform a series of reformulations of the original equation: 2*u = 8, so (2*u)*½ = 8*½, so (u*2)*½ = 4, so u*(2*½) = 4, so u*1 = 4, so u = 4.

Seven Ways to Solve the two Proportionality Questions

The need to change units has mad the two proportionality questions the most frequently asked questions in the outside world, thus calling for multiple solutions.

With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g. Q1: "7 meters need how many seconds?", and Q2: "How many meters is covered in 12 seconds?"

● Europe used 'Regula-de-tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.

● Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter.

● Equating the rates. The velocity rate is constantly 2meter/5second. So we can set up an equation equating the rates. Q1: 2/5 = 7/x, where cross-multiplication gives 2*x = 7*5, which gives x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, where cross-multiplication gives 5*x = 12*2, which gives x = (12*2)/5 = 4.8.

● Recount in the per-number. Double-counting produces the per-number 2m/5s used to recount the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.

● Multiply with the per-number. Using the fact that T = 2m, and T = 5s, division gives T/T = 2m/5s = 1, and T/T = 5s/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.

● Modeling a linear function f(x) = c*x, with f(2) = 5, f(7) = ?, and f(x) =12.

A Case: Peter, about to Peter Out of Teaching

As a new middle school teacher, Peter is looking forward to introducing fractions to his first-year class coming directly from primary school where the four basic operations have been taught so that Peter can build upon division when introducing fractions in the traditional way. However, Peter is shocked when seeing many students with low division performance, and some even showing dislike when division is mentioned. So, Peter soon is faced with a class divided in two, a part that follows his introduction of fractions, and a part that transfers their low performance or dislike from divisions to fractions.

The following year seeing his new class behaving in the same way, Peter is about to give up teaching when a colleague introduces him to a different approach where division is used for bundle-counting instead of sharing called 'Recounting fingers with flexible bundle-numbers'. The colleague also recommends some YouTube videos to watch and some material to download from the MATHeCADEMY.net to try it yourself.

Inspired by this, Peter designs a micro-curriculum for his class aiming at introducing the class to bundle-counting leading to the recount-formula leading to double-counting in two units leading to per-numbers having fractions as the special case with like units.

“Welcome class, this week we will not talk about fractions!” “?? Well, thank you Mr. teacher, then what will we do?” “We will count our five fingers.” “Ah, Mr. teacher we did that in preschool!” “Correct, in preschool we counted our fingers in ones, now we will bundle-count them in 2s and 3s and 4s using bundle-writing. In this way we will see that a total can be counted in three different ways: overload, standard and underload. Look here:

Outside we have IIIII = ǂ I I I = ǂ ǂ I = ǂ ǂ ǂ

Inside we write: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s

We will call this to recount 5 with flexible bundle-numbers. Now count the five fingers in 3s and 4s in the same way. Later, we will count all ten fingers.”

The following class, Peter began by rehearsing.

“Welcome class. Yesterday we saw that an outside total can be recounted in different units, and that the result inside can be bundle-written in three ways, with overload, standard and underload. Today we will begin by recounting twenty in hands, in six-packs and in weeks. Why twenty? Because counting in twenties was used by the Vikings who also gave us the words eleven and twelve, meaning one-left and two-left in Viking language.”

Later, Peter introduced the recount-formula:

“Here we have 6 cubes that we will count in 2s. We do that by pushing away 2-bundles, and write the result as T = 6 = 3B 2s. We see that the inside division stroke looks like an outside broom pushing away the bundles. And asking the calculator, 6/2, and we get the answer 3 predicting it can be done 3 times. We can illustrate this prediction with a recount formula ‘T = (T/B)xB’ saying that ‘from the total T, T/B times, B can be pushed away’. So, from now on, 6/2 means 6 recounted in 2s; and 3x2 means 3 bundles of 2s. And since it is counted in tens, 42 is seen as 4B2 or 3B12 or 5B-8 using flexible bundle-numbers.

