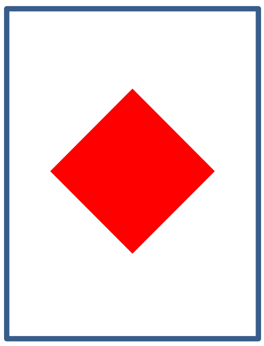
Math

with

Playing Cards



Allan.Tarp@MATHeCADEMY.net

Mathematics with playing cards

This booklet contains short articles, most of which have been printed in the LMFK member magazine for Danish upper secondary school math teachers. Thus, (2013.6) indicates that the article has been published in magazine nr. 6 from 2013. The goal is to show how mathematics formulas may be discovered by working with ordinary playing cards. Some formulas are limited by the fact that cards only have positive numbers, so the question if the formulas also apply to negative numbers may be partly answered by testing.

The article on Heron's formula is the only one not using playing cards.

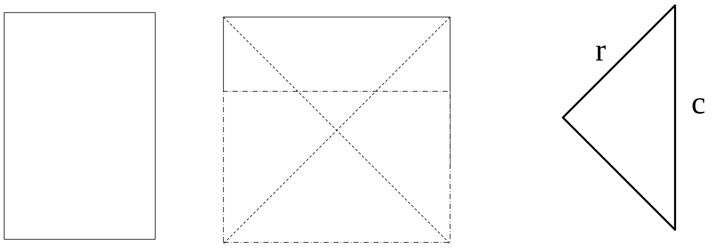
Allan Tarp, Aarhus, January 2016

Content

01. The Little, Medium and Large Pythagoras with 3, 4 and 5 Playing Cards

I. The Little Pythagoras

A third playing card can show how much two others should be moved to form a square with the side length c. The two diagonals, each with length 2*r, form four like isosceles right-angled triangles, each with the area ½*r^2. So, c ^ 2 = 4*½*r ^ 2 = r ^ 2 + r ^ 2.

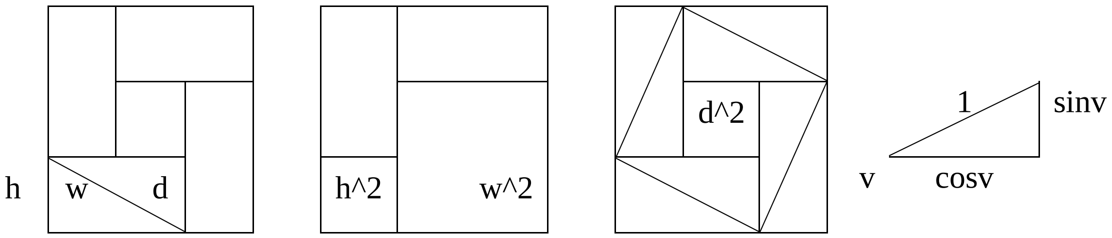


II. The Medium Pythagoras

Four playing cards have the width w and height h. The deck is rotated a quarter turn and placed to the right of the lower card, which remains unturned. This is repeated 3 times thus forming a shape that covers the area h^2 + w^2 + two cards.

During this process, also the diagonals with length d also make a quarter turn, and they now form the area d^2, which covers the shape above together with four half cards. Since four half-cards is the same as two whole cards, d^2 = h^2 + w^2.

In particular, sin v ^ 2 + cos v ^ 2 = 1 in a right triangle with diagonal 1 and sides sin v and cos v.

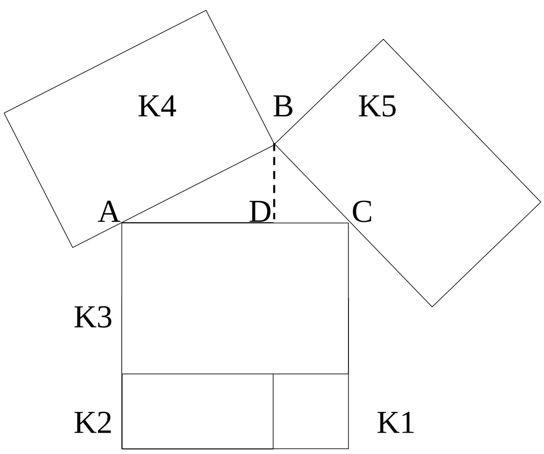


III. The Large Pythagoras

The card K1 is placed horizontally. K2 is placed on-top and makes a quarter turn so the lower left corners are congruent. K3 is placed on-top so that K1 and K3 form a square. The cards K4 and K5 are used to generate the triangle ABC.

Here, the height BD is an extension of K2 's right side. Also, BD divides ABC into two right-angled triangles, BDA and BDC.

