Math Ed & Research 2017

A Goal Displacement Makes a Means a fake Goal.

The Goal of Teaching Math is ~~to Learn Math~~ to Master Many.

- so, Count & Multiply before you Add.

OnTop & NextTo Addition Root Proportionality & Integral Calculus.

Reversed Addition Roots Equations and Differential Calculus.

As PerNumbers, Fractions are Operators, needing Numbers to become Numbers

- so, PerNumbers and Fractions Add by their Areas as Integral Calculus.

Allan.Tarp@MATHeCADEMY.net

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Preface

01-02. To celebrate the 500year anniversary of the 95 Luther theses I decided to write two feature articles to a Danish newspaper. The first asks If Europe really need Compulsory School Classes, arguing that the North American self-chosen half-year blocks might be a better way to support adolescents in their complicated identity work after puberty. The second asks why mathematics, which was created as a straight forward natural science about the physical fact Many has to be presented as a metaphysical self-referring science that transforms many potential users to losers.

03. The first conference was the Congress of the European Society for Research in Mathematics Education, CERME, 10th congress in Ireland in February. I chose the thematic working group on Arithmetic and Number Systems, and wrote the paper 'CupCounting and Calculus in Early Childhood Education'. The paper points to the decreased PISA performance despite increased research, and the uses a method called 'institutional skepticism' to explain it. The paper then looks at mathematics as essence, and as existence containing chapters on creating number-icons, counting in icons, Re-counting in the same unit and in a different unit, Double-counting creates proportionality as per-numbers, added on-top or next-to, reversed adding on-top and next-to, and how schools use ten-counting only. The paper then describes a micro-curriculum designed for the outside goal 'to master Many'. Finally, literature on cup-counting is addressed before the conclusion and recommendation.

04. At the conference, a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is "Only one: to improve, mathematics education should ask, not what to do, but what to do differently." Thus, to be successful, research should not study problems but look for hidden differences that might make a difference. Research that is skeptical towards institutionalized traditions could be called difference research or contingency research or Cinderella research making the prince dance by looking for hidden alternatives outside the ruling tradition. The French thinker Lyotard calls it 'paralogy' inventing dissension to the reigning consensus. To give a more detailed answer I wrote the academic essay 'Fifty Years of Research without Improving Mathematics Education, Why?' Looking at the Outside Roots of Mathematics, the essay suggests a Rethinking of Mathematics from Below. Then it presents how School Teaches Mathematics in 12 points, and in 25 points how School Could Teach Mathematics. The conclusion is followed by an extract of my personal contributions to mathematics education research.

05. The second conference was the 9th International Mathematics Education and Society Conference or MES9 that took place in Greece in April. The conference focused on the social theorizing of mathematics education and the theme of MES9 was 'Mathematics Education and Life at Times of Crisis'. My paper proposal was called 'A 1year pre-engineer course for Young migrants, a job for critical or civilized math education.' It contained chapters on Background and question, Critical and civilized thinking, A historical background, Critical versus civilized mathematics education, Criticizing and civilizing rational numbers, 'Preschool calculus and multiplication before addition' as a 1year pre-engineer math course; and a final discussion and conclusion as well as an appendix showing: a critical and a civilized math curriculum.

06. My project proposal was called 'Online teacher training for curing math dislike: cup & re-counting & multiplication before addition. It contained chapters on the background, A Case: Peter, stuck in division and fractions, and a conclusion. And as appendices, 1day Skype Seminar on how to cure Math-dialike by BundleCounting, ReCounting & BundleWriting before adding, and a summary of the 4 primary and secondary 4 study units at the MATHeCADEMY.net

07. My symposium proposal was called a 'Debate on how to improve mathematics education.' It contained chapters on Mathematics Itself, Education in General, Mathematics Education, the Learner, the Teacher, the Political System, Research, and Conflicting Theories.

08. My poster proposal called ‘MigrantMath as CupCounting & PreSchool Calculus’ contained chapters on the background, Mathematics as an Essence, as well as Mathematics as ManyMath, a Natural Science about Many before the conclusion.

09. At the International Congress on Mathematics Education, ICME, in Germany in 2016 my contribution to the topic study group on Philosophy of mathematics education was called ‘From Essence to Existence In Mathematics Education’. My proposal for a book following the topic study group was called ‘A Heidegger View on How to Improve Mathematics Education.’ It contains chapters on introduction, What does ‘is’ Mean, the Heidegger Universe, Meeting Many celebrating the 500year anniversary for Luther’s 95 theses by describing meeting Many in 12 theses; and a conclusion.

10. Now the time had come to summarize the 2017 work in an article to be sent to a journal. The proposal is called ‘Count and Multiply Before You Add: Proportionality and Calculus for Early Childhood and Migrants.’ It contains chapters on Decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Institutional Skepticism, Mathematics as Self-Referring Gossip, Meeting Many, Comparing Many-matics with Mathe-matics, Testing a Many-Matics Micro-Curriculum, Ending the Dienes Era, as well as a Conclusion and a Recommendation.

11. Each second year the Swedish ‘MatematikBiennale’ takes place in relation to the MADIF conference. I send in 11 proposals for the 2018 Biennale. They were all rejected. The proposals were called as follows. Start-math for children and migrants: bundle-count and re-count before adding, Multiplication before addition strengthens the number sense in children and migrants, Dislike towards division cured with 5 sticks and 1 cup and bundle-writing, Fractions and percentages as per-numbers, Fractions and per-numbers add as integration, Proportionality as double-counting, with per-numbers, Equations solved by moving, reversing or re-counting, Calculus: Addition of and division into locally constant per-numbers, Calculus in primary, middle and high school, Stem-based core-math makes migrants pre-engineers, The teacher as a difference-researcher. At the end a comment was added called ‘Fifty years of research without improving mathematics education, why?’

12-13. To help Sweden cope with OECD report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change’ I wrote an article sent to the Swedish government and a university in southern Sweden. Swedish educational shortages challenge traditional mathematics education offered to migrants. Mathematics could be taught in its simplicity instead of as ‘meta-matsim’, a mixture of ‘meta-matics’ defining concepts as examples of inside abstractions instead of as abstractions from outside examples; and ‘mathe-matism’ true inside classrooms but seldom outside as when adding numbers without units. Rebuilt as ‘many-matics’ from its outside root, Many, mathematics unveils its simplicity to be taught in a STEM context at a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries. The article is called ‘The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants.’ It contains chapters on decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Meeting Many, Meeting Many in a STEM Context with subchapters on a short World History, Nature Obeys Laws, but from Above or from Below?, Counting and DoubleCounting Time and Space and Matter and Force and Energy, Warming and Boiling water, Letting Steam Work, An Electrical circuit, How high up and how far out, How many turns on a steep hill, Dissolving material in water. Finally, a chapter on the Simplicity of Mathematics as well as a discussion: How does Traditional MatheMatics differ from ManyMatics was included before the conclusion. A shorter 10 page version for the MADIF11 Conference is included.

14. Introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015

OECD-report ‘Improving Schools in Sweden’. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

15. The KomMod report provides an alternative response to KOM-project terms of reference, in the expectation that the Science Board of education and the Ministry of education want to respect a common democratic IDC-tradition with Information and Debate between alternatives before a Choice is made. It was written in Danish in 2002 And translated into English in 2017.

16-18, 20. The third conference was the 9th conference of the Classroom Teaching Research for all Students, CTRAS, held in China in July. Having presented my contribution ‘Decreasing PISA Performance in spite of increasing research’ I was asked to give a keynote-presentation. I sent in twelve proposals formulated so they could also become proposals for 1day Skype Seminars: The Root of Mathematics, Many, dealt with by Block-Numbers, Bundle-Counting & Preschool Calculus; 12 Luther-like Theses about how ManyMath can Improve Math Education; Curing Math Dislike with one Cup and five Sticks; DoubleCounting rooting Proportionality - and Fractions and Percentages as PerNumbers; Algebraic Fractions made easy by Block-Numbers with Units; Algebra and Geometry, always Together, never Apart; Calculus in Middle School and High School; Mathematics, the Grammar of the Number-Language. But why teach Grammar before Language?; Quantitative Literature also has three Genres: Fact and Fiction and Fiddle; Distance Teacher Education in Mathematics by the CATS method: Count & Add in Time & Space; 50 years of Sterile Mathematics Education Research, Why?; and Difference-Research, a more Successful Research Paradigm?

The conference chose the latter, so I wrote the article ‘Difference-Research Powering PISA Performance: Count & Multiply before You Add’, containing chapters on Decreased PISA Performance Despite Increased Research, Difference-research Searching for Hidden Differences, Social Theory Looking at Mathematics Education, a philosophical Background for Difference Research, Meeting Many, Examples of Difference-research, Remedial Curricula, a Macro STEM-based Core Curriculum, Teaching Differences to Teachers, Being a Difference-Researcher, and a conclusion. Likewise, the keynote PowerPoint Presentation at the CTRAS 2017 Conference is included. At the end of the conference I summarized my reflections in a paper called: Reflections from the CTRAS 2017 Conference in China, Examples of Goal Displacements in Mathematics Education.

19. At the 8th ICMI-East Asia Regional Conference on Mathematics Education, the theme of the Conference was ‘Flexibility in Mathematics Education’. The website writes:

“Flexibility in Mathematics Education” has been chosen as the theme of the conference. Flexibility is highly related to creativity, multiplicity, and adaptation. In the current era, rapid changes in economy, environment and society have been facilitated by the rapid development of technology and engineering. Flexibility in mathematical thinking, problem solving, teaching methods, evaluation, teacher education and mathematics education research is a key to empowering learners, teachers, educators and researchers to tackle the complexity and uncertainty, and to giving them the capacity and motive to change in the innovative era.

The Topic Study Group themes were TSG 1: Flexibility in Mathematics Curriculum and Materials; TSG 2: Flexibility in Mathematics Classroom Practices; TSG 3: Flexibility in Mathematics Assessment; TSG 4: Flexibility in Mathematics Teacher Education and Development; TSG 5: Flexibility in the Use of ICT in Mathematics; TSG 6: Flexibility in the Use of Language and Discourse in Mathematics; and TSG 7: Flexibility in Mathematics Learning.

21-22. Next, an overview on how to cure Math Dislike is included together with concrete material called ‘Migrant Math’ containing 20 exercises inspired by the paper ‘The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants.’ The exercises are: From Sticks to Icons, Counting-sequences in Icons, BundleCount in Icons, BundleCount with Dices, ReCount in the Same Unit, ReCount in a New Unit, ReCount in BundleBundles, ReCount in Tens on Squared Paper or an Abacus, ReCount from Tens, ReCount Large Numbers in Tens, DoubleCount with PerNumbers, DoubleCount with Fractions and Percentages, ReCount PerNumbers, Fractions, Add OnTop, Reversed Adding OnTop, Add NextTo, Reversed Adding NextTo, Add Tens, Reversed Adding Tens, Recounting Solves Equations.

Allan Tarp, Aarhus Denmark, December 2017

01. Does Europe really need Compulsory School Classes?

Compulsory classes force children and young people to follow the year group and its schedule. Compulsory classes made sense when created in Prussia about 200 years ago in an agricultural society; and also in industrial society with its permanent life jobs. In an IT-society, compulsory classes make sense in primary school: with both mother and father in changing self-realizing jobs, the first 3-4 school-years children need a warm and loving nanny with only one class, quickly getting a gaze of each child's characteristics and needs.

On the other hand, compulsory classes mean disaster in secondary school with young people who have left childhood and started an extensive identity work to uncover and develop their personal potential and talent. Here a compulsory class is the last thing they need, which is evident when observing the seven sins of compulsory classes.

Noise. Having an activity imposed that you do not master or find interesting, you quickly switch to other activities, surfing the Web or chatting with others in the same situation. The result is noise, which can be so violent that the rest of the class must wear hearing protectors.

Absence. Once you have given up on learning you feel a desire for absence, perhaps even to drop out. But that will hurt the school’s economy, so you will not be allowed to leave the class regardless of your extent of absence.

Bullying. When you finally meet up again after an absence, it is tempting to bully those who meet every day.

Drinking. Especially if they do not want to participate in the extended weekend drinking starting in lower secondary school and coming to full expression in das Gymnasium, where many are sent to the hospital at the annual welcome parties or get hurt under excessive drinking on study tours.

Substitute teachers. Once you have conquered the territory, it is natural to bully also the various teachers who come to visit. Some can take it, others cannot and take a long-term sick leave. Skilled substitute teachers are expensive, so often a recent high school graduate is selected instead, or cleaning personnel.

Bottom marks. The extent of mental absence is shown by the written marks. Thus, in Denmark with 5 passing marks, the three lowest are given when answering correctly 16%, 33% or 50% at the final exam in mathematics at the end of lower secondary school. And here the international passing level at 70% gives the second-highest mark. The low level of learning can, however, be hidden by replacing written tests with oral, which is much more effective to increase the marks with floods of leading questions. Denmark is virtually the only country in the world maintaining an oral exam. Its credibility is illustrated by the joke, which is often exchanged over coffee table during an exam: With a friendly external examiner, a good teacher can examine a chair to a passing mark, provided the chair stays quiet.

War against boys. In a compulsory class, girls and boys are forced to go along, although the girls are two years ahead in development. It provides both with a skewed impression of the opposite sex, and school dislike makes boys leave school before upper secondary school, where there are two girls for every boy. In short, compulsory classes pump boys out of school to remain in the outskirts, while girls are pumped into the juggernaut universities in Copenhagen; and in Aarhus, where they then move to Copenhagen after graduation, since that is where the jobs are. With the absence of boys, girls find another girl and a sperm bank so that together they can get a single child.

Which creates the compulsory class’ most fatal consequence, a birth-rate in Europe at 1½ child per family. A quick calculation shows that with 0.75 child per woman, Europe's population will halve twice over the course of 100 years. A population decline unprecedented in history.

Unlike in the North American republics. Here young people do not have multi-year compulsory classes. Instead, they are welcomed to high school with recognition: “Inside, you carry a talent that it is our mutual job to uncover and develop through daily lessons in self-chosen half-year academical or practical blocks together with a teacher who only teaches one subject. If successful we say ‘good job, you have talent, try out more blocks’. If not we say ‘good try, you have courage to try out the unknown, now let’s find another block for you to try out. And at the last year you can try out college blocks.”

Thus, the absence of multi-year educational defeats allows you to enter a local block-organized college at 18 and get a two-year practical diploma degree or continue at a regional college and get a four-year job-directed bachelor's degree.

Without compulsory classes, Europe could do the same, so that every other boy could be an engineer at the age of 22; and at the age of 25 have a well-paid job, a family, and three children ensuring state survival: one for mother, one for father, and one for the state.

As demonstrated in North America, compulsory classes are not a biological necessity.

As mammals, we are equipped with two brains, one for routines and one for feelings. When we raised up on our hind legs, we developed a third brain to keep balance; and to hold concepts since we could now use the front legs to grab the food and eat it or share it with others. In this way, grapping could provide the holes in our head with our two basic needs, food for the body and information to the brain. For by assigning sounds to what we grasp, we develop language to transfer information between brains.

In fact, we have two languages, a word language and a number language. At home children learn to talk and to count. Then as an institution, the school takes over and teaches children to read and to write and to calculate, and to live together with others in a democracy.

The ancient Greek sophists saw enlightenment as a prerequisite for democracy: knowing the difference between nature and choice, we can avoid hidden patronization in the form of choice presented as nature. The philosophers had the opposite view: Choice does not exist, since all physical things are but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Consequently, people should give up democracy and accept the patronage of the philosophers.

The Christian Church eagerly took over the idea of metaphysical patronage and converted the academies into monasteries, until the Reformation recovered the academies. Likewise, nor emperors nor kings had anything against being inserted by the Lord's grace.

Metaphysical patronage ended with Newton’s three times no. “No, the moon does not move among the stars, it falls to the ground like an apple. No, moons and the apples do not follow a metaphysical unpredictable will; instead they follow their own will, which is predictable because it follows a formula. And no, a will does not maintain order, it changes it.”

Once Newton discovered the existence of a non-metaphysical changing will, this created the foundation for the 1700 Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. The result was two republics, one in the United States and one in France. The United States still has its first Republic, France its fifth, since Prussia tried to overthrow the French Republic again and again.

France first got upper hand by mobilizing the population with enlightenment and democracy. As a counter measure, Prussia created a strong central administration with an associated ‘Bildung’ education with three goals: The population must be kept unenlightened so it will not demand democracy. Instead, Bildung must install nationalism transforming the population into a 'people', Germans, obeying the almighty Spirit by fighting other 'people', especially the French with their democracy. Finally, from the population, its elite must be sorted out to form a new central administration; and receive classical Bildung to become a new knowledge-nobility to replace the old blood-nobility, which was unable to strangle French enlightenment and democracy.

The rest of Europe eagerly took over the Prussian Bildung education. One might expect that when Europe became republics, its school form would follow. Here is only to say that still it is not too late. But it requires a comprehensive school reform, for the two school forms are very different.

In continental Europe, compulsory classes are replaced by a mess of competing compulsory lines in upper secondary school and with a confusion of more or less coordinated lines at the tertiary level leading to a 3year bachelor degree, usable only if supplemented with a 2year university directed master degree.

In the North American republics, compulsory classes stop after primary school. With self-chosen half-year blocks, learners can try something new each half year and continue if the trial was successful; and, as important, get out if it turns out to be an area outside your personal talent.

At the same time, the mark reflects the personal effort. Thus, at a half-year math block you can collect 700 points. The daily assignments give 100 points based on neatness, completeness and correctness. Late delivery does not count. The final test counts 200 points; and 400 points come from five tests, of which the lowest is neglected.

The 700 points corresponds to 100%, and the characters A, B and C correspond to 90%, 80% and 70% of the points. A score below 70% means that the block must be retaken or be replaced by another block.

At 18 you can continue at a regional four-year college, or a local two-year community college, which is divided into quarters so it's easy to take blocks while you work or during summer holidays. Likewise, the block system makes it easier to change job in case of unemployment or a desire for new challenges.

But why don’t Europe do the same? Because Europe is so over-institutionalized, that it cannot imagine a society without institutions. And once you have chosen institutions, the school is used to create public servants through compulsory classes in primary school and in a myriad of compulsory lines at the secondary and university level.

And compulsory classes mean disappearance of the freedom to develop your personal potential. Instead school struggle with its well documented seven sins. Sins, Europe believes it can eliminate through its political system. If it has not died out before.

02. Mathematics, Banality or Evilness

Mathematics is steeped in evilness right from the first to the last class in the 12-year school, which we leave our children and young people to in the belief that the school will prepare them to master their environment and its two languages, the word language, and the number language called ‘math’ by the school. Strange, for we master our world through actions, by reading and writing and by counting and adding, so why is it necessary to learn to ' math'?

Thus the evilness of mathematics begins with its name; and by claiming that counting and adding are mere applications of mathematics, which, as such, of course, must first be learned before it can be applied; and which, unfortunately, is so difficult to learn, that it requires an extra effort leading still more to fail.

Also mathematics hides its origin. The ancient Greek Pythagoreans used the word as a common name for their four knowledge areas, music and stars and shapes and numbers, that constitutes ancient and medieval basic training, quadrivium, as recommended by the Greek philosopher Plato.

With music and astronomy out, today mathematics is just a common name for the two remaining areas, geometry, which in Greek means earth-measuring; and algebra, which in Arabic means to reunite numbers, and again hidden by the school, claiming instead that algebra means to search for patterns.

Algebra followed when the Renaissance replaced Roman numbers as CCXXXIV with the Arabic number 234 = 2 ten-tens and 3 tens and 4 ones = 2*10*10 + 3*10 + 4*1 showing algebra’s four ways to unite numbers. Addition unites unlike numbers such as 3 + 4. Multiplication unites like plus-numbers such as 3 + 3 + 3 + 3 = 3*4; power unites like multipliers such as 3*3*3*3 = 3^4; and the three number-blocks 200, 30 and 4 are united by next-to addition, also called times-plus calculation, or integration, the Latin word for uniting.

And blocks is exactly what children bring to school. Asking a three-year child "how old will you next time?" the answer is four with four fingers shown. But displaying four fingers held together two and two will prompt an immediate protest: "No, it's not four, that is two twos!"

So children come to school with two-dimensional block-numbers all carrying a unit, corresponding to Lego-blocks that stack as 1, 2, 3 or more 4ere. By combining geometry and algebra in their shapes and buds, blocks are highly suitable as a basis for connecting the starting point, children's block-numbers, with the final goal: algebra’s uniting block-numbers illustrated by geometrical shapes.

However, the school is ignoring this and instead it teaches one-dimensional line-numbers located on a number line with each their name; and where the system will only be visible in the late twenties, where many children count over by saying 'ten-and-twenty' instead of 'thirty'. This then allows the school to pass a dyscalculia-diagnose and to institutionalize a corresponding dyscalculia-treatment supported by a growing dyscalculia-research with an associated dyscalculia-industry.

Evilness occurs when the school itself installs dyscalculia in the child by teaching line-numbers instead of block-numbers, thus teaching today's two-dimensional Arabic numbers, used by communities and kids, as if they were one-dimensional ancient Roman numbers.

Both number systems count by bundling.

Roman numbers use linear bundling: in a row of sticks, 5 1s are bundled to a V, 2 V'er to an X, 5 X's to a L, 2 Ls to a C, and so on. So a Roman number remains a one-dimensional string of letters as I, V, X, L, C etc.

Arabic numbers use rectangular bundling: in a row of sticks, twelve 1s are bundled to 1 ten-bundle and 2 unbundled, written as 12. Bundles then stack to a block of e.g. 4 10s, until ten bundles of 10s create a new block with the unit ten-ten or hundred, which then again stack in a block until ten of them create the unit ten-ten-ten or one thousand, etc.

So, where Roman numbers never have units, Arabic numbers always have, just as in children's own number system.

Nevertheless, the school teaches only in numbers without units. Likewise, the school does not distinguish between 2 * 3 = 6 and 2+3 = 5. The former is always true since 2 3s can be recounted to 6 ones. The latter is true only if the omitted units are the same: 2 days + 3 days is 5 days, but 2 weeks + 3 days is 17 days, and 2 days + 3 weeks is 23 days. Mathema-tics without units should be called 'mathema-tism', something that is true inside, but seldom outside a classroom. This would allow seeing if its diagnoses are created by teaching mathematics as mathematism.

Its evilness begins when mathematics neglects children's own Arabic number system and impose on them a Roman number system. It continues by forcing children to add before counting; and by forcing upon children the four operations in the order addition, subttraction, multiplication and division, where the last is presented so difficult that it triggers new dyscalculia diagnoses.

It is in fact the opposite order that is the natural. We count by bundling, so 7 sticks are counted in 3s by removing 3s many times, which is division predicted by a calculator as '7/3 = 2. something'. Then the 2 3s are stacked, which is multiplication. Removing the stack to look for unbundled is subtraction, predicted by a calculator as '7 – 2*3 = 1'. So, the calculator prediction holds true: 7 = 2.1 3s. Which shows that a natural number is a decimal number with a unit where the decimal point separates bundles form the unbundled. In contrast to the school that writes 5.6 tens as 56, i.e. without a unit and with a misplaced decimal point, and even calls such a number a natural number. An effective way to create even more diagnoses.

So counting includes the three operations division, multiplication and subtraction, and in that order.

After counting, it is natural to learn re-counting, back-counting and double-counting to change unit, or to create or remove an overload occurring when removing or adding. Thus, 7 can be recounted in the same unit 3s with or without an overload as 1.4 3s or 2.1 3s.

Recounting in a new unit means asking e.g. ‘how many 4s is 2 3s?'. We get the answer by a manual recounting, or by asking the calculator for a prediction: 2*3/4 = 1. something and 2*3 – 1*4 = 2, so 2 3s = 1.2 4ere.

Recounting the tens is done by pure multiplication: 3 8ere = 3*8 = 24 = 2.4 tens.

Back-counting from tens leads to solving equations. The question '5 tens is how many 4ere?' becomes the equation 50 = 4*x. The solution is obtained by recounting 50 in 4s, x = 50/4. So an equation is just another word for a back-counting, which means using the opposite operation, i.e. moving a number to the opposite side with the opposite sign. A natural approach easy to understand.

But, again silenced by the school, instead postponing equations to later grade levels. Here equations are presented as examples of open statements expressing equivalence between two numbers-names, and which teachers learn to solve using an abstract neutralization method.

Double-counting in different colors leads directly to the most important numbers, 'per-numbers', used to change units: If 3 red corresponds to 4 blue then 5 red correspond to how many blue? Or later: If 3 kg cost 4 $ then what is the cost of 5 kg? To answer we use the per-number 4$/3kg to recount the kilo-number 5 in 3s, 5/3, so many times we must pay 4$.

Changing unit is one of the two core areas of mathematics. However, the school does not recognize words as re-counting, back-counting, double- counting, or per-numbers. Instead, it uses the word 'proportionality', and again postpones it to later grades and makes it so difficult that new diagnoses are issued.

Why must children not learn the different ways of counting already in pre-school, where they count by themselves, time after time? Why does the school hide the great advantages in counting before adding? After all, totals must be counted before they can be added?

In addition, addition is not well defined: Should two blocks be added on-top or next-to each other, also called integration, the Latin word for uniting?

On-top addition means recounting to a common unit. But the school insists on using a so-called carry-method, which creates new diagnoses.

At the same time, the school only works with totals counted in tens. It is therefore unnecessary to change unit and to do next-to addition, the second main area of mathematics, and therefore more important than on-top addition; and that can be learned as early as pre-school by posing Lego-blocks next to each other and ask '3 2s plus 5 4s total how many 6s?' Nevertheless, school postpones it to the last school year with the claim that only the very best can learn next-to addition.

Reversing next-to addition is called differentiation. It asks e.g. '3 2s plus how many 4s gives 7 6s?'. Here we first remove the 3 2s with a minus before we recount the rest in 4s by division. So in reversed next-to addition subtraction comes before division. Of course, for in next-to addition, multiplication comes before addition.

But, the school does not recognize the words next-to addition or reversed next-to addition, nor does it recognize the word times/plus calculation or minus/division calculation. Instead, it introduces the Latin words integral and differential calculus and postpone both to the upper-secondary level where they are presented in reversed order, i.e. reversed next-to addition before next-to addition. Which makes both hard to understand with a high failure rate as a consequence.

A sly way to sabotage any high school reform. The parliament would like everyone to learn forward and reversed next-to addition, but both teachers and their teachers, the university professors, protest loudly: It cannot be done!

Of course it can, you just need to teach what is in the world, blocks to be united or split, and in that order, i.e. integration before differentiation. It is that simple to make calculus accessible to all.

So if the school allowed children and young people to meet its root Many as it naturally occurs in the world, i.e. as block-numbers that are counted, re-counted, back-counted, and double-counted, to be added on-top or next-to and forward or reversed, then everyone would learn everything in mathematics.

However, then no longer can mathematics be used for exclusion, which is precisely the school's main task, according to the sociologist Bourdieu. We think we got rid of the nobility with its privileges, but instead of a blood-nobility we got a knowledge-nobility protecting its monopoly on today's most important capital form, knowledge capital, by using the school to exercise what he calls symbolic violence.

The word-language cannot be used for exclusion since it is learnt before school. In contrast to the number-language that school can make so hard that is will be accessible to the nobility's own children alone. In other words, the same technique as the mandarin class used when they made the Chinese alphabet so difficult that only their children could pass the state's official exams.

But why do teachers accept to teach evil mathematics? Because of the banality of evil as described by Arendt in her book about Eichmann in Jerusalem. Here Arendt points to the lurking evil stored in blindly following orders in institutions originally created to ensure that good thing happens.

To keep your job, you must obey orders, 'conform or die'. Institutions do not compete as does the private labor market where 'compete or die' ensures control by the users' needs.

Together with skeptical post-modern thinking, also Arendt finds inspiration in the last century’s great philosopher, Heidegger, who points out that to realize your existential potential you must have an authentic relationship with the surrounding things. To ensure this, we continually must ask if a thing’s true existence is shown or hidden by institutionalized essence claims.

So, as an institution, mathematics education should continually ask whether it mediates an authentic image of its subject, the physical fact Many. Or, whether the institution is caught in what the sociologist Baumann calls a ‘goal displacement’, where the initial goal is transformed into a subordinate instrument to a new target: the institution's self-preservation.

Mathematics education could be a framework for children's and young people's authentic meetings with its physical root, Many. Instead, it has become an attempt to cure self-created diagnoses.

To deal with Many is simple and banal, so why drown the banality of mathematics in evilness?

Sensory perception, experience and common sense are the worst enemies of evil mathematics. So practice existence before essence, also in mathematics education. Which instead should comply with the international PISA-intention: To equip a population with knowledge and skills for the realization of their individual potentials.

Consequently, please drop the evil mathematics. Allow the child to develop its existing number language through guided learning meetings with its root, Many. Remove the evil textbooks on line-numbers and addition before counting. Use blocks and playing cards to illustrate block-numbers and activities such as counting, re-counting, back-counting and double-counting followed by forward and reversed on-top and next-to addition; and swap differential and integral calculus in high school, so all young people learn next-to addition both forth and back.

Again, Luther is right: Contact can be established individually without an institutionalized intermediary.

03. CupCounting and Calculus in Early Childhood Education

To improve PISA results, institutional skepticism rethinks mathematics education to search for hidden alternatives to choices institutionalized as nature. Rethinking preschool and primary school mathematics uncovers cup-counting in bundles less than ten; as well as re-counting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, information and communications technology, a calculator can predict re-counting results before being carried out manually. By allowing overloads and underloads when re-counting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary classes only allowing ten-counting and on-top addition to take place.

Decreased PISA performance despite increased research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

> The highest performing education systems across OECD countries are those that combine excellence with equity. A thriving education system will allow every student to attain high level skills and knowledge that depend on their ability and drive, rather than on their social background. Sweden is committed to a school system that promotes the development and learning of all its students, and nurtures within them a desire for lifelong learning. PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD 2015, p. 3).

Created to help students cope with the outside world, schools institutionalize subjects as inside means to outside goals. To each goal there are many means, to be replaced if not leading to the goal; unless a means becomes a goal itself, thus preventing looking for alternative means that could lead to the real goal if difficult to access. So we can ask: Does mathematics education have a goal-means exchange seeing inside mathematics as the goal and the outside world as a means?

Once created as a means to solve an outside problem, not solving the problem easily becomes a means to necessitate the institution. So to avoid a goal/means exchange, an institution must be reminded constantly about its outside goal. Institutional skepticism is created to do precisely that.

Institutional skepticism

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by Plato philosophy presenting choices as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, symbolic interactionism and Grounded theory (Glaser & Strauss, 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget, 1970). In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnoses, and education all presenting patronizing choices as nature (Lyotard, 1984; Tarp, 2004).

Building on Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers ‘existence precedes essence, or (..) that subjectivity must be the starting point’ (Marino, 2004, p. 344). Kierkegaard was skeptical to institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone ‘may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.’ (Marino, 2004, pp. 186–187). Inspired by Heidegger, Arendt divided human activity into labor and work aiming at survival and reproduction, and action focusing on politics, creating institutions to be treated with utmost care to avoid the banality of evil by turning totalitarian (Arendt, 1963).

Since one existence gives rise to many essence-claims, the existentialist distinction between existence and essence offers a perspective to distinguish between one goal and many means.

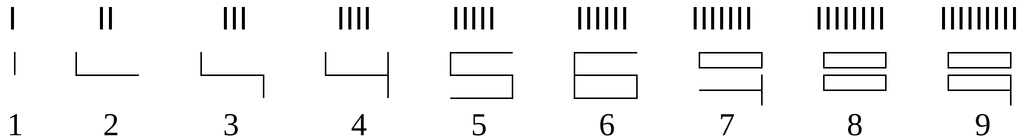
Mathematics as essence

In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra (Freudenthal, 1973) both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite Many’ in Arabic.

Then the invention of the concept SET allowed mathematics to be a self-referring collection of ‘well-proven’ statements about ‘well-defined’ concepts, i.e. as ‘MetaMatics’, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract essence. Thus SET transformed grounded mathematics into a self-referring ‘MetaMatism’, a mixture of MetaMatics and ‘MatheMatism’ true inside a classroom but not outside where claims as ‘1 + 2 IS 3’ meet counter-examples as e.g. 1 week + 2 days is 9 days. And, as expected, teaching numbers without units and meaningless self-reference creates learning problems.

Mathematics as existence

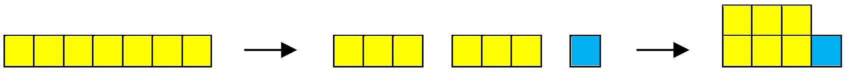
Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry. Algebra contains four ways to unite as shown when writing out fully the total T = 342 = 3*B^2 + 4*B + 2*1 = 3 bundles of bundles and 4 bundles and 2 unbundled singles = 3 blocks. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reversing splitting operation. So, with a human need to describe the physical fact Many, algebra was created as a natural science about Many. To deal with Many, we count by bundling and stacking. But first we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written less sloppy. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc.



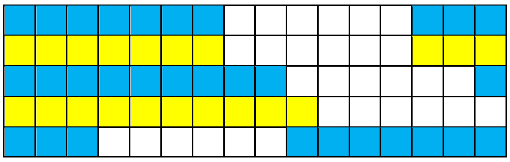
Holding 4 fingers together 2 by 2, a 3year old child will say ‘That is not 4, that is 2 2s. This inspires ‘cup-counting’ bundling a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1 where the bundles are placed in a bundle-cup with a stick for each bundle, leaving the unbundled outside. Then we describe by icons, first using ‘cup-writing’, T = 2]1, then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Moving a stick outside or inside the cup changes the normal form to overload or underload form. Also, we can use plastic letters as B and C for the bundles.

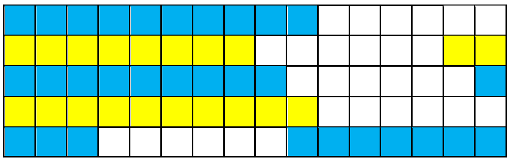
T = 7 = IIIIIII → III III I → II] I → 2]1 3s = 1]4 3s = 3]-2 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a cup-number or a decimal number, T = 7 = 2 3s & 1 = 2]1 3s = 2.1 3s.



We live in space and in time. To include both when counting, we introduce two different ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.





To predict the result we use a calculator. A stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s. As for the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once.

Thus by entering '7/3' we ask the calculator 'from 7 we can take away 3s how many times?' The answer is '2.some'. To find the leftovers we take away the 2 3s by asking '7 – 2*3'. From the answer '1' we conclude that 7 = 2]1 3s. Likewise, showing '7 – 2*3 = 1', a display indirectly predicts that 7 can be recounted as 2 3s and 1, or as 2]1 3s.

7 / 3	2.some
7 – 2 * 3	1

A calculator thus uses a 'recount-formula', T = (T/B)*B, saying that 'from T, T/B times Bs can be taken away'; and a 'restack-formula', T = (T–B)+B, saying that 'from T, T–B is left if B is taken away and placed next-to'. The two formulas may be shown by using LEGO blocks.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3]2 2s with an outside overload; or as 5]-2 2s with an outside underload thus leading to negative numbers:

Letters	Sticks	Total T =	Calculator	
B B B B	II II II II	4]0 2s	4*2 – 4*2	0
B B B I I	II II II I I	3]2 2s	4*2 – 3*2	2
B B B B B B	II II II II II II	5]-2 2s	4*2 – 5*2	-2

To recount in a different unit means changing unit, also called proportionality or linearity. Asking '3 4s is how many 5s?' we can use sticks or letters to see that 3 4s becomes 2]2 5s.

IIII IIII IIII → IIIII IIIII I I → 2]2 5s. With letters, C = BI so that BBB → BB IIII → CC II

A calculator can predict the result. Entering '3*4/5' we ask 'from 3 4s we take away 5s how many times?' The answer is '2.some'. To find the leftovers we take away the 2 5s and ask '3*4 – 2*5'. Receiving the answer '2' we conclude that 3 4s can be recounted as 2 5s and 2, or as 2]2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘6$ = ?kg’ we use the per-number to recount 6 in 2s: 6$ = (6/2)*2$ = 3*3kg = 9kg. And vice versa: Asking ‘?$ = 12kg’, the answer is 12kg = (12/3)*3kg = 4*2$ = 8$.

Once counted, totals can be added on-top or next-to

Asking ‘3 5s and 2 3s total how many 5s?’ we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s as 1]1 5s that added to the 3 5s gives a total of 4]1 5s.

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4]1 5s. With letters: 3B + 2C = 3B III III = 4BI.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. Thus we get 4]1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2]5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2]5 8s → 2.5 8s

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2]5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing adding on-top and next-to

Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2]6 8s?’, using sticks will give the answer 2]1 3s:

IIIII IIIII IIIII III III I ← IIIII III] IIIII III] IIIIII ← 2]6 8s

Using a calculator to predict the result the remaining is bracketed before counted in 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration means subtracting before dividing, as shown in the gradient formula $y' = \Delta y/t = (y2 – y1)/t$.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Primary schools use ten-counting only

In primary school numbers are counted in tens to be added, subtracted, multiplied and divided. This leads to questions as ‘3 4s = ? tens’. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount- and restack-formula above is impossible since the calculator has no ten button. Instead it is programmed to give the answer directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

3 * 4	12

Recounting icon-numbers in tens is called doing times tables to be learned by heart. So from grade 1, 3*4 is not 3 4s any more but has to be recounted in tens as 1.2 tens, or 12 in the short form.

Recounting tens in icons by asking ‘38 = ? 7s’ is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens, 0.3 7s must be written in fraction form as 3/7 and calculated as 0.428…, shown directly by the calculator, 38/7 = 5.428…

38 / 7	5.some
38 – 5 * 7	3

Without recounting, primary school labels the problem ‘38 = ? 7s’ as an example of a division, 38/7, which is hard to many, or as an equation ’38 = x*7’ to be postponed to secondary school.

Designing a micro-curriculum

With curriculum architecture as one of its core activities, the MATHeCADEMY.net was asked to design a micro-curriculum understandable and attractive to teachers stuck with division problems; and allowing special need students to return to their ordinary class. Two were designed.

In the ‘1 cup and 5 sticks’ micro-curriculum, 5 is cup-counted in 2s as 1]3 2s or 2]1 2s or 3]-1 2s to show that a total can be counted in 3 ways: overload, normal or underload with an inside and an outside for the bundles and singles. So to divide 336 by 7, 5 bundles are moved outside as 50 singles to recount 336 with an overload: 336 = 33]6 = 28]56, which divided by 7 gives 4]8 = 48.

Besides the ‘Cure Math Dislike by 1 cup and 5 sticks’, 8 extra micro-curricula were designed (mathecademy.net/preschool/) where cup-counting involves division, multiplication, subtraction and later next-to and on-top addition, in contrast to primary school that turns this order around and only allows on-top addition using carrying instead of overloads. Thus, if using cup-writing with overloads or underload instead of carrying, the order of operations can be turned around to respect that totals must be counted before being added.

	Carry	Bundle-writing	Words
Add	1 4 5 <u>1 7</u> 6 2	4]5 <u>1]7</u> 5]12 6]2 = 62	4 ten 5 <u>1 ten 7</u> 5 ten 12 5 ten 1 ten 2 6 ten 2 = 62
Subtract	1 4 5 <u>1 7</u> 2 8	4]5 <u>1]7</u> 3]-2 2]10-2 = 2]8 = 28	4 ten 5 <u>1 ten 7</u> 3 ten less2 2 ten 8 = 28
Multiply	4 <u>2 6 * 7</u> 1 8 2	7 * 2]6 14]42 18] 2 = 182	7 times 2 ten 6 14 ten 42 14 ten 4 ten 2 18 ten 2 = 182
Divide	<u>2 4 rest 1</u> 3 | 7 3 <u>6</u> 1 3 <u>1 2</u> 1	7]3 counted in 3s 6]13 6]12 + 1 2 3s]4 3s + 1 24 3s + 1 73 = 24*3 + 1	7ten3 6ten 13 6ten12 + 1 3 times 2ten4 + 1 3 times 24 + 1

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using a cup for the bundles; and reporting first with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and underloads. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other. In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eights, the learner reverses next-to addition. Finally, the learner sees how double-counting creates per-numbers.

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2]1 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	'2.7 5s is also how many 5s?' IIIII IIIII IIIIIII = V V V II = V V V V III 2]7 = 2+1]7–5 = 3]2 = 3+1]2–5 = 4]-3 So 2.7 5s = 3.2 5s = 4.-3 5s	(2*5+7)/5 (2*5+7) – 3*5 (2*5+7) – 4*5	3.some 2 -3
M4	2 5s is how many 4s?' IIIII IIIII = IIII I IIII I = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2*4	2.some 2
M5	'2 5s and 4 3s total how many 5s?' IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2
M6	'2 5s and 4 3s total how many 8s?' IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 (2*5+4*3) – 2*8	2.some 6
M7	'2 5s and ? 3s total 4 5s?' IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1
M8	'2 5s and ? 3s total how 2.1 8s?' IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(2*8+1 – 2*5)/3 (2*8+1 – 2*5) – 2*8	2.some 1

One curriculum used silent education where the teacher demonstrates and guides through actions only, not using words; and one curriculum was carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator quickly took over the communication. For further details watch www.youtube.com/watch?v=IE5nk2YEQIA.

After these micro-curricula a learner went back to her grade 6 class where proportionality created learning problems. The learner suggested renaming it to double-counting but the teacher insisted on following the textbook. However, observing that the class took over the double-counting method, he finally gave in and allowed proportionality to be renamed and treated as double-counting. When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integration.

Thus cup-counting and a calculator for predicting recounting results allowed the learner to reach the outside goal, mastering Many, by following an alternative to the institutionalized means that because of a goal-means exchange had become a stumbling block to her; and performing and reversing next-to addition introduced her to and prepared her for later calculus classes.

Literature on cup-counting

No research literature on cup-counting was found. Likewise, it is not mentioned by Dienes (1964).

Conclusion and recommendation

As to theory, two genres exist; a master genre exemplifying existing theory, and a research genre developing new theory by including a question and a theoretical guidance to a valid answer based upon analyzing reliable data. To avoid indifference, this paper addresses the OECD report 'Improving schools in Sweden' by asking if mathematics education might have a goal-means exchange. As theoretical guidance, institutional skepticism allows using the existentialist existence-versus-essence distinction to distinguish outside goals from inside means, which leads to asking when mathematics is respectively existence and essence. Analyzing traditional math shows that by being set-based and by adding numbers without units, its concepts and statements are unrooted and little applicable to the outside world, thus being primarily essence. Then grounded theory helps showing how mathematics looks like if grounded in its physical root, Many. To tell the difference, two names are coined, 'ManyMatics' versus 'MetaMatism' mixing 'MetaMatics' defining concepts as examples of abstractions instead of as abstractions from examples, with 'MatheMatism' valid only inside classrooms. To validate its findings and again to avoid indifference, the paper includes a classroom test of a micro curriculum described in details to allow it to be tested in other classrooms. Its originality should welcome the paper for publishing since no literature on ManyMatics exists.

So, if a research conference fails to accept the paper for presentation or as a poster, an extra exchange can be added to help solving the paradox that the Swedish problems occur despite increased research and funding: Neglecting a genre analysis might exchange the master and research genres with the consequence that peer-review becomes unable to accept groundbreaking new paradigms. Such research conferences include master papers that, although career promoting, are unable to uncover alternative, hidden ways to guide solving problems in mathematics education.

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04. Fifty Years of Research without Improving Mathematics Education, Why?

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by the PISA studies organised by the Organisation for Economic Co-operation and Development, OECD, showing a low level and a continuing decline in many countries. Thus, to help the former model country Sweden, OECD wrote a critical 2015 report ‘Improving Schools in Sweden, an OECD Perspective’: “PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.”

Researchers in mathematics education meet in different fora. On a world basis, the International Congress on Mathematical Education, ICME, meets each four year. And on a European basis, the Congress of the European Society for Research in Mathematics Education, CERME, meets each second year.

At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is “Only one: to improve, mathematics education should ask, not what to do, but what to do differently.” Thus, to be successful, research should not study problems but look for hidden differences that might make a difference. Research that is skeptical towards institutionalized traditions could be called difference research or contingency research or Cinderella research making the prince dance by looking for hidden alternatives outside the ruling tradition. The French thinker Lyotard calls it ‘paralogy’ inventing dissension to the reigning consensus. Difference research scarcely exists today since it is rejected at conferences for not applying or extending existing theory that is able to produce new researchers and to feed a growing research industry, but unable to reach its goal, to improve mathematics education.

To elaborate, mathematics education research is sterile because its three words are not well defined.

As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist: a continental European education serving the nation’s need for public servants though multi-year compulsory classes and lines at the secondary and tertiary level; and a North American education aiming at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers.

As to research, academic articles can be written at a master level applying or exemplifying existing theories, or at a research level questioning them. Just following ruling theories is especially problematic in the case of conflicting theory as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible respectively.

Consequently, you cannot know what kind of mathematics and what kind of education has been studied, and you cannot know if research is following ruling traditions or searching for new discoveries. So, seeing education as an institutional help to children and youngsters master outside phenomena leads to the question: What outside phenomena roots mathematics?

The Outside Roots of Mathematics

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep balance and to store sounds assigned to what we grasped with our forelegs, thus providing the holes in the head with our two basic needs, food for the body and information for the brain.

The sounds developed into languages. In fact, we have two languages, a word-language and a number-language. Children learn to talk and to count at home. Then, as an institution, school takes over and teaches children to read and to write and to calculate.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair'. Observing the existence of many chairs, we ask 'how many totally?' and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, 'the total is 3 chairs' or, if counting legs, 'the total is 3 fours', which we abbreviate to 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is an operation'.

And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

And since we master outside phenomena through actions, learning the word-language means learning actions as how to listen, to read, to write and to speak. Likewise, learning the number-language means learning actions as how to count and to add. You cannot learn how to math, since math is not an action word, it is a label as is grammar. Thus, mathematics may be seen as the grammar of the number-language.

Using the phrasing 'the number-language is an application of mathematics' implies that then 'of course mathematics must be taught and learned before it can be applied'. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

So, one way of improving mathematics education is to respect that language comes before meta-language. Which was also the case in continental Europe before the arrival of the 'New Math' that turned mathematics upside down to become a 'meta-matics' presenting its concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically and which would present mathematics as 'many-matics', a natural science about Many.

Thus, Euler defined a function as a common name for calculations with unspecified numbers, in contrast to calculations without that could be calculated right away without awaiting numbers to be specified. Defining all concepts as examples of the mother concept set, New Math turned a function into an example of a set-product where first-component identity implies second-component identity, which learners heard as 'bublibub is an example of bablibab'.

Before New Math, Germanic countries taught counting and reckoning in primary school. Then the lower secondary school taught algebra and geometry, which are also action words meaning to reunite totals and to measure earth in Arabic and in Greek. 50 years ago, New Math made all these activities disappear. This means that what research has studied is problems coming from teaching how to math. So, one alternative presents itself immediately: Forget about New Math and, once again, teach mathematics as rooted in numbers and reckoning and reuniting totals and measuring earth.

Re-rooting mathematics resonates with its historic origin as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many.

As to New Math, its idea of deriving definitions from the mother concept set leads to meaningless self-reference as in the classical liar paradox 'This sentence is false', being true if false and false if true. This was shown by Russell looking at the set of sets not belonging to itself. Here a set belongs to the set if it doesn't, and does not belong if it does.

To avoid self-reference, Russell created a hierarchical type theory in which fractions could not be numbers if defined by numbers as done by New Math defining fractions as equivalence classes in a set of number-pairs. Insisting that fractions are numbers, New Math invented a new set-theory that by mixing sets and elements also mixes concrete examples and their abstract names, thus mixing concrete apples that can feed humans and the word 'apple' that cannot. By mixing things and their names, New Math and its meta-matics ceases to be a language about the real world. Still, it has entered universities worldwide as the only true version of mathematics.

So, to improve its education, mathematics should stop teaching top-down meta-matics from above and begin teaching bottom-up many-matics from below instead.

Rethinking Mathematics from Below

To improve it we must rethink mathematics. To rethink we seek guidance by one of the greatest thinkers of the 20th century, Heidegger, being very influential within existentialist thinking and French skeptical post-structural thinking.

Heidegger holds that to exist fully means to establish an authentic relationship to the things around us. To allow a thing to open its 'Wesen' and escape its gossip-prison created by reigning essence-claims we must use constant questioning. So, returning to the fundamental goal of education, preparing humans for what is outside, we must keep on asking to the Wesen of the root of the number language, the physical fact Many, and allow Many to escape from its New Math gossip, 'Gerede'.

With 2017 as the 500year anniversary for Luther's 95 theses, we can describe meeting Many in theses.

1. Using a folding ruler we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent.

2. Using a cup for the bundles we discover that a total can be 'cup-counted' in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and 'less 1' outside; or, if using 'cup-writing' to report cup-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s. We discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s.

On a folding ruler, distances are counted in tens. Here one centimeter is a bundle of ten millimeters, and ten centimeters gives a bundle of one decimeter. If the length of a hand is counted to 6 strokes after 1.7 tens, we write the length as T = 1.76 tens centimeters = 17.6 centimeters leaving the 6 unbundled millimeters outside.

3. Using recounting a total in the same unit by creating or removing overloads or underloads, we discover that bundle-writing offers an alternative way to perform and write down operations:

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92 ; and T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336 ; and T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

4. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved in counting by bundling and stacking. To count 7 in 3s we take away 3 many times iconized by an uphill stroke showing the broom wiping away the 3s. Showing

7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, dividing and subtracting a multiple, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate e.g. T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.

5. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. A ‘recount formula’ T = (T/B)*B saying that T/B times B can be taken away from T, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and a ‘restack formula’ T = (T–B)+B saying that T–B is left when B is taken away from T and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Thus we discover the nature of formulas: formulas predict.

6. Wanting to recount a total in a new unit, we discover that again a calculator can predict the result by bundling and stacking and dragging away the stack:

T = 4 5s = ? 6s. First (4*5)/6 = 3.some. Then (4*5) – (3*6) = 2. Finally T = 4 5s = 3.2 6s

Also, we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.

7. Wanting to recount a total in tens, we discover that a calculator can predict the result directly by multiplication. Only, the calculator leaves out the unit and misplaces the decimal point:

T = 3 7s = ? tens. Answer: T = 21 = 2.1 tens

Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.

And wanting to recount a total from tens to icons, we discover that this again is an example of recounting to change the unit:

T = 3 tens = ? 7s. First 30/7 = 4.some. Then 30 – (4*7) = 2. Finally T = 30 = 4.2 7s

Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged

8. Using the letter u for an unknown number, we can rewrite recounting from tens, e.g. 3 tens = ? 7s, as 30 = u*7 with the answer 30/7 = u. Officially this is called to solve an equation, so here we discover a natural way to do so: Move a number to the opposite side with the opposite sign. The equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, thus predicted by the restack-formula as 8 = (8–2)+2. Thus, the equation 8 = u + 2 has the solution is 8–2 = u, again moving a number to the opposite side with the opposite sign.

9. Once counted, totals can be added. But we discover that addition is not well defined. With two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to get the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. Thus next-to addition geometrically means to add areas, and algebraically it means to combine multiplication and addition. Officially this is called integration, a core part of traditional mathematics in high school and at the first year of university.

10. Also we discover that addition can be reversed. Thus, the equation above restacking 8 by moving 2, 8 = u+2, can also be read as reversed addition: u is the number that added to 2 gives 8,

which is precisely the formal definition of u = 8–2. So, we discover that subtraction is reversed addition. And, again we see that the equation u+2 = 8 is solved by u = 8–2, i.e. by moving to the opposite side with the opposite sign. Likewise, the equation recounting 8 in 2s, 8 = u*2, can be read as reversed multiplication: u is the number that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2? So, we discover that division is reversed multiplication. And, again we see that the equation u*2 = 8 is solved by u = 8/2, i.e. by moving to the opposite side with the opposite sign. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root and logarithm as u = 3√20 and u = log3(20). So, again we solve the equations by moving to the opposite side with the opposite sign. Reversing next-to addition, we can ask e.g. 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer, first we remove the initial total T1, then we count the rest in 5s: u = (T–T1)/5. Combining subtraction and division in this way is called differentiation. By observing that this is reversing multiplication and addition we discover that differentiation is reversed integration.

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

12. Observing that a quantity may be double-counted in the same unit, we discover that per-numbers may take the form of fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. Thus, to find 3 per 5 of 20, 3/5 of 20, we just recount 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3. To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. Thus, calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

Writing out a total T as we say it, T = 345 = 3*ten*ten + 4*ten + 5*1, shows a number as blocks united next-to each other. Also, we see algebra's four ways to unite numbers: addition, multiplication, repeated multiplication or power, and block-addition also called integration. Which is precisely the core of mathematics: addition and multiplication together with their reversed operations subtraction and division in primary school; and power and integration together with their reversed operations root, logarithm and differentiation in secondary school. Including the units, we see there can only be four ways to unite numbers: addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers.

How School Teaches Mathematics

Before addressing how school guides children on their way to mastering Many let us look at the number-language children bring to school. Asking a three-year old child "how old will you next time?" the answer is four with four fingers shown. But displaying four fingers held together two and two will prompt an immediate protest: "No, that is not four, that is two twos!"

So, children come to school with two-dimensional 'block-numbers' all carrying a unit, corresponding to LEGO-bricks that stack as 1, 2, 3 or more 4s. Thus, by combining geometry and algebra in their shapes and knobs, they are an excellent basis for connecting the starting point, children's block-numbers, with the final goal, the Arabic numbers also being a collection of blocks of 1s, tens, ten-tens etc.

To emphasize that we count by bundling and stacking, the school could tell children that eleven and twelve is a special ‘Viking-way’ to say ten-1 and ten-2. Then they probably would count ‘2ten9, 3ten, 3ten1’ instead of saying ‘ten-and-twenty’ and risk being diagnosed with dyscalculia. In Danish, eleven and twelve mean ‘one left’ and ‘two left’, implying that the ten-bundle has been counted already. And, except from some French additions because of the Norman conquest, English is basically Anglish, a dialect from Harboøre on the Danish west coast where the ships left for Angland.

Now let us see how school prepare children and youngsters to meet Many by offering them what is called mathematics education. Again, we use the form of theses.

1. School could respect the origin of the word mathematics as a mere name for algebra and geometry both grounded in the physical fact Many and created to go hand in hand. Instead, school teaches mathematics as a self-referring ‘meta-matics’ defining concepts as examples of abstractions, and not as abstractions from examples. Likewise, school teaches algebra and geometry separately.

2. School could respect that a digit is an icon containing as many sticks as it represents. Instead, school presents numbers as symbols like letters. Seldom it tells why ten does not have an icon or why ten is written as 10; and seldom it tells why ten1 and ten2 is called eleven and twelve.

3. School could follow the word-language and use full sentences ‘The total is 3 4s or T = 3 4s or T = 3*4’. Instead, by only saying ‘3’, school removes both the subject and the unit from number-language sentence, thus indicating that what children should learn is not a number-language but a one-dimensional number system claimed to be useful later when meeting life’s two-dimensional numbers.

4. School could develop the two-dimensional block-numbers children bring to school and are supposed to leave school with. Instead, school teaches its one-dimensional line-numbers as names for the points along a number line, using a place-value system. Seldom numbers are written out as we say them with the unit ones, ten, ten-tens, etc. Seldom a three-digit number is taught as a short way to report three countings: of ones, of bundles, and of bundles of bundles. Seldom tens is called bundles; seldom hundreds is called ten-tens or bundles of bundles.

5. School could respect that a number is a horizontal union of vertical blocks of 1s, bundles, bundles of bundles etc., and that counting-on means going up one step in the 1-block until we reach the bundle level where a bundle of 1s is transformed into 1 extra bundle making the bundle block go up 1 while the 1-block falls back to zero; and school could respect that a natural number is a decimal number with a unit. Instead school represents numbers by a horizontal number-line, where counting-on means moving one step to the right and where a natural number is presented without unit and with a misplaced decimal point.

6. School could respect that totals must be counted and sometimes recounted in a different unit before being added. Instead, without first teaching counting, school teaches addition from the beginning regardless of units, thus transforming addition to mere counting-on. Seldom school teaches real on-top and next-to addition respecting the units.

7. School could respect that also operations are icons showing the three basic counting activities: division as bundling, multiplication as stacking the bundles, and subtraction as removing the stack to look for unbundled singles; and school could respect the natural order of operations: division before multiplication before subtraction before addition. Instead school reverses this order without respecting that addition has two meanings, on-top and next-to, or that division has two meanings, counted in and split between.

8. School could respect that 3*8 means 3 8s that may or may not be recounted in tens. Instead school insists the 3*8 IS 24 and asks children to learn the multiplication tables by heart. Seldom the geometrical understanding is included showing that recounting in tens means the stack increases its width and therefore must decrease its height to leave the total unchanged.

9. School could respect that basic calculations become understandable by recounting a total in the same unit to create or remove over- or underloads. Instead school does not allow over- and underloads and insists on using specific algorithms with a carry-technique.

10. School could respect that proportionality is just another word for per-numbers coming from double-counting, and that per-numbers are operators that need a number to become a number. Instead school renames per-numbers to fractions, percentages and decimal numbers and teach them as numbers that can be added without considering the unit, and teaches proportionality as an example of a linear function, which isn't linear since the b in y = a*x+b makes it an affine function instead.

11. School could respect that equations are just another name for reversed calculation rooted in recounting tens in icons and solved by moving to the opposite side with the opposite sign. Instead school teaches equations as statements expressing equivalence between two different number-names to be solved by performing the same operation to both sides aiming at using the laws of abstract algebra to neutralize the numbers next to the unknown.

12. School could respect that integrating means adding non-constant per-numbers to be taught in primary school as next-to addition of block-numbers, and in middle school as mixture tasks; and respect that reversed integration is called differentiation made relevant since adding many differences boils down to one single difference between the end- and start-number. Instead school neglects primary and middle school calculus; and it teaches differentiation before integration, that is reduced to finding an antiderivative to the formula to be integrated. Seldom continuity and differentiability are introduced as formal names for local constancy and local linearity. Seldom the units are included to make clear that per-numbers are integrated, and that differentiation cerates per-numbers.

How School Could Teach Mathematics

Seeing the goal of mathematics education as preparing students for meeting Many, doing so in a Heideggerian gossip-free space offers many differences to be tried out and studied. Again, we use a list form.

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to teach bundle-counting, recounting in the same and in a different unit, calculator prediction, on-top and next-to addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling and tripling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to Here a difference is to talk about 8/2 as ‘8 counted in 2s’ instead of as ‘8 divided between 2’; and to rewrite the number as ‘10 or 5 times less something’ and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 33-5B6+50 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of ‘mathe-matism’ true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes ‘middle school integration’ later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of reverse operations to establish the basic rule for solving equations ‘move to the opposite side with the opposite sign’: In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn’t have a reverse sign. This gives 3*u = 14 – 2. Since u doesn’t have a reverse sign, 3 moves to the other

side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. ‘transform the formula T = a/(b-c) so that all letters become subjects.’ A hymn can be created: “Equations are the best we know / they’re solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don’t want to stop.”

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to ‘pre-dict’ a number, instead statistics can ‘post-dict’ its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5th root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times

gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio Δy/Δx can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children's own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched, and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden

learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format.

The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial:

T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, with at least two alternative meanings for all three words, at least 2*2*2 i.e. 8 different forms of mathematics education research exist. The past 50 years has shown the little use of the present form applying theory to study meta-matics taught in compulsory multi-year classes or lines. So, one alternative presents itself directly as an alternative for future studies: to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many, and to teach it in self-chosen half-year block at the secondary and tertiary level; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining and trying to understand the misery coming from teaching meta-matism in compulsory classes. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research must search for differences and test if they make a difference, not in compulsory classes, but with daily lessons in self-chosen half-year blocks. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a sentence, the subject exists, but the sentence about it may be gossip; so, stop teaching essence and start experiencing existence.

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05. A 1year pre-engineer course for Young migrants, a job for critical or civilized math education

UN population forecasts predict a continuing migrant flow to Europe to benefit from its socialist welfare and educational systems. But a critical question could ask: Is European education ready to benefit from the engineer potential in young migrants allowing them to build up welfare and education in their own country? Is critical socialist thinking able to reform its European line-organized office directed education dating back to the Napoleon wars? A recent OECD report saying that Sweden should urgently reform its school system to improve quality and equity suggests that a solution might instead be provided by the civilized thinking of the North American Enlightenment republics, historically created to receive and integrate migrants through its half-year block-organized talent developing education.

Background and question

According to the numbers of hours spend there, education is by far the most extensive public intervention in private life; and with the basic human need for a word- and a number-language for communication, mathematics is one of its core subjects. Consequently, research in math education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Centre for Mathematics Education in Sweden that with its positive attitude to receiving male migrants now beats China with 123 boys/ 100 girls of the 16-17 years old. However, despite increased research and funding, Sweden has seen its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change' (OECD, 2015).

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life." (p. 3)

In the report OECD writes

> Sweden has the highest percentage of students arriving late for school among all OECD countries, especially among socio-economically disadvantaged and immigrant students, and the lack of punctuality has increased between 2003 and 2012. There is also a higher-than-average percentage of students in Sweden who skip classes, in particular among disadvantaged and immigrant students. Arriving late for school and skipping classes are associated highly negatively with mathematics performance in PISA and can have serious adverse effects on the lives of young people, as they can cut into school learning and also distract other students. (p. 69) The reforms of recent years are important, but evidence suggests they are also somewhat piecemeal, and simply too few, considering the serious situation of the Swedish school system. (p. 55) Sweden faces a serious deterioration in the quality and status of the teaching profession that requires immediate system-wide attention. This can only be accomplished by building capacity for teaching and learning through a long-term human resource strategy for the school sector. (p. 112)

Inspired by the OECD report we can ask: How to improve mathematics and its education to better serve the population and migrants? And more specifically: How to design a 1year pre-engineer course for young migrants beginning from scratch?

Critical and civilized thinking provide two kinds of answers.

Critical and civilized thinking

As to the content of critical thinking, the Oxford Dictionary of Philosophy writes:

> The title is specifically applied to the philosophical approach of the Frankfurt school. This owed its philosophical background to Hegel and to Marx, seeing social and cultural imperfections as defects of rationality, and comparing them with an ideal to which the progress of reason, embodied in pure and undistorting social arrangements, would ideally tend (pp. 88-89)

Civilized thinking mixes existentialism, seeing existence as preceding essence, with the thinking of the two Enlightenment republics, American pragmatism being skeptical towards any philosophical is-claim, and French post-structuralism warning against hidden patronization in choices presented as nature. But to more clearly see the difference between the two we need to go back in history.

A historical background

The distance from its energy source allows water in all three forms: solid, liquid and gas. Thus a continuous flow of incoming high order energy from the sun and outgoing low order waste energy to space during the night allows green cells to store energy to be exploited by grey cells coming in three forms: reptiles, mammals and humans. That by standing up allowed the brain to develop language by remembering sounds given to what the forelegs transformed to hands was grasping. Thus meeting the two fundamental needs shown by the holes in the head: to supply the stomach and the lungs and the brain with food and oxygen and information.

When humans left Africa some went east to the fertile river valleys, some went west to the mountains. Trade took place exchanging eastern silk and pepper with western silver. Its silver mines allowed ancient Greece to develop a culture where men could leave the daily routine work to women and slaves to discuss social matters as 'can adults live together on equal terms or is patronization needed as with children?'

Social theory thus has human interaction as its main focus. As to communication, the most basic interaction, Berne (1964) has developed a transactional analysis describing three different ego-states called Parent and Adult and Child to reflect the social fact that human interaction can be patronized and non-democratic, or it can be non-patronized and democratic. In a family the interaction between children and parents will typically be one of patronization. In a society adult interaction typically will be non-patronized, unless the society is a non-democratic autocracy where patronization is carried on into adulthood. In this way Berne describes the main problem in human interaction, the choice between patronization and self-determination or 'Mündigkeit'. The fact that the German word 'Mündigkeit' does not have an English equivalent indicates that social interaction is quite different outside continental Europe and inside where the presence of and resistance against patronization created the label 'Mündigkeit'.

The debate on patronization runs all the way though the history of social theory (Russell, 1945). In ancient Greece the sophists warned against hidden patronization coming from choices presented as nature. Hence to practice the three ingredients of a democracy, information and debate and decision, a population should be enlightened to tell choice from nature. Seeing the physical as examples of metaphysical forms only visible to philosophers from his academy, Plato labelled democratic debate as ignorance. Instead social power should be given to the philosophers who could make wise decisions based upon information coming from insight and knowledge, thus needing neither debate nor democracy. In this way Plato instituted the patronization that Foucault calls 'pastoral power' to be continued first by the Christian church and later by modern universities still using the scholastic research method of only allowing late opponents to already defended texts to be accepted as researchers.

The Greek silver mines lasted about hundred years. Then the Romans took over, financing their empire by silver mines in Spain, eventually captured by the Vandals and by the Arabs. The lack of silver made Europe descend into the dark Middle Age. Here the patronization question reappeared in the controversy on universals between the realists and the nominalists. The realist took the Plato standpoint by renaming his metaphysical forms to universals claimed to have independent existence and to be exemplified in the physical world, and consequently waiting to be discovered by philosophers. In contrast to this the nominalist saw universals as names invented to facilitate human interaction.

Then German silver transported to Italy reopened east-west trade financing the Renaissance, seeing a protestant uprising against the patronization of the Roman Catholic Church resulting in the bloody

30year war from 1618. To avoid the chaos of war, Hobbes in his book ‘Leviathan’ argued that to protect themselves against their natural egoistic state, humans would have a much better life if accepting the patronization of an autocratic monarch.

Seeing the laboratory as preceding the library, Brahe retrieved data for the motion of planets, which together with Kepler’s interpretation allowed Newton to discover that the moon doesn’t move among the stars, instead it falls towards the earth as does the apple, both following their own physical will and not the will of a metaphysical patronizor. This inspired Locke to argue against patronization. His chief work, the ‘An Essay Concerning Human Understanding’, was highly inspirational in the Enlightenment 1700-century, which resulted in two democracies being installed, one in the US and one in France. American sociology sees human interaction as based upon enlightenment and freed from patronization. Its ‘it is true if it works’ pragmatism expressed by Peirce and James leads on to symbolic interactionism and to the natural observation rooted research paradigm Grounded Theory resonating with the principles of natural learning expressed by Piaget. In harmony with this, the US enlightenment school, being organized in half-year blocks and aiming at developing the talent of the individual has set the international standard followed worldwide outside Europe.