Now let us read 42/3 as 4bundle2 tens recounted in 3s; and let us use flexible bundle-numbers to rewrite 4B2 with an overload as 3B12. Then we have T = 42 /3 = 4B2 /3 = 3B12 / 3 = 1B4 = 14. We notice that squeezing a box from base 10 to base 3 will increase the height, here from 4.2 to 14.

And, by the way, flexible bundle-numbers also come in handy when multiplying: Here 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 bundles and 56 unbundled singles, which can be recounted to remove the overload:

T = 7 x 4B8 = 28B56 = 33B6 = 336.”

The third day Peter repeated the lesson with 7 cubes counted in 3s to show that where the unbundled single was placed would decide if the total should be written using a decimal number when placed next-to as separate box of ones, T = 2B1 3s = 2.1 3s. Placed on-top means missing 2 to form a bundle, thus written as T = 3B-2 3s = 3.-2 3s. Or it means recounting 1 in 3s as 1 = (1/3)x3 = 1/3 3s, a fraction.

Later, Peter introduced per-numbers and fractions as described above, which allowed Peter to work with fractions and ratios and proportionality at the same time; and later to introduce calculus as adding fractions and per-numbers by areas.

Observing the increase of performance and the disappearance of dislike, the headmaster suggested to the headmaster of the nearby primary school that Peter be used as a facilitator for in-service teacher training. This would allow primary school children to meet fractions and negative numbers and proportionality when recounting and double-counting a total in a new bundle-unit.

Discussion and Recommendation

This paper asked "Is there a hidden different way to see and teach core middle school concepts as fractions, quotients ratios, rates and proportionality?" The answer is yes: per-numbers includes them all as examples, as well as integral calculus and equations.

So introducing per-numbers through double-counting the same total in two units makes a difference by allowing fractions, quotients, rates and ratios to be seen and taught as examples of per-numbers, and by allowing integral calculus to be introduced in middle school, and by allowing a more natural way to solve multiplication equations, and by allowing STEM examples in the classroom since most STEM formulas are proportional formulas.

Furthermore, introducing recounting with flexible bundle-numbers allows math dislike to be cured by taking the hardness out of division, seen traditionally as the basis for fractions but becoming a tumbling stone instead if not learned well.

Consequently, it is recommended that primary school accepts and develops the double-numbers children bring to school. And that middle school introduces students to recounting in flexible bundle-numbers from the start to provide a strong division foundation for fractions that becomes connected with quotients, ratios, rates, proportionality, equations and calculus if introduced as per-numbers coming from double-counting in two units that may be the same.

So yes, mathematics is hard, not by nature, but by a choice replacing it with a mixture of top-down meta-matics and mathe-matism seldom true outside the class.

References

Derrida, J. (1991). *A Derrida Reader: Between the Blinds.* P. Kamuf (ed.). New York: Columbia University Press.

ICMI study 24 (2018). School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities. *Pre-conference proceedings*. Editors: Yoshinori Shimizu and Renuka Vithal.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Mills, C. W. (1959). *The sociological imagination.* UK: Oxford University Press.

OECD. (2015). *Improving schools in Sweden: an OECD perspective.* Retrieved 07/01/19 from: www.oecd.org/edu/ school/improving-schools-in-sweden-an-oecd-perspective. htm.

Rezat, S., Fan, L., Hattermann, M., Schumacher, J., & Wuschke, H. (Eds.). (2019). *Proceedings of the Third International Conference on Mathematics Textbook Research and Development*. 16–19 September 2019, Paderborn, Germany. Paderborn: Universitätsbibliothek Paderborn.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations

Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

A need for curricula for all students

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

Traditionally, a school system is divided into a primary school for children and a secondary school for adolescents, typically divided into a compulsory lower part, and an elective upper part having precalculus as its only compulsory math course. So, looking for a change we ask: how can precalculus be sustainably changed?