In the right triangle BDC we see that DB = a*sinC and DC = a*cosC.



AC's outer square consists of two squares formed by AD and DC, as well as two small strips. AD's square will be AC's square minus the two large strips, plus DC's square, which is deducted twice:

AD ^ 2 = AC^2 + DC ^ 2 – 2*DC*AC = b ^ 2 + (a*cosC) ^ 2 – 2*a*b*cosC

Using that

a^2*sinC^2 + a^2*cosC^2 = a^2*(sinC^2 + cosC^2) = a^2 *1 = a^2

We see that, in the left triangle ABD

AB^2 = DB^2 + AD^2.

= a^2*sinC^2 + a^2*cosC^2 + b^2 – 2*a*b*cos C

= a^2 + b^2 – 2*a*b*cos C

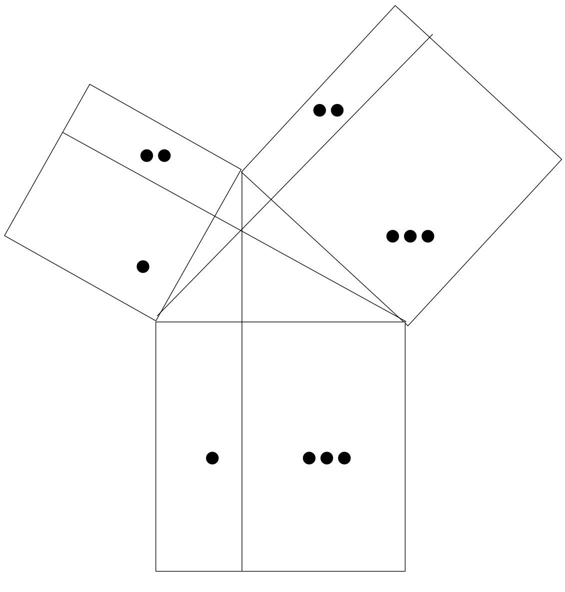
This implies that c^2 = a^2 + b^2 – 2*a*b*cosC (the large Pythagoras).

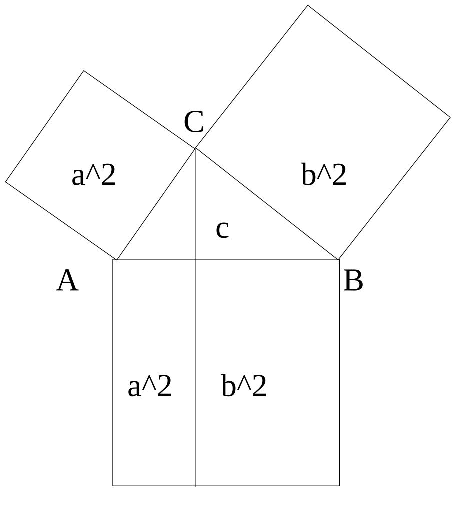
At the same time, we see that the height from B divides b's outer square in two parts with the areas b*c*cosA to the A side, and a*b*cosC to the C side. Likewise, we see that the height from A divides a's outer square in two parts with the areas a*c*cosB to B's side, and a*b*cosC to C's side. As well as to height from C divides c's outer square in two parts with the areas a*c*cosB to B's side and area b*c*cosA to a's side.

Consequently, the following two rules apply:

In a triangle without obtuse angles, the heights divide the opposite side’s outer squares in parts that are pairwise identical at the three angles, with a*b*cosC at angle C, and so on.

In a right triangle, the height divides the diagonal’s outer square into parts corresponding to the short side’s squares.





02. Pi with three playing cards

Three playing cards placed horizontally have the length b. The two top cards are rotated 90 degrees and placed so the three cards lower left corners are congruent. The top card is shifted to the right until it covers the lower card. The 2 top cards now form a square with side length b and diagonal d.

This square can be inscribed in a circle with the center at the intersection of diagonals and with the diagonal as diameter. Divided into 4 identical (red) isosceles triangles, their outer sides can be considered as a first approximation A1 to the circle circumference.

The outsides may be calculated by halving the center angles

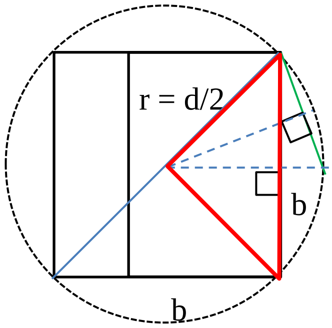
A1 = 4*2*d/2*sin (360/4/2) = d*4*sin (180/4).

The next approximation A2 is available as the outsides of the 8 (green) isosceles triangles, which comes when halving the center angles and keeping the radius r as inside:

A2 = 8*2*(d/2)*sin (180/4/2) = d*8*sin (180/8).

