# Math Ed & Research 2018

Good Math & Goofy Math, which is Truer?

Conflicting Theories in Mathematics Education

Rethinking Line-Number Arithmetic as Block-Number Algebra

Addition Free Mathematics Rooted in STEM Re-Counting Formulas

Mathematics: Useful Abstractions or an Undiagnosed Compulsory Cure?

Fifty Years of Ineffective Math Education Research, Why? Oops, Wrong Numbers, Sorry

Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum

Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions

A Twin Curriculum Since Contemporary Mathematics May Block

the Road to its Educational Goal, Mastery of Many

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## Contents

## Preface

Mathematics Biennial 2018. Teachers, school administrators, teacher trainers, researchers from all over Sweden attended the mathematics biennial at KARLSTAD University on 25-26/1. It is a recurring Conference carried out every two years at different sites in Sweden. Participants have the opportunity to choose among 200 different program points and five of the lecturers are international researchers.

01. At my stand I presented 19 posters.

The 25th Adults Learning Mathematics international conference, ALM25, was held in London, in July, 2018 with the theme: Boundaries and Bridges: adults learning mathematics in a fractured world. The conference asked for 1000word submissions together with a 200word extended abstract. I sent in 4 proposals.

02-03. Migrants Master Many by Re-counting in Block- & Per-numbers, having the abstract: Children show a surprising mastery of Many when re-counting totals in the same or in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Solving equations when re-counting from tens to blocks, and practicing proportionality and calculus when adding on-top and next-to, block-numbers offer a direct way to a quantitative competence that allows migrants help rebuild their original country.

04-05. Math: Useful Abstractions or an Undiagnosed Compulsory Cure? Conflicting Theories in Mathematics Education. The chapters are: Philosophical Controversies, Psychological Controversies, Sociological Controversies, and Mathematical Controversies.

06-07. A STEM-based Mathematics without Addition, having the abstract: Its many applications make mathematics useful. But of course, it must be learned before it can be applied. And since it is hard, there is no alternative to hard work? Well, observing the quantitative competence children bring to school, we discover as an alternative to the present set-based mathe-matics a Many-based 'many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount formula occurring all over the STEM subjects.

08-09. Cure Math Dislike with 1 Cup & 5 Sticks: New Meanings to Numbers, Counting, Operations & Fractions

10-11. Good Math & Goofy Math, which is Truer? The chapters are Good and Goofy Statements, Good and Goofy Concepts Good and Goofy Textbooks Good and Goofy Math

12-13. A Core STEM Curriculum for Young Migrants, having the abstract: Observing children's quantitative competence uncovers as an alternative to the present set-derived mathe-matics a Many-based 'many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. Results are predicted by a recount formula occurring all over the STEM subjects.

To the 42th PME Conference in Sweden I sent in a short oral communication and a poster.

14. Fifty Years of Ineffective Math Education Research, Why? Oops, Wrong Numbers, Sorry

15. Rethinking Line-Number Arithmetic as Block-Number Algebra

I sent in three proposals for the Mathematics Education in the Digital Age Conference 5-7 September 2018 - University of Copenhagen

16. A Count-before-Adding Curriculum for Preschool and Migrants, having the abstract: Children show a surprising mastery of Many with a quantitative competence where totals are re-counted in the same and in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional

LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Allowed to keep their block-numbers, children and migrants will be practising proportionality and calculus when adding block-number on-top and next-to; and will be solving equations when re-counting from tens to blocks.

17. Difference-research Saving Dropout Ryan with a TI-82 Calculator, having the abstract: At principal asked for ideas to lower the number of dropouts in pre-calculus classes. The author proposed using a cheap TI-82, but the teachers rejected saying students weren't even able to use a TI-30. Still the principal allowed buying one for a class. A compendium called 'Formula Predict' replaced the textbook. A formula's left- and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects.

18. Conflicting Theories Help Teachers Improve Mathematics Education, having the abstract: Traditionally, education is seen as teachers transferring institutionalized knowledge to individual learners. As such, education involves several choices. Shall teachers teach or guide? Is mathematics an eternal truth or a social construction? Is it knowledge about, or knowing how to? How to motivate learning? Should a class be optional or mandatory? To answer, teacher education refers to theory from philosophy, psychology and sociology. Including the existence of conflicting theories will allow teachers try out alternatives if wanting to improve mathematics education.

To the 2018 CTRAS Conference in China I sent three proposals

19. Addition-free Core STEM Curriculum for Late Learners along the Silk Road, having the abstract: Its many applications make mathematics useful. But to solve core STEM tasks you need no addition, thus calling for an addition-free curriculum. Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount-formula being a core in all STEM subjects.

20. Good, Bad & Evil Mathematics - Sociological Imagination in Math Education

21. Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum, having the abstract: Its many applications make mathematics useful; and of course, it must be learned before applied. Or, can it be learned through its original roots? Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. Asking 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a recount-formula occurring all over mathematics. Once counted, totals can be added next-to or on-top rooting calculus and proportionality. From this 'Count-before-Adding' curriculum, Many-matics offers remedial micro-curricula for classes stuck in a traditional curriculum.

The next article was published in Journal of Mathematics Education, March 2018, Vol. 11, No. 1,

22. Mastering Many by Counting, Re-counting and Double-counting before Adding On-top and Next-to, having the abstract: Observing the quantitative competence children bring to school, and by using difference-research searching for differences making a difference, we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

At the EARCOME8 conference in Taiwan I presented a lecture and a paper

23-24. Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions, plus a PPP

25-26. The Simplicity of Math reveals a Core Curriculum, plus a PPP

The next paper addresses the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11), the Thematic Working Group 26, Mathematics in the Context of STEM Education. The paper was rejected for presentation.

27. Addition Free Migrant-Math Rooted in STEM Re-Counting Formulas, having the abstract: STEM typically contain multiplication formulas expressing re-counting in different units, thus calling for an addition-free curriculum. The mastery of Many children bring to school uncovers a Many-based 'Many-matics' as an alternative to the present self-referring set-based mathe-matics. To answer the question 'How many in total?' we count and re-count totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a re-count formula as a core formula in all STEM subjects.

The next two paper address the ICMI Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, Tsukuba, 26-30 November 2018. It was accepted for presentation.

28-29. A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many, having the abstract: Mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum. Plus a PPP with 77 slides.

30. A New Curriculum - But for Which of the 3x2 Kinds of Mathematics Education, an essay on observations and reflections at the ICMI study 24 curriculum conference, having the abstract: As part of institutionalized education, mathematics needs a curriculum describing goals and means. There are however three kinds of mathematics: pre-, present and post-'setcentric' mathematics; and there are two kinds of education: multi-year lines and half-year blocks. Thus, there are six kinds of mathematics education to choose from before deciding on a specific curriculum; and if changing, shall the curriculum stay within the actual kind or change to a different kind? The absence of federal states from the conference suggests that curricula should change from national multi-year macro-curricula to local half-year micro-curricula; and maybe change to post-setcentric mathematics.

Allan Tarp, Aarhus Denmark, December 2018

# 01. Posters at the 2018 Swedish Biennale Stand

## ●Math Dislike Cured by 1 Cup & 5 Sticks

## ●Migrant Math for STEM Teachers/Engineers

**INTRO:** Saving the Princess with BundleNumbers

**LEFT**

Good Math: MANY-Math, Tales about Totals
Bad Math: SET-Math, Tales about LineNumbers
Evil MATH: Fraction-Math, Tales about Operators
Good Math: Icons, Bundling, ReCounting & PerNumbers
Core Math from Childhood
Grand Theory in Math Ed Research & Difference Research

**MIDDLE**

Math Dislike CURED by 1 Cup & 5 Sticks
Improving Schools in Sweden
Migrant-Math making migrants STEM-Teachers or Engineers
Count before you Add
Kids own Math
Activities

**RIGHT**

1Year online CATS-Course
1Week STEM-Course
Is Math True always or sometimes? Is Mathematics well-defined?
PYRAMIDeDUCATION & Material
Beware of Institutions & Teachers & Research & Forced Classes
Good & Bad & Evil Education
Rejected Research Papers

**Rejected Research Papers:** "Math Competenc(i)es- Catholic or Protestant." "The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants."

***(STEM: Science + Technology + Engineering + Mathematics)***

# MATHeCADEMY.net

# Saving the Princess with BundleNumbers

Once upon a time, a Princess was stuck in division. She simply could not do 336/7 and locked herself in behind a bush of thorns. The King summoned all the Wise who agreed that the Princess should be motivated by reformulating the task to split 336 among 7. Only a newcomer objected that the task was to recount 336 in 7s. “Here we all count in tens, so please wait at the lawn outside.” To solve the disagreement whether 7 should be above or below or to the right or left of 336, the Wise recommended all methods tried out together with an alternative method saying no method at all allowing the Princess to invent her own method. But nothing helped.

“Are there no other methods? Who is out on the lawn?” the King asked. “Just a newcomer with crazy ideas”. But in spite of strong protests from the Wise the King let him in.

“You also want to teach me division?”, the Princess asked. “No, I bring you a cup with 5 sticks that we will count.” “But they are already counted?” “ We will count them in bundles of 2s. As we see on our hand, this can be done in three ways: as 1 bundle & 3, as 2 bundles & 1, and as 3 bundles less 1. Using the cup for the bundles, we see that all numbers have inside bundles and outside singles; and that a total can be counted in the standard way or with an outside overload or an underload.”

“But isn’t 336 a name for a point far out on a number-line?” the Princess asked.

“No. 336 is not a line-number as everyone claims, it is a bundle-number. Asking 3year-olds “How old next time?” they say 4; but object to 4 fingers held together 2 by 2: “That is not 4 that is 2 2s.”


Children both see and count the bundles; and come to school with 2dimensional bundle-numbers or block-numbers with the core of mathematics inside them: 3 2s may be added to 1 4s in two ways; on-top, the units must be the same, and changing units is just another word for proportionality; and next-to means adding areas which is just another word for integral calculus.

So 336 is a bundle-number with 33 bundles inside and 6 singles outside. Wanting 28 bundles inside we move 5 bundles outside; so 33B6 and 28B56 is the same number just recounted with an overload. Counted in 7s we have 4 inside and 8 outside.