A Traditional Precalculus Curriculum

Typically, basic math is seen as dealing with numbers and shapes; and with operations transforming numbers into new numbers through calculations or functions. Later, calculus introduces two additional operations now transforming functions into new functions through differentiation and integration as described e.g. in the ICME-13 Topical Survey aiming to "give a view of some of the main evolutions of the research in the field of learning and teaching Calculus, with a particular focus on established research topics associated to limit, derivative and integral." (Bressoud et al, 2016)

Consequently, precalculus focuses on introducing the different functions: polynomials, exponential functions, power functions, logarithmic functions, trigonometric functions, as well as the algebra of functions with sum, difference, product, quotient, inverse and composite functions.

Woodward (2010) is an example of a traditional precalculus course. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

A Different Precalculus Curriculum

Inspired by difference research (Tarp, 2018) we can ask: Can this be different; is it so by nature or by choice?

In their ‘Principles and Standards for School Mathematics’ (2000), the US National Council of Teachers of Mathematics, NCTM, identifies five standards: number and operations, algebra, geometry, measurement and data analysis and probability, saying that “Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century (p. 2).” In the chapter ‘Number and operations’, the Council writes: “Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number (p. 7).”

Their biological capacity to adapt to their environment make children develop a number-language allowing them to describe quantity with two-dimensional block- and bundle-numbers. Education could profit from this to teach children primary school calculus that adds blocks (Tarp, 2018). Instead, it imposes upon children one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must be learned before it can be applied to operate on the functions introduced at the precalculus level.

However, listening to the Ausubel (1968) advice “The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly (p. vi).”, we might want to return to the two-dimensional block-numbers that children brought to school.

So, let us face a number as 456 as what it really is, not a one-dimensional linear sequence of three digits obeying a place-value principle, but three two-dimensional blocks numbering unbundled singles, bundles, bundles-of-bundles, etc., as expressed in the number-formula, formally called a polynomial:

T = 456 = 4*B^2 + 5*B + 6*1, with ten as the international bundle-size, B.

This number-formula contains the four different ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant ‘double-numbers’ or ‘per-numbers’. We might call this beautiful simplicity ‘the algebra square’ inspired by the Arabic meaning of the word algebra, to re-unite.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $log_a(T) = b$

Figure 01. The ‘algebra-square’ has 4 ways to unite, and 5 to split totals

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers, creating the operations integration and its reverse operation, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power, and its two reverse operations, root and logarithm.

Precalculus, building on or rebuilding?

In their publication, the NCTM writes “High school mathematics builds on the skills and understandings developed in the lower grades (p. 19).”

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they may have from the lower grades?

Furthermore, what kind of mathematics has been taught? Was it ‘grounded mathematics’ abstracted ‘bottom-up’ from its outside roots as reflected by the original meaning of ‘geometry’ and ‘algebra’ meaning ‘earth-measuring’ in Greek and ‘re-uniting’ in Arabic? Or was it ‘ungrounded mathematics’ or ‘meta-matics’ exemplified ‘top-down’ from inside abstractions, and becoming ‘meta-matism’ if mixed with ‘mathe-matism’ (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept ‘function’, Euler saw it as a bottom-up name abstracted from ‘standby calculations’ containing specified and unspecified numbers. Later meta-matics defined a function as an inside-inside top-down example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may also be seen as an outside-inside bottom-up number-language sentence containing a subject, a verb and a predicate allowing a value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group operation applying the associative and commutative law as well as the neutral element and inverse elements, thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2:

2*u =6, so (2*u)*½ =6*½, so (u*2)*½ =3, so u*(2*½) =3, so u*1 =3, so u =3.

However, the equation 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula ‘T = (T/B)*B’ (Tarp, 2018), present all over mathematics as the proportionality formula, thus solved in one step: 2*u = 6 = (6/2)*2, giving u = 6/2.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology.

Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as a number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar.

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction for an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

A Grounded Outside-Inside Fresh-start Precalculus from Scratch

Let students see that both the word-language and the number-language provide 'inside' descriptions of 'outside' things and actions by using full sentences with a subject, a verb, and an object or predicate, where a number-language sentence is called a formula connecting an outside total with an inside number or calculation, shortening 'the total is 2 3s' to 'T = 2*3';

Let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation. Writing 'y = f(x)' means that the y-number is found by specifying the x-number and the f-calculation. So with x = 3, and f(x) = 2+x, we get y = 2+3 = 5.