Continuing in this way, the approximation An = d*n*sin(180/n) will approach more and more to the circle circumference, which then can be written as d*pi where pi = n*sin(180/n) for n sufficiently large:

n	100	1000	10000	table value
n*sin(180/n) ≈ π	3.141076	3.141587	3.141593	3.141593…



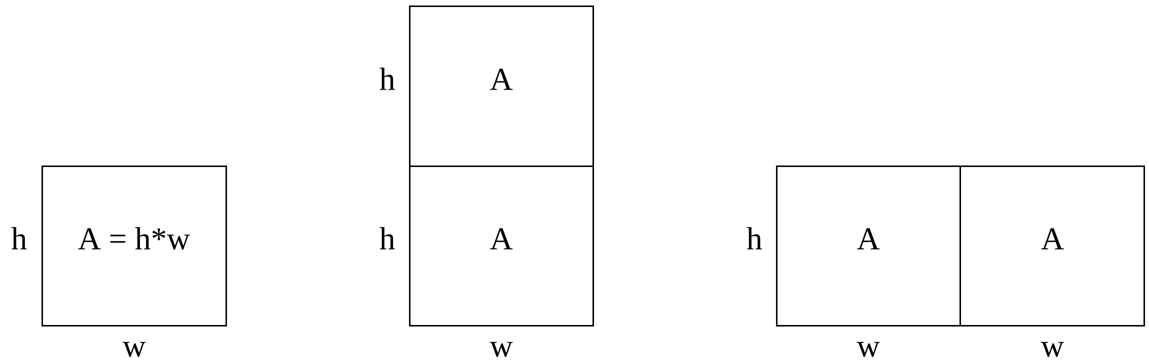
03. Proportionality with 2 playing cards

The product formulas A = h*w is very common, often called the proportionality formula or a bridging formula with the per-number h connecting the two unit-numbers A and w. The per-numbers may be $/kg, meter/second, or mol/liter in economics, physics or chemistry.

A product formula is illustrated by a playing card where the height h and width w will give the area A = h*w. Placed after the other vertically, the width w is constant, and doubling the height h implies doubling the area A. Thus, the height and the area are proportional.

Likewise, we see that the width and the area are proportional when placing the cards after each other horizontally.

Moving the top card from vertical to horizontal position keeps the surface area A constant whereas the width w and height h will be doubled and halved respectively, thus being inverse proportional.



04. The Product Rules with 2-4 playing cards

A playing card with width a and height b has the area a*b.

A. Card 1 is placed horizontally with card 2 turned and placed on top so the bottom left corners are congruent. In this way, the upper-right corner becomes a square with the area (a-b) ^ 2, obtained by removing two cards from the area a ^ 2 and add b ^ 2, since this area is removed twice:

$(a – b)^2 = a^2 – 2*a*b + b^2$

Card 1 divides card 2 in an upper part with the area (a-b)*a, and a lower part with the area b ^ 2:

$(a – b)*b = a*b – b^2$

B. Card 2 moves vertically up until leaving card 1. Card 2 will then split card 1 into two parts, where the left part with the area b ^ 2 together with card 2 covers the area (a + b)*b:

$(a + b)*b = a*b + b^2$

C. Card 3 is placed horizontally on top of card 2 so the upper left corners are congruent. Card 2 divides card 1 and card 3 into two parts, where the right parts are included in a rectangle with area (a+b)*(a-b). And where card 1 's share and the visible part of card 2 both have the area (a-b)*b. The bent area then is the difference between a^2 and b^2:

$(a + b)*(a – b) = a^2 – b^2$

D. Card 3 is shifted to the right. A vertical card 4 is added below so that the four cards form a square with the side length a + b, combined to the left by the area a ^ 2 at the bottom, and b^2 at the top, together with the two cards C3 and C4 to the right.