Inside Europe counter-enlightenment came from Germany where Hegel reinstalled metaphysical patronization in the form of a Spirit expressing itself through the history of the people. Trying to end the French Republic by war resulted in French occupation of Berlin. To get Napoleon out, the king realized that as the French he could no more depend on the blood nobility. So he asked Humboldt to use Hegel to design a line-organized Bildung education with three goals: Bildung must not enlighten to keep the population from demanding democracy as in France; instead, by imposing upon it a feeling of nationalism, Bildung should transform the population into a people following the will of the Spirit by fighting other people, especially the French. Finally Bildung should use the Sprit expressing itself in romanticism to sort out a knowledge nobility among the people for a central administration (Berglar, 1970).

Opposing Hegel, Nietzsche argued that only by freeing itself from metaphysical philosophical hegemony, western individuals would be able to realize their full potentials. Following Hegel, Marx claimed that until a socialist utopia has been established, a socialist party serving the interest of the working people should patronize people through a dictatorship of the proletariat. Once in power, Hegel-based socialism saw no reason to replace the Hegel-based counter-enlightening line-organized education with the enlightening block- organized education of the American republics. Marxist thinking developed into the critical theory of the Frankfurter school infiltrating the 1968 student revolt to secure that Europe’s Bildung education could carry on its Hegel-based patronization.

Today, the sophist warning against unrooted is-claims is carried on by the existentialism of Kierkegaard and Nietzsche and Heidegger and Sartre, defining existentialism as holding that ‘existence precedes essence, or (..) that subjectivity must be the starting point’ (Marino, 2004: 344); and by French post- structuralism with Derrida and Lyotard and Foucault and Bourdieu showing skepticism towards hidden patronization in our most fundamental institutions: words, correctness, cures and education (Lyotard, 1984), (Tarp, 2004, 2). Foucault thus says:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006: 41)

In Germany, Arendt carried his Heidegger’s work further by dividing human activity into labor and work focusing on the private sphere and action focusing on the political sphere thus accepting as the first philosopher political action as a worthy human activity creating institutions that should be treated with care to avoid ‘the banality of evil’ if turning totalitarian by the sheer banality of just following orders (Arendt, 1963). Likewise, Bauman points out that by following authorized routines modernity can create both gas turbines and gas chambers (Baumann, 1989).

As to their meanings, the word 'critical' comes from Greek 'kritike' meaning to pass judgement; and civilized comes from latin 'civis' meaning a free citizen. So civilized thinking means republican thinking always being skeptical towards false is-claims; and critical thinking means passing judgements; but to pass a judgement you must be elected judge by a democratic process, or have insight in the difference between right and wrong as e.g. believing in the Hegel assumption that instead of being free to create their own history, humans are puppets on a string playing out the manuscript of the Spirit. So basically the contradiction between critical and civilized thinking is a replay of the ancient controversy between the Greek philosophers and sophists.

Critical versus civilized mathematics education

The difference between critical and civilized mathematics education is seen in a paper describing how to deal with teaching and learning problems in a Brazilian math class (Tarp, 2004, 1)

> In Brazil there is a research group, which has focused on issues related to new technologies and mathematics education. This research group has developed software and work with students at different levels and with teachers. A teacher from a nearby school approached the group (..) From the teacher perspective, she had some tough problems to face and she foresaw that the computers would be able to help her. The teacher was teaching a class of 5th graders, which in her view was really problematic. The kids were older (15 years average) than the expected age for this grade: 11. The kids felt humiliated somehow as they were put in a school with kids much younger than them and they had flunked many times, and at several instances they had to repeat all the subjects of a given school year because their 'failure in mathematics'. The students transformed this humiliation into violence in class. The teacher was in fact considering the possibility of just quitting the job since she could not work with those kids in a way she found effective. (..) The teacher was enthusiastic about a software, which deals with rational numbers. (..) both researchers and teacher had the 'intuition' that the computer might have a positive effect in this class and maybe could avoid that the students had to repeat this grade again. (Sec. 2, par. 2-4)

The teacher is supposed to teach rational numbers to a class with a mixture of 11year old students and 15year old repeaters having given up rational numbers and turning to violence. The research group could have asked critical questions as 'is rational numbers defined from below as an abstraction from concrete examples or from above as an example of an abstraction?' and ' why teach addition when it is meaningless to add fractions without units?' Instead the group uncritically assumed 'that the computer might have a positive effect'.

The paper also describes how civilized thinking would work differently:

> The research group is working halftime in classrooms and halftime at the university. It focuses on the concerns of typical classrooms as expressed by students, teachers in their stories of complaints. The teacher complains about the violence in the class tempting her to quit the job since she cannot work in a way she finds effective. And the students complain about having to repeat the class because they don't want to learn about fractions, since the teacher by just echoing the textbook is unable to explain to the students, why they shall learn fractions, and what they are useful for. Asked to comment this, the teacher says that mathematics education means education in mathematics, and since rational numbers is part of the mathematics textbook it must be taught and learned. Mathematics is difficult to learn, so the students have to work harder, or be supported by computers. Hence the problems will not disappear before schools can afford computers, or the students decide to become more engaged in mathematics.
>
> Based upon the motto "echo-phrasing is freezing, re-phrasing is freeing" postmodern thinking sees modern institutions frozen in echo-phrasings, that have to be discovered and rephrased. Since the teacher is echoing the textbook, the echoes can be found here. The textbook presents fractions as examples of rational numbers, being example of number sets, being examples of sets. This is the typical way of presentation within modern set-based mathematics explaining concepts as examples of more abstract concepts. This phrasing conflicts with the student demand for explanations relating fractions to their use.
>
> So instead of developing software to supplement, and thus support the existing top-down echo-phrasing of fractions, the group begins to look for alternative bottom-up approaches in journals, other textbooks, other countries, and in other time periods. Also they use their imagination by accessing the silent part of their 'knowledge-iceberg' developed through years of classrooms experience as mathematics educators. Using curriculum architecture they design examples of micro curricula, where fractions emerges from dividing

problems, that can be introduced into the ordinary classroom as e.g. games, where students work in pairs throwing dices and splitting the profit, or loss, proportional to their stakes shown by their dice-numbers.

This 'proportional splitting' approach leads to (and thus shows the authenticity and necessity of) fractions, and multiplication of fractions and integers. (Sec. 5, par. 2-5)

So where critical thinking shows no criticism towards the actual mathematics education tradition, civilized thinking asks if this tradition is nature or choice presented as nature and thus hiding alternatives.

Criticizing and civilizing rational numbers

In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, (Freudenthal, 1973) both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite Many' in Arabic.

Meeting Many we ask 'how many?' Counting and adding gives the answer. We count by bundling and stacking as seen when writing a total T in its block form: T = 354 = 3·B^2 + 5·B + 4·1 where the bundle B is ten typically. This illustrates the four ways to unite: On-top addition unites variable numbers, multiplication unites constant numbers, power unites constant factors and per-numbers, and next-to addition, also called integration, unites variable blocks. As indicated by its name, uniting can be reversed to split a total into parts predicted by the reversed operations: subtraction, division, root & logarithm and differentiation.

Operations **unite**/*split* Totals in	Variable	Constant
Unit-numbers m, s, kg, $	**T = a + b** *T – b = a*	**T = a·b** *T/b = a*
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫a·db** *dT/db = a*	**T = a^b** $b\sqrt{T} = a$ $\log_a T = b$

Although presented as nature, ten-bundling is a choice. Bundling Many in a 'icon-bundles' less than ten means asking e.g. 'T = 7 = ? 4s'. The answer is predicted on a calculator by two formulas, a recount-formula 'T = (T/B)·B' telling that from a total T, T/b times B s can be taken away, and a restack-formula 'T = (T–B)+B' telling that from a total T, T–B is left when B is taken away and placed next-to. First T = 7/4 gives 1.some. Then T = 7 – 1·4 leaves 3. So the prediction is T = 7 = 1 4s & 3 = 1.3 4s = 1 ¾ 4s. Thus with icon-counting, a natural number is a decimal number with a unit where the decimal point separates singles from bundles (Tarp, 2016)

Double-counting physical units creates per-numbers as 3$/4kg. With this, units can be changed by recounting $s in 3s or kgs in 4s: 15$ = (15/3)·3$ = (15/3)·4kg = 20kg. So as per-numbers, fractions are not numbers, but operators, needing a number to become a number. To add, per-numbers must be multiplied to unit-numbers, thus adding as areas, called integration: ½ of 4 + 2/3 of 3 = (1/2*4 + 2/3*3) of (4+3) = 4 of 7.

The root of geometry is the standard form, a rectangle, that halved by a diagonal becomes two right-angled triangles with sides and angles connected by three laws, A+B+C = 180, a^2+b^2 = c^2 and tanA = a/b. Being filled from the inside by triangles, a circle with radius r gets the circumference 2·π·r where π = n·tan(180/n) for n large.

Thus, as a label for algebra and geometry, mathematics is a natural science about the physical fact Many. However, the invention of the concept SET allowed mathematics to become a self-referring collection of 'well-proven' statements about 'well-defined' concepts, i.e. as 'MetaMatics', defined from above as examples from abstractions instead of from below as abstractions from examples. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory tries to avoid self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract essence. To avoid self-reference Russell introduced a type theory allowing reference only to lower degree types. Consequently, fractions cannot be numbers since they refer to numbers in their modern definition: In a set-product of integers, a fraction is an equivalence set created by the equivalence relation $a/b \sim c/d$ if $a \cdot d = b \cdot c$.

Thus SET transformed grounded mathematics into a self-referring 'MetaMatism', a mixture of MetaMatics and 'MatheMatism' true inside a classroom but seldom outside where claims as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So rational numbers is pure MetaMatism by also being MatheMatism: Inside a classroom, 1/2 + 2/3 = 7/6. Outside 1 coke out of 2 bottles and 2 cokes out of 3 bottles add up to 3 cokes out of 5 bottles, and not 7 cokes out of 6 bottles as taught inside.

Not criticizing rational numbers shows that critical thinking has taboos and that it lacks self-criticism by showing no criticism towards its own un-criticalness.

'Preschool calculus and multiplication before addition' as a 1year pre-engineer math course

As a label, mathematics has no content itself, only it ingredients have, algebra and geometry both rooted in the physical fact Many. To deal with Many we count & add. By counting a total T in bundles, bundle-counting creates numbers as blocks of bundles and unbundled occurring in three different ways, normal and overload and underload as in T = 2B1 3s = 1B4 3s = 3B-2 3s when recounted in the same unit. Recounted in a different unit roots proportionality through the recount formula $T = (T/B) \cdot B$ allowing a calculator to predict the result. Recounting in and from tens means resizing blocks where the height and the base are inversely proportional as in 3 7s = 2)1 tens or 4 tens = 5 8s. Reversed addition is called equations solved by recounting: $2 \cdot x = 8 = (8/2) \cdot 2$ so $x = 8/2$, showing the solving method 'move to opposite side with opposite sign'. With counting before adding, division and multiplication comes before addition.

Once counted, totals can be added on-top if the units are made the same by proportionality, and next-to as areas also called integration. A composite area always changes with the last block added: change in Area = height* change in base, or $\Delta A = h \cdot \Delta b$ or $h = \Delta A/\Delta b$. So areas can be found by developing $\Delta/\Delta x$-calculations, also called differentiation in the case of replacing interval changes with local changes: $y' = dy/dx = \Delta y/\Delta x$ for Δx arbitrarily small; as when the per-number is neither globally nor piecewise but locally constant (continuous) (Tarp, 2013).

Finally, double counting a physical quantity in two different units creates pre-numbers or fractions as 2\$/3kg = 2/3 \$/kg that must be multiplied to areas before being added. The difference between a full critical and civilized mathematics education curriculum is illustrated in the appendix.

Discussion and conclusion

We asked: wanting to design a 1year pre-engineer course for migrants beginning from scratch, should we use critical and civilized thinking?

Investigating its theoretical background shows that critical thinking is based on Marx, again based on Hegel counter-enlightenment going back to Greek Plato philosophy resonating with the Greek meaning of the word 'kritike', to pass judgement. For Plato, that was precisely what the philosophers were able to do since to them all physical was but examples of metaphysical forms only visible to them. Hegel replaced the forms with a Spirit expressing itself through the history of different people thinking they can decide their future themselves, but in reality just being puppets on a string playing out the masterplan of the Spirit. To Marx, the means to the Spirit's goal, a socialist society, was a proletarian dictatorship with a democracy in the form of a representative pyramid where the top central committee decided the correctness code that justified the judgement passed by critical thinking. Consequently, rational numbers cannot be criticized if part of this code. Likewise, criticizing

Hegel-based line-organized office directed education is out of the question. With its lack of self-criticism and dependence on the will of a metaphysical Spirit, critical thinking reminds of a totalitarian religion preaching political correctness instead of teaching enlightenment.

Being skeptical towards ungrounded is-claims, civilized thinking unmasks false nature by uncovering hidden alternatives to choices presented as nature. So categories and correctness are grounded in the outside world; and as means avoiding the banality of evil, its institutions accommodate to resistance from the outside goals they are created to meet. Consequently, mathematics is ManyMath, a natural science accommodating to the physical fact Many; and education must be organized in flexible half-year blocks aiming at uncovering and developing the talent of the individual learner.

So as a 1year pre-engineer course for migrants from scratch we will get to different answers. Uncritically accepting mathematics as meaningless MetaMatism, critical thinking will say it is impossible to learn a pre-engineer background in one year since mathematics is difficult to learn thus taking many hours of hard dedicated work.

Civilized thinking welcomes a course showing that while MetaMatism is difficult, ManyMath is quickly learned: To deal with many, we count and recount and double-count before performing next-top and on-top addition and reversed addition. First we count in ones to produce icons, then we bundle-count in normal, overload and underload form by recounting in the same unit thus realizing that numbers are 2dimensional blocks and not names to the points on a 1dimensional cardinality line as claimed by MetaMatism. Then we recount in a new unit to proportional numbers. Then we recount in and from tens to resize the number blocks. Then we double-count to create per-numbers and fractions. Then to add on-top we must change the unit by proportional recounting; and to add per-numbers we must add next-to as areas where a composite area changes with the last block added. And finally reversed addition leads to solving equations presenting ‘opposite side with opposite sign’ as a natural method.

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Appendix: a critical and a civilized math curriculum

Primary school

Critical mathematics curriculum	Civilized mathematics curriculum (Tarp, 2016)
1dim. Number-line with number-names No counting, only adding and next-to Addition & Subtraction before Multiplication & Division Multiplication tables to be memorized No calculator	2dim. Number-blocks with units Counting before adding, next-to before on-top Multiplication & Division before Subtraction & Addition Multiplication tables recount to & from tens Calculator from the start as predictors
One and two digit numbers Addition Subtraction Multiplication Division Simple fractions	BundleCount Many in BundleCups ReCount Many in same Unit & in new Unit (Proportionality) ReCount: In Tens & From Tens (Multiplication & Division) Calculator Prediction: RecountFormula Addition: NextTo (Integration) & OnTop Reversed addition: Equations

Middle school

Fractions are numbers that can be added without units. Letter-fractions must be factorized before added	Fractions are PerNumbers (operators needing a number to become a number) and added by areas (integration)
Negative numbers Fractions Percentages & Decimals Proportionality LetterNumbers Algebraic fractions Solve a linear equation Solve 2 equations w. 2 unknowns	DoubleCounting produces PerNumbers & PerFives (fractions) & PerHundreds (%) Geometry and algebra go hand in hand when working with letter-numbers and letter-formulas; and with lines and forms The coordinate system coordinates geometry and algebra so that length can be translated to D-change, and vice versa

High school

Functions are set-relations	Functions are formulas with two variables
Squares and square roots Solve quadratic equations Linear functions Quadratic functions Exponential functions Logarithm Differential Calculus Integral Calculus Statistics & probability	Integral Calculus as adding PerNumbers Change & Global/Piecewice/Local constancy Root/log as finding/counting change-factors Constant change: Proportional, linear, quadratic, exponential, power Simple and compound interest Predictable Change: Integral Calculus & Differential Calculus Unpredictable Change: Statistics & probability

06. Online Teacher Training for Curing Math Dislike: Cup&Re-Counting & Multiplication Before Addition

Set transformed Mathematics from a mere label for Algebra and Geometry into a self-referring subject changing the two from example-containers to examples of set, causing massive learning problems as shown by PISA. Re-rooting mathematics in the physical fact Many, the MATeCADEMY.net offers an alternative teacher training.

Background

In spite of increased mathematics education research, Swedish PISA results decrease as witnessed by the OECD 2015 report 'Improving Schools in Sweden'. Mathematics seems to be hard, but we could ask: Maybe it is not mathematics that is taught, and maybe there is a hidden mathematics that rooted in the outside world becomes meaningful? And if so, where can teachers learn about it? Existentialist thinking might provide an answer. Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism as holding that 'existence precedes essence' (Marino, 2004 p. 344). But how does essence-math differ from existence-math?

A Case: Peter, stuck in division and fractions

Being a mathematics teacher in a class of ordinary students and repeaters flunking division and fractions, Peter is about to give up teaching when he learns about the '1cup & 5sticks' method to cure mathematics dislike by watching 'CupCount and ReCount before you Add' (https://www.youtube.com/watch?v=IE5nk2YEQIAxx).

Here 5 sticks are CupCounted in 2s using a cup for bundles. He sees that a total can be recounted in the same unit in 3 different forms: overload, standard and underload:

T = 5 = I I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = II II II ɨ = 3B-1 2s

So counted in bundles, a total has an inside number of bundles and an outside number of singles; and moving a stick out or in creates an over-load or an under-load.

When multiplying, 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 inside and 56 outside as an overload that can be recounted: T = 7 x 4B8 = 28B56 = 33B6 = 336.

And when dividing, 336/7 is bundle-written as 33B6 /7 recounted to 28 inside and 56 outside according to the multiplication table. So 33B6 /7 = 28B56 /7 = 4B8 = 48.

To try it himself, Peter downloads the 'CupCount & ReCount Booklet'. He gives a copy to his colleagues and they decide to arrange a free 1day Skype seminar.

In the morning they watch the PowerPoint presentation 'Curing Math Dislike', and discuss six issues: first the problems of modern mathematics, MetaMatism; next the potentials of postmodern mathematics, ManyMath; then the difference between the two; then a proposal for a ManyMath curriculum in primary and middle and high school; then theoretical aspects; and finally where to learn about ManyMath.

Here MetaMatism is a mixture of MatheMatism, true inside a classroom but rarely outside where '2+3 = 5' is contradicted by 2weeks+3days = 17days; and MetaMatics, presenting a concept TopDown as an example of an abstraction instead of BottomUp as an abstraction from many examples: A function IS an example of a set-product.

In the afternoon the group works with an extended version of the CupCount & ReCount Booklet where Peter assists newcomers. At the seminar there are two Skype sessions with an external instructor, one at noon and one in the afternoon.

Bringing ManyMath to his classroom, Peter sees that many difficulties disappear, so he takes a 1year distance learning education at the MATHeCADEMY.net teaching teachers to teach MatheMatics as ManyMath, a natural science about Many. Peter and 7 others experience PYRAMIDeDUCATION where they are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. An external coach assists the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. In a pair each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays by coaching a new group of 8 teachers.

At the academy, the 2x4 sections are called CATS for primary and secondary school inspired by the fact that to deal with Many, we Count & Add in Time & Space.

At the academy, primary school mathematics is learned through educational sentence-free meetings with the sentence subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic 'grasp-to-grasp' learning.

Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic 'gossip-learning': Thank you for telling me something new about something I already knew.

Conclusion

An existentialist distinction between essence and existence shows that what is taught in schools in not mathematics, but a self-referring MetaMatism turning mathematics upside down and containing some statements that do not apply outside. As a common label for Algebra and Geometry meaning reuniting Many and measuring Earth in Arabic and Greek, mathematics should let existence precede essence and become ManyMatics, a natural science about how to divide the earth and its Many products.

Reference

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Cure MathDislike: CupCount 'fore you Add

1Day Skype Seminar on CupCounting, ReCounting & CupWriting

Action Learning on the child's own 2D NumberLanguage as observed when holding 4 fingers together 2 by 2 makes a 3-year-old child say 'No, that is not 4, that is 2 2s.'

<table>
<tr><td>09-11</td><td>Listening and Discussing: Curing Math Dislike, a PowerPointPresentation
To master Many, we Math?? No, first we Count, then we Add. Math is a label, not an action word.</td></tr>
<tr><td></td><td>1. The problems of Modern MatheMatics, or MetaMatism
2. The potentials of PostModern MatheMatics, or ManyMath
3. The Difference between MetaMatism and ManyMath
4. A ManyMath Curriculum for Primary and Middle and High school
5. Theoretical aspects, and 6. Where to learn about ManyMath?
Bad Math: MatheMatism, true inside but rarely outside classes: 2+3 IS 5, but 2weeks+3days = 17d? Adding 1D Line Numbers without units may create MathDislike.
Evil Math: MetaMatics, presenting a concept TopDown as an example of an abstraction instead of BottomUp as an abstraction from many examples: A function IS an example of a set-product.
Good Math: ManyMatics, a natural science about Many mastering Many by CupCounting, ReCounting & CupWriting: T = 5 = I I I I I = II I I I = 1]3 2s = II II I = 2]1 2s = II II II I = 3]-1 2s.</td></tr>
<tr><td>11-13</td><td>Skype Conference. Lunch</td></tr>
<tr><td>13-15</td><td>Doing: Trying out the CupCount 'fore you Add booklet to see proportionality and calculus and solving equations as golden LearningOpportunities in Cup- & Re-Counting and NextTo Addition.</td></tr>
<tr><td></td><td>RECOUNTING, in the same unit creates over- or under-load, in a new unit creates proportionality
Question: T = 2.1 3s = ? 3s. Answer: T = 2.1 = 2]1 = 1]4 = 3]-2 3s
Q: T = 2 3s = ? 4s A: T = 2 3s = III III = IIII II = 1]2 4s = 1]1 5s = 3] 2s = 1]1] 2s = 11.0 2s</td></tr>
<tr><td></td><td>CalculatorPrediction. Q: T = 2 4s = ? 5s. A: T = 1.3 5s since
RecountFormula T = (T/B)*B says 'From T, T/B times, Bs can be taken away'
2*4/5   1.some
2*4 – 1*5   3</td></tr>
<tr><td></td><td>RECOUNTING in and from Tens resizes blocks meaning teaching multiplication before addition:
Q: T = 3 7s = ? tens. A: T = 3*7 = 21 = 2.1 tens. Q: T = 47 = ? 6s. A: T = (47/6)*6 = 7 6s & 5</td></tr>
<tr><td></td><td>Multiply & Divide with CupWriting creating or removing overloads
Q: T = 7 * 463 = ?   A: T = 7 * 4]6]3 = 28]42]21 = 28]44]1 = 32]4]1 = 3241
Q: T = 3241 / 7 = ?   A: T = 32]4]1 / 7 = 28]44]1 / 7 = 28]42]21 / 7 = 4]6]3 = 463</td></tr>
<tr><td></td><td>ADD NextTo.   Q: T = 2 4s + 3 5s = ? 9s. A: T = 2.5 9s (integration)
ADD OnTop.   Q: T = 2 4s + 3 5s = ? 5s. A: T = 1.3 5s + 3 5s = 1]3 + 3] = 4]3 = 4.3 5s</td></tr>
<tr><td></td><td>DoubleCounting in two units creates PerNumbers
Q: T = 10$ = ?kg with 4$ per 5kg. A: T = 10$ = (10/4) * 4$ = (10/4) * 5 kg = 12.5 kg</td></tr>
<tr><td></td><td>Reversed Addition: Solving Equations by moving to Opposite Side with Opposite Sign
<table><tr><td>2 x ? = 8 = (8/2) x 2</td><td>2 + ? = 8 = (8-2) + 2</td><td>T = 2 3s + ? 5s = 3.2 8s</td></tr><tr><td>? = 8/2, ReCounting</td><td>? = 8-2, ReStacking</td><td>? = (3.2 8s – 2 3s)/5 = ΔT/5, Differentiation</td></tr></table></td></tr>
<tr><td>15-16</td><td>Coffee. Skype Conference.</td></tr>
</table>

Background

The effect of MathDislike is seen in the 2015 OECD report *Improving Schools in Sweden*: 'PISA 2012, however, showed a stark decline in the performance of 15-year-old students with more than one out of four students not even achieving the baseline Level 2in mathematics at which students begin to demonstrate competencies to actively participate in life'.

MATHeCADEMY.net offers UK or DK online Teacher Training based upon Action Learning and Research papers on CupCounting published at the ICME 2004-2012 *(mathecademy.net/papers/icme-trilogy)*. More details on MrAlTarp YouTube videos:

Summary of the 4 primary and secondary 4 study units at the MATHeCADEMY.net

	QUESTIONS	ANSWERS
C1 COUNT	How to count Many? How to recount 8 in 3s: T = 8 = ? 3s How to recount 6kg in \$: T = 6kg = ?\$ How to count in standard bundles?	By bundling and stacking the total T predicted by T = (T/b)*b T = 8 = ?*3 = ?3s, T = 8 = (8/3)*3 = 2)2 3s = 2.2 3s = 2*3 +2 = 2 2/3*3 If 4kg = 2\$ then 6kg = (6/4)*4kg = (6/4)*2\$ = 3\$ Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten 2ten 3 = 4*B^2 + 2*B + 3
C2 COUNT	How can we count possibilities? How can we predict unpredictable numbers?	By using the numbers in Pascal's triangle We 'post-dict' that the average number is 8.2 with the deviation 2.3. We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation)
A1 ADD	How to add stacks concretely? T = 27+16 = 2ten7+1ten6 = 3ten13 = ? How to add stacks abstractly?	By restacking overloads predicted by the restack-equation T = (T-b)+b T = 27+16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43 Vertical calculation uses carrying. Horizontal calculation uses FOIL
A2 ADD	What is a prime number? What is a per-number? How to add per-numbers?	Fold-numbers can be folded: 10 = 2fold5. Prime-numbers cannot: 5 = 1fold5 Per-numbers occur when counting, when pricing and when splitting. The \$/day-number a is multiplied with the day-number b before added to the total \$-number T: T2 = T1 + a*b
T1 TIME	How can counting & adding be reversed ? Counting ? 3s and adding 2 gave 14. Can all calculations be reversed?	By calculating backward, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign. x*3+2 = 14 is reversed to x = (14-2)/3 Yes. x+a = b is reversed to x = b-a, x*a = b is reversed to x = b/a, x^a = b is reversed to x = a√b, a^x = b is reversed to x=logb/loga
T2 TIME	How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable?	By using constant change-equations: If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44 If Ko = 30 and ΔK/K = r = 2%, then K7= Ko*(1+r)^n= 30*1.02^7= 34.46 By solving a variable change-equation: If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx
S1 SPACE	How to count plane and spatial properties of stacks and boxes and round objects?	By using a ruler, a protractor and a triangular shape. By the 3 Greek Pythagoras', mini, midi & maxi By the 3 Arabic recount-equations: sinA=a/c, cosA=b/c, tanA=a/b
S2 SPACE	How to predict the position of points and lines? How to use the new calculation technology?	By using a coordinate-system: If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14) Computers can calculate a set of numbers (vectors) and a set of vectors (matrices)

07. Debate on how to improve mathematics education

In this symposium, the author invites opponents to debate how to improve mathematics education inspired by the Chomsky-Foucault debate on human nature. The main question is: 'If research cannot improve Math education, then what can?

Bo: Today we discuss Mathematics education and its research. Humans communicate in languages, a word-language and a number-language. In the family, we learn to speak the word language, and we are taught to read and write in institutionalized education, also taking care of the number-language under the name Mathematics, thus emphasizing the three r's: Reading, Writing and Arithmetic. Today governments control education, guided by a growing research community. Still international tests show that the learning of the number language is deteriorating in many countries. This raises the question: If research cannot improve Mathematics education, then what can? I hope our two guests will provide some answers. I hope you will give both a statement and a comment to the other's statement before the floor will comment.

1. Mathematics Itself

Bo: We begin with Mathematics. The ancient Greeks Pythagoreans used this word as a common label for what we know, which at that time was Arithmetic, Geometry, Astronomy and Music. Later Astronomy and Music left, and Algebra and Statistics came in. So today, Mathematics is a common label for Arithmetic, Algebra, Geometry and Statistics, or is it? And what about the so-called 'New Math' appearing in the 1960s, is it still around, or has it been replaced by a post New-Math, that might be the same as pre New-Math? In other words, has pre-modern Math replaced modern Math as post-modern Math? So, I would like to ask: 'What is Mathematics, and how is it connected to our number-language?'

2. Education in General

Bo: Now let us talk about education in general. On our planet, life takes the form of single black cells, or green or grey cells combined as plants or animals. To survive, plants need minerals, pumped in water from the ground through their leaves by the sun. Animals instead use their heart to pump the blood around, and use the holes in the head to supply the stomach with food and the brain with information. Adapted through genes, reptiles reproduce in high numbers to survive. Feeding their offspring while it adapts to the environment through experiencing, mammals reproduce with a few children per year. Humans only need a few children in their lifetime, since transforming the forelegs to hands and fingers allows humans to grasp the food, and to share information through communication and education by developing a language when associating sounds to what they grasp. Where food must be split in portions, information can be shared. Education takes place in the family and in the workplace; and in institutions with primary, secondary and tertiary education for children, for teenagers and for the workplace. Continental Europe uses words for education that do not exist in the English language such as Bildung, unterricht, erziehung, didactics, etc. Likewise, Europe still holds on to the line-organized office preparing education that was created by the German autocracy shortly after 1800 to mobilize the population against the French democracy, whereas the North American republics have block-organized talent developing education from secondary school. As to testing, some countries use centralized test where others use local testing. And some use written tests and others oral tests. So, my next question is 'what is education?'

3. Mathematics Education

Bo: Now let us talk about education in Mathematics, seen as one of the core subjects in schools together with reading and writing. However, there seems to be a difference here. If we deal with the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. However, you cannot Math, you can reckon. At the European continent reckoning, called 'Rechnung' in German, was an independent subject until the arrival of the so-called new Mathematics around 1960. When opened up, Mathematics still contains subjects as fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc. Today, Europe only offers classes in Mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. Therefore, I ask, 'what is Mathematics education?'

4. The Learner

Bo: Now let us talk about at the humans involved in Mathematics education: Governments choose curricula, build schools, buy textbooks and hire teachers to help learners learn. We begin with the learners. The tradition sees learning taking place when learners follow external instructions from the teacher in class and from the textbook at home. Then constructivism came along suggesting that instead learning takes place through internal construction. Therefore, I ask 'what is a learner?'

5. The Teacher

Bo: Now let us talk about the teacher. It seems straightforward to say that the job of a teacher is to teach learners so that learning takes place, checked by written tests. However, continental Europe calls a teacher a 'Lehrer' thus using the same word as for learning. In addition, a Lehrer is supposed to facilitate 'unterrichtung and erziehung and to develop qualifications and competences. In teacher education, the subject didactics, meant to determine the content of Bildung, is unknown outside the continent. And until lately, educating lehrers took place outside the university in special lehrer-schools. Thus, being a teacher does not seem to be that well-defined. Therefore, my next question is 'what is a teacher?'

6. The Political System

Bo: Now let us talk about governments. Humans live together in societies with different degrees of patronization. In the debate on patronization, the ancient Greek sophists argued that humans must be enlightened about the difference between nature and choice to prevent patronization by choices presented as nature. In contrast, the philosophers saw choice as an illusion since physical phenomena are but examples of metaphysical forms only visible to philosophers educated at Plato's Academy who consequently should be accepted as patronizors. Still today, democracies come in two forms with a low and high degree of institutionalized patronization using block-organized education for individual talent developing or using line-organized education for office preparation. As to exams, some governments prefer them centralized and some prefer them decentralized. As to curricula, the arrival of new Mathematics in the 1960s integrated its subfields under the common label Mathematics. Likewise, constructivism meant a change from lists of concepts to lists of competences. However, these changes came from Mathematics and education itself. So my question is: 'Should governments interfere in Mathematics education?'

7. Research

Bo: Now let us talk about research. Tradition often sees research as a search for laws built upon reliable data and validated by unfalsified predictions. The ancient Greek Pythagoreans found three metaphysical laws obeyed by physical examples. In a triangle, two angles and two sides can vary freely, but the third ones must obey a law. In addition, shortening a string must obey a simple ratio-law to create musical harmony. Their findings inspired Plato to create an academy where knowledge meant explaining physical phenomena as examples of metaphysical forms only visible to philosophers educated at his academy by scholasticism as 'late opponents' defending their comments on an already defended comment against three opponents. However, this method discovered no new metaphysical laws before Newton by discovering the gravitational law brought the priority back to the physical level, thus reinventing natural science using a laboratory to create reliable data and test library predictions. This natural science inspired the 18th century Enlightenment period, which again created counter-enlightenment, so today research outside the natural sciences still uses Plato scholastics. Except for the two Enlightenment republics where American Pragmatism used natural science as an inspiration for its Grounded Theory, and where

French post-structuralism has revived the ancient Greek sophist skepticism towards hidden patronization in categories, correctness and institutions that are ungrounded. Using classrooms to gather data and test predictions, Mathematics education research could be a natural science, but it seems to prefer scholastics by researching, not Math education, but the research on Math education instead. To discuss this paradox I therefore ask, ‘what is research in general, and within Mathematics education specifically?’

8. Conflicting Theories

Bo: Of course, Mathematics education research builds upon and finds inspiration in external theories. However, some theories are conflicting. Within Psychology, constructivism has a controversy between Vygotsky and Piaget. Vygotsky sees education as building ladders from the present theory regime to the learners’ learning zones. Piaget replaces this top-down view with a bottom-up view inspired by American Grounded Theory allowing categories to grow out of concrete experiences and observations. Within Sociology, disagreement about the nature of knowledge began in ancient Greece where the sophists wanted it spread out as enlightenment to enable humans to practice democracy instead of allowing patronizing philosophers to monopolize it. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. As counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe’s line-organized office preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the post-structuralism of the French Enlightenment republic. Likewise, the two extreme examples of forced institutionalization in 20th century Europe, both terminated by the low institutionalized American Enlightenment republics, made thinkers as Baumann and Arendt point out that what made termination camps work was the authorized routines of modernity and the banality of evil. Reluctant to follow an order, you can find another job in the private sector, but not in an institution. Here the necessity of keeping a job forces you to carry out both good and evil orders. As an example of a forced institution, this also becomes an issue in Mathematics Education. So I ask: What role do conflicting theories play in Mathematics education and its research?

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08. Poster: MigrantMath as CupCounting & PreSchool Calculus

Europe receives a continuing migrant flow to benefit from its welfare and educational systems. To benefit from the engineer potential in young migrants allowing them to build up welfare and education in their own country, Europe must rethink its line-organized office directed education dating back to the Napoleon wars; and must replace meaningless top-down MetaMatism with bottom-up ManyMath.

Background

Increased mathematics education research seems to create a decrease in Nordic PISA results as witnessed by the latest PISA study and the OECD 2015 report ‘Improving Schools in Sweden’. We ask: Can existentialism point to a possible solution?

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism as holding that ‘existence precedes essence’ (Marino, 2004 p. 344). Thus a hypothesis can be formulated: Mathematics performance will increase if replacing essence-math with existence-math.

Mathematics as an Essence

The Pythagoreans labeled their four knowledge areas by a Greek word for knowledge, mathematics. With astronomy and music now as independent areas, today mathematics is a common label for its two remaining activities both rooted in Many: Geometry meaning to measure earth in Greek, and Algebra meaning to reunite numbers in Arabic and replacing Greek Arithmetic (Freudenthal, 1973).

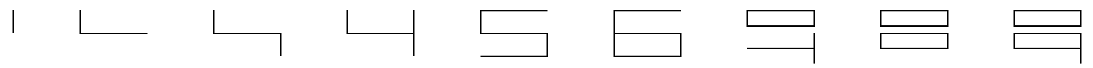
Then the set-concept transformed mathematics to ‘MetaMatics’ defining its concepts by self-reference as examples from internal abstractions instead of as abstractions from external examples. Looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence it false’ being false if true and true if false: If M = $\{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

‘MetaMatism’ means mixing MetaMatics with ‘MatheMatism’ true inside but seldom outside the classroom as e.g. ‘the fraction paradox’ where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

Mathematics as ManyMath, a Natural Science about Many

A number as 345 = 3*B^2+4*B+5*1 shows that to deal with Many, first we iconize then we bundle and stack. Until ten we count in 1s by iconizing, i.e. by rearranging sticks in icons so five ones becomes one five-icon 5 with five sticks, etc.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII



1	2	3	4	5	6	7	8	9

With icons, a total can be ‘bundle-counted’ in icon-bundles so a total T of 7 is bundled in 3s as T = 2 3s & 1 shown with 2 sticks in a in a bundle-cup and 1 stick outside; reported with ‘bundle-writing’, T = 2B1 3s, then with ‘decimal-writing’ where a decimal point separates the bundles from the singles, and including the unit 3s, T = 2.1 3s.

A calculator can predict a counting result. A stack of 2 3s is iconized as 2x3 showing a lift used 2 times to lift the 3s. Taking away is iconized with ‘/3’ or ‘–3’ showing the broom or the trace when wiping away 3 several times or just once, called division and subtraction. Entering ‘7/3’, we ask the calculator ‘from 7 take away 3s’ and get the answer ‘2.some’. Entering ‘7 – 2x3’ we ask ‘from 7 take away 2 3s’ and get the answer 1 leftover. Thus the calculator predicts that 7 = 2B1 3s = 2.1 3s.

Once bundle-counted, totals are re-counted, double-counted or added next-to or on-top. To recount in the same unit, changing a bundle to singles creates over- or under-load as when recounting 4 2s as 3.2 2s, or as 5 less 2 2s leading to negative numbers:

T = 4 2s = 3.2 2s, or T = 4 2s = 5.-2 2s

To recount in a different unit means changing unit, called proportionality. Asking '3 4s is how many 5s?' sticks give the result 2.2 5s as predicted by a calculator.

T = 3 4s = IIII IIII IIII → IIIII IIIII I I → 2B2 5s → 2.2 5s

Recounting in and from tens means resizing number-blocks where the height and the base are inversely proportional as in 3 7s = 2B1 tens or 4 tens = 5 8s.

Double-counting a physical quantity creates 'per-numbers' as 4$/5kg allowing 16$ to be recounted in 4s to bridge to the kg-numbers: 16$ = (16/4)*4$= (16/4)*5kg = 20kg.

Next-to addition of 2 3s and 4 5s as 3.2 8s means adding areas, called integration. To add on-top the units are made the same by recounting as 1.1 5s and 4 5s = 5.1 5s. Reversed addition is called equations solved by recounting: 2·x = 8 = (8/2)·2 so x = 8/2, showing the solving method 'move to opposite side with opposite sign'.

The root of geometry is a rectangle that halved by a diagonal becomes two right-angled triangles where the sides and the angles are connected by three laws, A+B+C = 180, a^2+b^2 = c^2 and tanA = a/b. Being filled from the inside by such triangles, a circle with radius r gets the circumference 2·π·r where π = n·tan(180/n) for n large.

Conclusion

There is a fundamental difference between essence- and existence-math, MetaMatism and ManyMath. This means the latter has to be tested outside traditional school in preschool or in special courses for young migrants wanting to become engineers or teachers to help building welfare and education systems in their own country.

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09. A Heidegger View on How to Improve Mathematics Education

After 50 years of research, mathematics education still has learning problems as witnessed by the PISA studies. So, a suspicion arises: Can we be sure that what has been undertaken is mathematics and education and research? We seek an answer in philosophy by listening to Heidegger that, wanting to establish its meaning, finds two forms of Being: that what is, and how it is. In a Heidegger universe, the core ingredients are I and It and They, where I must neglect the gossip from They to establish an authentic relationship to It. Bracketing mathematics' gossip will allow its root, Many, to open itself and disclose a 'many-matics' as a grounded natural science different in many ways from the traditional self-referring set-based mathe-matics. So to improve its educational sentences, math should bring its subjects to the classroom, but leave its gossip outside.

Introduction

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by PISA studies showing a low level and a continuing decline in many countries. Thus, the former model country Sweden face that 'more than one out of four students not even achieving the baseline level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life." (OECD 2015, p. 3)

Researchers in mathematics education meet in different fora. In Europe, the Congress of the European Society for Research in Mathematics Education, CERME, meets each second year. At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? (http://cerme10.org/scientific-activities/plenary-sessions/)

By questioning its success, maybe the short answer is: How can mathematics education research be successful when its three words are not that well defined? As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist. In continental Europe, education serves the nation's need for public servants through multi-year compulsory classes and lines at the secondary and tertiary level. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers at the secondary level, and the tertiary level also has a flexible block organization allowing additional blocks to be taken in the case of unemployment or change of job.

As to research, academic articles can be written at a master level applying or exemplifying existing theories, or at a research level questioning them. Just following ruling theories is especially problematic in the case of conflicting theory as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible respectively.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and we cannot know if research is following ruling traditions or searching for new discoveries. So to answer the question 'How to improve mathematics education research', first we must try to make the three words well defined by asking: What is meant by Mathematics, what is meant by education and what is meant by research?

Common for all three questions is the word 'is', so let us begin by asking 'what is meant by 'is'?'.

What does 'is' Mean

'To be or not to be', 'Cogito, ergo sum', 'What is 'is'?'. Three statements about the nature of being that may or may not have been formulated by Hamlet, Descartes and Heidegger. Still they direct our attention to reflecting and discussing the most used word in sentences, to be.

In his book 'Being and Time', 'Sein und Zeit' in the original German version, Heidegger writes:

> Do we in our time have an answer to the question of what we really mean by the word 'being'? Not at all. So it is fitting that we should raise anew the question of the meaning of Being. (..) Our aim in the following treatise is to work out the question of the meaning of Being and to do so concretely. (Heidegger 1962, p. 1)

Going back in time, Heidegger says that the question 'provided a stimulus for the researches of Plato and Aristotle only to subside from then on as a theme for actual investigation. (p. 2).' Furthermore, Heidegger says, '(..) a dogma has been developed which not only declares the question about the meaning of Being to be superfluous, but sanctions its complete neglect. It is said that Being is the most universal and the emptiest of concepts. As such it resists every attempt at definition (p. 2).'

Heidegger sees this dogma based upon three presuppositions. As to seeing Being as the most universal concept, Heidegger writes 'In medieval ontology Being is designated as a 'transcendens'. Aristotle himself knew the unity of this transcendental 'universal' as a unity of analogy in contrast to the multiplicity of the highest generic concepts applicable to things (..) So if it is said that Being is the most universal concept, this cannot mean that it is the one which is clearest or that it needs no further discussion. It is rather the darkest of all (p. 3).'

As to seeing the concept of Being is indefinable Heidegger says that 'Being cannot be derived from higher concepts by definition, nor can it be presented through lower ones (..) We can infer only that Being cannot have the character of an entity (..) The indefinability of Being does not eliminate the question of its meaning (p. 4).'

As to seeing Being as a concept that of all concepts is the one that is self-evident, Heidegger says 'The very fact that we already live in understanding of Being and that the meaning of Being is still veiled in darkness proves that it is necessary in principle to raise this question again (p. 4).'

Heidegger concludes by saying that

> By Considering these prejudices, however, we have made plain not only that the question of Being lacks an answer, but that the question itself is obscure ad without direction. So if it is to be revived, this means that we must first work out an adequate way of formulating it (p. 4).

To do so, Heidegger says that 'We must therefore explain briefly what belongs to any question whatsoever, so that from this standpoint the question of Being can be made visible as a very special one with its own distinctive character (p. 5).'

Then Heidegger addresses the nature of a general question aiming at establishing a definition of M by answering the question 'What is M?'. Heidegger assigns to a question the word inquiry and says that 'Every enquiry is a seeking. Every seeking gets guided beforehand by what is sought. Inquiry is a cognizant seeking for an entity both with regard to the fact that it is and with regard to its Being as it is (p. 5).'

Here Heidegger describes the two different uses of being, one that establishes existence, 'M is', and one that establishes 'how M is' to others, since what exists is perceived by humans that begin to categorize it by naming or characterizing or analogizing it, in all three cases using the word 'is'.

Heidegger points to four different uses of the word 'is'. 'Is' can claim a mere existence of M, 'M is'; and 'is' can assign predicates to M, 'M is N', but this can be done in three different ways. 'Is' can point down as a 'naming-is' ('M is for example N or P or Q or …') defining M as a common name for its volume of more concrete examples. 'Is' can point up as a 'judging-is' ('M is an example of N') defining M as member of a more abstract category N. Finally, is can point over as an 'analogizing-is' ('M is like N') portraying M by a metaphor carrying over known aspects from another N.

Heidegger stresses the double meaning of being, 'that M is & how M is' by saying 'Everything we talk about, everything we have in view, everything towards which we comport ourselves in any way, is being; what we are is being and so is how we are. Being lies in the fact that something is and in its Being as it is (p. 6-7).'

To separate that which is from how it is, Heidegger coins the word 'Dasein' by saying 'This entity which each of us is in himself and which includes inquiring as one of the possibilities of its Being, we shall denote by the term “Dasein” (p.7).’

So here Heidegger transforms the ‘cogito ergo sum’ into ‘Ich bin da, und Ich frage’ (I exist here and I question). By connecting the word ‘da’ to existence, Heidegger places existence in time and space since ‘da’ can mean both there and then. Also, Heidegger sees questioning as the most important ability of Dasein.

Within existentialist thinking, existence and essence are core concepts (Marino 2004). Here Heidegger says

> [Dasein’s] Being-what-it-is (essentia) must, so far as we can speak of it at all, be conceived in terms of its Being (existentia). (..) To avoid getting bewildered, we shall always use the Interpretative expression “presence-at-hand” for the term “existentia”, while the term “existence”, as a designation of Being, will be allotted solely to Dasein. The essence of Dasein lies in its existence. (p. 42)

Here Heidegger reformulates his basic statement ‘that M is and how M is’ to ‘by existing, M has existentia described (by Others) by essentia’; or ‘existing, M exists together with presence-at-hand.’

To tell if the essentia of existentia, that is, the characteristics of presence-at-hand, is determined by the Others or by Dasein itself, Heidegger later introduces the concept ‘ready-at-hand’

> Equipment can genuinely show itself only in dealings cut to its own measure (hammering with a hammer, for example) (..) In dealings such as this, where something is put to use, our concern subordinates itself to the “in-order-to” which is constitutive for the equipment we are employing at the time; the less we just stare at the hammer-Thing, and the more we seize hold of it and use it, the more primordial does our relationship to it become, and the more unveiledly is it encountered as that which it is - as equipment. (..) The kind of Being which equipment possesses - in which it manifests itself in its own right - we call “readiness to-hand”. (p. 69)

As to existence, Heidegger talks about authentic an unauthentic existence.

> In each case Dasein is its possibility, and it ‘has’ this possibility, but not just as a property, as something present-at-hand would. And because Dasein is in each case essentially its own possibility, it can, in its very Being, ‘choose’ itself and win itself; it can also lose itself and never win itself; or only ‘seem’ to do so. But only in so far as it is essentially something which can be authentic - that is, something of its own - can it have lost itself and not yet won itself. As modes of Being, authenticity and inauthenticity (these expressions have been chosen terminologically in a strict sense) are both grounded in the fact that any Dasein whatsoever is characterized by mineness. (p. 42-43)

As to the Other, Heidegger talks about a dictatorship.

> We have shown earlier how in the environment which lies closest to us, the public ‘environment’ already is ready-to-hand and is also a matter of concern. In utilizing public means of transport and in making use of information services such as the newspaper, every Other is like the next. This Being-with-one-another dissolves one’s own Dasein completely into the kind of Being of ‘the Others’, in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the “they” is unfolded. (p. 126)

As to describing the present-at-hand, Heidegger warns against gossip in the form of idle talk, ‘Gerede’ in German.

> Discourse, which belongs to the essential state of Dasein’s Being and has a share in constituting Dasein’s disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (..) Thus, by its very nature, idle talk is a closing-off, since to go back to the ground of what is talked about is something which it leaves undone. (..) Because of this, idle talk discourages any new inquiry and any disputation, and in a peculiar way suppresses them and holds them back. (p. 169)

The Heidegger Universe

Summing up, from a Heidegger viewpoint the question 'what is 'is'?' leads to two forms of being: that what is; and how it is. Which depends on how They see it: sentenced by a judging-is as an example of an above category, or accepted by a naming-is as a difference among other examples below, or facetted by an analogizing-is as artistically metaphorized by parallel examples.

By his two-fold statement 'that what is; and how it is', Heidegger suggests that an ordinary sentence as 'Peter destroys the apple' is in fact two sentences, on stating existence, 'Peter is', and one stating a judgement 'destroys the apple', that might be gossip since it can be questioned: Is Peter destroying the apple, or preparing it for food, or transforming it in an artistic process, or …?

As to existence statements, the language has seven basic is-statements: I am, you are, he/she is, it is, we are, you are, they are. Heidegger sees three of these as more basic, I am and it is and they are, describing the core of the meaning of being: I exist in a world together with Things and Others.

So, the core of a Heidegger universe is I and It and They. Or, using Heidegger's terms, Dasein is in a world together with Things and They; and to escape unauthenticity, Dasein must constantly question what is present-at-hand to set it free from its prison of ruling They-gossip, so it becomes ready-at-hand, allowing Dasein an authentic existence. Thus, Dasein should be sceptical towards the essence-claims produced by They using judging-is to trap existence in a predicate-prison. Instead, Dasein should ask the judged to open itself to allow alternative authentic terms to arise using naming-is and analogizing-is.

Traditionally, education means teaching learners about the outside world. Here Heidegger sees a learner as a Dasein having as possibility to transform the surrounding presence-at-hand to ready-at-hand; but being hindered by They, teaching presence-at-hand as examples of textbook gossip instead of arranging meetings allowing the transformation to take place.

As to mathematics education, Heidegger sees Dasein in a world with numbers as entities present-at-hand, but caught in essence-claims of idle talk called mathematics. So to establish an authentic ready-at-hand relationship to them, Dasein must meet numbers directly and replace the gossip's judgment statements pointing up with naming statements pointing down.

However, numbers come in different forms. Buildings often carry roman numbers, and number plates carry Arabic numbers in two versions, an Eastern and a Western. Apparently, numbers are local gossip about something behind, to be seen in the first three Roman numbers, I and II and III, that is, about different degrees of 'Many'.

So, in the sentence 'here are three apples', three is not in the world by itself, apples are, as well as other units as oranges, chairs, days, hours etc. all having the form of plural to signal the presence of Many. Consequently, what is in the world is Many, and it is Many that Dasein should ask to open itself to establish an authentic relationship free of the restrictions of the gossip called mathematics.

Meeting Many

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep balance and to store sounds assigned to what we grasped with our forelegs, freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into languages. In fact, we have two languages, a word-language and a number-language.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair'. Observing the existence of many chairs, we ask 'how many totally?' and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, 'the total is 3 chairs' or, if counting legs, 'the total is 3 fours', abbreviated to 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

With 2017 as the 500year anniversary for Luther’s 95 theses, we can choose to describe meeting Many in 12 theses.

1. Using a folding ruler we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. (Thus, there are four sticks in the four icon, and five sticks in the five icon, etc. Transforming four ones to one fours allows counting with fours as a unit also.)

2. Using a cup for the bundles we discover that a total can be ‘bundle-counted’ in three ways: the normal way or with an overload or with an underload. (Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘bundle-writing’ to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Finally, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s.)

3. Using recounting a total in the same unit by creating or removing overloads or underloads, we discover that bundle-writing offers an alternative way to perform and write down operations. (Thus,

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48)

4. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved in counting by bundling and stacking. (Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate e.g. T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.)

5. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. (Thus, the ‘recount formula’ T = (T/B)*B says that T/B times B can be taken away from T, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the ‘restack formula’ T = (T–B)+B says that T–B is left when B is taken away from T and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict.)

6. Wanting to recount a total in a new unit, we discover that a calculator can predict the result when

bundling and stacking and dragging away the stack. (Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Also, we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.)

7. Wanting to recount a total in tens, we discover that a calculator predicts the result directly by multiplication; only leaving out the unit and misplacing the decimal point. (Thus, asking T = 3 7s = ? tens, the calculator predicts: T = 21 = 2.1 tens. Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.)

And wanting to recount a total from tens to icons, we discover that this again is an example of recounting to change the unit. (Thus, asking T = 3 tens = ? 7s. the calculator predicts: First 30/7 = 4.some; then 30 – (4*7) = 2; and finally T = 30 = 4.2 7s. Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged.)

8. Using the letter u for an unknown number, we can rewrite recounting from tens as 3 tens = ? 7s, as 30 = u*7 with the answer 30/7 = u, officially called to solve an equation; hereby discovering a natural way to do so: Move a number to the opposite side with the opposite sign. (Thus, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to; predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, again moving a number to the opposite side with the opposite sign.)

9. Once counted, totals can be added, but addition is ambiguous. (Thus, with two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to get the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. So next-to addition geometrically means to add areas, and algebraically it means to combine multiplication and addition. Officially this is called integration, a core part of traditional mathematics in high school and at the first year of university.)

10. Also we discover that addition and other operations can be reversed. (Thus, in reversed addition, 8 = u+2, we ask: what is the number u that added to 2 gives 8, which is precisely the formal definition of u = 8–2. And in reversed multiplication, 8 = u*2, we ask: what is the number u that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root, the factor-finder, and logarithm, the factor-counter, as u = 3√20 and u = log3(20). In all cases we solve the equations by moving to the opposite side with the opposite sign. Reversing next-to addition we ask 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer u, from the terminal total T we remove the initial total T1 before we count the rest in 5s: u = (T–T1)/5 = ΔT/5. Combining subtraction and division in this way is called differentiation, the reverse operation to integration combining multiplication and addition.)

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. (Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.)

12. Observing that a quantity may be double-counted in the same unit, we discover that per-numbers may take the form of fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. (Thus, to find 3 per 5 of 20, 3/5 of 20, we just recount 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3. To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and

fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.)

Conclusion

To answer the questions 'what is mathematics, education and research' we looked for an answer in a Heidegger universe by allowing the root of mathematics, the physical fact Many, to open itself for us. This disclosed a 'many-matics' with digits as icons containing as many sticks as they represent; and where counting and recounting and double-counting totals come before adding them next-to and on-top, thus creating a natural order for the four basic operations, also being icons present in the counting process: first division draws away bundles then multiplication lift them to a stack that subtraction takes away to look for unbundled singles. This shows that natural numbers are two-dimensional blocks with a counting-number and a bundle-number as a unit, and with a decimal point to separate the bundles from the unbundled. Once counted, blocks can be added where next-to addition means adding areas, also called integration; and where on-top addition means recounting in the same unit to remove or create overloads. And where reversed addition next-to and on-top leads to differentiation and equations. Double-counting in different units leads to per-numbers being added or calculated in calculus, present in primary school as adding blocks, and in middle and high school, as adding piecewise and locally constant per-numbers. Finally, letters and functions are used for unspecified numbers and calculations.

Many-matics differs in many respects from traditional mathematics; that presents digits as symbols and numbers as names for points along a one-dimensional number-line; that neglects counting and recounting and double-counting and next-to addition and goes directly to on-top addition first, then subtraction, then multiplication and in the end division leading on to fractions that by being added without units becomes an example of 'mathe-matism' true inside but seldom outside classrooms: ½ + 2/3 is claimed to be 7/6 in spite of the fact that 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and certainly not 7 red of 6 apples. Being set-based, definitions use self-referring judging-is statements from above instead of naming-is statements from below, thus defining a concept as 'meta-matics', that is, as an example of an abstraction instead of as an abstraction from examples, as it was created historically. Thus a function is defined as an example of a set-relation where first-component identity implies second-component identity, instead of as a placeholder for an unspecified calculation with unspecified numbers. A closer look thus discloses traditional set-based mathematics as 'meta-matism', a mixture of meta-matics and mathe-matism.

Meta-matism as '2+3 = 5' adding numbers without units contradicts observations as 2weeks + 3 days = 17 days. And it makes a syntax error in the number-language sentence 'T = 2+3' by silencing the subject and the verb. By keeping the gossip part and leaving out the existence part, meta-matism ceases to be a number-language describing the real world. This contradicts the historic origin of mathematics as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many.

In Greek, geometry means earth measuring, which is done by dividing earth into triangles. In Arabic, algebra means to reunite numbers. Writing out a total T as we say it, T = 345 = 3*ten*ten + 4*ten + 5*1, shows a number as blocks united next-to each other. Also, we see algebra's four ways to unite numbers: addition, multiplication, repeated multiplication or power, and block-addition also called integration. Which is precisely the core of mathematics: addition and multiplication together with their reversed operations subtraction and division in primary school; and power and integration together with their reversed operations root, logarithm and differentiation in secondary school. Including the units, we see there can only be four ways to unite numbers: addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers.

As to traditional set-based mathematics, its idea of deriving definitions from the mother concept set leads to meaningless self-reference as in the classical liar paradox 'This sentence is false', being true if false and false if true. This was shown by Russell looking at the set of sets not belonging to itself. Here a set belongs to the set if it doesn't, and does not belong if it does.

To avoid self-reference, Russell created a hierarchical type theory in which fractions could not be numbers if defined by numbers, e.g. as equivalence classes in a set of number-pairs as done by set-based mathematics that consequently invented a new set-theory that by mixing sets and elements also mixes concrete examples and their abstract names, thus mixing concrete apples that can feed humans and the word 'apple' that cannot. By mixing things and their names, existence and gossip, set-based mathematics and its meta-matism fill the number-language with both semantic and syntax errors. Still, this language has entered universities worldwide as the only true version of mathematics to be transmitted through education that is improved using research to produce solid findings.

In a Heidegger universe, education means allowing I to meet It directly without They and its patronizing gossip; and to replace judging-is with naming-is when choosing how to label It. Likewise with research seen as a collective education replacing ungrounded categories with grounded ones.

So, maybe the answer to the question about solid findings in mathematics education research is 'Only one: to improve, mathematics education should ask, not what to do, but what to do differently.' Maybe research should not study problems but look for hidden differences that make a difference.

However, difference research scarcely exists today since it is rejected at conferences (Tarp 2015) for not applying or extending existing theory that might produce new researchers and feed a growing appliance industry, but being unable to reach its goal, to improve mathematics education.

In short, to be successful, mathematics education research must stop studying the misery coming from teaching meta-matism in compulsory classes. Instead, mathematics must respect its origin as a natural science grounded in Many. And research must search for differences and test if they make a difference, not in compulsory classes, but with daily lessons in self-chosen half-year blocks. Then learning the word-language and the number-language together may not be that difficult, so that all will leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a sentence, the subject exists, but the sentence about it may be gossip; so stop preaching essence and start teaching existence; or, bring the subject to the classroom and leave the sentence outside.

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10. Count and Multiply Before You Add: Proportionality and Calculus for Early Childhood and Migrants

Disappointing PISA results might be caused by a goal displacement seeing mathematics as the goal and its outside root, Many, as a means. Meeting Many free of a self-referring mathe-matics uncovers an alternative 'many-matics'; with digits and operations as icons containing as many sticks as they represent and showing the counting process; with multiplication before addition; with numbers as two-dimensional decimal numbers with units created by bundle-counting, and ready to be re-counted in the same unit to remove or create overloads to make operations easier, or in a new unit, later called proportionality, or to and from tens. Addition now occurs both on-top and next-to, later called integration. So, to improve, mathematics should again be a natural science about Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, where it was the lowest in the Nordic countries and significantly below the OECD average. This caused OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other countries also experience declining PISA results. Since mathematics education is a social institution, social theory might be able to explain the 50 years of unsuccessful research.

Social Theory Looking at Mathematics Education

As to the nature of sociology, Bauman talks about its role and about organizations:

> Sociological thinking (..) renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. (..) **Rational** action is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right: the new end against which the organization tends to measure the rationality of its performance. (Bauman, 1990, pp. 16, 79, 84)

As an institution, mathematics education is a public organisation with a 'rational action in which the end to be achieved is clearly spelled out', apparently aimed at educating students in mathematics: The goal of mathematics education is to teach mathematics. On the other hand, such a goal is self-referring; as is 'the goal of bublibub education is to teach bublibub'. So we ask: in mathematics education, **is** mathematics the end or **is** it a means; and if so then what **is** the end? Or, in other words: **is** there a goal displacement in mathematics education?

The answer to these is-questions may be found in what Bauman calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'? (Bauman, 1992, p. ix)

Institutional Skepticism

On the first page of his book 'Being and Time', Heidegger writes that his aim is 'to work out the question of the meaning of *Being* and to do so concretely'. As to looking for an answer, Heidegger says:

> Every enquiry is a seeking. Every seeking gets guided beforehand by what is sought. Inquiry is a cognizant seeking for an entity both with regard to the fact that it is and with regard to its Being as it is. (Heidegger, 1962, p. 5)

Here Heidegger describes two uses of 'is'. One claims existence, 'M is', one claims 'how M is' to others, since what exists is perceived by humans categorizing it by naming or characterizing or analogizing it to create 'M is N'-statements. This gives a total of four uses:

'Is' can claim a mere existence, 'M is'. 'Is' can point down as a 'naming-is' ('M is for example P or Q or …') defining M as an abstract name for its volume of concrete examples. 'Is' can point up as a 'judging-is' ('M is an example of N') defining M as member of a more abstract category N. Finally, 'is' can point over as an 'analogizing-is' ('M is like N') portraying M by a metaphor carrying over known aspects from a similar N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are', or, I exist in a world together with It and with They, with Things and with Others. The 'I' Heidegger calls 'Dasein', the 'It' he calls 'present-at-hand', and 'They' he calls 'the Others'. To have real existence, Dasein must create an authentic relationship to 'It' by transforming what is 'present-at-hand' to 'ready-at-hand'. However, this is made difficult by the dictatorship of 'They', shutting 'It' up in a predicate-prison of idle talk, gossip:

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

In France, Heidegger inspired the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu pointing out that society forces words upon you to diagnose you so it can offer you cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world. (Lyotard, 1984)

From a Heidegger view, education preparing learners for the outside world is seen as Dasein having the transformation from present-at-hand to ready-at-hand hindered by They hiding presence-at-hand in textbook sentences. To create an authentic relationship, Dasein therefore should insist on meeting the subject and neglect the rest of the sentence.

As to mathematics education, a Heidegger view sees Dasein in a world with numbers present-at-hand, but caught up in idle talk called mathematics. So, to establish an authentic ready-at-hand relationship to them, Dasein must meet numbers directly and replace the gossip's judgments pointing up with naming definitions pointing down.

However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. Apparently, numbers are local gossip about something behind, to be seen in the first three Roman numbers, I and II and III, that is, different degrees of 'Many'.

Consequently, what is in the world is Many, so Many should be the subject in the questions Dasein poses to obtain an authentic relationship free of the restrictions of the potential gossip called mathematics.

Mathematics as Self-Referring Gossip

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time; and together forming the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here the invention of the concept SET created a set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts, self-referring defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false:

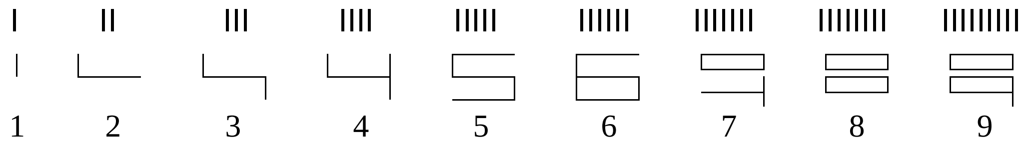
If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts. In this way SET transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside a classroom where adding numbers without units as ‘1 + 2 IS 3’ meet counter-examples as e.g. 1 week + 2 days is 9 days.

Meeting Many

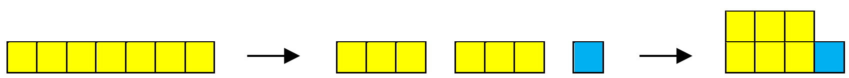
Many exists all over time and space: Many days and years, many houses and cars etc. To deal with Many we ask, ‘how many in total?’ To answer, we count and add. We count by bundling and stacking as seen when holding 4 fingers together 2 by 2 makes a 3-year-old child say: ‘That is not 4, that is 2 2s.’; and when writing out fully the total T = 456 = 4*B^2 + 5*B + 6*1 showing three stacks or blocks next-to each other: one with 4 bundles of bundles, one with 5 bundles, and one with 6 unbundled singles.

Digits occur by rearranging sticks into icons so that five ones become one five-icon 5 with five sticks, if written less sloppy. In this way, we create icons until ten since the bundle-number is counted as 10, one bundle and no unbundled, followed by eleven and twelve meaning ‘one left’ and ‘two left’ in ‘Anglish’, a western Danish dialect around Harboøre from where the Vikings sailed to ‘Angland’.



A total may be counted in several ways. Some gather-hunter cultures count ‘one, two, many’. Agriculture needed to differentiate between degrees of many: ‘1, 2, 3, 4, .., 10, 11, 12’, etc. To include the bundle-size we can count ‘01, 02, .., B, 1B1, 1B2’, etc.; or ‘0.1 tens, 0.2 tens’, etc. To signal closeness to the bundle we can count ‘1, 2, …, 7, ten less 2, ten less 1, ten’, etc.

A total is re-counted in another unit when asking ‘7 is how many 3s?’ Using squares or LEGO-blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to so that a total T of 7 1s can be counted in 3s as T = 2 3s and 1:



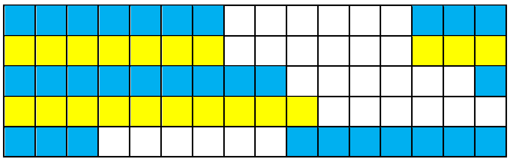
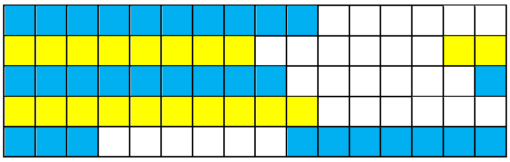
With ‘bundle-counting’ we place the bundles in a bundle-cup with a stick for each bundle, leaving the unbundled outside. Then, with icons, we report by ‘bundle-writing’, T = 2B1 3s, and by ‘decimal-writing’, T = 2.1 3s where a decimal point separates the bundles from the unbundled, always including the unit 3s. A stick moves outside the cup as a bundle of 1s, and inside as 1 bundle. This will change the ‘normal’ form to an ‘overload’ or an ‘underload’:

T = 7 = IIIIIII → III III I → II] I → 2B1 3s = 1B4 3s = 3B-2 3s

Using a plastic letter B for the bundles, we get

T = 7 = IIIIIII → III III I → BBI → 2BI 3s

We include space and time in the two ways of counting: ‘geometry-counting’ in space, and ‘algebra-counting’ in time. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.

To predict the result, we use a calculator. A stack of 2 3s is iconized as 2x3 (or 2*3) showing a lift used 2 times to stack the 3s. As for taking away, subtraction shows the trace left when taking away just once, and division shows the broom wiping away several times.