Let students see how calculations predict: how 2+3 predicts what happens when counting on 3 times from 2; how 2*5 predicts what happens when adding 2$ 5 times; how 1.02^5 predicts what happens when adding 2% 5 times; and how adding the areas 2*3 + 4*5 predicts adding the 'per-numbers' when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Solving Equations by Moving to Opposite Side with Opposite Sign

Let students see the subtraction 'u = 5-3' as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers 'to opposite side with opposite calculation sign'; a rule that applies also to the other reversed operations:

- the division u = 5/3 is the number u that multiplied with 3 gives 5, thus solving the equation u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, thus solving the equation u^3 = 5, and making root a 'factor-finder'
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, thus solving the equation 3^u = 5, and making logarithm a 'factor-counter'.

Let students see multiple calculations reduce to a single calculation by unhiding 'hidden brackets' where 2+3*4 = 2+(3*4) since, with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s.

This prevents solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5. Allowing to unhide the hidden bracket we get:

2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4

This solution is verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy a 'Hymn to Equations': "Equations are the best we know, they're solved by isolation. But first the bracket must be placed, around multipli-cation. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don't want to stop!"

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let students place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, three terms disappear if adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = b^2/4 – c.

Moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c, so u + b/2 = $\pm\sqrt{\text{b^2/4 – c}}$, so u = -b/2 $\pm\sqrt{\text{b^2/4 – c}}$

Recounting Grounds Proportionality

Let students see how recounting in another unit may be predicted by a recount-formula 'T = (T/B)*B' saying "From the total T, T/B times, B may be pushed away" (Tarp, 2018). In primary school this formula recounts 6 in 2s as 6 = (6/2)*2 = 3*. In secondary school the task is formulated as an equation u*2 = 6 solved by recounting 6 in 2s as u*2 = 6 = (6/2)*2 giving u = 6/2, thus again moving 2 'to opposite side with opposite calculation sign'.

Thus an inside equation u*b = c can be 'de-modeled' to the outside question 'recount c from ten to bs', and solved inside by the recount-formula: u*b = c = (c/b)*b giving u = c/b.

Let students see how recounting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. And see how filling a circle with right triangles from the inside allows phi to be found from a formula: circumference/diameter = $\pi \approx$ n*tan(180/n) for n large.

Double-counting Grounds Per-numbers and Fractions

Let students see how double-counting in two units create 'double-numbers' or 'per-numbers' as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number:

Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg.

Asking '9kg = ?$' we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Let students see how double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 40%.

To find 40% of 20$ means finding 40$ per 100$, so we re-count 20 in 100s:

T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$:

T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

Let students see how double-counting physical units create per-numbers all over STEM (Science, Technology, Engineering and mathematics):

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter;

meter = (meter/second) * second = velocity * second;

joule = (joule/second) * second = watt * second

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, T = 456 = 4*B^2 + 5*B + 6*1, containing the formulas for constant change:

T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change. So where constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics to 'post-dict' what we can't 'pre-dict'; and using confidence for predicting intervals.

Combining linear and exponential change by n times depositing a$ to an interest percent rate r, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest percent rate R is predicted by the formula 1+R = (1+r)^n. This saving may be used to neutralize a debt Do, that in the same period changes to D = Do*(1+R).

This formula and its proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Uniting Constant Per-Numbers as Factors

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$, changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, the formula for change with a constant change-factor, also called exponential change.

Reversing the question, this formula entails two equations. Asking 600 = 300*a^5, we look for an unknown change-factor. So here the job is ‘factor-finding’ which leads to defining the fifth root of 2, i.e. 5√2, found by moving the exponent 5 to opposite side with opposite calculation sign, root.

Asking instead 600 = 300*1.2^n, we now look for an unknown time period. So here the job is ‘factor-counting’ which leads to defining the 1.2 logarithm of 2, i.e. log1.2(2), found by moving the base 1.2 to opposite side with opposite calculation sign, logarithm.