Consequently, a block of 33 tens & 6, or 336, can be recounted as a block of 48 7s; which makes sense since a shorter width calls for a larger height.”

All of a sudden, the thorns changed to roses. The Newcomer got the Princess and half the kingdom where they lived happily ever after.

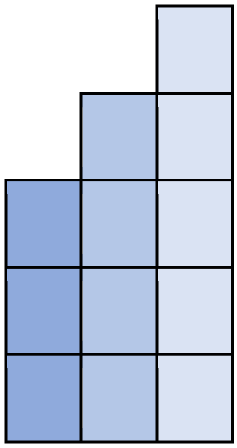
# Good Math: MANY-Math

## Tales about Totals

*MANY-matics: A Natural Science about MANY*

**Meeting MANY we ask:**

***"How Many in Total?".***



To answer, we Bundle and Stack in Blocks

$T = 345 = 3*BB + 4*B + 5*1$, or

$T = 345 = 3*B^2 + 4*B + 5*1$,

**The SIMPLICITY of MANY-Math**

***First Iconize & Count & ReCount,***

***then Add OnTop & NextTo***

**4 ways to add: + , * , ^ , ∫**

| Algebra unite/*split* | Variable | Constant |
|---|---|---|
| Unit-numbers | $T = a+n$<br>*$T-a = n$* | $T = a*n$<br>*$T/a = n$* |
| Per-numbers | $T = \int a\, dn$<br>*$dT/dn = a$* | $T = a^n$<br>*$\log_a(T)=n$, $\sqrt[n]{T}=a$* |

# Bad Math: SET-Math

## Tales about Numbers

*META-matics: concepts as examples of abstractions, not as abstractions of examples*

**00. Digits are symbols, not icons**

**01. Numbers are 1dimensional linear names, not 2dimensional blocks**

**02. Only ten-counting, no icon-counting**

**03. No 'T=', only 42; not 'T = 4.2 tens'**

**04. Add & Subtract before Multiply & Divide**

**05. Only OnTop addition - no NextTo addtion**

**06. 6*7 IS 42 – not 6 7s or 4.2 tens**

**07. 8/4 IS 8 split by 4; not 8 counted in 4s**

**08. No recounting to create or remove over- or underloads when operating on numbers**

**09. Solving equations by neutralizing; not by recounting in icons or reversed operations**

**10. Functions as set-relations; not as number-language sentences about the Total**

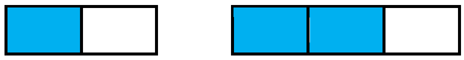
**11. Plane before coordinate geometry; not trigonometry before coordinate geometry**

**12. Differential before Integral Calculus.**

# Evil MATH: Fraction-Math

## Tales about Operators

*Mathema-TISM: True inside, but seldom outside*



**Claim: 1/2+2/3 IS 7/6**

*But 1 blue of 2 + 2 of 3 is 3 blues of 5,*
*and not 7 blues of 6?*

**Claim: 2+3 IS 5**

*But 2weeks + 3days is 17days?*

**Never ADD without units**

**00. Fractions are numbers; not operators needing numbers to become numbers**

**01. Fractions add without units; not with units making it integral calculus**

**02. Fractions before percentages and decimals; not the inverse order**

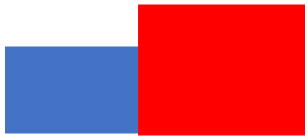
**03. Fractions are equivalence classes in a set product; not per-numbers from double-counting in the same unit**

# Core Math from Childhood

## Proportionality in Primary & Middle School

- Recount to change unit: 2 **3s** = (2*3/5)*5 = 1.1 **5s**
- With 2**$**/5**kg**, 20**kg** = (20/5)*5**kg** = (20/5)*2**$** = 8**$**
- Adding OnTop, 2 **3s** + 4 **5s** = ? **5s**

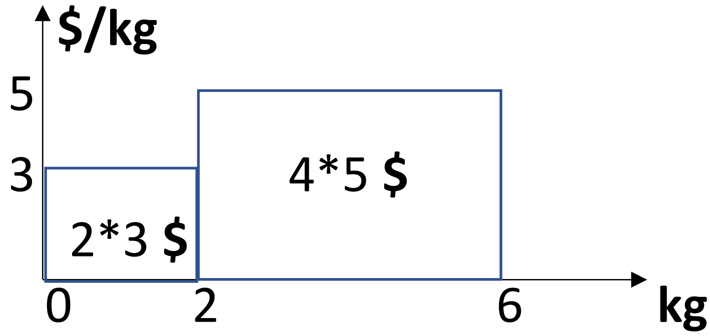
## Calculus in Primary & Middle School



- Adding NextTo 2 **3s** + 4 **5s** = ? **8s**

| | | | |
|---|---|---|---|
| | 2 **kg** | at | 3 **$/kg** |
| + | 4 **kg** | at | 5 **$/kg** |
| | (2+4)**kg** | at | ~~(3+5)~~ **$/kg** |

**Unit-numbers** add on-top.
**Per-numbers** add next-to as areas under the per-number graph.




## Formulas in Primary & Middle School

- NumberLanguage Sentences about the Total, T = 4.2 tens = 42
- The general NumberFormula T = 3***B**^2+4***B**+5 with its examples
- T = a*x^2 + b*x + c; T = m*x, T = m*x + c, T = a*x^n, T = a*n^x

## Equations in Primary & Middle School

**Recount icons ← tens: x*7 = 42** = (42/7)*7, x = 42/7 (***opposite side & sign***)

### The ReCount Formula is all over Mathematics

**ReCount-formula: <u>T = (T/B)*B</u>** ‘from T, T/B times, B is taken away’

| | | |
|---|---|---|
| ReCounting | T = (T/B)*B | T = 8 = (8/2)*2 = 4*2 = 4 2s |
| Proportionality | $ = ($/kg)*kg | $ = price*kg |
| Coordinate Geometry | Δy = (Δy/Δx)*Δx | Δy = m*Δx |
| Differential Calculus | dy = (dy/dx)*dx | dy = y’ * dx |
| Trigonometry | a = (a/c)*c = sinA*c; | a = (a/b)*b = tanA*b |
| Linearity | y = k*x | F = m*a, dist = vel*time, … |
| Eigenvalues | HΨ = E*Ψ | Schroedinger Equation |

# Grand Theory in Math Ed Res.

**BAUMAN & WEBER:** ***Beware of Goal-Means exchanges***
**ARENDT:** ***- and of the Banality of Evil***
**HEIDEGGER:** ***Respect the Subject & question the Predicate***
**FOUCAULT:** ***- also question Cures and Institutions***

## Difference Research

### Finding Differences making a Difference

***Almost Everything can be Different***

| | |
|---|---|
| **DIGITS:** | **Icons vs. symbols** |
| **NUMBERS:** | **2D blocks vs. 1D lines** |
| **OPERATIONS:** | **Icons vs. inter-set mappings** |
| **ADDITION:** | **OnTop/NextTo vs. after** |
| **MULTIPLICATION:** | **ReCounting to tens vs. tables** |
| **DIVISION:** | **ReCounting from tens vs. splitting** |
| **RE-COUNTING:** | **Changing units vs. neglect** |
| **DOUBLE-COUNTING:** | **Proportionality vs. neglect** |
| **PER-NUMBERS:** | **Core numbers vs. neglect** |
| **FRACTIONS:** | **PerNumbers vs. rational numbers** |
| **FRACTIONS:** | **Operators vs. rational numbers** |
| **FORMULAS:** | **Total statements vs. inter-set relations** |
| **EQUATIONS:** | **ReCount in icons vs. open statements** |
| **GEOMETRY:** | **Trigonometry before coord. geometry vs. plane geometry first** |
| **POLYNOMIALS:** | **Number-formulas vs. functions** |
| **CALCULUS:** | **Integrate bef. differentiate vs. inverse** |

# 1Year online CATS-Course

## CATS: Count & Add in Time & Space

| | **Self Instructing QUESTIONS** 1: Primary, 2: Secondary School |
|---|---|
| **C1 COUNT** | How to count Many?<br>How to recount 8 in **3s**: T= 8 = ? **3s**<br>How to count in standard bundles?<br>How to recount 6**kg** in **$ with 2$/4kg**: T = 6**kg** = ?**$** |
| **C2 COUNT** | How can we count possibilities?<br>How can we predict unpredictable numbers? |
| **A1 ADD** | How to add blocks concretely?<br>T = 27 + 16 = 2**ten**7 + 1**ten**6 = 3**ten**13 = ?<br>How to add blocks abstractly? |
| **A2 ADD** | What is a prime and a folding number?<br>What is a per-number?<br>How to add per-numbers? |
| **T1 TIME** | How can counting & adding be reversed ?<br>Counting ? **3s** and adding 2 gave 14.<br>Can all calculations be reversed? |
| **T2 TIME** | How to predict the terminal number<br>• If the change is constant?<br>• If the change is variable, but predictable? |
| **S1 SPACE** | How to count plane and spatial properties of blocks and round objects? |
| **S2 SPACE** | How to predict the position of points and lines?<br>How to use the new calculation technology? |
| **QL** | What is quantitative literature? Does it also have the 3 different genres: fact, fiction and fiddle? |

# 1Week STEM-Course

## The Simplicity of Math:

*First Count & ReCount - then Add OnTop & NextTo*

**Day 01. Good & Bad & Evil Math in General**

***The root of math: MANY or SET***

**Day 02. Good & Bad & Evil Math in Primary School**

***Iconize & Count & ReCount before you ADD***

**Day 03. Good & Bad & Evil Math in Middle School**

***DoubleCounting and PerNumbers vs. Fractions***

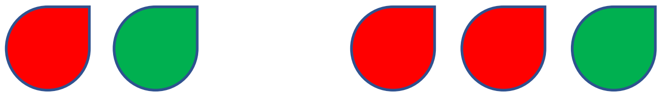
**Day 04. Good & Bad & Evil Math in High School**

***Calculus: adding locally constant PerNumbers***

**Day 05. Good Math in a STEM setting**

***PerNumbers Predicting Matter in Time and Space***

# Is Math True always or sometimes?

| *Is this True* | **Always** | **Never** | **Sometimes** |
|---|---|---|---|
| **2 + 3 = 5** | | | X<br>*2weeks + 3days = 17days; only with the same unit* |
| **2 x 3 = 6** | X<br>*2x3 is 2 **3s** III III that can always be recounted as 6 **1s*** | | |
| $\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$ | <br>*1 red of 2 apples + 2 of 3 is 3 of 5, and not 7 of 6* | | X |
| $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$ | *Only if taken of the same total* | | X |
| **A FUNCTION is** | • **<u>for example 2+x, but not 2+3</u>;** i.e. a name for a calculation with an unspecified number (before SET, 1750-1900)<br>• **<u>an example of a set relation</u>,** where first component identity implies second component identity (after SET, 1900) | | |

# Is Mathematics Well Defined?

## Ancient Greece

A common **LABEL** for **Quadrivium**: Arithmetic, Geometry, Music & Astronomy (Many by itself and in space & time). **Trivium**: Grammar, Logic, Rhetoric

## PreModern

A common **LABEL** for Arithmetic & Geometry; different from "Rechnung".