So, by entering ‘7/3’ we ask the calculator ‘from 7, 3s can be taken away how many times?’ The answer is ‘2.some’. To find the leftovers we take away the stack of 2 3s by asking ‘7 – 2*3’. From the answer ‘1’ we conclude that 7 = 2B1 3s. Showing ‘7 – 2*3 = 1’, a display indirectly predicts that 7 can be re-counted as 2 3s and 1, or as 2.1 3s.

7 / 3	2.some
7 – 2 * 3	1

A calculator thus uses a ‘recount-formula‘, T = (T/B)*B, saying that ‘from T, T/B times, Bs can be taken away’; and a ‘restack-formula’, T = (T–B)+B, saying that ‘from T, T–B is left if B is taken away and placed next-to’. The two formulas may be shown by using LEGO blocks.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 4 2s as 3B2 2s with an outside overload; or as 5B-2 2s with an outside underload thus leading to negative numbers:

Letters	Sticks	Total T	Calculator	
B B B B	II II II II	4B0 2s	4*2 – 4*2	0
B B B I I	II II II I I	3B2 2s	4*2 – 3*2	2
B B B B B B	II II II II II II	5B-2 2s	4*2 – 5*2	-2

To re-count in a different unit means changing unit, also called proportionality. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2B2 5s.

T = 3 4s = IIII IIII IIII → IIIII IIIII I I = 2B2 5s.

A calculator can predict the result. Entering ‘3*4/5’ we ask ‘from 3 4s, 5s can be taken away how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be re-counted as 2 5s and 2, or as 2B2 5s, or as 2.2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Re-counting in and from tens

Re-counting icon-numbers in the standard bundle-size tens leads to questions as ‘3 4s is how many tens’. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens.

T = 3 4s = IIII IIII IIII → IIII IIII II I I → 1B2 tens = 1.2 tens

Using the recount- and restack-formula above is impossible since the calculator has no ten button. Instead it gives the answer directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

3 * 4	12

Re-counting icon-numbers in tens is called multiplication tables to be learned by heart. However, seeing 3*4 as 3 4s means seeing multiplication as a geometrical block that re-counted in tens will increase its width and therefore decrease its height to keep the total unchanged. Furthermore, the ten-by-ten table can be reduced to a small 4-by-4 table since 5 is half of ten and 6 is ten less 4, 7 is ten less three etc.

Thus T = 4*7 = 4 7s that re-counts in tens as T = 4*7 = 4*(10 – 3) = 40 – 12 = 28 = 2.8 tens; and T = 6*9 = (10 – 4)*(10 – 1) = 100 – 40 – 10 + 4 = 54.

These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

Re-counting tens in icons by asking ‘38 = ? 7s’ is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens, 0.3 7s must be written as what it is, 3 counted in 7s, also called a fraction 3/7, and calculated as 0.428…, shown directly by the calculator:

38 / 7	5.428

Once counted, totals can be added on-top or next-to

Asking ‘3 5s and 2 3s total how many 5s?’ we see that to add on-top, the units must be the same so the 2 3s must be re-counted in 5s as 1B1 5s that added to the 3 5s gives 4B1 5s.

T = 3 5s + 2 3s = IIIII IIIII IIIII + III III → IIIII IIIII IIIII + III II I = 4B1 5s = 4.1 5s.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. So we get 4B1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas, called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2B5 8s.

T = 3 5s + 2 3s = IIIII IIIII IIIII + III III → ~~IIIIIIII~~ ~~IIIIIIII~~ IIIII = 2B5 8s = 2.5 8s

Using a calculator to predict the result we use a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away 2 8s leaves 5. So we get 2B5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing adding on-top and next-to

Reversed addition may be called backward calculation or solving equations. Reversing next-to addition may be called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2B6 8s?’, using sticks will give the answer 2B1 3s:

IIIII IIIII IIIII + III III I ← IIIII III IIIII III IIIII I ← II] IIIIII = 2B6 8s

Using a calculator to predict the result, the remaining is bracketed before counted in 3s.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Adding or integrating two areas next-to each other means multiplying before adding. Reversing integration, i.e. differentiation, then means subtracting before dividing, as shown in the gradient formula y' = Δy/t = (y2 – y1)/t.

Double-counting in different units creates per-numbers and proportionality

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 5kg, or 2$/5kg. To answer the question '6$ = ?kg' we use the per-number to re-count 6 in 2s, that many times we have 5kg: 6$ = (6/2)*2$ = (6/2)*5kg = 3*5kg = 15kg. And vice versa: Asking '?$ = 20kg', the answer is 20kg = (20/5)*5kg = (20/5)*2$ =4*2$ = 8$.

Double-counting in the same unit creates fractions and percentages as per-numbers

Double-counting a quantity in the same unit, per-numbers take the form of fractions, 3$ per 5$ = 3/5, or percentages as 3$ per 100$ = 3/100 = 3%.

Thus, to find 3$ per 5$ of 20$, or 3/5 of 20, we just re-count 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3.

To find what 3$ per 5$ is per 100$, or 3/5 = ?%, we just re-count 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So 3$ per 5$ is the same as 60$ per 100$, or 3/5 = 60%.

Adding 3kg at 4$/kg to 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8$/kg. Likewise with adding fractions. Thus, per-numbers and fractions are not numbers, but operators needing a number to become a number, and adding by their areas; thus becoming integration as when adding block-numbers next-to each other. Thus, calculus appears at all school levels: at primary level, at lower and at upper secondary level, and at tertiary level.

Using letters and functions for unspecified numbers and calculations

We can set up a calculation with an unspecified number u, as T = 2 + ? = 2 + u. Also, we can set up an unspecified calculation with an unspecified number u, as T = 2 ? u = f(u), called a formula or a 'function'. Although we can write it, T = f(2) is meaningless since 2 is not an unspecified number.

Comparing Many-matics with Mathe-matics

Meeting Many free of gossip discloses a 'many-matics' quite different from traditional set-based mathe-matics:

As to digits, the former sees them as icons containing as many sticks as they represent; the latter sees them as symbols like letters.

As to natural numbers, the former sees them as two-dimensional blocks described as decimal numbers with a unit and a decimal point to separate inside bundles from outside singles; the latter sees them as examples of a place value system naming points along a one-dimensional number-line.

As to operations, the former sees them as icons describing the three parts of a counting process: bundling by division, stacking by multiplication, removing stacks by subtraction, and uniting stacks by on-top or next-to addition; the latter sees them as mappings from a set-product to a set, and in the opposite order: addition first, then subtraction, then multiplication, then division.

As to calculators, the former sees them as means to predict a re-counting result; the latter typically sees them as hindering understanding.

As to the different forms of counting (bundle-counting, re-counting in the same and a different unit, re-counting to and from tens) the former sees them as means to describe a total by a number to answer the basic question 'how many in total?'; the latter sees bundle-counting and re-counting as irrelevant because of the place value system with base ten.

As to addition, the former sees addition to be postponed till after the total has been bundle-counted and re-counted, and sees on-top and next-to addition to be treated at the same time; the latter sees addition as the first operation to introduce and in no need of re-counting or overloads or next-to addition since numbers are counted in tens only.

As to multiplication, the former sees a product as a number with a unit that may or may not be re-counted in tens, and sees overloads as a natural way to report the result; the latter sees it as a calculation following a specific algorithm.

As to division, the former uses bundle-writing to re-count a number with an overload to make the division easier; the latter sees it as a calculation following a specific algorithm.

As to fractions, the former sees them as per-numbers, both being operators needing a number to produce a number, thus being added by their areas, i.e. by integration; the latter sees fractions as rational numbers that can be added without considering units.

As to proportionality, the former sees it as double-counting in two units creating a per-number as a bridge between the units; the latter sees it as an example of a linear function.

As to equations, the former sees it as a name for reversed calculation to be solved by moving to opposite side with opposite sign; the latter sees it as an equivalence statement to be changed by performing the same operations to both sides aiming at neutralizing the numbers next to the unknown.

As to calculus, the former sees three types, preschool calculus adding blocks next-to each other, middle school calculus adding piecewise constant per-numbers and fractions by their areas, and high school calculus finding the are under a locally constant (continuous) graph; the latter sees differential calculus as preceding integral calculus.

Testing a Many-Matics Micro-Curriculum

A '1 cup and 5 sticks' micro-curriculum can be designed to help a class stuck in division. The intervention begins by bundle-counting 5 sticks in 2s, using the cup for the bundles. The results, 1B3 2s and 2B1 2s and 3B-1 2s, show that a total can be counted and written in 3 ways, overload and normal and underload. So, to divide 336 by 7, 5 bundles are moved outside as 50 singles to re-count 336 with an overload: 336 = 33B6 = 28B56, which divided by 7 gives 4B8 = 48. With multiplication singles move inside as bundles: 7* 4[8 = 28[56 = 33[6 = 336.

Ending the Dienes Era

No research literature on bundle-counting was found. However, similar ideas were found at Dienes, the inventor of Multi-base blocks. As to the place value system, Dienes says:

> I have been suggesting, for the past half century, that different bases be used at the start, and to facilitate understanding of what is going on, physical materials embodying the powers of various bases should be made available to children. Such a system is a set of multibase blocks (..) Educators today use the "multibase blocks", but most of them only use the base ten, yet they call the set "multibase". These educators miss the point of the material entirely. (Dienes, 2002, p. 1)

Dienes wants children to use physical blocks to understand counting with different bases and the role of power. So here mathematics is the goal and blocks are a means. Which Dienes makes clear by his 6 stages to real understanding of set-based mathematics: Free play, following rules, comparison, representation, symbolization, formalization. (Clouthier, 2010)

A Dienes-approach to mathematics thus is an example of a goal displacement having self-referring set-based mathematics as its goal, and dedicated blocks as a means.

Conclusion

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject and a verb and an object or predicate, 'This is a chair'. As does the number-language assigning numbers to like things, 'the 3 chairs have 4 legs each', abbreviated to 'the total is 3 fours', or 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is an operation'. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, and with the number-language in its original form where mathematics was a common name for algebra and geometry both rooted in the physical fact Many. But in its present self-referring set-based form, mathematics has turned into 'meta-matism', a grammar for the number-language seen as the goal in mathematics education, using both the outside world and the number-language as means. So, by its self-reference and by its difference from many-matics, mathe-matics has a goal displacement.

Recommendation

There is no second chance to make a first impression. So how learners meet mathematics matters, both in early childhood and as migrants. Consequently, education preparing for the outside world should bring into the classroom examples of Many, becoming Totals when bundle-counted as block-numbers with units, thus ready to be re-counted in the same unit or in another unit or in or from tens, always illustrated algebraically using bundle-writing and geometrically using LEGO-blocks. Keeping algebra and geometry together introduces negative numbers and proportionality before introducing addition, that might even be postponed to after double-counting in physical units have introduced per-numbers and fractions and percentages.

The core of mathematics education, proportionality and calculus and equations, lies in counting. So, to improve mathematics education, counting and multiplication come before adding; number-sentences are written out fully as 'T = 6*7' instead of just '6*7 = 42' depriving the total of its true identity as 6 7s by forcing it to be re-counted in tens right away; bundle-counting and re-counting in tens precede the place value system introduced as a sloppy but quicker way to write natural numbers without units and with misplaced decimal point; addition includes both the on-top and next-to version, both followed by reversed addition leading to equations. Finally, double-counting leads to per-numbers and fractions and percentages, all added by their areas thus creating the right order in calculus by letting integration precede differentiation.

Thus, mathematics education should begin with number-language sentences as T = 6*7 = 6 7s, or its general form as the recount-formula T = (T/B)*B occurring all over mathematics: when re-counting in another unit; when solving equations by re-counting, leading to the 'opposite side & sign'-method, u*3 = 24 = (24/3)*3 giving u = 24/3 = 8; when double-counting to change unit; when relating proportional quantities; in trigonometry as a = a/c*c = sinA*c; and in calculus as dy = dy/dx*dx = y'*dx. And addition should begin as calculus when adding blocks next-to before on-top.

In short, fixation of mathematics as a goal in itself, hides the real goal, its outside root Many. So stop teaching self-referring meta-matism and start teaching many-matics, a natural science about the physical fact Many.

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11 Proposals for the Mathematics Biennale 2018

01) Start-math for children and migrants: Bundle-count and re-count before adding

A 3year old sees 4 fingers held together 2 by 2: "It is not 4, it is 2 2s". So a child counts in the block-numbers as we do: 456 = 4 bundle-bundles + 5 bundles + 6 unbundled. The child's block-numbers lead directly to proportionality and equations. So we should count before we add.

Digits units many sticks in one icon: Five sticks in the 5-icon etc.; up to ten = 1bundle0 = 1B0 = 10.

With a cup for bundles, a total T of 7 sticks is bundle-counted in icon-bundles as T = 7 = 2 3s & 1 = 2B1 3s. Next, the total can be re-counted in the same unit to create overload or underload:

T = 7 = 2B1 3s = 1B4 3s = 3B-2 3s.

A total can also be re-counted in a new unit (proportionality), e.g. 2 4s = ? 5s, predicted by a calculator as 2*4/5 = 1 and 2*4 – 1*5 = 3, so T = 2 4s = 1B3 5s.

We count by bundling and stacking predicted by operations, also being icons: Counting a total 8 in the 2s, 8/2 shows the broom that from 8 sweeps 2s away. Multiplication 4x2 shows the lift that stacks the 4 2s, and subtraction 8 – 2 shows the trace created by from 8 dragging 2 away. The result may therefore be predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

Re-counting from icon-bundles to 10s leads to the multiplication table:

T = 3 4s = 3*4 = 12 = 1ten2 = 1B2 10s.

Reversing by re-counting from 10s to icons creates equations to be solved by re-counting: 'How many 5s give 40' leads to the equation ‘x*5 = 40’ solved by recounting 40 in 5s: 40 = (40/5)*5, giving x = 40/5. So an equation is solved by moving to the opposite side with the opposite sign.

02) Multiplication before addition strengthens the number sense in children and migrants

We master Many using a number-language with number-language sentences, formulas, e.g. T = 4 5s = 4*5, showing how we master Many by bundling and stacking. So 4*5 is 4 5s that may be re-counted in another unit, e.g. in 7s. Or in tens, the international bundle-size.

Viewing numbers as bundle-formulas makes math easy and prevents math-problems and dyscalculia; therefore, to be practiced with various counting rhymes where '5, 6, 7, 8, 9, 10' is counted also as ' 5, bundle less 4, B-3, B-2, B-1, bundle'; and as ' ½bundle, ½bundle & 1, ½, ½bB&2, ½B &3, ½B &4, bundle. Likewise, '10, 11, 12, 13, 14, 15 ' can be counted as 'bundle, 1bundle & 1, 1B & 2, 1B & 3, 1B & 4, 1B & 5 ', and as 'Bundle, 1left, 2left, 3left, 4left, 5left ' in order to show that 'eleven' and 'twelve' is derived from the Viking age.

Digits unite many sticks into one icon: Five sticks in the 5-icon, etc., up to ten = 1bundle0 = 1B0 = 10. With a cup for bundles, a total T of 7 sticks is bundle-counted in icon-bundles as T = 7 = 2 3s & 1 = 2B1 3s. Next, the total can be re-counted in the same unit to create overload or underload: T = 7 = 2B1 3s = 1B4 3s = 3B-2 3s. Likewise with totals counted in tens, T = 68 = 6B8 = 5B18 = 7B-2.

Before adding, the number sense is trained by the multiplication table, reduced to a 5 x 5 table by rewriting number above 5, e.g. 6 = ½bundle & 1 = bundle-4. First doubling, e.g. T = 2*7 = 2*(½ bundle & 2) = bundle & 4 = 14, or T = 2*7 = 2*(bundle-3) = 20-6 = 14. Then with bundle-counting, e.g. T = 2*38 = 2*3B8 = 6B16 = 7B6 = 76. Then halving, e.g. ½*38 = ½*3B8 = ½*4B-2 = 2B-1 = 19.

Multiplying with 5 means multiplying with half-bundles, 5*7 = ½bundle*7 = ½*70 = ½* 6B10 = 3B5 = 35.

03) Dislike towards division cured with 5 sticks and 1 cup and bundle-writing

A class has problems with division, e.g. 336/7. The solution is to see 336/7, not as 336 divided among 7, but as 336 counted in 7s; and to use bundle-writing 336 = 33B6, where the cup splits the total in bundled within the cup and unbundled outside.

And to bundle-count totals in three ways: normal and with overload or underload.

First with 5 sticks bundle-counted in 2s with a cup to the bundles.

Normal: T = IIIII = II II I = 2B1 2s.

With overload: T = IIIII = II III = 1B3 2s.

With underload: T = IIIII = II II II = 3B-1 2s.

In the same way we count in 10s: T = 74 = 7B4 = 6B14 = 8B-6.

So, with a total of 336 (i.e. 33.6 tens) there are 33 bundles inside the cup and 6 unbundled outside. But we prefer 28 within, so 5 bundles move outside as 50 giving 56 outside that divided by 7 gives 4 inside and 8 outside:

T = 336 = 33B6 = 28B56, and T/7 = 4B8 = 48.

Bundle-writing can be used by all operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65-27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7 * 48 = 7 * 4B8 = 28B56 = 33B6 = 336

T = 7 * 48 = 7 * 5B-2 = 35B-14 = 33B6 = 336

T = 336/7 = 33B6/7 = 28B56/7 = 4B8 = 48

T = 338/7 = 33B8/7 = 28B58/7 = 4B8 + 2/7 = 48 2/7

Bundle-writing can also be used with the multiplication table:

T = 4*8 = 4*1B-2 = 4B-8 = 32 and 7*8 = 7*2 = 1B-7B-14 = 6B-4 = 5B6 = 56

04) Fractions and percentages as per-numbers

A class has trouble with fractions. Both to find a fraction of a total, and to expand and shorten fractions, where many add and subtract instead of multiplying and dividing.

The solution is to see a fraction as a per-number obtained by a double-counting in the same unit, 2/3 = 2$ per 3$ or as percentage 2% = 2/100 = 2$ per 100$.

Investment is expected to give a return, which may be higher or lower, e.g. 7$ per 5$ or 3$ per 5$.

With re-counting and double-counting we use a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

Now 2/3 of 12 is found as 2$ per 3$ of 12$. So we re-count 12 in 3s as 12$ = (12/3)*3$ giving (12/3)*2$ = 8$. So 2/3 of 12 is 8.

The task 'what is 3 in percent of 5? ' is solved by re-counting 100 in 5s and replace 5$ with 3$: T = 100$ = (100/5)*5$ giving (100/5)*3$ = 60$. So 3/5 is the same as 60 per 100, or 3/5 = 60%.

To expand or shorten a fractions is done by inserting or removing the same unit above and below the fraction bar: T = 2/3 = 2 4s/3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6.

Fractions and decimal numbers should be introduced in grade 1 relating to counting in icons under ten. 7 counted in 3s gives a stack on the 2 3s & 1. The unbundled 1 can be placed next-to as its own stack, a decimal number, T = 7 = 2.1 3s. Or it can be placed on-top counted as 3s, i.e. as a fraction: T = 7 = 2 1/3 3s.

05) Fractions and per-numbers add as integration

A class have problems with adding fractions. Many adds the numerators and the denominators separately.

The solution is to view a fraction as a per-number obtained from double-counting in the same unit, 3/5 = 3$ per 5$, or as the percentage of 3% = 3/100 = 3$ per 100$. As well as to begin with adding fractions with units, such as ½ of 2 + 2/3 of 3, that just gives 1 + 2 of 2 + 3, so 3/5 of 5. Here, then, ½ + 2/3 = 3/5, which is obtained by adding the numerators and denominators separately.

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg. So here is 3$/kg + 5$/kg = 4.33 $/kg, called the weighted average.

Geometrically, adding products means adding areas, called integration. So per-numbers add by their areas under the piecewise constant per-number graph. Corresponding with fractions.

Adding two fractions a/b and c/d without units is meaningless, but can be given meaning if taken of the same total, b*d: a/b of b*d + c/d of b*d gives a total on a*d + c*b of b*d.

So a/b + c/d = (a*d + c*b)/b*d.

Adding fractions and per-numbers with units provides a good introduction to calculus. As shown, multiplication before addition is the same as integration. And inverted integration is the same as differentiation:

The task '2kg at 3$/kg + 4kg at ? $/kg = 6kg á 5$/kg' leads to the equation 6 + 4*x = 30 or T1 + 4*x = T2, solved with subtraction before division, called differentiation: x = (T2-T1)/4 = ΔT/4.

06) Proportionality as double-counting, with per-numbers

A class has a problem with proportionality. The price is 2$/3 kg. All will find the $-number for 12 kg, but only a few will find kg-number for 16$. The solution is to rename proportionality to 'shifting units' by 'double-counting', leading to 'per-numbers' such as 2$ per 3 kg or 2$/3kg or 2/3 $/kg. The units are connected by re-counting the known part of the per-number.

With re-counting and double-counting we use a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

This allows re-counting 16$ in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24kg. Likewise, the 12kg re-counts in 3s as T = 12 kg = (12/3)*3 kg = (12/3)*2$ = 8$. Will this difference make a difference? In theory, yes, since proportionality is associated with counting, a basic physical activity.

In fact, proportionality takes place in grade 1 when counting totals in icon-bundles different from the standard bundle ten and by afterwards re-counting the total in a different unit. This leads directly to the re-count formula, which has the same shape as y = k*x.

Thus, a total of 8 re-counts in 2s as T = (8/2)*2 = 4*2 = 4 2s.

And a total of 3 4s re-counts in 5s as T = (3*4/5)*5 = 2*5 + 2.

And per-numbers lead directly on to the fractions, obtained by double-counting in the same unit, e.g. 2$ per 3$ = 2$/3$ = 2/3 = 2 per 3.

Getting 2/3 of 15 means getting 2$ per 3$ of 15$ found by re-counting 15 in 3s and take 215 thereof: T = 15$ = (15/3)*3$ giving (15/3)*2$ = 10$. So 2/3 of 15 is 10.

Likewise, 20% of 15 is found by re-counting 15 in 100s:

T = 15 = (15/100)*100 giving (15/100)*20 = 3.

07) Equations solved by moving, reversing or re-counting

A class has problems with the equations 2+3*u = 14 and 25 – u = 14 and 40/d = 5, where the equation is composite or where the unknown has a inverse sign. The solution is to use the definitions of the inverse operations to create the basic solution rule: 'move to the opposite side with the opposite sign'.

In u+3 = 8 we seek a numbers u that added to 3 gives 8, which is u = 8-3 by definition; so + 3 moves to the opposite side with the opposite sign. Corresponding with u*2 = 8, solved by u = 8/2; and with u^3 = 12, solved by u = 3√12, where the root is a factor-finder; and with 3^u = 12, solved by u = log3 (12), where the logarithm is a factor-counter.

The equation 2+3*u = 14 can be seen as a double calculation that is reduced to a single by a bracket around the stronger operation, 2+(3*u). Moving 2 to the opposite side with the opposite sign gives 3*u = 14–2. Then 3 moves to the opposite side with opposite sign, but first a bracket is placed around what first must be calculated: u = (14–2)/3 = 12/3 = 4.

Equations can also be solved by walking forward and backward: Forward we first multiply with 3 and then add 2. Backwards, we first subtract 2 and then divide by 3, so u = (14–2)/3 = 4.

In the equation 25 – u = 14, u has opposite sign and therefore moves to the opposite side to get a normal sign before 14 moves to the opposite side with opposite sign:

25 = 14 + u; 25–14 = u; 11 = u.

Corresponding with 40/u = 5 giving 40 = 5*u and 40/5 = u or 8 = u.

Having learned re-counting this can also be used:
40 = (40/u)*u = 5*u and 40 = (40/5)*5, giving u = 40/5.

08) Calculus: Addition of and division into locally constant per-numbers

A class has problems with calculus. The solution is to postpone differential calculus until after integral calculus is taught as a means of adding piecewise or locally constant per-numbers by their areas.

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg.

Geometrically, adding products means adding areas, called integration. So per-numbers add by their areas under the piecewise constant per-number graph, i.e. by adding a few area strips, S = Σp*Δx.

A non-constant per-number is locally constant (continuous), meaning adding of countless many area strips, S = ∫p*dx. Unless we can rewrite the strips as changes, p*dx = dy or dy/dx = p. For when adding changes, all middle terms disappear leaving just the total change from the start to the end point.

This motivates differential calculus: If the strip 2*x*dx can be rewritten as a change, d (x^2), then the sum ∫ 2*x*dx is the change of x^2 from the start to the end point.

Change-formulas can be observed in a rectangle, where changes Δb and Δh in the base b and height h gives the total change of the area ΔT as the sum of a vertical strip, Δb*h and a horizontal strip, b*Δh; and a corner, Δb*Δh that can be neglected with small changes.

Therefore, d(b*h) = db*h + b*dh, or, if counted in Ts:

dT/T = db/b + dh/h, or with T' = dT/dx, T'/T = b'/b + h'/h.

So with (x^2)'/x^2 = x'/x + x'/x = 2*x'/x, (x^2)' = 2*x since x' = dx/dx = 1.

So differentiation is a smart way to add many numbers; but also useful to describe growth and decay and optimization.

09) Calculus in primary, middle and high school

Mathematics means knowledge in Greek, who chose the word as a common name for their four areas of knowledge, arithmetic and geometry and music and astronomy, which they saw as the study of many by itself, in space, in time and in space and time.

With music and astronomy gone, today mathematics is just a common name for algebra and geometry, both rooted in Many as evidenced by their meaning in Arabic and Greek: to reunite numbers and to measure the earth. Meeting Many we ask 'How many in total?' The answer we get by counting, before we add. Counting is done by bundling and stacking,

predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B', e.g. T = 3 4s = (3*4)/5*5 = 2 5s & 2.

Once counted, stacks can be added, but on-top or next-to?

Next-to addition of the stacks 2 3s and 4 5s as 8s means adding their areas, i.e. by integration, where multiplication comes before addition.

Reversed, we ask '2 3s +? 5s gives 3 8s', now letting subtraction come before division, called differentiation.

So in primary school, calculus occurs with next-to addition of stacks.

In middle school calculus occurs with blending and average tasks:

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg.

Geometrically, adding products means adding areas, called integration.

Thus per-numbers add by the area under the piecewise constant per-number graph, i.e. by adding area strips, $S = \Sigma p*\Delta x$, or $S = \int p*dx$ in high school, where per-numbers are locally constant (continuous), and where per-numbers are added before they can be subtracted by differentiation.

10) Stem-based core-math makes migrants pre-engineers

We master the outside world by a word-language and a number-language, describing it by sentences and formulas containing a subject, a verb, and a predicate: 'the table is yellow' and 'the total is 3 4s'. The two languages both have a meta-language, a grammar and a mathematics, that should be learned after the language, otherwise causing dyslexia and dyscalculia.

Young migrants get direct access to the number-language with core-math curriculum:

A) Digits are the icons with the number of sticks, it represents.

B) Operations are icons for counting by bundling and stacking: division removes bundles, multiplication stack bundles, subtraction removes a stack to look for unbundled, addition unites stacks on-top or next-to.

C) Bundle-counting and bundle-writing shows the bundles inside the cup and the un-bundled outside, e.g. T = 4B5 = 4.5 tens = 45.

D) Totals must be bundle-counted and re-counted and double-counted before they can add.

E) Re-counted in the same unit, a total can be written in 3 ways: normal, with overload or with underload, e.g. T = 46 = 4B6 = 3B16 = 5B-4.

F) Re-counting in a new unit (proportionality) be predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B', e.g. T = 3 4s = (3*4)/5*5 = 2 5s & 2.

G) Re-counting from tens to icons creates equations, e.g. x*5 = 40 = (40/5)*5 with solution x = 40/5. Double-counting gives per-numbers and proportionality with re-counting in the per-number: with 2$ per 3kg, 12 kg = (12/3)*3 kg = (12/3)*2$ = 8$.

H) After counting comes addition, on-top and next-to, leading to proportionality and integration.

I) Reverse addition leads to equations and differentiation.

J) Per-numbers lead to fractions, both operators needing to be multiplied to become numbers, and therefore added by their areas, i.e. by integration.

K) Calculus means adding and splitting into locally constant per-numbers.

L) Core STEM-areas become applications under the theme 'water in movement'.

Details: ' A STEM-based Core Math Curriculum for Outsiders and Migrants ', http://mathecademy.net/papers/miscellaneous/

11) The teacher as a difference-researcher

When traditions give problems, difference research uncovers hidden differences that make a difference. For example, the tradition says that ‘a function is an example of a set relation where first component identity implies second component identity’, which the learner hears as ‘bublibub is an example as bablibab ', which nobody finds meaningful. A difference is to use Euler's original definition accepted by all: 'a function is a common name for calculations with both known and unknown numbers'.

Difference research can be used by teachers to solve problems in class, or by teacher-researchers sharing their time between academic work at a university and intervention research in a class. Or by full-time researchers, working with teachers to apply difference research: the teacher observes the problems, the researcher identifies the differences. Together they establish a micro-curriculum to be tested by the teacher and reported by the researcher after a pretest-posttest study. A typical difference researcher begins as an ordinary teacher who reflects on whether alternatives can solve learning problems observed.

A difference-researcher combines elements of action learning and action research and intervention research and design research. First a difference is identified, then a micro-curriculum is designed to be tested to see what kind of difference it makes. The effect will be reported internally and discussed with colleagues. After repeating this cycle of design, teaching and internal reporting, it is time for an external reporting of the difference and its effect in magazines or journals or at conferences.

Research should provide knowledge to explain nature and to improve social conditions. But as an institution in runs the danger of the what the sociologist Bauman calls a goal displacement, so research will be self-referencing instead of finding differences. Hargreaves, write for example: ‘What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads’ (Hargreaves, 1996, p. 7).

Hargreaves, D.H. (1996). *Teaching as a Research-based Profession: Possibilities and Prospects.* Cambridge: Teacher Training Agency Lecture.

Fifty years of research without improving mathematics education, why?

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by the PISA studies showing a low level and a continuing decline in many countries. Thus, to help the former model country Sweden, OECD wrote a critical 2015 report ‘Improving Schools in Sweden, an OECD Perspective’.

At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is “Only one: to improve, mathematics education should ask, not what to do, but what to do differently.”

Thus, to be successful, research should not study problems but look for differences that make a difference. Research that is skeptical towards institutionalized traditions could be called difference research. In France, Lyotard calls it ‘paralogy’ inventing dissension to the reigning consensus.

Difference research scarcely exists today since it is rejected at conferences for not applying or extending existing theory.

To elaborate, maybe mathematics education research is sterile because its three words are not that well defined?

As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist: a continental European education serving the nation's need for public servants though multi-year compulsory classes and lines at the secondary and tertiary level; and a North American education aiming at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers.

As to research, academic articles can be written at a master level applying existing theories, or at a research level questioning them. Just following theories is problematic in the case of conflicting theories as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible.

Consequently, you cannot know what kind of mathematics and what kind of education has been studied, and you cannot know if research is following ruling traditions or searching for alternatives. So, if institutionalized education should help children and youngsters to master outside phenomena we must ask: What outside phenomena roots mathematics?

We master the outside world with two languages, a word-language and a number-language. Children learn to talk and to count at home. Then, as an institution, school takes over and teaches children to read and to write and to calculate.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair', and 'T = 3*4'. Both languages have a meta-language, a grammar, describing the language, describing the world. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

So, one way of improving mathematics education is to respect that language comes before meta-language. Which was also the case in continental Europe before the arrival of the 'New Math' that turned mathematics upside down to become a 'meta-matics' presenting its concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically and which would present mathematics as 'many-matics', a natural science about Many.

Before New Math, Germanic countries taught counting and reckoning in primary school. Then the lower secondary school taught algebra and geometry, which are also action words meaning to reunite totals and to measure earth in Arabic and in Greek. 50 years ago, New Math made all these activities disappear.

Thus, one alternative immediately presents itself: Re-root mathematics in its historic origin as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many by meaning 'reuniting numbers' and 'measuring earth' in Arabic and Greek respectively.

Consequently, to improve its education, mathematics should stop teaching top-down meta-matics from above and begin teaching bottom-up many-matics from below instead.

For details, see 'Difference-Research Powering PISA Performance', Fifty Years of Research without Improving Mathematics Education', and 'A STEM-based Core Math Curriculum for Outsiders and Migrants' at http://mathecademy.net/papers/miscellaneous/.

12. The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants

Swedish educational shortages challenge traditional mathematics education offered to migrants. Mathematics could be taught in its simplicity instead of as 'meta-matsim', a mixture of 'meta-matics' defining concepts as examples of inside abstractions instead of as abstractions from outside examples; and 'mathe-matism' true inside classrooms but seldom outside as when adding numbers without units. Rebuilt as 'many-matics' from its outside root, Many, mathematics unveils its simplicity to be taught in a STEM context at a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

Other countries also experience declining PISA results. Since mathematics education is a social institution, social theory might be able to explain 50 years of unsuccessful research in mathematics education.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16). A wish to uncover unnoticed alternatives motivates a 'difference-research' (Tarp, 2017) asking two questions: 'Can this be different – and will the difference make a difference?' If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

Natural sciences use difference-research to keep on searching until finding what cannot be different. Describing matter in space and time by weight, length and time intervals, they all seem to vary. However, including per-numbers will uncover physical constants as the speed of light, the gravitational constant, etc. The formulas of physics are supposed to predict nature's behavior. They cannot be proved as can mathematical formulas, instead they are tested as to falsifiability: Does nature behave different from predicted by the formula? If not, the formula stays valid until falsified.

Social sciences also use difference-research beginning with the ancient Greek controversy between two attitudes towards knowledge, called 'sophy' in Greek. To avoid hidden patronization, the sophists warned: Know the difference between nature and choice to uncover choice presented as nature. To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy. The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord deciding world behavior.

Today's democracies implement common social goals through institutions with means decided by parliaments. As to rationality as the base for social organizations, Bauman says:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called *goal displacement*. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

As an institution, mathematics education is a public organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts. In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about shapes and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers, operations and calculations. However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. And, being sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens, might speed up writing but might also slow down learning, together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, in Lincolns Gettysburg address, ‘Four scores and ten years ago’ shows that not all count in tens.

To get an answer to the questions ‘What is mathematics?’ and ‘How is mathematics education improved?’ we might include philosophy in the form of what Bauman calls ‘the second Copernican revolution’ of Heidegger asking the question: What is ‘is’? (Bauman, 1992, p. ix).

> Inquiry is a cognizant seeking for an entity both with regard to the fact that it is and with regard to its Being as it is. (Heidegger, 1962, p. 5)

Heidegger here describes two uses of ‘is’. One claims existence, ‘M is’, one claims ‘how M is’ to others, since what exists is perceived by humans wording it by naming it and by characterizing or analogizing it to create ‘M is N’-statements.

Thus, there are four different uses of the word ‘is’. ‘Is’ can claim a mere existence of M, ‘M is’; and ‘is’ can assign predicates to M, ‘M is N’, but this can be done in three different ways. ‘Is’ can point down as a ‘naming-is’ (‘M is for example N or P or Q or …’) defining M as a common name for its volume of more concrete examples. ‘Is’ can point up as a ‘judging-is’ (‘M is an example of N’) defining M as member of a more abstract category N. Finally, ‘is’ can point over as an ‘analogizing-is’ (‘M is like N’) portraying M by a metaphor carrying over known characteristics from another N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: ‘I am’ and ‘it is’ and ‘they are’; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the ‘I’ (Dasein) must create an authentic relationship to the ‘It’. However, this is made difficult by the ‘dictatorship’ of the ‘They’, shutting the ‘It’ up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one’s own Dasein completely into the kind of Being of ‘the Others’, in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the “they” is unfolded. (..) Discourse, which belongs to the essential state of Dasein’s Being and has a share in constituting Dasein’s disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

In France, Heidegger inspired the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu, pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Lyotard, 1984. Bourdieu, 1970. Chomsky et al, 2006).

From a Heidegger view a sentence contains two things: a subject that exists, and the rest that might be gossip. So, to discover its true nature hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its predicate-prison. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help rebuilding their original countries.

So, to restore its authenticity, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many

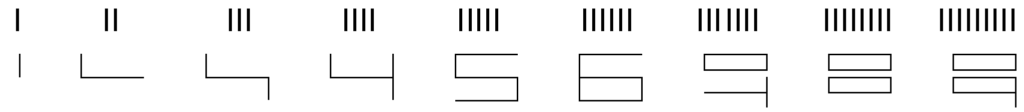
As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep the balance and to store sounds assigned to what we grasped with our forelegs, now freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into two languages, a word-language and a number-language.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’. Observing the existence of many chairs, we ask ‘how many in total?’ and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, ‘the total is 3 chairs’ or, if counting legs, ‘the total is 3 fours’, abbreviated to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

With 2017 as the 500year anniversary for Luther’s 95 theses, we can choose to describe meeting Many in theses.

01. Using a folding ruler, we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the four icon, and five sticks in the five icon, etc. Counting in 5s, the counting sequence is 1, 2, 3, 4, Bundle, 1-bundle-1, etc. This shows, that the bundle-number does not need an icon. Likewise, when bundling in tens. Instead of ten-1 and ten-2 we use the Viking numbers eleven and twelve meaning 1 left and 2 left in Danish.



1 2 3 4 5 6 7 8 9

02. Transforming four ones to a bundle of 1 4s allows counting with bundles as a unit. Using a cup for the bundles, a total can be ‘bundle-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘bundle-writing’ to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s. Next, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Or, with tens: T = 234 = 23B4 = 2BB3B4.

03. Recounting in the same unit by creating or removing overloads or underloads, bundle-writing offers an alternative way to perform and write down operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

04. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved when counting by bundling and stacking. Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication, iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.

05. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. The 'recount formula' T = (T/B)*B says that 'from T, T/B times B can be taken away' as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the 'restack formula' T = (T–B)+B says that from T, T–B is left when B is taken away and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. The recount or proportionality formula turns out to a very basic formula. It turns up in proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right-angled triangles, and in calculus as dy = (dy/dx)*dx = y'*dx when counting steepness on a curve.

06. Wanting to recount a total in a new unit, we discover that a calculator can predict the result when bundling and stacking and dragging away the stack. Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Also we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.

07. Wanting to recount a total in tens, we discover that a calculator predicts the result directly by multiplication, only leaving out the unit and misplacing the decimal point. Thus, asking T = 3 7s = ? tens, the calculator predicts: T = 21 = 2.1 tens. Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged. With 5 as half of ten, and 8 as ten less 2, a 10x10 multiplication table can be reduced to a 3x3 table including the numbers 2, 3 and 4. Thus, 4*8 = 4*(ten less 2) = 4ten less 8 = 32; 5*8 = half of 8ten = 4ten = 40; 7*8 = (ten less 3)*(ten less 2) = tenten, less 3ten, less 2ten, plus 6 = 56.

Wanting to recount a total from tens to icons, we discover this as another example of recounting to change the unit. Thus, asking T = 3 tens = ? 7s, the calculator predicts: First 30/7 = 4.some; then 30 – (4*7) = 2; and finally T = 30 = 4.2 7s. Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged.

08. Using the letter u for an unknown number, we can rewrite the recounting question '? 7s = 3tens' as 'u*7 = 30' with the answer 30/7 = u since 30 = (30/7)*7, officially called to solve an equation. Here we discover a natural way to do so: Move a number to the opposite side with the opposite calculation sign. Thus, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, obtained again by moving a number to the opposite side with the opposite calculation sign.

09. Once counted, totals can be added, but addition is ambiguous. Thus, with two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to have the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 =

3.2 8s. So next-to addition geometrically means adding areas, and algebraically it means combining multiplication and addition. Officially, this is called integration, a core part of traditional mathematics in high school and at the first year of university.

10. Also we discover that addition and other operations can be reversed. Thus, in reversed addition, 8 = u+2, we ask: what is the number u that added to 2 gives 8, which is precisely the formal definition of u = 8–2. And in reversed multiplication, 8 = u*2, we ask: what is the number u that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root, the factor-finder, and logarithm, the factor-counter, as u = 3√20 and u = log3(20). Again we solve the equation by moving to the opposite side with the opposite calculation sign. Reversing next-to addition we ask 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer u, from the terminal total T we remove the initial total T1 before we count the rest in 5s: u = (T–T1)/5 = ΔT/5, using Δ for the difference or change. Letting subtraction precede division is called differentiation, the reverse operation to integration letting multiplication precede addition.

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

12. Economy is based upon investing money and expecting a return that might be higher or lower than the investment, e.g. 7$ per 5$ or 3$ per 5$. Here when double-counting in the same unit, per-numbers become fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. Thus, to find 3 per 5 of 20, or 3/5 of 20, as before we just recount 20 in 5s and replace 5 with 3, T = 20 = (20/5)*5 giving (20/5)*3 = 12.

To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s and replace 5 with 3: T = 100 = (100/5)*5 giving (100/5)*3 = 60. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly, but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise when adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

13. Halved by its diagonal, a rectangle splits into two right-angled triangles. Here the angles are labeled A and B and C at the right angle. The opposite sides are labeled a and b and c.

The height a and the base b can be counted in meters, or in diagonals c creating a sine-formula and a cosine-formula: a = (a/c)*c = sinA*c, and b = (b/c)*c = cosA*c. Likewise, the height can be recounted in bases, creating a tangent-formula: a = (a/b)*b = tanA*b

As to the angles, with a full turn as 360 degrees, the angle between the horizontal and vertical directions is 90 degrees. Consequently, the angles between the diagonal and the vertical and horizontal direction add up to 90 degrees; and the three angles add up to 180 degrees.

An angle A can be counted by a protractor, or found by a formula. Thus, in a right-angled triangle with base 4 and diagonal 5, the angle A is found from the formula cosA = a/c = 4/5 as cos-1(4/5) = 36.9 degrees.

The three sides have outside squares with areas a^2 and b^2 and c^2. Turning a right triangle so that the diagonal is horizontal, a vertical line from the angle C split the square c^2 into two rectangles. The rectangle under the angle A has the area (b*cosA)*c = b*(cosA*c) = b*b = b^2. Likewise, the rectangle under the angle B has the area $(a*\cos B)*c = a*(\cos B*c) = a*a = a^2$. Consequently $c^2 = a^2 + b^2$, called the Pythagoras formula.

This allows finding a square-root geometrically, e.g. $x = \sqrt{24}$, solving the quadratic equations $x^2 = 24 = 4*6$, if transformed into a rectangle. On a protractor, the diameter 9.5 cm becomes the base AB, so we have 6units per 9.5cm. Recounting 4 in 6s, we get 4units = (4/6)*6units = (4/6)*9.5 cm = 6.33 cm. A vertical line from this point D intersects the protractor's half-circle in the point C. Now, with a 4x6 rectangle under BD, BC will be the square-root $\sqrt{24}$, measured to 4.9, which checks: $4.9^2 = 24.0$.

A triangle that is not right-angled transforms into a rectangle by outside right-angled triangles, thus allowing its sides and angles and area to be found indirectly. So, as in right-angled triangles, any triangle has the property that the angles add up to 180 degrees and that the area is half of the height times the base.

Inside a circle with radius 1, the two diagonals of a 4sided square together with the horizontal and vertical diameters through the center form angles of 180/4 degrees. Thus the circumference of the square is $2*(4*\sin(180/4))$, or $2*(8*\sin(180/8))$ with 8 sides instead. Consequently, the circumference of a circle with radius 1 is $2*\pi$, where $\pi = n*\sin(180/n)$ for n large.

14. A coordinate system coordinates algebra with geometry where a point is reached by a number of horizontally and vertically steps called the point's x- and y-coordinates.

Two points A(xo,yo) and B(x,y) with different x- and y-numbers will form a right-angled change-triangle with a horizontal side $\Delta x = x-xo$ and a vertical side $\Delta y = y-yo$ and a diagonal distance r from A to B, where by Pythagoras $r^2 = \Delta x^2 + \Delta y^2$. The angle A is found by the formula $\tan A = \Delta y/\Delta x = s$, called the slope or gradient for the line from A to B. This gives a formula for a non-vertical line: $\Delta y/\Delta x = s$ or $\Delta y = s*\Delta x$, or $y-yo = s*(x-xo)$. Vertical lines have the formula $x = xo$ since all points share the same x-number.

In a coordinate system three points A(x1,y1) and B(x2,y2)and C(x3,y3) not on a line will form a triangle that packs into a rectangle by outside right-angled triangles allowing indirectly to find the angles and the sides and the area of the original triangle.

Different lines exist inside a triangle: Three altitudes measure the height of the triangle depending on which side is chosen as the base; three medians connect an angle with the middle of the opposite side; three angle bisectors bisect the angles; three line bisectors bisect the sides and are turned 90 degrees from the side. Likewise, a triangle has two circles; an outside circle with its center at the intersection point of the line bisectors, and an inside circle with its center at the intersection point of the angle bisectors.

Since Δx and Δy changes place when turning a line 90 degrees, their slopes will be $\Delta y/\Delta x$ and $-\Delta x/\Delta y$ respectively, so that $s1*s2 = -1$, called reciprocal with opposite sign.

Geometrical intersection points are predicted algebraically by solving two equations with two unknowns, i.e. by inserting one into the other. Thus with the lines $y = 2*x$ and $y = 6-x$, inserting the first into the second gives $2*x = 6-x$, or $3*x = 6$, or $x = 2$, which inserted in the first gives $y = 2*2 = 4$, thus predicting the intersection point to be $(x,y) = (2,4)$. The same answer is found on a solver-app; or using software as GeoGebra.

Finding possible intersection points between a circle and a line or between two circles leads to a quadratic equation $x^2 + b*x + c = 0$, solved by a solver. Or by a formula created by two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter round clockwise. With $k = m-x$, this creates 4 areas combining to $(x + k)^2 = x^2 + 2*k*x + k^2$. With $k = b/2$ this becomes $(x + b/2)^2 = x^2 + b*x + (b/2)^2 + c - c = (b/2)^2 - c$ since $x^2 + b*x + c = 0$. Consequently the solution formula is $x = -b/2 \pm\sqrt{(b/2)^2 - c}$.

Finding a tangent to a circle at a point, its slope is the reciprocal with opposite sign of the radius line.

15. A formula predicts a total before counting it. A formula typically contains both specified and unspecified numbers in the form of letters, e.g. $T = 5+3*x$. A formula containing one unspecified number is called an equation, e.g. $26 = 5+3*x$, to be solved by moving to opposite side with opposite calculation sign, $(26-5)/3 = x$. A formula containing two unspecified numbers is called a function, e.g. $T = 5+3*x$. An unspecified function containing an unspecified number x is labelled f(x), T = f(x). Thus f(2) is meaningless since 2 is not an unspecified number. Functions are described by a table or a graph in a coordinate system with y = T = f(x), both showing the y-numbers for different x-numbers. Thus, a change in x, Δx, will imply a change in y, Δy, creating a per-number $\Delta y/\Delta x$ called the gradient.

16. In a function y = f(x), a small change x often implies a small change in y, thus both remaining 'almost constant' or 'locally constant', a concept formalized with an 'epsilon-delta criterium', distinguishing between three forms of constancy. y is 'globally constant' c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is 'piecewise constant' c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is 'locally constant' c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in the latter case written as dy/dx. Formally, local constancy and linearity is called continuity and differentiability.

17. As to change, a total can change in a predictable or unpredictable way; and predictable change can be constant or non-constant.

Constant change comes in several forms. In linear change, T = b + s*x, s is the constant change in y per change in x, called the slope or the gradient of its graph, a straight line. In exponential change, T = b*(1+r)^x, r is the constant change-percent in y per change in x, called the change rate. In power change, T = b*x^p, p is the constant change-percent in y per change-percent in x, called the elasticity. A saving increases from two sources, a constant $-amount per month, c, and a constant interest rate per month, r. After n months, the saving has reached the level C predicted by the formula C/c = R/r. Here the total interest rate after n months, R, comes from 1+R = (1+r)^n. Splitting the rate r = 100% in t parts, we get the Euler number e = (1+100%/t)^t = (1+1/t)^t if t is large.

Also the change can be constant changing. Thus in T = c + s*x, s might also change constantly as s = c + q*x so that T = b + (c + q*x)*x = b + c*x + q*x^2, called quadratic change, showing graphically as a line with a curvature, a parabola.

If not constant but still predictable, we have a change formula $\Delta T/\Delta x = f(x)$ or dT/dx = f(x) in the case of interval change or local change. Such an equation is called a differential equation which is solved by calculus, adding up all the local changes to a total change being the difference between the end and start number: $T2-T1 = \Sigma\, \Delta T = \int dT = \int f(x)*dx$. Thus, with dT/dx = 2*x, $T2-T1 = \Delta(x\hat{}2)$. Change formula come from observing that in a block, changes Δb and Δh in the base b and the height h impose on the total a change ΔT as the sum of a vertical strip $\Delta b*h$ and a horizontal strip $b*\Delta h$ and a corner $\Delta b*\Delta h$ that can be neglected for small changes; thus d(b*h) = db*h + b*dh, or counted in T's: dT/T = db/b + dh/h, or with T' = dT/dx, T'/T = b'/b + h'/h. Therefore (x^2)'/x^2 = x'/x + x'/x = 2/x, giving (x^2)' = 2*x since x' = dx/dx = 1.

18. Unpredictable change can be exemplified by throwing a dice with two results: winning, +1, if showing 4 or above, and losing, 0, if showing 3 or below. Throwing a dice 5 times thus have 6 outcomes, winning from 0 to 5 times. The outcome is called an unpredictable or stochastic or random number or variable. Per definition, random numbers cannot be pre-dicted, instead they can be 'post-dicted' using statistics and probability.

Thus the outcome '0,0,0,1,1' can be described by three numbers. The mode is 0 since this number has the highest frequency, 3 per 5, or 3/5. The median is 0 since this is the middle number when aligned in increasing order. The mean u is the fictional number had all numbers been the same: u*5 = 0+0+0+1+1 with the solution u = 2/5 = 0.4. With the outcome '0,0,1,1,1', the mode and median and mean is 1 and 1 and 3/5 = 0.6.

To find the three numbers if the experiment is repeated many times we look at a 'possibly tree'. The first toss has two results, win or lose, both occurring ½ of the times. Likewise with the following tosses: After two tosses we have three outcomes: 2 wins, 1 win and 0 wins. Here 2 wins and 0 wins occur half of half of the times, i.e. with a probability ¼. 1 win occurs twice, as win-lose or as lose-win, both with a probably of ¼, so the total probability for 1 win is 2*1/4 = ½. Continuing in this way we find that with 5 tosses there are 6 outcomes, winning from 0 to 5 times with the probabilities ½^5 a certain number of times: 1, 5, 10, 10, 5, 1. By calculations we find that the mode is 2 and 3, and that the median and the mean is 2.5, also found by multiplying the number of repetition with the probability for winning.

A spreadsheet random generator can show examples of other outcomes.

19. A sphere may be distorted into a cup. Even if distorted, a rectangle will still divide a sphere into an inside and an outside needing a bridge to be connected. And a sphere with a bridge may be distorted into a cup with a handle or into a donut. Distortion geometry is called topology, useful when setting up networks, thus able to prove that connecting three houses with water, gas and electricity is impossible without a bridge.

20. As qualitative literature, also quantitative literature has three genres, fact and fiction and 'fiddle', used when modeling real world situations. Fact is 'since-then' calculations using numbers and formulas to quantify and to predict predictable quantities as e.g. 'since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20'. Fact models can be trusted once the numbers and the formulas and the calculation has been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter. Fiction is 'if-then' calculations using numbers and formulas to quantify and to predict unpredictable quantities as e.g. 'if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20'. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made. Fiddle models is 'what-then' models using numbers and formulas to quantify and to predict unpredictable qualities as e.g. 'since a graveyard is cheaper than a hospital, then a bridge across the highway is too costly.' Fiddle models should be rejected and relegated to a qualitative description.

Meeting Many in a STEM Context

Having met Many by itself, now we meet Many in time and space in the present culture based upon STEM, described by OECD as follows:

> The New Industrial Revolution affects the workforce in several ways. Ongoing innovation in renewable energy, nanotech, biotechnology, and most of all in information and communication technology will change labour markets worldwide. Especially medium-skilled workers run the risk of being replaced by computers doing their job more efficiently. This trend creates two challenges: employees performing tasks that are easily automated need to find work with tasks bringing other added value. And secondly, it propels people into a global competitive job market. (..) In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone?

No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles, mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves. Nitrates and carbon-dioxide and water is waste for grey cells, but food for green cells producing proteins and carbon-hydrates and oxygen as food for the grey cells in return.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. To master plane or spatial shapes, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent.

So, a core STEM curriculum could be about cycling water. Heating transforms it from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we must build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

A short World History

When humans left Africa, some went west to the European mountains, some went east where the fertile valleys in India supplied everything except for silver from the mountains. Consequently, rich trade took place sending pepper and silk west and silver east. European culture flourished around the silver mines, first in Greece then in Spain during the Roman Empire. Then the Vandal conquest of the mines brought the dark middle age to Europe until silver was found in the Harz valley (Tal in German leading to thaler and dollar), transported through Germany to Italy. Here silver financed the Italian Renaissance, going bankrupt when Portugal discovered a sea route to India enabling them to skip the cost of Arab middlemen. Spain looked for a sea route going west and found the West Indies. Here there was neither pepper nor silk but silver in abundance e.g. in the land of silver, Argentine. On their way home, slow Spanish ships were robbed by sailing experts, the Vikings descendants living in England, now forced to take the open sea to India to avoid the Portuguese fortification of Africa's coast.

In India, the English found cotton that they brought to their colonies in North America, but needing labor they started a triangle-trade exchanging US cotton for English weapon for African slaves for US cotton. In the agricultural South, a worker was a cost to be minimized, but in the industrial North a worker was a consumer needed at an industrial market. During the civil war, no cotton came to England that then conquered Africa to bring the plantations to the workers instead. Dividing the world in closed economies kept new industrial states out of the world market that it took two world wars to open for free competition.

Nature Obeys Laws, but from Above or from Below?

In the Lord's Prayer, the Christian Church says: 'Thy will be done, on earth as it is in heaven'. Newton had a different opinion.

As experts in sailing, the Viking descendants in England had no problem stealing Spanish silver on its way across the Atlantic Ocean. But to get to India to exchange it for pepper and silk, the Portuguese fortification of Africa's cost forced them to take the open sea and navigate by the moon. But how does the moon move? The church had one opinion, Newton meant differently.

'We believe, as is obvious for all, that the moon moves among the stars,' said the Church, opposed by Newton saying: 'No, I can prove that the moon falls to the earth as does the apple.' 'We believe that when moving, things follow the unpredictable metaphysical will of the Lord above whose will is done, on earth as it is in heaven,' said the Church, opposed by Newton saying: 'No, I can prove they follow their own physical will, a physical force, that is predictable because it follows a mathematical formula.' 'We believe, as Aristotle told us, that a force upholds a state,' said the Church, opposed by Newton saying: 'No, I can prove that a force changes a state. Multiplied with the time applied, the force's impulse changes the motion's momentum; and multiplied with the distance applied, the force's work changes the motion's energy.' 'We believe, as the Arabs have shown us, that to deal with formulas you just need ordinary algebra,' said the Church, opposed by Newton saying: 'No. I need to develop a new algebra of change which I will call calculus.'

Proving that nature obeys its own will and not that of a patronizor, Newton inspired the Enlightenment century realizing that if enlightened we don't need the double patronization of the physical Lord at the Manor house and the metaphysical Lord above. Citizens only need to inform themselves, debate and vote. Consequently, to enlighten the population, two Enlightenment republics were created, in the US in 1776 and in France in 1789. The US still have their first republic allowing its youth to uncover and develop their personal talent through daily lessons in self-chosen half-year blocks, whereas the Napoleon wars forced France and the rest of continental Europe to copy the Prussians line-organized education forcing teenagers to follow their year-group and its schedule, creating a knowledge nobility (Bourdieu, 1970) for public offices, and unskilled workers, good for yesterday's industrial society, but bad for today's information society where a birth rate at 1.5 child per family will halve the population each 50 years since (1.5/2)^2 = 0.5 approximately.

Counting and DoubleCounting Time, Space, Matter, Force and Energy

Counting time, the unit is seconds. A bundle of 60 seconds is called a minute; a bundle of 60 minutes is called an hour, and a bundle of 24 hours is called a day, of which a bundle of 7 is called a week. A year contains 365 or 366 days, and a month from 28 to 31 days.

Counting space, the international unit is meter, of which a bundle of 1000 is called a kilometer; and if split becomes a bundle of 1000 millimeters, 100 centimeters and 10 decimeters. Counting squares, the unit is 1 square-meter. Counting cubes, the unit is 1 cubic-meter, that is a bundle of 1000 cubic-decimeters, also called liters, that split up as a bundle of 1000 milliliters.

Counting matter, the international unit is gram that splits up into a bundle of 1000 milligrams and that unites in a bundle of 1000 to 1 kilogram, of which a bundle of 1000 is called 1 tons.

Counting force and energy, a force of 9.8 Newton will lift 1 kilogram, that will release an energy of 9.8 Joule when falling 1 meter.

Cutting up a stick in unequal lengths allows the pieces to be double-counted in liters and in kilograms giving a per-number around 0.7 kg/liter, also called the density.

A walk can be double-counted in meters and seconds giving a per-number at e.g. 3 meter/second, called the speed. When running, the speed might be around 10 meter/second. Since an hour is a bundle of 60 bundles of 60 seconds this would be 60*60 meters per hour or 3.6 kilometers per hour, or 3.6 km/h.

A pressure from a force applied to a surface can be double-counted in Newton and in square meters giving a per-number Newton per square-meter, also called Pascal.

Motion can be double-counted in Joules and seconds producing the per-number Joule/second called Watt. To run properly, a bulb needs 60 Watt, a human needs 110 Watt, and a kettle needs 2000 Watt, or 2 kiloWatt. From the Sun the Earth receives 1370 Watt per square meter.

Warming and Boiling water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

Heating 1000 gram water 80 degrees in 167 seconds in a 2000 Watt kettle, the per-number will be 2000*167/80 Joule/degree, creating a double per-number 2000*167/80/ 1000 Joule/degree/gram or 4.18 Joule/degree/gram, called the specific heat of water.

Producing 100 gram steam in 113 seconds, the per-number is 2000*113/100 Joule/gram or 2260 J/g, called the heat of evaporation for water.

Letting Steam Work

A water molecule contains two Hydrogen and one Oxygen atom weighing 2*1+16 units. A collection of a million billion billion molecules is called a mole; a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. The volume should increase accordingly. But, if kept constant, instead the inside pressure will increase.

Inside a cylinder, the ideal gas law, p*V = n*R*T, combines the pressure, p, and the volume, V, with the number of moles, n, and the absolute temperature, T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

So, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus decreasing and increasing the pressure by letting steam in and out of the two holes. The leaving steam the is visible on steam locomotives. In the third generation used in power plants, two cylinders, a hot and a cold, connect with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back to the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical power to industries and homes.

An Electrical circuit

To work properly, a 2000 Watt water kettle needs 2000 Joule per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per charge-unit.

Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2 Volt/Ampere called Ohm.

Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula Watt = Volt*Ampere.

Thus, with a 60 Watt and a 120 Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So, the 120 Watt bulb only receives half of the energy of the 60 Watt bulb.

How high up and how far out

A ping-pong ball is send upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second.

To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the speed is split into a vertical and a horizontal part, both reducing it with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed by the formula: Horizontal distance to the top = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

The vertical distance is halved, but the vertical speed changes from 9.92 to 9.92*0.707 = 7.01 only. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the distance four times.

How many turns on a steep hill

On a 30-degree hillside, a 10 degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2.

In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2.

In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Dissolving material in water

In the sea, salt is dissolved in water. The tradition describes the solution as the number of moles per liter. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59gram = (100/59)*1mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

The Simplicity of Mathematics

Meeting Many, we ask 'How many in total?' To answer, we count and add. To count means to use division, multiplication and subtraction to predict unit-numbers as blocks of stacked bundles, but also to recount to change unit, and to double-count to get per-numbers bridging the units, both rooting proportionality.

Adding thus means uniting unit-numbers and per-numbers, but both can be constant or variable, so to predict, we need four uniting operations: addition and multiplication unite variable and constant unit-numbers; and integration and power unite variable and constant per-numbers. As well as four splitting operations: subtraction and division split into variable and constant unit-numbers; and differentiation and root/logarithm split into variable and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite. And it appears in Arabic numbers written out fully as T = 456 = 4 bundles-of-bundles & 5 bundles & 6 unbundled, showing all four uniting operations, addition and multiplication and power and next-to addition of stacks; and showing that the word-language and the number-language share the same sentence form with a subject and a verb and a predicate or object.

Shapes can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

So, in principle, mathematics is simple and easy and quick to learn if institutionalized education wants to do so; however, to preserve and expand itself, the institution might want instead to hide the simplicity of mathematics by leaving out the subject and the verb in the number-language sentences; and by avoid counting to hide the block-nature of numbers as stacked bundles in order to impose linear place-value numbers instead; and by reversing the natural order of operations by letting addition precede subtraction, preceding multiplication, preceding division; and by hiding the double nature of addition by silencing next-to addition; and by silencing per-numbers and present fractions as numbers instead of operators needing numbers to become numbers; and by adding fractions without units to hide the true nature of integration as adding per-numbers by their areas; and by postponing trigonometry to after ordinary geometry and coordinate geometry; and by forcing equations to be solved by obeying the commutative and associative laws of abstract algebra; and by hiding that a function is but another name for a number-language sentence; and by forcing differential calculus to precede integral calculus.

Discussion: How does Traditional MatheMatics differ from ManyMatics

But in the end, how different is traditional mathematics from ManyMatics? As their base they have Set and Many, but isn't that just two different words for the same? Not entirely. Many exists in the world, it is physical, whereas Set exists in a description, it is meta-physical. Thus, traditional mathematics defines its concepts top-down as examples, whereas ManyMatics defines its concepts bottom-up as abstractions. Still, the concepts might be the same, at least when taught? But a comparison uncovers several differences between the Set-derived tradition and its alternative grounded in Many.

The tradition sees the goal of mathematics education as teaching numbers and shapes and operations. In numbers, digits are symbols like letters, ordered according to a place value system, seldom renaming '234' to '2tentens 3tens 4'. There are four kinds of numbers: natural and integers and rational and real. The natural numbers are defined by a successor principle making them one dimensional placed along a number line given the name 'cardinality'. The integers are defined as equivalence classes in a set of ordered number-pairs where (a,b) is equivalent to (c,d) if a+d = b+c. Likewise, the rational numbers are defined by (a,b) being equivalent to (c,d) if a*d = b*c. Finally, the real numbers are defined as limits of number sequences.

The alternative sees the goal of mathematics education as teaching a number-language describing the physical fact Many by full sentences with the total as the subject, e.g. T = 2*3, thus having the same structure as the word-language, both having a language level describing the world, and a meta-language level describing the language. Digits are icons containing as many sticks as they represent if written less sloppy. Numbers occur when counting Many by bundling and stacking produces a block of bundles and unbundled, using bundle- or decimal-writing to separate the inside bundles from the outside unbundled. The bundle-number, typically ten, does not need an icon since it is counted as '1 bundle'. Thus, a natural number is a decimal number with a unit, illustrated geometrically as a row of blocks containing the unbundled, the bundles, the bundle of bundles etc. Counting includes recounting in the same unit to create overload or underload, as well as recounting in another unit, especially in and from tens. Double-counting in different units gives per-numbers and fractions; however, these are not numbers but operators needing a number to become a number. A diagonal divides a block into two like right-angled triangles where the base and the altitude can be recounted in diagonals or in each other. Real numbers as √2 are calculations with as many decimals as needed, since a single can always be seen as a bundle of parts.

The tradition sees operations in a number set as mappings from a set-product into the set. Addition is the basic operation allowing number sets to be structured with an associative and a commutative and a distributive law as well as a neutral element and inverse elements. Addition is defined as repeating the successor principle, and multiplication is defined as repeated addition. Subtraction and division is defined as adding or multiplying inverse numbers. Standard algorithms for operations are introduced using carrying. Electronical calculators are not allowed when learning the four basic operations. The full ten-by-ten multiplication tables must be learned by heart.

The alternative sees operations as icons describing the counting process. Here division is an uphill stroke showing a broom wiping away the bundles; multiplication is a cross showing a lift stacking the bundles into a block, to be dragged away to look for unbundled singles, shown by a horizontal track called subtraction. Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other. To add on-top, the blocks must be recounted in the same unit, thus grounding proportionality. Next-to addition means adding areas, thus grounding integration. Reversed adding on-top or next-to grounds equations and differentiation. A calculator is used to predict the result by two formulas, a recount-formula T = (T/B)*B, and a restack-formula T = (T-B)+B. A multiplication table shows recounting from icons to tens, and is used when recounting from tens to icons introduces equations as reversed calculations. When recounting a total to or from tens, increasing the base means decreasing the altitude, and vice versa. As to multiplication, the commutative law says that the total stays unchanged when turning over a 3 by 4 block to a 4 by 3 block. The associative law says that the total stays unchanged when including or excluding a factor from the unit, T = 2*(3*4) = (2*3)*4. The distributive law says that before adding, recounting must provide a common unit to bracket out, T = 2 3s + 4 5s = 1.1 5s + 4 5s = (1.1 + 4) 5s.

The tradition sees fractions as rational numbers to which the four basic operations can be applied. Thus, fractions can be added without units by finding a common denominator after splitting the numerator and the denominator into prime factors. Fractions are introduced after division, and is followed by ratios and percentages and decimal numbers seen as examples of fractions.

The alternative sees fractions as per-numbers coming from double-counting in the same unit. As per-numbers, fractions are operators needing a number to become a number, thus added by areas, also called integration. Double-counting is introduced before addition. With factors as units, splitting a number in prime factors just means finding all possible units.

After working with number sets, the tradition introduces working with letter sets and polynomial sets to which the four basic operations can be applied once more observing that only like terms can be added, but not mentioning that this is because it means the unit is the same.

The alternative sees letters as units to bracket out during addition or subtraction, and that when multiplied or divided gives a composite unit.

The tradition sees an equation as an open statement expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*x = 8	an open statement
(2*x)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(x*2)*(1/2) = 4	since multiplication is commutative
x*(2*(1/2)) = 4	since multiplication is associative
x*1 = 4	by definition of an inverse element
x = 4	by definition of a neutral element

As to the equation 2 + 3*x = 14, the same procedure as above is carried out twice, first with addition then with multiplication.

The alternative sees an equation as another name for a reversed calculation, to be reversed once more by recounting. Thus in the equation ‘2*x = 8’, recounting some 2s in 1s resulted in 8 1s, which recounted back into 2s gives 2*x = 8 = (8/2)*2, showing that x = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the opposite side & sign method.