Calculus Deals with Uniting Changing Per-Numbers as Areas

In mixture problems we ask e.g. ‘2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?’ Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before added, thus adding by areas. So here multiplication precedes addition.

Asking inversely ‘2kg at 3$/kg + 4kg at how many $/kg gives 6kg at 5 $/kg?’, we first subtract the areas 6*5 – 2*3 before dividing by 4, a combination called differentiation, ΔT/4, thus meaningfully postponed to after integration.

Modeling in Precalculus Exemplifies Quantitative Literature

Furthermore, graphing calculators allows authentic modeling to be included in a precalculus curriculum thus giving a natural introduction to the following calculus curriculum, as well as introducing ‘quantitative literature’ having the same genres as qualitative literature: fact, fiction and fiddle (Tarp, 2001).

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator thus using a natural science proof, valid until finding counterexamples.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A’ = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A’ = x’ = 1; and that if A = ½*x^2, then y = A’ = (½*x^2)’ = x; and that if A = 1/3*x^3, then y = A’ = (1/3*x^3)’ = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to 3, instead of adding small strips we just calculate the change in the area over this distance, later called the fundamental theorem of calculus.

A Literature Based Compendium

An example of an ideal precalculus curriculum is described in ‘Saving Dropout Ryan With a Ti-82’ (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren’t even able to use a TI-30.

A compendium called ‘Formula Predict’ (Tarp, 2019) replaced the textbook. A formula’s left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by ‘solve Y1-Y2 = 0’. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Other projects show how a market price is determined by supply and demand, how a saving may be used for paying off a debt or for paying out a pension. Finally, it includes projects on linear programming and zero-sum two-person games, as well as finding the dimensions of a wine box, how to play golf, how to maximize a collection fund, all to provide a short practical introduction to calculus.

An Example of a Fresh/start Precalculus Curriculum

This example was tested in a Danish high school around 1980. The curriculum goal was stated as: ‘the students know how to deal with quantities in other school subjects and in their daily life’. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

An Example of an Exam Question

Authentic material was used both at the written and oral exam. The first had specific, the second had open questions as the following asking ‘What does the table tell?

Agriculture: Number of agricultural farms allocated over agricultural area

	1968	1969	1970	1971	197?	1973	1974	1975	1976	1977
Farms in total	**161142**	**154 694**	**148 512**	**144 070**	**143093**	**141 137**	**137712**	**13424S**	**130 7S3**	**127117**
0,0- 4,9 ha	25 285	23 493	21 533	21623	22123	21872	21093	19915	18 852	17 833
5.0- 9.9-	34 644	32129	30 235	28 404	27693	26 926	26109	25072	24066	23152
10,0-19.9-	48 997	46482	43 971	41910	40850	39501	38261	36 702	35 301	34 343
20.0-29.9-	25670	25 402	25181	24 472	24 195	23 759	23 506	23134	22737	22376
30,0-49.9-	18 505	18 779	18 923	18 705	18 968	18 330	19 095	19 304	10 305	19 408
50,0-99.9-	6 552	6 852	7 076	7 275	7 549	7956	7 847	8247	8 556	8723
100.0 ha and over	1489	1 557	1611	1681	1 715	1791	1801	1871	1934	1882

Figure 02. A table found in a statistical survey used at an oral exam.

Discussion and Conclusion

Asking “how can precalculus be sustainably changed?” an inside answer would be: “By its nature, precalculus must prepare the ground for calculus by making all function types available to operate on. How can this be different?”

An outside answer could be to see precalculus, not as a goal but as a means, an extension to the number-language allowing us to talk about how to unite and split into changing and constant per-numbers. This could motivate renaming precalculus to per-numbers calculations.

In this way, precalculus becomes sustainable by dealing with adding, finding and counting change-factors using power, roots and logarithm. Furthermore, by including adding piecewise constant per-numbers by their areas, precalculus gives a natural introduction to calculus by letting integral calculus precede and motivate differential calculus since an area changes with the last strip, thus connecting the unit number, the area, with the per-number, the height.