$(a + b)^2 = a^2 + b^2 + 2*a*b$

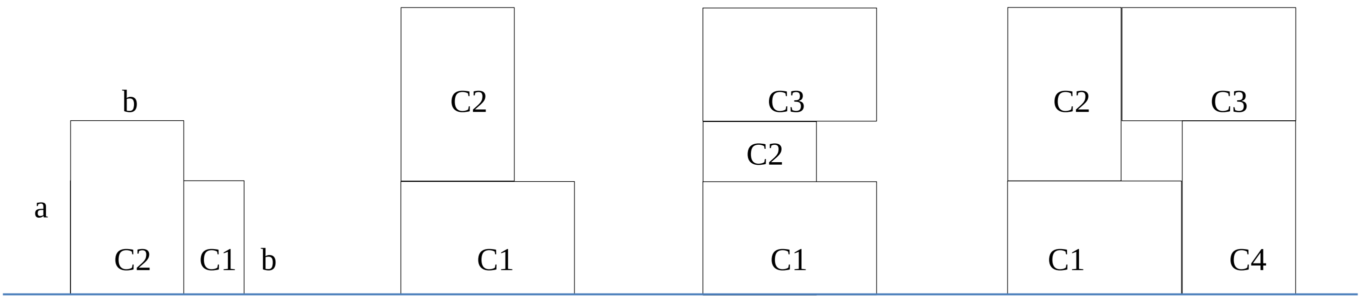


figure A figure B figure C figure D

05. A quadratic equation solved with two playing cards

The equation $x + 2 = 8$ asks: what is the number x that added to 2 gives 8? To answer, we invent an opposite operation to addition, called subtraction, where the number $x = 8 – 2$ by definition is the number x that added to 2 gives 8. Thus, we see that an equation is solved when the unknown number is isolated by moving a number to the opposite side with the opposite calculation sign.

Likewise, by definition, the number $x = 8/2$ is the solution to the equation $x*2 = 8$ asking for the number x that multiplied with 2 gives 8. Likewise, the number $x = \pm \sqrt{8}$ is the solution to the equation $x ^ 2 = 8$ asking for the number x that multiplied with itself gives 8. Again, we see that an equation is solved by moving a number to the opposite side with the opposite calculation sign.

In the quadratic equation $x ^ 2 + 6*x + 8 = 0$ there are two unknown x's so it needs to be rewritten, so there is only one x.

Two playing cards has the width k and the height x + k. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that

$(x+k)^2 = x^2 + 2*k*x + k^2$, or, have the unknown x only once on the right side:

$(x+k)^2 – k^2 = x^2 + 2*k*x$, or ‘x plus k squared, minus k squared gives x squared + double-k x

We can now rewrite the equation $x^2 + 6x + 8 = 0$ first to $(x^2 + 2*3*x) + 8 = 0$, then to $(x+3)^2 – 3^2 + 8 = 0$, and finally to $(x+3)^2 – 1 = 0$ that is solved by three times moving to the opposite side:

k = ½b	½b*x	b*b/4	3	3*x	3*3
x	x*x	½b*x	x	x*x	3*x
	x	k=½b		x	3

$x^2 + 6x + 8 = 0$

$(x+6/2)^2 – (6/2)^2 + 8 = 0$

$(x+3)^2 – 1 = 0$;

Now 3 times, we move to opposite side:

$(x+3)^2 = 1$

$x+3 = \pm\sqrt{1}$

$x = -3+1 = -2$, and $x = -3-1 = -4$

With unspecified letter-numbers, the quadratic equation is solved in the same way:

$x^2 + b*x + c = 0$, giving

$(x^2 + 2*b/2*x) + c = 0$, giving

$(x + b/2)^2 – (b/2)^2 + c = 0$, giving

$(x + b/2)^2 = b^2/4 – c = D/4 = 0$, where $D = b^2 – 4*c$ is called the discriminant.

Thus, the quadratic equation $x^2 + b*x + c = 0$ has two solutions, or one, or none, depending on if the value of the discriminant D is positive, zero, or negative.

$$(x + \frac{b}{2})^2 = \frac{D}{4}$$

$$x + \frac{b}{2} = \pm \frac{\sqrt{D}}{2}$$

$$x = \frac{-b}{2} \pm \frac{\sqrt{D}}{2}$$

$$x = \frac{-b \pm \sqrt{D}}{2}$$

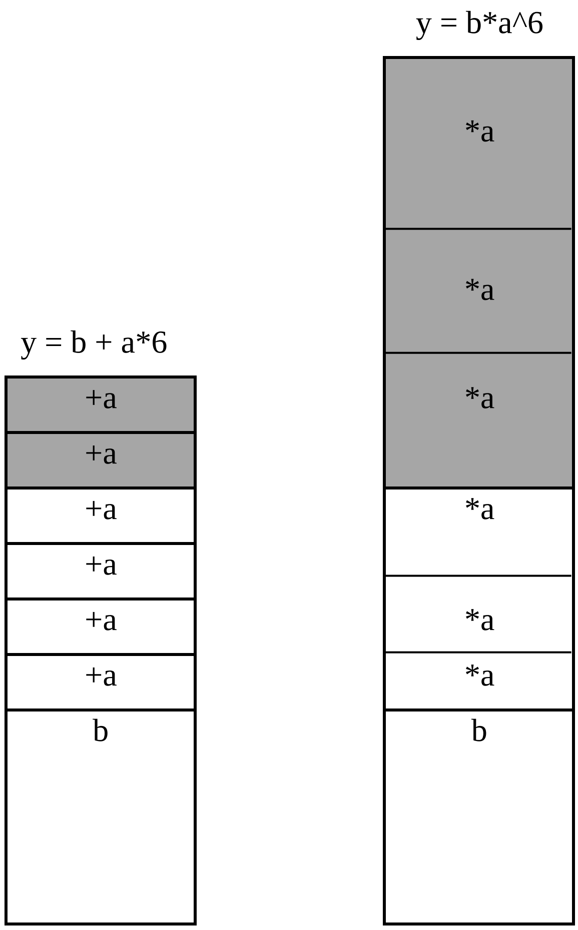
06. Change by adding or multiplying with playing cards