The equation 2 + 3*x = 14, can be seen in two ways. As reversing a next-to addition of the two blocks, thus solved by differentiation, first removing the initial block and then recounting the rest in 3s: x = (14-2)/3 = 4. Or as a walk that multiplying by 3 and then adding by 2 gives 14,

x *(*3→)* 3*x *(+2→)* 3*x+2 = 14.

Reversing the walk by subtracting 2 and dividing by 3 gives the initial number:

x = 4 = (14-2)/3 *(←/3)* 14-2 *(←-2)* 14

The answer is tested by once more walking forward, 3*4 + 2 = 12 + 2 = 14.

The tradition sees a quadratic equation x^2 + b*x + c = 0 as a pure algebraic problem to be solved, first by factorizing, then by completing the square, and finally by using the solution formula.

The alternative sees solving a quadratic equation as a problem combining algebra and geometry, where a square with the sides x+b/2 creates fives areas, x^2 and b/2*x twice and c and (b^2/4-c) where the first four disappear and leaves (x+b/2)^2 to be the latter, b^2/4-c.

The tradition sees a function as an example of a relation between two sets where first-component identity implies second-component identity. And it gives the name 'linear function' to f(x) = a*x+b even if this is an affine function not satisfying the linear condition f(x+y) = f(x)*f(y), as does the proportionality formula f(x) = a*x.

The alternative sees a function as a name for a formula containing two unspecified numbers or variables, typically x and y. Thus, a function is a fiction showing how the y-numbers depends on the x numbers as shown in a table or by a graph.

The tradition sees proportionality as an example of a function satisfying the linear condition. The alternative sees proportionality as a name for double-counting in different units creating per-numbers.

The tradition sees geometry to be introduced in the order: plane geometry, coordinate geometry and trigonometry.

The alternative has the opposite order. Trigonometry comes first grounded in the fact that halving a block by its diagonal allows the base and the altitude to be recounted in diagonals or in each other. This also allows a calculator to find pi from a sine formula. Next comes coordinate geometry allowing geometry and algebra to always go hand in hand so that algebraic formula can predict intersection points coming from geometrical constructions.

The tradition has quadratic functions following linear functions, both examples of polynomials.

The alternative sees affine functions as one example of constant change coming in five forms: constant y-change per x-change, constant y-percent-change per x-change, constant y-percent-change per x-percent-change, constant y-change per x-change together with constant y-percent-change per x-change, and finally constantly changing y- change.

The tradition sees logarithm as defined as the integral of the function y = 1/x.

The alternative sees logarithm and root combined both solving power equations. Thus a^x = b gives x = loga(b); and x^a = b gives x = a√b. This shows the logarithm as a factor-counter and the root as a factor-finder.

The tradition sees differential calculus as preceding integral calculus, and the gradient y' = dy/dx is defined algebraically as the limit of Δy/Δx for Δx approaching 0, and geometrically as the slope of a tangent being the limit position of a secant with approaching intersection points. The limit is defined by an epsilon-delta criterium.

The alternative sees calculus as grounded in adding blocks next-to each other. In primary school calculus occurs when performing next-to addition of 2 3s and 4 5s as 8s. In middle school calculus occurs when adding piecewise constant per-numbers, as 2m at 3m/s plus 4m at 5m/s. In high school calculus occurs when adding locally constant per-numbers, as 5seconds at 3m/s changing constantly to 4m/s. Geometrically, adding blocks means adding areas under a per-number graph. In the case of local constancy this means adding many strips, made easy by writing them as differences since many differences add up to one single difference between the terminal and initial numbers, thus showing the relevance of differential calculus. The epsilon-delta criterium is a straight forward way to formalize the three ways of constancy, globally and piecewise and locally, by saying that constancy means an arbitrarily small difference.

Conclusion

With 50 years of research, mathematics education should have improved significantly. Its lack of success as illustrated by OECD report 'Improving Schools in Sweden' made this paper ask: Applying sociological imagination when meeting Many without having predicates forced upon it by traditional mathematics, can we design a STEM-based core math curriculum aimed at making migrants pre-teachers and pre-engineers in two years? This depends on what we mean by mathematics. And, looking back, mathematics has meant different things through its long history, from a common label for knowledge to today's 'meta-matism' combining 'meta-matics' defining concepts by meaningless self-reference, and 'mathe-matism' adding numbers without units thus lacking outside validity. So, inspired by Heidegger's 'always question sentences, except for its subject' we returned to the original Greek meaning of mathematics: Knowledge about Many by itself and in time and space.

Observing Many by itself allows rebuilding mathematics as a 'many-matics', i.e. as a natural science about the physical fact Many, where counting by bundling leads to block-numbers that recounted in other units leads to proportionality and solving equations; where recounting sides in triangles leads to trigonometry; where double-counting in different units leads to per-numbers and fractions, both adding by their areas, i.e. by integration; where counting precedes addition taking place both on-top and next-to involving proportionality and calculus; where using a calculator to predict the counting result leads to the opposite order of operations: division before multiplication before subtraction before next-to and on-top addition; and where calculus occurs in primary school as next-to addition, and in middle and high school as adding piecewise and locally constant per-numbers; and where integral calculus precedes differential calculus.

With water cycles fueled by the sun and run by gravity as exemplary situations, STEM offers various examples of Many in space and time since science and technology and engineering basically is about double-counting physical phenomena in different units.

The designed STEM-based core math curriculum has been tested in parts with success at the educational level in Danish pre-university classes. It might also be tested on a research level if it becomes known through publishing, i.e., if it will be accepted at the review process. It will offer a sociological imagination absent from traditional research seen by many teachers as useless because of its many references.

Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality used to collect reliable data and to test the validity of its findings by falsification attempts.

In a Swedish context, obsessive self-referencing has been called the 'irrelevance of the research industry' (Tarp, 2015, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics education and research are two examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead thus marginalizing or forgetting its original outside goal.

So, if a society as Sweden really wants to improve mathematics education, extra funding might just produce more researchers more eager to follow inside traditions than solving outside problems. Instead funding should force the universities to arrange curriculum architect compositions to allow alternatives to compete as to creativity and effectiveness, thus allowing the universities to rediscover their original outside rational goals and to change its routines accordingly. A situation described in several fairy tales; the Sleeping Beauty hidden behind the thorns of routines becoming rituals until awakened by the kiss of an alternative; and Cinderella making the prince dance, but only found when searching outside the established nobility.

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13. The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants

Educational shortages described in the OECD report 'Improving Schools in Sweden' challenge traditional math education offered to young male migrants wanting a more civilized education to return help develop and rebuild their own country. Research offers little help as witnessed by continuing low PISA scores despite 50 years of mathematics education research. Can this be different? Can mathematics and education and research be different allowing migrants to succeed instead of fail? A different research, difference-research finding differences making a difference, shows it can. STEM-based, mathematics becomes Many-based bottom-up Many-matics instead of Set-based top-down Meta-matics.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Math Education. But, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

To find an unorthodox solution let us pretend that a university in southern Sweden arranges a curriculum architect competition: 'Theorize the low success of 50 years of mathematics education research, and derive from this theory a STEM-based core mathematics curriculum for young male migrants.'

Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is a rational organization, 'in which the *end* is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Such a goal displacement occurs if saying 'The goal of mathematics education is to teach and learn mathematics'. Furthermore, by its self-reference such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts. In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2x3=6' stating that 2 3s can be re-counted as 6 1s.

So, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about shapes and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement, according to what Bauman (1992, p. ix) calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'?

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. (Heidegger, 1962, pp. 126, 169)

Heidegger has inspired existentialist thinking, described by Sartre (2007, p. 22) as holding that 'existence precedes essence'. In France, Heidegger inspired Derrida, Lyotard, Foucault and Bourdieu in poststructuralist thinking pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and to your world (Lyotard, 1984; Bourdieu, 1970; Foucault, 1995).

As to the political aspects of research, Foucault says:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006, p. 41; also on YouTube)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics and education and research are examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as an inside goal instead, thus marginalizing or forgetting its original outside goal.

To avoid this, difference-research is based upon the Greek sophists, saying ‘Know nature form choice to unmask choice masked as nature.’; and Heidegger saying ‘In sentences, trust the subject but question the rest.’; and Sartre saying ‘Existence precedes essence’; and Foucault, seeing a school as a ‘pris-pital’ mixing power techniques of a prison and a hospital by keeping children and adolescents locked up daily to be cured without being properly diagnosed. For it is differences that unmask false nature, and unmask prejudice in predicates, and uncover alternative essence, and cure an institution from a goal displacement.

Furthermore, difference-research knows the difference between what can be different and what cannot. From a Heidegger view an is-sentence contains two things: a subject that exists and cannot be different, and a predicate that can and that may be gossip masked as essence, provoking ‘the banality of Evil’ (Arendt, 2006) if institutionalized. So, to discover its true nature, we need to meet the subject, the total, outside its predicate-prison of traditional mathematics. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help develop or rebuild their original countries.

So, to restore its authenticity, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many, Children use Block-numbers to Count and Share

How to deal with Many can be learned from preschool children. Asked ‘How old next time?’, a 3year old will say ‘Four’ and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, ‘That is not 4, that is 2 2s. Children also use block-numbers when talking about Lego bricks as ‘2 3s’ or ‘3 4s’. When asked ‘How many 3s when united?’ they typically say ‘5 3s and 3 extra’; and when asked ‘How many 4s?’ they say ‘5 4s less 2’; and, placing them next-to each other, they say ‘2 7s and 3 extra’.

You don’t need research to observe how children love digital counting by bundling, replacing a bundle of 2 1s with 1 Lego Brick with 2 knobs to be placed in a cup for the bundles; and they don’t mind exchanging 2 2s with 1 Lego brick with 4 knobs to be placed in a cup for 4s. And they have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 1 4s &3, etc. And children don’t mind writing a total of 7 using ‘bundle-writing’ as T = 7 = 1B5 = 2B3 = 3B1 = 1BB0B3 = 1BB1B1. And with 1 plastic S for 1 borrowed, some children even writes T = 7 = 3B1 = 4BS = 5BSSS. Also, children love to count in 3s and 4s. Recounting in 5s is unfortunately not possible since Lego refuses to produce bricks with 5 knobs.

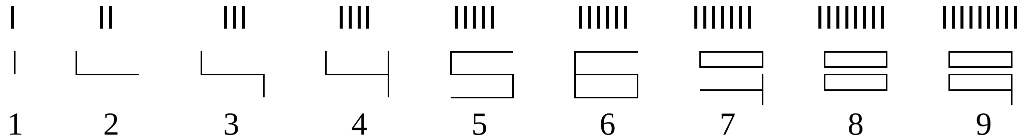
Sharing 9 cakes, 4 children takes one by turn as long as possible; with 4s taken out they say ‘I take 1 of each 4’, and with 1 left they say ‘let’s count it as 4’. And they smile when seeing that sharing 4 5s by 3 is predicted by asking a calculator 4*5/3. Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. So children master sharing, taking parts and splitting into parts before having learned about division and counting- and splitting-fractions, which they would like to learn before being forced to add.

Children thus show core mastery of Many before coming to school, allowing school to build upon this knowledge instead of rejecting it. So, school could ask research to design a curriculum, that counts totals in two-dimensional block-numbers instead of one-dimensional line-numbers; that counts and re-counts and double-counts totals before they are added, and then both on-top and next-to; that teaches 8/4 as 8 counted in 4s giving 2 4s instead of as 8 split between 4 giving 4 2s; and that root counting-fractions and splitting-fractions in per-numbers and re-counting. Difference-research gladly takes on such a curriculum design.

Meeting Many Creates a Count&Multiply&Add Curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding to create a number-language sentence, T = 2 3s, containing a subject, a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents, icons can be used as units when counting: four strokes in the 4-con, five in the 5-icon, etc.



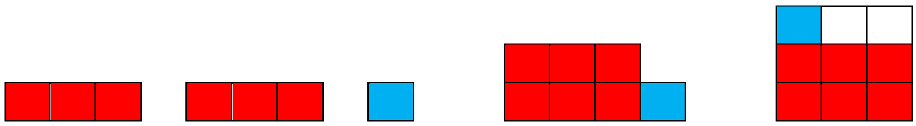
We count in bundles to be stacked as block-numbers to be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count we take away bundles (thus rooting division as a broom wiping away the bundles) to be stacked (thus rooting multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting subtraction as the trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)xB saying that ‘from T, T/B times, B can be taken away’:

7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

Finally, bundle- or bundle-writing double-counts the bundles inside the bundle-cup and the singles outside, where an overload or underload is removed or created by re-counting in the same unit, T = 7 = 2B1 3s = 2B1 3s = 1B4 3s = 3B-2 3s.

Likewise, placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations to be solved by re-counting, resulting in moving numbers to the opposite side with the opposite sign: u x 7 = 42 = (42/7)x7 gives u = 42/7.

Double-counting in physical units creates per-numbers bridging the units, thus rooting proportionality. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. Then both on-top and next-to addition can be reversed, thus rooting equations and differential calculus.

In a rectangle split by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Meeting Many in a STEM Context

Having met Many by itself, now we meet Many in time and space in the present culture based upon STEM, described by OECD as follows:

> In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature’s physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature’s three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground since motion transfers through collisions, now present as increased motion in molecules; so the motion has lost its order and can no longer work.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making it combine to stars and planets and galaxies. Some planets have a size and a distance from its star that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles and mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves.

Technology is knowledge about ways to satisfy human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for motion and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language for predicting Many by calculation sentences, formulas, expressing counting and adding processes. First Many is double-counted in bundles and singles to create a total T that might be re-counted in the same or in a new unit or into or from tens; or double-counted in two physical units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. Finally, triangulation predicts spatial forms.

So, a core STEM curriculum could be about cycling water. Heating pumps in motion transforming water from solid to liquid to gas, i.e. from ice to water to steam; and cooling pumps motion out. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform moving water to moving electrons, electricity. To get to the dam, we build roads on hillsides.

The Electrical circuit, an Example

To work properly, a 2000Watt water kettle needs 2000Joule per second. The socket delivers 220Volts, a per-number double-counting Joules per charge-unit.

Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle has a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per charge-unit) *(charge-unit per second) we also have a second formula Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So surprisingly, the 120Watt bulb only receives half of the Joules of the 60Watt bulb.

Difference-research Differing from Critical and Postmodern Thinking

Together with difference-research, also critical thinking and postmodernism show skepticism towards knowledge claims, so how does difference-research differ?

As to critical thinking, Skovsmose & Borba (2000) describes a Brazilian research group that, focusing on issues related to new technologies and mathematics education, has developed software and work with students at different levels and with teachers. The group was approached by a teacher from a nearby school where she had some tough problems to face and hoped that the computers would be able to help her. She was teaching rational numbers to a class of 5th graders, with a mixture of 11year old students and 15year old repeaters having given up rational numbers and turning to violence.

> The teacher was enthusiastic about a software, which deals with rational numbers. (..) Both researchers and teacher had the ‘intuition’ that the computer might have a positive effect in this class and maybe could avoid that the students had to repeat this grade again. (p. 7)

By recommending computers, the researchers showed criticism, not towards fractions as such, but towards how they are taught. Critical thinking thus sees mathematics as an unquestionable goal, only how it is taught can be questioned.

Contrary to this, difference-research sees fractions as a means rooted in double-counting, and recommends fractions introduced as per-numbers via the ‘fraction-paradox’: 1 red of 2apples and

2red of 3apples total 3red of 5apples and not 7red of 6apples as says the textbook. Fractions thus add by their areas as integral calculus. Adding fractions of the same total can be treated later. Introducing fractions via per-numbers and separating core-mathematics from 'footnote-mathematics' will side the teacher with the learner against the textbook.

As to postmodern thinking, the book 'Mathematics Education within the Postmodern' (Walshaw, 2004) contains 12 chapters divided into three parts: thinking otherwise for mathematics education, postmodernism within classroom practices, and within the structures of mathematics education. The preface says:

> It is a groundbreaking volume in which each of the chapters develops for mathematics education the importance of insights from mainly French intellectuals of the post: Foucault, Lacan, Lyotard, Deluze. (p. vii)

Although the book wants to be skeptical towards both mathematics and its education, it is only the educational part that is scrutinized; and most authors describes how what is labeled postmodern thinking can be exemplified in educational contexts, they don't see mathematics itself as a social construction that could be questioned also. A central thinker as Derrida is mentioned only in the two survey chapters, and the core concept of deconstruction is not mentioned at all despite its fundamental importance to a postmodern perspective to mathematics education (Tarp, 2012).

By going behind French thinking to its root in Heidegger existentialism, difference-research is the only skeptical thinking raising the basic sociological question about a possible goal displacement in mathematics itself.

Conclusion and Recommendation

The task of the curriculum architect competition was 'Theorize the low success of 50 years of mathematics education research, and derive from this theory a STEM-based core mathematics curriculum for young male migrants.'

One explanation sees the situation caused by mathematics itself as very hard to teach and learn. This paper, however, sees it caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many. The two views lead to different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying 'To master Many, count to produce block-numbers and per-numbers that might be constant or variable, to be united by adding or multiplying or powering or integrating.

Thus, this simplicity of mathematics as expressed in a Count&Multiply&Add curriculum allows learners to keep their own block-numbers, and to acquire core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young male migrants learn core STEM subjects at the same time, thus allowing them to become pre-teachers or pre-engineers after two years to return help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

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14. Math Competenc(i)es - Catholic or Protestant?

Introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015 OECD-report 'Improving Schools in Sweden'. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other Scandinavian countries also have experienced declining PISA results. Which came as a surprise since they all adopted the idea of the eight mathematics competencies introduced by Niss (2003) as a means to solve poor mathematics performance. Of course, new ideas cannot work overnight, but after close to two decades it is time to ask: Why does math competencies not work?

Since education and textbooks are social constructions meant to solve important problems by common social institutions, maybe sociology can provide an answer to the lacking success of the eight mathematics competencies.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959), and by Bauman (1990) saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16). As to rationality as the base for social organizations, Bauman says (pp. 79, 84):

> **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right.

As an institution, mathematics education is a public organization with a rational action 'in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas, arithmetic and geometry and music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time, i.e. as a 'Many-matics'. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught 'reckoning' (Rechnung in German) in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming a meaningless language by mixing concrete examples and abstract concepts. In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1week + 2days is 9 days. So, mathematics has meant different things during its long history.

Defining Mathematics Competencies

In the paper 'Mathematical Competencies and the Learning of Mathematics: The Danish Kom Project' Niss writes (2003, p. 1):

> The fundamental idea of the project is to base the description of mathematics curricula primarily on the notion of a "mathematical competency", rather than on syllabi in the traditional sense of lists of topics, concepts, and results. This allows for an overarching conceptual framework which captures the perspectives of mathematics teaching and learning at whichever educational level.

Niss writes (pp. 4-5) that the project was initiated in 2000 by the Danish Ministry of Education asking the following questions:

- To what extent is there a need for innovation of the prevalent forms of mathematics education?
- Which mathematical competencies need to be developed with students at different stages of the education system?
- How do we ensure progression and coherence in mathematics teaching and learning throughout the education system?
- How do we measure mathematical competence?
- What should be the content of up-to-date mathematics curricula?
- How do we ensure the ongoing development of mathematics as an education subject as well as of its teaching?
- What does society demand and expect of mathematics teaching and learning?
- What will mathematical teaching materials look like in the future?
- How can we, in Denmark, make use of international experiences with mathematics teaching?
- How should mathematics teaching be organised in the future?

Next, Niss defines what it means to master mathematics (pp. 5-6, 8):

> The Committee based its work on an attempt to answer the following question: *What does it mean to master mathematics?*' (..) To master mathematics means to possess mathematical competence. (..)

> To possess a competence (to be competent) in some domain of personal, professional or social life is to master (to a fair degree, modulo the conditions and circumstances) essential aspects of life in that domain. *Mathematical competence* then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. (..) *A mathematical competency* is a clearly recognisable and distinct, major constituent of mathematical competence. (..) There are eight competencies which can be said to form two groups. The first group of competencies are to do with the ability to *ask and answer questions in and with mathematics*. (..) The other group of competencies are to do with the ability to deal with and *manage mathematical language and tools*:

Before writing that 'Possessing a mathematical competency (to some degree) consists in being prepared and able to act mathematically on the basis of knowledge and insight (p. 10)' Niss lists (pp. 7-9) and specify the two groups of four mathematical competencies

1. Thinking mathematically (mastering mathematical modes of thought)
2. Posing and solving mathematical problems
3. Modelling mathematically (i.e. analysing and building models)
4. Reasoning mathematically
5. Representing mathematical entities (objects and situations)
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools (IT included)

Discussing Mathematics Competencies

As to the definition of mathematics competencies, Niss is very clear: Mathematics competencies are the eight constituents of mathematics competence, defined as the ability to master mathematics. What is not so clear is what Niss means with these two words, mathematics and master.

What kind of mathematics

As to mathematics, at least two kinds of mathematics exits as shown above, a bottom-up and a top-down version, the original Greek grounded Many-matics and the modern self-referring meta-matism. Likewise, on the background of the science wars and mathematics wars in the previous decades, it would be relevant to clarify what kind of mathematics Niss is talking about: the original Greek version, the 'back to basics' pre-NewMath version, the set-based NewMath version, or a post-NewMath version in its constructivist or postmodern forms (Tarp, 1998, 2000).

Instead Niss refers to the fact that in Denmark, as one of the few countries if not the only, teacher education is not allowed to take place at universities where only research directed set-based mathematics is taught forcing students to include a master degree before being allowed to teach in upper secondary school.

Niss describes this difference in teacher background by saying that before upper secondary school, teachers 'are ambassadors of the student to the subject', whereas 'the university graduates who end up teaching mathematics see themselves as ambassadors of mathematics to the student' (pp. 2-3).

> A further aspect of the cultural and institutional differences that exist in Danish mathematics education is that mathematics is perceived and treated so differently at the different levels that one can hardly speak of the same subject, even if it carries the same name throughout the system. (..) The main problem is that the different educational levels tend to see themselves as competitors rather than as agents - acting at different sections of the education system - of the same overall endeavour and a common project, namely to increase and strengthen the mathematical competence of all students who receive some form of mathematics education.

On this background it seems clear that what Niss means with mathematics is the set-based university mathematics introduced with the NewMath. So what Niss points out is which competences are needed to master inside set-based university mathematics, not which are needed to master its outside root, Many. Thus, the question about what could be called quantitative competence is left unanswered.

What kind of Mastering

In the final report Niss left out two of the original Ministry questions, 'How can education take into account the new student type?' and 'What impact will a modified education have for teacher training?'. And in two questions, 'Which competences and qualifications can be acquired at the various stages of the education' and 'How can competences and qualifications be measured?', the word qualification is left out and the word mathematics is added. Likewise, the original term competence has replaced by his own term, competency (Tarp, 2002).

The difference between qualifications and competence might be illustrated by the fact that learning is a process shared by all three kinds of animals, reptiles and mammals and humans, all producing offspring to reproduce, but in different numbers since the chances of survival are different because of different learning abilities. Darwin's 'survival of the fittest' principle points to the fact that to survive you must fit to the surrounding outside world. Reptiles survive by their genes that might change over generations through mutations. Mammals feed their offspring until sexual maturity so they can adapt to the outside surroundings by guidance from their parents in an informal learning setting that could be called apprenticeship or learning from the master, providing the learner with tacit knowledge, also called abilities or know-how or competences. Likewise, humans learn basic living skills and the mother language as competences through apprenticeship guided by caring parents and adults. However, humans benefit from an additional learning possibility occurring when expanding the brain to keep the balance when standing up freed the forelegs to become graspers. Now the brain was also able to store sounds to mentally grasp what was grasped physically (in German: 'greifen & begreifen'), thus developing a word-language and a number-language for outside qualities and quantities allowing for life-long learning.

Language allowing information to be transferred between brains thus creates more competences quicker and more effective. And creates a formal learning setting called education or schooling using rational goal-means descriptions to qualify the learners to obtain the goal by following the means.

Thus, where animals develop competences from 'ex-ducational' informal learning outside school, humans learn additional qualification from 'in-ducational' formal learning inside schools. So human knowledge comes from two channels, from inside school as qualifications and from outside school as competences.

Inside teaching can take place through mediation to qualify or through guidance to develop competences. This discussion takes place between traditional teaching and constructivism; and within constructivism, between a social and a radical version where Vygotsky points to teaching, and Piaget to guidance.

Competence versus Capital

Niss uses no theoretical reference to mathematics or education, but points out that the report is supposed to be a response to question posed by the Ministry (p. 6).

Thus, there is no discussion of parallel and more developed or used concepts describing the same reality as does competences. As an example, Bourdieu (1977) has developed a theory on habitus and capital describing how in a social filed, your social or knowledge capital depends on your habitus within the field. Thus, it seems as if competence is a parallel concept to capital. If that is the case then, according to Bourdieu, capital is only obtainable by informal learning processes.

The Counter KomMod report

The KomMod report (Tarp, 2002) shows the original 12 Ministry questions and how they can be answered in a different way. In the end it compares the two reports by talking about a catholic and a protestant version of mathematics with eight and two competences respectively (p. 3):

> Defining competence as insight-based, the report assumes that mathematics is already learned, after which the rest of the time can be used to apply mathematics, not on the outside world, but on mathematics itself through eight internal competencies leading to exercising mathematical

professionalism. This makes it a report on 'catholic mathematics' with eight sacraments, through which the encounter with science can take place. In contrast to this, the counter-report portrays a 'protestant mathematics' that emphasizes the importance of a direct meeting between the individual and the knowledge root, Many, through two sacraments, count and add.

Quantitative Competence

In the outside world, Many often occurs in time and space. To master Many, you must have quantitative competence from informal learning or quantitative qualifications from formal learning.

Meeting Many, we ask 'How many in total?' To answer, we count and add to get a number for a number-language sentence telling that the total is e.g. T = 456, thus containing a subject and a verb and a predicate as in the word-language. By counting and adding you build different know-how as to how to master Many:

- A digit has as many strokes as it represents, e.g. four strokes in the 4-icon, etc.
- Counting the fingers on a hand, the total cannot be different, but how to count it can be different, e.g. T = 5 1s = 2 2s & 1 = 1 3s & 2 1s = 1 3s & 1 2s etc.
- The sentence T = 456 is a short way of writing T = 4*BB + 5*B + 6*1, describing what exists, three blocks with 6 1s and 5 bundles and 4 bundles-of- bundles, typically using ten as the bundle-size and therefore needing no icon since ten then is 1*B. This shows that a number is the result of several countings: of unbundled ones, of bundles, of bundles-of-bundles etc.; and shows that all numbers have units: ones, bundles, bundles-of-bundles, etc.
- Writing out fully, T = 456 also shows the four ways to unite totals: on-top addition creating a block described by multiplication as repeated addition, power describing repeated multiplication when forming bundles-of-bundles, and finally integration as next-to addition when juxtaposing blocks.
- Operations are icons also: division is iconized as a broom wiping away the bundles; multiplication as a lift stacking the bundles into a block; subtraction as a trace left when dragging away the blocks to look for unbundled singles; and addition as a cross since blocks may be added both on-top or next-to.
- To deal with leftover singles when bundling we introduce a decimal point to separate the bundles from the singles, e.g. T = 7 = 2B1 3s = 2.1 3s, or we count the singles in bundles also even if a part only, T = 7 = 2B1 3s = 2 1/3 3s.
- A total can be recounted to change unit. Recounting in the same unit creates overload or underload e.g. T = 42 = 4B2 = 3B12 = 5B-8. This is useful when performing standard operations as e.g. T = 5*43 = 5*4B3 = 20B15 = 21B5 = 215. Or, we just move the decimal point separating the bundle from the unbundled, e.g. T = 4.3 hundreds = 43 tens = 0.43 thousands.
- To recount in another bundle size we use a 'recount formula' T = (T/B)*B saying that 'from T, T/B times B can be taken away' as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the 'restack formula' T = (T–B)+B saying that 'from T, T–B is left when B is taken away and placed next-to', as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. The recount formula turns out to be a very basic formula turning up repeatedly: In proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right-angled triangles, and in calculus as dy = (dy/dx)*dx = y'*dx when counting steepness on a curve.
- To recount icons in tens we use the multiplication table, e.g. T = 6 7s = 6*7 = 42. To recount tens in icons we solve equations, e.g. T = 42 = ? 7s = x*7 solved by x = 42/7, i.e. by moving numbers to opposite side with opposite sign.
- Double-counting a quantity in physical units creates per-numbers as e.g. 4$/5kg or 4/5 $/kg allowing the two units to be bridges by recounting in the per-number: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$, etc. With like units we get fractions, or percentages.

- Adding means uniting unit- and per-numbers, that can be constant or variable. So to predict, we need four uniting operations: addition and multiplication uniting variable and constant unit-numbers; and integration and power uniting variable and constant per-numbers. As well as four splitting operations: subtraction and division splitting into variable and constant unit-numbers; and differentiation and root/logarithm splitting into variable and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite.
- Blocks can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

Proportionality, an Example of Different Quantitative Competences

A question asks 'If 5kg costs 30$ what does 8kg cost; and what does 54$ buy?

A 1867 reguladetri 'long way-method' says: 'Make the outer units like, then multiply and divide, but from behind'. So, after reformulating the second question to '30$ buys 5kg, what does 54$ buy?' the first answer is 8*30/5$ = 48$; and the second answer is 54*5/30kg = 9kg.

A 1917 unit-method says: 1kg costs 30/5 = 6$, so 8 kg costs 6*8 = 48$.

A 1967 function-method says: With f(5) = 30, the linear function f(x) = c*x becomes f(x) = 6*x. So f(8) = 6*8 = 48. And 54 = 6*x is an equation. To neutralize 6, both sides are multiplied with its inverse element, 1/6, giving x = 54*1/6 = 9.

A 2017 back-to-basics method says 'cross-multiply' the price equation: 30/5 = x/8 gives 5*x = 8*30, so x = 48. And 30/5 = 54/x gives 30*x = 5*54, so x = 9.

A 2067 double-counting method recounts in the per-number 5kg/30$. So 8kg = (8/5)*5kg = (8/5)*30$ = 48$. And 54$ = (54/30)*30$ = (54/30)*5kg = 9kg.

Conclusion

Invented to improve mathematics education, the eight mathematics competencies inspired Scandinavian educational reforms that failed as witnessed by low PISA results decreasing until 2015. This paper asked why the competencies failed.

Formal education can use mediation to qualify or constructivism to create competences by guided meetings with the outside subjects for which education is supposed to prepare the learner. With Niss we can discuss which competences to create and how, but only in a constructivist setting that accepts the original Greek meaning of mathematics as knowledge about Many in time and space.

Niss may be right that his eight mathematical competences are needed to survive at a university that holds on to the original set-based version of mathematics introduced with the NewMath and recommended by Bruner to also be mediated in schools. But to master the outside goal Many, two competences will do, count & add, since they allow answering the standard question 'How many in total' by producing a number created by counting and adding as shown when writing out fully a number as a combination of blocks.

So the eight mathematics competences failed because university mathematics and school mathematics have different goals. At the university, education prepares you for the inside goal of staying at the university as a researcher; and in school, education prepares you for the task of mastering Many as it appears outside school in time and space.

Recommendation: Expand the Existing Quantitative Competence

By distinguishing between 4 and 2 2s at the 4th birthday, a child shows that before formal learning begins in school, the informal learning of growing up makes the child develop the two core quantitative competences, counting and adding. By counting in 2dimensional block-numbers supplied with some leftovers, children show a basic competence in double-counting a total in bundles and unbundled. And, when adding blocks, they answer by using one of the units or by uniting the units, thus showing a basic competence in proportionality and calculus.

Seeing expanding the learner's quantitative competence as the goal of mathematics education, school may choose to use guiding 'footnote-teaching':

- Show that digits are icons with as many strokes as they represent by inviting the child to build up a 5-icon with five dolls or cars or animals, etc.
- Ask the child to use cups for the bundles when re-counting a total in icons thus emphasizing that counting means double-counting, first bundles to be placed in a bundle-cup, then unbundled singles to be left outside, allowing a total to be counted in three ways: normal, and with outside overload or underload.
- Show that the four operations are icons as well, created to allow a calculator to predict the result when recounting a total in another unit; especially from icons to tens predicted directly by the multiplication table; or from tens icons, becoming equations solved by recounting in the icon, and technically by moving numbers to opposite side with opposite sign.
- Accept overload or underload, quickly created or removed by recounting, with standard operations as adding, subtracting, multiplying or dividing.
- Show that totals can be added both on-top after recounting them in the same unit thus rooting proportionality, and next-to recounting them in the united unit thus rooting integral calculus.
- Show that reversed on-top addition roots equations, again solved by recounting, i.e. by moving to the opposite side with opposite sign; and that reversed next-to addition roots differential calculus by using subtraction to remove the initial block, and division to recount the rest.

Once school has allowed the child to use and develop its own quantitative competence, it will be possible to expand this by introducing double-counting in physical units to create per-numbers, becoming fractions if using the same physical unit. Adding per-numbers and fractions by their areas then becomes just another example of adding blocks next-to each other, also by their areas. (Tarp, 2017)

So, formal school mathematics education can choose to expand the child's existing two quantitative competences, to count and to add. Or it can choose to discard them and force upon the child eight mathematics competencies about one-dimensional number-names arranged in a place-value system, and about more or less obscure algorithms when adding, subtracting, multiplying and dividing, and about fractions as numbers that can be added without considering the units.

In short, the school can choose to strengthen or weaken the mastery of Many that the child brings to school. Wanting to improve mathematics education, maybe it would be a good idea to choose the former and stop practising the latter.

So, we can celebrate the 500year Luther anniversary by saying: The subject of mathematics education, Many, we can meet directly without being mediated by its 'latinized' version in the form of a self-referring meta-matism.

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15. The ‘KomMod Report’, a Counter Report to the Ministry’s Competence Report

Allan Tarp, 2002, translated into English in 2017.

The KomMod report provides an alternative response to KOM-project terms of reference, in the expectation that the Science Board of education and the Ministry of education want to respect a common democratic IDC-tradition with Information and Debate between alternatives before a Choice is made. The report replies to the following questions relating to mathematics education:

(a) What is the society's requirements for the education?

(b) To what extent is there a need to renew the existing education?

(c) How can education take into account the new student type?

(d) What content can contemporary mathematics education have?

(e) How can the future of education be organized?

(f) How to secure progression and consistency in education?

(g) What impact will a modified education have for teacher training?

(h) Which competences and qualifications can be acquired at the various stages of the education?

(i) How can competences and qualifications be measured?

(j) How can future teaching materials look like?

(k) How to secure a continuous development of the education?

(l) How can Denmark exchange educational experience with other countries?

Ad a. Our democratic society needs citizens and specialists to have a common number-language to communicate about quantities and calculations. Society needs mathematics as a human right, both as a discursive qualification and as silent competence.

Ad b. There is a need to renew the current mathematics education in order to solve its three main problems: 1. There is a widespread number-languages illiteracy, where many citizens are reluctant to use the number-language. 2. There are major transition issues between primary, secondary and tertiary education. 3. There is a decreasing enrolment to math-based education in science, technology and economy, as well as a large shortage of new secondary school teachers in mathematics.

Ad c. Future mathematics teaching should respect today's democratic, anti-authoritarian youth and its requirements on meaning and authenticity. This can be achieved if the subject respects its historical roots, and re-humanizes itself by presenting abstractions as abstractions and not as examples, i.e. as abstractions from examples (a function is a name for a formula with variable numbers), and not as examples of even more abstract abstractions (a function is an example of a set-relation). In short, the subject should portray itself as mathe-matics, recognizing its outside roots from which it has grown bottom up through abstractions. And the subject must say goodbye to the current ‘meta-matics’ and its belief that it has meta-physical roots and has grown top down as examples. Finally, the subject should respect the fact that people learn differently. Children learn by touching the world, i.e. by building competences. Young people learn by listening to the world, i.e. by building narratives and skills from the learning question "tell me something I don't know about something I know" (gossip-learning).

Ad d. Mathematics must respect its history as grown through abstractions, and thus also its construction as a number-language grammar, which can only be introduced after the number-language has been developed. The number-language has grown out of the meeting with quantity in time (repetition) and in space (many-ness). This meeting constructed numbers to describe the total, either through counting in pieces, bundles, bundles of bundles, bundles of bundles of bundles etc. Or faster by means of calculations to unite and divide unit-numbers (3$) and per-numbers (3$/day, 3%): Plus and minus unite and divide in variable unit-numbers (3 + 5 =?, 3 +? = 8). Multiplication and division unite and divide in constant unit-numbers (3 * 5 =?, 3 *? = 15). Potency and root & logarithm unite and divide in constant per numbers (3 times 5% =?%, 3 times ?% = 20%, ? times

5% = 20%). Integration and differentiation unite and divide variable per-numbers (5 seconds at 2 m/s growing evenly to 4 m/s =? m, 5 seconds of 2 m/s growing to 4 m/s = 18 m?). In short, the subject must respect the fact that geometry has grown out of what the word means in Greek, earth measurement; and respect that algebra has grown out of what the word means in Arabic, reunion, i.e. uniting and dividing constant and variable unit-numbers and per-numbers. Geometry and algebra must therefore respect their historical roots in an agricultural culture with two main questions: "How to share the Earth, and what it produces?" The number-language has a number of typical applications: Geometry deals with forms and shapes. Formulas deal with number levels. Growth deals with predictable change. Statistics/probability deals with change that is not pre-dictable but post-dictable. It is important to clean teaching of 'killer-Mathematics' (i.e. mathematics, that does not occur outside of the classroom, and that can only be used for one thing, killing students' interest). Addition should only occur within the parentheses, which ensures that the units are equal (T = 2 * 3 + 5 * 3 = (2 + 5) * 3 = 7 * 3 = 21). Fractions should only act together with their totals (1/2 of 2 plus 2/3 of 3 = 3/5 of 5). Equations should be solved by reversed calculation. Since the set concept cannot be well-defined it should be removed, and functions be postponed until it pops up historically after differential calculus.

Ad e. Future mathematics lessons can be organized in two main areas: Child math and youth math from respectively grade 1-7 and 8-12. Meeting the roots of mathematics roots, Many in time and space, will develop the learner's two core competences: to count and to add.

Ad f. Progression and consistency in teaching can be ensured by letting the child's math grow out of the local examples of Many, and of agricultural examples from rural and urban areas, and by letting the youth's math grow out of industrial culture and its global diversity. As well as by the child primarily working with unit-numbers, and young people primarily with per-numbers.

Ad g. By dividing education into the child's mathematics and the youth's mathematics, it will also be natural to divide teacher education in primary school teacher and secondary school teacher, as in the rest of the world approximately. This means that all future teacher-training takes place at a university. In the end, this will coincide with the division of the school into a primary school and secondary school that will take place within the next decade in connection with the high school collapse due to increased teacher retirement and decreasing enrollment of new teachers in mathematics and natural science.

Ad h. By meeting Many in time and space, the child develops competences in uniting and dividing constant and variable unit-numbers. In the countryside, bundling and re-bundling leads to multiplication and division. In the city, stacking and re-stacking leads to addition and subtraction. Calculating repetition and diversity develops the skills of young people to unite and divide constant and variable per-numbers. Totaling interest rates leads to power, root and logarithm. Totaling distances leads to integral and differential calculus.

Ad i. Competences are tacit knowledge and can therefore be neither described nor measured, but will evolve automatically through the meeting with meaningful and authentic situations, and grow from the many concrete experiences with Many in time and space, bundling and stacking, uniting and splitting, unit-numbers and per-numbers. Qualifications is measured as now through three types of tasks: Routine tasks, text tasks and projects.

Ad j. Future teaching materials should be short and concise so that time could be dedicated for student learning through self-activity. The material should respect that students have two brains, a reptile's brain for routines and a human brain for conceptual understanding. There should therefore be training tasks with responses, so learners can progress at their own pace and do as many exercises as wanted. As well as textbooks telling how mathematics has grown from practice through layers of abstractions, and accepting different names so concept may be named both bottom-up and top-down, as e.g. growth by adding and linear function etc.

Ad k. A continuous development of education can be ensured by continuously relating mathematics to its roots and not to the current political correctness.

Ad l. Exchange of experience with foreign countries can be done through establishing a Danish development research, in which practitioners can combine being researcher at a University with being attached to a teacher team at a school. This will avoid the current barren ‘ghost research’ performed by researchers without experience background in teaching practice. Development research should be difference-research (Cinderella-research) using practice based and sociological imagination to discover and try out hidden alternatives.

The Difference between the KOM- and KOMMOD Reports

In mathematics education, the two main question are: ‘How do concepts enter into the world and into the student's head - from the outside or from the inside?’ These questions give rise to different answers. Secondary school structuralism says ‘outside-outside’: Concepts exist in the meta-physical world, they are discovered by researchers and mediated by teachers. Primary school constructivism says ‘outside-inside’: Concepts exist in the meta-physical world, but are discovered through experimentation, in which each student construct their own knowledge and abilities (schemata and competences), both being silent and only to be observed through use. Post-structuralism says the ‘inside-outside’: Concepts are created through invention and social construction, and should be presented as such. Apprenticeship says ‘inside-inside’: Concepts are constructed by the apprentice during the participation in the master's practice.

Worldwide, two knowledge wars rage, a math-war between structuralism and constructivism, and a science-war between structuralism and post- structuralism. Instead of acknowledging this diversity, the report is trying to conceal it by taking over the core constructivist concept, competence, but giving it a structuralist content (insight-based action-readiness). The French philosopher Foucault has shown how new words create new clients: ‘Qualification’ creates the unqualified, and ‘competent’ creates the incompetent. But where the unqualified can cure themselves by qualifying themselves, the incompetent cannot cure themselves by ‘competencing’ themselves, and are thus left to be cured by others, the competence-competent. Adoption and modification of the word competence can therefore be interpreted as a structuralist attempt to win the math-war by a coup, instead of using it to a fruitful dialogue with equal partners.

First structuralism tried to solve the math-crisis through the wording ‘responsibility for your own learning’. Students took this seriously and turned their back to ‘meta-matics’ with its meaningless self-reference (a function is an example of a set-relation: bublibub is an example of bablibab).

Now instead the teachers are disciplined and incapacitated by constructing them as incompetent, with a consequent need for competence development through massive in-service training. Omitting the competence ‘experimenting’ shows that the report only respects science as an end-product, and neither the process nor its roots in the outside world. Neither does it respect the way in which young people and especially children acquire knowledge through self-activity and learning.

Defining competence as insight-based, the report assumes that mathematics is already learned, after which the rest of the time can be used to apply mathematics, not on the outside world, but on mathematics itself through eight internal competencies leading to exercising mathematical professionalism. This makes it a report on ‘catholic mathematics’ with eight sacraments, through which the encounter with science can take place. In contrast to this, the counter-report portrays a ‘protestant mathematics’ that emphasizes the importance of a direct meeting between the individual and the knowledge root, Many, through two sacraments, count and add; and emphasizes that linguistic competence precedes grammatical competence. Meaning that also with quantitative competence, the number-language comes before its grammar, mathematics; and as with the word-language, grammar remains a silent competence for most.

Will the math-war end with a KOM-coup? Or will it be settled through a democratic negotiation between opposing views? The choice is yours, and the KomMod report gives you an opportunity to validate the arguments, not from above from political correctness, but from below from the historic roots of mathematics. Best of luck.

SET-based 'MetaMatics', or Many-based ManyMatics: Learning by Meeting the Sentence or by meeting its Subject

Class 1-2

SETS are united: addition
2 + 3 = 5
47 + 85 = 135
82-65 = 17
PROBLEM:
Addition is a false abstraction: 2 m + 3 cm = 203 cm
2 weeks + 3 days = 17 days
2 C + 3 D = 23 D
3 stones = stone + stone + stone

Country: Bundle & ReBundle
Multiplication is true abstraction:
3 stones = 3 times stone = 3·stone
2·3·days = 6·days
2·m·3·cm = 6·m·cm = 600 cm ^ 2
Bundling and ReBundling:
Total = 6 1s =? 2s
Response: 6·1 = 6 = (6/2)·2 = 3·2
ReBundling-rule: T = (T/b)·b

6/2: Counted in 2s
6·2: Counting 2s

To find the total, count or calculate: ReBundling (division)

Multiplication rebundles in tens:
T = 8·3 = 24 = 2·ten & 4·1
= 2·D & 4·1
Multiplication is division!

Max-height 3:
T = 8 3s = overload
T = 8·3 = 2·3 ^ 2 & 2·3
Unbundled can also bundled in parts, for example in 5s:
T = 8·3 = (24/5)·5 = 4·5 & 4·1
= 4·5 & (4/5)·5 = (4 4/5)·5

Class 3-4

SETS are repeated: multiplication
2·3 = 6
7·85 = 595
372/7 = 53 1/7

City: Stacks & ReStack
T = 653 + 289 =?
653 = 6·C & 5·D & 3·1
279 = 2·C & 7·D & 9·1
T = 8·C & 13·D & 12·1
T = 8·C & (13 + 1)·D & (12-10)·1
T = (8 + 1)·C & (14-10)·D & 2·1
T = 9·C & 4·D & 2·1 = **942**
<u>ReStack rule: T = (T – b) + b</u>

T = 654-278 =?
653 = 6·C & 5·D & 4·1
278 = 2·C & 7·D & 8·1
T = 4·C & -2·D & -4·1
= (4-1)·(C) & (-2 + 10)·D & -4·1
= 3·(C) & (8-1)·(D) & (-4 + 10)·1
= 3·C & 7·D & 6·1 = 376

T = 7·653 =?
T = 7·(6·C & 5·D & 3·1)
= 42·C & 35·D & 21·1
= 42·(C) & (35+2)·(D) & (21-20) ·1
= (42 + 3)·(C) & (37-30)·D & 1·1
= 45·C & 7·D & 1·1 = 4571

T = 653/7 =?
T = 6/7·C & 5/5·D & 3/7
= 65/7·D & 3/7
= (65-2)/7·(D) & (20 + 3)/7
= 9·D & 23/7
= 9·D & 3 2/7 = 93 2/7
(double book-keeping)

Class 5-6

SETS are divided: fractions
$\frac{1}{2}+\frac{2}{3}=?$
$\frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$
PROBLEM:
$\frac{1}{2}+\frac{2}{3}=\frac{1+2}{2+3}=\frac{3}{5}$ if
1 coke of 2 bottles plus 2 cokes of 3 bottles is (1 + 2) cokes of (2 + 3) bottles.

City: weighted average
$T=\frac{1}{2}\cdot 2+\frac{2}{3}\cdot 3=3=\frac{3}{5}\cdot 5$, or
$T=\frac{1}{2}\cdot 4+\frac{2}{3}\cdot 3=4=\frac{4}{7}\cdot 7$
So there are many different answers to the question
$\frac{1}{2}+\frac{2}{3}=?$
But NEVER more than 1!

Trade calculations
5 kg cost 60 \$, 3 kg cost ? \$

ReBundle \$	ReBundle kg
\$ = (\$/kg)·kg	3 kg = (3/5)·5kg
\$ = (60/5)·3	3 kg = (3/5)·60\$
\$ = 36	3 kg = 36\$

Percentages part 1
• 8 has 2, so 100 has ?
100 = (100/8)·8 has (100/8)·2 = 25
• 100 has 25, so 8 has ?
8 = (8/100)·100 has (8/100)·25 = 2
• 100 has 25,so ? has 2
2 = (2/25)·25 had by (2/25)·100 = 8

Class 6-7

Solution-**SETS**: open statements (equations)

2 + 3·x = 8	3·x = 6
(2+3·x)-2 = 8-2	(3·x)/3 = 6/3
(3·x + 2)-2 = 6	(x·3)/3 = 2
3·x + (2-2) = 6	x·(3/3) = 2
3·x + 0 = 6	x·1 = 2

L = {x ∈ R | 2+3·x = 8} = {2}
PROBLEM:
The weight-metaphor hides the count process, and creates many error possibilities as e.g.
If 2 + 3·x = 8, then 5·x = 8

Castle & Monastery: Coding 2 + (3·5) = 17 → 2 + (3·x) = T
DeCoding (solving an equation):
ReStacking 8 in two stacks:
2 + (3·x) = 8 = (8-2) + 2
3·x = 8-2 = 6
ReBundling from 1s to 3s:
3·x = 6 = (6/3)·3
x = 6/3 = 2
Forward- & back calculations:
To opposite side with opp. sign

<u>Forward</u> <u>Back</u>
2 + 3·x = 8
+ 2 ↑↓ *-2*
3·x = 8-2 = 6
·3 ↑↓ */3*
x = 6/3 = 2

Percentage part 2
• 25% of 8 is ?
0.25·8 = x
• 25% of ? is 2
0.25·x = 2, så x = 2/0.25 = 8
• ? % of 8 is 2
x·8 = 2, so x = 2/8 = 0.25 = 25%

Class 8-9

SETS are connected: functions
Function: an example of a many-one set-relation
E.g. f (x) = 2 + 3·x
A function's value and graph
PROBLEM:
The function came after calculus!
A syntax error to confuse the language and meta-language: the function's value corresponds to the verb's tie.

City: Trade and Tax
Per-numbers: Tax, custom, exchange and interest rates, profit, loss, bonds, assurance.
Adding per-numbers:
3 kg at 4\$/kg + 5 kg at 6\$/kg gives 8 kg at ? \$/kg
Geometry: area and volume of plane and spatial forms. Right-angled triangles: Pythagoras, sine, cosine & tangent.
Linear funct.: growth by adding:
T = b + a + a + a + … = b + a·n
A function is a name for a calculation with variable numbers, such as. T = 2 + 3 x.
(Euler 1748)
Calculations give fixed and functions give variable number.
The change of a function can be shown in tables or on curves.
The Inn: Redistribution by games
Winn on pools, lotto, roulette.
Statistics counts number of wins.
Risk = Consequence·propability.

Class 10	Class 11	Class 12
Set theory Function theory: Domains & values. Algebraic functions: Polynomials and polynomial fractions. First- & second-degree polynomials. Trigonometry. Analytical geometry.	Function theory: reverse and composite function. Non-algebraic functions: trigonometric functions. Logarithm- & exponential functions as homomorphisms: $f(x*y) = f(x) \# f(y)$ Stochastic functions. Core calculus.	Vector spaces. Main calculus. Simple differential equations.
The Renaissance: Constant per-numbers Numbers as many-bundles (polynomials): $T = 2345 = 2\cdot B^3 + 3\cdot B^2 + 4\cdot B + 5\cdot 1$ Reversed calculations with powers: $B^4 = 81$ $4^n = 1024$ $B = \sqrt[4]{81}$ $n = \log 1024/\log 4$ Interest rates: Single r, total R, compound RR $(1 + r)^n – 1 = R = n\cdot r + RR$ Change with constant per-number and percentage: x: + 1 → T: + a$ linear change $T = b + a\cdot x$ x: + 1 → T: + r% exponential $T = b\cdot(1+r)^x$ x: + 1% → T: + r% power change $T = b\cdot x^r$ x: + 1 → T: + r% + a$ savings $T = a\cdot R/r$ Change with unpredictable (random) variation $\Delta T =?$ $T = MID \pm 2\cdot SPR$ Adding percentages by their areas (integration): 300$ at 4% and 500$ at 6% is 800$ at ? %. Change percentage: $T = a\cdot b$: $\Delta T/T \approx \Delta a/a + \Delta b/b$ $T = a/b$: $\Delta T/T \approx \Delta a/a – \Delta b/b$ Trigonometry: SIN & COS: short sides in percent of the long. TAN: the one short side in percent of the other.	**Industry: Variable per-numbers** Coordinate geometry: Geometry & algebra, always together, never apart. Curve fitting with polynomials: $T = A + B\cdot x + C\cdot x^2 + D\cdot x^3$ (or $y = A + B\cdot x + C\cdot x^2$) A: level, B: rise, C: curvature, D: counter-curvature Variable, predictable change: Differential calculus: $dT = (dT/dx)\cdot dx = T'\cdot dx$ The non-linear is locally linear: $(1+r)^n \approx 1 + n\cdot r$ ($= 1 + n\cdot r + RR$: with a small interest, the compound-interest can be neglected) $T = x^n$: $dT/T = n\cdot dx/x$, $dT/dx = n\cdot T/x = n\cdot x^{(n-1)}$ Optimization tasks in engineering and economics. Integral calculus: $\Delta T = T2 – T1 = \int dT = \int f\cdot dx$, Total change = terminal – start = the sum of single changes, regardless of their number or size. Integration is done by rewriting to change form: Since $6\cdot x^2 + 8\cdot x = d/dx\,(2\cdot x^3 + 4\cdot x^2) = d/dx(T)$ then $\int(6\cdot x^2 + 8\cdot x)\,dx = \int d(2\cdot x^3 + 4\cdot x^2)$ $= \int dT = \Delta T = T2 – T1$ Accumulation tasks in engineering and economics.	**Major works in the Quantitative Literature:** Geometry, Trade, Economics, Physics, Biology. The three genres for quantitative literature: - *Fact or since-then calculations* quantifies the quantifiable, and calculates the calculable: since the price is 4$/kg, then the cost of 6 kg is 6·4$ = 24$. - *Fiction or if-then calculations* quantifies the quantifiable, and calculate the incalculable: if my income is 4m$/year, then 6 years of income will be 6·4m$ = 24 million $. - *Fiddle or so-what calculations* quantify the non-quantifiable: If the consequence ‘broken leg’ C is taken to be 2 million $, and if the probability p is taken to 30%, then the risk R will be R = C·p = 2m$·0.3 = 0.6 million $. The three courses of action: fact models are controlled especially for the units; fiction models are supplemented with alternative scenarios; fiddle models are referred to a qualitative treatment. Change equations solved by numerical integration. Functions of two variables. Differentiation and integration. Optimization and accumulation. Vectors used in trade and in the movement on a surface and in space.

16. Twelve Proposals for 1day Skype Seminars

01) The Root of Mathematics, Many, dealt with by Block-Numbers, Bundle-Counting & Preschool Calculus

”How old next time?” I asked the child. The answer was four with four fingers shown. But held together two by two created a protest: “That is not four, that is two twos!”. That opened my eyes. Children come to school with two-dimensional block-numbers where all numbers have units. Instead, school teaches cardinality as a one-dimensional line with different number-names; thus disregarding the fact that numbers are two- dimensional blocks all having a unit as shown when writing out fully a total T = 345 = 3 BundleBundles + 4 Bundles + 5 Singles = 3*10^2 + 4*10 + 5*1. So, a number is blocks united (integrated) next-to each other, showing the four ways to unite numbers presented by Algebra, meaning reuniting in Arabic: Power and multiplication and ‘on-top’ and ‘next-to’ addition (integration).

Consequently, mathematics education should develop the two-dimensional block-numbers that children bring to school and allow them to practice counting before adding.

To master Many, we ask ‘how many?’ To answer, we bundle-count using a cup for the bundles. So, a number always has some bundles inside and some unbundled outside the cup.

Recounting in the same unit creates overloads or underloads by moving in or out of the cup, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. This makes calculation easy: T = 4 x 56 = 4 x 5B6 = 20B24 = 22B4 = 224.

Once counted and recounted, totals may de added. To have 2 3s and 4 5s added on-top as 5s, a unit must be changed, called proportionality. To add them next-to as 8s means adding their areas, called integration; which becomes differentiation when reversed by saying 2 3s + ? 5s = 3 8s’, thus allowing calculus to take place in preschool.

02) 12 Luther-like Theses about how ManyMath can Improve Math Education

1. Digits are icons with as many sticks as they represent.

2. A total T can be ‘bundle-counted’ in the normal way or with an overload or underload: T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s.

3. ‘Bundle-writing’ makes operations easy: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

4. Counting T by bundling, T = (T/B)xB = (5/2)x2 = 2.1 2s, shows a natural number as a decimal number with a unit.

5. Operations are icons showing counting by bundling and stacking. -2 takes away 2. /2 takes away 2s. x2 stacks 2s. +2 adds 2 on-top or next-to.

6. A calculator predicts. Asking T = 4 5s = ? 6s, first (4x5)/6 = 3.some; then (4x5) – (3x6) = 2. So T = 4 5s = 3.2 6s

7. Recounting in tens, calculators leave out the unit and misplace the decimal point: T = 3 7s = 3*7 = 21 = 2.1 tens.

8. Recounting from tens, ‘? 7s = 3 tens’, or ‘u*7 = 30 = (30/7)x7’, the answer u = 30/7 is found by ‘move to opposite side with opposite sign’.

9. Adding totals is ambiguous: OnTop using proportionality, or NextTo using integration?

10. Operations are reversed with reverse operations: With u+3 = 8, u = 8–3; with ux3 = 8, u = 8/3; with u^3 = 8, u = 3√8; with 3^u = 8, u = log3(8); with T1 + u*3 = T2, u = ΔT/3.

11. Double-counting in different units gives ‘per-numbers’ as 4$/5kg, bridging the two units by recounting: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$

12. Double-counting in the same unit, per-numbers become fractions as operators, needing a

number to become a number, thus adding by their areas as integration.

03) Curing Math Dislike with one Cup and five Sticks

A class is stuck in division and gives up on 234/5. Having heard about '1cup & 5 sticks', the teacher says 'Time out. Next week, no division. Instead we do bundle-counting'. Teacher: 'How many sticks?' Class: '5.' Teacher: 'Correct, 5 1s, how many 2s?' Class: '2 2s and 1 left over'. Teacher: 'Correct, we count by bundling. The cup is for bundles, so we put 2 inside the cup and leave 1 outside. With 1 inside, how many outside? And with 3 inside, how many outside?' Class: '1 inside and 3 outside; and 3 inside and 1 lacking outside.' Teacher: 'Correct. A total of 5 sticks can be counted in 3 ways. The normal way with 2 inside and 1 outside. With overload as 1 inside and 3 outside. With underload as 3 inside and less 1 outside.' Class: 'OK'. Teacher. 'Now 37 means 3 inside and 7 unbundled 1s outside. Try recounting 37 with overload and underload. Class: '2 inside and 17 outside; and 4 inside and less 3 outside.'

Teacher: 'Now let us multiply 37 by 2, how much inside and outside?' Class: 6 inside and 14 outside. Or 7 inside and 4 outside. Or 8 inside and less 6 outside.'

Teacher: 'Now to divide 74 by 3 we recount 7 inside and 4 outside to 6 inside and 14 outside. Dividing by 3 we get 2 inside and 4 outside; plus 2 leftovers that still must be divided by 3. So 74/3 gives 24 and 2/3.'

Class: 'So to divide 234 by 5 we recount 234 as 20 inside and 34 outside. Dividing by 5 we get 4 inside and 6 outside; plus and 4 leftovers that still must be divided by 5. Thus 234/5 gives 46 and 4/5?'

Teacher: 'Precisely. Now try multiplication using bundle-counting'.

04) DoubleCounting rooting Proportionality - and Fractions and Percentages as PerNumbers

A class is stuck in fractions and percentages and gives up on ¾ = 75%. Having heard about 'per-numbers', the teacher says: Time out. Next week, no fractions, no percentage. Instead we do double-counting. First counting: 42 is how many 7s? The total T = 42 = (42/7)*7 = 6*7 = 6 7s. Then double-counting: Apples double-counted as 3 $ and 4 kg have the per-number 3$ per 4 kg, or 3$/4kg or ¾ $/kg. Asking how many $ for 10kg, we recount 10 in 4s, that many times we have 3$: The total T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Asking how many kg for 18$, we recount 18 in 5s, that many times we have 4kg: The total T = 18$ = (18/5)*5$ = (18/5)*4kg = 14.4kg. Double-counting in the same unit gives fractions and percentages as 3 per 4, ¾; and 75 per hundred, 75/100 = 75%.

¾ of 200$ means finding 3$ per 4$, so we recount 200 in 4s, that many times we have 3$: The total T = 200$= (200/4)*4$ gives (200/4)*3$ = 150$. 60% of 250$ means finding 60$ per 100$, so we recount 250 in 100s, that many times we have 60$: The total T = 250$= (250/100)*100$ gives (250/100)*60$ = 150$.

To find 120$ in percent of 250$, we introduce a currency # with the per-number 100# per 250$, and then recount 120 in 250s, that many times we have 100#: The total T = 120$ = (120/250)*250$ = (120/250)*100# = 48#. So 120$/250$ = 48#/100# = 48%. To find the end-result of 300$ increasing with 12%, the currency # has the per-number 100# per 300$. 12# increases 100# to 112# that transforms to $ by the per-number. The total T = 112# = (112/100)*100# = (112/100)*300$ = 336$.

05) Algebraic Fractions made easy by Block-Numbers with Units

A class is stuck in algebraic fractions insisting that (2b+4)/2b is 4. Having heard about 'Block-Numbers with units, the teacher says: 'Time out. Next week, no algebraic fractions. Instead we count totals with units.' Teacher, showing six sticks: 'How many sticks?' Class: '6.' Teacher: 'Correct, 6 1s, how many 2s?' Class: '3 2s'. Teacher: 'Correct, we count in 2s by taking away 2s, that is by dividing by 2, so T = 6 = (6/2) 2s = 3 2s = 3*2. So, factorizing 2b as 2*b, 2b is 2 bs or b

2s. Can 4 be written with a unit?' Class: '4 is 2 2s'. Teacher: 'Correct, so 2b and 4 can be written as b 2s and 2 2s totalling b+2 2s or (b+2)*2.' Class: 'OK'. Teacher: 'Now, 6 2s divided by 3 2s gives 6 divided by 3 or 2. And c 2s divided by 3 2s gives c divided by 3.' Class: 'OK'. Teacher: 'So, b+2 2s divided by b 2s gives b+2 divided by b.' Class: 'OK, and that gives 2?' Teacher: 'Well, division means removing a common unit. So, with b as b 1s and 2 as 2 1s we can remove the 1s. But b+2 1s divided by b 1s still gives b+2 divided by b, which is the result.' Class: 'OK'. Teacher: 'Now try (3c+9)/6c.' Class: 'We factorize to find a common unit 3: 3c is c 3s, 9 is 3 3s, and 6c is 2x3xc or 2c 3s. Removing the common unit we get (3c+9)/6c = (c+3)/2c.' Teacher: 'Correct. Now try (b^2c+ bd^3)/bc'. Class: 'We factorize to find a common unit b: b^2c is bxbxc or bc bs , bd^3 is d^3 bs, and bc is c bs. Removing the common unit, we get (b^2c+ bd^3)/bc = (bc + d^3)/c.'

06) Algebra and Geometry, always Together, never Apart

The ancient Greeks used mathematics as a common label for their four knowledge areas, arithmetic, geometry, music and astronomy, seen as many by itself, many in space, many in time and many in space and time. With music and astronomy gone, mathematics was a common label for algebra and geometry until the arrival of the 'New Math' that insisted that geometry must go and that algebra should be defined from above as examples of sets instead of from below as abstractions from examples. Looking at the set of sets not belonging to themselves, Russell showed that set-reference means self-reference as in the classical liar paradox 'this sentence is false' being true if false and vice versa. Still, the new set-based 'meta-matics' entered universities and schools as the only true mathematics; except for the US going 'back to basics', that by separating algebra and geometry crates learning problems that disappear if they are kept together as advocated by Descartes. Thus, in primary school, numbers should be two dimensional LEGO-blocks as 2 3s. And 3x6 should be a block of 3 6s that if recounted in tens must widen its width and shorten its height, so that 3 6s becomes 1.8 tens. And in secondary school bxc should mean b cs; and fractions should be operators needing a number to become a number thus by multiplication becoming areas that are added by integration. Likewise, Euclidean geometry should be introduced in a coordinate system allowing equations to predict the exact position of intersection points of lines in triangles before being constructed with ruler and compasses. And the quadratic equation x^2+bx+c=0 geometrically tells that since x^2+bx = -c, the four parts of a (x+b/2) square reduce to (b/2)^2-c = D, allowing x to be found easily as x = -b/2 ±√D.

07) Calculus in Middle School and High School

A class is stuck in differential calculus and gives up on d/dx(x^2) = 2x. Having heard about 'per-numbers', the teacher says 'Time out. Next week, no differentiation. Instead we go back to middle school and look at per-numbers.' Class: 'Per-numbers, what is that?' Teacher: 'Per-numbers are for example meter per second, dollar per kilo, or dollar per hour. Here is an example: What is the total of 2 kg at 3 $/kg + 4 kg at 5$/kg?'. Class: 'The kg-numbers add to 6, but how do we add per-numbers?' Teacher: 'Can we change $/kg-numbers to $-numbers?'. Class: 'We can multiply 2 and 3 to 6$, and 4 and 5 to 20$ that add up to 26$. But multiplication means adding areas?' Teacher: 'Precisely. Adding per-numbers by their areas is called integral calculus, also called finding the area under the per-number-graph.'

Class: 'But what if the per-number graph is not constant? Then there are too many strips to add!' Teacher: 'We use a trick. Adding 1000 numbers is difficult, but adding 1000 differences is easy since the middle numbers cancel out, so we are left with the difference between the end and the start number.' Class: 'But how can we write area-strips as differences?' Teacher: 'Well, if p is the per-number, then the area-strip with width dx is close to p*dx; but it is also the difference between the end area A2 and the start area A1, so p*dx = A2-A1 = dA, or p = d/dx(A).' Class: 'But that is differentiation?' Teacher: 'Precisely, so if we know that d/dx(x^2) = 2x, then we know that the area under the 2x graph is A2-A1 with A = x^2. So to find a quick way to area-formulas we need to learn to differentiate.' Class: 'OK.'

08) Mathematics, the Grammar of the Number-Language. But why teach Grammar before Language?

Humans have two languages, a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’ and ‘3 chairs have a total of 3x4 legs’, abbreviated to ‘T = 3x4’. Both languages have a meta-language, a grammar, that describes the language that describes the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3x4’ leads to a meta-sentence ‘’x’ is an operation’. We master outside phenomena through actions, so learning a word-language means learning actions as how to listen, to read, to write and to speak. Likewise, learning the number-language means learning actions as how to count and to add. We cannot learn how to math, since math is not an action word, it is a label, as is grammar. Thus, mathematics can be seen as the grammar of the number-language. Since grammar speaks about language, language should be taught and learned before grammar. This is the case with the word-language, but not with the number-language. Saying ‘the number-language is an application of mathematics’ implies that then ‘of course mathematics must be taught and learned before it can be applied’. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language. Instead school should follow the word-language and use full sentences ‘The total is 3 4s’ or ‘T = 3x4’. By saying ‘3x4’ only, school removes both the subject and the verb from number-language sentence, thus depriving it of its language nature.