Finally, graphing calculators allows authentic modeling to take place so that precalculus may be learned through its use, and through its outside literature.

References

Ausubel, D. (1968). *Educational Psychology: A Cognitive View.* New York: Holt, Rinehart & Winston.

Bauman, Z. (1990). *Thinking sociologically.* Oxford, UK: Blackwell.

Bressoud, D.; Ghedamsi, I.; Martinez-Luaces, V. and Törner, G. (2016). *Teaching and Learning of Calculus.* Hamburg, Germany: Springer Open.

Halliday, M. A. K. (1973). *Explorations in the function of language.* London, UK: Edward Arnold.

Kuhn, T.S. (1962). *The structure of scientific revolutions.* Chicago: Univ. of Chic. Press.

Mills, C. W. (1959). *The sociological imagination.* UK: Oxford University Press.

OECD. (2015). *Improving schools in Sweden: an OECD perspective.* Retrieved 07/01/19 from: www.oecd.org/edu/ school/improving-schools-in-sweden-an-oecd-perspective.htm.

Tarp, A. (2001). Fact, fiction, fiddle - three types of models. in J. F. Matos, W. Blum, K. Houston & S. P. Carreira (Eds.), Modelling and mathematics education: ICTMA 9: Applications in Science and Technology. *Proc. 9th Int. Conf. on the Teaching of Mathematical Modelling and Applications* (pp. 62-71). Chichester, UK: Horwood Publishing.

Tarp, A. (2012). *Saving dropout Ryan with a TI-82.* Paper presented in ICME 12 at Topic Study Group 18: Analysis of uses of technology in the teaching of mathematics. Retrieved 07/01/19 from http://mathecademy.net/papers/icme-trilogy/, pp. 229-237.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Tarp, A. (2019). *Mathematics predicts, precalculus, compendium & projects.* Retrieved 29/09/19 from http://mathecademy.net/various/us-compendia/.

The National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics an overview.* Reston, VA.

Widdowson, H. G. (1978). *Teaching language as communication.* Oxford, UK: Oxford University Press.

Woodward, E. (2010). *Pre-calculus.* New Jersey, US: Research & Education Association.

A Lyotardian Dissension to the Early Childhood Consensus on Numbers and Operations

Can Sociological Imagination Improve Mathematics Education?

Decreasing Swedish PISA results made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)"

As a social institution, mathematics education might improve by inspiration form sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology; and also emphasized in Lyotard's report on knowledge in a postmodern digitalized condition (1984):

> "We no longer have recourse to grand narratives (..) But as we have seen, the little narrative remains the quintessential form of imaginative invention most particularly in science. In addition, the principle of consensus as a criterion of validation seems to be inadequate. (..) consensus is a component of the system, which manipulates it in order to maintain and improve its performance. It is the object of administrative procedures (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. The problem is therefore to determine whether it is possible to have a form of legitimation based solely on paralogy. Paralogy must be distinguished from innovation: the latter is under the command of the system, or at least used by it to improve its efficiency; the former is a move (the importance of which is often not recognized until later) played in the pragmatics of knowledge. (..) It is necessary to posit the existence of a power that destabilizes the capacity for explanation, manifested in the promulgation of new norms for understanding (p. 60-61)."

As a number-language, mathematics would create a paradigm shift (Kuhn, 1962) if copying the communicative turn in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by connecting to its outside world before its inside grammar,

In the workshop we focus on early childhood mathematics education as described in the ICME study 23 (Sun et al, 2015); and with a dissension by Tarp (2018).

Time Table for the Workshop

A 20minutes introduction will focus on the core question: As to the goal of mathematics education, is it to master inside mathematics as the means to later master outside Many; or is it to master outside Many by choosing among its three inside versions; the present setcentric Skemp-based 'meta-matics' defining concepts as examples of abstractions instead of as abstractions from examples, the pre setcentric Skinner-based 'mathe-matism' true inside but seldom outside classrooms by adding numbers and fractions without units; and the post setcentric Lyotard-based 'many-math', accepting the number-language children develop when adapting to Many before school.