Change by adding or multiplying occurs when x times an initial value b will be respectively added or multiplied by the same number a. This gives the terminal values respectively y = b + a*x and y = b*a ^ x, also called linear and exponential growth. The two change forms may be illustrated with two decks of playing cards.

To the left is placed a deck of 7 cards. The top card stays. The next six cards are shifted upwards with a quarter of the length of the first card to show the terminal number after six times adding with the same change-number a.

Next to we place a deck of 6 cards. The top card stays. The next three cards are shifted upwards with a quarter of the length of the previous stack to show that the terminal number after 3 changes is roughly the same as with four changes when changing by adding since as 125%*125*125%% = 195.3%, i.e. almost 200%. Thus, a change-percent at 25% means doubling after three changes. So, after 6 changes, the initial number has been doubled twice, thus having 4 as the total change-factor or change-multiplier.

Linear change with b = 100$, and a = 25$.   Exponential change with b = 100$, and a = 125%.



07. The saving formula with nine playing cards

With a monthly deposit a $ and interest percent, a saving combines change by adding and by multiplying. A saving is also called an annuity.

A saving occurs if a bank creates two accounts, K1 and K2. K2 receives the one-time deposit a/r $. Each month, K1 receives first the monthly interest percent r of its own amount, and then the monthly interest amount of the amount in K2, i.e. a fixed deposit of a/r $ * r = a $.

After n months, K1 will contain a saving A growing monthly from a deposit of a$ and an interest percent r. But at the same time K1 will contain the total interest percent R of the initial amount a/r $ in K2, so A = a/r*R or A/R = a/r, where 1+R = (1 + r) ^ n.

This can be illustrated with nine playing cards placed on a A4 paper divided into two with K1 to the left, and K2 to the right.

After the first month, K1 receives the interest percent r of its deposit, 0$; as well as a$ from K2, shown with a playing card placed horizontally with the backside up.

After the second month, K1 receives the interest percent r of its deposit, r*a $, shown with a vertical card push up a little; as well as a $ from K2, shown with a playing card placed horizontally with the backside up.

We will continue until the end of the fifth month. We gradually increase the pushing up of the vertical interest card because of the growing deposit in K1. Finally, we cover the right part of the two last cards with a white paper strip.

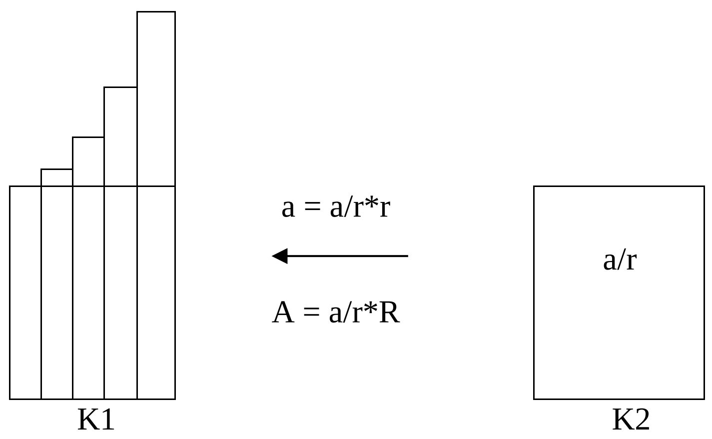
K1 now contains a saving, A, consisting of a series of constant $-deposits, the horizontal cards, and the interest on these amounts, the vertical cards.

At the same time, the horizontal cards represent the simple interest of K2, whereas the vertical cards represent the compound interest of K2.

So, again we have that the saving is A = a/r*R, giving A/R = a/r.

As to the relation between the total interest rate R, single interest rate r, and compound-interest rate RR, the cards show that R = n*r + RR or RR = R – n*r.

Thus, the simple and the compound interest might be taxed differently.