## Modern

A self-referring **SET** of Proofs about **SET**-derived Concepts.

## PostModern

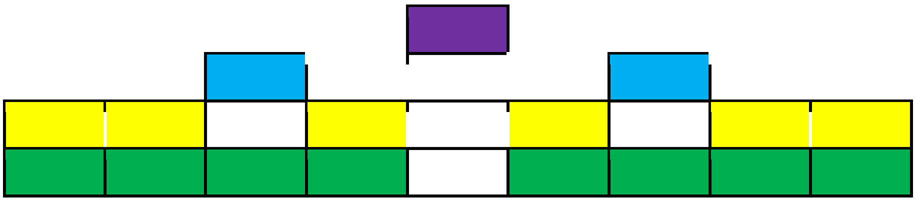
**Many**-math: A **Natural SCIENCE** Counting & Adding & Predicting Many.

# PYRAMIDeDUCATION

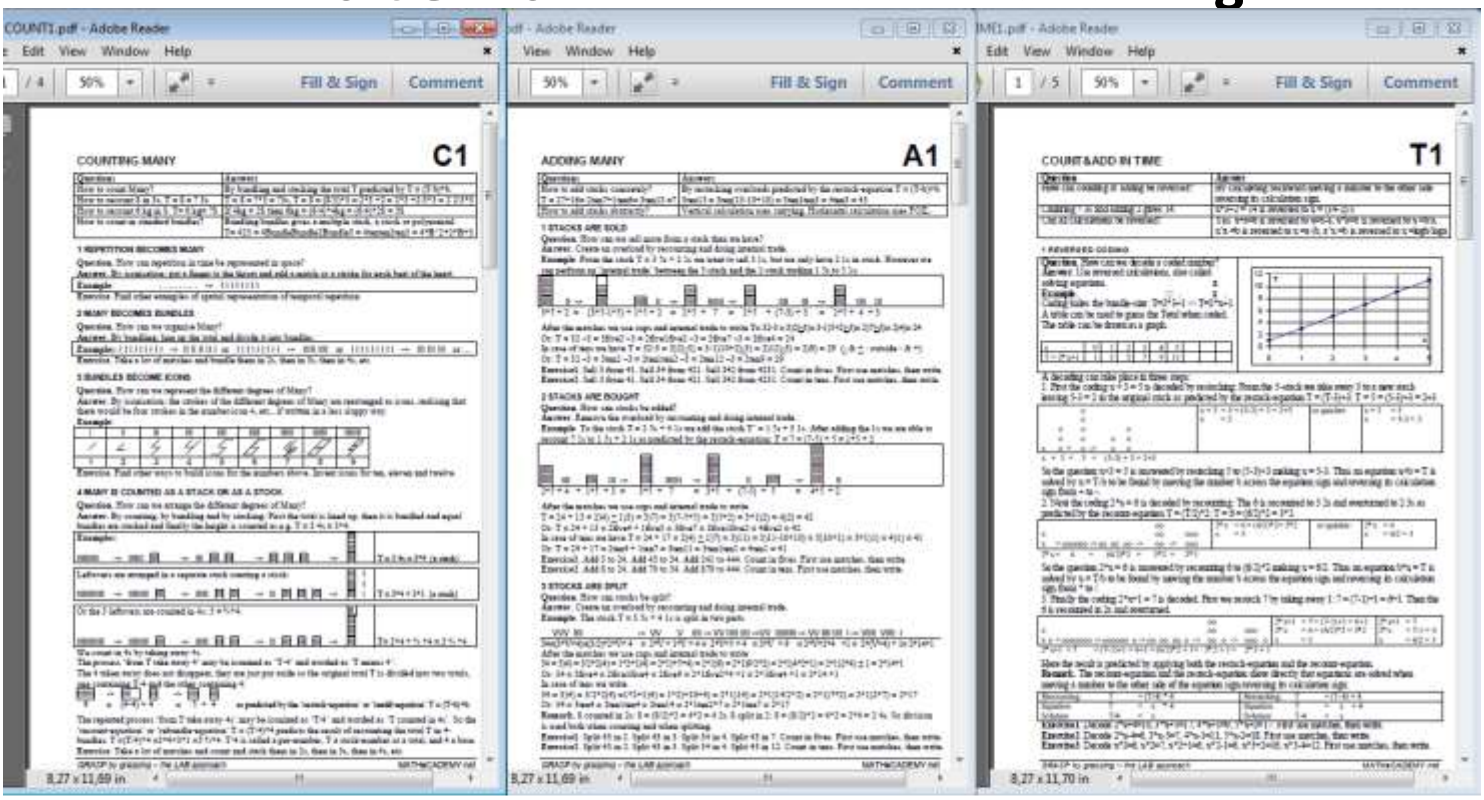
8 learners organized in 2 teams with 2 instructors and 3 pairs by turn.

- Each pair works together to solve **C**ount&**A**dd problems.
- The coach assists the instructors when instructing their team and when correcting the **C**ount&**A**dd assignments.
- Each learner pays by coaching a new group of 8 learners.

1 Coach
2 Instructors
3 Pairs
2 Teams



## Material: short & self-instructing




## YouTube Videos

- Postmodern Mathematics Debate
- CupCounting removes Math Dislike
- IconCounting & NextTo-Addition
- PreSchool Mathematics
- Fractions
- PreCalculus & Calculus
- Mandarin Mathematics
- World History

# Beware of

**INSTITUTIONS** exchanging goals and means

- The goal of math education is to ~~learn math~~ master Many with quantitative competence

**TEACHERS** teaching Bad & Evil math

- LINE-numbers instead of BLOCK-numbers
- Addition before Counting & Multiplication
- Adding Fractions without units
- Differential before Integral Calculus

**RESEARCH** uncritically

- researching itself instead of math education
- exemplifying instead of questioning itself
- accepting math as self-referring MetaMath
- accepting 50 years of unsuccessful research

**FORCED CLASSES**

- Constraining young people to stay with their age-group for several years - instead of choosing their own daily ½year blocks in order to uncover and develop their personal talent

# Rejected Research Papers

Allan Tarp, MATHeCADEMY.net, the 2018 MADIF Conference

## The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Young Male Migrants

*Educational shortages described in the OECD report 'Improving Schools in Sweden' challenge traditional math education offered to young male migrants wanting a more civilized education to return help develop and rebuild their own country. Research offers little help as witnessed by continuing low PISA scores despite 50 years of mathematics education research. Can this be different? Can mathematics and education and research be different allowing migrants to succeed instead of fail? A different research, difference-research finding differences making a difference, shows it can. STEM-based, mathematics becomes Many-based bottom-up Many-matics instead of Set-based top-down Meta-matics.*

REJECTED

## Math Competenc(i)es - Catholic or Protestant?

*Introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015 OECD-report 'Improving Schools in Sweden'. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.*

REJECTED

## 02. Migrants Master Many by Re-counting in Block- & Per-numbers - Short

Observing the quantitative competence children bring to school, and using difference-research, finding differences making a difference, we discover as a difference to the present set-based mathematics a Many-based: ‘many-matics’.

Here digits are icons with as many sticks as they represent. As are operations where division, multiplication and subtraction allow a total of seven to be bundle-counted as T = 2B1 3s, i.e. as a number-language sentence with a subject, a verb and a predicate as in the word-language; thus producing two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads or underloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations.

Here double-counting in two units creates per-numbers, becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers.

Here addition occurs on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes plane and coordinate geometry.

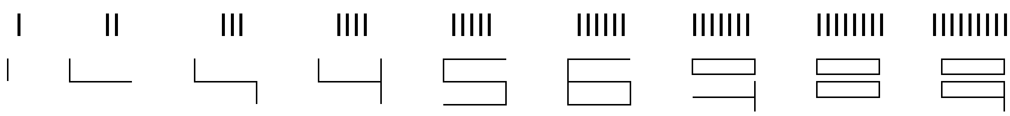
Solving equations when re-counting from tens to blocks, and practising proportionality and calculus when adding on-top and next-to, block-numbers offer a direct way to a quantitative competence that allows migrants help rebuild their original country.

## 03. Migrants Master Many by Re-counting in Block- & Per-numbers - Long

*Children show a surprising mastery of Many when re-counting totals in the same or in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Solving equations when re-counting from tens to blocks, and practising proportionality and calculus when adding on-top and next-to, block-numbers offer a direct way to a quantitative competence that allows migrants help rebuild their original country.*

### Meeting Many

Difference-research, finding differences making a difference (Tarp, 2017), uncovers a different ‘Many-matics’, mastering Many by bundling and stacking. To count, first we rearrange sticks into icons with e.g. five sticks in the 5-icon if written less sloppy. Counted as ‘one bundle’, ten needs no icon.