A 30minutes group discussion on the three questions below is followed by 20 minutes in exchange-groups, and a 20minutes plenum for summing up.

Consensus and Dissension on Early Childhood Numbers & Operations

Question 01: There seems to be a consensus saying 'Of course numbers must be learned before being applied in numbering. And as one-dimensional, numbers are names for points along a number line obeying a place value principle when containing more digits'. Thus, a dissension may ask: 'From the age around four, children seem to distinguish between four ones and two twos thus developing double-numbers with units when adapting to outside quantity. So, why not develop the double-numbers with units children bring to school?'

Question 02: There seems to be a consensus saying ‘Of course addition must be learned before subtraction, multiplication and division since they are all defined from addition’. Thus, a dissension may ask: “Counting an outside total in bundle-counted by a broom pushing away the bundles, iconized as division, to be stacked by a lift iconized as multiplication, to be pulled away by a rope iconized as subtraction, thus finding unbundled singles that placed next-to or on-top the block roots decimals, fractions and negative numbers. This creates a ‘recount-formula’ T = (T/B)xB saying ‘From T, T/B times, B is pushed away’, present all over mathematics and science. Once counted, blocks may be added, but on-top needing units to be changed by recounting, or next-to as areas as in integral calculus? This ambiguity leaves addition not that well defined. So, why not accept the opposite order of the operations as the natural?’

Question 03: There seems to be a consensus saying ‘Of course functions are postponed to secondary school since their algebra builds upon the algebra of letter expressions.’ Thus, a dissension may ask: ‘The word- and the number-language both offer an inside description of an outside object or action by using sentences with a subject, a verb and a predicate, abbreviating ‘the total is 2 3s’ to ‘T = 2x3’. So, why not use functions as number-language sentences from the start?’

References

Bauman, Z. (1990). *Thinking sociologically.* Oxford, UK: Blackwell.

Halliday, M. A. K. (1973). *Explorations in the function of language*. London, UK: Edward Arnold.

Kuhn T.S. (1962). *The structure of scientific revolutions.* Chicago: University of Chicago Press.

Lyotard, J. (1984). *The postmodern Condition.* Manchester: Manchester University Press.

Mills, C. W. (1959). *The sociological imagination.* UK: Oxford University Press.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics.* Middlesex, UK: Penguin Books.

Skinner, B. F. (1953). *Science and Human Behaviour.* New York, NY: The Free Press.

Sun, X., Kaur, B., & Novotna, J. (Eds.). *Conference proceedings of the ICMI study 23: Primary mathematics study on whole numbers.* Macao, China University of Macao.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Widdowson, H. G. (1978). *Teaching language as communication.* Oxford, UK: Oxford University Press.

Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis?

Does Mathematics Education Research have an Irrelevance Paradox?

The Swedish Centre for Mathematics Education is meant to mediate research findings and facilitate their implementation. Still, decreasing Swedish PISA results made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life' (p. 3).

Increasing research together with decreasing student performance points to an 'irrelevance paradox' in mathematics education research, possibly caused by peer reviewing failing to assure research quality. The so-called 'replication crisis' suggests that this might indeed be the case. First noticed in medical science, the crisis may also occur in schools seen by Foucault (1995) as 'pris-pitals', i.e. prison-like hospitals using education to cure humans from the diagnose 'uneducated'.

Consequently, there is a need for a workshop discussing this hypothesis, as well as ways to make peer reviewed conferences produce more quality. We may ask: When mathematics itself has abandoned peer review, why shouldn't also mathematics education?

The Replication Crisis in Science

In the article "How Science goes Wrong", The Economist writes:

> A rule of thumb among biotechnology venture-capitalists is that half of published research cannot be replicated. Even that may be optimistic. Last year researchers at one biotech firm, Amgen, found they could reproduce just six of 53 "landmark" studies in cancer research. (..) The most enlightened journals are already becoming less averse to humdrum papers. (..) But these trends need to go much further. Journals should allocate space for "uninteresting" work, and grant-givers should set aside money to pay for it. Peer review should be tightened - or perhaps dispensed with altogether, in favour of post-publication evaluation in the form of appended comments. That system has worked well in recent years in physics and mathematics (The Economist, 19 Oct. 2013).