09) Quantitative Literature also has three Genres: Fact and Fiction and Fiddle

Humans communicate in languages: A word language with sentences assigning words to things and actions. And a number language with equations assigning numbers or calculations to things and actions. ‘Word stories’ come in three genres: Fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like ‘This sentence is false’. ‘Number stories’ are often called mathematical models. They come in the same three genres. Fact models can be called a ‘since-then’ models or ‘room’ models. Fact models quantify quantities and predict predictable quantities: “What is the area of the walls in this room?”. Since the model’s prediction is what is observed, fact models can be trusted. Algebra’s four basic uniting models are fact models: T = a+b, T = axb, T = a^b and T = ∫y dx; as are many models from basic science and economy. Fiction models can be called ‘if-then’ models or ‘rate’ models. Fiction models quantify quantities but predict unpredictable quantities: “My debt is gone in 5 years at this rate!”. Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based upon alternative assumptions. Models from statistics calculating averages assuming variables to be constant are fiction models; as are models from economic theory showing nice demand and supply curves. Fiddle models can be called ‘then-what’ models or ‘risk’ models. Fiddle models quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence x Probability. It has meaning in insurance but not when quantifying casualties where it is cheaper to stay in a cemetery than at a hospital.

10) Distance Teacher Education in Mathematics by the CATS method: Count & Add in Time & Space

The MATHeCADEMY.net teaches teachers teach mathematics as ‘many-math’, a natural science about Many. It is a virus academy saying: To learn mathematics, don’t ask the instructor, ask Many. To deal with Many, we Count and Add in Time and Space. The material is question-based.

Primary School. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives

14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

Secondary School. COUNT: How can we count possibilities? How can we predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? QUANTITATIVE LITERATURE, what is that? Does it also have the 3 different genres: fact, fiction and fiddle?

PYRAMIDeDUCATION organizes 8 teachers in 2 teams of 4 choosing 3 pairs and 2 instructors by turn.

The instructors instruct the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation.

The instructors correct the count&add assignments. In a pair, each teacher corrects the other teacher’s routine-assignment. Each pair is the opponent on the essay of another pair.

11) 50 years of Sterile Mathematics Education Research, Why?

PISA scores are still low after 50 years of research. But how can mathematics education research be successful when its three words are not that well defined? Mathematics has meant different things in its 5000 years of history, spanning from a natural science about Many to a self-referring logic.

Within education, two different forms exist at the secondary and tertiary level. In Europe, education serves the nation’s need for public servants through multi-year compulsory classes and lines. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks with one-subject teachers.

As to research, academic articles can be at a master level exemplifying existing theories, or at a research level questioning them. Also, conflicting theories create problems as within education where Piaget and Vygotsky contradict each other by saying ‘teach as little and as much as possible’.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and if research is following traditions or searching for new discoveries. So to answer the question ‘How to improve mathematics education research’, first we must make the three words well defined by asking: What is meant by mathematics, and by education, and by research? Answers will be provided by the German philosopher Heidegger, asking ‘what is ‘is’?’

It turns out that, instead of mathematics, schools teaches ‘meta-matism’ combining ‘meta-matics’, defining concepts from above as examples of abstractions instead of from below as abstractions from examples; and ‘mathe-matism’ true inside but seldom outside class, such as adding fractions without units, where 1 red of 2 apples plus 2 red of 3 gives 3 red of 5 and not 7 red of 6 as in the textbook teaching $1/2 + 2/3 = 7/6$.

So, instead of meta-matism, teach ‘many-math’ in self-chosen half-year blocks.

12) Difference-Research, a more Successful Research Paradigm?

Despite 50 years of research, many PISA studies show a continuing decline. Maybe, it is time for difference-research searching for hidden differences that make a difference:

1. The tradition teaches cardinality as one-dimensional line-numbers to be added without being counted first. A difference is to teach counting before adding to allow proportionality and integral calculus and solving equations in early childhood: bundle-counting in icon-bundles less than ten,

recounting in the same and in a different unit, recounting to and from tens, calculator prediction, and finally, forward and reversed on-top and next-to addition.

2. The tradition teaches the counting sequence as natural numbers. A difference is natural numbers with a unit and a decimal point or cup to separate inside bundles from outside singles; allowing a total to be written in three forms: normal, overload and underload: T = 5 = 2.1 2s = 2B1 2s = 1B3 2s = 3B-1 2s.

3. The tradition uses carrying. A difference is to use bundle-writing and recounting in the same unit to remove overloads: T = 7x 48 = 7x 4B8 = 28B56 = 33B6 = 336. Likewise with division: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

4. Traditionally, multiplication is learned by heart. A difference is to combine algebra and geometry by seeing 5x6 as a stack of 5 6s that recounted in tens increases its width and decreases its height to keep the total unchanged.

5. The tradition teaches proportionality abstractly. A difference is to introduce double-counting creating per-number 3$ per 4kg bridging the units by recounting the known number: T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Double-counting in the same unit transforms per-numbers to fractions and percentages as 3$ per 4$ = ¾; and 75kg per 100kg = 75/100 = 75%.

17. Difference-Research Powering PISA Performance: Count & Multiply before You Add

To explain 50 years of low performing mathematics education research, this paper asks: Can mathematics and education and research be different? Difference-research searching traditions for hidden differences provides an answer: Traditional mathematics, defining concepts from above as examples of abstractions, can be different by instead defining concepts from below as abstractions from examples. Also, traditional line-organized office-directed education can be different by uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks. And traditional research extending its volume of references can be different, either as grounded theory abstracting categories from observations, or as difference-research uncovering hidden differences to see if they make a difference. One such difference is: To improve PISA performance, Count and Multiply before you Add.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA performance decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

Other countries also experience low and declining PISA performance. And apparently research can do nothing about it. Which raises the question: Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? So, it is time to seek guidance by difference-research.

Difference-research Searching for Hidden Differences

Difference-research asks two questions: 'Can this be different – and will the difference make a difference?' If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

Natural sciences use difference-research to keep on searching until finding what cannot be different. Describing matter in space and time by weight, length and time intervals, they all seem to vary. However, including per-numbers will uncover physical constants as the speed of light, the gravitational constant, etc. The formulas of physics are supposed to predict nature's behavior. They cannot be proved as can mathematical formulas, instead they are tested as to falsifiability: Does nature behave different from predicted by the formula? If not, the formula stays valid until falsified.

Social sciences can also use difference-research; and since mathematics education is a social institution, social theory might be able to explain 50 years of unsuccessful research in mathematics education.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman (1990, p. 16) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.' Also, he talks about rationality as the base for social organizations:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called *goal displacement*. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

As an institution, mathematics education is a public organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

Thus, SET has transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So looking back, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about forms and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers, operations and calculations. However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. And, being sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens, might speed up writing but might also slow down learning, together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, in Lincolns Gettysburg address, ‘Four scores and ten years ago’ shows that not all count in tens.

So, despite being presented as universal, many things can be different in mathematics, apparently having a tradition to present its choices as nature that cannot be different. And to uncover choice presented as nature is the aim of difference research.

A philosophical Background for Difference Research

Difference research began with the Greek controversy between two attitudes towards knowledge, called ‘sophy’ in Greek. To avoid hidden patronization, the sophists warned: Know the difference between nature and choice to uncover choice presented as nature. To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy. The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord using an unpredictable will to choose how the world behaves.

However, in the Renaissance difference research returns with Brahe, Kepler and Newton. Observations showed Brahe that planetary orbits are predictable in a way that did not falsify the church’s claim that the earth is the center of the universe. Kepler pointed to a different theory with the sun in the center. To falsify the Kepler theory a new planet had to be launched, which was impossible until Newton showed that planets and apples obey the same will, and a falling apple validates Kepler’s theory.

As experts in sailing, the Viking descendants in England had no problem stealing Spanish silver on its way home across the Atlantic Ocean. But to get to India to exchange it for pepper and silk, the Portuguese fortification of Africa’s cost forced them to take the open sea and navigate by the moon. But how does the moon move? The Church had one opinion, Newton had a different.

‘We believe, as is obvious for all, that the moon moves among the stars,’ said the Church; opposed by Newton saying: ‘No, I can prove that the moon falls to the earth as does the apple.’ ‘We believe that when moving, things follow the unpredictable metaphysical will of the Lord above whose will is done, on earth as it is in heaven,’ said the Church; opposed by Newton saying: ‘No, I can prove they follow their own physical will, a force that is predictable because it follows a mathematical formula.’ ‘We believe, as Aristotle told us, that a force upholds a state,’ said the Church; opposed by Newton saying: ‘No, I can prove that a force changes a state. Multiplied with the time applied, the force’s impulse changes the motion’s momentum; and multiplied with the distance applied, the force’s work changes the motion’s energy.’ ‘We believe, as the Arabs have shown us, that to deal with formulas we use algebra,’ said the Church; opposed by Newton saying: ‘No, we need a different algebra of change which I will call calculus.’

By discovering a physical predictable will Newton inspired a sophist revival in the Enlightenment Century: With moons and apples obeying their own physical will instead of that of a metaphysical patronizer, once enlightened about the difference between nature and choice, humans can do the same and do without a double patronization by the Lord at the manor house and the Lord above. Thus, two Enlightenment republics were installed, one in North America in1776 and one in France in 1789.

The US still has its first republic showing skepticism towards philosophical claims by developing American pragmatism, symbolic interactionism and grounded theory; and by allowing its citizens to uncover and develop talents through daily lesson in self-chosen half-year blocks in secondary and tertiary education.

France now has its fifth republic turned over repeatedly by their German neighbors seeing autocracy as superior to democracy and supporting Hegel’s anti-enlightenment thinking reinventing a metaphysical Spirit expressing itself through the history of different national people. To protect the republic, France established line-organized and office-directed elite schools, copied by the Prussia wanting to prevent democracy by Bildung schools meeting there criteria: The population must not be enlightened to prevent it asking for democracy as in France; instead a feeling of nationalism should be installed transforming the population into a people following the will of the Spirit by fighting other people especially the French; and finally the population elite should be extracted and receive Bildung to become a knowledge nobility for a new strong central administration to replace the inefficient blood nobility unable to stop democracy from spreading from France.

To warn against hidden patronization in institutions, France developed a post-structuralist thinking inspired by existentialist thinking (Tarp, 2016), especially as expressed in what Bauman (1992, p. ix). calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'?

> Inquiry is a cognizant seeking for an entity both with regard to the fact <u>that</u> it is and with regard to its Being <u>as</u> it is. (Heidegger, 1962, p. 5)

Heidegger here describes two uses of 'is'. One claims existence, 'M is', one claims 'how M is' to others, since what exists is perceived by humans wording it by naming it and by characterizing or analogizing it to create 'M is N'-statements.

Thus, there are four different uses of the word 'is'. 'Is' can claim a mere existence of M, 'M is'; and 'is' can assign predicates to M, 'M is N', but this can be done in three different ways. 'Is' can point down as a 'naming-is' ('M is for example N or P or Q or …') defining M as a common name for its volume of more concrete examples. 'Is' can point up as a 'judging-is' ('M is an example of N') defining M as member of a more abstract category N. Finally, 'is' can point over as an 'analogizing-is' ('M is like N') portraying M by a metaphor carrying over known characteristics from another N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

Inspired by Heidegger, the French poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu points out that society forces words upon you to diagnose you so it can offer curing institutions including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Derrida, 1991. Lyotard, 1984. Bourdieu, 1970. Tarp, 2012).

From a Heidegger view a sentence contains two things: a subject that exists, and the rest that might be gossip. So, to discover its true nature hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its 'predicate-prison'. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a different mathematics curriculum, e.g. one based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help rebuilding their original countries.

The philosophical and sociological background for difference research may be summed up by the Heidegger warning: In sentences, trust the subject but question the rest since it might be gossip. So, to restore its authenticity, we now return to the original subject in Greek mathematics, the physical fact Many, and use Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition

(Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep the balance and to store sounds assigned to what we grasped with our forelegs, now freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into two languages, a word-language and a number-language. The 'pencil-paradox' observes that placed between a ruler and a dictionary, a pencil can itself point to its length but not to its name. This shows the difference between the two languages, the word-language is for opinions, the number-language is for prediction.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair'. Observing the existence of many chairs, we ask 'how many in total?' and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, 'the total is 3 chairs' or, if counting legs, 'the total is 3 fours', abbreviated to 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is an operation'. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

Thus, we can ask: What happens if looking at mathematics differently as a number-language? Again, difference-research might provide an answer.

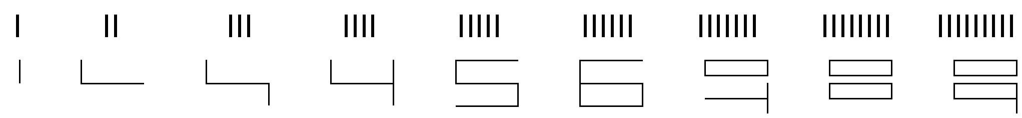
Examples of Difference-research

To prevent that mathematics becomes a meta-language that can be applied to describe and solve real-world problems, we must be careful with our language. Although it seems natural to talk about mathematics and its applications, this includes the logic that 'of course mathematics must be learned before it can be applied'. Which is equivalent to saying 'of course a grammar must be learned before it can be applied to describe a language'. This would lead to widespread illiteracy if applied to the word-language. And 'grammar before language' might be the cause of several problems in mathematics education. Of course, the subject must exist before the sentences can be made about it. So differences typically come from respecting that the number-language comes before its grammar and after meeting and experiencing the subject of its sentences, the total, describing the physical fact Many.

Digits as icons

A class of beginners, e.g. preschool or year 1 or migrants, is stuck in the traditional introduction of digits as symbols like letters. Some confuse the symbols, some have difficulties writing them, some can't see why ten is written 10, some ask why eleven and twelve is not called ten-1 and ten-2.

Here a difference is to use a folding ruler to discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the four-icon, and five sticks in the five-icon, etc. Counting in 5s, the counting sequence is 1, 2, 3, 4, Bundle, 1-bundle-1, etc. This shows, that the bundle-number does not need an icon. Likewise, when bundling in tens. Instead of ten-1 and ten-2 we use the Viking numbers eleven and twelve meaning '1 left' and '2 left' in Danish, understood that the ten-bundle has already been counted.



1 2 3 4 5 6 7 8 9

Will this difference make a difference? In theory, yes, since rearranging physical entities into icons, e.g. five cars into in a five-icon, makes the icons physically before being formally written down. In his genetic epistemology, Piaget expresses a ‘greifen-vor begreifen’ principle, grasping physically before mentally. Thus, going from unordered cars to cars ordered into an icon to writing down the icon includes three of the four parts of his stage theory, the preoperational and the concrete operational and the formal operational stage. In practice, it works on a pilot study level thus being ready for a more formal study.

Counting sequences in different forms

A class of beginners have problems with the traditional introduction of the counting sequence and the place value system. Some count ‘twenty-nine, twenty-ten, twenty-eleven’. Some mix up 23 and 32.

Here a difference is to count a total of a dozen sticks in fives using different counting sequences: ‘1, 2, 3, 4, bundle, 1-bundle-1, …, 2 bundles, 2-bundles-1, 2-bundles-2’. Or ’01, 02, 03, 04, 10, 11, …, 22’. Or ‘.1, .2, .3, .4, 1., 1.1, …, 2.2’. Or ‘1, 2, bundle less 2, bundle less 1, bundle, bundle&1, bundle&2, 2bundle less 2, 2bundle less 1, 2bundles, 2bundles&1, 2bundles&2.’

Using a cup for the bundles, a total can be ‘bundle-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘bundle-writing’ to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, shows that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s; and t = 3B1 tens = 3.1 tens = 31 if leaving out the unit and misplacing the decimal point

Will this difference make a difference? In theory, yes, since counting by taking away bundles and placing one stick in a cup per bundle again combines the three operational parts of Piaget’s stage-theory allowing the learner to see, that a number has three parts: a unit, and some bundles inside the cup, and some unbundled outside. In practice, it works on a pilot study level thus being ready for a more formal study.

Multiplication tables made simpler

A class is stuck in multiplication tables. Some add instead of multiplying, some tries to find the answer by repeated addition, some just give random answers, and some have given up entirely to learn the tables by heart.

Here a difference is to see multiplication as a geometrical stack or block that recounted in tens increases its width and therefore decreases its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21.

Another difference is to begin by reducing the full ten-by-ten table to a small 2-by-2 table containing doubling and tripling, using that 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc.

Thus, beginning with doubling visualized by LEGO bricks, T = 2 6s = 2*6 = 2*(5&1) = 10&2 = 12, or T = 2*6 = 2*(10-4) = 20-8 = 12. And T = 2 7s = 2*7 = 2*(5&2) = 10&4 = 14, or T = 2*7 = 2*(10-3) = 20 – 6 = 14. Doubling then can be followed by halving that by counting in 2s will introduce a recount-formula T = (T/B)*B saying that T/B times B may be taken away from T: So when halving 8, 8 = (8/2)*2 = 4 2s, and 9 = (9/2)*2 = (8&1/2)*2 = (4&1/2)*2 = 4 & ½ 2s.

As to tripling, T = 3*7 = 3*(10-3) = 30 – 9 = 21.

Proceeding with factors after 2 and 3, 2-by-2 Medieval multiplication squares can be used to see that e.g. T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Division using bundle-writing and recounting

A class is stuck in short and long division. Some subtract instead of dividing, some invent their own algorithms typically time-consuming and often without giving the correct answers, some give up because they never learned the multiplication tables.

Here a difference is to talk about 8/2 as '8 counted in 2s' instead of as '8 divided between 2'; and to rewrite the number as '10 or 5 times less something' and use the results from a multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

As to long division, here a difference is to combine renaming numbers using bundle names, e.g. sixty-five as 6ten5, with bundle-writing allowing recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

Once bundle-writing is introduced, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Or, with tens: T = 234 = 23B4 = 2BB3B4.

Thus, by recounting in the same unit by creating or removing overloads or underloads, bundle-writing offers an alternative way to perform and write down all operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Proportionality as double-counting creating per-numbers

A class stuck in proportionality. Nearly all find the $-number for 12kg at a price of 2$/3kg but some cannot find the kg-number for 16$. Here a difference is to see the price as a per-number, 2$ per 3kg, bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Will this difference make a difference? In theory, yes, since proportionality is translated to a basic physical activity of counting and recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Fractions and percentages as per-numbers

A class is stuck in fractions. Rewriting fractions by shortening or enlarging, some subtract and add instead of dividing and multiplying; and some add fractions by adding numerators and denominators.

Here a difference is to see a fraction as a per-number coming from double-counting in the same unit, 3/5 = 3$ per 5$, or as percentage 3% = 3/100 = 3$ per 100%. Thus 2/3 of 12 is seen as 2$ per 3$ of 12$ that recounts in 3s as 12$ = (12/3)*3$ giving (12/3)*2$ = 8$ of the 12$. So 2/3 of 12 is 8. Other examples are found in economy investing money and expecting a return that might be higher or lower than the investment, e.g. 7$ per 5$ or 3$ per 5$.

The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6.

To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100$ in 5s and replace 5$ with 3$: T = 100$ = (100/5)*5$ giving (100/5)*3$ = 60$. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%.

As per-numbers, also fractions are operators needing a number to give a number: a half is always a half of something as shown by the recount-formula T = (T/B)*B = T/B Bs. So also fractions must have units to be added.

If the units are different, adding fractions means finding the average fraction. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as the tradition teaches.

Taking fractions of the same quantity makes the unit the same, assumed to be already bracketed out, so that T = a/b + c/d really means T = (a/b + c/d) of (b*d). Thus adding 2/3 and 4/5 it is implied that the fractions are taken of the same total 3*5 = 15 that is bracketed out, so the real question is 'T = 2/3 of 15 + 4/5 of 15 = ? of 15, giving T = 10 + 12 = 22 = (22/15)*15 when recounted in 15s.

Thus, adding fractions is ambiguous. If taken of the same total, 2/3 + 4/5 is 22/15; if not, the answer depends on the totals: 2/3 of 3 + 4/5 of 5 is (2+4)/(3+5) of 8 or 6/8 of 8, and 2/3 of 3 + 4/5 of 10 is 10/13 of 13, thus providing three different answers, 22/15 and 6/8 and 10/13, to the question '2/3+4/5 = ?'

Hiding the ambiguity of adding fractions makes mathematics 'mathe-matism' true inside but seldom outside classrooms.

As to algebraic fractions, a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

As when adding fractions, adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly, but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So adding fractions as the area under a piecewise constant per-number graph becomes 'middle school integration' later to be generalized to high school integration finding the area under a locally constant per-number graph. Thus calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level. n practice, it works on a pilot study level thus being ready for a more formal study.

Will this difference make a difference? In theory, yes, if first performing double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand.

Equations as walking or recounting

A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. when equations are composite or with a reverse sign in front of the unknown.

Here a difference is to use the definitions of reverse operations to establish the basic 'OSS'-rule for solving equations, 'move to the Opposite Side with the opposite Sign'. Thus, in the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise, with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12).

As to 2+3*u = 14, a difference is to see it as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn't have a reverse sign. This gives 3*u = 14 – 2. Since u doesn't have a reverse sign, 3 moves to the opposite side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4 since 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14.

Another difference is to see 2+3*u =14 as a walk, first multiplying u by 3 then adding 2 to give 14. To get back to u we reverse the walk by performing the reverse operations in reverse order. Thus, first subtracting 2 and then dividing by 3 gives u = (14-2)/3 = 4, checked by repeating the walk now with a known staring number: 4*3+2 = 14. Seeing an equation as a walk motivates using the terms 'forward and backward calculation sides' for 2+3*u and 14 respectively.

With 25 – u = 14, u moves to the opposite side to have its reverse sign reversed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5 giving 40 = 5*u; 40/5 = u; 8 = u. Alternatively, recounting twice gives 40 = (40/u)*u = 5*u, and 40 = (40/5)*5, consequently u = 40/5.

Pure letter-formulas build routine as e.g. 'transform the formula T = a/(b-c) so that all letters become subjects.' When building a routine, students often have fun singing:

"Equations are the best we know / they're solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don't want to stop."

Another difference is to introduce equations the foist year in primary school as another name for recounting from tens to icons, e.g. asking 'How many 9s are 45' or 'u*9 = 45' giving u = 45/9 since recounting 45 in 9s, the recount formula gives 45 = (45/9)*9, again showing the OppositeSide&Sign rule.

Likewise, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, again obtained by moving a number to the opposite side with the opposite calculation sign.

Will this difference make a difference? In theory, yes, since equations are related to something concrete, walking or recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Geometry and algebra, always together, never apart

A class is stuck in geometry. Some mix up definitions, some find the theorems to abstract to understand, some find proofs difficult and hard to remember, some find geometry boring.

Here a difference is to use a coordinate system to coordinate geometry and algebra so they go hand in hand always and never apart, thus using algebra to predict geometrical intersection points, and vice versa, to use intersection points to solve algebraic equations. Both in accordance with the Greek meaning of mathematics as a common label for algebra and geometry.

In a coordinate-system a point is reached by a number of horizontally and vertically steps called the point's x- and y-coordinates. Two points A(xo,yo) and B(x,y) with different x- and y-numbers will form a right-angled change-triangle with a horizontal side Δx = x-xo and a vertical side Δy = y-yo and a diagonal distance r from A to B, where by Pythagoras r^2 = Δx^2 + Δy^2. The angle A is found by the formula tanA = Δy/Δx = s, called the slope or gradient for the line from A to B. This gives a formula for a non-vertical line: Δy/Δx = s or Δy = s*Δx, or y-yo = s*(x-xo). Vertical lines have the formula x = xo since all points share the same x-number.

In a coordinate system three points A(x1,y1) and B(x2,y2)and C(x3,y3) not on a line will form a triangle that packs into a rectangle by outside right triangles allowing indirectly to find the angles and the sides and the area of the original triangle.

Different lines exist inside a triangle: Three altitudes measure the height of the triangle depending on which side is chosen as the base; three medians connect an angle with the middle of the opposite side; three angle bisectors bisect the angles; three line bisectors bisect the sides and are turned 90 degrees from the side. Likewise, a triangle has two circles; an outside circle with its center at the intersection point of the line bisectors, and an inside circle with its center at the intersection point of the angle bisectors.

Since Δx and Δy changes place when turning a line 90 degrees, their slopes will be Δy/Δx and -Δx/Δy respectively, so that s1*s2 = -1, called reciprocal with opposite sign.

As mentioned, geometrical intersection points are predicted algebraically by equating formulas. Thus with the lines y = 2*x and y = 6-x, equating formulas gives 2*x = 6-x, or 3*x = 6, or x = 2, which inserted in the first gives y = 2*2 = 4, thus predicting the intersection point to be (x,y) = (2,4). The same answer is found on a solver-app; or using software as GeoGebra.

Finding possible intersection points between a circle and a line or between two circles leads to a quadratic equation x^2 + b*x + c = 0, solved by a solver. Or by a formula created by two x-by-(x+k) playing cards placed on top of each other with the bottom left corner at the same place and the top card turned a quarter round clockwise. This creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution formula is x = -b/2 ±√((b/2)^2 – c).

Finding a tangent to a circle at a point, its slope is the reciprocal with opposite sign of the radius line.

Will this difference make a difference? In theory, yes, since coordinating geometry and algebra gives equations a geometrical form and allows geometrical situations to be predicted by equations. In practice, it works on a pilot study level thus being ready for a more formal study.

Trigonometry as right triangles with sides mutually recounted

A class is stuck in trigonometry. Some find the ratios to abstract to understand, some mix up the formulas, some find the algebra difficult to use.

A difference is to introduce trigonometry as blocks halved in two by its diagonal, making a rectangle split into two right triangles. Here the angles are labeled A and B and C at the right angle. The opposite sides are labeled a and b and c.

The height a and the base b can be counted in meters, or in diagonals c creating a sine-formula and a cosine-formula: a = (a/c)*c = sinA*c, and b = (b/c)*c = cosA*c. Likewise, the height can be recounted in the base, creating a tangent-formula: a = (a/b)*b = tanA*b

As to the angles, with a full turn as 360 degrees, the angle between the horizontal and vertical directions is 90 degrees. Consequently, the angles between the diagonal and the vertical and horizontal direction add up to 90 degrees; and the three angles add up to 180 degrees.

An angle A can be counted by a protractor, or found by a formula. Thus, in a right triangle with base 4 and diagonal 5, the angle A is found from the formula cosA = a/c = 4/5 as cos-1(4/5) = 36.9 degrees.

The three sides have outside squares with areas a^2 and b^2 and c^2. Turning a right triangle so that the diagonal is horizontal, a vertical line from the angle C splits the square c^2 into two rectangles. The rectangle under the angle A has the area (b*cosA)*c = b*(cosA*c) = b*b = b^2. Likewise, the rectangle under the angle B has the area (a*cosB)*c = a*(cosB*c) = a*a = a^2. Consequently c^2 = a^2 + b^2, called the Pythagoras formula.

This allows finding a square-root geometrically, e.g. x = √24, solving the quadratic equations x^2 = 24 = 4*6, if transformed into a rectangle. On a protractor, the diameter 9.5 cm becomes the base AB, so we have 6units per 9.5cm. Recounting 4 in 6s, we get 4units = (4/6)*6units = (4/6)*9.5 cm = 6.33 cm. A vertical line from this point D intersects the protractor’s half-circle in the point C. Now, with a 4x6 rectangle under BD, BC will be the square-root √24, measured to 4.9, which checks: 4.9^2 = 24.0.

A triangle that is not right-angled transforms into a rectangle by outside right triangles, thus allowing its sides and angles and area to be found indirectly. So, as in right triangles, any triangle has the property that the angles add up to 180 degrees and that the area is half of the height times the base.

Inside a circle with radius 1, the two diagonals of a 4sided square together with the horizontal and vertical diameters through the center form angles of 180/4 degrees. Thus the circumference of the square is 2*(4*sin(180/4)), or 2*(8*sin(180/8)) with 8 sides instead. Consequently, the circumference of a circle with radius 1 is 2*π, where π = n*sin(180/n) for n large.

Will this difference make a difference? In theory, yes, since in Greek, geometry means to measure earth, typically by dividing it into triangles, again divided into right triangles, which can be seen as rectangles halved by their diagonals; and recounting totals in new units leads directly to mutual recounting the sides in a right triangle, which leads on to a formula for calculating pi. Furthermore, the many applications of trigonometry might increase the motivation for learning more geometry where coordinate geometry uses right triangles to increase any triangle to a rectangle with horizontal and vertical sides. In practice, it works on a pilot study level thus being ready for a more formal study.

PreCalculus as constant change

A class is stuck in precalculus. Some find the function concept to abstract to understand, some sees f(2) as a variable f multiplied by 2, some cannot make sense of roots and logarithm. The tradition defines a function top-down from above as a set-relation where first-component identity implies second component identity.

A difference is to return to the original Euler-meaning of a function defining it bottom-up from below as a name for a formula containing specified and unspecified numbers. And to see a formula as the core concept of mathematics respecting that, whatever it means, in the end mathematics is but a means to an outside goal, a number-language.

As a number-language sentence, a formula contains both specified and unspecified numbers in the form of letters, e.g. T = 5+3*x. A formula containing one unspecified number is called an equation, e.g. 26 = 5+3*x, to be solved by moving to opposite side with opposite calculation sign, (26-5)/3 = x. A formula containing two unspecified numbers is called a function, e.g. T = 5+3*x. An unspecified function containing an unspecified number x is labelled f(x), T = f(x). Thus f(2) is meaningless since 2 is not an unspecified number. Functions are described by a table or a graph in a coordinate system with y = T = f(x), both showing the y-numbers for different x-numbers. Thus, a change in x, Δx, will imply a change in y, Δy, creating a per-number Δy/Δx called the gradient of the formula.

As to change, a total can change in a predictable or unpredictable way; and predictable change can be constant or non-constant.

Constant change comes in several forms. In linear change, T = b + s*x, s is the constant change in y per change in x, called the slope or the gradient of its graph, a straight line. In exponential change, T = b*(1+r)^x, r is the constant change-percent in y per change in x, called the change rate. In power change, T = b*x^p, p is the constant change-percent in y per change-percent in x, called the elasticity. A saving increases from two sources, a constant $-amount per month, c, and a constant interest rate per month, r. After n months, the saving has reached the level C predicted by the formula C/c = R/r. Here the total interest rate after n months, R, comes from the formula 1+R = (1+r)^n. Splitting the rate r = 100% in t parts, we get the Euler number e = (1+100%/t)^t = (1+1/t)^t if t is large.

Also the change can be constant changing. Thus in T = c + s*x, s might also change constantly as s = c + q*x so that T = b + (c + q*x)*x = b + c*x + q*x^2, called quadratic change, showing graphically as a bending line, a parabola.

The difference seeing functions as predicting number-language sentences also suggests that functions in the form of formulas should be introduced from the first class of mathematics to predict counting results by a calculator, allowing the basic operations to be introduced as icons showing the three tasks involved when counting by bundling and stacking. Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication, iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. To also bundle bundles, power is iconized as a cap, e.g. 5^2, indicating the number of times bundles themselves have been bundled. Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other. To add on-top, the blocks must be recounted in the same unit, thus grounding proportionality. Next-to addition means adding areas, thus grounding integration. Reversed adding on-top or next-to grounds equations and differentiation. Also, the four basic operations uncover the original meaning of the word algebra, meaning ‘to reunite’ in Arabic: Addition unites unlike numbers, multiplication unites like numbers into blocks, power unites like factors, and integration unite unlike blocks.

Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s, using a cup or a decimal point to separate the ‘inside’ bundles from the ‘outside’ unbundled. This prediction holds at a manual counting:

T = 7 = I I I I I I I = III III I = 2 3s & 1.

Thus a calculator can predict a counting result by describing the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, i.e. with two formulas. The ‘recount formula’ T = (T/B)*B says that ‘from T, T/B times B can be taken away’ as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the ‘restack formula’ T = (T–B)+B says that ‘from T, T–B is left when B is taken away and placed next-to’, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. Wanting to recount a total in a new unit, the two formulas can predict the result when bundling and stacking and dragging away the stack. Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Recounting a total in a new unit means changing unit, also called proportionality or linearity, a core concept in mathematics at school and at university level. Thus the recount formula turns up in proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right triangles, and in calculus as dy = (dy/dx)*dx = y’*dx when counting steepness on a curve by recounting a vertical change in a horizontal.

Will this difference make a difference? In theory, yes, since describing mathematics as the grammar of the number-language is a powerful metaphor uncovering the real outside goal of mathematics education, to develop a number-language having the same sentence structure as the word-language, which will demystify the nature of mathematics to many students. In practice, it works on a pilot study level thus being ready for a more formal study.

Calculus as adding locally constant per-numbers

A class is stuck in calculus. Some find the limit concept too abstract. Some find the applications too artificial. For some, their hate to differential calculus prevents them from learning integral calculus.

Here a difference is to postpone differential calculus till after integral calculus is presented as a means to add piecewise or locally constant per-numbers by their areas. Thus, when adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 add by their areas as 3*2 + 5*4, meaning that per-numbers add by the area under the per-number-graph. With a piecewise constant per-number this mean a small number of area strips to add. But seeing a non-constant per-number as locally constant it means adding a huge amount of area strips, only possible if we can rewrite the strips as differences since the disappearance of the middle terms makes many differences add up to one single difference between the terminal and initial number. This of course makes rewriting a formula as a difference highly interesting, thus motivating a study of differential calculus. Thus, with the area strip 2*x*dx written as d(x^2), summing up the strips gives a single difference:

T2-T1 = Δ(x^2) = Σ ΔT = ∫ dT = ∫f(x)*dx = ∫2*x*dx .

Change formula come from observing that in a block, changes Δb and Δh in the base b and the height h impose on the total a change ΔT as the sum of a vertical strip Δb*h and a horizontal strip b*Δh and a corner Δb*Δh that can be neglected for small changes; thus d(b*h) = db*h + b*dh, or counted in T’s: dT/T = db/b + dh/h, or with T’ = dT/dx, T’/T = b’/b + h’/h. Therefore (x^2)’/x^2 = x’/x + x’/x = 2/x, giving (x^2)’ = 2*x since x’ = dx/dx = 1.

As to the limit concept, a difference is to rename it to ‘local constancy’: In a function y = f(x) a small change x often implies a small change in y, thus both remaining ‘almost constant’ or ‘locally constant’, a concept formalized with an ‘epsilon-delta criterium’, distinguishing between three forms of constancy. y is 'globally constant' c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is 'piecewise constant' c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is 'locally constant' c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in the latter case written as dy/dx. Formally, local constancy and linearity is called continuity and differentiability.

Finally, calculus allows presenting the core of the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Will this difference make a difference? In theory, yes, since presenting it as adding piecewise or locally constant per-numbers will ground integral calculus in meaningful real-world problems. Likewise, observing the enormous advantage in adding differences gives a genuine motivation for differential calculus that is lost if insisting that it comes before integral calculus. In practice, it works on a pilot study level thus being ready for a more formal study.

How Different is the Difference?

Difference research uses sociological imagination to revive the ancient sophist warning: Know nature from choice to discover choice presented as nature. Thus, true and false nature are separated by asking the tradition: Can this be different, and will the difference make a difference? Witnessed by 50 years of sterility, mathematics education research is a natural place to see if difference-research, DR, will make a difference.

The tradition says, 'To obtain its goal, to learn mathematics, mathematics education must teach mathematics!' DR objects, 'No, to obtain its goal, mastery of Many, mathematics is a means to be replaced by another means if not leading to the goal, e.g. by 'Many-matics', defining its concepts from below as abstractions from examples instead of from above as examples of abstractions as does the traditional 'meta-matics'.

The tradition says, 'The core of mathematics is to operate on numbers!' DR objects, 'No, the core of mathematics is number-language sentences describing how totals are counted and recounted before being added; and having the same sentence structure as the word-language: a subject, a verb and a predicate.'

The tradition says, 'Digits must be taught as symbols like letters!' DR objects, 'No, digits are icons containing as many strokes as they represent.'

The tradition says, 'To describe cardinality, numbers must be taught as a one-dimensional number-line!' DR objects, 'No, numbers are two-dimensional blocks counting a total in stacks of bundles and unbundled singles.'

The tradition says, 'Natural numbers must be taught as a place value system and ten-bundling is silently understood!' DR objects, 'No, numbers should be taught using bundle-writing to separate inside bundles from outside singles, making a natural number a decimal number with a unit. And ten-counting should be postponed until icon-counting and re-counting in the same and in a different unit has been experienced'.

The tradition says, 'There are four kinds of numbers, natural and integer and rational and real numbers!' DR objects, 'No, a number is a positive or negative decimal number with a unit. Rational numbers are per-numbers, i.e. operators needing a number to become a number; and real numbers are calculations to deliver as many decimals as wanted.'

The tradition says, ‘Operations must be taught as functions from a set-product to the set supplying it with a structure obeying associative, commutative and distributive laws as well as neutral and inverse elements allowing equations to be solved by neutralization!’ DR objects, ‘Operations are icons showing the three processes of counting, bundling and stacking and removing stacks to look for unbundled singles; and adding stacks or blocks on-top or next-to. Solving equations is another word for reversing the processes by re-bundling or re-stacking’

The tradition says, ‘The natural order of teaching operations is addition before subtraction before multiplication before division allowing fractions to be introduced as rational numbers to which the same operations can be applied!’ DR objects, ‘No, since totals must be counted before they can be added, the natural order is the opposite: first division to take away bundles many times, then multiplication to stack the bundles, then subtraction to take away the stack once to look for unbundled singles, and finally addition in its two versions, on-top and next-to. And counting also implies recounting in the same or another unit, to and from tens, and double-counting producing per-numbers as operators needing numbers to become numbers, thus being added by their areas, i.e. by integration.’

The tradition says, ‘Calculators should not be allowed before all four operations are taught and learned!’ DR objects, ‘Calculators should be used from the start to predict counting and recounting results.’

The tradition says, ‘Operations must be taught using carrying!’ DR objects, ‘No, operations should be taught using bundle-writing allowing totals to be recounted with overloads or underloads.’

The tradition says, ‘Multiplication tables must be learned by heart!’ DR objects, ‘No, multiplication tables describe recounting from icon-bundles to ten-bundles; geometrically seen as changing a block by increasing the width and decreasing the height to keep the total unchanged; and algebraically sees as doubling or tripling totals written with an overload or an underload.’

The tradition says, ‘Division is difficult and must be taught using constructivism to allow learners invent their own algorithms!’ DR objects, ‘No, division should be taught as recounting from ten-bundles to icon-bundles using bundle-writing and recounting in the same unit to benefit from the multiplication tables.’

The tradition says, ‘Arithmetic comes before geometry, and they must be held apart until the introduction of the coordinate system!’ DR objects, ‘No, arithmetic should be seen as algebra kept together with geometry all the time and from the beginning, where numbers are a collection of blocks as well as a collection of numbers in cups; where recounting and multiplication means changing block-sizes as well as changing bundle-numbers; and where addition means adding blocks as well as bundle-numbers.’

The tradition says, ‘Proportionality must be postponed until functions have been introduced!’ DR objects, ‘No, as another name for changing units, proportionality occurs from the beginning as recounting in another unit; and is needed when adding on-top and next-to. And reoccurring when double-counting creates per-numbers as bridges between physical units.’

The tradition says, ‘Fractions must be introduced first as parts of something then as numbers by themselves!’ DR objects, ‘No, created by double-counting in the same unit, fractions are per-numbers and as such operators needing a number to become a number.’

The tradition says, ‘Prime-factorizing must precede adding fractions by finding a common denominator!’ DR objects, ‘No, prime-factorizing comes with recounting to another unit to find the units allowing a total to be recounted fully without any unbundled singles. And fractions should be added as operators, i.e. by integrating their areas.’

The tradition says, ‘Equations must be taught as statements about equivalent number-names, solved by the neutralizing method obeying associative, commutative and distributive laws!’ DR objects, ‘No, equations occur when recounting totals from tens to icons, and when reversing on-top and next-to addition.’

The tradition says, ‘A function must be taught as an example of a set-relation where first-component identity implies second-component identity!’ DR objects, ‘No, a function should be taught as a formula with two unspecified numbers thus respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. Later polynomials can be introduced as the number-formula containing the different formulas for constant change: T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x.’

The tradition says, ‘Linear functions must be taught before quadratic functions!’ DR objects, ‘No, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d.’

The tradition says, ‘Quadratic equations must be solved by factorizing before introducing the solution formula!’ DR objects, ‘No, when solving the quadratic equation x^2+b*x+c = 0, algebra and geometry should go hand in hand to show that inside a square with the sides x+b/2, the equation makes three rectangles disappear leaving only (b/2)^2-c, allowing possible roots to be found and used in factorization if necessary.’

The tradition says, ‘Differential calculus must be taught before integral calculus since the integral is defined as the anti-derivate.’ DR objects, ‘No, integral calculus comes before differential calculus. In primary school, next-to addition means multiplying before adding when asking e.g. T = 2 3s + 4 5s = ? 8s’, while reversing the question by asking 2 3s + ? 5s = 6 8s, or T1 + ? 5s = T, leads to differential calculus subtracting before dividing to get the answer (T-T1)/5. In middle school, fractions and per-numbers add by their areas, i.e. by integration. And in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant predictable change.’

The tradition says, ‘The epsilon-delta definition is essential in order to understand real numbers and calculus and must be learned by heart!’ DR objects, ‘No, it needs not be learned by heart. With units, it can be grounded in formalizing three ways of constancy; globally constant needing only the epsilon, piecewise constant with delta before epsilon, and locally constant with epsilon before delta.’

The tradition says, ‘Statistics and probability must be taught separately!’ DR objects, ‘No, they should be taught together aiming at pre-dicting unpredictable numbers by intervals coming from ‘post-dicting’ their previous behavior.’

In continental Europe, the tradition says, ‘Education means preparing for offices in the public or private sector. Hence the necessity of line-organized education with forced year-group classes in spite of the fact that teenage girls are two years ahead of the boys in personal development. Of course, boys and dropouts are to pity, but they all had the chance.’ North American republics object: ‘No, Education means uncovering and developing the learner’s individual talents through daily lessons in self-chosen practical or theoretical half-year blocks together with a person teaching only one subject and praising the learner for having a talent or for having courage to test it.’

In mathematics education, the tradition says, ‘Education means connecting learners to the canonical correctness through scaffolding from the learner’s zone of proximal development as described in social constructivism by Bruner and Vygotsky.’ DR objects, ‘No, education means bringing outside phenomena inside a classroom to be assimilated or accommodated by the learners thus respecting that in a sentence, the subject is objective but the rest might be subjective as described in radical constructivism by Piaget and Grounded Theory and Heidegger existentialism.

In mathematics education, the tradition says, ‘Research means applying or extending existing theory.’ DR objects, ‘No, where master level work means applying existing theory, research level means questioning existing theory, e.g. by asking if it could be different.’

How to Improve PISA Performance

PISA performance (Tarp, 2015a) can be improved in three ways: by a different macro-curriculum from class one, by remedial micro-curricula when a class is stuck, and by a STEM-based core-curriculum for outsiders.

Improving PISA performance means improving mathematics learning which can be done by observing three basic facts about our human and mammal and reptile brains.

The human brain needs meaning, so what is taught must be a meaningful means to a meaningful outside goal, mastery of Many; thus mathematics must be taught as 'Many-matics' in the original Greek sense as a common name for algebra and geometry both grounded in an motivated by describing Many in time and space; and not as 'meta-matism' mixing 'meta-matics', defining concepts from above as examples of internal abstractions instead of from below as abstractions from external examples, with 'mathe-matism', true inside but seldom outside classrooms as adding numbers without units.

The mammal brain houses feelings, positive and negative. Here learning is helped by experiencing a feeling of success from the beginning, or of suddenly mastering or understanding something difficult.

The reptile brain houses routines. Here learning is facilitated by repetition and by concreteness: With mathematics as a text, its sentences should be about subjects having concrete existence in the world, and having the ability to be handled manually according to Piagetian principle 'through the hand to the head'.

Also, we can observe that allowing alternative means than the tradition makes it not that difficult to reach the outside goal, mastery of many. Meeting Many, we ask 'How many in total?' To get an answer we count and add. We count by bundling and stacking and removing the stack to look for unbundles leftovers. This gives the total the geometrical form of a collection of blocks described by digits also having a geometrical nature by containing as many sticks as they represent. Counting also includes recounting in the same or in a new unit; or double-counting to produce per-numbers. Once counted, totals can be united or split, and with four kinds of numbers, constant and variable unit-numbers and per-numbers, there are four ways to unite: addition, multiplication, power and integration; and four ways to split: subtraction, division, root/logarithm and differentiation.

Thus, the best way to obtain good PISA performance is to replace the traditional SET-based curriculum with a different Many-based curriculum from day one in school, and to strictly observe the warning: Do not add before totals are counted and recounted – so multiplication must precede addition. However, this might be a long-term project. To obtain short-term improvements, difficult parts of a curriculum where learners often are stuck might be identified and replaced by an alternative remedial micro-curriculum designed by curriculum architecture using difference-research and sociological imagination. Examples can be found in the above chapter 'Examples of difference-research'.

Finally, in the case of teaching outsiders as migrants or adults or dropouts with no or unsuccessful educational background, it is possible to design a STEM-based core curriculum as described above allowing the outsiders become pre-teachers and pre-engineers in two years. Thus, applying sociological imagination when meeting Many without predicates forced upon it, allows avoiding repeating the mistakes of traditional mathematics.

The Tradition's 3x3 mistakes

Choosing learning mathematics as the goal of teaching mathematics has serious consequences. Together with being set-based this makes both mathematics education and mathematics itself meaningless by self-reference. Here a difference is to accept that the goal of teaching mathematics is mastering Many by developing a number-language parallel to the word-language; both having a meta-language, a grammar, that should be taught after the language to respect that the language roots the grammar instead of being an application of it; and both having the same sentence structure with a subject and a verb and a predicate, thus saying 'T = 2*3' instead of just '2*3'.

This goal displacement seeing mathematics as the goal of mathematics education leads to 3x3 specific mistakes in primary, middle and high school:

In primary school, numbers are presented as 1dimensional line numbers written according to a place value convention; instead of accepting that our Arabic numbers like the numbers children bring to school are 2dimensional block numbers. Together with bundle-counting and bundle-writing this gives an understanding that a number really is a collection of numbers counting what exists in the world, first inside bundles and outside unbundled singles, later a collection of unbundled and bundles and bundles of bundles etc.

Furthermore, school skips the counting process and goes directly to adding numbers without considering units; instead of exploiting the golden learning opportunities in counting and recounting in the same or in another unit, and to and from tens. This would allow multiplication to be taught and learned before addition by accepting that 4*7 is 4 7s that maybe recounted in tens as T = 4 7s = 2.8 tens = 28, to be checked by recounting 28 back to 7s, T = 28 = (28/7)*7 = 4*7 = 4 7s, using the recount-formula reappearing in proportionality, trigonometry and calculus. And giving division by 7 the physical meaning of counting in 7s.

Finally, addition only includes on-top addition of numbers counted in tens only and using carrying, a method that neglects the physical fact that adding or subtracting totals might crate overloads or underloads to be removed by recounting in the same unit. And neglecting the golden learning opportunities that on-top addition of numbers with different unit roots proportionality, and that next-to addition roots integration, that reversed roots differentiation thus allowing calculus to be introduced in primary school.

In middle school, fractions are introduced as numbers that can be added without units thus presenting mathematics as ‘mathematism’ true inside but seldom outside classrooms. Double-counting leading to per-numbers is silenced thus missing the golden learning opportunities that per-numbers give a physical understanding of proportionality and fractions, and that both per-numbers and fractions as operators need numbers to become numbers that as products add as areas, i.e. by integration.

Furthermore, equations are presented as open statements expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*u = 8	an open statement about two equivalent number-names
(2*u)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(u*2)*(1/2) = 4	since multiplication is commutative
u*(2*(1/2)) = 4	since multiplication is associative
u*1 = 4	by definition of an inverse element
u = 4	by definition of a neutral element

The alternative sees an equation as another name for reversing a calculation that stops because of an unknown number. Thus the equation ‘2*u = 8’ means wanting to recount 8 in 2s: 2*u = 8 = (8/2)*2, showing that u = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the ‘opposite side&sign’ method. A method that allows the equation ‘20/u = 5’ to be solved quickly by moving across twice; 20 = 5*u’ and 20/5 = u’, or more thoroughly by recounting 20 = (20/u)*u = 5*u = (20/5)*5 = 4*5, so u = 4.

Finally, middle school lets geometry precede coordinate geometry, again preceding trigonometry; instead of respecting that in Greek, geometry means to measure earth, which is done by dividing it into triangles again divided into right triangles. Consequently, trigonometry should come first as a mutual recounting of the sides in a right triangle. And geometry should be part of coordinate geometry allowing solving equations predict intersection points and vice versa, thus experiencing repeatedly that the strength of mathematics is the fact that formula predict.

In high school, a function is presented as an example of a set-relation where first-component identity implies second-component identity; and the important functions are polynomials with linear functions preceding quadratic functions; instead of respecting that a function is a name for a formula with two unspecified numbers, again respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. As to polynomials, they should be introduced as the number-formula containing the different forms of formulas for constant change, T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x. Consequently, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d and parallel to the other examples of constant change. Thus emphasizing the double nature of formulas that the can predict both level and change.

Furthermore, differential calculus is presented before integral calculus, presenting an integral as an antiderivative; instead of postponing differential calculus until after integral calculus is presented as adding locally constant per-numbers, i.e. as a natural continuation of adding fractions as piecewise constant per-numbers in middle school and next-to addition of blocks in primary school. Only in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant change.

Finally, high school presents algebra as a search for patterns, instead of celebrating the fact that calculus completes the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Uniting/ *splitting*	Variable	Constant
Unit-numbers	T = a + n *T – a = n*	T = a * n *T/n = a*
Per-numbers	T = ∫ a dn *dT/dn = a*	T = a^n, *loga(T) = n* *n√T = a*

Remedial Curricula

A remedial micro-curriculum might be relevant whenever learning problems are observed. Since you never get a second chance to create a first impression, especially remedial curricula in primary school are important to prevent mathematics dislike.

Thus, as described above in the chapter ‘examples of difference-research’, in primary school, problems might be eased by

- with digits, using a folding ruler to observe that a digit contains as many sticks or strokes as it represents if written in a less sloppy way.
- with counting sequence, using sequences that shows the role of bundling when counting to indicate that a given total as e.g. seven can be named in different ways: 7, .7, 0.7, bundle less 3, ½bundle&2, etc.

- with recounting, using a cup and 5 sticks to experience that at total of 5 can be recounted in 2s in three ways: with an overload, normal, or with an underload: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s, or T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s if using decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles.
- when learning multiplication tables, letting 3*7 mean 3 7s recounted in tens, i.e. a block that when increasing its width must decrease its height to keep the total unchanged.
- when learning multiplication tables, beginning by doubling and halving and tripling; and to recount numbers using half-ten and ten as e.g. 7 = half-ten&2 = 10less3 so that 2 times 7 is 2 times half-ten&2 = ten&4 = 14, or 2 times 10less3 = 20 less 6 = 14.
- when multiplying, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336, or T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336
- when dividing, using bundle-writing to create overloads or underloads according to the multiplication table, as e.g. T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48
- when subtracting, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38
- when adding, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

In middle school, problems might be eased by keeping algebra and geometry together and by re-describing

- proportionality as double-counting in different units leading to per-numbers
- fractions as per-numbers coming from double-counting in the same unit
- adding fractions as per-numbers by their areas, i.e. by integration
- solving equations as reversing calculations by moving to the opposite side with the opposite calculation sign

In high school, problems might be eased by re-describing

- functions as number-language sentences, i.e. formulas becoming equations or functions with 1 or 2 unspecified numbers
- calculus as integration preceding differentiation
- integration as adding locally constant per-numbers
- pre-calculus, calculus and statistics as pre- or post-dicting constant, non-constant and non-predictable change

A Macro STEM-based Core Curriculum

A macro-curriculum (Tarp, 2017) was designed as an answer to a fictitious curriculum architect contest set up by a Swedish university wanting to help the increasing number of young male migrants coming to Europe each year: ‘The contenders will design a STEM-based core mathematics curriculum for a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries.’

The design was inspired by an article on STEM (Han et al, 2014). Thus te curriculum goal is mastery of Many in a STEM context for learners with no background. As to STEM, OECD writes:

> The New Industrial Revolution affects the workforce in several ways. Ongoing innovation in renewable energy, nanotech, biotechnology, and most of all in information and communication technology will change labour markets worldwide. Especially medium-skilled workers run the risk of being replaced by computers doing their job more efficiently. This trend creates two challenges: employees performing tasks that are easily automated need to find work with tasks bringing other added value. And secondly, it propels people into a global competitive job market. (..) In developed economies, investment in STEM disciplines (science,

> technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules, called heat; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles and mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves. Nitrates and carbon-dioxide and water is waste for grey cells, but food for green cells producing proteins and carbon-hydrates and oxygen as food for the grey cells in return.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. To master plane or spatial forms, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent. So, mastery of Many means counting and recounting and adding and reversing addition and describing change and spatial shapes.

A STEM-based core curriculum can be about cycling water. Heating transforms it from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

Teaching Differences to Teachers

A group of teachers wanting to bring difference-research findings to the classroom might want first to watch some YouTube videos at the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many.

Then to try out the 'Free 1day SKYPE Teacher Seminar: Cure Math Dislike by 1 cup and 5 sticks' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally, and at a Skype conference with a coach. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in the three genres of quantitative literature, fact and fiction and fiddle. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Being a Difference-Researcher

In mathematics education, difference-research can be used by teachers observing problems in the classroom, or by teacher-researchers splitting their time between academic work at a university and intervention research in a classroom. Or by full-time researchers cooperating with teachers both using difference-research, the teacher to observe problems, the researcher to identify differences, working out a different micro-curriculum together to be tested by the teacher and reported by the researcher conducting a pretest-posttest study.

Thus, a typical difference-researcher begins as an ordinary teacher observing learning problems in his classroom and wondering if he could teach differently. Personally, in a precalculus class I taught linear and exponential functions by following the textbook order presenting them as examples of functions, again presented as examples of relations between two sets assigning one and only one element in one set to each element in the other set. I realized that by defining concepts as examples of abstractions instead of as abstractions from examples, I basically taught that 'bublibub is an example of bablibab' which some learners just memorized while others refused to learn before I gave them some applications. Talking about the difference between saving at home and in a bank, some asked me: Instead of calling it linear and exponential functions, why don't you just call it change by adding and by multiplying since that is what it is?'

So here the students themselves invented a difference that makes sense since historically, functions came after calculus. And the difference made two differences. Nobody had problems with learning about change by adding and by multiplying. And the Ministry of Education followed my suggestion to replace functions with variables instead of making pre-calculus non-compulsory, which was the plan because of the high number of low marks.

So one way to become a difference-teacher is to combine elements from action learning and action research and intervention research and design research. First you identify a difference, then you design a micro-curriculum, then you teach it to learn what difference the difference makes, then you learn from reporting and discussing it internally with colleagues. After having repeated this cycle of teaching and reporting the difference, the difference and the difference it makes in a posttest or a pretest-posttest setting is reported externally to teacher magazines or to conferences or to research journals.

Research is an institution supposed to produce knowledge to explain nature and improve social conditions. But as an institution, research risks a goal displacement if becoming self-referring. This raises two questions: Can a teacher produce research, and can research produce teaching? (Hammersley, 1993, p. 215). Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality used to collect reliable data and to test the validity of its findings by falsification attempts.

Often sociological imagination (see e.g. Zybartas et al, 2005) seems to be absent from traditional research seen by many teachers as useless because of its many references. In a Swedish context, this has been called the 'irrelevance of the research industry' (Tarp, 2015b, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying in a YouTube debate with Chomsky on Human nature:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics education and research are two examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead thus marginalizing or forgetting its original outside goal.

Conclusion

With 50 years of research, mathematics education should have improved significantly. Its lack of success as illustrated by OECD report 'Improving Schools in Sweden' made this paper ask: Apparently half a century's research in mathematics education has not prevented low and declining PISA performance. Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? Seeking guidance by difference-research searching traditions for hidden differences that make a difference, the answer is: Yes, mathematics can be different, education can be different, and research can be different.

Looking back, mathematics has meant different things through its long history, from a common label for knowledge in ancient Greece to today's 'meta-matism' combining 'meta-matics' defining concepts by meaningless self-reference, and 'mathe-matism' adding numbers without units thus lacking outside validity. So, looking for a difference to traditional set-based meta-matism, one alternative is the original Greek meaning of mathematics: Knowledge about Many in time and space.

Observing Many, allows rebuilding mathematics as a 'many-matics', i.e. as a natural science about the physical fact Many, where counting by bundling and stacking leads to block-numbers that recounted in other units leads to proportionality and solving equations; where recounting sides in triangles leads to trigonometry; where double-counting in different units leads to per-numbers and fractions, both adding by their areas, i.e. by integration; where counting precedes addition taking place both on-top and next-to involving proportionality and calculus. And where using a calculator to predict the counting result leads to the opposite order of operations: division before multiplication before subtraction before next-to and on-top addition.

Observing classes in continental Europe and in North America shows that education can be line-organized with forced year-group classes aiming at fulfilling the nation's need for officials for the public or private sector; or education can be block-organized with self-chosen half-year classes aiming at uncovering and developing the learner's individual talent. In mathematics education, the tradition sees learning mathematics as the goal of teaching mathematics and defines its concepts from above as examples of abstractions, part of the ruling canonical correctness, to be reached by learners through scaffolding. Here a difference is to accept that concepts historically arose from below as abstractions from examples, thus allowing new concepts to connect to existing.

Observing conference proceedings, shows that research papers may instead be master level papers applying instead of questioning existing theory and aiming at explaining instead of solving educational problems. Here a difference is difference-research searching traditions for hidden differences that make a difference.

So yes, as to mathematics education research, all three components can be different. Bottom-up many-matics can replace top-down meta-matism. In teenage education, daily lessons in self-chosen half-year blocks can replace periodic lessons in forced year-group lines. And, searching for useable differences can replace attempts at understanding the lack of understanding non-understandable self-reference.

Consequently, PISA performance may increase instead of decrease, and Swedish schools might improve dramatically by respecting that education means preparing learners for the outside world, brought inside to change the classroom from a library with self-referring textbooks to be learned by hart into a laboratory allowing the learner to meet the educational subject directly instead of indirectly through textbook 'gossip'. And by avoiding a goal displacement seeing mathematics as the goal for mathematics education, thus hiding the real goal, a number-language about Many in time and space.

To teach many-matics instead of meta-matism, big-scale in-service teacher training is needed, e.g. through the MATHeCADEMY.net, designed to teach teachers to teach mathematics as a natural science about Many by the CATS-approach, Count & Add in Time & Space, using PYRAMIDeDUCATION, where learners learn by being taught by the subject directly instead of indirectly by a sentence.

So, if a society as Sweden really wants to improve mathematics education, extra funding should force its universities to arrange curriculum architect contests to allow differences to compete as to imagination, creativity and effectiveness, thus allowing universities to rediscover their original external goal and to change their internal routines accordingly. A situation described in several fairy tales: The Beauty Sleeping behind the thorns of routines becoming rituals; and Cinderella making the prince dance, but only found when searching outside the canonical correctness.

With 2017 as the 500^{th} anniversary of Luther's 95 theses, the recommendation of difference-research to mathematics education research could be the following theses:

- To master Many, count and multiply before you add
- Counting and recounting give block-numbers and per-numbers, not line-numbers
- Adding on-top and next-to roots proportionality and integration, and equations when reversed
- Beware of the conflict between bottom-up enlightening and top-down forming theories.
- Institutionalizing a means to reach a goal, beware of a goal displacement making the institution the goal instead
- To cure, be sure, the diagnose is not self-referring
- In sentences, trust the subject but question the rest

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18. Reflections from the CTRAS 2017 Conference in China

Examples of Goal Displacements in Mathematics Education

At the annual conference of the Classroom Teaching for All Students Research Working Group (CTRAS), the 2017 conference theme was to promote classroom teaching research on exploring effective teaching strategies to support all students' mathematics learning. The two conference days contained half a day of plenary lectures. The first day also contained four examples of classroom teaching where a class of 5x3x2 students were taught in 30-40 minutes to illustrate examples of classroom lessons in China and the US. This paper reflects upon the lessons and some of the plenary lectures from a difference-research perspective looking for differences making a difference.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4th year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA performance decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other countries also experience a low and declining PISA performance. And apparently research can do nothing about it. At a plenary discussion, it was mentioned that according to an American Educational Research Association report, many research studies on teacher education does not have value to classroom teachers and classroom teaching. So, to improve student performance, maybe a different kind of research is needed to rise questions as: Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? So, maybe it is time to seek guidance by difference-research, searching for differences making a difference.

Searching for Hidden Differences, Difference-Research looks at Mathematics Education

Difference-research (Tarp, 2017) asks two questions: 'Can this be different – and will the difference make a difference?' Difference-research is inspired by the ancient Greek sophists looking for hidden differences to unmask choice masked as nature. If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

As to mathematics education, education is a social institution, and perhaps the most intervening one considering the numbers of hours spent there per week and during childhood and adolescence. As to institutions, Bauman talks about rationality and goal displacements in social organizations:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) It may happen in effect that the task originally seen as the reason to establish it is relegated to a secondary position by the all-powerful interest of the organization in self-perpetuation and self-aggrandizement. The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

So, in a social institution, its goal cannot be different unless a means is masked as a fake goal, to be unmasked and replaced by the original goal by difference-research finding hidden differences.

As an institution, mathematics education is a social organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'we teach you mathematics so you can learn mathematics'. But this is a goal displacement created by meaningless self-reference (we teach you bublibub so you can learn bublibub). So, if mathematics isn't the goal in mathematics education, what is? And, how well-defined is mathematics after all?

How Well-Defined is Mathematics?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, algebra, replacing arithmetic because of smarter numbers, and geometry, both rooted in the physical fact Many through their original meanings, 'to reunite' in Arabic and 'to measure earth' in Greek. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when all were replaced by the 'New Mathematics'.

Here the invention of the concept Set created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant definition by self-reference, i.e. defining a concept top-down as examples of abstractions instead of bottom-up as abstractions from examples. Thus the concept 'function', originally labeling a calculation containing both specified and unspecified numbers, was turned into a subset of a set-product where first-component identity implies second-component identity.

However, looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts: You can eat an example of an apple, but not the word 'apple'.

Thus, SET has transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics, defining concepts as examples of abstractions instead of as abstractions from examples; and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units, as '1 + 2 IS 3', meets counter-examples as e.g. 1 week + 2 days is 9 days.

So, looking back, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about forms and numbers and operations? We all teach that 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the number-language sentence 'The total is 3 8s' or 'T = 3*8' we treat the predicate, 3 8s or 3*8, as if it was the subject, which is a clear indication of a goal displacement. Thus, the total of fingers on a hand cannot be different, but the way they are counted can: T = 5 1s = 2 2s &1 = 1 2s & 3 = 3 2s less 1 = 1 3s & 2 etc.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers and operations and calculations.

However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. Furthermore, we are sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens. This might speed up writing, but might also slow down learning; together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, Lincolns Gettysburg address, 'Four scores and ten years ago' shows that not all count in tens. Thus in Denmark, seventy is called 'half four' with scores understood.

So, despite being presented as universal, many things can be different in mathematics, apparently having a tradition to present its choices as nature that cannot be different. And to unmask choice presented as nature is precisely the aim of difference-research.

How to find Hidden Differences?

Research is an institution supposed to produce knowledge to explain nature and improve social conditions. But as an institution, research risks a goal displacement if becoming self-referring. Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality, which then is used to collect reliable data and to test the validity of its findings by falsification attempts.

Hidden differences might be found by sociological imagination, seen as the core of sociology by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman (1990, p. 16) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.'

However, often sociological imagination (see e.g. Zybartas et al, 2005) seems to be absent from traditional research, seen by many teachers as useless because of its many references. In a Swedish context, this has been called the 'irrelevance of the research industry' (Tarp, 2015, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying in a YouTube debate with Chomsky on Human nature:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics and education and research are three examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead, thus marginalizing or forgetting its original outside goal.

Here Heidegger gives a tool to tell goals from means by pointing out, that in defining is-statements we should trust the subject but question the predicate since the subject, by its existence, cannot be different whereas the predicate is a judgement that might be a prejudice, i.e. one among several means that can be different, as illustrated above when reporting on the number of fingers on a hand.

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

Inspired by Heidegger, the French poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu points out that society forces words upon you to diagnose you so it can offer you curing institutions including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to your world and yourself (Derrida, 1991. Lyotard, 1984. Bourdieu, 1970. Tarp, 2012).

Thus Foucault (1995) sees a school as a 'pris-pital', i.e. a mixture of a prison and a hospital. A school is prison-like by forcing students to stay together in classes for a long period of time, where continental Europe uses multi-year lines based upon age, in contrast to North America that from secondary school uses self-chosen half-year blocks.

And a school is hospital-like by wanting to cure the students by treating them for a diagnose that is not always that well-defined, and in many cases self-referring as when saying: we teach you mathematics so you can learn mathematics.

So, to make education a meaningful and civilized 'pris-pital', a diagnose must refer to a lack of or in knowledge about outside things or phenomena that students will meet when leaving school.

Thus, the original educational goal, to prepare children and adolescents for mastering the outside world, leads to two questions: What should the students meet in the classroom, the outside world brought inside, or descriptions of it in textbooks? And should all students meet the same in forced multi-year classes or be allowed to choose individually between half-year blocks?

And, to make its education a meaningful and civilized cure we must confront mathematics, seen as a collection of definitions and truth-claims, with two questions: Are the definitions self-referring or rooted in the outside goal, Many? Has the inside truth outside validity also?

Heidegger's warning 'In sentences, trust the subject but question the rest' implies that to discover the true nature of the subject hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its 'predicate-prison'. By opening us, Many will appear with its nature undisguised, thus allowing us to construct different mathematics micro- and macro-curricula.

So we now return to the original subject in Greek mathematics, the physical fact Many, and use sociological imagination and Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties (Tarp, 2017). We do so to answer the question: How to find differences debunking mathematics from a goal to an inside means to the real outside goal, mastery of Many.

Meeting Many Creates a Count&Multiply&Add Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we count and add to create a number-language sentence, T = 2 3s, containing a subject, a verb and a predicate as in a word-language sentence. We count in bundles to be stacked as block-numbers to be re-counted and double-counted and processed by on-top and next-to direct or reversed addition. Thus, to count we take away bundles (thus rooting division) to be stacked (thus rooting multiplication) to be moved away to look for unbundled singles (thus rooting subtraction); finally we answer using bundle-writing for the bundles inside the bundle-cup and the singles outside, possibly with an overload or an underload to be removed or created by re-counting in the same unit, T = 7 = 2B1 3s = 1B4 3s = 3B-2 3s = 2 1/3 3s = 2.1 3s (thus rooting fractions and decimals to describe the singles). The result is predicted by a re-count formula T = (T/B)*B saying that 'from T, T/B times B can be taken away'. Re-counting in another unit roots proportionality. A total counted in icons can be re-counted in tens (thus rooting multiplication tables), or a total counted in tens can be re-counted in icons (thus rooting equations).