1 2 3 4 5 6 7 8 9

Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, a number of bundles that may or may not be re-counted as ones. This inspires ‘bundle-counting’, re-counting a total in icon-bundles. Thus, a total $T$ of 5 1s is re-counted in 2s as $T$ = 2 2s & 1. Here the bundles can be placed inside a bundle-cup with a stick for each bundle, leaving the unbundled singles outside; and described by ‘bundle-writing’, $T$ = 2$B$1 2s, or ‘decimal-writing’, $T$ = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup:

$T$ = 5 = I I I I I → II II I → [II] I → 2$B$1 2s = 2.1 2s

Entering ‘5/2’, we ask a calculator ‘from 5 we take away 2s’ The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking ‘5 – 2*2’. The answer, 1, predicts that 5 = 2$B$1 2s = 2.1 2s as indirectly predicted on the bottom line.

| | |
|---|---|
| 5 / 2 | 2.some |
| 5 – 2 * 2 | 1 |

We see that also operations are icons: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away bundles, and subtraction shows the trace left when taking away a stack only once.

A calculator thus uses a 're-count formula', $T = (T/B)*B$, to predict that 'from $T$, $T/B$ times, $B$s can be taken away'. This re-count formula occurs all over mathematics: when relating proportional quantities as $y = c*x$; in trigonometry as *sine* and *cosine* and *tangent*, e.g. $a = (a/c)*c = sinA*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

***Re-counting in the same unit and in a different unit***

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 2*B*1 2s as 1*B*3 2s with an outside 'overload'; or as 3*B*-1 2s with an outside 'underload' thus rooting negative numbers.

Re-counting in a different unit means changing unit, also called proportionality or linearity. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2*B*2 5s.

Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s' The answer, 2.some, predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer, 2, predicts that 3 4s can be re-counted in 5s as 2*B*2 5s or 2.2 5s.

***Re-counting to and from tens***

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

Re-counting from tens to icons by asking '38 = ? 7s' is called an equation $x*7 = 38$. It is easily solved by re-counting 38 in 7s: $x*7 = 38 = (38/7)*7$. So $x = 38/7 = 5$ & 3/7 as predicted by a calculator showing that 38 = 5.3 7s = 5*7 + 3.

***Double-counting creates proportionality as per-numbers***

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg.

To answer the question '$T$ = 6$ = ?kg', we re-count 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa: Asking 'T = 12kg = ?$ ', the answer is 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

***Once counted, totals can be added on-top or next-to***

Adding on-top in 5s, '3 5s + 2 3s = ? 5s?', re-counting must make the units the same. Asking a calculator, the two answers, '4.some' and '1', predict the result as 4*B*1 5s. Since 3*5 is an area, adding next-to in 8s, '3 5s + 2 3s = ? 8s?', means adding areas, called integral calculus. Asking a calculator, the two answers, '2.some' and '5', predict the result as 2*B*5 8s.

***Reversing adding on-top and next-to***

Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking '3 5s and how many 3s total 2*B*6 8s?', using sticks will give the answer 2*B*1 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula $y' = \Delta y/t = (y2 – y1)/t$.

## References

Tarp, A. (2017). *Math ed & research 2017.* Retrieved from http://mathecademy.net/ 2017-math-articles/.

## 04. Math: Useful Abstractions or an Undiagnosed Compulsory Cure? Conflicting Theories in Mathematics Education - Short

Many countries face poor PISA results despite 50 years of research. Is this caused by conflicting theories?

Within philosophy, ancient Greece saw a controversy between the sophists, warning against choice presented as nature, and the philosophers seeing choice as an illusion since the physical is but examples of metaphysical forms. The natural science reinvention of scepticism inspired the two Enlightenment republics, the American and the French, transforming scepticism into pragmatism and post-structuralism.

Within psychology, a controversy exists between Vygotsky and Piaget recommending teaching as much and as little as possible.

Within sociology, a controversy exists between a structure and an actor viewpoint, reflecting societies with high or low degree of institutionalization. Thus, Foucault sees knowledge as socially constructed discourses; and describes schools as 'prispitals' mixing power techniques from a prison and a hospital: the learners are fixed in classrooms and diagnosed as ignorant to be cured.

Today two mathematics discourses exist. One is the institutionalized presentation of a body of self-referring knowledge being useful through its many applications.

The other is a silenced discourse seeing mathematics as a natural science about Many expressing itself in number-language sentences with a subject and a verb and a predicate as in the word-language.

## 05. Math: Useful Abstractions or an Undiagnosed Compulsory Cure? Conflicting Theories in Mathematics Education - Long

### Philosophical Controversies

Ancient Greece saw two forms of knowledge, called 'sophy'. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. The Christian Church eagerly took over a metaphysical patronage and changed the academies into monasteries, until the Reformation changed some back again.

By letting the laboratory precede the library, natural science reinvented scepticism. Newton discovered that falling objects obey their own will instead of that of a patronizor. This inspired the Enlightenment Century and it two republics, the American and the French, transforming scepticism into pragmatism and post-structuralism, based upon existentialism defined by Sartre as 'Existence preceding essence'; and with the Heidegger warning: in a sentence, respect the subject, but question the predicate since it might be gossip. Thus, post-structuralism deconstructs ungrounded diagnoses forcing humans to accept unfounded patronization.

### Psychological Controversies

As to how learners acquire knowledge, several constructivist theories exist among which are Vygotskian and Piagetian social and radical constructivism disagreeing by recommending teaching as much and as little as possible.

Vygotsky sees knowledge as essence to be transferred by good teaching. However, a learner can only take in unknown sentences about subjects already known, so the teacher must know the individual 'zone of proximal development' in order to successfully connect it to the institutionalized knowledge by scaffolding.

Whereas Piaget recommends meeting existence directly to allow learners form individual concepts and sentences to be negotiated and accommodated socially.

**Sociological Controversies**

As a social institution, education can be seen from a structure or an actor viewpoint, reflecting societies with high or low degree of institutionalization. Being highly institutionalized, continental Europa has developed a structure-based sociology seeing humans as bound by social structures. Thus, Foucault sees knowledge as socially constructed discourses; and describes a school as a 'prispital' mixing the power techniques of a prison and a hospital: the learners are fixed in classrooms and diagnosed as ignorant to be cured by discourses institutionalized as truth but instead exerting 'pastoral power'. Whereas their escape from Europe made Americans actors developing grounded theory, lifting Piagetian accommodation to a social level.

**Mathematical Controversies**

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about Many by itself, in space, in time, and in space and time. With astronomy and music gone, today mathematics should be a common label for geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about fifty years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics'. Here the invention of the concept Set created a 'meta-matics' defining concepts as examples of abstractions instead of as abstractions from examples. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: In the set M of sets not belonging to themselves, M belongs only if it does not.

So today two mathematics discourses exist. One discourse is the institutionalized presenting mathematics as a body of self-referring knowledge that shows its usefulness through its many applications; but of course, to be applied, first it must be learned, even if this may be hard as shown by poor PISA results.

The other is a silenced discourse seeing mathematics as a natural science about Many expressing itself in number-language sentences with a subject and a verb and a predicate as in the word-language. And showing the silenced differences:

Numbers could be icons & predicates in Tales of Many, T = 2 3s = 2*3. Instead they are changed from predicates to subjects by silencing the real subject, the total; and place-values hide the bundle structure.

Operations could be icons for the counting process as predicted by the re-count formula T = (T/B)*B, 'from T, T/B times, B can be taken away'. Instead they hide their icon-nature and their role in counting; and they are presented in the opposite order.

Addition could wait to after counting & recounting & double-counting have produced unit- and per-numbers. Instead it silences counting and next-to addition; and silences bundling; and uses carry instead of overloads; and assumes numbers as ten-based.

Fractions could be per-numbers coming from double-counting in the same unit and added with units by areas (integration) since they are, not numbers, but operators needing numbers to become numbers. Instead they are defined as rational numbers that can be added without units.

Equations could be re-counting from tens to icons and reversing on-top and next-to addition. Instead they are defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra.

Proportionality could be re-counting in another unit when adding on-top; or double-counting producing per-numbers and fractions. Instead it is defined as multiplicative thinking.

Trigonometry could be mutual recounting of the sides in a block halved by its diagonal. Instead it is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.

Functions could be number-language's sentences or formulas, T = 2*3, with subject & verb & predicate. Instead they are set-relations where first-component identity implies second-component identity.

Calculus could occur in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers. Instead differential calculus precedes integral calculus, presented as anti-differentiation.

### References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## 06. A STEM-based Mathematics without Addition - Short

Its many applications make mathematics useful. But of course, it must be learned before it can be applied. And since it is hard, there is no alternative to hard work.

Observing the quantitative competence children bring to school, we discover as an alternative to the present set-based mathe-matics a Many-based 'many-matics'.

with the number of sticks they represent, digits become icons. As do operations where division, multiplication and subtraction allow a total of seven to be bundle-counted as T = 2B1 3s, i.e. as a number-language sentence with a subject, a verb and a predicate as in the word-language. These two-dimensional block-numbers are ready to be re-counted in the same unit to remove or create overloads or underloads to ease operations; or in a new unit, called proportionality; or to and from tens rooting tables and solving equations.

Here double-counting in two units creates per-numbers, becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers.

Counting involves division and multiplication, predicted by a 'recount formula' T = (T/B)*B, occurring in all the STEM subjects science, technology, engineering and math, e.g. meter = (meter/sec)*sec = speed*sec.

So recounting and STEM tasks should precede addition.

## 07. A STEM-based Mathematics without Addition - Long

*Its many applications make mathematics useful. But of course, it must be learned before it can be applied. And since it is hard, there is no alternative to hard work? Well, observing the quantitative competence children bring to school, we discover as an alternative to the present set-based mathematics a Many-based 'many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount formula occurring all over the STEM subjects.*

### Meeting many

Meeting many, we ask 'How many in total?' To answer, we count. Holding 4 fingers together 2 by 2, a 3year-old will say 'That is not 4, that is 2 2s', thus describing what exists, a number of bundles that may or may not be recounted as ones.

This inspires 'bundle-counting', recounting a total in icon-bundles. Thus, a total *T* of 5 1s is recounted in 2s as *T* = 2 2s & 1; and is described by 'bundle-writing', *T* = 2*B*1 2s, or 'decimal-writing', *T* = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

Entering '5/2', we ask a calculator 'from 5 we take away 2s'. The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking '5 – 2*2'. The answer, 1, predicts that 5 = 2*B*1 2s = 2.1 2s as indirectly predicted on the bottom line.

| | |
|---|---|
| 5 / 2 | 2.some |
| 5 – 2 * 2 | 1 |

We see that also operations are icons: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away bundles, and subtraction shows the trace left when taking away a stack only once.