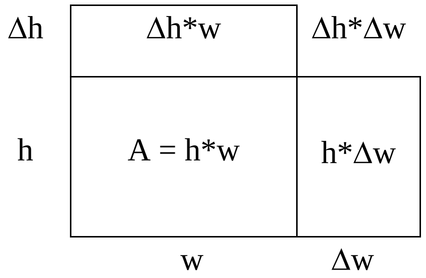
08. Product change with three playing cards

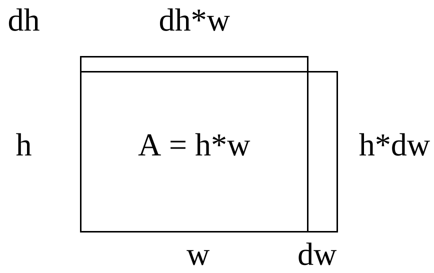
In geometry, the product of two numbers h and w occurs as the area A of a rectangle with height h and width w, $A = h*w$.

In economics the product occurs as the total $-number for w kg at h $/kg, $T = h*w$, or, more generally, each time a per-number is multiplied up to a quantity.

The question now is how changes in h and w will change the product.

Three playing cards has the height h and width w. The top card stays, the middle card is shifted off a piece Δw to the right, and the bottom card is shifted off a piece Δh upwards.





We see that the change in area, ΔA, consists of three parts, $\Delta h*w$ and $h*\Delta w$ and $\Delta h*\Delta w$.

With small changes, the last corner part may be neglected since the product of two small numbers gives a very small number: Assume that in the product 2*3 both numbers are changed with 0.01 to

$(2+0.01)*(3+0.01) = 2*3 + 2*0.01 + 0.01*3 + 0.01*0.01$

$= 6 + 0.02 + 0.03 + 0.0001$

$= 6.\ 0501$

$= 6.05$ with three significant figures.

Furthermore, the corner part has less and less influence, the smaller the change is.

Change t	0.1	0.01	0.001
h*w = (2+t)*(3+t)	6.51	6.0501	6.005001

Writing a small change as d instead of Δ will give the following rule for how a product is changed by small changes in its factors:

$dA = d(h*w) = dh*w + h*dw$, or as percentages:

$dA/A = d(h*w)/(h*w) = dh/h + dw/w$

Thus, with products, the change-percentages almost just add: Changing a kg-number with 3% and a $/kg-number with 5% will make the $-number change with approximately 3% + 5% = 8%. This rule applies to changes less than 10% with decreasing precision.

Since $A = h*w$, the per-number $h = A/w$.

Moving to opposite side with opposite calculation sign, we get

$dh/h*w = dA/A – dw/w$, or $d(A/w)/(A/w) = dA/A – dw/w$

Thus, with ratios, the change-percentages almost just subtract: Changing a $-number with 7% and a kg-number with 3% will make the $/kg-number change with approximately 7% – 3% = 4%. Again, this rule applies to changes less than 10% with decreasing precision.

So, with $y = x^n$ we get that $dy/y = n*dx/x$, or $dy/dx = n*y/x = n*x^n/x = n*x^{(n-1)}$

09. Integral- and differential calculus with 2 playing cards

Where unit-numbers add directly, per-numbers add by their area: 2 kg at 6$/kg plus 3 kg at 4$/kg gives a total (2 + 3) kg at (6*2 + 4*3)/(2 + 3) $/kg.

This can be shown with two cards placed side by side, card1 placed vertically, and card2 placed horizontally. Card1 has the per-number 6$/kg vertically and the unit-number 2 kg horizontally, giving the area 12 $.

The unit-numbers add directly, the dollar-numbers to 12+12 = 24, and the kilo-numbers to 2+3 = 5.

The per-numbers, the $/kg-numbers 6 and 4, add by their area 24/5 = 4.8. So Σ ($/kg) = Σ $ / Σ kg.

Graphing the cards in a coordinate system provides the rule: per-numbers add by the area under the per-number graph, i.e. by integration.

Integration uses multiplication before addition. In contrast, subtraction comes before division when reversing integration, also called differentiation:

the question "2 kg at 6$/kg plus 3 kg at u $/kg gives a total of 24$" is answered by first removing card1, and then recount the card2 area in 3s, thus applying subtraction before division:

$u = (24 – C1)/3 = \Delta C/3$

Integration may be seen as a process of change: Card1 specifies a starting area. Placed next-to, card2 makes the area change to a new width w and the new area A. But now the variable height means a variable per-number, that might be found with differential calculus:

Card1 contains the initial numbers: the height ho, the width wo and area number Ao. Card2 contains the change-numbers