Double-counting in physical units creates per-numbers (again rooting proportionality) becoming fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas (thus rooting calculus).

Once counted or re-counted, totals can be added on-top after being re-counted in the same unit (again rooting proportionality); or next-to as areas (again rooting integral calculus). Then both on-top and next-to addition can be reversed (thus rooting equations and differential calculus).

In a rectangle split by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Thus, a Count&Multiply&Add curriculum differs from the tradition by presenting counting and multiplication before addition, and by using calculus to add fractions as per-numbers (Tarp, 2017).

Classroom Lessons

At the CTRAS 2017 conference, the first day contained four example of classroom lessons where a class of 5x3x2 students were taught in 30-40 minutes to illustrate examples of classroom teaching in China and the US.

B. China Teacher Lesson Display, Grade 5

The first lesson was a China teacher lesson display with a grade 5 class. The task was to fill a 3x3 square with the numbers 1-9 are so that they add up to 15 horizontally, vertically and on the diagonals, motivated by a video sequence from a fairy tale showing that this would lift a spell.

Personally, I found this an interesting task allowing the children to use their imagination and creativity. Likewise, a motivating video was a good idea. I observed that some students seemed to find the task difficult. This raises the question: ‘Will a different approach make a difference as to how many students succeed?’ So, from the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

Based upon the principle ‘algebra & geometry, always together, never apart’, symmetry is present on the geometry part, so it ought also to be present on the algebra part, e.g. by applying a counting sequence for the numbers 1-9 that counts the numbers as ‘Bundle less or plus’ using five as the bundle-number: Bundle less 4, B-3, B-2, B-1, B+0, B+1, B+2, B+3, B+4, inspired by the Roman numbers and a Chinese or Japanese abacus.

By its geometry, each sum will contain three numbers, so we can leave out the bundle B and redesign the task to ‘adding up to zero’. Because of the symmetry in geometry and algebra, 0 must be in the middle. Seeing zero as an even number, the three terms must be odd+odd+even, so the corners must be odd numbers.

Thus, the task could split up in several subtasks:

1. Starting by 5, find a symmetrical way to count from 1 to 9. Describe the symmetry.

2. Reformulate the task using these new numbers. Which number must be placed in the middle?

3. Adding two numbers to an odd number, how can the result be an even number?

4. Which numbers must be placed in the corners?

5. Show and test the answer using the numbers 1-9.

In a Count&Multiply&Add curriculum, re-counting the numbers from 1 to 9 in 5s is a routine task since the fingers on a hand is counted as ‘1 or bundle less 4; 2 or B-3 etc.’. Bundle-counting implies that you chose a bundle-size for the cup. In this case the sum 15 is obtained by three numbers, so 5 would be a natural choice as bundle-size allowing re-counting as T = 3 = 1B-2 5s, and T = 8 = 1B3 5s, etc. In such a class, the first subtask would be: ‘1. With the sum 15 obtained by three numbers, chose a bundle-size and reformulate the task.’

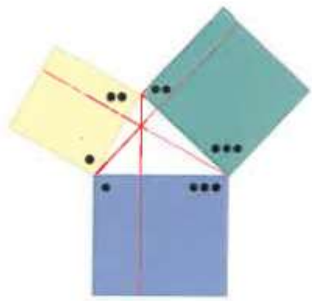
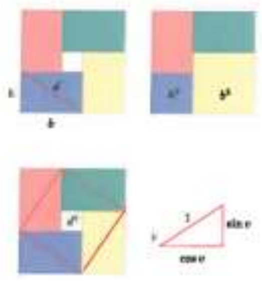
B. China Teacher Lesson Display, Grade 8

The second lesson was a China teacher lesson display with a grade 8 class. The task was to give a geometrical proof of the Pythagoras Theorem

Personally, I found this an interesting task allowing the adolescents to use their imagination and creativity. A proof is a core task in classical geometry; and choosing the Pythagoras Theorem as a core theorem is a good idea. I observed that some students seemed to find the task difficult. This raises the question: ‘Will a different approach make a difference as to how many students succeed?’ So, from the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if the Pythagoras Theorem is a goal or a means. Thus the Pythagoras Theorem may be seen as an inside means to the outside goal of adding travel-distances. If parallel, two distances add or subtract directly. But if perpendicular, they add by their squares: 3 steps over plus 4 steps up total 5 steps, since $3^2 + 4^2 = 5^2$.

Based upon the principle ‘algebra & geometry, always together, never apart’, the task could contain both a geometrical and an algebraical proof. If it is correct that the theorem can be proved in more than 100 ways, two easy proofs could be used first to include all students, and two more difficult proofs could be added later, as could a proof using trigonometry.

As to the background, three cases can be mentioned: an isosceles right-angled triangle, a right-angled triangle, and an arbitrary triangle leading to $a^2 + a^2 = c^2$ (or $c = a*\sqrt{2}$), $a^2 + b^2 = c^2$; and $a^2 + b^2 – 2*a*b*\cos C = c^2$, in its algebraic versions; or in its geometrical version: the three heights split the opposite squares in parts that are like to its outside neighbors. An easy algebraic proof is the one showing that the height splits the opposite square in parts that are like to its outside neighbors, which also holds for triangles that are not right-angled.	
An easy geometrical proof could be the one presented the next day in the plenary lecture ‘The wisdom of Traditional Mathematical Teaching in China’, shown with playing cards: Place four h-by-b playing cards after each other after turning them a quarter turn. The diagonals c also turn and now form a square with the area c^2. The full area can be expressed in two ways, as c^2 + 4 half cards, or as $h^2 + b^2$ plus two full cards. Consequently $h^2+b^2 = c^2$.	

Thus, the task splits up in several subtasks:

1. A 1.4-by-1.4 square is split into four triangles by the two diagonals. Prove that the triangles are isosceles and right-angled. Prove geometrically and algebraically that the Pythagoras Theorem $a^2+b^2 = c^2$ applies here. Measure the length of the diameter - are you surprised?

2. Draw a triangle with three angles less than 90 degrees. The three heights split the opposite squares in two parts. What can be said about the areas of two outside neighbors? Does this also apply to a right-angled triangle?

3. A geometrical proof of the Pythagoras Theorem uses four h-by-b playing cards placed after each other after turning them a quarter turn. The diagonals c also turn and now form a square with the area c^2. How can the total area be expressed?

4. Give an algebraic proof of the Pythagoras Theorem by using the result from question 2 and by splitting c in c1 and c2.

5. Tossing two dices gives the number of steps horizontally and vertically on a squared paper. Predict the length of the shortcut and test by measuring.

6. A 2meter bar is carried around a right-angled corner. How wide must the corridor be?

In a Count&Multiply&Add curriculum, counting includes a mutual re-counting of the sides in a right-angled triangle, seen as a rectangle halved by a diagonal. This allows trigonometry to be taught before geometry in accordance with the Greek meaning, earth-measuring. Thus, in a triangle ABC with C as the right angle and the side c split in c1 and c2 by the height, $\cos A = c1/b = b/c$, or $b^2 = c*c1$ and likewise with cosB. This shows that the height splits the opposite square in parts that are like to its outside neighbors, which also holds for triangles that are not right-angled. So in this case $c^2 = a^2 + b^2 – 2*a*b*\cos C$. The subtasks would be the same.

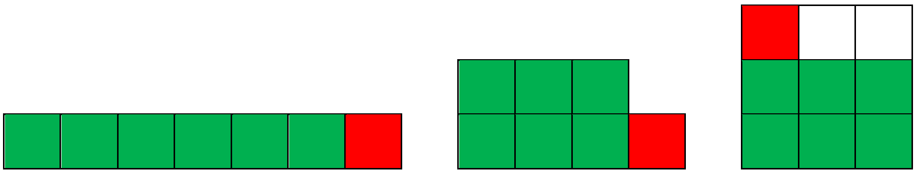
C. American Teacher Lesson Display, Grade 3

The third lesson was an American teacher lesson display with a grade 3 class. The task was to learn about and apply fractions, a core concept in algebra. I observed that some students seemed to find the task difficult. This raises the question: ‘Will a different approach make a difference as to how many students succeed?’ So, from the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if fractions is a goal or a means. Looking for the outside root of fractions we find double-counting in various contexts as e.g. icon-counting and switching units and parting.

Thus ‘icon-counting fractions’ occur when counting a total by bundling and stacking, which creates a double-counting of bundles and unbundled leftovers that can be placed in a separate stack for unbundled ones, separated by a decimal point, or on-top of the stack counted in 3s as a fraction creating a mixed number, since counting in 3s means taking away 3s, i.e. divide by 3.

Thus, a total of 7 can be counted as:



T = 7 = 2.1 **3s** = 2 1/3 **3s**

Re-counting 6 7s in tens gives T = 6 7s = 6*7 = 4 2/10 tens = 4.2 tens = 42 if leaving out the unit and misplacing the decimal point.

‘Per-number fractions’ or ‘unit switching fractions’ occur when double-counting something in the same or in different units. Counting in different units, per-numbers as 4$/5kg or 4/5 $/kg allows bridging the units by re-counting in the per-number:

10$ = (10/4)*4$ = (10/4)*5kg = 12.5kg; and 20kg = (20/5)*5kg = (20/5)*4$ = 16$.

‘Parting fractions’ are per-numbers coming from double-counting a part and the total in the same unit: If 5 of 7 apples are green, the fraction 5/7 of the 7 apples is the green part. Splitting a total in the ratio 2:3 means getting the fractions 2/5 and 3/5 of the total.

An outside sharing-situation can be a root or an application.

Sharing 8 apples between 4 persons not knowing division, they will repeat taking one each by turn as long as possible, e.g. by letting a mediator take away a bundle of 4s several times. In each bundle, a person then takes 1 of 4, or 1 per 4 or 1/4. In this case, the outside goal sharing roots the inside means icon-counting and per-numbers.

Or, sharing 8 apples between 4 persons may be presented as an application of getting the fraction ¼ of 8, found by dividing 8 by 4. Postponed to after division and fractions have been taught and learned, this is an example of a goal displacement, where the inside means, divisions and fractions, are treated as goals in need of outside applications as means for student motivation.

In education, a choice should be made as to which fraction should be taught first. In the actual lesson, the choice seemed to be teaching parting fractions as 2/7 by double-counting the part and the total, and to apply them to describe a self-designed packman, although the lesson also contained examples of sharing fractions when dividing a geometrical figure.

Observing the Piaget principle ‘through the hand to the head (greifen vor begreifen)’, one way to introduce parting fractions could be using the biological counters, the fingers: On my left hand, the fingers can be straight or bent. If 2 of the 5 fingers are bent I will say that the fraction 2 of 5 or 2/5 of my fingers are bent. If no fingers are bent, the fraction is 0/5. If all five fingers are bent, the fraction is 5/5. Thus, a fraction is used to describe a double-counting of a part in the total. In the fraction 2/5, 2 is called a numerator since it numbers the specials; and 5 is called a denominator since it names the total, and ‘nomen’ is ‘name’ in latin.

Later both hands can be used to illustrate fractions as 7/10, or 5/8 if excluding the thumbs.

Next step could be to discuss what is meant by saying that 3/5 of my ten fingers are bent. Here a choice must be made between parting and per-number fractions.

As to parting fractions, looking at the ten fingers I must apply my mathematical knowledge to say: I find the fraction 1/5 by splitting the total in 5 equal parts, which is done by dividing 10 by 5 giving 2. Now I can multiply with 3 to get 6. So I bend 6 fingers’.

As to per-number fractions, looking at the ten fingers I reformulate the task: the fraction 3/5 means taking 3 per 5, and with ten as two 5s, I just bend 3 fingers on both hands, i.e. 6 fingers’

As alternative means to the same goal, both should be presented in the class to observe differences as to effect.

Another option is to introduce parting fractions in a symmetry context using a dice and writing a cross if the dice shows an even number. Such a task splits up in several subtasks:

1. Put your left hand flat on the table with all finger straight.

2. On a dice, which numbers are even and why? The rest are called odd.

3. On this paper you find 10 rows with 5 squares in each row. Throw a dice five times. If even, bend a finger and write a cross; else leave the finger straight and write nothing. And report the number of crosses as a fraction 3 of 5 and as 3/5. Each time, mention which is the numerator and which is the denominator.

4. Please do the same with the next 9 rows.

5. At the bottom line, please fill in the report saying: The result 0/5 I got ?/10 times, etc.

6. Among the 10 rows, how many are identical? How many are symmetrical?

A third option is to introduce parting fractions in a probability context and continue with the following subtasks:

6. Use centi-cubes or double centi-cubes to show the answer to question 5.

7. In groups of fours, build your centi-cubes together vertically and write a report: The result 0/5 we got ?/40 times, etc.

8. In the class, arrange all the structures behind each other horizontally. Are they like?

In a Count&Multiply&Add curriculum, counting by bundling and stacking implies fractions and decimals to account for the unbundled singles placed on-top of or next-to the stack thus creating mixed numbers as e.g. T = 7 = 2 1/3 3s. Later fractions occur as per-numbers coming from double-counting in the same unit, as e.g. 2$/5$ = 2/5 = 2 per 5. Taking the fraction 2/5 of 20 means taking 2$ per 5$ of 20$, so we just re-count 20 in 5s as T = 20$ = (20/5)*5$ giving (20/5)*2$ = 8$.

As to addition, fractions add as per-numbers, both being operators needing a number to become a number. Multiplying before adding creates areas to be added, thus rooting integral calculus.

D. American Teacher Lesson Display, Grade 8

The fourth lesson was an American teacher lesson display with a grade 8 class. The task was to find a formula connecting the number of angles with the angle sum in a polygon.

Personally, I found this a core task in geometry, allowing the adolescents to use their imagination and creativity. I observed that some students seemed to find the task difficult. This raises the question: ‘Will a different approach make a difference as to how many students succeed?’ So, from the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if finding the angle sum in a polygon is a goal or a means. Looking for the outside root of angles we find changing direction under a closed journey with many turns. Thus the lesson could focus on a paper with three closed journeys with 3 and 4 and 5 turning points labeled A and B and C and D and E.

The triangle allows showing that the angle sum is 180 degrees from a new perspective: Inserting an extra point P between A and B transforms the line segment AB into a tri-angle APB where P adds 180 degrees to the angle sum zero. Pulling P out makes P decrease with what A and B increase, so the angle sum remains 180 degrees.

Likewise, on a triangle ABC, inserting an extra point P between A and B transforms the triangle into a four-angle APBC where B adds 180 degrees to the angle sum. Pulling P out makes P decrease with what A and B increase, so the angle sum remains added with 180.

And again the angle sum is increased by 180 degrees by inserting an extra point Q between A and P in the four-angle APBC. So each time an angle is added to the original 3, the angle sum gets 180 added to the original 180 degrees. Consequently, the total angel sum is 180 + 180*(angle number – 3).

Thus, as to teaching, the task could split up in several subtasks:

On this sheet, you see three different polygons. We would like to find a formula connecting the angle number with the angle sum in a polygon.

1. The word ‘polygon’ is Greek. What does it mean in English? In German a triangle is called a ‘Dreieck’. Are the words describing the same?

2. On a line segment AB, insert an extra point P between A and B to transform the line segment into a 3-angle APB. What is the angle sum in APB?

3. Pulling P away from the line segment makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APB?

4. On a triangle ABC, insert an extra point P between A and B to transform the 3-angle into a 4-angle APBC. What is the angle sum in APBC?

5. Pulling P away from the triangle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBC?

6. On a 4-angle polygon ABCD, insert an extra point P between A and B to transform the 4-angle into a 5-angle APBCD. What is the angle sum in APBCD?

7. Pulling P away from the four-angle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBCD?

8. On the 5-angle polygon ABCDE, insert an extra point P between A and B to transform the 5-angle into a 6-angle APBCDE. What is the angle sum in APBCDE?

9. Pulling P away from the four-angle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBCDE?

10. Try formulating a formula connecting the angle number n with the angle sum S.

11. Are any of these formulas correct?

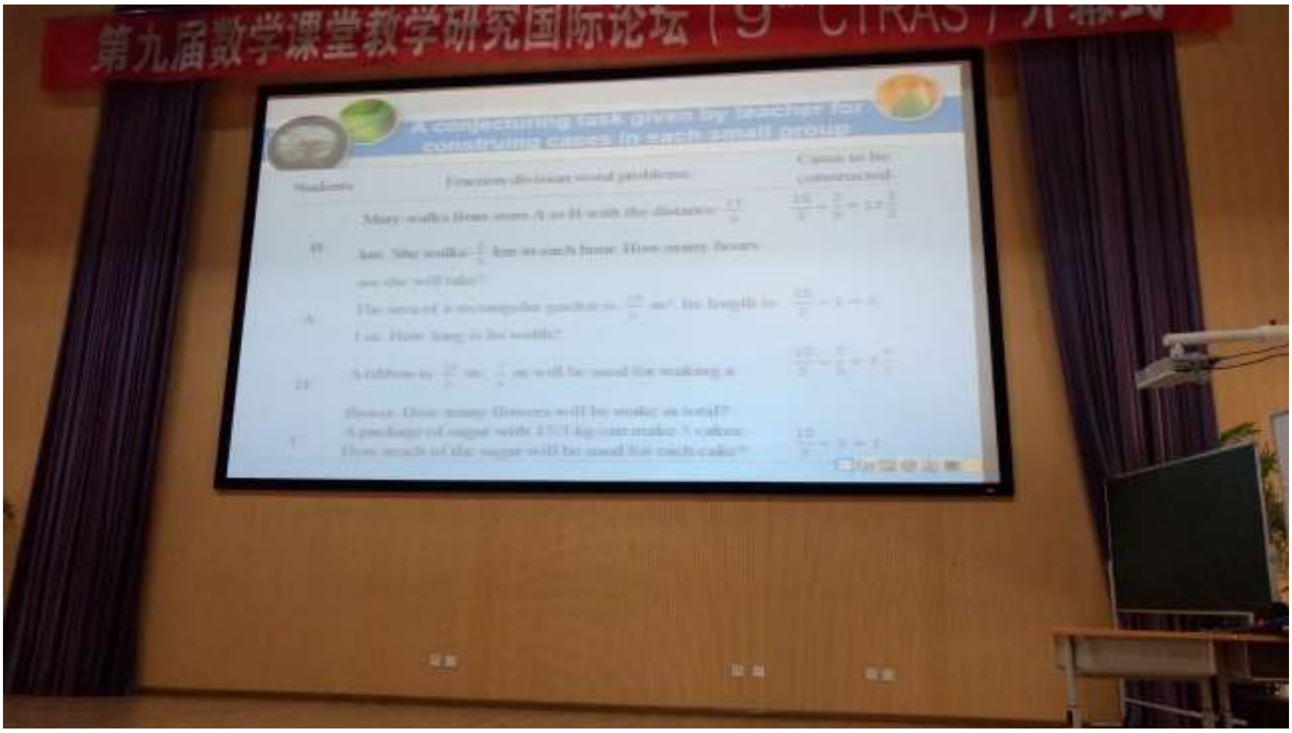
S = n*180 -3	S = n*180 - 360	S = 180 + 180*(n-2)

In a Count&Multiply&Add curriculum, the ‘Geometry & algebra, always together, never apart’ principle is observed. Thus a polygon will be lines connecting angles with given coordinates. So an angle is found by solving the equation tanA = slope. If all angles are to be found, in the end the rule for the angle sum can be used for checking.

Fractions and Mixed Numbers

Two plenary presentations contained mixed numbers. The ‘Using sharing brownies task for mixed number concept development’ presentation discussed the task: How to split 13 cookies between 4 children? The ‘The conjecturing contributing to the group argumentation in primary classrooms’ presentation contained a slide with three parts.

Students	Fraction division word problems	Cases to be constructed
B	Mary walks from store A to B with the distance 15/3 km. She walks 2/5 km in each hour. How many hours are she will take.	15/3 ÷ 2/5 = 12½
A	The area of a rectangular garden is 15/3 m^2. Its length is 1 m. How long is its width?	15/3 ÷ 1 = 15/3
C	A ribbon is 15/3 m. 7/5 m will be used for making a flower. How many flowers will be make in total?	15/3 ÷ 7/5 = 3 4/7
D	A package with 15/3 kg can make 5 cakes. How much of the sugar will be used for each cake?	15/3 ÷ 5 = 1

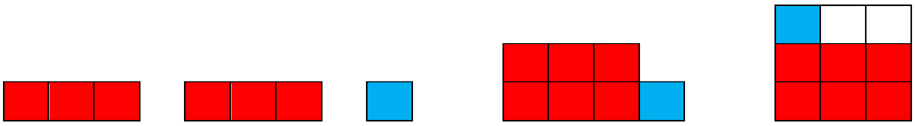


From the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if mixed numbers is a goal or a means. Looking for the outside root of mixed numbers we look for the roots of fractions, again rooted in division.

Seeing ‘Mastering Many’ as the outside root of mathematics, meeting Many leads to the question ‘How many in total?’. To answer we total by counting and adding. We count by bundling and stacking, predicting the resulting number-block by a re-count formula, T = (T/B)*B, saying ‘from T, T/B times B can be taken away’, thus rooting division and multiplication. Thus, a total of 8 can be re-counted in 4s as T = 8 = (8/4)*4 = 2*4 = 2 4s. So the root of division is counting by bundling.

Counting 7 in 3s gives T = 7 = 2B1 = 2.1 3s if the singles are placed next-to the stack of 3s as a stack of 1s; or T = 2 1/3 3s if the singles are placed on-top of the stack of 3s, counted in 3s as part of a 3-bundle.



T = 7 = 2 3s & 1 = 2.1 3s = 2 1/3 3s

So the root of mixed numbers is double-counting a total in bundles and parts, expressing the part as a fraction or by a decimal point. Counting in icon-bundles different from ten, the fraction remains unchanged, T = 7 = 2 1/3 3s. But counting in tens, the fraction is changed into decimals: T = 6 7s = 4 2/10 tens = 4.2 tens = 42 if leaving out the unit and the decimal point.

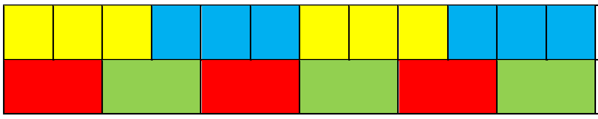
Sharing 8 apples between 4 persons not knowing division, they will repeat taking one each by turn as long as possible, e.g. by letting a mediator take away a bundle of 4s several times. In each bundle, a person then takes 1 of 4, or 1 per 4 or 1/4. Thus the root of fractions is per-numbers.

So, the sharing question can be reformulated to ‘How manty times can 4 be served by 8 items?’ or ‘8 = ?*4’ or ‘8 = u*4’ which is an equation solved by re-counting 8 in 4s: u*4 = 8 = (8/4)*4 giving u = 8/4 = 2, showing that an equation is solved by moving a number to the opposite side with the opposite sign.

Seeing 8/4 as ‘8 counted in 4s’ thus reflects what takes place psychically when sharing. However, the tradition says that ‘8/4 = 2’ means ‘8 shared between 4’ giving 4 2s and not 2 4s.

Thus, seeing division as THE sharing tool will exclude students unable to learn division, normally considered the difficult of the four operations; and introduced as the last operation, despite introducing it as the first is the natural approach if respecting that the natural way to share is to count the total in shares.

Furthermore, the sharing-understanding of division does not allow problems as ‘4 shared between 1/3’, to which the counting-understanding has the natural answer T = 4/(1/3) = 4*3 = 12. This resonates with the re-count formula saying T = (4/(1/3))*1/3, so 4/(1/3) must mean 4*3. Likewise, counting 4 in 2/3s halves the result, so T = 4/(2/3) = 4*3/2 = 6, or k/(2/3) = k*(3/2).



	4 counted in 1/3s gives 12, so 4/(1/3) = 4*3
	4 counted in 2/3s gives 6 , so 4/(2/3) = 4*3/2

Here the difference-research question is ‘Will presenting division as a counting means instead of as a sharing means make a difference?’

Returning to the discussion about outside goals and inside means we can ask: With ten as the international standard for bundles, does mixed numbers occur outside or only inside classrooms?

Sharing 13 brownies between 4, each get 3 ¼ brownie, which makes sense since a brownie can split in 4 equal parts. However, the answer could also be ‘3 each and 1 leftover’ as would be the case if sharing 13 cats instead. So whether 13 shared by 4 is 3 ¼ or ‘3 & 1 left’ depends on the unit.

To study the difference in concept development, difference-research would arrange two additional introductions ’13 brownies are arranged in boxes of fours; how many boxes are needed?’; and ’13 brownies are served in quarters, how many can be served? How many boxes are needed?’

Measuring lengths in inches, it makes sense to talk about 3 ¼ inch since 1inch splits into parts by repeated halving. Whereas 3 1/5 inch makes no sense.

However, internationally, length is measured in meters that splits into ten-parts, that split into ten-parts etc., making fractions of tens transform into decimals.

In the first slide task, a distance of 15/3 km only makes sense if sharing 15 km between 3 persons or parts. And presenting a velocity as 2/5 km per hour only makes sense if presented as a per-number 2 km per 5 hours. But both are rare cases that should be presented as footnotes to the typical outside problems using decimal numbers.

The next slide tasks also contain mixed numbers: 'The area of a rectangular garden is 15/3 m^2'; and 'A ribbon is 15/3 m'; and '7/5 m will be used for making a flower'; and 'A package with 15/3 kg can make 5 cakes'. By geometrical constructions it is possible to construct 15/5 in the case of a garden and a ribbon. It is however not possible to find precisely 1/3 of 15 kg without first calculating 15/3. So, again we can ask: Are these typical situations in need of mixed numbers, or will decimal numbers be more frequently used in such situations?

Likewise, we can ask if problems describing outside phenomena with mixed numbers are examples of a goal displacement where the outside goal has become a means to motivate learning an inside means presented as a goal?

Basically, a mixed number as 23 ¼ is a mixture of two different bundle-sizes, 23 is counted in tens and ¼ is counted in 4s. This only has meaning when measuring length in inches but since meters has become the international unit, maybe mixed numbers should play only a minor role as footnotes to decimal numbers especially if mathematics education should include all.

Adding mixed numbers directly have meaning when adding inches. Elsewhere, by containing fractions, which are not numbers but operators needing a number to become a number, they should be added by areas, i.e. by integration.

In a Count&Multiply&Add curriculum, mixed numbers thus occur from day one when counting a total by bundling leaves some unbundles singles described by a fraction or a decimal point. Later when ten bundling takes over, mixed numbers become decimal numbers. And from the beginning, all numbers are seen as mixed decimal numbers in disguise, T = 43 = 4.3 tens just as it is said, 4ten3, and as it is written in China.

Fractions: Numbers or Operators

At the end of the first day plenary presentation a discussion took place about the nature of fractions. Arguing that fractions are not numbers but operators needing a number to become a number, I used 5 water bottles for illustration: 'To my right I have 2 bottles, 1 is horizontal since it is empty; to my left I have 3 bottles, 2 are empty. So to the right ½ of my 2 bottles are empty, and to the left 2/3 of my 3 bottles are empty. In total 1+2 = 3 of my 2+3 = 5 bottles are empty, so in this case, adding ½ and 2/3 gives 3/5 of my bottles, the same answer as many students give when adding fractions by adding the numerators and adding the denominators. But the school teaches that ½ + 2/3 = 7/6, meaning that when added, I have 6 bottles and 7 of them are empty. This is perhaps why students dislike fractions. We teach fractions as if they are numbers. But fractions are not numbers, fractions are operators needing a number to become a number. So maybe we should teach fractions that way.'

A gentleman gave as a counter argument that I was mixing fractions with ratios: The example should be described, not by fractions but by ratios, to the right the ratio of empty bottles is 1:2, and to the left it is 2:3, and since ratios do not add it was meaningless to ask for the total. I replied that ratios describe sharing situations which was not the case here. But I thanked him for disagreeing and asked the conference organizers to include in the next conference a debate between persons with different views on mathematics and its education, e.g. on the nature of fractions, or on other issues. In the break, we continued the discussion and agreed on considering writing a common paper on fractions and ratios.

Decimal Multiplication in Grade 5

The second day, a plenary presentation presented a study om fifth graders' learning of decimal multiplication, a core task in algebra, but causing problems to some students when asked to do the multiplication ‘110*2.54’. From the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if multiplying and decimal numbers is a goal or a means. Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding. To count, we take away bundles to be stacked, thus rooting division and multiplication, allowing the result to be predicted by a re-count formula T = (T/B)*B saying ‘from T, T/B times, B can be taken away’. A total of e.g. 8 can be re-counted in 4s as a block-number T = (8/4)*4 = 2*4 = 2 4s.

Multiplication thus is a means to stack six 7s as T = 6 7s = 6*7; and a means to re-count 6 7s in tens: T = 6 7s = 6*7 = 42 = 4.2 tens if including the unit and the decimal point. So, looking for the outside root of multiplication we find stacking and shifting units. Thus, the present task is to re-count 110 2.54s in tens, or to re-count 2,54 110s in tens.

Based upon the principle ‘algebra & geometry, always together, never apart’, this task can be reformulated to changing the size of a number block: Re-counted in tens, a block of 110 2.54s will increase the base 2.54 with a factor close to 4 and decrease the height with the same factor, so the result will be close to 110/4 tens or 27.5 tens or 275. Or Re-counted in tens, a block of 2.54 110s will decrease the base 110 with a factor 11 and increase the height with the same factor, so the result will be close to 2.54*11 tens close to 27.5 tens or 275.

Using pure algebra, the ten-units can be shown as factors: 110*2.54 = 11 tens*2.54 = 11*25.4 = 1.1 tens 25.4 = 1.1*254 = 279.4’

Thus, the task could split up in several subtasks:

1. Geometrically, show the product 110*2.54 as two number-block with a base and a height.

2. In each case, what is the factor needed to change the base to tens.

3. How will this factor change the height?

4. Factorize the product to show the ten-units.

6. Include the ten-factors in the other factor.

7. Write the product with and without the unit tens.

Recommending counting and multiplying before adding, multiplication and decimal numbers are part of counting in a Count&Multiply&Add curriculum seeing mastering Many as the outside goal of mathematics. Meeting Many, we ask ‘How many in Total?’ Counting 7 in 3s gives T = 7 = 2B1 = 2.1 3s if the singles are placed next-to the stack of 3s as a stack of 1s, or 2 1/3 3s if the singles are placed on-top of the stack of 3s, counted in 3s as part of a 3-bundle.

To answer the question ‘How many in Total?’ we use a number-language sentence with a subject and a verb and a predicate as has word-language sentences. Thus T = 6*7 means that the total is counted by bundling and stacking as a block of 6 7s, that may or may not be re-counted in tens as T = 6 7s = 6*7 = 4ten2 = 4Bundle2 = 4B2 = 4.2 tens = 4 2/ten tens, or 42 if we ask a calculator, leaving out the unit and the decimal point.

We see that a decimal point is an inside means to the outside goal of separating parts from bundles. Thus, counting in 3s, 1 single is described by a decimal number or a fraction as 0B1 or 0.1 or 1/3. And, when counted in tens, 1 single becomes 0B1 or 0.1 or 1/10.

Counting in tens, a bundle-of-bundles, a BB, is called a ten-tens or a hundred; and a bundle-of-bundles-of-bundles, a BBB, is called a ten-ten-tens or a thousand. A bundle-of-bundles-of-bundles-of-bundles, a BBBB, is called a wan in Chinese probably describing a standard army unit of hundred hundreds.

Thus, a total of 2 thousands and 3 hundreds and 4 tens and 5 ones, written shortly as T = 2345 with 1s as the unit, can also be written as T = 2BBB3BB4B5 = 234.5 tens = 234.5*10. We see that multiplying with the bundle-number 10 moves the decimal point one place to the right. And reversely, dividing with (or counting in) the bundle-number 10 moves the decimal point one place to the left.

Using cups for the bundles and the bundles-of-bundles etc. allows a total to be reported by bundle-writing, where T = 2345 = 2BBB3BB4B5 tens = 234.5 tens.

Changing the unit to hundreds where H = BB, we get T = 2345 = 2BH3H4B5 = 2BB3B45 hundreds = 23.45 hundreds = 23.45*100. Changing the unit to thousands where M = BBB, we get T = 2345 = 2M3BB4B5 = 2B345 thousands = 2.345 thousands = 2.345*1000. Again, we see that the decimal point moves one place to the right each time we multiply with the bundle-number 10.

With a ten-bundle as a ten-part of a hundred-bundle we can write T = 10 = 0H1P = 0.1 hundreds, again using the decimal point to separate the parts. And with 1 as a ten-part of a ten-part, we can write T = 1 = 0H0P1PP = 0.01 hundreds. So counting in hundred-bundles, T = 345 = 3B4P5PP = 3.45 hundreds = 3.45*100.

Some physical units can be divided in parts. The length 1meter divides into ten ten-parts called a decimeter, dm, that divides into ten ten-parts called a centimeter, cm, that divides into ten ten-parts called a millimeter, mm. Thus T = 2345 mm = 234.5 cm = 23.45 dm = 2.345 m. Or counted in decimeters, T = 23.45 dm = 2B3.4P5PP, again using a decimal point to separate the parts.

So a number can change to a number between 1 and 10 by factoring ten-units in or out:

T = 2.3*75.6 = 2.3*7.56*10 = 17.388*10 = 173.88

T = 0.023*7560 = 2.3/10/10*7.65*10*10*10 = 17.388*10 = 173.88

The multiplication table is an inside means to the outside goal to change the unit from icons to tens by asking e.g. T = 6 7s = ? tens, or T = 6*7 = ?*10.

One way is to memorize the full ten-by-ten table, another way is to reduce it to a small 2-by-8 table containing doubling (and halving) and tripling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus

T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14, or 2*(½B2) = 1B4 = 14.

T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21, or 3*(B-3) = 3B-9 = 21.

T = 6*7 = 6*(½B2) = 3B12 = 4B2 = 42, or 6*7 = 6*(B-3) = 6B-18 = 4B2 = 42.

T = 6*7 = (5+1) * (10-3) = 50 – 15 +10 – 3 = 42, or

T = 6*7 = (10-4)*(10-3) = 100 – 30 – 40 + 12 = 42.

These results generalize to a*(b – c) = a*b – a*c and vice versa; and to (a – d)*(b – c) = a*b – a*c – b*d + d*c.

5	1	
50	10	B
-15	-3	-3

Multiplying often creates an overload to be removed by stepwise bundling

T = 3 57s = 3*57 = 3* 5ten7 = 15ten21 =15ten2ten1 = 17ten1 = 171, or

T = 3 57s = 3*57 = 3* 5B7 = 15B21 = 15B2B1 = 17B1 = 171

T = 13 57s = 13*57 = 13* 5B7 = 65B91 = 74B1 = 741, or

T = 13 57s = 13*57 = 1B3 * 5B7 = 5BB+7B+15B+21 = 5BB22B21 = 5BB24B1 = 7BB4B1 = 741.

The same result comes from using Renaissance-multiplication, also useful with multi-digit multiplication and when multiplying polynomials.

	1	3	
	/ 5	1 / 5	5
	/ 7	2 / 1	7
7	4	1	

	2a	-3b	
	8a^2	-12ab	4a
	10ab	-15b^2	+5b
8a^2	-2ab	-15b^2	

*Renaissance-mult. showing that 13*57 = 741 and that (2a-3b)*(4a+5b) = 8a^2-2ab-15b^2*

Creating and removing overloads also applies for decimal numbers as 523.47 = 5BB2B3.4P7PP.

T = 6* 523.47 = 6* 5BB2B3.4P7PP = 30BB12B18.24P42PP = 30BB12B18.28P2PP = 30BB12B20.8P2PP = 30BB14B0.8P2PP = 3140.82

Or, with bundle-writing: T = 6* 523.47 = 6* 5]2]3.4]7] = 30]12]18.24]42] = 30]12]18.28]2] = 30]12]20.8]2] = 30]14]0.8]2] = 3140.82

The same when multiplying multi-digit numbers:

T = 2.3*75.6 = 2.3P * 7B5.6P = 14B10.12P + 21BP+15P+18PP = 14B + (10+21) + (12+15)P + 18PP = 14B31.27P18PP = 7B3.8P8PP= 173.88 since BP = 1

The same result comes from using Renaissance multiplication also useful with many-digit multiplication and multiplying polynomials.

		2	.3	
	/	1 / 4	2 / 1	7
	/	1 / 0	1 / .5	5
	/	1 / .2	.1 / 8	.6
1	7	3	.8	8

The same using B for bundles and P for parts, and where 1BP = 1:

T = 2.3*75.6 = 75.6*2.3 = 7B5.6P * 2.3P

		7B	5	6P	
		14B	10	12P	2
		21	15P	18PP	3P
	14B	31	27P	18PP	
	14B	31	28P	8PP	
	14B	33	8P	8PP	
	17B	3	8P	8PP	
1BB	7B	3	8P	8PP	
1	7	3.	8	8	

Difference-Research Presentation

My own plenary presentation (Tarp, 2017) was called ‘Difference-Research Powering PISA Performance: Count and Multiply before you Add’. Seeing poor PISA performance as the result of 50 years of low-performing Mathematics Education Research, I asked if this could be different.

First I talked about different education, comparing two types of classrooms: Half-year self-chosen blocks in North America versus multi-year forced lines in Continental Europe.

Then I talked about different kinds of mathematics, comparing bottom-up Many-based ‘Many-matics’ from below with top-down Set-based ‘meta-matics’ from above.

Next, I pointed to ancient Sophism, Renaissance natural science, and (post)modern existentialism as the inspiration for difference-research searching for differences making a difference.

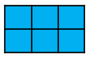
Finally, I talked about a different mathematics education, showing the beauty of the simplicity of mathematics: To master Many, count and re-count and multiply before you add; and when you add forwards & reverse, add block-numbers next-to & on-top, and add per-numbers and fractions by their areas, i.e. by calculus present in both primary and middle and high school.

Inspired by The Greek Sophist saying ‘Beware of choice masked as nature’, I warned against a Goal Displacement in mathematics education, occurring when a means becomes the goal; and unmasking means masked as goals is what difference-research is aiming at.

As to the main finding of difference-research, I showed the following slide unveiling the simplicity of mathematics when presented as tales of Many:

Difference-Research, Main Finding: The Simplicity of Math – Math as Tales of Many

Meeting Many we ask: 'How Many in Total'



- To answer, we math. *Oops, sorry, math is not an action word but a predicate.*
- Take II. To answer, we **Count & Add**. And report with Tales of Many (Number-Language sentences): T = 2 **3s** = 2*3

Three ways to Count: CupCount & ReCount & DoubleCount

- CupCount gives units. ReCount changes units. Double-count bridges units by per-numbers as 2$/3kg

Recount to & from tens gives Multiplication & Equations, coming before Addition

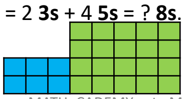
- To tens: T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens** . From tens: T = ? **7s** = u*7 = 42 = (42/7)*7 = 6 **7s** (ReCount-Formula)

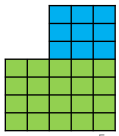
Counting gives variable or constant unit- or per-numbers, to be Added in 4 ways

- Addition & multiplication unites variable & constant unit-numbers.
- Integration & power unites variable & constant per-numbers.

Adding NextTo & OnTop roots Early Childhood Calculus & Proportionality

- EarlyChildhood-Calculus: T = 2 **3s** + 4 **5s** = ? **8s**. EarlyChildhood-Proportionality: T = 2 **3s** + 4 **5s** = ? **5s**





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As to the main warning of difference-research, the following slide shows the 3x3 goal displacements in mathematics education in primary, middle and high school:

Difference-Research, Main Warning: The 3x3 Goal Displacements in Math Education

Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
	Equations	Could: be introduced in primary as recounting from ten-bundles to icon-bundles; and as reversed on-top and next-to addition Instead: Defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra
	Proportionality	Could: be introduced in primary as recounting in another unit when adding on-top; be double-counting producing per-numbers Instead: defined as linear functions, or as multiplicative thinking supporting the claim that fractions and ratios are rational numbers
High	Trigonometry	Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.
	Functions	Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate Instead: are introduced as set-relations where first-component identity implies second-component identity
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

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As to a different mathematics, the following slide shows the beauty of the simplicity of mathematics in 8 areas:

20. Different Mathematics

The Beauty of the Simplicity of Mathematics

21. The Goal & Means of Mathematics Education
22. Totals as Blocks. Digits as Icons. Operations as CupCounting Icons
23. ReCounting gives Proportionality & Multiplication & Equations
24. Multiplication tables simplified by ReCounting
25. DoubleCounting in different & same units creates PerNumbers & Fractions
26. Geometry: Counting Earth in HalfBlocks
27. Once Counted, Totals can be Added. But counting and double-counting gives 4 number-types (constant & variable unit-numbers & per-numbers) to add in 4 ways
28. How Different is the Difference? Set-based versus Many-based Mathematics

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As to the goals and means of mathematics education, the following slide shows the difference between the Set-based top-down tradition and the Many-based bottom-up difference:

21. Different Mathematics

The Goal and Means of Mathematics Education

The Set-based Top-Down Tradition:

- Mathematics exists as a collection of well-proven statements about well-defined concepts, all derived from the mother concept SET
- Mathematics is surprisingly useful to modern society
- Consequently, mathematics must be taught and learned

The Many-based Bottom-Up Difference:

- Many exists; to master Many we develop a number-language with Tales of Many, a 'ManyMatics'.
- **Many-matics**, defining concepts from below as **abstractions from examples**, is a more successful means to the goal of mastering Many than
- '**Meta-matics**' defining concepts from above as **examples from abstractions**

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The following slide compares the Set-based top-down tradition and the Many-based bottom-up difference:

28a. Different Mathematics

How Different is the Difference?

Set-based Math versus Many-based Math

	SET-based Tradition	Many-based Difference
Goal/Means	Learn Mathematics / Teach Mathematics	Learn to master Many / Math as Tales of Many
Digits	Symbols as letters	Icons with as many sticks as they represent
Numbers	Place-value number line names. Never with units	A union of blocks of stacked singles, bundles, bundle-bundles etc. Always with units
Number-types	Four types: Natural, Integers, Rational, Real	Positive and negative decimal numbers with units
Operations	Mapping from a set-product to the set	Counting-icons: /,*,-,+ (bundle, stack, remove, unite)
Order	Addition, subtraction, multiplication, division	The opposite
Fractions	Rational numbers, add without units	Per-numbers, not numbers but operators needing a number to become a number, so added by integration
Equations	Statement about equvalent number-names	Recounting from tens to icons, reversing operations
Functions	Mappings between sets	Number-language sentences with a subject, a verb and a predicate
Proportionality	A linear function	A name for double-counting to different units
Calculus	Differential before integral (anti-differentiation)	Integration adds locally constant per-numbers.

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Finally, a slide showed the main parts of a curriculum in 'ManyMath' seeing mathematics as a natural science about the physical fact Many

28b. Different Mathematics

Main Parts of a ManyMath Curriculum

Primary School – respecting and developing the Child's own 2D NumberLanguage

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- CupCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: / x - +

Middle school – integrating algebra and geometry, the content of the label math

- DoubleCounting produces PerNumbers as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always so length becomes change and vv.

High School – integrating algebra and geometry to master CHANGE

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

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The PowerPoint presentation was supplemented with a paper (Tarp, 2017) carrying the same title, describing in detail how PISA performance can improve in three ways: by a different macro-curriculum from class one, by remedial micro-curricula when a class is stuck, and by a STEM-based (Han et al, 2014) core-curriculum for outsiders. The next three chapters are extracts from the paper.

How to Improve PISA Performance

Improving PISA performance means improving mathematics learning which can be done by observing three basic facts about our human and mammal and reptile brains.

The human brain needs meaning, so what is taught must be a meaningful means to a meaningful outside goal, mastery of Many; thus mathematics must be taught as ‘Many-matics’ in the original Greek sense as a common name for algebra and geometry both grounded in an motivated by describing Many in time and space; and not as ‘meta-matism’ mixing ‘meta-matics’, defining concepts from above as examples of internal abstractions instead of from below as abstractions from external examples, with ‘mathe-matism’, true inside but seldom outside classrooms as adding numbers without units.

The mammal brain houses feelings, positive and negative. Here learning is helped by experiencing a feeling of success from the beginning, or of suddenly mastering or understanding something difficult.

The reptile brain houses routines. Here learning is facilitated by repetition and by concreteness: With mathematics as a text, its sentences should be about subjects having concrete existence in the world, and having the ability to be handled manually according to Piagetian principle ‘through the hand to the head’.

Also, we can observe that allowing alternative means than the tradition makes it not that difficult to reach the outside goal, mastery of many. Meeting Many, we ask ‘How many in total?’ To get an answer we count and add. We count by bundling and stacking and removing the stack to look for unbundles leftovers. This gives the total the geometrical form of a collection of blocks described by digits also having a geometrical nature by containing as many sticks as they represent. Counting also includes recounting in the same or in a new unit; or double-counting to produce per-numbers. Once counted, totals can be united or split, and with four kinds of numbers, constant and variable unit-numbers and per-numbers, there are four ways to unite: addition, multiplication, power and integration; and four ways to split: subtraction, division, root/logarithm and differentiation.

Thus, the best way to obtain good PISA performance is to replace the traditional SET-based curriculum with a different Many-based curriculum from day one in school, and to strictly observe the warning: Do not add before totals are counted and recounted – so multiplication must precede addition. However, this might be a long-term project. To obtain short-term improvements, difficult parts of a curriculum where learners often are stuck might be identified and replaced by an alternative remedial micro-curriculum designed by curriculum architecture using difference-research and sociological imagination. Examples can be found in the above chapter ‘Examples of difference-research’.

Finally, in the case of teaching outsiders as migrants or adults or dropouts with no or unsuccessful educational background, it is possible to design a STEM-based core curriculum as described above allowing the outsiders become pre-teachers and pre-engineers in two years. Thus, applying sociological imagination when meeting Many without predicates forced upon it, allows avoiding repeating the mistakes of traditional mathematics.

The Tradition’s 3x3 mistakes

Choosing learning mathematics as the goal of teaching mathematics has serious consequences. Together with being set-based this makes both mathematics education and mathematics itself meaningless by self-reference. Here a difference is to accept that the goal of teaching mathematics is mastering Many by developing a number-language parallel to the word-language; both having a meta-language, a grammar, that should be taught after the language to respect that the language roots the grammar instead of being an application of it; and both having the same sentence structure with a subject and a verb and a predicate, thus saying ‘T = 2*3’ instead of just ‘2*3’.

This goal displacement seeing mathematics as the goal of mathematics education leads to 3x3 specific mistakes in primary, middle and high school:

In primary school, numbers are presented as 1dimensional line numbers written according to a place value convention; instead of accepting that our Arabic numbers like the numbers children bring to school are 2dimensional block numbers. Together with bundle-counting and bundle-writing this gives an understanding that a number really is a collection of numbers counting what exists in the world, first inside bundles and outside unbundled singles, later a collection of unbundled and bundles and bundles of bundles etc.

Furthermore, school skips the counting process and goes directly to adding numbers without considering units; instead of exploiting the golden learning opportunities in counting and recounting in the same or in another unit, and to and from tens. This would allow multiplication to be taught and learned before addition by accepting that 4*7 is 4 7s that maybe recounted in tens as T = 4 7s = 2.8 tens = 28, to be checked by recounting 28 back to 7s, T = 28 = (28/7)*7 = 4*7 = 4 7s, using the recount-formula reappearing in proportionality, trigonometry and calculus. And giving division by 7 the physical meaning of counting in 7s.

Finally, addition only includes on-top addition of numbers counted in tens only and using carrying, a method that neglects the physical fact that adding or subtracting totals might crate overloads or underloads to be removed by recounting in the same unit. And neglecting the golden learning opportunities that on-top addition of numbers with different unit roots proportionality, and that next-to addition roots integration, that reversed roots differentiation thus allowing calculus to be introduced in primary school.

In middle school, fractions are introduced as numbers that can be added without units thus presenting mathematics as ‘mathematism’ true inside but seldom outside classrooms. Double-counting leading to per-numbers is silenced thus missing the golden learning opportunities that per-numbers give a physical understanding of proportionality and fractions, and that both per-numbers and fractions as operators need numbers to become numbers that as products add as areas, i.e. by integration.

Furthermore, equations are presented as open statements expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*u = 8	an open statement about two equivalent number-names
(2*u)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(u*2)*(1/2) = 4	since multiplication is commutative
u*(2*(1/2)) = 4	since multiplication is associative
u*1 = 4	by definition of an inverse element
u = 4	by definition of a neutral element

The alternative sees an equation as another name for reversing a calculation that stops because of an unknown number. Thus the equation ‘2*u = 8’ means wanting to recount 8 in 2s: 2*u = 8 = (8/2)*2, showing that u = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the ‘opposite side&sign’ method. A method that allows the equation ‘20/u = 5’ to be solved quickly by moving across twice; 20 = 5*u’ and 20/5 = u’, or more thoroughly by recounting 20 = (20/u)*u = 5*u = (20/5)*5 = 4*5, so u = 4.

Finally, middle school lets geometry precede coordinate geometry, again preceding trigonometry; instead of respecting that in Greek, geometry means to measure earth, which is done by dividing it into triangles again divided into right triangles. Consequently, trigonometry should come first as a mutual recounting of the sides in a right triangle. And geometry should be part of coordinate geometry allowing solving equations predict intersection points and vice versa, thus experiencing repeatedly that the strength of mathematics is the fact that formula predict.

In high school, a function is presented as an example of a set-relation where first-component identity implies second-component identity; and the important functions are polynomials with linear functions preceding quadratic functions; instead of respecting that a function is a name for a formula with two unspecified numbers, again respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. As to polynomials, they should be introduced as the number-formula containing the different forms of formulas for constant change, T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x. Consequently, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d and parallel to the other examples of constant change. Thus emphasizing the double nature of formulas that the can predict both level and change.

Furthermore, differential calculus is presented before integral calculus, presenting an integral as an antiderivative; instead of postponing differential calculus until after integral calculus is presented as adding locally constant per-numbers, i.e. as a natural continuation of adding fractions as piecewise constant per-numbers in middle school and next-to addition of blocks in primary school. Only in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant change.

Finally, high school presents algebra as a search for patterns, instead of celebrating the fact that calculus completes the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Uniting/ *splitting*	Variable	Constant
Unit-numbers	T = a + n *T – a = n*	T = a*n *T/n = a*
Per-numbers	T = ∫ a dn *dT/dn = a*	T = a^n, *loga(T) = n* *n√T = a*

Remedial Curricula

A remedial micro-curriculum might be relevant whenever learning problems are observed. Since you never get a second chance to create a first impression, especially remedial curricula in primary school are important to prevent mathematics dislike.

Primary school. Here problems might be eased by

- with digits, using a folding ruler to observe that a digit contains as many sticks or strokes as it represents if written in a less sloppy way.
- with counting sequence, using sequences that shows the role of bundling when counting to indicate that a given total as e.g. seven can be named in different ways: 7, .7, 0.7, bundle less 3, ½bundle&2, etc.

- with recounting, using a cup and 5 sticks to experience that at total of 5 can be recounted in 2s in three ways: with an overload, normal, or with an underload: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s, or T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s if using decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles.
- when learning multiplication tables, letting 3*7 mean 3 7s recounted in tens, i.e. a block that when increasing its width must decrease its height to keep the total unchanged.
- when learning multiplication tables, beginning by doubling and halving and tripling; and to recount numbers using half-ten and ten as e.g. 7 = half-ten&2 = 10less3 so that 2 times 7 is 2 times half-ten&2 = ten&4 = 14, or 2 times 10less3 = 20 less 6 = 14.
- when multiplying, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336, or T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336
- when dividing, using bundle-writing to create overloads or underloads according to the multiplication table, as e.g. T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48
- when subtracting, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38
- when adding, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

Middle school. Here problems might be eased by keeping algebra and geometry together and by re-describing

- proportionality as double-counting in different units leading to per-numbers
- fractions as per-numbers coming from double-counting in the same unit
- adding fractions as per-numbers by their areas, i.e. by integration
- solving equations as reversing calculations by moving to the opposite side with the opposite calculation sign

High school. Here problems might be eased by re-describing

- functions as number-language sentences, i.e. formulas becoming equations or functions with 1 or 2 unspecified numbers
- calculus as integration preceding differentiation
- integration as adding locally constant per-numbers
- pre-calculus and calculus as describing constant and variable predictable change; and statistics as post-dicting non-predictable change allowing it to be predicted by confidence intervals.

Conclusion

Mathematics education is a social institution with one goal and many means; and as such running the risk of a goal displacement where the original goal becomes a means to a means becoming the goal instead, seduced by a persuasive logic: Mathematics is highly applicable to the outside world, but of course, mathematics must be learned before it can be applied. So of course, mathematics, as defined by the mathematicians, is the goal, and outside applications may be included as a means to motivate the students for learning mathematics even if it is a hard subject demanding a serious commitment, as witnessed by poor PISA results even after 50 years of mathematics education research.

To this compelling argument, difference-research, searching for differences making a difference, will ask: maybe it is the other way around. Maybe there are several forms of mathematics and has been so during its long history, all leading to the same outside goal described in ancient Greece as four knowledge areas about Many in time and space, together labeled ‘mathematics’.

So maybe mathematics becomes simple and easy to learn for all, if once again it accepts itself as a means to the outside goal, mastery of Many, accessible through a Many-matics answering the basic question ‘How many in total?’ by number-language sentences with a subject and a verb and a predicate in the form of a calculation uniting constant or variable unit- or per-numbers.

Therefore, if mathematics for all is a social goal, society must remind mathematics about its role as a means serving the outside goal, mastery of Many, by constantly asking the basic question from the fairy tale Cinderella: Are there other alternatives outside the saloons of present correctness? This precisely is the aim of difference-research searching for differences making a difference. This entails two tasks, to find differences and to test them in a classroom. In this paper only the first task was conducted. In doing so, hidden differences were located within:

- Number sequences. The tradition counts the fingers on a hand as 1, 2, 3, 4, 5. A difference is to count 1, 2, 3, 4, B (bundle); or 1, 2, 3, B less 1, B; or B less 4, B-3, B-2, B-1, B. Emphasizing the word ‘bundle’ allows showing the nature of counting as bundling, might make a difference in micro-studies.
- Multiplication. The tradition says that 6*7 is 42. A difference is to say that 6*7 is 6 7s that may stay as it is or be recounted in another unit. If recounted in tens, 6 7s is 4.2 tens, shown geometrically as a block where an increase of the base from 7 to ten means a decrease of the height from 6 to 4.2 to keep the total unchanged. Multiplication thus becomes an inside means for two outside goals, to stack bundles and to change the unit to tens. Presenting multiplication before addition as a means to stack and change unit might make a difference in micro-studies.
- Multiplication tables. The tradition says that 6*7 is 42, which is a part of a ten-by ten multiplication table. A difference is to include the total behind and to recount 6 and 7 by saying T = 6*7 = 6 7s to be recounted in tens = (ten less 4)*(ten less 3) = tenten, less 4ten, less 3ten, and 4 3s = 100 – 40 – 30 + 12 = 42; or T = 6*7 = (½ten & 1)*(ten less 3) = ½tenten, ten, less ½ 3ten, less 3 = 50 + 10 – 15 – 3 = 42; or with bundle-writing, T = 6*(1B-3) = 6B-18 = 4B2 = 42; or counting in 5s, T = 6*(½B2) = 3B12 = 4B2 = 42. Allowing numbers to be recounted before multiplied might make a difference in micro-studies.
- Multiplying decimal numbers. The tradition says that multiplying decimal numbers is like multiplying numbers, only keeping track of the place of the decimal point. A difference is to see both factors as numbers between 1 and 10 with ten-units factored in or out. Another difference is to use bundle-writing and allow overloads in the different cups by gradual re-counting. Presented in this way it might make a difference in micro-studies.
- Division. The tradition says that 9/4 is 9 shared by 4 giving each the mixed number 2 ¼ . A difference is to say that 9/4 is 9 counted in 4s giving a total of T = (9/4)*4 = 2*4 +1 = 2 ¼ 4s = 2.1 4s. And to realize that sharing 9 between 4 involves two take-steps. First 4-bundles are taken away from 9 to re-count 9 in 4s; then, in a 4-bundle, each takes 1 part of 4, i.e. ¼. Sharing thus does not root the traditional division-understanding, instead sharing roots both counting in icons and taking fractions. Presented in this way it might make a difference in micro-studies.
- Fractions. The tradition says that the fraction 3/5 is a rational number describing 3 as a part of 5. A difference is to say that the fraction 3/5 is a per-number coming from double-counting in the same unit; and as per-numbers, fractions are not numbers but operators needing a number to become a number thus adding by their areas as in calculus. Using the fingers on both hand, you quickly learn about 2/5 of 5 and 2/5 of ten. Presenting fractions as per-numbers occurring in sharing situations might make a difference in micro-studies
- Pythagoras. The tradition says that the Pythagoras Theorem allows calculating the hypotenuse from the two other sides in a right-angled triangle. A difference says that parallel distances add

directly but perpendicular distances add by their areas. Presented in this way it might make a difference in micro-studies.

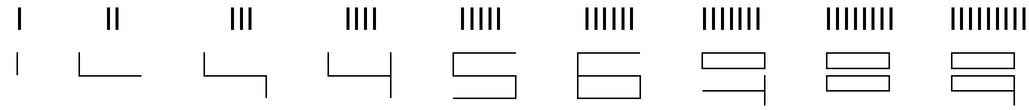
When teaching children to obtain mastery of Many, two options are available.

One option is to see mathematics as an unavoidable means that therefore might be a goal as well, leading to traditional teaching of line-numbers to be added, subtracted, multiplied and divided; and to fractions as rational numbers to be added directly without units, etc.

Another option is to build on what the children already know about mastering Many from being exposed to Many for several years before beginning school. Asking ‘How old will you be next time?’ a 3year old child answers ‘four’ with four fingers shown; but reacts to four fingers held together 2 by 2 with a ‘That is not four, that is two twos.’

So children come to school with 2dimensional number-blocks where all numbers have a unit as with the Arabic numbers they are supposed to learn, T = 345 = 3*10^2 + 4*10 + 5*1.

This allows school to practice guided discovery so the child can see that the digits are, not symbols as letters, but icons with as many strokes as they represent if written less sloppy, thus allowing the child to discover the transition from 4 1s to 1 4s, that can serve as a bundle when counting and re-counting.



1 2 3 4 5 6 7 8 9

Then school can practice double representation of totals using Lego blocks and bundle-writing with full number-language sentences as T = 2*3; and practice re-counting in the same unit to create or remove overloads, allowing the child to see that a total can be counted in different ways, as e.g. T = 7 = 5&2 = ten less 3 = ½bubndle & 2 = bundle less 3; or T = 12 = bundle & 2 = 2 ½bundles & 2 = 1 ½bundle & 7 = 3 ½bundles less 3 = 2 left (twelve = two left, ‘twe levnet’ in Viking language)

Then school can practice re-counting in a different unit so the child can experience the operations as means for a calculator-prediction using the re-count formula T = (T/B)xB, saying that ‘from T, T/B times B can be taken away’, presenting division as a broom wiping away the bundles, and multiplication as a lift stacking the bundles in a block to be removed by subtraction to count the unbundled singles: Asking ‘7 is how many 3s’, first we take 3s a number of times, predicted by 7/3 as 2. Then we take away the stack of 2 3s to count the leftovers, predicted by 7 – 2x3 as 1:

T = 7 = ? 3s. First 7/3 gives 2.some; next 7 – 2x3 gives 1; so T = 7 = 2.1 3s = 2 1/3 3s

Then school can practice recounting between icon-bundles and ten-bundles. Recounting in ten-bundles allows the multiplication table to be built slowly by beginning with doubling and halving and tripling. And recounting from ten-bundles to icon bundles allows the child to solve equation by recounting. So to answer the question ‘how many 8s is 24’ we juts re-count 24 in 8s to get the answer 3, thus moving a number to the opposite side with opposite sign:

?*8 = 24 = (24/8)*8 = 3*8 = 3 8s; so ? = 24/8 = 3

Then school can practice double-counting to create per-numbers bridging countings in different units, and becoming fractions if the units are the same.

Finally, once counted and re-counted, totals can add; either on-top after being re-counted to the same unit, later called proportionality, or next-to as areas also used when adding per-numbers and fractions, later called integral calculus. And then addition can be reversed, later called equations and differential calculus.

Thus, if the school allows children to develop their own number-language they will learn core subjects as proportionality and calculus and solving equations in the first year or two.

So why not celebrate the beauty of the simplicity of the child's own mathematics? Why replace the child's own 'Many-matics' with the school's traditional 'meta-matism,' mixing 'meta-matics', defining concepts as example of abstractions instead of as abstraction form examples, with 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units meet countless counterexamples: T = 2weeks + 3days is not 5 but 17 days; in contrast to this, T = 2*3 = 6 says that 2 3s can be re-counted as 6 1s which is universally true by including the unit 3.

Of course, an ethical issue occurs when depriving the child of its natural number-language, and forcing upon the child an alien language consisting of self-referring definitions and statements with uncertain validity.

In the second enlightenment republic France, Bourdieu calls this 'symbolic violence'; and Foucault, seeing the school as a 'pris-pital' mixing power techniques from a prison and a hospital, would warn against curing children not properly diagnosed, and against accepting self-reference when diagnosing (Bourdieu, 1970. Foucault, 1995).

So, wanting mathematics education to be for all, Many-based Many-matics from below should be preferred to set-based meta-matism from above.

This is how the Count&Multiply&Add curriculum was designed to allow children to develop their own number-language by the natural tasks of counting and re-counting and double-counting and multiplying before performing on-top and next-to addition and reversed addition.

So, a conjecture to be tested and researched is: PISA-like testing will improve if letting a Many-based Bottom-up Count&Multiply&Add curriculum replace the traditional Set-based top-down curriculum presenting 1dimensional numbers to be treated by addition firsts, then subtraction, then multiplication, then division leading to fractions added without units.

Of course, it will take many years to see the effect of a full curriculum, so in the meantime micro-curricula can be designed and tested via intervention research. Or the full curriculum can be tested as a 1year 'migrant-mathematics' course allowing young male migrants coming to Europe in high numbers to acquire competence as a pre-teacher or a pre-engineer to return help develop or rebuild their homeland after two years (Tarp, 2017).

As to online in-service teacher education, the MATHeCADEMY.net has been designed to teach teachers to teach mathematics as Many-matics, a natural science about Many, using the CATS-approach, Count&Add in Time&Space, partly described in DrAlTarp YouKu and YouTube videos; and organizing learners in groups of 8 using PYRAMIDeDUCATION.

Recommendation

With only a small percentage of mathematics education research having value to the classroom we must ask if research can be conducted differently. Here difference-research is a difference that might make a difference. Difference-research goes to the classroom to observe problems, allowing it to ask its basic question: Find a difference that makes a difference. Seeing education as preparing students for the outside world leads to accepting mathematics as it arose historically, an inside means to an outside goal, mastery of Many. This allows using intervention research to construct a different micro-curriculum to be tested and adapted in the classroom to see if it makes a difference.

Becoming a difference-researcher is straight forward. You begin as a teacher wanting to teach mathematics for all. At the master level, you read conflicting theory within sociology, philosophy and psychology. In sociology, you focus on the difference between patronizing and enlightening societies as described e.g. by Bauman and Giddens. In philosophy, you focus on the difference between a Platonic top-down view and a sophist bottom-up view as described e.g. by existentialism and post-structuralism. In Psychology, you focus on the difference between mediation and discovery as described e.g. by Vygotsky and Piaget. At the research level, you focus on the difference between testing existing theory and generating new theory as described e.g. by top-down deductive operationalization and a bottom-up inductive grounded theory. And you conduct intervention research by deigning different micro-curricula inspired by thinking differently within sociology, philosophy and psychology.

So, to improve mathematics education worldwide, China could educate ten-ten difference-researchers to spread along the coming new silk road where they each educate ten difference-researchers to help the local population implement a mathematics education for all, rooted in everyday experiences, thus allowing all to enjoy the beauty of the simplicity of mastering many.

With 2017 as the 500year anniversary for Luther's 95 theses, the recommendation can be given as 12 or 20 theses (Tarp, 2017), here reduced to 7 theses:

- To master Many, count and multiply before you add
- Counting and recounting give block-numbers and per-numbers, not line-numbers
- Adding on-top and next-to roots proportionality and integration, and equations when reversed
- Beware of the conflict between bottom-up enlightening and top-down forming theories.
- Institutionalizing a means to reach a goal, beware of a goal displacement making the institution the goal instead
- To cure, be sure, the diagnose is not self-referring
- In sentences, trust the subject but question the rest

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19. Sixteen Proposals for the 8th ICMI-East Asia Regional Conference on Mathematics Education

Theme of the Conference

"Flexibility in Mathematics Education" has been chosen as the theme of the conference. Flexibility is highly related to creativity, multiplicity, and adaptation. In the current era, rapid changes in economy, environment and society have been facilitated by the rapid development of technology and engineering. Flexibility in mathematical thinking, problem solving, teaching methods, evaluation, teacher education and mathematics education research is a key to empowering learners, teachers, educators and researchers to tackle the complexity and uncertainty, and to giving them the capacity and motive to change in the innovative era.

The Topic Study Group themes are:

TSG 1: Flexibility in Mathematics Curriculum and Materials

TSG 2: Flexibility in Mathematics Classroom Practices

TSG 3: Flexibility in Mathematics Assessment

TSG 4: Flexibility in Mathematics Teacher Education and Development

TSG 5: Flexibility in the Use of ICT in Mathematics

TSG 6: Flexibility in the Use of Language and Discourse in Mathematics

TSG 7: Flexibility in Mathematics Learning

The Simplicity of Mathematics Revealing a Core Curriculum (TSG 01)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add. First we take away bundles, thus rooting division; then we stack the bundles, thus rooting multiplication; then we move the stack away to look for singles, thus rooting subtraction; finally we answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences.

A calculator predicts the result by the recount-formula T = (T/B)*B saying 'from T, T/B times, B can be taken away', thus rooting fractions and decimals to describe the singles, e.g. T = 7 = 2 1/3 3s = 2.1 3s. Recounting in another unit roots proportionality. Changing units between icons and tens roots multiplication tables and equations.

Once counted, totals add on-top after being recounted in the same unit, again rooting proportionality; or totals add next-to, thus rooting integration. Reversing on-top and next-to addition roots equations and differentiation.

Double-counting in different physical units creates per-numbers, again rooting proportionality, where per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, again rooting integration.