A calculator thus uses a 'recount formula', $T = (T/B)*B$, to predict that 'from *T*, *T/B* times, *B*s can be taken away'. This recount formula occurs all over mathematics: when relating proportional quantities as $y = c*x$; in trigonometry as *sine* and *cosine* and *tangent*, e.g. $a = (a/c)*c = sinA*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/ \Delta x)* \Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

***Recounting in the same unit and in a different unit***

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2*B*1 2s as 1*B*3 2s with an outside 'overload'; or as 3*B*-1 2s with an outside 'underload' thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48.

Recounting in a different unit means changing unit, also called proportionality or linearity. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2*B*2 5s.

Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s' The answer, 2.some, predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer, 2, predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

***Recounting to and from tens***

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $x$*7 = 35. It is easily solved by recounting 35 in 7s: $x$*7 = 35 = (35/7)*7. So $x$ = 35/7, showing that equations are solved by moving to opposite side with opposite calculation sign.

***Double-counting creates proportionality as per-numbers***

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘$T$ = 6$ = ?kg’, we recount 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

***Mathematics in STEM subjects***

STEM (Science, Technology, Engineering and Mathematics) combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature’s physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots will help transforming nature into human necessities.

Nature consists of things in motion, combined in the momentum = mass*velocity. Things contain mass and molecules and electric charge. Thus, nature is counted in meter and second and kilogram and mole and coulomb.

Looking at the list of formulas we see that nature is predictable by recounting & per-numbers. Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus (Tarp, 2017).

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
meter = (meter/second) * second = velocity * second
Δ momentum = (Δ momentum/second) * second = force * seconds
Δ energy= (Δ energy/meter) * meter = force * meter = work
energy = (energy/kg/degree) * kg * degree = heat * kg * degree
force = (force/square-meter) * square-meter = pressure * square-meter
gram = (gram/mole) * mole = molar mass * mole
mole = (mole/litre) * litre = molarity * litre

## References

Tarp, A. (2017). *Math ed & research 2017.* Retrieved from http://mathecademy.net/ 2017-math-articles/.

## 08. Cure Math Dislike with 1 Cup & 5 Sticks: New Meanings to Numbers, Counting, Operations & Fractions - Short

With a cup for the bundles, 5 sticks are ‘bundle-counted’ in 2s as 2B1 2s.

Moving bundles outside or inside creates ‘overload’ or ‘underload’: 5 = 2B1 = 1B3 = 3B-1 2s.

This eases operations: 336/7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

With bundle-counting, numbers are 2D block-numbers with units, instead of 1D line-numbers.

Counting 5 in 2s, division takes away 2s, multiplication stacks the bundles, and subtraction takes away the stack to find unbundled singles.

Operations are icons, as are digits, containing as many sticks as they represent.

A calculator predicts with a ‘re-count formula’ T = (T/B)*B’ saying that ‘from T, T/B times we take B away from T’; and occurring all over mathematics.

Re-counting in a new unit means changing units, called proportionality.

Re-counting from icons to tens, multiplication predicts the result directly.

Re-counting from tens to icons is called an equation.

Double-counting in different units creates per-numbers as 4$/5kg, becoming fractions with like units.

Once counted, totals add on-top or next-to, rooting proportionality and integral calculus.

Reversing on-top or next-to addition roots solving equations and differential calculus.

Thus, calculus also appears as next-to addition of block-numbers and as adding fractions with units.

## 09. Curing Math Dislike with 1 Cup & 5 Sticks: New Meanings to Numbers, Counting, Operations & Fractions - Long

Division seems hard, but not with 1cup & 5sticks. Using a cup for the bundles, a total T of 5 sticks is ‘BundleCounted’ in 2s as T = 5 = I I I I I = II II I = 2)1 2s = 2B1 2s = 2.1 2s using a decimal point to separate the bundles from the singles. A total thus has an inside number of bundles and an outside number of singles.

Bundle-counting 7 in 3s thus leads to a 2dimensinal LEGO-like block-numbers T = 2B1 3s, different from the 1dimensional line-numbers taught in school; and containing three digits:  the size, the inside bundles and the outside singles.

Including bundles in the counting sequence, we count 0Bundle1, 0B2, …, 0B9, 1B0, 1B1. Or 1B-1, 1B0, 1B1.

To ease operations, we re-count in the same unit by moving a bundle outside or inside the bundle-cup to create an ‘overload’ or ‘underload’: T = 5 = 2)1 2s = 1)3 2s = 3)-1 2s. This also applies when counting in tens: T = 42 = 4B2 = 3B12 = 5B-8.

Adding: T = 35 + 47 = 3B5 + 4B7 = 7B12 = 8B2 = 82

Subtracting: T = 75 – 47 = 7B5 – 4B7 = 3B-2 = 2B8 = 28

Multiplying: T = 7 x 48 = 7 x 4B8 = 28B56 = 33B6 = 336.

Dividing: T = 336/7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

Sticks or a folding ruler show digits as icons with as many sticks as they represent if written les sloppy. Operations are icons also: To count 7 in 3s we take away 3 many times iconized by a broom wiping away the 3s. Showing 7/3 = 2.some, a calculator predicts that 2 times we can take 3 away from 7. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1.

A calculator thus predicts results by a ‘re-count formula’ T = (T/B)*B’ saying that ‘from T, T/B times we can take B away from T’, as e.g. T = 8 = (8/4)*4 = 2*4 = 2 4s. It occurs all over mathematics: as y = c*x; as a = (a/c)*c = sinA*c; as lie gradients, delta-y = (delta-y/delta-x)* delta-x = c* delta-x; and as derivatives, dy = (dy/dx)*dx = y’*dx.

Re-counting in a new unit means changing units, called proportionality. Again the re-count formula predicts the result: with T = 4 5s = ? 6s; first (4*5)/6 = 3.some; then (4*5) – (3*6) = 2. So: T = 4 5s = 3.2 6s.

Re-counting from icons to tens, multiplication predicts the result directly. Only, the calculator leaves out the unit and misplaces the decimal point: Asking T = 3 7s = ? tens, the answer is 3x7 = 21 i.e. 2.1 tens. Geometrically it makes sense that increasing the width of a block from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.

Re-counting from tens to icons is called an equation, e.g. T = 35 = ? 7s. Using u for the unknown number, we solve the equation by re-counting 35 in 7s: u*7 = 35 = (35/7)*7, so u = 35/7 = 5. Geometrically it makes sense that decreasing the width of the block from ten to 7 means increasing its height from 3 to 5 to keep the total unchanged.

The ‘move to opposite side with opposite sign’ method applies to all equations: u+2 = 8 is solved by u = 8-2; u^3 = 20 is solved by u = 3√20, and 3^u = 20 is solved by u = log3(20), where root and logarithm is introduced as a factor-finder and a factor-counter.

Double-counting in different units creates per-numbers as 4$/5kg or 4/5 $/kg. With 20 kg, we recount 20 in 5s: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. With 60$, we recount 60 in 4s: T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

Double-counting in the same unit creates fractions and percentages as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%.

We see that per-numbers and fractions are not numbers, but operators needing a number to become a number.

Counting thus involves dividing and multiplying and subtracting to predict that 7 = 2B1 3s = 2.1 3s. Geometrically, placing the unbundled single next-to the block of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the block makes it 1/3 3s, so 1/3 3s = 0.1 3s.

Once counted, totals can be added on-top or next-to. To add on-top, two totals T1 = 2 3s and T2 = 4 5s must be re-counted to have the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. Thus next-to addition geometrically means adding areas called integral calculus.

Reversing on-top or next-to roots solving equations and differential calculus.

Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6, i.e. by integration. Likewise with adding fractions.

Thus, calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

## References

Tarp, A. (2017). *Math ed & research 2017.* Retrieved from http://mathecademy.net/ 2017-math-articles/.

## 10. Good Math & Goofy Math, which is Truer? - Short

Looking at four statements we ask if they are true always, sometimes or never: 2+3 = 5, 2*3 = 6, 1/2+2/3 = 3/5, 1/2+2/3 = 7/6.

Looking at concepts we ask which definition is correct: A function is an example of a set-product where first-component identity implies second-component identity; or a function is e.g. y = 2+x but not y = 2+3, i.e. a name for a calculation with an unspecified number.

Looking at textbooks we ask: should we write ‘3*4’ only, or ‘T = 3*4’ as a full sentence with a subject and a verb and a predicate to show the similarity of the number-language and the word-language. We then discuss how to apply the definitions below to numbers, operations, equations, functions, trigonometry, and calculus.

In good mathematics, concepts are defined bottom-up as abstractions from examples; and statements are always true inside and outside classrooms. In goofy mathematics, concepts are defined top-down as examples from abstractions; and statements are always true inside and sometimes outside classrooms. Thus, fractions are goofy mathematics if treated as numbers without units; but good mathematics if treated as per-numbers, i.e. as operators, needing a number to become a number.

## 11. Good Math & Goofy Math, which is Truer? - Long

To make true statements about the outside world, we must distinguish between good and goofy mathematics being true always and sometimes.

### Good and Goofy Statements

Asking if they are true always, sometimes or never, we look at four statements:  2*3 = 6, and 2+3 = 5, and 1/2+2/3 = 3/5, and 1/2+2/3 = 7/6.

Saying ‘2*3 = 6’ is stating that 2 3s can be recounted as 6 1s, which is always true. Saying ‘2+3 = 5’ is true if the unit is the same; but may be false with different units, e.g. 2 weeks + 3 days = 17 days. So to be always true, addition must include the units as exemplified by the Mars Orbiter that crashed because of adding cm and inches. Multiplying, the units need not be the same as exemplified by physics where e.g. 2 Newton*3 meter = 6 Newton-meter = 6 Joule.

Adding 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and not 7 reds of 6 apples as taught in school; and true only if taken of the same total. So, depending on the units, 1/2+2/3 can take on many different values. Where goofy fractions neglect units, good fractions accept being operators needing numbers to become numbers, thus added by area as integration.