The replication crisis thus comes from the 'metascience' observation that many research studies are difficult or impossible to replicate or reproduce. It applies to different fields, e.g. psychology where Pashler and Wagenmakers (2012) writes:

> Is there currently a crisis of confidence in psychological science reflecting an unprecedented level of doubt among practitioners about the reliability of research findings in the field? It would certainly appear that there is (p. 528).

The authors refer among others to Ioannidis (2005) who writes:

> Scientists in a given field may be prejudiced purely because of their belief in a scientific theory or commitment to their own findings. Many otherwise seemingly independent, university-based studies may be conducted for no other reason than to give physicians and researchers qualifications for promotion or tenure. (..) Prestigious investigators may suppress via the peer review process the appearance and dissemination of findings that refute their findings, thus condemning their field to perpetuate false dogma (p. 0698).

As to the peer review process, LeBel (2015) writes:

> In recent years, there has been a growing concern regarding the replicability of findings in psychology (..) I propose a new replication norm that aims to further boost the dependability of findings in psychology (p. 1).

Addressing case series studies, Horton (1996) writes:

> The importance of the case series in surgical research is beyond doubt. Therefore, it seems reasonable to ask whether we can trust this study method to yield a valid result. According to conventional epidemiological wisdom, the answer is no (p. 984).

The quality of research was also questioned by Lyotard (1984) distinguishing between consensus and dissension:

> Consensus is a component of the system, which manipulates it (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. (..) Returning to the description of scientific pragmatics, it is now dissension that must be emphasized (p. 60-61).

Time Table for the Workshop

A 20minute introduction to the replication crisis and to conflicting theories within sociology, psychology and philosophy also includes examples on peer-reviews from MADIF 10, CERME 11, ICMT 3, and a journal (Tarp, 2018); and a proposal for a 'Salon des Refusés' created in France in 1863 to display rejected paintings later inspiring important innovation.

Then a 30minutes group discussion will use a short reader with excerpts of the authors cited above to discuss questions as: What kind of dissension risks being silenced by a peer review consensus? Will master-level papers applying existing theory oust research-level papers questioning or expanding it? Also, the groups are invited to exchange experiences on peer reviews; and to exchange opinions on how to increase the quality of the peer review process.

The next 20minutes, the groups split up to join the other groups to exchange views. Finally, a 20minutes plenum will sum up and formulate recommendations as to how to add quality to the coming MADIF sessions.

References

Foucault, M. (1995). *Discipline & punish.* New York: Vintage Books.

Horton, R. (1996). Surgical research or comic opera: Questions, but few answers. *The Lancet. 347* (9007): 984–985. doi:10.1016/S0140-6736(96)90137-3.

Ioannidis, J.P.A. (2005). Why most published research findings are false. *PLoS Med 2*(8): e124.

LeBel, E. P. 2015. A new replication norm for psychology. *Collabra, 1*(1): 4, pp. 1–13, DOI: http://dx.doi.org/10.1525/collabra.23.

Lyotard, J. (1984). *The postmodern condition: a report on knowledge.* Manchester, UK: Manchester University Press.

OECD. (2015). *Improving schools in Sweden: An OECD perspective.* Retrieved from http://www.oecd.org/education/school/improving-schools-in-sweden-an-oecd-perspective.htm.

Pashler, H.; Wagenmakers, E. J. (2012). Editors' introduction to the special section on replicability in psychological science: A crisis of confidence? *Perspectives on psychological science. 7* (6): 528–530. doi:10.1177/174569161246 5253.

Tarp, A. (2018). Mastering Many by Counting, Recounting and Double-counting before Adding On-top and Next-to. *Journal of mathematics education, March 2018, Vol. 11*(1), 103-117.

The Economist, 19 Oct. 2013: 13(US). Business insights: essentials. *How science goes wrong.*