Now in a rectangle split by a diagonal, recounting the side mutually creates the per-numbers sine, cosine and tangent. And traveling in a coordinate system, parallel distances add directly whereas perpendicular distances add by their squares. Recounting the y-change in the x-change creates linear formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Looking at constant and variable predictable change roots pre-calculus and calculus; and looking at unpredictable change roots statistics to post-dict the behavior of numbers by a mean and a deviation, again allowing probability to predict, not numbers but intervals. (Tarp, 2017)

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A STEM-based Math Core-Curriculum for migrants (TSG 01)

Seeing 'Mastery of Many' as the outside goal, we can construct a core math curriculum based upon exemplary situations of Many in a STEM context, having a positive effect on learners with a non-standard background (Han et al, 2014), thus allowing young male migrants to help their original countries as pre-teachers or pre-engineers.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles, mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells, and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or variable. To master plane or spatial shapes, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent. So, a core STEM-based curriculum could be about formulas controlling cycling water cycles (Tarp, 2017).

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50 years of Sterile Mathematics Education Research, Why? (TSG 01)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

PISA scores still are low after 50 years of research. But how can mathematics education research be successful when its three words are not that well defined? Mathematics has meant different things in its 5000 years of history, spanning from a natural science about Many to a self-referring logic.

Within education, two different forms exist at the secondary and tertiary level. In Europe, education serves the nation’s need for public servants through multi-year compulsory classes and lines. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks with one-subject teachers.

Academic articles can be written at a master-level exemplifying existing theories, or at a research-level questioning them. Also, conflicting theories create problems as within education where Piaget and Vygotsky contradict each other by saying ‘teach as little and as much as possible’.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and if research is following traditions or searching for new discoveries. So to answer the question ‘How to improve mathematics education research’, first we must make the three words well defined by asking: What is meant by mathematics, and by education, and by research? Answers will be provided by the German philosopher Heidegger (1962), asking ‘what is ‘is’?’

It turns out that, instead of mathematics, schools teach ‘meta-matism’ combining ‘meta-matics’, defining concepts from above as examples of abstractions instead of from below as abstractions from examples; and ‘mathe-matism’ true inside but seldom outside class, such as adding fractions without units, where 1 red of 2 apples plus 2 red of 3 gives 3 red of 5 and not 7 red of 6 as in the textbook teaching 1/2 + 2/3 = 7/6.

So, instead of meta-matism, teach mathematics as ‘many-math’, a natural science about Many, in self-chosen half-year blocks.

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The Center of Math Education: Its Sentences or its Subjects? (TSG 02)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal?

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences. However, a controversy exists as to what is the center of mathematics education, the predicate 2*3 or the subject T.

Seeing reproducing textbook knowledge as the goal, Vygotsky points to good teaching as the best means and recommends teaching as much as possible. Seeing individual sentences about the outside fact Many as the goal, Piaget points to good guidance as the best means and recommends teaching as little as possible.

Thus, where a Vygotsky class follows a textbook strictly, a Piaget class brings the subject of its sentences to the class to allow the learner to create individual sentences to be adapted through sharing, thus respecting Many as the outside goal of mathematics. Which resonates with Heidegger (1962) saying: In a sentence, the subject exists, but the rest might be gossip.

Flexibility in a primary classroom thus means using full sentences where the total exists as sticks and where the predicate can be flexible by using bundle-counting to count inside bundles and outside singles, e.g. T = IIIIIII = III III I = 2B1 3s or T = 1B4 3s = 3B-2 3s if allowing overloads and underloads outside the cup; which becomes useful when multiplying, T = 5* 67 = 5* 6B7 = 30B35 = 33B5 = 335; and when dividing: T = 335 /5 = 33B5 /5 = 30B35 /5 = 6B7 = 67.

The 'geometry and algebra, always together, never apart' principle allows learners to develop a flexible double number-concept, seeing the total T = 2*3 geometrically as number-block with 2 3s, that may or may not be recounted as 6 1s. Recounting 4*5 = 2 tens says that doubling its width, a block of 4 5s must halve its height to keep the total unchanged. Likewise, equations become tangible when recounting from tens to icons.

That totals must be counted and recounted before they add allows multiplication to precede addition.

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DoubleCounting roots Proportionality - and Fractions and Percentages as Per-Numbers (TSG 02)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

A school sees fractions as goals instead of means making a class stuck. Having heard about difference-research and per-numbers (Tarp, 2017), the teacher says: Time out. Next week, no fractions. Instead we do double-counting. First counting: 42 is how many 7s? The total T = 42 = (42/7)*7 = 6*7 = 6 7s. Then double-counting: Apples double-counted as 3 $ and 4 kg have the per-number 3$ per 4 kg, or 3$/4kg or ¾ $/kg. Asking how many $ for 10kg, we recount 10 in 4s, that many times we have 3$: The total T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Asking how many kg for 18$, we recount 18 in 5s, that many times we have 4kg: The total T = 18$ = (18/5)*5$ = (18/5)*4kg = 14.4kg. Double-counting in the same unit gives fractions and percentages as 3 per 4, ¾; and 75 per hundred, 75/100 = 75%.

¾ of 200$ means finding 3$ per 4$, so we recount 200 in 4s, that many times we have 3$: The total T = 200$= (200/4)*4$ gives (200/4)*3$ = 150$. 60% of 250$ means finding 60$ per 100$, so we recount 250 in 100s, that many times we have 60$: The total T = 250$= (250/100)*100$ gives (250/100)*60$ = 150$.

To find 120$ in percent of 250$, we introduce a currency # with the per-number 100# per 250$, and then recount 120 in 250s, that many times we have 100#: The total T = 120$ = (120/250)*250$ = (120/250)*100# = 48#. So 120$/250$ = 48#/100# = 48%. To find the end-result of 300$ increasing with 12%, the currency # has the per-number 100# per 300$. 12# increases 100# to 112# that transforms to $ by the per-number. The total T = 112# = (112/100)*100# = (112/100)*300$ = 336$.

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Assessing Goals Instead of Means (TSG 03)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask ‘How many in Total?’. To answer, we count and add and compare totals; and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences.

Counting includes bundle-counting to separate a total in bundles inside the cup and singles outside; and recounting in the same unit to create an outside overload or underload needed to ease operations, e.g. T = 4*56 = 4* 5B6 = 20B24 = 22B4 = 224.

Recounting in another unit, called proportionality, is predicted by a recount-formula T = (T/B)*B saying ‘from T, T/B times, B can be taken away’, thus rooting fractions and decimals to describe the singles, e.g. T = 7 = 2 1/3 3s = 2.1 3s. Changing units between icons and tens roots multiplication tables and equations.

Once counted, totals add on-top after being recounted in the same unit, again rooting proportionality; or next-to thus rooting integration. Reversing on-top and next-to addition roots equations and differentiation.

Double-counting in different physical units creates per-numbers, becoming fractions if the units are the same. Since per-numbers and fractions are operators needing a number to become a number, they add by their areas, again rooting integration.

In a rectangle split by a diagonal, recounting the side mutually creates the per-numbers sine, cosine and tangent. And traveling in a coordinate system, parallel distances add directly whereas perpendicular distances add by their squares. Recounting the y-change in the x-change creates linear formulas, algebraically predicting geometrical intersection points.

To avoid a goal displacement, assessment should test goals instead of means; and always use totals with units. With proportionality formulas in science as a core root for mathematics, several tasks should include per-numbers, e.g. taken from classical word problems. Numbers without units should be excluded, since adding numbers and fractions without units are examples of ‘mathematism’ true inside but seldom outside classrooms, where claims as 2+3 = 5 meet counterexamples as 2 weeks + 3 days = 1 7days. And where 1 red of 2 apples + 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as taught in school.

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The 2 Core Math Competences, Count & Add, in an e-learning Teacher Development (TSG 04)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal?

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3. Counting and double-counting in two units creates 4 number-types: variable and constant unit- and per-numbers that unite by addition, multiplication, integration and power.

That this simplicity typically is unknown to teachers created the MATHeCADEMY.net, teaching teachers to teach mathematics as 'ManyMath', a natural science about Many suing the CATS-approach: Count & Add in Time & Space. It is a virus academy saying: To learn mathematics, don't ask the instructor, ask Many. The material is question-based.

Primary School. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives 14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

Secondary School. COUNT: How to count possibilities? How to predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? Quantitative Literature, what is that? Does it also have the 3 genres: fact, fiction and fiddle?

PYRAMIDeDUCATION organizes 8 teachers in 2 teams of 4 choosing 3 pairs and 2 instructors by turn. The instructors instruct the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The instructors correct the count&add assignments. In a pair, each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair.

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12 Theses not Taught in Teacher Education (TSG 04)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, mastering Many is the outside goal. As means, we iconize and bundle by digits, operations and formulas, becoming goals if forgetting the real goal.

1. Digits are icons with as many sticks as they represent.

2. A total T can be ‘bundle-counted’ in the normal way or with an overload or underload: T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s.

3. ‘Bundle-writing’ makes operations easy: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

4. Counting T by bundling, T = (T/B)xB = (5/2)x2 = 2.1 2s, shows a natural number as a decimal number with a unit.

5. Operations are icons showing counting by bundling and stacking. -2 takes away 2. /2 takes away 2s. x2 stacks 2s. +2 adds 2 on-top or next-to.

6. A calculator predicts. Asking T = 4 5s = ? 6s, first (4x5)/6 = 3.some; then (4x5) – (3x6) = 2. So T = 4 5s = 3.2 6s

7. Recounting in tens, calculators leave out the unit and misplace the decimal point: T = 3 7s = 3*7 = 21 = 2.1 tens.

8. Recounting from tens, ‘? 7s = 3 tens’, or ‘u*7 = 30 = (30/7)x7’, the answer u = 30/7 is found by ‘move to opposite side with opposite sign’.

9. Adding totals is ambiguous: On-top using proportionality, or next-to using integration?

10. Operations are reversed with reverse operations: With u+3 = 8, u = 8–3; with ux3 = 8, u = 8/3; with u^3 = 8, u = 3√8; with 3^u = 8, u = log3(8); with T1 + u*3 = T2, u = □T/3.

11. Double-counting in different units gives ‘per-numbers’ as 4$/5kg, bridging the two units by recounting: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$

12. Double-counting in the same unit, per-numbers become fractions as operators, needing a number to become a number, thus adding by their areas as integration.

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Difference-Research at Work in a Classroom (TSG 04)

The CTRAS (Classroom Teaching Research for All Students) wants all students to benefit. The 2017 conference contained example of classroom lessons. Difference-research (Tarp, 2017) looks for a different approach based upon outside goals to see if more students benefit. Inspired by Greek sophists looking for hidden differences to unmask choice masked as nature, e.g. means presented as goals, difference-research asks two questions: 'Can this be different – and will the difference make a difference?'

The first task in a grade 5 class was to fill a 3x3 square with the numbers 1-9 so that they add to 15 horizontally, vertically and on both diagonals. Based upon the principle 'algebra & geometry, always together, never apart', the outside goal could be to give symmetry to both, e.g. by applying a counting sequence for the numbers 1-9 that counts the numbers as 'Bundle less or plus' using 5 as the bundle-number: Bundle less 4, B-3, B-2, B-1, B+0, B+1, B+2, B+3, B+4. By its geometry, each sum will contain three numbers, so we can leave out the bundle B and redesign the task to 'add to zero'. Thus, each sum must contain 2 odd numbers, placed in the corners.

The second task in a grade 8 class was to give a geometrical proof of the Pythagoras Theorem. Here an outside goal could be to add travel-distances. If parallel, two distances add or subtract directly. If perpendicular, they add by their squares: 3 steps over plus 4 steps up total 5 steps, since 3^2 + 4^2 = 5^2.

The third task in a grade 3 class was to learn about and apply fractions. Looking for the outside root of fractions we find double-counting in various contexts as e.g. icon-counting, statistics, splitting, per-numbers, changing. Double-counting bent and unbent fingers roots fractions as 2/5 of 5 and 2/5 of 10.

The fourth task in a grade 8 class was to find a formula connecting the number of angles to the angle sum in a polygon. Looking for the outside root of angles we find changing direction under a closed journey with many turns. Thus, the lesson could focus on a paper with three closed journeys with 3 and 4 and 5 turning points labeled from A to E. On the triangle, inserting an extra point P between A and B transforms the triangle ABC into a four-angle APBC where B adds 180 degrees to the angle sum. Pulling P out makes P decrease with what A and B increase, so the angle sum remains added with 180.

A plenary address discussed decimal multiplication in a grade 5 class exemplified by 110*2.54. Here a difference is to see multiplication as shifting units. Here a total of 110 2.54s is to be recounted in tens. Factorizing will show how the ten-units can change place: T = 110*2.54 = 1.1*10*10*2.54 = 1.1*254 = 279.4

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Pre-schoolers and Migrants Predict Recounting by a Calculator (TSG 05)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask ‘How many in Total?’. To answer we count and add. Asking T = 7 = ? 3s, first we take away bundles, thus rooting division iconized as a broom wiping away the bundles; then we stack the bundles, thus rooting multiplication iconized as a lift stacking the bundles; then we move the stack away to look for unbundled singles, thus rooting subtraction iconized as a trace left by the stack; and finally, we answer with a number-language sentence, containing a subject and a verb and a predicate as does word-language sentences.

To have the calculator predict the result we enter ‘7/3’. The answer 2.some tells us that 2 times 3s can be taken away. To look for unbundled singles we stack the 2 3s as 2*3 to be removed, so we enter ‘7 – 2*3’. The answer 1 tells us that 7 can be counted in 3s as 2 3s and 1, written as T = 7 = 2 1/3 3s if the single is placed on-top of the stack counted in 3s, or as T = 7 = 2.1 3s if the single is placed next-to the stack as a stack of unbundled.

This shows that a natural number is decimal number with a unit where the decimal point separates the bundles from the unbundled.

A calculator thus predicts the result by the recount-formula T = (T/B)*B saying ‘from T, T/B times, B can be taken away’.

Recounting in tens means just multiplying. Recounting from tens to icons means asking 30 = ? 6s. Here we use the recount-formula to recount 30 in 6s, T = 30 = (30/6)*6 = 5*6. This shows, that an equation is solved by moving to the opposite side with opposite sign.

The totals 2 3s and 4 5s can add on-top as 3s or 5s, or next-to as 8s. Again, a calculator can predict the result: Entering (2*3+4*5)/8 gives 3.some and then (2*3+4*5) – 3*8 gives 2 so the prediction is T = 3.2 8s.

Also, the recount-formula can bridge units when double-counting has created a per-number as s4$/5kg. Here T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. Likewise with 18$.

References

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Mathematics as a Number-Language Grammar (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra, to measure earth and to reunite numbers. Thus the outside goal of mathematics is to master Many.

Humans describe qualities and quantities with a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, 'This is a chair' and 'The total is 3x4 legs', abbreviated to 'T = 3x4'. Both are affected by the Heidegger (1962) warning: 'In is-sentences, trust the subject but question the predicate'.

Both languages also have a meta-language, a grammar, that describes the language that describes the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3x4' leads to a meta-sentence ''x' is an operation'.

We master outside phenomena through actions, so learning a word-language means learning actions as how to listen and read and write and speak. Likewise, learning the number-language means learning actions as how to count and add. We cannot learn how to math, since math is not an action word, it is a label, as is grammar. Thus, mathematics can be seen as the grammar of the number-language. Since grammar speaks about language, language should be taught and learned before grammar. This is the case with the word-language, but not with the number-language.

Saying 'the number-language is an application of mathematics' implies that 'of course mathematics must be taught and learned before it can be applied'. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

Instead school should follow the word-language and use full sentences 'The total is 3 4s' or 'T = 3x4'. By saying '3x4' only, school removes both the subject and the verb from number-language sentence, thus committing a goal displacement.

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Deconstructing the Vocabulary of Mathematics (TSG 06)

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Humans describe qualities and quantities with a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, 'This is a chair' and 'The total is 3x4 legs', abbreviated to 'T = 3x4'.

Inspired by the Heidegger (1962) warning 'In is-sentences, trust the subject but question the predicate', Derrida (1991) to recommends deconstructing labels by destructing and reconstructing them inspired by the subject itself.

Thus Mathematics could be renamed to Many-matics, Many-math, Many-ology, or number-language. Geometry could be renamed to 'earth-measuring'; and algebra to 'reuniting numbers' according to its Arabic meaning.

Counting could split into its different forms: bundle-counting, using a cup for the bundles; re-counting to change the unit; and double-counting to bridge two units by a per-number.

In division, 'divided between 5' could be renamed to 'counted in 5s'; and 'to multiplied by 3' could be renamed 'to change the unit from 3s to tens' by reshaping the number block, widening the base and shorting the height.

Addition could split into on-top addition using proportionality to change the units, and next-to addition adding by areas as in integration.

Solving equations could be renamed to reversing calculations.

Fractions could be renamed to per-numbers coming from double-counting in the same unit.

Proportionality could be renamed changing units; and proportional could be renamed to 'the same except for units'.

Linear and exponential functions could be renamed change by adding and multiplying.

A function y = f(x) could be renamed to a formula or a 'number-language sentence'.

A root and a logarithm could be renamed to a factor-finder and a factor-counter.

Continuous could be renamed locally constant, and differentiable could be renamed locally linear

Integration could be renamed added by area.

In a right-angled triangle, the hypotenuse could be renamed the diagonal.

Finally, mathematical models could be named quantitative literature, having the same genres as qualitative literature, fact and fiction and fiddle (Tarp, 2017).

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Will Difference-Research Make a Difference? (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015). So maybe it is time for a different research approach, e.g. Difference-Research (Tarp, 2017).

Inspired by Greek sophists looking for hidden differences to unmask choice masked as nature, e.g. means presented as goals, difference-research asks two questions: 'Can this be different – and will the difference make a difference?' The philosophical background is the Heidegger warning 'In is-sentences, trust the subject but question the rest since it might be gossip.'

Looking for outside goals to inside means presented as goals, we see:

1. The tradition teaches cardinality as one-dimensional line-numbers to be added without being counted first. A difference is to teach counting before adding to allow proportionality and integral calculus and solving equations in early childhood: bundle-counting in icon-bundles less than ten, recounting in the same and in a different unit, recounting to and from tens, calculator prediction, and finally, forward and reversed on-top and next-to addition.

2. The tradition teaches the counting sequence as natural numbers. A difference is natural numbers with a unit and a decimal point or cup to separate inside bundles from outside singles; allowing a total to be written in three forms: normal, overload and underload: T = 5 = 2.1 2s = 2B1 2s = 1B3 2s = 3B-1 2s.

3. The tradition uses carrying. A difference is to use bundle-writing and recounting in the same unit to remove overloads: T = 7x 48 = 7x 4B8 = 28B56 = 33B6 = 336. Likewise with division: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

4. Traditionally, multiplication is learned by heart. A difference is to combine algebra and geometry by seeing 5x6 as a stack of 5 6s that recounted in tens increases its width and decreases its height to keep the total unchanged.

5. The tradition teaches proportionality abstractly. A difference is to introduce double-counting creating per-number 3$ per 4kg bridging the units by recounting the known number: T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Double-counting in the same unit transforms per-numbers to fractions and percentages as 3$ per 4$ = ¾; and 75kg per 100kg = 75/100 = 75%.

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Calculus in Primary and Middle and High School (TSG 07)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3 = 2 3s, seeing that natural numbers are block-numbers with units.

Once counted, totals can be added, but addition is not well-defined: Two totals T1 = 2 3s and T2 = 4 5s may add on-top or next-to as 8s: T1 + T2 = 2 3s + 4 5s = 3.2 8s. Thus next-to addition means adding areas by combining multiplication and addition, called integration.

Reversing next-to addition, we ask e.g. 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer, first we remove the initial total T1, then we count the rest in 5s: u = (T–T1)/5. Combining subtraction and division in this way is called differentiation or reversed integration.

'Double-counting' a total in two physical units creates 'per-numbers' as 4$/5kg, or fractions as 4$/5$ = 4/5 if the units are the same. Per-numbers and fractions are not numbers, but operators needing a number to become a number: Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other.

In high school calculus occurs when adding locally constant per-numbers, as 5seconds at 3m/s changing constantly to 4m/s. This means adding many strips under a per-number graph, made easy by writing the strips as differences since many differences add up to one single difference between the terminal and initial numbers, thus showing the relevance of differential calculus, and that integration should precede differentiation.

The epsilon-delta criterium is a straight forward way to formalize the three ways of constancy, globally and piecewise and locally, by saying that constancy means that the difference can be made arbitrarily small. (Tarp, 2013)

References

Bauman, Z. (1990). *Thinking Sociologically.* Oxford, UK: Blackwell.

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Curing Math Dislike With 1 Cup and 5 Sticks (TSG 07)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra, to measure earth and to reunite numbers. Thus the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences, both affected by the Heidegger (1962) warning: 'In is-sentences, trust the subject but question the predicate'. However, by neglecting the subject and presenting the predicate as the goal, the tradition creates widespread dislike in math classes especially with division. To get the class back on track, the total must be reintroduced physically and in the sentence.

A class is stuck in division and gives up on 237/5. Having heard about '1cup & 5 sticks', the teacher says 'Time out. Next week, no division. Instead we do bundle-counting'. Teacher: 'How many sticks?' Class: '5.' Teacher: 'Correct, and how many 2s?' Class: '2 2s and 1 left over'. Teacher: 'Correct, we count by bundling. The cup is for bundles, so we put 2 inside the cup and leave 1 outside. With 1 inside, how many outside? And with 3 inside, how many outside?' Class: '1inside-3outside; and 3inside-less1outside.' Teacher: 'Correct. A total can be counted in 3 ways. The normal way with 2inside-1outside. With overload as 1inside-3outside. With underload as 3inside-less1outside.' Class: 'OK'. Teacher. 'Now 37 means 3inside-7outside if we count in tens. Try recounting 37 with overload and underload. Class: '2inside-17outside; and 4inside-less3outside.'

Teacher: 'Now let us multiply 37 by 2, how much inside and outside?' Class: 6inside-14outside. Or 7inside-4outside. Or 8inside-less6outside.'

Teacher: 'Now to divide 78 by 3 we recount 7inside-8outside to 6inside-18outside. Dividing by 3 we get 2inside-6outside or 26. With 79 we get 1 leftover that still must be divided by 3. So 79/3 gives 28 and 1/3.'

Class: 'So to divide 235 by 5 we recount 235 as 20 inside and 35 outside. Dividing by 5 we get 4 inside and 7 outside, or 47; With 237 we get 2 leftovers that still must be divided by 5. Thus 237/5 gives 47 and 2/5?'

Teacher: 'Precisely. Now let us go back to multiplication and division and use bundle-counting'.

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Quantitative Literature Also has 3 Genres: Fact and Fiction and Fiddle (TSG 07)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Humans communicate in languages: A word-language with sentences assigning words to things and actions; and a number-language with formulas assigning numbers or calculations to things and actions. ‘Word stories’ come in three genres: Fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like ‘This sentence is false’. ‘Number stories’ are often called mathematical models. They come in the same three genres.

Fact models can be called a ‘since-then’ models or ‘room’ models. Fact models quantify quantities and predict predictable quantities: “What is the area of the walls in this room?”. The model’s prediction is what is observed, so fact models can be trusted when units are checked. Algebra’s four basic uniting models are fact models: T = a+b, T = axb, T = a^b and T = ∫y dx; as are many models from basic science and economy.

Fiction models can be called ‘if-then’ models or ‘rate’ models. Fiction models quantify quantities but predict unpredictable quantities: “My debt is gone in 5 years at this rate!”. Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based upon alternative assumptions. Models from statistics calculating averages assuming variables to be constant are fiction models; as are models from economic theory showing nice demand and supply curves.

Fiddle models can be called ‘then-what’ models or ‘risk’ models. Fiddle models quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence x Probability. It has meaning in insurance but not when quantifying casualties where it is cheaper to stay in a cemetery than at a hospital.

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20. Plenary PowerPointPresentation at the CTRAS 2017 July Conference in China

Difference-Research
Powering PISA Performance:
Count and Multiply before you Add

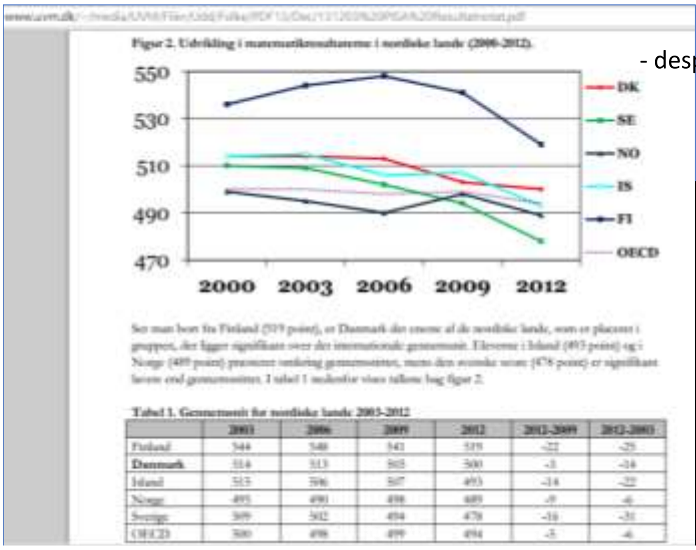
5 ~~Luther~~ Tarp Theses
II II ~~= 4~~ = 2 **2s**
3*5 ~~= 15~~ = 3 **5s**
8/4 ~~= 2 each~~ = 2 **4s**
2_w+3_d ~~= 5~~ = 17_d
1/2+2/3 ~~= 7/6~~ = 3/5

Curriculum Architect, Allan.Tarp@MATHeCADEMY.net
Teaches Teachers to Teach MatheMatics as ManyMatics, Tales of Many
a Heidegger-inspired VIRUS-Academy:
To learn, ask the subject, not the instructor
Full 31 page article: http://mathecademy.net/difference-research/

(Fractions ~~= numbers~~ = per-numbers = operators needing numbers to become numbers)

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 1

Poor Pisa Performance in Scandinavia



All go down, Sweden especially
- despite increased research funding
Can Difference-Research
make a Difference
by finding a Difference?

Different Differences

Background

- Poor PISA Performance, witnessing 50 years of low-performing Math Education Research

10. Different Education

- Classroom: Half-Year Self-Chosen Blocks versus Multi-Year Forced Lines

20. Different Mathematics

- BottomUp Many-based Math from Below, versus TopDown Set-based Math from Above

30. Different Research

- Ancient Sophism, Renaissance Natural Science, (Post)Modern Existentialism

40. Different Math Education, showing the Beauty of the Simplicity of Math

- To master Many, Count & Multiply before you Add, Add next-to & on-top, and forwards & backwards

Powering PISA Performance - in a Nutshell

The Greek Sophists: Beware of choice masked as nature.

A Number-Language Sentence (a Tale of Many): **the Total is five, T = 5**

I I I I I $\boxed{T = 5}$ = 1Bundle3 **2s** = 2B1 **2s** = 3B-1 **2s,** or 1B2 **3s** = …

The **predicate** can be different (choice with alternatives)

The **subject** cannot be different (nature without alternatives)

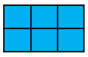
One Goal - many Means; Goal Displacement: When <u>a</u> Means becomes <u>the</u> Goal

Difference-Research unmasks Means masked as Goals, and says:

<u>Use Full Sentences</u>, if not, predicates becomes subjects and a means the goal

Difference-Research, Main Finding:
The Simplicity of Math – Math as Tales of Many

Meeting Many we ask: 'How Many in Total'



- To answer, we math. *Oops, sorry, math is not an action word but a predicate.*
- Take II. To answer, we **Count & Add**. And report with Tales of Many (Number-Language sentences): T = 2 **3s** = 2*3

Three ways to Count: CupCount & ReCount & DoubleCount

- CupCount gives units. ReCount changes units. Double-count bridges units by per-numbers as 2$/3kg

Recount to & from tens gives Multiplication & Equations, coming before Addition

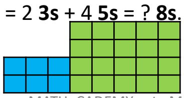
- To tens: T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens** . From tens: T = ? **7s** = u*7 = <u>42 = (42/7)*7</u> = 6 **7s** (<u>ReCount-Formula</u>)

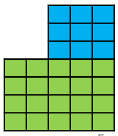
Counting gives variable or constant unit- or per-numbers, to be Added in 4 ways

- Addition & multiplication unites variable & constant unit-numbers.
- Integration & power unites variable & constant per-numbers.

Adding NextTo & OnTop roots Early Childhood Calculus & Proportionality

- EarlyChildhood-Calculus: T = 2 **3s** + 4 <u>**5s** = ? **8s**</u>. EarlyChildhood-Proportionality: T = 2 **3s** + 4 **5s** = ? **5s**





Education & Mathematics & Research

Education: a Social Institution

- In sociology, Bauman warns against ‘the danger of so-called *goal displacement*. The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right.’

Mathematics & Research: Truth claims

- In philosophy, Sartre says: ‘In existentialism, existence precedes essence.’
- In philosophy, Heidegger warns against true sentences with a subject & verb & predicate: ‘Trust the subject; but doubt the predicate, it could be different.’
- In counter-philosophy, the Greek sophists said: ‘Beware of choice masked as nature.’

Difference-Research asks 1 Question only: find a Difference that makes a Difference
- to **unmask** claimed goals, existence, subjects, nature as masked means, essence, predicates, choice.

Difference-Research, Main Recommendation: Visible and Tangible BUNDLES in Tales of Many

China: Educate Wans of DifferenceResearch Professors for the New SilkRoad & Africa

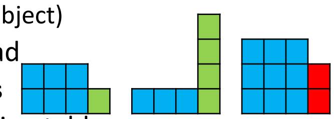
To improve PISA Performance, the Outsider (Child, Migrant) must touch & see & write the BUNDLE and use full number-language sentences in Tales of Many. (Bundles = units)

And must Count & Multiply before Adding.

- Several counting sequences:

T = I I I I I I I = 7 = B-3 (BUNDLE less 3) = ½B&2 (The Total is the goal, the subject)

- Recount in the same unit, **3s**, to create/remove over- or underload



T = 7 = I I I I I I I = III III I = 2B1 or T = III I I I I = 1B4 or T = III III III II = 3B-2 **3s**

Seeing T = 47 = 4B7 = 3B17 = 5B-3 makes a difference in multiplication tables:

T = 2*7 = 2*(½B&2) = B&4 = 14, or T = 2*7 = 2*(B-3) = 20-6 = 14

- A calculator predicts by the <u>RecountFormula</u>, where the operations (/, *, –) are icons for bundling & stacking & removing stacks to find unbundled: <u>T = 7 = (7/3)*3</u> = 2B1 **3s**

7/3	2.some
7 – 2*3	1

Difference-Research, Main Warning:
The 3x3 Goal Displacements in Math Education

Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
	Equations	Could: be introduced in primary as recounting from ten-bundles to icon-bundles; and as reversed on-top and next-to addition Instead: Defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra
	Proportionality	Could: be introduced in primary as recounting in another unit when adding on-top; be double-counting producing per-numbers Instead: defined as linear functions, or as multiplicative thinking supporting the claim that fractions and ratios are rational numbers
High	Trigonometry	Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.
	Functions	Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate Instead: are introduced as set-relations where first-component identity implies second-component identity
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

11. Different Education
EU: Line-organized & Office-directed Schools

From secondary school, continental Europe uses **line-organized** education with forced classes and forced schedules making teenagers stay together in age groups even if girls are two years ahead in mental development.

The classroom belongs to the class. This forces teachers to change room and (in lower secondary school) to teach several subjects outside their training.

Tertiary education is also **line-organized** preparing for offices in the public or private sector. This makes it difficult to change line in the case of unemployment, and it forces the youth to stay in education until close to 30 making reproduction fall to 1.5 child/family, causing the European population to die out very quickly by decreasing it to 25% in 100 years.

12. Different Education

US: Block-organized & talent-directed

Alternatively, North America uses **block-organized** education saying to teenagers:

“Welcome, inside you carry a **talent**! Together we will uncover and develop your personal talent through daily lessons in self-chosen half-year blocks, academical or practical, together with 1subject teachers. If successful the school will say ‘**good job**, you have a **talent**, you need some more’. If not, the school will say ‘**good try**, you have **courage** to try out the unknown, now try something new’”.

The classroom belongs to the teacher teaching one subject only.

Likewise, college is **block-organized** easy to supplement with additional blocks in the case of unemployment.

At the age of 25, most students have an education, a job and a family with three children, 1 for mother, 1 for father, and 1 for the state to secure reproduction.

20. Different Mathematics

The Beauty of the Simplicity of Mathematics

21. The Goal & Means of Mathematics Education
22. Totals as Blocks. Digits as Icons. Operations as CupCounting Icons
23. ReCounting gives Proportionality & Multiplication & Equations
24. Multiplication tables simplified by ReCounting
25. DoubleCounting in different & same units creates PerNumbers & Fractions
26. Geometry: Counting Earth in HalfBlocks
27. Once Counted, Totals can be Added. But counting and double-counting gives 4 number-types (constant & variable unit-numbers & per-numbers) to add in 4 ways
28. How Different is the Difference? Set-based versus Many-based Mathematics

21. Different Mathematics

The Goal and Means of Mathematics Education

The Set-based Top-Down Tradition:

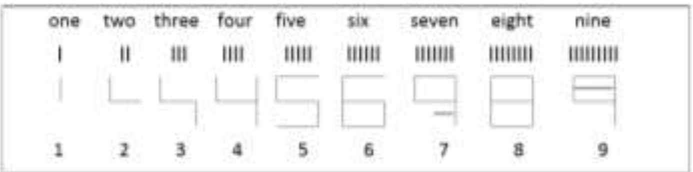
- Mathematics exists as a collection of well-proven statements about well-defined concepts, all derived from the mother concept SET
- Mathematics is surprisingly useful to modern society
- Consequently, mathematics must be taught and learned

The Many-based Bottom-Up Difference:

- Many exists; to master Many we develop a number-language with Tales of Many, a 'ManyMatics'.
- **Many-matics**, defining concepts from below as **abstractions from examples**, is a more successful means to the goal of mastering Many than
- '**Meta-matics**' defining concepts from above as **examples from abstractions**

22a. Different Mathematics

Digits as Icons. Totals as Blocks to be Cup-Counted



Icon-numbers. A folding ruler shows: digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the 4-icon, etc.

Counting-sequences. A total of a dozen sticks counted in **5s** gives different counting sequences:

'1, 2, 3, 4, Bundle, 1B1, …, 2 Bundles, 2B1, 2B2', or

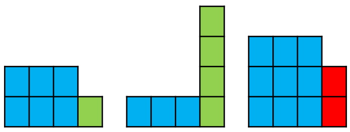
'01, 02, 03, 04, 10, 11, …, 22' , or '.1, .2, .3, .4, 1., 1.1, …, 2.2' , or

'1, 2, Bundle less 2, B-1, Bundle, B&1, B&2, 2B-2, 2B-1, 2Bundles, 2B&1, 2B&2.'

Cup-Counting. With a cup for the bundles, a total can be 'cup-counted' with inside bundles & outside singles in 3 ways: normal, with Overload or with Underload: T = 7 = 2]1 **3s** = 1]4 **3s** = 3]-2 **3s**

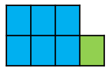
Or, when counting in tens :

T = 37 = 3]7 **tens** = 2]17 **tens** = 4]-3 **tens**



22b. Different Mathematics

Operations as CupCounting Icons



We count by bundling and stacking: T = I I I I I I I = ~~III~~ ~~III~~ I = $\boxed{\begin{matrix}\text{~~III~~}\\ \text{~~III~~}\end{matrix}}$ I = 2B1 **3s** = 2]1 **3s** = 2.1 **3s**

Thus, to count 7 in **3s** we take away 3 many times, iconized by an uphill stroke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times.

To stack the 2 **3s** we use multiplication, iconizing a lift, 2x3 or 2*3, transforming the bundles into a stack.

To look for unbundled singles, we drag away the stack of 2 **3s** iconized by a horizontal trace: 7 – 2*3 = 1.

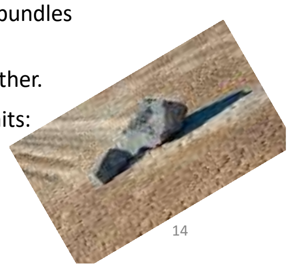
The prediction 'T = 7 = 2 **3s** & 1 = 2B1 **3s** = 2]1 **3s**' provides the

7/3	2.some
7 – 2*3	1

ReCount-formula: T = (T/B)*B saying 'from T, T/B times, B can be taken away'.

To also bundle bundles, power is iconized as a cap, e.g. 5^2, indicating the number of times bundles themselves have been bundled.

Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other.



Counting thus provides the number-formula called a polynomial, where all numbers have units:

T = 456 = 4*BundleBundle + 5*Bundle + 6*1 = 4*B^2 + 5*B + 6*1

So counting creates 3 operations: to divide & to multiply & to subtract.

22c. Different Mathematics

The ReCount Formula is all over Mathematics

ReCount-formula: T = (T/B)*B saying 'from T, T/B times, B can be taken away'

ReCounting	T = (T/B)*B	8 = (8/2)*2 = 4*2 = 4 **2s**	
Proportionality	$ = ($/kg)*kg	$ = p*kg	= price*kg, Economy
Coordinate Geometry	Δy = (Δy/Δx)*Δx	Δy = m*Δx	= slope*Δx
Differential Calculus	dy = (dy/dx)*dx	dy = y'*dx	= gradient*dx
Trigonometry	a = (a/c)*c; a = (a/b)*b	a = sinA*c; a = tanA*b	
Linearity	y = k*x	F = m*a, dist = vel*time, ..	Physics
Eigenvalues	Hψ = E*ψ	Schroedinger Equation in	Quantum mechanics

23. Different Mathematics

ReCounting gives Proportionality & Multiplication & Equations

ReCounting in the same unit creates overloads & underloads

- T = IIIIIII = III III I = 2]1 **3s** = 1]4 **3s** (Overload III I I I I) = 3]-2 **3s** (Underload III III III II)

ReCounting in different units means changing units (**Proportionality**)

- T = 4 **5s** = ? **6s**. Calculator predicts with ReCount-formula T = (T/B)*B, T = 3]2 **6s**

4*5/6	3.some
4*5 – 3*6	2

ReCounting from icons to tens gives **Multiplication** :

- T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens**, predicted by multiplication

ReCounting from tens to icons gives **Equations** :

- T = ? **7s** = u*7 = 42 = (42/7)*7 = 6 **7s** with solution

u*7 = 42 = (42/7)*7	
u	= 42/7 = 6

An equation is solved by moving to opposite side with opposite sign

24. Different Mathematics

Multiplication Tables Simplified by ReCounting

Geometry: Multiplication means that, recounted in tens, a block increases its width and therefore decreases its height to keep the total unchanged.

Thus T = 3*7 means 3 **7s** that may be recounted in tens as T = 2.1 **tens** = 21.

Algebra: The full ten-by-ten table can be reduced to a small 2-by-2 table containing doubling and tripling, using that 4 is doubling twice, 5 is ½Bundle, 6 is 5&1 or Bundle less 4, 7 is 5&2 or Bundle less 3, etc.

Beginning with doubling and halving visualized by CentiCubes

- T = 2 **6s** = 2*6 = 2*(½B&1) = B&2 = 12, or
- T = 2 **6s** = 2*6 = 2*(B-4) = 20-8 = 12.
  - T = 5 **7s** = 5*7 = 5*(B-3) = 5B – 15 = 50 – 15 = 35
  - T = 8 **7s** = 8*7 = (B-2)*(B-3) = BB – 2B – 3B + 6 = 100 – 20 – 30 + 6 = 56

25. Different Mathematics

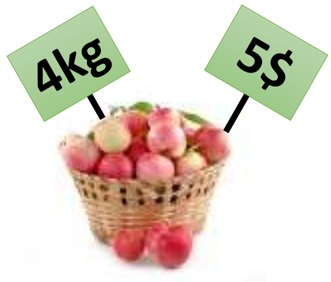
DoubleCounting in 2 units creates PerNumbers (Proportionality) DoubleCounting in the same unit creates Fractions

Apples are double-counted in kg and in $.

With **4kg = 5$** we have 4kg/5$ = 4/5 kg/$ = a per-number

Questions: *4kg/100kg = 4/100 = 4%*

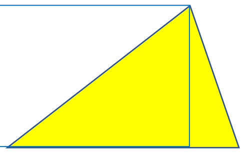
7kg = ?$	**8$ = ?kg**
7kg = (7/4)*4kg = (7/4)*5$ = 8.75$	8$ = (8/5)*5$ = (8/5)*4kg = 6.4kg



Answer: *Recount in the per-number*

26. Different Mathematics

Geometry: Counting Earth in HalfBlocks



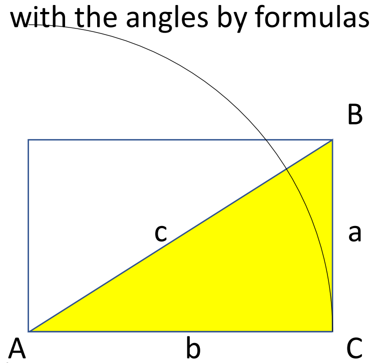
Geometry means to count earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras theorem. And connected with the angles by formulas recounting the sides in sides or in the diagonal:

A+B+C = 180 and a*a + b*b = c*c

a = (a/c)*c = sinA*c

a = (a/b)*b = tanA*b

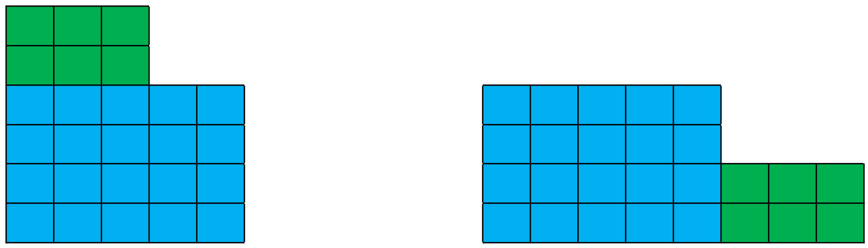
Circle: circum./diam. = **π = n*tan(180/n)** for n large



27a. Different Mathematics

Once Counted & ReCounted, Totals can be Added

OnTop	NextTo
2 **3s** + 4 **5s** = 1]1 **5s** + 4 **5s** = 5]1 **5s**	2 **3s** + 4 **5s** = 3]2 **8s**
The units are changed to be the same. *Change unit = Proportionality*	The areas are added. *Adding areas = Integration*



27b. Different Mathematics

Adding PerNumbers as Areas (Integration)

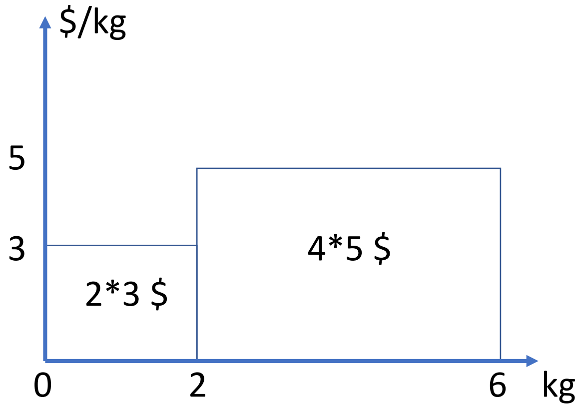
2 kg at 3 $/kg

+ 4 kg at 5 $/kg

(2+4)kg at (2*3 + 4*5)/(2+4) $/kg

Unit-numbers add on-top.

Per-numbers add next-to as **areas** under the per-number graph.



27c. Different Mathematics

With 2x2 different number-types we Add in 4 ways

Counting produces variable or constant **unit-numbers** or **per-numbers**

- Addition & Multiplication unites variable & constant unit-numbers
  - Subtraction & division splits into variable & constant unit-numbers
- Integration & Power unites variable & constant per-numbers
  - Differentiation & root/logarithm splits into variable & constant unit-numbers

Operations **unite** / *split into*	Variable	Constant
Unit-numbers *m, s, \$, kg*	**T = a + n** *T – a = n*	**T = a*n** *T/n = a*
Per-numbers *m/s, \$/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $\log_a T = n$, $\sqrt[n]{T} = a$

28a. Different Mathematics

How Different is the Difference?

Set-based Math versus Many-based Math

	SET-based Tradition	Many-based Difference
Goal/Means	Learn Mathematics / Teach Mathematics	Learn to master Many / Math as Tales of Many
Digits	Symbols as letters	Icons with as many sticks as they represent
Numbers	Place-value number line names. Never with units	A union of blocks of stacked singles, bundles, bundle-bundles etc. Always with units
Number-types	Four types: Natural, Integers, Rational, Real	Positive and negative decimal numbers with units
Operations	Mapping from a set-product to the set	Counting-icons: /,*,-,+ (bundle, stack, remove, unite)
Order	Addition, subtraction, multiplication, division	The opposite
Fractions	Rational numbers, add without units	Per-numbers, not numbers but operators needing a number to become a number, so added by integration
Equations	Statement about equvalent number-names	Recounting from tens to icons, reversing operations
Functions	Mappings between sets	Number-language sentences with a subject, a verb and a predicate
Proportionality	A linear function	A name for double-counting to different units
Calculus	Differential before integral (anti-differentiation)	Integration adds locally constant per-numbers.

28b. Different Mathematics

Main Parts of a ManyMath Curriculum

Primary School – respecting and developing the Child's own 2D NumberLanguage
- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- CupCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: / x - +

Middle school – integrating algebra and geometry, the content of the label math
- DoubleCounting produces PerNumbers as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always so length becomes change and vv.

High School – integrating algebra and geometry to master CHANGE
- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

31. Different Research

Ancient Greece: Sophist vs. Philo-Sophists

Difference research began with the Greek controversy between two attitudes towards knowledge, called 'sophy' in Greek. To avoid hidden patronization, the sophists warned: 'Know the difference between nature and choice to uncover choice presented as nature.'

To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy.

The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord using an unpredictable will to choose world behavior.

32. Different Research

Renaissance Natural Science

Background: Viking descendants in UK know how to sail, how to steal Spanish silver, how to follow the moon to go to India on open sea to buy silk and pepper:

How does the moon move?

Tradition: Between the stars. Newton: No, falling

Why does moons and apples fall?

Tradition : Following an metaphysical unpredictable will. Newton: No, a physical will predictable, following formulas.

What is the effect of a will or force

Tradition : Aristotle: a force maintains order . Newton: No, a force changes order.

How to use formulas?

Tradition : Arabic algebra. Newton: No, different algebra about change, Calculus

33. Different Research

Enlightenment Century 1700-1800

Newton's physical will inspired the Enlightenment century (Locke) with its two republics

The US: Skepticism towards philosophy, US pragmatism, Symbolic Interactionism, Grounded Theory, Action Learning & Research

The French 5th : post-structuralism inspired by German thinking:

- Counter-enlightenment: Hegel's metaphysical Spirit, the basis for Marxism and EU line-organized office-directed Bildung-education
- Existentialism: (Kierkegaard), Nietzsche, Heidegger, (Sartre: In Existentialism, existence precedes essence)

34. Different Research

French Post-Structuralism

Inspired by Heidegger's: 'In sentences, trust the subject & doubt the predicate'

- Derrida: Words can be different (DeConstruction)
- Lyotard: Truth can be different (PostModern skepticism towards meta-narratives)
- Foucault: Diagnoses can be different, Curing institutions also (a school is really a 'pris-pital' mixing power techniques from a prison and a hospital by fixing and diagnosing students at the same time)
- Bourdieu: Education can be different, and stop using symbolic violence and mathematics especially to create outsiders accepting power be given to a new knowledge-nobility

35. Different Research

Difference-Research finds Differences making a Difference

Difference-Research, inspired by its historical roots,

- Questions traditional words & truths & institutions
- Designs different micro-curricula & macro-curricula
- Reports if a difference makes a difference

<u>Examples</u>

Micro-curricula: MATHeCADEMY.net with YouTube/YouKu videos (MrAlTarp/DrAlTarp)

Macro-curriculum: 'The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants', http://mathecademy.net/stem-based-core-math-for-migrants/

36. Different Research

Difference-Research: For whom?

- For teachers observing problems in the classroom
- For teacher-researchers splitting their time between academic work at a university and intervention research in a classroom.
- For full-time researchers cooperating with teachers both using difference-research, the teacher to observe problems, the researcher to identify differences, together working out a different micro-curriculum, to be tested by the teacher, and reported by the researcher conducting a pretest-posttest study.
- Difference-research begins by observing learning problems and wondering if we could teach differently, e.g. a child saying 'II II, that is not 4, but 2 **2s**', showing that children bring 2dimensional block-numbers to school where 1dimensional cardinal line-numbers then are forced upon them.

Conclusion

The 3x3 Goal Displacements in Math Education

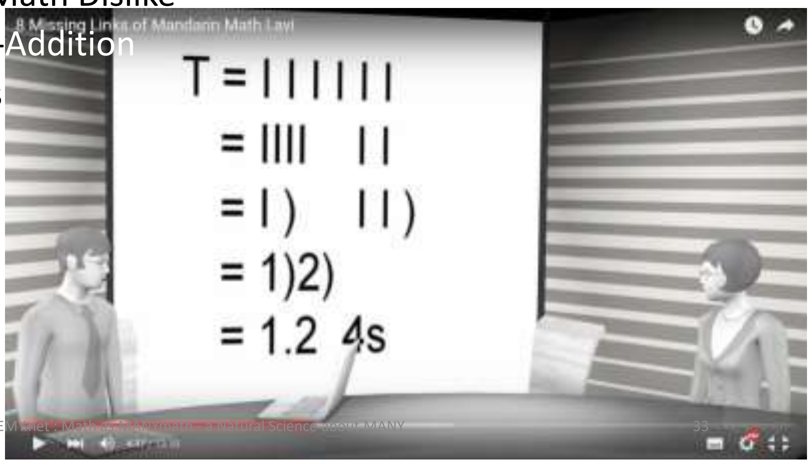
Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
	Equations	Could: be introduced in primary as recounting from ten-bundles to icon-bundles; and as reversed on-top and next-to addition Instead: Defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra
	Proportionality	Could: be introduced in primary as recounting in another unit when adding on-top; be double-counting producing per-numbers Instead: defined as linear functions, or as multiplicative thinking supporting the claim that fractions and ratios are rational numbers
High	Trigonometry	Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.
	Functions	Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate Instead: are introduced as set-relations where first-component identity implies second-component identity
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

ManyMath is Different
But does it make a Difference? Try it out.

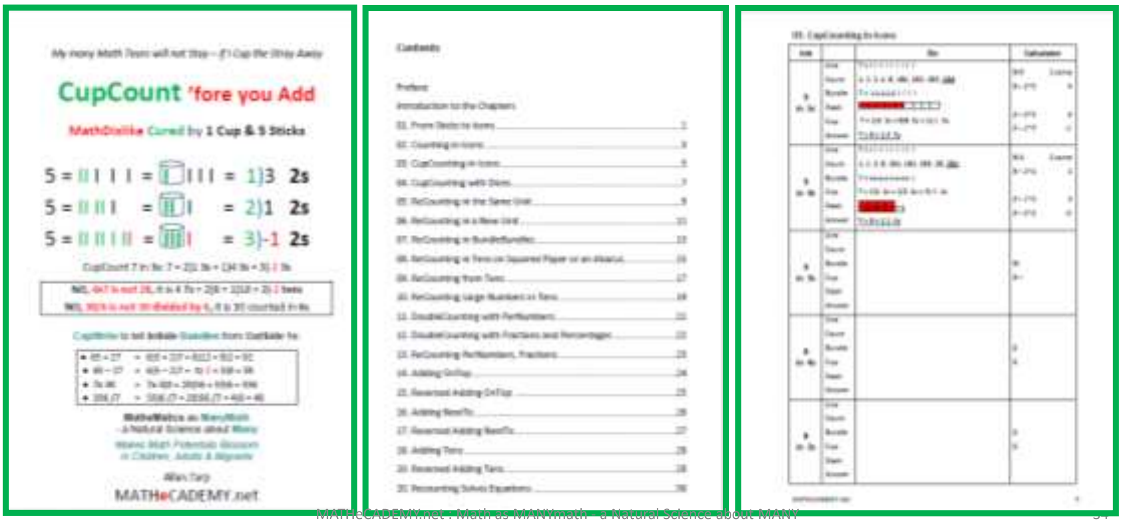
- Watch some YouTube or YouKu videos (MrAlTarp/DrAlTarp)
- Try the **CupCount before you Add** Booklet
- Try a 1day free Skype seminar **How to Cure Math Dislike**
- Try Action Learning and Action Research, e.g. **1Cup, 5Sticks**
- Collect data and Report on its 8 **MicroCurricula**, M1-M8
- Try a 1year online InService TeacherTraining at the MATHeCADEMY.net using PYRAMIDeDUCATION to teach teachers to teach MatheMatics as **ManyMatics**, a Natural Science about the root of mathematics, **Many**

Some MrAlTarp YouTube Videos
Screens & Scripts on MATHeCADEMY.net

- Postmodern Mathematics Debate
- CupCounting removes Math Dislike
- IconCounting & NextTo-Addition
- PreSchool Mathematics
- Fractions
- PreCalculus
- Calculus
- Mandarin Mathematics
- World History



CupCount 'fore you Add Booklet, free to Download


CupCount 'fore you Add

MATHeCADEMY.net

1day free Skype Seminar:
To Cure Math Dislike, CupCount before you Add

Action Learning based on the Child's own 2D NumberLanguage

09-11. Listen and Discuss the PowerPointPresentation
To Cure MathDislike, replace MetaMatism with ManyMath

- **MetaMatism** = MetaMatics + MatheMatism
- **MetaMatics** presents a concept TopDown as an example instead of BottomUp as an abstraction
- **MatheMatism** is true inside but rarely outside classrooms
- **ManyMath**, a natural science about Many mastering Many by CupCounting & Adding NextTo and OnTop.

11-13. Skype Conference. Lunch.

13-15. Do: Try out the CupCount before you Add booklet to experience proportionality & calculus & solving equations as golden LearningOpportunities in CupCounting & NextTo Addition.

15-16. Coffee. Skype Conference.

336/7 =
? ? ?
☹ 💣 ☠

A Primary School Test Curriculum, before
Math Dislike CURED by 1 Cup & 5 Sticks

Having problems in a division class, the teacher says: “Timeout, class. Next week no division, instead we take a field trip back to day 1 to learn CupCounting”

Let’s recount 5 in 2s by bundling, using a cup for the bundles:

5 = II I I I = I I I I = 1)3 **2s** = 1 Bundle & 3 **2s** *overload*

5 = II II I = II I = 2)1 **2s** = 2 Bundles & 1 **2s** *normal*

5 = II II II = III I = 3)-1 **2s** = 3 Bundles less 1 **2s** *underload*

Now we know that numbers can be ReCounted in 3 ways:
Normal, overload or underload if we move a stick **OUTSIDE** or **INSIDE**.
Now CupCount 7 in **3s**:
7 = I I I I I I I = 2)1 **3s** = 1)4 **3s** = 3)-2 **3s**

336**/7**
= 33)6 **/7**
= 28)56 **/7** = 4)8
☺ ✌ 🏳

A Primary School Test Curriculum, after
Math Dislike CURED by 1 Cup & 5 Sticks

When counting in TENS, before calculating, we cup-write the number to separate the **INSIDE** bundles from the **OUTSIDE** singles. Later we recount.

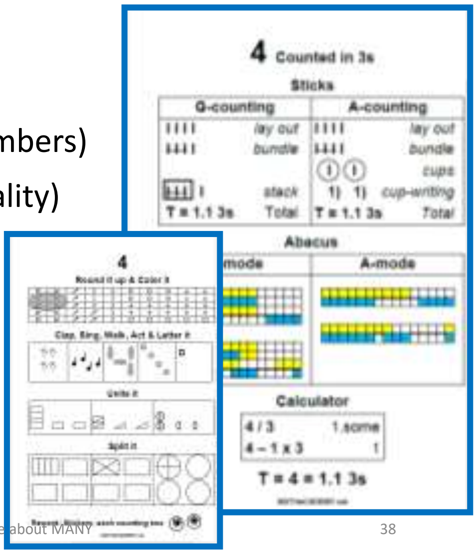
- **65 + 27** = 6)5 + 2)7 = 8)12 = 9)2 = **92**
- **65 – 27** = 6)5 – 2)7 = 4)-2 = 3)8 = **38**
- **7x 48** = 7x 4)8 = 28)56 = 33)6 = **336**
- **336 /7** = 33)6 /7 = 28)56 /7 = 4)8 = **48**

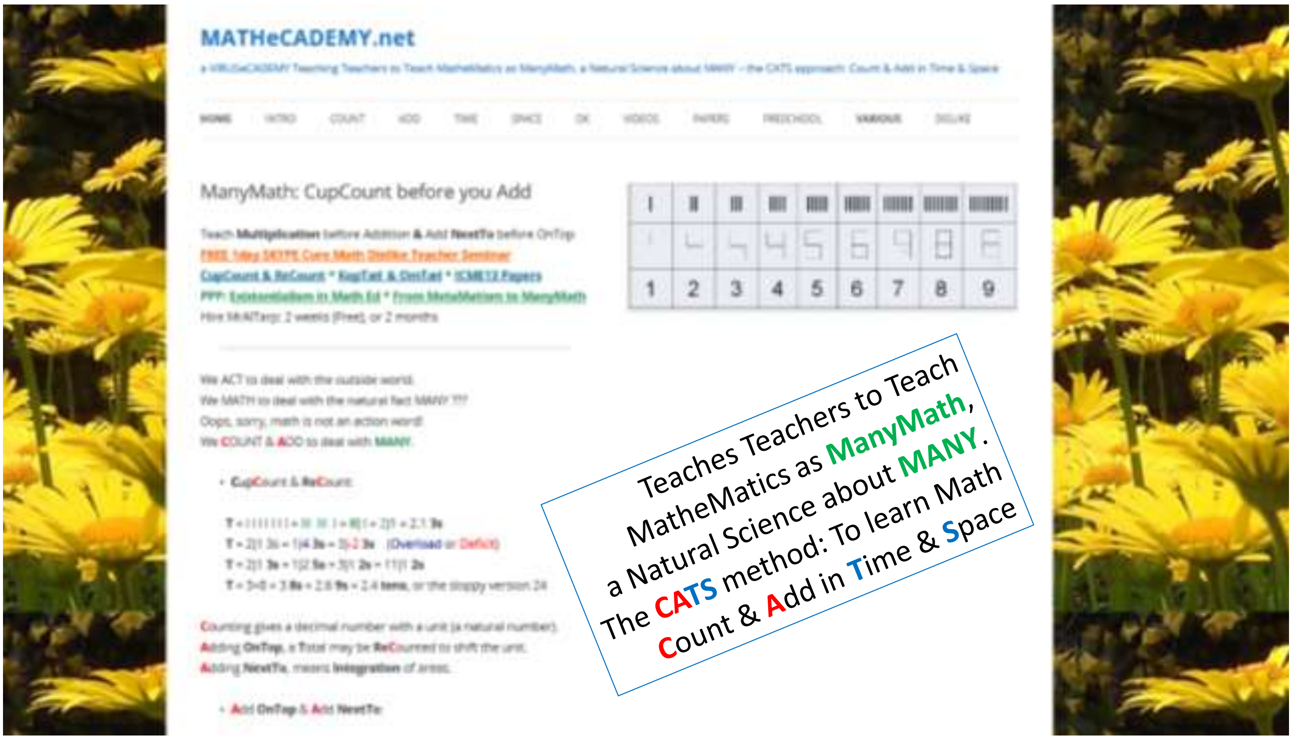
*With 336 we have 33 **INSIDE**, so to get 28, so we move 5 **OUTSIDE** as 50.*
Now try 456 / 7.

- **456 /7** = 45)6 /7 = 42)36 /7 = 6)5 + 1 = **65 1/7**

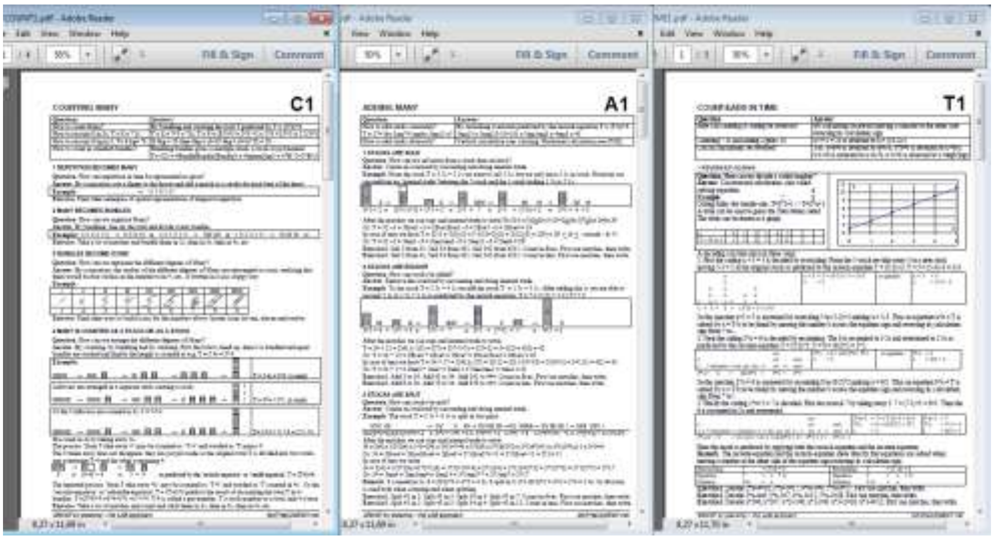
8 MicroCurricula for Action Learning & Research

C1. Create Icons
C2. Count in Icons (Rational Numbers)
C3. ReCount in the Same Icon (Negative Numbers)
C4. ReCount in a Different Icon (Proportionality)
A1. Add OnTop (Proportionality)
A2. Add NextTo (Integrate)
A3. Reverse Adding OnTop (Solve Equations)
A4. Reverse Adding NextTo (Differentiate)





Teacher Training in CATS ManyMath
Count & Add in Time & Space



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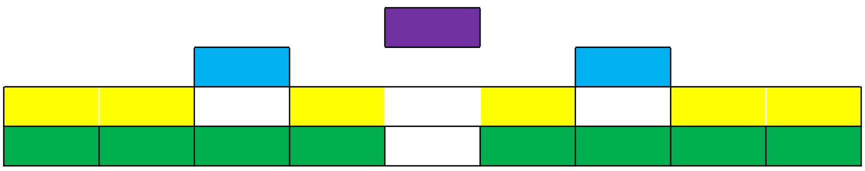
PYRAMIDeDUCATION

To learn MATH: Count&Add MANY
Always ask Many, not the Instructor
MATHeCADEMY.net - a VIRUSeCADEMY

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

- Each pair works together to solve Count&Add problems.
- The coach assists the instructors when instructing their team and when correcting the Count&Add assignments.
- Each teacher pays by coaching a new group of 8 teachers.

1 Coach
2 Instructors
3 Pairs
2 Teams



MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY

41

Main Main Point: Respect the Child's own 2D BlockNumbers
allowing ReCounting & Multiplying before Adding

42

Difference-Research
Powering PISA Performance:
Count and Multiply before you Add
Think Things

III III III III = T = 4 **3s** = 1 **dozen** = 1.2 **tens** = 12 = twelve = 'two left' in Wiking Danish

This talk has been in Anglish, a dialect from the Wiking area on the Danish WestCost

Full 31 page article: http://mathecademy.net/difference-research/

Thank you for listening (Tak do for lytningen)

Allan.Tarp@MATHeCADEMY.net, Denmark

21. Math Dislike

CURED

by 1 Cup & 5 Sticks

My Many Math Tears will not Stay – if I Cup the Stray Away

CupCOUNT before you ADD

5 = II I I I = I I I I = 1]3 2s, 5 = II II I

= II I = 2]1 2s

5 = II II I II = III I = 3]-1 2s

3 ways to CupCount: Overload, Normal, Underload

ReCount 7 in **3s**: 7 = 2]1 **3s** = 1]4 **3s** = 3]-2 **3s**

NO, 4x7 is not 28, it is 4 **7s** = 2]8 = 1]18 = 3]-2 **tens**

NO, 30/6 is not 30 divided by 6, it is **3tens** counted in **6s**

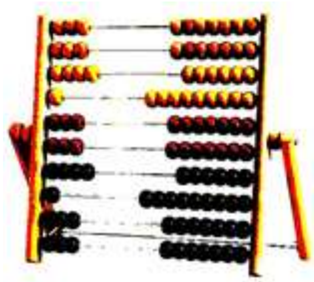
CupWriting tells **InSide Bundles** from **OutSide 1s**

- **65 + 27** = 6]5 + 2]7 = 8]12 = 9]2 = **92**
- **65 – 27** = 6]5 – 2]7 = 4]-2 = 3]8 = **38**
- **7x 48** = 7x 4]8 = 28]56 = 33]6 = **336**
- **336 /7** = 33]6 /7 = 28]56 /7 = 4]8 = **48**

MatheMatics as ManyMath - a Natural Science about Many

Makes Math Potentials Blossom in Children, Adults & Migrants

MATHeCADEMY.net



CupCount 'fore you Add

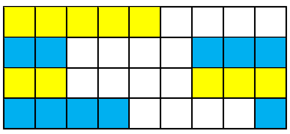
MatheMatics as **Many**Math

a Natural Science about **MANY**

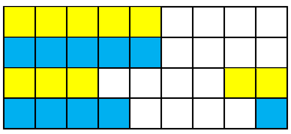
MATHeCADEMY.net Cure **Math Dislike**: Use Children's own 2D Numbers with Units

Count In *Icons* In *BundleCups*	T = I I I I = 4 = 4 T = 7 = I I I I I I I = III III I = 2]1 **3s** = 2 **Bundles** & 1 **3s**
ReCount In same Unit In new Unit	**ReBundle** to create Overload & Underload T = 7 = I I I I I I I = 2]1 **3s** = 1]4 **3s** = 3]-2 **3s** T = 2]1 **3s** = 1]3 **4s** = 1]2 **5s** = 3]1 **2s** = 1]1]1 **2s** = 11]1 **2s**
ReCount In Tens From Tens	3 **7s** = ? **tens** Answer: 3x7 = 21 = 2]1 **tens** ? **7s** = 3 **tens** Answer: (30/7)x7 = 4]2 **7s**
DoubleCount in *PerNumbers* in *PerFive, 3/5* in *PerHundred, %*	With 4$ per 5kg, T = 20kg = (20/5) x 5kg = (20/5) x 4$ = 16$ 3 per 5 of 200$ = ?$. 200$ = (200/5) x 5$ gives (200/5) x 3$ = 120$ 70% of 300$ = ?$. 300$ = (300/100) x 100$ gives (300/100) x 70$ = 210$
Calculator Prediction RecountFormula	T = 2 **4s** = ? **5s** = 1]3 **5s** since 2x4/5 1.some 2x4 – 1x5 3 T = (T/B) x B = T/B **Bs**
Add NextTo OnTop	T = 2 **3s** + 4 **5s** = 3]2 **8s** *Integration* T = 2 **3s** + 4 **5s** = 1]1 **5s** + 4 **5s** = 5]1 **5s** *Proportionality*
Multiply, Divide Use *CupWriting*	**7 x** 463 = **7 x** 4]6]3 = 28]42]21 = 28]44]1 = 32]4]1 = 3241 3241 **/7** = 32]4]1 **/7** = 28]44]1 **/7** = 28]42]21 **/7** = 4]6]3 = 463

T = 7 = 2]1 **3s** on an **Abacus**:



Geometry-mode



Algebra-mode

MrAlTarp

YouTube Videos

Allan.Tarp@MATHeCADEMY.net

MATHeCADEMY.net **FREE** 1day Skype Seminar

*Teaching Teachers to Teach MatheMatics as **Many**Math*

Piaget: Grasping with Fingers leads to Grasping Mentally

Four as an icon built by four cars, four rhinos, four sticks, a ruler folded in four parts, four beads on an abacus, LEGO blocks, pearls on a pearl board, etc.