### Good and Goofy Concepts

Looking at concepts we ask which definition is correct: A function is an example of a set-product where first-component identity implies second-component identity; or, a function is e.g. y = 2+x but not y = 2+3, i.e. a name for a calculation with an unspecified number.

To answer, we look at the history of mathematics. In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite Many’ in Arabic.

Then the invention of the set-concept transformed mathe-matics to ‘meta-matics’ with ‘well-defined’ self-referring concepts defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false; and to goofy mathematics.

### Good and Goofy Textbooks

Looking at textbooks we ask: should we write ‘3*4’ only, or ‘T = 3*4’ as a full sentence with a subject and a verb and a predicate to show the similarity of the number- and the word-language.

The word-language assigns words to things through sentences, ‘This is a chair’. Asking ‘How many in total?’ we use the number-language to assign numbers to like things in sentences as ‘The total of legs on three chairs is 3 fours’, abbreviated to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

With the total we include both what exist and what essence we claim, in accordance with the existentialist philosophy saying that existence should precede essence. Thus, the fingers on the left hand exist, but how they are counted can vary: T = 5 1s = 2 2s & 1 = 1 3s &2 = 2 3s less 1.

### Good and Goofy Math

Good numbers are two-dimensional block-numbers carrying units as seen when writing out fully a total T of 345 as T = 3*B*B + 4*B + 5*1, also showing the four ways to unite numbers: multiplication, power and on-top and next-to block addition, also called integration. Including the unit when counting by bundling, a natural number as T = 2B3 4s has 3 digits: a size-number 4, a bundle-number 2 and a single-number 3, which add in different ways.

Goofy numbers are one-dimensional line-numbers silencing the unit and misplacing the decimal point by writing 23 instead of 2.3 tens.

Good addition asks for the units before adding on-top or next-to. And waits until bundle-counting and re-counting and double-counting has introduced division and multiplication and subtraction as well as the ‘recount formula’ T = (T/B)*B, saying that ‘T/B times B can be taken away from T’, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s. Goofy addition add digit without considering the units.

Good division sees 8/2 as 8 split in 2s where goofy division sees it as 8 split in 2.

Good multiplication sees 3*4 as 3 4s that may or may not be re-counted in tens. Goofy multiplication sees 3*4 as 1.2 tens.

Good fractions are per-numbers coming from double-counting in the same unit. Goofy fractions neglect units.

Good functions names a difference between examples. Goofy functions state a banality: of course, measuring produces on number only.

Good equations are reversed calculations solved by moving to opposite side with opposite sign. Goofy equations use neutralizing performing identical operations to both sides; but silencing the group definition of abstract algebra applied.

Good trigonometry occurs before plane and coordinate geometry as mutual re-counting of the sides in a block halved by its diagonal. Goofy trigonometry occurs after.

Good calculus occurs in primary school as next-to addition of block-numbers, and in middle and high school as adding piecewise and locally constant per-numbers. Goofy calculus is postponed to the end of high school and lets differential calculus precede integral calculus.

### References

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## 12. A Core STEM Curriculum for Young Migrants - Short

Seeing mathematics and education both as social institutions we can ask: is poor PISA performance and widespread learner dislike caused by a Baumanian goal displacement making a monopolized means, mathematics, the goal even if not leading to the original goal, to master Many with number-language sentences about how totals are counted, united and changed.

To master Many, the Foucauldian truth regime of 1D line-numbers should give way to what children bring to school, 2D block-numbers as 2 3s or 4 5s that when added on-top and next-to root core mathematics as proportionality to make the units like, and calculus to integrate the areas.

Block-numbers include decimals and negative numbers when recounting a total of 5 in 2s in three ways, overload and standard and underload: T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s.

With addition postponed, division and multiplication and subtraction will allow counting, re-counting and double-counting totals in STEM multiplication formulas.

In the next number of the Philosophy of Mathematics Education Journal, my article will show the full potential of a STEM based ‘recount before adding’, curriculum allowing young migrants to become pre-STEM teachers or engineers in half a year starting from scratch.

## 13. A Core STEM Curriculum for Young Migrants - Long

*Observing children’s quantitative competence uncovers as an alternative to the present set-derived mathe-matics a Many-based ‘many-matics’. To answer the question ‘How many in total?’ we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. Results are predicted by a recount formula occurring all over the STEM subjects.*

### Meeting many

Meeting many, we ask ‘How many in total?’ To answer, we count. Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, a number of bundles forming a 2D ‘block-number’ that may or may not be recounted as 1D line-numbers.

Using ‘bundle-counting’, a total $T$ of 5 1s is recounted in 2s as $T$ = 2 2s & 1; and is described by ‘bundle-writing’, $T$ = 2$B$1 2s, or ‘decimal-writing’, $T$ = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

Entering ‘5/2’, we ask a calculator ‘from 5 we take away 2s’. The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking ‘5 – 2*2’. The answer, 1, predicts that 5 = 2$B$1 2s = 2.1 2s.

Operations thus are icons also: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away bundles, and subtraction shows the trace left when taking away a stack only once.

To changes units (called proportionality), a calculator uses a ‘recount formula‘, $T = (T/B)*B$, to predict that ‘from $T$, $T/B$ times, $B$s can be taken away’. It occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as *sine* and *cosine* and *tangent*, e.g. $a = (a/c)*c = sinA*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/ \Delta x)* \Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

### *Recounting in the same unit and in a different unit*

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2$B$1 2s as 1$B$3 2s with an outside ‘overload’; or as 3$B$-1 2s with an outside ‘underload’ thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48; as well as multiplication, subtraction and addition.

### *Recounting to and from tens*

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right. Multiplication thus is a special form of division.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation, $x*7 = 35$. It is easily solved by recounting 35 in 7s: $x*7 = 35 = (35/7)*7$. So $x = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

### *Double-counting creates proportionality as per-numbers*

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question '*T* = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

### *Mathematics in STEM subjects*

STEM (Science, Technology, Engineering and Mathematics) allow humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots will help transforming nature into human necessities. Nature consists of things in motion, combined in the momentum = mass*velocity. Things contain mass and molecules and electric charge. Thus, nature is counted in meter and second and kilogram and mole and coulomb. Looking at the list of formulas we see that nature is predictable by recounting & per-numbers. Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus (Tarp, 2018).

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
meter = (meter/second) * second = velocity * second
Δ momentum = (Δ momentum/second) * second = force * seconds
Δ energy= (Δ energy/meter) * meter = force * meter = work
energy = (energy/kg/degree) * kg * degree = heat * kg * degree
force = (force/square-meter) * square-meter = pressure * square-meter
gram = (gram/mole) * mole = molar mass * mole

## Social theory looking at math education

In social theory, Bauman (1990) warns against the danger of goal displacement where 'The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84).'

Foucault (1995) talks about 'truth regimes': 'A corpus of knowledge, techniques, 'scientific' discourses is formed and becomes entangled with the practice of the power to punish (p. 23).'

Let's accept mastery of Many as the real educational goal; and let's replace the truth regime of 1D line-numbers with 2D block-numbers. This will allow all young migrants develop a STEM-based quantitative competence benefitting both themselves and the society.

## References


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## 14. Fifty Years of Ineffective Math Education Research, Why? Oops, Wrong Numbers, Sorry

Many countries face poor PISA results. Does its nature make mathematics so hard to learn despite 50 years of research? We need to read again the two founding fathers.

Freudenthal sees set-based university mathematics as so important to the outside world that it must be taught in schools. Skemp sees a true understanding of mathematics as based upon sets compared as to cardinality by, not counting them, but by establishing a correspondence between them.

Sociology points to a different explanation: maybe mathematics education has a goal displacement where it sees itself as the goal, and its outside root, Many, as a means.

So we may ask: as an alternative to the set-based tradition, is there is a different way to the outside goal of mathematics education, mastery of Many?

By observing the quantitative competences children bring to school, and by using difference-research searching for differences making a difference, we discover an alternative to the present set-based mathematics that was introduced some 50 years ago as 'New Math': a 'many-matics' seeing mathematics as a natural science about Many.

Here digits are icons with as many sticks as they represent. Also operations are icons where bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads or underloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations.

Here double-counting in two units creates per-numbers, becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers.

Addition here occurs on-top and next-to rooting proportionality, and integral calculus by adding areas; and here trigonometry precedes plane and coordinate geometry.

So, we need to research what happens if two-dimensional block-numbers replace one-dimensional line-numbers; if the order of operations is reversed; if bundle- counting, re-counting and double-counting precedes adding next-to and on-top; and, if using full sentences about the total in the number-language, as T = 2.1 3s with a subject, a verb and a predicate as in the word-language.

**References**


Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht-Holland: D. Reidel Publishing Company.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics.* Middlesex, UK: Penguin Books.

## 15. Rethinking Line-Number Arithmetic as Block-Number Algebra

'Early Algebra' recommends further research in seven areas. Apart from rethinking the examples included, this poster addresses area 2 and 4, curricula activity and theorizing numbers and operations. Using Difference-research, searching for differences making a difference, the poster asks: Will a different block-number algebra allow rethinking traditional line-number arithmetic?

As sceptical thinking from the French and American Enlightenment republics, Foucault Concept Archaeology and Existentialism and Grounded Theory is used to look at the roots of Algebra.

In Arabic, Algebra means to reunite. Numbers as T = 345 = 3*B^2+4*B+5*1 show the four ways to unite a total: addition, multiplication, power and integration of juxtaposed blocks. They also show that asking 'How many in total?', the answer is expressed as a 'number-language' sentence containing, as does a word-language sentence, a subject and a verb and a predicate, thus rooting a formula with an equation sign.

Using grounded theory we observe that, before receiving formal education, preschool children use 2dimensional Bundle-numbers or Block-numbers as T = 2 5s & 1. Re-counting a total into a new unit by asking T = 3 4s = ? 5s, children quickly accept division as an icon for a broom wiping away 5-bundles, and multiplication as an icon for stacking the bundles into a block, and subtraction for the trace left when dragging away the block to look for unbundled leftover singles.