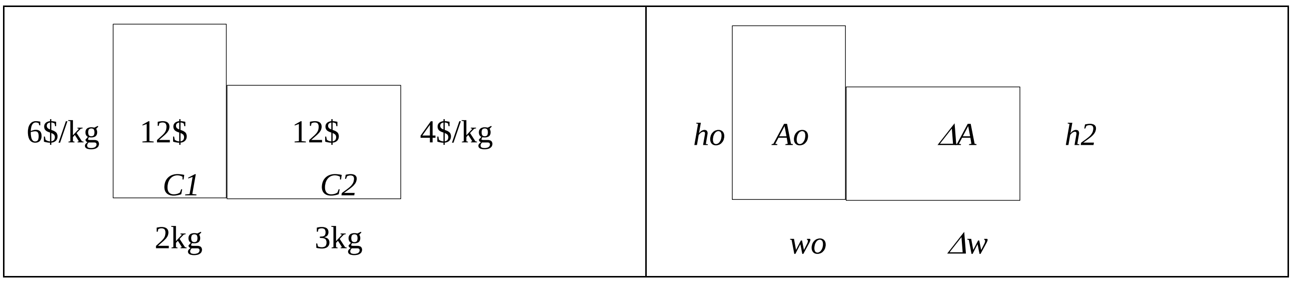
$\Delta w = w – wo$, and

$\Delta A = A – Ao$, which is also $h2 * \Delta w$.

We then find the per-number for card2 as

$h2 = (A – Ao)/(w – wo) = \Delta A/\Delta w = \Delta y/\Delta x = \Delta f/\Delta x$

Here the width-number and the area-number are graphed as a line in an x-y coordinate system, where the per-number will be the slope of the area or unit-number curve y, typically given as A = y = f (x), i.e. as a formula with a variable number x.



10. How to differentiate sine and cosine with three playing cards

Three playing cards have the height h and width w. Card 1 rotates so w forms the angle v with the horizontal direction. Card 2 rotates 90 degrees and is placed at the end of card 1, so w here forms the angle v with the vertical direction. Card 3 remains horizontal and kicks under card 2 and over card 1 until forming a triangle with h as the long side.

As known, sine and cosine may be read as the first and second coordinate in a unit circle.

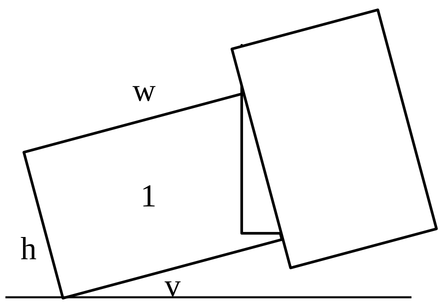
If the angle v gets a very small change dv, the circle (left w on card 2) is approximately linear.

Furthermore, the two left legs from v and v + dv (lower and upper w on card 1) are approximately parallel.

If v is measured in radian, the triangle’s vertical, horizontal and long side will be three increment sides d(sin v) and -d (cos v) and dv. The long side forms the angle v with vertical, so

$\cos v = \frac{d(\sin v)}{dv}$, and $\sin v = -\frac{d(\cos v)}{dv}$,

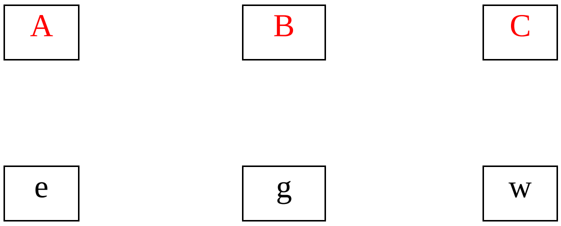
or, $\frac{d(\sin v)}{dv} = (\sin v)' = \cos v$, and $\frac{d(\cos v)}{dv} = (\cos v)' = -\sin v$



11. Topology with six playing cards

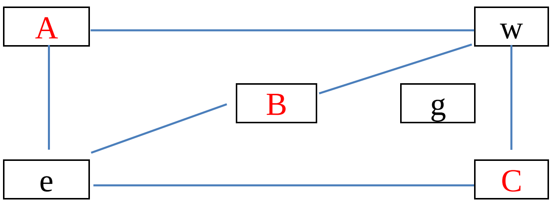
Six playing cards may illustrate a supply problem, a classic problem in topology, i.e. geometry, where neither distances nor angles, but only the relative positions between the points matter.

Problem: How can three houses A, B and C be provided with electricity, gas and water without the wires crossing?



We notice that the connection from house A to electricity to house C to water forms a closed ring that splits the plane in two areas, inside and outside. To be connected, house B and gas must be on the same side.