Seven sticks bundle-counted as 1]2 5s, or as 2]1 3s or as 3]1 2s

The MATHeCADEMY.net stand at the MatematikBiennale in Sweden, 2014

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Migrant Math

Core Math
for
Late Beginners

Mathematics as ManyMath

a Natural Science about Many

MATHeCADEMY.net

Preface

”How old will you be next time?” I asked the child. “Four”, he answered and showed me four fingers. “Four, you said?” I asked and showed him four fingers held together two by two. “No, that is not four, that is two twos!” the child replied. That opened my eyes. Children come to school with two-dimensional block numbers where all numbers have units. However, the school does not allow the children to count the numbers before being added. Instead the school teaches cardinality as a one-dimensional number line where numbers have different names; thus disregarding the fact that numbers are two dimensional blocks where all numbers have a unit as shown when writing out fully

T = 345 = 3 BundleBundles + 4 Bundles + 5 Singles = 3*10^2 + 4*10 + 5*1.

This booklet allows schools and parents to choose an education that develops the 2D number blocks that the children bring to school instead of forcing a 1D number line upon them. Also, the booklet allows the children to practice ‘counting before adding’ and to include bundle-counting and re-counting to different units. The booklet thus is an answer to the question ‘How to Save and Develop a Child’s Math Potential?’

To master Many we ask ‘how many?’ To answer, we count by bundling and stacking to get a total T. Once counted, first a total can be recounted in the same unit to create overload or underload, or to create a different unit; next totals can be added NextTo, or OnTop if the units are the same.

Counting a total T of 7 ones in 3s we get the result T = 7 = 2 **3s** & 1 = 2B1 **3s** where B means Bundles.

We separate the *inside* bundles from the *outside* unbundled singles by a *cup* becoming a bracket when reporting the result with *bundle-writing*: T = III III I = II B I = 2B1 **3s**

Once counted, a total can be *recounted* to create *overload* or *underload*, deficit. To create an overload, we move a stick from the inside to the outside: T = II B I = I B III I = 1B4 **3s**.

To create an underload, we borrow foreign sticks to move a bundle from the outside to the inside

T = II B I = II B I II II = III B II = 3B-2 **3s**.

Thus a given total can be *recounted* in three ways: normal, with overload and with underload.

T = 7 = 2B1 **3s** = 1B4 **3s** = 3B-2 **3s**.

A total of 68 can be recounted in four different ways as T = 68 = 6B8 **tens** = 5B18 **tens** = 7B-2 **tens**.

Recounting and bundle-writing come in handy when we add, subtract, multiply or divide numbers:

Using bundle-writing to add 65 and 27 we get an overload outside the bundle cup allowing us to move 10 **1s** from the outside to the inside as 1 **tens**

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

Using bundle-writing to subtract 27 from 65 we get an underload outside the bundle cup allowing us to move a bundle of 1 **tens** from the inside to the outside as 10 **1s** to remove the underload.

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

Alternatively, before subtracting we can create an overload outside by moving 1 **tens** from the inside to the outside as 10 **1s**

T = 65 – 27 = 6B5 – 2B7 = 5B15 – 2B7 = 3B8 = 38

Using bundle-writing to multiply 48 with 7 we get an overload outside the bundle cup allowing us to move 50 **1s** from the outside to the inside as 5 **tens**

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

Alternatively, before multiplying we can create an underload outside by borrowing 2 **1s**. Later the underload can be removed by moving 2 **tens** outside as 20 **1s**

T = 7* 48 = 7* 4B8 = 7* 5B-2 = 35B-14 = 33B6 = 336

Using bundle-writing to divide 336 with 7 we prefer to have 28 instead of 33 inside the bundle cup, so we create an overload outside by moving 5 bundles outside as 50 **1s**

T = 336 = 33B6 = 28B56; so T /7 = 4B8 = 48

Alternatively, we can create an underload outside before dividing

T = 336 = 33B6 = 35B-14; so T /7 = 5B-2 = 4B8 = 48

To divide 338 by 7 we get 2 single leftovers that counted in 7s becomes a fraction 2/7

T = 338 = 33B8 = 28B58 = 28B56 + 2; so T /7 = 4B8 + 2/7 = 48 2/7

Introduction to the Chapters

Chapter 01, From Sticks to Icons, shows how rearranging four sticks creates a 4-icon with as many sticks as it represents; likewise with the other icons until ten having a name but no icon.

Chapter 02, Counting-sequences in Icons, shows that when counting by bundling, the bundle-icon is not used. Hence, when counting in tens, ten does not need an icon. A natural counting sequence will report both the bundles and the unbundled: 01, 02, …, 10, 11; or 0.1, 0.2, …, 1.0, 1.1 always including the bundle-name as the unit. Each bundle-size has its own counting sequence, but the standard is ten-counting in a sloppy version leaving out the unit and misplacing the decimal point by saying 23 instead of 2.3 tens.

Chapter 03, BundleCount in Icons, shows how a total T can be recounted in icon-bundles. Thus a total of nine things, represented by a line of sticks or beads on an abacus, can be counted in fours by a counting sequence. Also, they can be represented by a stack of bundles placed with one stick per bundle in a bundle cup that can written as a bracket (bundle-writing) and reported as a decimal number with a unit where the decimal point separates the bundles from the unbundled singles, T = 9 = 2B1 4s = 2.1 4s. Alternatively, a calculator can be asked to predict the counting result. Entering '9/4', we ask 'from 9, taking away 4s can be done how many times?' The calculator answers '2.some' so by entering '9 – 2x4' we ask 'from 9, once taking away 2 4s leaves what?' The answer '1' gives the calculator prediction T = 9 = 2.1 4s. Thus also operations are icons: /4 shows the broom wiping away 4 many times, – 4 shows the trace left when dragging away 4 only once, 2x shows the lifting needed to create a stack of 2 bundles, and +3 shows the juxtaposition of 3 singles left next to a stack of bundles. Moving 1 stick outside the bundle cup creates an overload T = 1B5 4s; and moving an extra stick in gives an underload, a deficit, T = 3B-3 4s. Thus by recounting, a total T of nine can be recounted in 4 different ways: T = nine = 9 1s = 2B1 4s = 1B5 4s = 3B-3 4s. This comes in handy when totals are added, subtracted, multiplied or divided. A good calculator says 2+3*4 = 14; a bad says 2+3*4 = 20.

Chapter 04, BundleCount with dices, shows how a total T can be recounted in icon-bundles where the total is shown on two similar dices and the icon-number is shown on a third dice.

Chapter 05, ReCount in the same Unit, shows how to recount a total T in the same unit by unbundling a bundle to singles thus creating an overload, or by borrowing extra singles that then has be counted for as a deficit. Thus a total of 2.1 5s can be written with overload as T = 1B6 5s or as T = 1.6 5s, or with borrowing as T = 3B-4 5s or as T = 3.-4 5s

Chapter 06, ReCount in a new Unit, shows how once counted in one unit, a total T can be recounted in another unit. Thus a total of 2 9s can be recounted in 6s as in chapter 3, again by lining, counting, bundling, stacking, bundle-writing and answering; and again checked by a calculator prediction using two formulas. The ReCount formula **T = (T/B)*B** saying that 'from T, T/B times Bs can be taken away'; and the ReStack formula **T = (T–B)+B** saying that 'From T, T–B is left when B is placed next to'. To change a unit is also called **proportionality**.

Chapter 07, ReCount in BundleBundles, shows how an overload in a bundle-cup can be removed by an extra cup for bundles-of-bundles. Thus counting a total T of 4 8s in 5s gives T = 6B2 5s. However, with 5 as the bundle-size, 5 bundles can be recounted as 1 bundle-of-bundles of 5s so that
T = 6B2 5s = B1B2 5s = 1BB1B2 5s or T = 6.2 5s = 11.2 5s.

Chapter 08, ReCount in Tens on Squared Paper or an Abacus, shows how easy totals counted in icon-bundles can be recounted in tens since the calculator is programmed to give the answer directly in its sloppy version. Thus to recount 3 8s in tens we enter 3*8 and get the answer 24, so T = 3 8s = 2.4 tens. Recounting icon-numbers in tens systematically will provide the multiplication tables showing individual patterns in a ten by ten square or on an abacus.

Chapter 09, ReCount from Tens, shows, as in chapter 3, that we can get the answer through a calculator prediction or through lining, rebundling, and bundle-writing. Only this time we shorten the line by using Roman numbers as icons. Recounting large numbers from tens, we save time by using a multiplication table. Thus to recount a total T of 253 in 7s we use bundle-writing to create overloads guide by the table:
T = 253 = 25B3 = 21B43 = 21B 42+1 = 3B6 *7 +1, so T = 253 = 36 7s + 1.

Chapter 10, ReCount Large Numbers in Tens, show how bundle-writing may be used to create overloads later to be removed to get the final answer. Thus to recount 7 43s in tens gives a total
T = 7 43s = 7*43 = 7*4B3 = 28B21 = 30B1 = 301 as confirmed by a calculator.

Chapter 11, DoubleCount with PerNumbers, shows that counting a quantity in two different physical units will provide a per-number to be used as a bridge connecting the two units. Thus counting a quantity as 4$ and as 5 kg gives the per-number 4$/5kg or 4/5 $/kg. Asking '8$ = ? kg', the answer comes from recounting the 8s in 4s to be able to use the per-number as a bridge between the two units:
T = 8$ = (8/4)*4$ = (8/4)*5kg = 10kg. Likewise when asking e.g.'?$ = 12kg'
Chapter 12, DoubleCount with Fractions and Percentages, shows that fractions and percentages can be treated as per-numbers. Thus asking '3/5 of 200$' is the same as asking '3 per 5 of 200$ gives ?'. So we recount the 200 in 5s to get the answer: T = 200$ = (200/5)*5$ giving (200/5)*3$ = 120$. And asking '3% of 250$' is the same as asking '3 per 100 of 250$'. So we recount the 250 in 100s to get the answer: T = 250$ = (250/100)*100$ gives (250/100)*3$ = 7.5$ as confirmed by writing '3/100*250' on a calculator.
Chapter13, ReCount PerNumbers, Fractions, shows how changing unit transforms per-numbers.
Chapter 14, Adding OnTop, shows that to add two totals T1 and T2 OnTop the units must be the same so recounting may be needed to change a unit. Thus adding 2 3s and 4 5s as 3s, the 4 5s must be recounted as 3s to give a total of 8.2 3s as confirmed by a calculator.
Chapter 15, Reversed Adding OnTop, shows that to reverse OnTop addition, the known total must be taken away before counting the rest in the unit of the second total. Thus asking '2 3s + ? 5s total 5 3s, we take away the 2 3s from the 5 3s before recounting the rest, T – T1, in 5s by saying (T–T1)/5 = ΔT/5 = 1.4 5s as confirmed by a calculator. Subtraction followed by division is called differentiation.
Chapter 16, Adding NextTo, shows that adding two totals T1 and T2 NextTo means adding their areas, also called integration. Thus adding 2 3s and 4 5s NextTo each other as 8s on a ten by ten square or on an abacus gives 3.2 8s as confirmed by a calculator.
Chapter 17, Reversed Adding NextTo, shows that to reverse NextTo addition, the known total must be taken away before counting the rest in the unit of the second total. Thus asking '2 3s + ? 5s total 3 8s, we take away the 2 3s from the 3 8s before recounting the rest, T – T1, in 5s by saying (T–T1)/5 = ΔT/5 = 3.3 5s as confirmed by a calculator. Together, integration and differentiation is called **calculus**.
Chapter 18, Adding Tens, shows that when adding tens, bundle-writing can be used to create and remove overloads. Thus adding totals as 27 and 85 creates an overload that can be removed by bundle-writing,
T = 27 + 85 = 2B7 + 8B5 = 10B12 = 11B2 = 112 as confirmed by a calculator.
Chapter 19, Reversed Adding Tens, the number added must be taken away which might result in a deficit calling for a unbundling a bundle, unless this is done first resulting in an overload that allows taking the number away without creating a deficit. Thus asking '? + 27 = 85' or '85 – 27', bundle-writing is used to remove the deficit, 85 – 27 = 8B5 – 2B7 = 6B-2 = 5B8 = 58; or used to create an overload,
85 – 27 = 8B5 – 2B7 = 7B15 – 2B7 = 5B8 = 58, both confirmed by a calculator.
Chapter 20, Recounting Solves Equations, shows that equations expressing a reversed calculation can be solved by recounting and restacking. Thus to solve the equation u*2 = 8, 8 is recounted in 2s as 8 = (8/2)*2 = 4*2, so that u = 4, checked by a calculator by entering 4*2. With u*2 = 8 solved by u = 8/2 we get a shortcut for solving equations: *Move to the opposite side with the opposite sign.*

u*2 = 8 = (8/2)*2 = 4*2	Here we recount 8 in 2s as 8 = (8/2)*2 = 4*2	**u = 4**
u+2 = 9 = (9-2)+2 = 7+2	Here we restack 9 to 9-2+2 = 7+2	**u = 7**
u/3 = 2	Here we recount 2 in 3s as 2 = (2/3)*3 = 2*3/3 = 6/3	**u = 6**
u-2 = 6	Here we restack 6 to 6-2+2 = 6+2-2 = 8-2	**u = 8**
2*u+3 = 15	Here we restack 15 to 15-3+3 = 12+3, and 2*u = 12 = 12/2*2 = 6*2	**u = 6**
2*u-3 = 15	Here we restack 15 to 15-3+3 = 15+3-3 = 18-3, and 2*u = 18 = 18/2*2 = 9*2	**u = 9**
u/2+3 = 15	Here we restack 15 to 15-3+3 = 12+3, and u/2 = 12 = 12/2*2 = 12*2/2 = 24/2	**u = 24**
2/u-3 = 15	Here we restack 15 to 15-3+3 = 15+3-3 = 18-3, and 2/u = 18 = 18/2*2 = 18*2/2 = 36/2	**u = 36**

Count & Color Squares, Odd & Even

1		2			3			4			5			6			7			8			9		

9					8			7			6			5			4			3		2			1

2			4			6			8			1		3			5			7			9		

1		2			4			8			3			5			9				6			7	

3			6			9				8			5			4			7			1		2	

Migrant Math 01

From Sticks to Icons

I I I I →  → 4 → FOUR

Many sticks can be arranged in a row of for example four ones.
Four ones can be rearranged to 1 icon with four sticks.
Written sloppy, the icon becomes a digit.
Icons are created for all numbers until ten.
Ten, eleven, twelve etc. has no icon because we count in tens.
Ten is counted as 1 bundle and no unbundles, ten = 10
Eleven is counted as 1 bundle and 1 unbundled, eleven = 11
Twelve is counted as 1 bundle and 2 unbundled, twelve = 12
In Danish, eleven and twelve means one left and two left, understood that a bundle has already been counted.

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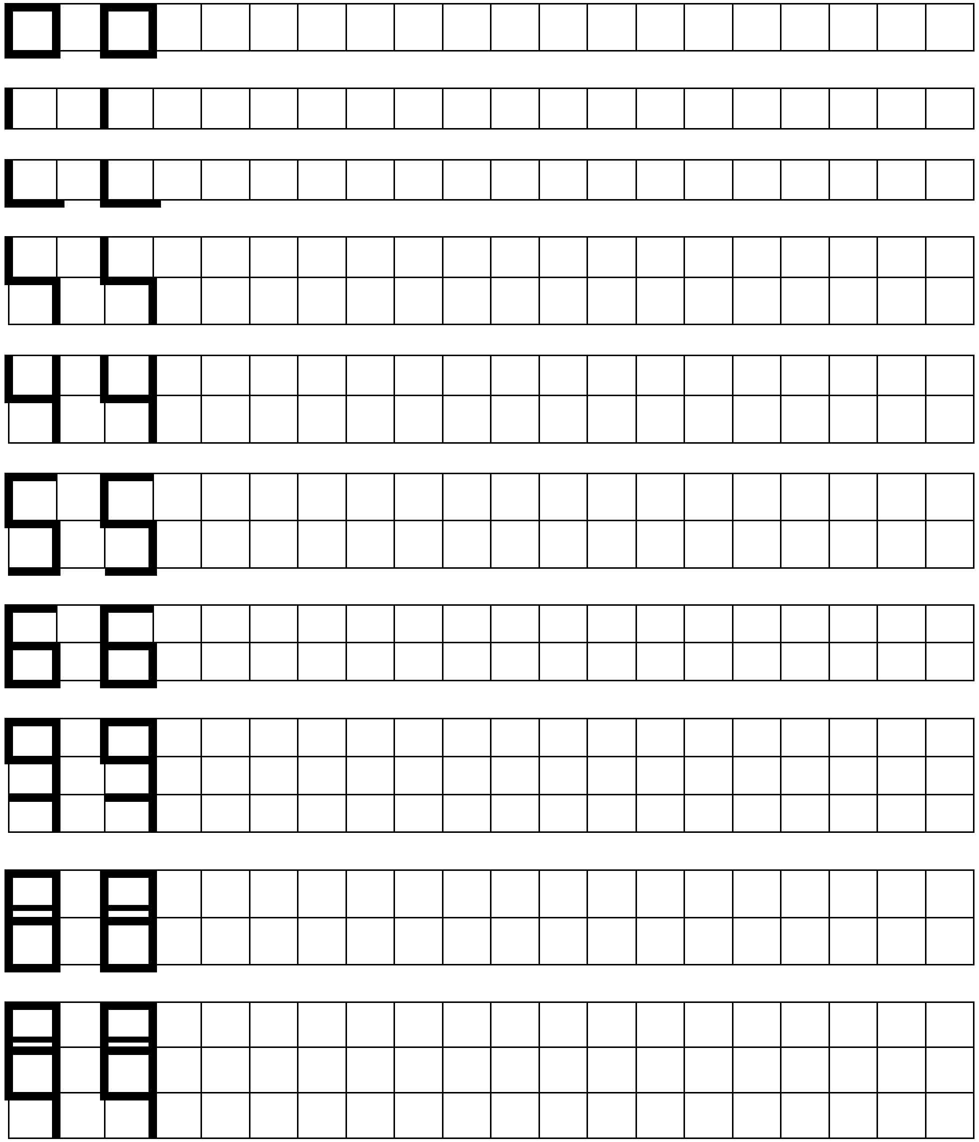
Job	Do
I	
II	
III	
IIII	
IIIII	
IIIII I	
IIIII II	
IIIII III	
IIIII IIII	



Write & Say

0											Zero
1											One
2											Two
3											Three
4											Four
5											Five
6											Six
7											Seven
8											Eight
9											Nine
10											Ten
11											Ten1
12											Ten2
13											Ten3
14											Ten4
15											Ten5
16											Ten6

Count 1s & 2s & 3s in Icons

	I	II	III	I	II	III	I	II	III	I	II	III
ten	01	03	06			1B2					2B1	
ten	.1	.3										
9												
9												
8												
8												
7												
7												
6												
6												
5												
5												
4												
3												
3												

Migrant Math 02

Counting-sequences in Icons

1, 2, 3, 4, 5, …

We count by bundling, so the bundle-icon is not used.

If we count in tens, ten does not need an icon.

A counting-sequence reports the bundles and the unbundled:

1, 2, 3, 4, …, 10, 11, 12

or 01, 02, …, 10, 11, 12

or 0.1, 0.2, …, 1.0, 1.1, 1.2 tens, with the bundle-name as the unit.

Each bundle-size has its own counting sequence.

The standard is ten-counting in a sloppy version leaving out the unit and misplacing the decimal point by saying 23 instead of 2.3 tens.

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Count in Icons

	I	I	I	I	I	I	I	I	I	I	I	I
ten	1	2	3	4	5	6	7	8	9	10	11	12
ten	01	02	03	04	05	06	07	08	09	1B	1B1	1B2
ten	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.	1.1	1.2
9	01	02	03	04	05	06	07	08	1B	1B1	1B2	1B3
9	.1	.2	.3	.4	.5	.6	.7	.8	1.	1.1	1.2	1.3
8												
8												
7												
7												
6												
6												
5												
5												
4												
4												

Count 2s in Icons

	II	II	II	II	II	II	II	II	II	II	II	II
ten	02				1B							2B2
ten												
9												
9												
8												
8												
7												
7												
6												
6												
5												
5												
4												
3												
3												

Job		Do	Calculator
7 in 4s	Line Count Bundle Cup Stack Answer		 7 7
6 in 5s	Line Count Bundle Cup Stack Answer		 6 6
5 in 4s	Line Count Bundle Cup Stack Answer		 5 5
4 in 5s	Line Count Bundle Cup Stack Answer		 4 4
3 in 5s	Line Count Bundle Cup Stack Answer		 3 3

Migrant Math 03

BundleCount in Icons

T = 9 = IIIIIIIII = IIII IIII I = 2B1 **4s** = 2.1 **4s**

A total T is counted in icon-bundles that are stacked.
A total of nine sticks can be counted in fours by a counting sequence.
Also, we can place one stick per bundle in a bundle cup that can written as a bracket (bundle-writing) and reported as a decimal-number with a unit where the decimal point separates the bundles from the unbundled singles, T = 9 = 2B1 **4s** = 2.1 **4s**.
A calculator can predict the counting result.
With ‘9/4’ we ask ‘from 9, taking away 4s how many times?’
The answer is ‘2.some’
With ‘9 – 2x4’ we ask ‘from 9, taking away 2 4s leaves what?’
The answer ‘1’ gives the calculator prediction T = 9 = 2.1 **4s**.
Moving 1 stick outside the bundle cup gives an overload, T = 1B5 **4s**.
Moving 1 stick inside gives an underload, a deficit, T = 3B-3 **4s**.
Thus a total T of nine can be recounted in 4 different ways:
T = nine = 9 1s = 2B1 **4s** = 1B5 **4s** = 3B-3 **4s**.
This is handy when totals are added, subtracted, multiplied or divided.
A good calculator says 2+3*4 = 14; a bad calculator says 2+3*4 = 20.

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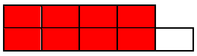
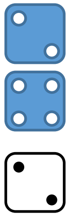
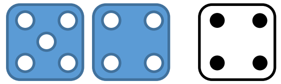
03. BundleCount in Icons

Job		Do	Calculator
9 **in 5s**	Line Count Bundle Stack Cup Answer	T = I I I I I I I I I 1, 2, 3, 4, B, 1B1, 1B2, 1B3, **1B4** T = ~~IIIII~~ I I I I  T = 1B4 5s = 0B9 5s = 2B-1 5s T = 9 = 1.4 5s	9/5 1.some 9 – 1*5 4 9 – 0*5 9 9 – 2*5 -1
9 **in 4s**	Line Count Bundle Cup Stack Answer	T = I I I I I I I I I 1, 2, 3, B, 1B1, 1B2, 1B3, 2B, **2B1** T = ~~IIII~~ ~~IIII~~ I T = 2B1 4s = 1B5 4s = 3B-3 4s  T = 9 = 2.1 4s	9/4 2.some 9 – 2*4 1 9 – 1*4 5 9 – 3*4 -3
9 **in 3s**	Line Count Bundle Cup Stack Answer		9/ 9 –
8 **in 4s**	Line Count Bundle Cup Stack Answer		8 8
8 **in 3s**	Line Count Bundle Cup Stack Answer		8 8

Job		Do	Calculator
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		

Migrant Math 04

BundleCount with Dices



T = 9 = ? **4s**

A total T can be recounted in icon-bundles.
The total is shown by two like dices.
The bundle-number is on a third dice where 1 counts as 7.

Calculator prediction:

9/4	2.some
9 – 2*4	1

Answer: T = 9 = 2.1 **4s**

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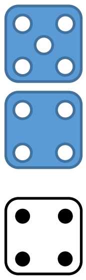
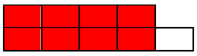
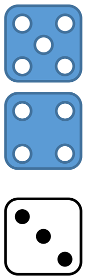
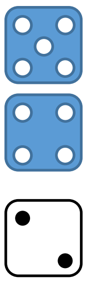
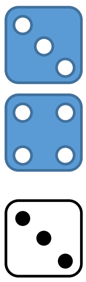
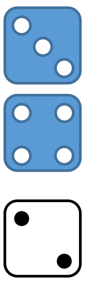
04. BundleCount with Dices

Job		Do	Calculator
	Line	T = I I I I I I I I I	
	Count	1, 2, 3, B, 1B1, 1B2, 1B3, 2B, <u>2B1</u>	9/4 2.some
	Bundle	T = ~~IIII~~ ~~IIII~~ I	9 – 2*4 1
	Cup	T = 2B1 4s = 1B5 4s = 3B-3 4s	
	Stack		*9 – 1*4 5*
9 i 4s	Answer	<u>T = 9 = 2.1 4s</u>	*9 – 3*4 -3*
	Line		
	Count		
	Bundle		9/
	Cup		9 –
	Stack		
	Answer		
	Line		
	Count		
	Bundle		9
	Cup		9
	Stack		
	Answer		
	Line		
	Count		
	Bundle		7
	Cup		7
	Stack		
	Answer		
	Line		
	Count		
	Bundle		7
	Cup		7
	Stack		
	Answer		

Job		Do	Cup	Answer
4.3 6s in 6s	Line UnBundle Borrow			
4.3 5s in 5s	Line UnBundle Borrow			
4.3 4s in 4s	Line UnBundle Borrow			
5 7s in 7s	Line UnBundle Borrow			
5 6s in 6s	Line UnBundle Borrow			
5 4s in 4s	Line UnBundle Borrow			
3 7s in 7s	Line UnBundle Borrow			
3 5s in 5s	Line UnBundle Borrow			
1.3 6s in 6s	Line UnBundle Borrow			
1.3 5s in 5s	Line UnBundle Borrow			

Migrant Math 05

ReCount in the same Unit

T = 2B1 **5s** = 1B6 **5s** = 3B-4 **5s**
T = 2.1 **5s** = 1.6 **5s** = 3.-4 **5s**

A total T is recounted in the same unit in two ways:

- create an overload: unbundle a bundle to singles

- create an underload:

borrow extra singles that becomes a deficit

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05. ReCount in the Same Unit

Job		Do	Cup	Answer
2.1 5s in 5s	Line UnBundle Borrow	T = IIIII IIIII I T = IIIII I I I I I I T = IIIII IIIII IIIII	2B1 1B6 3B-4	T = 2.1 **5s** T = 1.6 **5s** T = 3.-4 **5s**
2.1 4s in 4s	Line UnBundle Borrow			
2.1 3s in 3s	Line UnBundle Borrow			
2.1 6s in 6s	Line UnBundle Borrow			
2.1 7s in 7s	Line UnBundle Borrow			
3.2 7s in 7s	Line UnBundle Borrow			
3.2 6s in 6s	Line UnBundle Borrow			
3.2 5s in 5s	Line UnBundle Borrow			
3.2 4s in 4s	Line UnBundle Borrow			
3.2 3s in 3s	Line UnBundle Borrow			

Job		Do	Calculator
2 7s in 5s	Line Count Bundle Stack Cup Answer		 2*7 2*7
2 6s in 5s	Line Count Bundle Stack Cup Answer		 2*6 2*6
2 6s in 4s	Line Count Bundle Stack Cup Answer		 2*6 2*6
2 6s in 3s	Line Count Bundle Stack Cup Answer		 2*6 2*6
2 5s in 4s	Line Count Bundle Stack Cup Answer		 5 5

Migrant Math 06

ReCount in a new Unit

T = 3 **5s** = ? **6s**

Once counted in one unit, a total T can be recounted in another unit.
A total of 3 **5s** can be recounted in **6s** as in chapter 04
- by lining, counting, bundling, stacking, bundle-writing and answering
- by asking a calculator to predict the result using two formulas:

The ReCount formula **T = (T/B)*B** saying that
'from T, T/B times Bs can be taken away'
The ReStack formula **T = (T–B)+B** saying that
'from T, T–B is left when B is placed next to'.

To change a unit is also called **proportionality**.

Calculator prediction:

3*5/6	2.some
3*5 – 2*6	3

Answer: T = 3 5s = 2.3 **6s**

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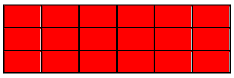
06. ReCount in a New Unit

Job		Do	Calculator
2 9s in 6s	Line Count Bundle Stack Cup Answer	T = I I I I I I I I I I I I I I I I I I 1, 2, 3, 4, B, 1B1, 1B2, 1B3, 1B4, …, **3B** T = ~~IIIIII~~ ~~IIIIII~~ ~~IIIIII~~  T = 3B T = 2 9s = 3 6s	2*9/6 3 2*9 – 3*6 0
2 9s in 5s	Line Count Bundle Stack Cup Answer		2*9/ 2*9 –
2 8s in 6s	Line Count Bundle Stack Cup Answer		2*8 2*8
2 8s in 5s	Line Count Bundle Stack Cup Answer		2*8 2*8
2 7s in 6s	Line Count Bundle Stack Cup Answer		2*7 2*7

Job		Do	Calculator
7 in 2s	Cup Ans.	T = 7 = 3]1 = 1BB1B1 T = 7 = 3.1 **2s** = 11.1 **2s**	7/2 3.some 7 – 3*2 1
9 in 2s	Cup Ans.	T = 9 = 4B1 = 2BB0B1 = 1BBB0BB0B1 T = 9 = 4.1 **2s** = 20.1 **2s** = 100.1 **2s**	9/2 4.some 9 – 4*2 1
3 4s in 2s	Cup Ans.		
3 5s in 2s	Cup Ans.		
5 4s in 2s	Cup Ans.		
4 7s in 3s	Cup Ans.		
4 8s in 3s	Cup Ans.		
4 9s in 3s	Cup Ans.		
5 7s in 3s	Cup Ans.		
5 8s in 3s	Cup Ans.		
5 9s in 3s	Cup Ans.		
6 8s in 3s	Cup Ans.		
7 8s in 3s	Cup Ans.		

Migrant Math 07

ReCount in BundleBundles

T = 9.3 5s = 9B3 **5s** = 1BB4B3 **5s** = 14.3 **5s**

An overload in a bundle-cup can be removed
by an extra cup for bundles-of-bundles.
Counting a total T of 6 **8s** in **5s** gives T = 9.3 5s.
However, with 5 as the bundle-size,
5 bundles can be recounted as
1 bundle-of-bundles of **5s** so that
T = 6 **8s** = 9.3 **5s** = 14.3 **5s**.

Calculator prediction:

6*8/5	9.some
6*8 – 9*5	3

9/5	1.some
9 – 1*5	4

Answer: T = 6 **8s** = 9.3 **5s** = 14.3 **5s**

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07. Recount in BundleBundles

Job		Do	Calculator
4 8s **in 5s**	Cup Answer	T = 4 **8s** = 6B2 **5s** = 1BB1B2 **5s** <u>T = 4 **8s** = 6.2 **5s** = 11.2 **5s** = 12.-3 **5s**</u>	4*8/5 6.some 4*8 – 6*5 2
5 8s **in 6s**	Cup Answer		
6 9s **in 7s**	Cup Answer		
9 9s **in 8s**	Cup Answer		
3 9s **in 4s**	Cup Answer		
4 5s **in 3s**	Cup Answer		
6 8s **in 5s**	Cup Answer		
6 8s **in 4s**	Cup Answer		
7 8s **in 5s**	Cup Answer		
7 8s **in 4s**	Cup Answer		
8 8s **in 5s**	Cup Answer		
8 8s **in 4s**	Cup Answer		

Job		Do	Calculator
5s in tens			10 * 5 = 9 * 5 = 8 * 5 = 7 * 5 = 6 * 5 = 5 * 5 = 4 * 5 = 3 * 5 = 2 * 5 = 1 * 5 =
4s in tens			10 * 4 = 9 * 4 = 8 * 4 = 7 * 4 = 6 * 4 = 5 * 4 = 4 * 4 = 3 * 4 = 2 * 4 = 1 * 4 =
3s in tens			10 * 3 = 9 * 3 = 8 * 3 = 7 * 3 = 6 * 3 = 5 * 3 = 4 * 3 = 3 * 3 = 2 * 3 = 1 * 3 =

Migrant Math 08

ReCount in Tens on Squared Paper or an Abacus

T = 3 **8s** = ? **tens**   T = 3 **8s** = 3*8 = 24 = 2.4 **tens**

Totals counted in icon-bundles can easily be recounted in tens.
A calculator gives the answer directly in its sloppy version.
To recount 3 **8s** in tens, we enter 3*8 and get the answer 24.
So T = 3 **8s** = 24 = 2.4 **tens**.
Recounting icon-numbers in tens systematically gives the **multiplication tables**, showing individual patterns in a ten by ten square or on an abacus.

Calculator prediction:

3*8	24

Answer: T = 3 **8s** = 24 = 2.4 **tens**

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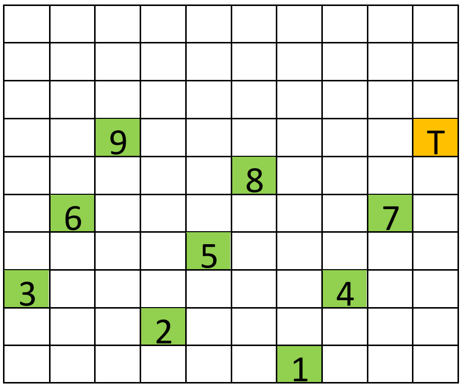
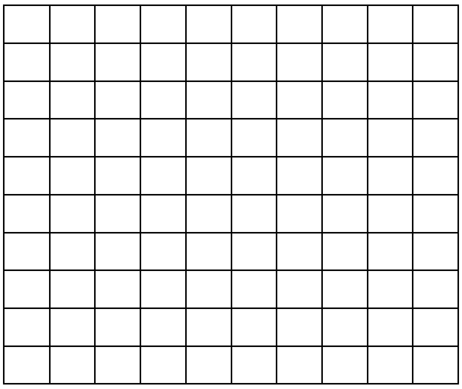
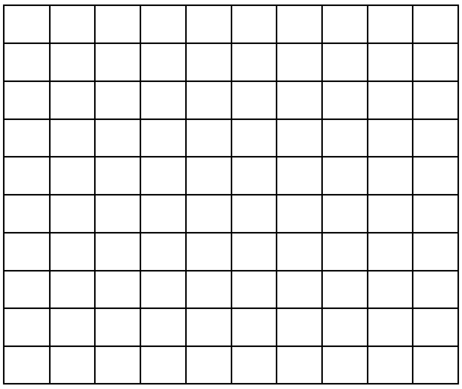
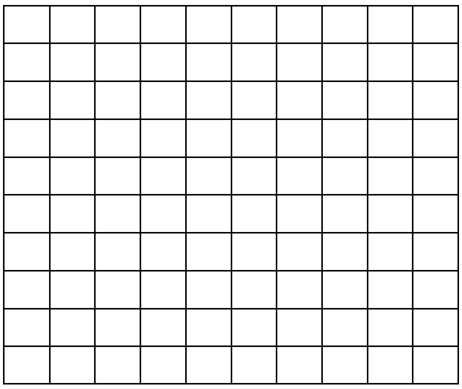
08. ReCount in Tens on Squared Paper or an Abacus

Job		Do	Calculator
7s **in tens**			10 * 7 = 70 9 * 7 = 63 8 * 7 = 56 7 * 7 = 49 6 * 7 = 42 5 * 7 = 35 4 * 7 = 28 3 * 7 = 21 2 * 7 = 14 1 * 7 = 7
8s **in tens**			10 * 8 = 9 * 8 = 8 * 8 = 7 * 8 = 6 * 8 = 5 * 8 = 4 * 8 = 3 * 8 = 2 * 8 = 1 * 8 =
9s **in tens**			10 * 9 = 9 * 9 = 8 * 9 = 7 * 9 = 6 * 9 = 5 * 9 = 4 * 9 = 3 * 9 = 2 * 9 = 1 * 9 =
6s **in tens**			10 * 6 = 9 * 6 = 8 * 6 = 7 * 6 = 6 * 6 = 5 * 6 = 4 * 6 = 3 * 6 = 2 * 6 = 1 * 6 =

Job		Do	Calculator
253 in 7s	Cup Ans.	T = 2BB5B3 = 25B3 = 21B43 = 21B42 + 1 T = 3B6 * 7 + 1 = 36 * 7 + 1 = <u>36 1/7 7s</u>	253/7 36.some 253 – 36*7 1
253 in 9s	Cup Ans.		
253 in 5s	Cup Ans.		
253 in 3s	Cup Ans.		
842 in 7s	Cup Ans.		
842 in 5s	Cup Ans.		
842 in 4s	Cup Ans.		
842 in 2s	Cup Ans.		
904 in 8s	Cup Ans.		
904 in 7s	Cup Ans.		
904 in 5s	Cup Ans.		
904 in 3s	Cup Ans.		
789 in 8s	Cup Ans.		
789 in 5s	Cup Ans.		
789 in 4s	Cup Ans.		

Migrant Math 09

ReCount from Tens

T = 3 **tens** = ? **7s**

A total of 3 **tens** can be recounted in **7s** as in chapter 06
- by lining (we shorten with Roman numbers as icons), counting, bundling, stacking, bundle-writing and answering
- by asking a calculator to predict the result using the two formulas

Calculator prediction:

30/7	4.some
30 – 4*7	2

Answer: T = 3 **tens** = 4.2 **7s** = <u>4 2/7 **7s**</u> *(fraction form)*

Recounting large numbers from tens, we save time using a multiplication table. So to recount a total T of 253 in 7s we use bundle-writing to create an overload guided by the table:

T = 253 = 25B3 = 21B43 = 21B 42 +1 = 3B6 *7 +1
T = 253 = 36 **7s** + 1 = 36 1/7 **7s**.

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09. Recount From Tens

Job		Do	Calculator
37 in 9s	Line ReBundle Cup Answer	X X X V II 9I 9I 9I V II -> 9 9 9 X -> 9 9 9 9 1 3B 7 = B37 = B36 + 1 = B4*9 + 1 T = 37 = 4*9 + 1 = 4.1 **9s** = 4 1/9 **9s**	 37/9 4.some 37 – 4*9 1
37 in 7s	Line ReBundle Cup Answer		
37 in 5s	Line ReBundle Cup Answer		
42 in 7s	Line ReBundle Cup Answer		
42 in 5s	Line ReBundle Cup Answer		
26 in 7s	Line ReBundle Cup Answer		
26 in 5s	Line ReBundle Cup Answer		

Job		Do	Calculator
17 43s	Cup Ans.	T = 17 * 4B3 = 68B51 = 73B1 = 731 T = 17 **43s** = 73.1 **tens** = 731	17*43 731
27 43s	Cup Ans.		
37 43s	Cup Ans.		
47 43s	Cup Ans.		
57 43s	Cup Ans.		
67 43s	Cup Ans.		
77 43s	Cup Ans.		
87 43s	Cup Ans.		
32 243s	Cup Ans.	T = 32 * 2BB4B3 = 64BB128B96 = 64BB137B6 = 77BB7B6 = 777.6 **tens** = 7776	32*243 7776
35 413s	Cup Ans.		
43 343s	Cup Ans.		
56 453s	Cup Ans.		
62 637s	Cup Ans.		
74 843s	Cup Ans.		
87 543s	Cup Ans.		
92 493s	Cup Ans.		

Migrant Math 10

ReCount Large Numbers in Tens

T = 7 **43s** = 7*43 = 7*4B3 = 28B21 = 30B1 = 301

To reCount large numbers in Tens, bundle-writing is used to create an overload, later to be removed to get the final answer.
To recount 7 **43s** in tens gives a total
T = 7 **43s** = 7*43 = 7*4B3 = 28B21 = 30B1 = 301 = 30.1 **tens**
This makes sense: Shrinking the width of the stack from 43 to ten means increasing the height to keep the same total.

Calculator prediction:

7*43	301

Answer: T = 3 **8s** = 24 = 2.4 **tens**

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10. Recount Large Numbers in Tens

Job		Do	Calculator
7 43s	Cup Answer	T = 7 * 4B3 = 28B21 = 30B1 = 301 T = 7 **43s** = 30.1 **tens** = 301	7*43 301
8 43s	Cup Answer		
9 43s	Cup Answer		
6 43s	Cup Answer		
5 62s	Cup Answer		
4 62s	Cup Answer		
3 62s	Cup Answer		
2 62s	Cup Answer		
27 436s	Cup Answer		
3 436s	Cup Answer		
4 436s	Cup Answer		
5 436s	Cup Answer		
6 436s	Cup Answer		
7 436s	Cup Answer		
8 436s	Cup Answer		

Job	Do	Formula
With 4 $ per 5 kg 8$ = ?kg ?$ = 12 kg	8$ = (8/4)*4$ = (8/4)*5kg = 10kg 12kg = (12/5)*5kg = (12/5)*4$ = 9.6$	Kg = (kg/$)*$ Kg = (5/4)*8 = 10 $ = ($/kg)*kg $ = (4/5)*12 = 9.6
With 3 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 6 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 8 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 3 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 6 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 8 kg 8$ = ?kg ?$ = 12 kg		
With 2 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 2 $ per 7 kg 8$ = ?kg ?$ = 12 kg		

Migrant Math 11

DoubleCount with PerNumbers

With 4$/5kg, T = 8$ = (8/4)*4$ = (8/4)*5kg = 10kg

Counting a quantity in two different physical units provides a **per-number** to be used as a bridge connecting the two units.

Thus counting a quantity as 4$ and as 5 kg

gives the per-number 4$/5kg or 4/5 $/kg.

Asking '8$ = ? kg', the answer comes from recounting the 8s in 4s to be able to use the per-number as a bridge between the two units:

T = 8$ = (8/4)*4$ = (8/4)*5kg = 10kg.

Likewise when asking e.g.'? $ = 12kg'

T = 12kg = (12/5)*5kg = (12/5)*4$ = 9.6$

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11. DoubleCount with PerNumbers

Job	Do	Formula
With 4 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**	8$ = (8/4)*4$ = (8/4)*5kg = 10kg 12kg = (12/5)*5kg = (12/5)*4$ = 9.6$	Kg = (kg/$)*$ Kg = (5/4)*8 = 10 $ = ($/kg)*kg $ = (4/5)*12 = 9.6
With 3 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 6 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 8 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 3 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 6 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 8 kg **8$ = ?kg** **?$ = 12 kg**		
With 2 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 2 $ per 7 kg **8$ = ?kg** **?$ = 12 kg**		

Job	Do	Calculator
3 per 5 of 200$	200$ = (200/5)*5$ Giving (200/5)*3$ = 120$	3/5*200 120
3 per 5 of 400$		
2 per 5 of 200$		
1 per 5 of 200$		
3 per 6 of 240$		
2 per 6 of 240$		
5 per 6 of 300$		
3 per 100 of 250$ or 3% of 250$	250$ = (250/100)*100$ Giving (250/100)*3$ = 7.5$	3/100*250 7.5
8 per 100 of 200$ or 8% of 200$		
20 per 100 of 200$ or 20% of 200$		
3 per 100 of 560$ or 3% of 560$		
8 per 100 of 560$ or 8% of 560$		
12 per 100 of 560$ or 12% of 560$		
20 per 100 of 560$ or 20% of 560$		
60 per 100 of 560$ or 60% of 560$		

Migrant Math 12

DoubleCount with Fractions & Percentages

With 4/5, T = 30$ = (30/5)*5$ gives (30/5)*4$ = 24$

Fractions and percentages can be treated as per-numbers.

Asking ‘3/5 of 200$’ is the same as asking ‘3 per 5 of 200$ gives ?’.

So we recount the 200 in 5s to get the answer:

T = 200$ = (200/5)*5$ giving (200/5)*3$ = 120$.

Asking ‘3% of 250$’ is the same as asking ‘3 per 100 of 250$’.

So we recount the 250 in 100s to get the answer:

T = 250$ = (250/100)*100$ gives (250/100)*3$ = 7.5$

as confirmed by writing ‘3/100*250’ on a calculator.

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12. DoubleCount with Fractions & Percentages

Job	Do	Calculator
3 per 5 of 200$	200$ = (200/5)*5$ Giving (200/5)*3$ = 120$	3/5*200 120
3 per 5 of 400$		
2 per 5 of 200$		
1 per 5 of 200$		
3 per 6 of 240$		
2 per 6 of 240$		
5 per 6 of 300$		
3 per 100 of 250$** **or 3% of 250$	250$ = (250/100)*100$ Giving (250/100)*3$ = 7.5$	3/100*250 7.5
8 per 100 of 200$** **or 8% of 200$		
20 per 100 of 200$** **or 20% of 200$		
3 per 100 of 560$** **or 3% of 560$		
8 per 100 of 560$** **or 8% of 560$		
12 per 100 of 560$** **or 12% of 560$		
20 per 100 of 560$** **or 20% of 560$		
60 per 100 of 560$** **or 60% of 560$		

Job	Do	Do	Calculator	Calculator
2/3 = ?	2/3 = 2 2s / 3 2s = 4/6 2/3 = 2 3s / 3 3s = 6/9	2/3 = 2 4s / 3 4s = 8/12 2/3 = 2 5s / 3 5s = 10/15	2/3 = 0.66.. 4/6 = 0.66..	8/12 = 0.66.. 10/15 = 0.66..
1/3 = ?				
1/5 = ?				
2/5 = ?				
3/5 = ?				
4/5 = ?				
4/6 2/6 6/8 2/8	4/6 = 2 2s / 3 2s = 2/3 2/6 = 1 2s / 3 2s = 1/3	6/8 = 3 2s / 4 2s = 3/4 2/8 = 1 2s / 4 2s = 1/4	4/6 = 0.66.. 2/3 = 0.66.. 2/6 = 0.33.. 1/3 = 0.33..	6/8 = 0.75 3/4 = 0.75 2/8 = 0.25 1/4 = 0.25
2/10 4/10 6/10 8/10				
2/12 4/12 6/12 8/12 10/12				
2/14 4/14 6/14 8/14 10/14 12/14				
2/16 4/16 6/16 8/16 10/16 12/16 14/16				

Migrant Math 13

ReCount PerNumbers & Fractions

$$\frac{2}{3} = \frac{2\ \mathbf{2s}}{3\ \mathbf{2s}} = \frac{2*2}{3*2} = \frac{4}{6}$$

Changing unit transforms per-numbers.
With 2 per 3, the per-number does not depend of the unit.
So we can always change unit to the same unit on both numbers.

$2 \text{ per } 3 = \frac{2}{3} = \frac{2\ \mathbf{2s}}{3\ \mathbf{2s}} = \frac{2*2}{3*2} = \frac{4}{6} = 4 \text{ per } 6$

Or we can remove the same unit from both numbers

$4 \text{ per } 6 = \frac{4}{6} = \frac{2*2}{3*2} = \frac{2\ \mathbf{2s}}{3\ \mathbf{2s}} = \frac{2}{3} = 2 \text{ per } 3$

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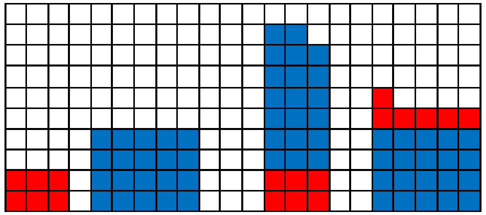
13. ReCount PerNumbers & Fractions

Job	Do	Do	Calculator	Calculator
2/3 = ?	2/3 = 2 2s / 3 2s = 4/6 2/3 = 2 3s / 3 3s = 6/9	2/3 = 2 4s / 3 4s = 8/12 2/3 = 2 5s / 3 5s = 10/15	2/3 = 0.66.. 4/6 = 0.66..	8/12 = 0.66.. 10/15 = 0.66..
1/3 = ?				
1/5 = ?				
2/5 = ?				
3/5 = ?				
4/5 = ?				
4/6 2/6 6/8 2/8	4/6 = 2 2s / 3 2s = 2/3 2/6 = 1 2s / 3 2s = 1/3	6/8 = 3 2s / 4 2s = 3/4 2/8 = 1 2s / 4 2s = 1/4	4/6 = 0.66.. 2/3 = 0.66.. 2/6 = 0.33.. 1/3 = 0.33..	6/8 = 0.75 3/4 = 0.75 2/8 = 0.25 1/4 = 0.25
2/10 4/10 6/10 8/10				
2/12 4/12 6/12 8/12 10/12				
2/14 4/14 6/14 8/14 10/14 12/14				
2/16 4/16 6/16 8/16 10/16 12/16 14/16				

Job	Do	Calculator
2 3s + 4 5s = ? 3s ? 5s		(2*3+4*5)/3 8.some (2*3+4*5) - 8*3 2 2 3s + 4 5s = 8.2 3s (2*3+4*5)/5 5.some (2*3+4*5) - 5*5 1 2 3s + 4 5s = 5.1 5s
2 4s + 3 5s = ? 4s ? 5s		
3 2s + 4 6s = ? 2s ? 6s		
2 5s + 4 3s = ? 5s ? 3s		
5 2s + 3 4s = ? 2s ? 4s		

Migrant Math 14

Add OnTop

T = 2 **3s** + 4 **5s** = ? **5s**

T = (2*3 + 4*5)/5 **5s** = 5.1 **5s**

To add two totals T1 and T2 OnTop, the units must be the same.
so recounting may be needed to change a unit.
To add 2 **3s** and 4 **5s** as **5s**,
the 2 **3s** must be recounted as 5s as (2*3)/5 **5s** = 1.1 **5s.**
T = 2 **3s** + 4 **5s** = 1.1 **5s** + 4 **5s** = 5.1 **5s**
as confirmed by a calculator.

Calculator prediction:

(2*3+4*5)/5	5.some
(2*3+4*5) – 5*5	1

Answer: T = 2 **3s** + 4 **5s** = 5.1 **5s**

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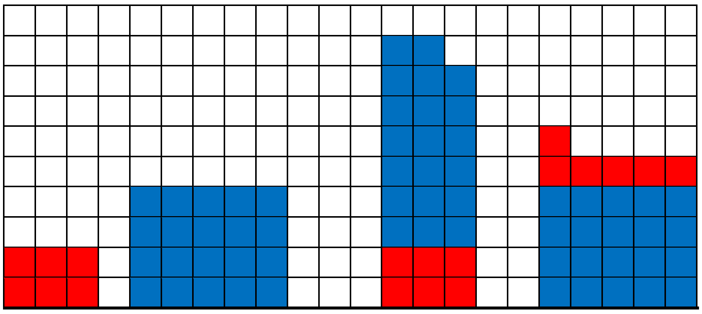
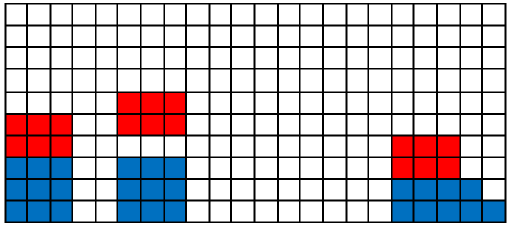
14. Add OnTop

Job	Do	Calculator
2 3s **+** **4 5s** **=** **? 3s** **? 5s**		(2*3+4*5)/3 8.some (2*3+4*5) - 8*3 2 <u>2 3s + 4 5s = 8.2 3s</u> (2*3+4*5)/5 5.some (2*3+4*5) - 5*5 1 <u>2 3s + 4 5s = 5.1 5s</u>
2 4s **+** **3 5s** **=** **? 4s** **? 5s**		
3 2s **+** **4 6s** **=** **? 2s** **? 6s**		
2 5s **+** **4 3s** **=** **? 5s** **? 3s**		
5 2s **+** **3 4s** **=** **? 2s** **? 4s**		

2 3s + ? 5s = 5 3s		(5*3-2*3)/5 1.some (5*3-2*3) - 1*5 4 2 3s + 1.4 5s = 5 3s
2 4s + ? 5s = 5 4s		
2 6s + ? 5s = 4 6s		
2 7s + ? 5s = 6 5s		
2 6s + ? 5s = 4 5s		

Migrant Math 15

Reversed Adding OnTop

T = 2 **5s** + ? **3s** = 6 **3s**

T = (6*3 – 2*5)/3 **3s** = 2.2 **3s**

To reverse OnTop addition, the known total must be taken away before counting the rest in the unit of the second total.
Asking ‘2 **5s** + ? **3s** total 6 **3s**, we take away the 2 **5s** from the 6 **3s** before recounting the rest, T – T1, in 3s by saying
(T–T1)/3 **3s** = ΔT/3 **3s** = 2.2 **3s** as confirmed by a calculator.
Subtraction followed by division is differentiation, part of calculus.

Calculator prediction:

(6*3 – 2*5)/3	2.some
(6*3 – 2*5) – 1*3	2

Answer: T = 2 **5s** + 2.2 **3s** = 6 **3s**

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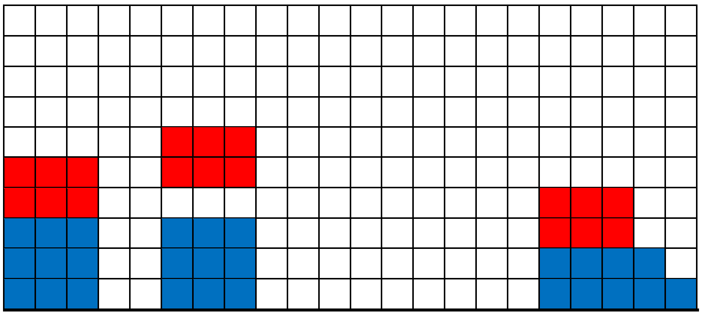
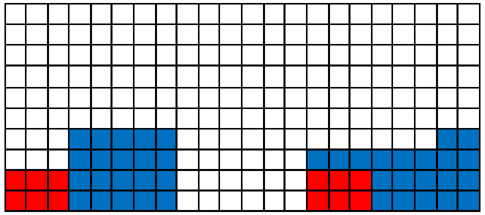
15. Reversed Adding OnTop

2 3s **+** **? 5s** **=** **5 3s**		(5*3-2*3)/5 1.some (5*3-2*3) - 1*5 4 2 3s + **1.4 5s** = 5 3s
2 4s **+** **? 5s** **=** **5 4s**		
2 6s **+** **? 5s** **=** **4 6s**		
2 7s **+** **? 5s** **=** **6 5s**		
2 6s **+** **? 5s** **=** **4 5s**		

Job	Do	Calculator
2 3s + 4 5s = ? 8s		(2*3+4*5)/8 3.some (2*3+4*5) - 8*3 2 2 3s + 4 5s = 3.2 8s
3 2s + 4 5s = ? 7s		
2 3s + 4 6s = ? 9s		
2 4s + 4 5s = ? 9s		
4 3s + 2 4s = ? 6s		

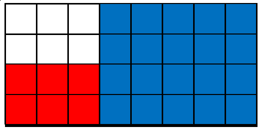
Migrant Math 16

Add NextTo

$$T = 2\ \mathbf{3s} + 4\ \mathbf{5s} = ?\ \mathbf{8s}$$

$$T = (2*3 + 4*5)/8\ \mathbf{8s} = 3.2\ \mathbf{8s}$$

To add two totals T1 and T2 NextTo means adding their areas.
Adding areas is called integration, a part of calculus.
To add 2 **3s** and 4 **5s** next to each other as **8s** on a ten by ten square or on an abacus gives 3.2 **8s** as confirmed by a calculator.



Calculator prediction:

(2*3+4*5)/8	3.some
(2*3+4*5) – 3*8	2

Answer: T = 2 **3s** + 4 **5s** = 3.2 **8s**

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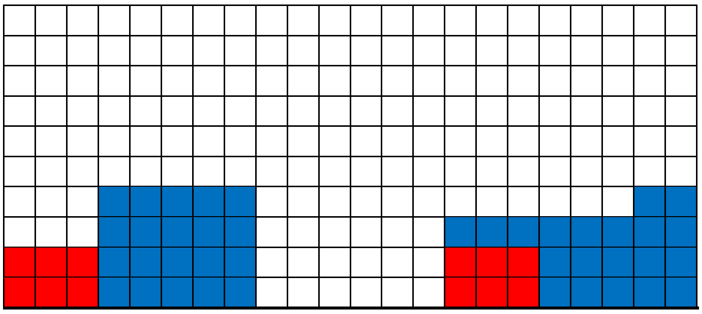
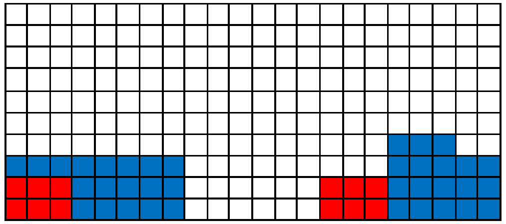
16. Add NextTo

Job	Do	Calculator
2 3s + 4 5s = ? 8s		(2*3+4*5)/8 3.some (2*3+4*5) - 8*3 2 <u>2 3s + 4 5s = 3.2 8s</u>
3 2s + 4 5s = ? 7s		
2 3s + 4 6s = ? 9s		
2 4s + 4 5s = ? 9s		
4 3s + 2 4s = ? 6s		

2 3s + ? 5s = 3 8s		(3*8-2*3)/5 3.some (3*8-2*3) - 3*5 3 2 3s + 3.3 5s = 3 8s
2 4s + ? 5s = 3 9s		
2 3s + ? 4s = 3 7s		
4 3s + ? 5s = 3 8s		
5 2s + ? 5s = 3 7s		

Migrant Math 17

Reversed Adding NextTo

T = 2 **3s** + ? **5s** = 3 **8s**

T = (3*8 – 2*3)/5 **5s** = 3.3 **5s**

To reverse NextTo addition, the known total must be taken away before counting the rest in the unit of the second total.
Asking '2 **3s** + ? **5s** total 3 **8s**, we take away the 2 **3s** from the 3 **8s** before recounting the rest, T – T1, in 5s:
(T–T1)/5 **5s** = ΔT/5 **5s** = 3.3 **5s** as confirmed by a calculator.
Subtraction followed by division is differentiation, part of calculus.

Calculator prediction:

(3*8 – 2*3)/5	3.some
(3*8 – 2*3) – 3*5	3

Answer: T = 2 **3s** + 3.3 **5s** = 3 **8s**

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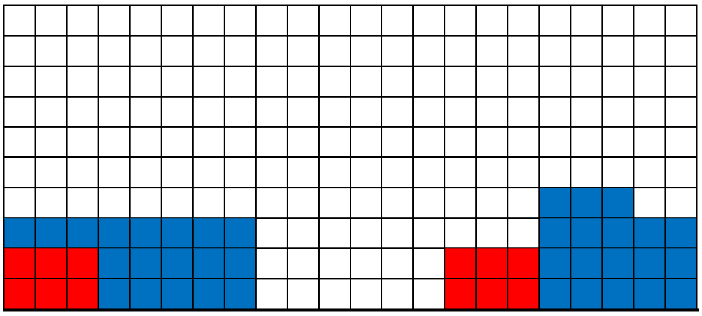
17. Reversed Adding NextTo

2 3s + ? 5s = 3 8s		(3*8-2*3)/5 3.some (3*8-2*3) - 3*5 3 <u>2 3s + **3.3 5s** = 3 8s</u>
2 4s + ? 5s = 3 9s		
2 3s + ? 4s = 3 7s		
4 3s + ? 5s = 3 8s		
5 2s + ? 5s = 3 7s		

Job		Do	Calculator
27 + 85	Cup Answer	T = 2]7 + 8]5 = 10]12 = 11]2 = 112 T = 27 + 85 = 11.2 tens = 112	27+85 112
27 + 85	Cup Answer		
33 + 78	Cup Answer		
39 + 71	Cup Answer		
45 + 67	Cup Answer		
58 + 57	Cup Answer		
57 + 49	Cup Answer		
27 + 205	Cup Answer		
33 + 198	Cup Answer		
39 + 191	Cup Answer		
45 + 187	Cup Answer		
58 + 177	Cup Answer		
57 + 169	Cup Answer		
127 + 385	Cup Answer		
433 + 578	Cup Answer		

Migrant Math 18

Add tens

T = 27 + 85 = 2B7 + 8B5 = 10B12 = 11B2 = 112

Adding tens might create an overload in a bundle-cup or outside.
Bundle-writing is used to remove overloads.
Adding 27 and 85 creates an overload outside the bundle-cup.
The overload is removed by bundle-writing moving bundles inside.
T = 27 + 85 = 2B7 + 8B5 = 10B12 = 11B2 = 112
as confirmed by a calculator.

Calculator prediction:

27+85 112

Answer: T = 27 + 85 = 112

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18. Add Tens

Job		Do	Calculator
27 + 85	Cup Answer	T = 2B7 + 8B5 = 10B12 = 11B2 = 112 T = 27 + 85 = 11.2 tens = 112	27+85 112
27 + 85	Cup Answer		
33 + 78	Cup Answer		
39 + 71	Cup Answer		
45 + 67	Cup Answer		
58 + 57	Cup Answer		
57 + 49	Cup Answer		
27 + 205	Cup Answer		
33 + 198	Cup Answer		
39 + 191	Cup Answer		
45 + 187	Cup Answer		
58 + 177	Cup Answer		
57 + 169	Cup Answer		
127 + 385	Cup Answer		
433 + 578	Cup Answer		

Job		Do	Calculator
27 + ? = 85 85 – 27	Cup Answer	D = 8]5 – 2]7 = 6]-2 = 5]8 = 58 D = 8]5 – 2]7 = 7]15 – 2]7 = 5]8 = 58 T = 85 – 27 = 5.8 tens = 58	85 – 27 58
63 – 17	Cup Answer		
55 – 36	Cup Answer		
35 – 17	Cup Answer		
185 – 27	Cup Answer		
235 – 128	Cup Answer		
242 – 128	Cup Answer		
245 – 167	Cup Answer		
312 – 159	Cup Answer		
421 – 268	Cup Answer		

Migrant Math 19

Reversed adding tens

T = 85 – 27 = 8B5 – 2B7 = 6B-2 = 5B8 = 58

Reversing adding tens, the known number must be taken away.
This might give a deficit calling for unbundling a bundle.
Unless this is done first to create an overload that allows
taking the number away without creating a deficit.
Thus asking '? + 27 = 85' or '85 – 27', bundle-writing is used
to remove the deficit, or to create an overload
T = 85 – 27 = 8B5 – 2B7 = 6B-2 = 5B8 = 58
T = 85 – 27 = 8B5 – 2B7 = 7B15 – 2B7 = 5B8 = 58
both confirmed by a calculator.

Calculator prediction:

85 – 27	58

Answer: T = 85 – 27 = 58

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19. Reversed Adding Tens

Job		Do	Calculator
27 + ? = 85 **85 – 27**	Cup Answer	D = 8B5 – 2B7 = 6B-2 = 5B8 = 58 D = 8B5 – 2B7 = 7B15 – 2B7 = 5B8 = 58 T = 85 – 27 = 5.8 **tens** = 58	85 – 27 58
63 – 17	Cup Answer		
55 – 36	Cup Answer		
35 – 17	Cup Answer		
185 – 27	Cup Answer		
235 – 128	Cup Answer		
242 – 128	Cup Answer		
245 – 167	Cup Answer		
312 – 159	Cup Answer		
421 – 268	Cup Answer		

Do	Equation	Calculator
ReCount ReCount Answer	**2*u+3 = 15** = (15-3)+3 = 12 + 3 2*u = 12 = (12/2)*2 = 6*2 u = 6	2*6+3 15
ReCount ReCount Answer	**3*u+4 = 19**	
ReCount ReCount Answer	**4*u+6 = 38**	
ReCount ReCount Answer	**2*u-3 = 15** = (15-3)+3 = 15+3-3 = 18 - 3 2*u = 18 = (18/2)*2 = 9*2 u = 9	2*9-3 15
ReCount ReCount Answer	**3*u-4 = 8**	
ReCount ReCount Answer	**4*u-5 = 23**	
ReCount ReCount Answer	**u/2+3 = 15** = (15-3)+3 = 12 + 3 u/2 = 12 = (12/2)*2 = (12*2)/2 = 24/2 u = 24	24/2+3 15
ReCount ReCount Answer	**u/3+4= 12**	
ReCount ReCount Answer	**u/2-3 = 15** = (15-3)+3 = (15+3)-3 = 18 - 3 u/2 = 18 = (18/2)*2 = (18*2)/2 = 36*2 u = 36	36/2-3 15
ReCount ReCount Answer	**u/4-7 = 5**	
ReCount ReCount Answer	**u/5-8 = 2**	

Migrant Math 20

Recounting Solves Equations

u*2 = 8 = (8/2)*2 so u = 8/2 = 4

A reversed calculation is called an equation.
An equation can be solved by recounting and restacking.
In both cases an equation is solved by a moving-method:
Move to the opposite side with the opposite sign
In the end, the solution is tested.

To solve the equation u*2 = 8 8 is recounted as 8 = (8/2)*2	To solve the equation u+2 = 8 8 is restacked as 8 = (8–2)+2
u*2 = 8 = (8/2)*2 u = 8/2 = 4 Test: 4*2 = 8 ☺	u + 2 = 8 = (8–2)+2 u = 8–2= 6 Test: 6+2 = 8 ☺

A calculator with a solver will confirm the answer:

Solve(u*2 = 8) 4

Answer: u*2 = 8 is solved by u = 4

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20. ReCounting solves Equations

Do	Equation	Calculator
ReCount	**u*2 = 30** = (30/2)*2 = 15*2	15*2 30
Answer	u = 15	
ReCount	**u*3 = 15**	
Answer		
ReCount	**u*4 = 32**	
Answer		
ReCount	**u*5 = 40**	
Answer		
ReCount	**u/3 = 12** = (12/3)*3 = 12*3/3 = 36/3	36/3 12
Answer	u = 36	
ReCount	**u/3 = 10**	
Answer		
ReCount	**u/4 = 8**	
Answer		
ReCount	**u/5 = 6**	
Answer		
ReCount	**u+2 = 30** = (30-2)+2 = 28 + 2	28+2 30
Answer	u = 28	
ReCount	**u+3 = 24**	
Answer		
ReCount	**u+4 = 20**	
Answer		
ReCount	**u+5 = 12**	
Answer		
ReCount	**u-2 = 30** = (30-2)+2 = 30+2-2 = 32- 2	32-2 30
Answer	u = 32	
ReCount	**u-3 = 20**	
Answer		
ReCount	**u-5 = 10**	
Answer		