Likewise, children find it natural to formulate the recounting process as 'from a total T, T divided by the bundle B gives the number of times Bs can be taken away', shortened to a 'recount-formula' T = (T/B)*B. This formula allows using a calculator to predict the result of a re-counting process: Recounting 7 in 3s, we enter 7/3. The answer '2.some' predicts it can be done 2 times. Taking away the stack of 2 3s, the answer '7-2x3 = 1' shows the prediction: 7 can be re-counted as 2 3s & 1.

The recount-formula T = (T/B)*B leads directly to the heart of Algebra by allowing children to use formulas as a natural way to communicate in math education. The commutative, associative and distributive laws follow directly from watching 2- and 3-dimensinal blocks.

And solving equations takes place when re-counting from tens to icons: asking T = 40 = ? 8s the solution is found by re-counting 40 in 8s as 40 = (40/8)*8 = 5*8 = 5 8s.

**References**


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## 16. A Count-before-Adding Curriculum for Preschool and Migrants

*Children show a surprising mastery of Many with a quantitative competence where totals are re-counted in the same and in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Allowed to keep their block-numbers, children and migrants will be practising proportionality and calculus when adding block-number on-top and next-to; and will be solving equations when re-counting from tens to blocks.*

*Keywords: preschool, add, equation, proportional, calculus.*

### Decreased PISA performance despite increased research

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life' (p. 3).

The ineffectiveness of mathematics education research may prove that, by its very nature, mathematics is indeed hard to learn. On the other hand, since mathematics education is a social institution, social theory may provide a different reason.

### Social Theory Looking at Mathematics Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'.

However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84).

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference. So, inspired by sociology we can ask the 'Cinderella question': 'as an alternative to the tradition, is there is a different way to the outside goal of mathematics education, mastery of Many?'

### How well-defined is mathematics?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. Algebra thus contains the four ways to unite as shown when writing out fully the total T = 342 = 3*B^2 + 4*B + 2*1 = 3 bundles of bundles and 4 bundles and 2 unbundled singles = 3 blocks. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reverse splitting operation.

So, as a label, mathematics has no existence itself, only its content has, algebra and geometry; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek ‘many-matics’ rooted in Many was replaced by the ‘New Mathematics’.

Here the invention of the concept Set created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meets counter-examples as 2weeks + 3days is 17days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

### Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists that wanting to avoid being patronized by choices presented as nature (Russell, 1945), ‘difference-research’ is searching for hidden differences making a difference (Tarp, 2017). So, to avoid a goal displacement in mathematics education, difference-research asks the grounded theory question: How will mathematics look like if grounded in its outside root, the physical fact Many?

To answer, we will use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1970) from a personal to a social level, to allow Many to open itself for us and create its own categories and properties. In short, we will search for an alternative to the ruling tradition by returning to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space.

### Meeting many

We live in space and in time. To include both when counting Many, we use two different ways of counting. Counting in space, we count blocks and report the result with LEGO-blocks or on a ten-by-ten abacus in ‘geometry-mode’. Counting in time, we count bundles and report the result on a ten-by-ten abacus in ‘algebra-mode’.

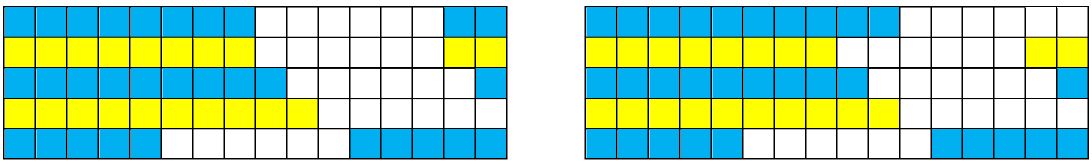

*Table 1: 5 re-counted as 2 2s & 1 on an abacus in geometry- and in algebra-mode*

Thus, to master Many, we count by bundling and stacking. But first we rearrange sticks into icons with e.g. five sticks in the five-icon 5 if written less sloppy. Counted as ‘1 bundle’, ten does not need an icon when used as the bundle-size.

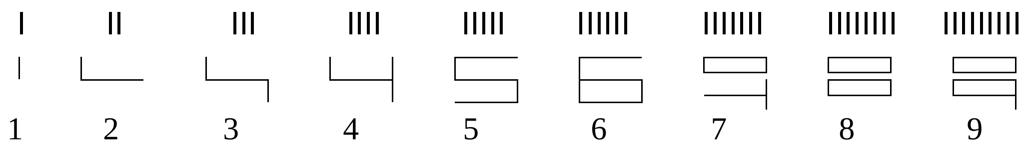


*Table 2: Sticks rearranged as icons with as many sticks as they represent*

Holding 4 fingers together 2 by 2, a 3year old will say ‘That is not 4, that is 2 2s’, thus describing how Many presents itself, as a number of bundles that may or may not be re-counted as four ones. This inspires ‘bundle-counting’ bundling a total in icon-bundles. Thus, a total *T* of 5 1s is bundled in 2s as *T* = 2 2s and 1 where the bundles are placed inside a bundle-cup with a stick for each bundle, leaving the unbundled outside; and described by ‘bundle-writing’, *T* = 2*B*1, or ‘decimal-writing’, *T* = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

*T* = 5 = I I I I I → II II I → [II] I → 2*B*1 2s = 2.1 2s

Entering ‘5/2’ we ask a calculator ‘from 5 we can take away 2s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 2s by asking ‘5 – 2*2’. From the answer ‘1’ we conclude that 5 = 2*B*1 2s. Likewise, showing ‘5 – 2*2 = 1’, a display indirectly predicts that 5 can be re-counted as 2 2s and 1, or as 2*B*1 2s.

| | |
|---|---|
| 5 / 2 | 2.some |
| 5 – 2 * 2 | 1 |

**Table 3: A calculator predicts that 5 re-counts in 2s as 2.1 2s**

We see that also operations are icons: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away 2s several times, and subtraction shows the trace left when taking away the stack once.

A calculator thus uses a ‘re-count formula‘, *T* = *(T/B)*B*, saying that ‘from *T*, *T/B* times *B*s can be taken away’. This re-count formula occurs all over mathematics: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = sinA*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

***Re-counting in the same unit and in a different unit***

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 5 as 1*B*3 2s with an outside ‘overload’; or as 3*B*-1 2s with an outside ‘underload’ thus leading to negative numbers:

*T* = 5 = I I I I I I → II I I I → 1*B*3 2s, or *T* = 5 = I I I I I I → II II II → 3*B*-1 2s

To re-count in a different unit means changing unit, also called proportionality or linearity. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2B2 5s.

*T* = 3 4s = IIII IIII IIII → IIIII IIIII I I → 2*B*2 5s.

A calculator can predict the result. Entering ‘3*4/5’ we ask ‘from 3 4s we take away 5s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be re-counted in 5s as 2 5s and 2, or as 2*B*2 5s.

| | |
|---|---|
| 3 * 4 / 5 | 2.some |
| 3 * 4 – 2 * 5 | 2 |

*Table 4: A calculator predicts that 3 4s re-counts in 5s as 2.2 5s*

***Double-counting creates proportionality as per-numbers***

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘6$ = ?kg’ we use the per-number to re-count 6 in 2s: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa: Asking ‘?$ = 12kg’, the answer is 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Double-counting in the same unit creates fractions and percentages: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

***Re-counting to and from tens***

Asking ‘3 4s = ? tens’ is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, i.e. in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

Re-counting from tens to icons by asking ‘38 = ? 7s’ is called an equation *x**7 = 38. It is easily solved by re-counting 38 in 7s: *x**7 = 38 = (38/7)*7. So *x* = 38/7 = 5 & 3/7 as predicted by a calculator showing that 38 = 5.3 7s = 5*7 + 3.

| | |
|---|---|
| 38 / 7 | 5.some |
| 38 – 5 * 7 | 3 |

*Table 5: A calculator predicts that 38 re-counts in 7s as 5.3 7s*

***Once counted, totals can be added on-top or next-to***

Asking ‘3 5s and 2 3s total how many 5s?’ we see that to add on-top, the units must be the same, so the 2 3s must be re-counted in 5s as 1*B*1 5s that added to the 3 5s gives a total of 4*B*1 5s.

IIII IIII IIII & III III → IIII IIII IIII IIII I → 4*B*1 5s.

For a calculator prediction, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. Thus, we get 4*B*1 5s.

| | |
|---|---|
| (3 * 5 + 2 * 3)/ 5 | 4.some |
| (3 * 5 + 2 * 3) – 4 * 5 | 1 |

*Table 6: A calculator predicts that the sum of 3 5s and 2 3s re-counts in 5s as 4.1 5s*

Since 3*5 is an area, adding next-to means adding areas, called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2*B*5 8s.

IIII IIII IIII & III III → IIII III IIII III I I I I I → 2*B*5 8s = 2.5 8s

For a calculator prediction, we include the two totals in a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2*B*5 8s.

| | |
|---|---|
| (3 * 5 + 2 * 3)/ 8 | 2.some |
| (4 * 5 + 2 * 3) – 2 * 8 | 5 |

*Table 7: A calculator predicts that the sum of 3 5s and 2 3s re-counts in 8s as 2.5 8s*

***Reversing adding on-top and next-to***

Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2*B*6 8s?’, using sticks will give the answer 2*B*1 3s:

IIII IIII IIII III III I ← IIII III IIII III I I I I I I ← 2*B*6 8s

For a calculator prediction, the remaining is bracketed before being counted in 3s.

| | |
|---|---|
| (2 * 8 + 6 – 3 * 5)/ 3 | 2 |
| (2 * 8 + 6 – 3 * 5) – 2 * 3 | 1 |

*Table 8: A calculator predicts that 2.6 8s comes from next-to addition of 2.1 3s to 3 5s*

Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula

$y' = \Delta y/t = (y2 - y1)/t$.

#### Designing a count-before-adding curriculum

To study the effect of re-counting, an eight-part ‘count-before-adding’ curriculum can be designed where bundle-counting involves division, multiplication, subtraction before next-to and on-top addition to respect that totals must be counted before being added; in contrast to school that turns this order around and insists numbers be counted in tens only to be added on-top only, using carrying instead of overloads.