Suppose House B and gas is inside. Then the connection from electricity to house B to water will be splitting the inside in two closed areas with houses A and C in different areas. Located in one area, gas cannot be connected to the house located in the other area.



Now, suppose that house B and gas is outside. Then the connection from electricity to house B to water to another house back to electricity will enclose the third House, and the argument above can now be repeated.

Conclusion: the task cannot be solved unless we add a bridge whereby the plan changes its topology to a torus which is a plane with a handle.

Gluing a strip together at the ends, you can get from the outside to the inside in two ways: You can turn the strip a half turn (a Möbius strip), or you can punch a 'wormhole' (crosscap) in the strip.

In network analysis, topology is used to describe the number of bridges or handles in a given network.

12. Heron's formula, a triangle's circles, and Pythagoras on factor form

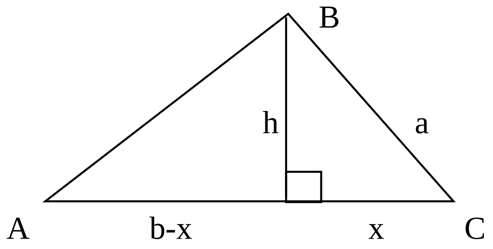
Heron's area formula T = √(s*(s-a)*(s-b)*(s-c)) comes from repeating the factor formula

p^2 – q^2 = (p+q)*(p-q).

If 2*s = a+b+c, then 2*s – 2*a = -a+b+c, 2*s – 2*b = a-b+c and 2*s – 2*c = a+b-c

Proof.

Let B be the largest angle in the triangle ABC. Then the height h from B will divide the triangle into two right triangles, and it will divide the side b into two parts, x towards C and b – x towards A.



The lengths h and x are found from the two right triangles: x^2 + h^2 = a^2, so h^2 = a^2 – x^2.

Inserting this in (b-x)^2 + h^2 = c^2 gives c^2 = (b-x)^2 + a^2 – x^2.

Moving to opposite side with opposite sign gives

c^2 – a^2 = (b-x)^2 – x^2 = (b-x+x)*(b-x-x) = b*(b-2*x) = b^2 – 2*b*x.

Consequently, 2*b*x = a^2 + b^2 – c^2.

Now T = ½*h*b, so 16*T^2 = 4*h^2*b^2 = 4*(a^2 – x^2)*b^2 = 4*a^2*b^2 – 4*x^2*b^2 = (2*a*b)^2 – (2*x*b)^2 = (2*a*b + 2*b*x)* (2*a*b – 2*b*x)

Now we insert that 2*b*x = a^2 + b^2 – c^2:

16*T^2 = (2*a*b + a^2 + b^2 – c^2)* (2*a*b – a^2 – b^2 + c^2) = ((a+b)^2 - c^2)*(-(a-b)^2 + c^2) = ((a+b+c)*(a+b-c))*((a-b+c)*(-a+b+c)) = 2*s*(2s-2c)*(2s-2b)*(2s-2a) = 16*s*(s-a)*(s-b)*(s-c).

Consequently T^2 = s*(s-a)*(s-b)*(s-c).

Et viola: T = √(s*(s-a)*(s-b)*(s-c)).

Let the inscribed circle have the radius r. Then

T = ½*r*a + ½*r*b + ½*r*c = ½*r*2s = r*s, which gives

r = √((s-a)*(s-b)*(s-c)/s).

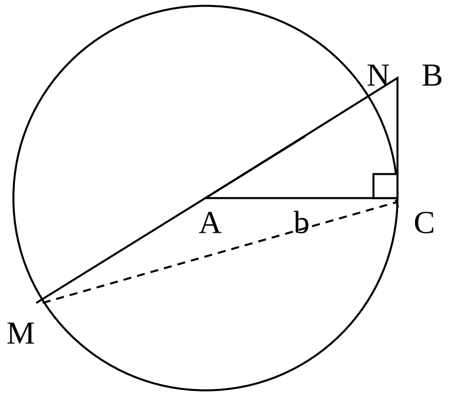
Let the circumscribed circle have the radius R. this allows extending the sinus relations

2*R = a/sinA.

Inserting this into the area formula T = ½*b*c*sinA gives the formula 4*R*T = a*b*c, "4 Round Turns teaches you abc"

The Pythagorean theorem can be found both on term form, a^2 + b^2 = c^2, and on factor form, a^2 = (c + b)*(c-b).

The factor form is found by drawing a circle with center A and radius b. Let M and N be the points where c intersect the circle. Then BM = c + b and BN = c-b. The factor form follows from of the two similar triangles BCN and BMC, where the angles BCN and BMC are like, as they span the same arc; or from calculating the point B's 'power-of-a-point theorem'.


N
B
A
b
C
M