In the first micro-curriculum, the learner uses sticks and a folding ruler to build the number-icons up to nine; and uses strokes to draw the icons thus realizing there are as many sticks or strokes in the icon as it represents if written less sloppy. In the second, the learner counts a given total in icons by bundling sticks and using a cup for the bundles; and reports, first with bundle-writing and decimal-writing with a unit; then by using an abacus in geometry- and algebra-mode; always using a calculator to predict the counting result. In the third, the learner re-counts a total in the same unit thus creating or removing overloads and underloads.

In the fourth micro-curriculum, the learner re-counts a total in a different unit. In the fifth, the learner adds two block-numbers on-top of each other. In the sixth, the learner adds two block-numbers next-to each other. In the seventh, the learner reverses on-top addition. And in the eighth, the learner reverses next-to addition. Finally, the learner sees how double-counting creates per-numbers.

| | Examples | Calculator prediction | |
|---|---|---|---|
| M2 | 7 1s is how many 3s?<br>I I I I I I I → III III I → 2B1 3s<br>So 7 1s = 2*B*1 3s = 2.1 3s | 7/3<br>7 – 2*3 | 2.some<br>1 |
| M3 | ‘2.7 5s is also how many 5s?’<br>V V IIIIIII = V V V II = V V V IIIII<br>2*B*7 = 2+1*B*7–5 = 3*B*2 = 3+1*B*2–5 = 4*B*-3<br>So 2.7 5s = 3.2 5s = 4.-3 5s | (2*5+7)/5<br>(2*5+7) – 3*5<br>(2*5+7) – 4*5 | 3.some<br>2<br>-3 |
| M4 | 2 5s is how many 4s?’<br>V V = IIIII IIIII = IIII IIII I I<br>So 2 5s = 2.2 4s | 2*5 / 4<br>2*5 – 2*4 | 2.some<br>2 |
| M5 | ‘2 5s and 4 3s total how many 5s?’<br>V V III III III III = V V V V I I<br>So 2 5s + 4 3s = 4.2 5s | (2*5+4*3) /5<br>(2*5+4*3) – 4*5 | 4.some<br>2 |
| M6 | ‘2 5s and 4 3s total how many 8s?’<br>V V III III III III = IIIIIIII IIIIIIII III III<br>So 2 5s + 4 3s = 2.6 8s | (2*5+4*3) /8<br>(2*5+4*3) – 2*8 | 2.some<br>6 |
| M7 | ‘2 5s and ? 3s total 4 5s?’<br>V V V V = V V III III III I<br>So 2 5s + 3.1 3s = 4 5s | (4*5 – 2*5)/3<br>(4*5 – 2*5) – 3*3 | 3.some<br>1 |
| M8 | ‘2 5s and ? 3s total 2.1 8s?’<br>IIIIIIII IIIIIIII I = IIIII III IIIII III I<br>So 2 5s + 2.1 3s = 2.1 8s | (2*8+1 – 2*5)/3<br>(2*8+1–2*5) – 2*3 | 2.some<br>1 |

*Table 9: A ‘count-before-adding’ curriculum shown as micro-curricula*

To be tested in special education, an additional ‘1 cup & 5 sticks’ micro-curriculum was added. Here using a cup for the bundles allows 5 sticks to be bundle-counted in 2s as 1*B*3 or 2*B*1 or 3*B*-1 to show that a total can be counted in three ways, overload and normal and underload, with an inside and an outside number for the bundles and the singles. So, to divide 336 by 7, 5 bundles move outside as 50 singles to re-count 336 with an overload: 336 = 33*B*6 = 28*B*56, which divided by 7 gives 4*B*8 or 48.

When tested, one curriculum used silent education where the teacher demonstrates and guides by actions only, not using words; and in one curriculum the teacher spoke a foreign language not understood by the learners. In both cases the abacus and the calculator quickly took over the communication. For further details watch the video www.youtube.com/watch?v=IE5nk2YEQIA.

After the micro-curricula a learner went back to her grade 6 class where proportionality created learning problems. The learner suggested renaming it to double-counting but the teacher insisted on following the textbook. However, observing that the class took over the double-counting method, the teacher gave in and allowed proportionality to be renamed and treated as double-counting. When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integral calculus.

Thus bundle-counting together with a calculator for predicting re-counting results allowed the learner to reach the outside goal, mastering Many, by following an alternative to the institutionalized means that because of a goal displacement had become a stumbling block to her; and performing and reversing next-to addition introduced her to and prepared her for later calculus classes.

### Literature on bundle-counting and block-numbers

No research literature on bundle-counting and block-numbers was found. Nor is it mentioned by Dienes (1964).

### Conclusion and recommendations

To avoid a goal displacement in mathematics education, this paper searched for a different way to the goal of mathematics education, mastery of Many. Difference-research and Grounded Theory showed how mathematics looks like if grounded in its physical root, Many. To tell the difference, two names were coined, ‘many-matics’ versus ‘meta-matism’ mixing ‘meta-matics’, defining concepts as examples of abstractions instead of as abstractions from examples, with ‘mathe-matism’ valid only inside classrooms. To validate its findings, the paper includes a classroom test of a ‘Count-before-adding’ curriculum described in detail to allow it to be tested in other classrooms also. To improve mathematics education, the curriculum can be used in early childhood and in special education to give a physical understanding of how numbers come from counting by bundling and stacking before using the short way of writing numbers counted in tens without decimal points and units.

Likewise, it can be used for young migrants wanting to help rebuild their country as STEM-teachers or STEM-engineers (Science, Technology, Engineering, Math). A six months STEM-based many-matics curriculum has been designed allowing migrants use the first month to learn to reunite constant and variable unit- and per-numbers by addition and multiplication and integration and power, and by their reverse operations subtraction, division, differentiation and root/logarithm. The next five months then consist of modelling situations in science, technology, and engineering (Tarp, 2017). Finally, as an alternative to line-numbers, block-numbers opens up a completely new research paradigm (Kuhn, 1962).

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## 17. Difference-research Saving Dropout Ryan with a TI-82 Calculator

*At principal asked for ideas to lower the number of dropouts in pre-calculus classes. The author proposed using a cheap TI-82, but the teachers rejected saying students weren't even able to use a TI-30. Still the principal allowed buying one for a class. A compendium called 'Formula Predict' replaced the textbook. A formula's left- and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects.*

*Keywords: precalculus, calculus, calculator, model, dropout.*

### The Task

The headmaster asked the mathematics teachers: "We have too many pre-calculus dropouts. What can we do?" I proposed buying the cheap TI-82 graphing calculator, but the other teachers rejected this proposal arguing that students weren't even able to use a simple TI-30. Still I was allowed to buy this calculator for my class allowing me to replace the textbook with a compendium emphasizing modeling with TI-82.

### How Well-Defined is Mathematics After All?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

So, as a label, mathematics has no existence itself, only its content has, algebra and geometry; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics' wanting concepts to be well-defined by being derived from the set-concept. This despite that Russell, by looking at the set of sets not belonging to itself, showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

Thus, the set-concept turned mathematics upside down to a 'meta-matism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathe-matism' true in the library, but not in the laboratory, as e.g. 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest that when counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

Despite being neither well-defined nor well-proved, mathematics still teaches meta-matism, thus creating huge problems to mathematics education.

### Difference Research

Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists emphasized telling choice from nature to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should become the natural patronizing rulers (Russell, 1945).

Inspired by the ancient Greek sophists, ‘difference-research’ is searching for hidden differences making a difference (Tarp, 2017). So, it asks the grounded theory question: How will mathematics look like if grounded in its outside root, the physical fact Many? To answer, we will use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1970) from a personal to a social level, to allow Many to open itself for us and create its own categories and properties. In short, we will search for an alternative to the ruling tradition by returning to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space.

### The Case of Teaching Math Dropouts

Being our language about quantities, mathematics is a core part of education in both primary and secondary school. Students generally accept the importance of learning mathematics, but many fail to see the meaning of doing so. Consequently, special core courses for math dropouts are developed.

### Traditions of Core Precalculus Courses for Dropouts

A typical core course for math dropouts is halving the content and doubling the text volume. So, in a slow pace the students work their way through a textbook once more presenting mathematics as a subject about numbers, operations, equations and functions applied to space, time, mass and money. To prevent spending time on basic arithmetic, a TI-30 calculator is handed out, typically without instruction.

As to numbers, the tradition focuses on fractions and how to add fractions.

Then solving equations is introduced using the traditional balancing method, isolating the unknown by performing identical operations to both sides of the equation. Typically, the unknown occurs in fractions as $5 = 40/x$; or on both sides of the equation as $2*x + 3 = 4*x – 5$

Then relations between variables are introduced using tables, graphs and functions with special emphasize on the linear function $y = f(x) = a*x + b$.

In a traditional curriculum, a linear function is followed by the quadratic function. But a core course might instead go on to the exponential and power functions $y = b*a$^$x$ and $y = b*x$^$a$. To avoid solving its equations, the solutions are given as formulas.

#### *Problems in Traditional Core Courses*

A traditional core course wants to give a second chance to learners having dropped out of the traditional math course. However, from a sceptical viewpoint trying to avoid presenting choice as nature, several questions can be raised.

As to numbers, are fractions numbers or calculations that can be expressed with as many decimals as we want, typical asking for three significant figures? Is it meaningful to add fractions without units as shown by the fraction-paradox above?

As to equations, is the balancing method nature or choice presented as nature? The number $x = 8 – 3$ is defined as the number that added to 3 gives 8, $x + 3 = 8$. This can be restated as saying that the equation $x + 3 = 8$ has the solution $x = 8 – 3$; suggesting that the natural way to solve equations is the ‘move to opposite side with opposite sign’ method. This method applies to all reversed calculation, defining a root as a factor-finder, and a logarithm as a factor-counter.

| | | | |
|---|---|---|---|
| $x + 3 = 15$ | $x * 3 = 15$ | $x$^3 = 125 | 3^$x$ = 243 |
| $x = 15 – 3$ | $x = 15/3$ | $x = \sqrt[3]{125}$ | $x$ = ln243/ln3 |

*Table 1: the basic equations solved by the ‘opposite side & sign’ method*

As to relations between variables, is the function nature or choice presented as nature? A basic calculation as $3 + 5 = 8$ contains three numbers. If one of these is unknown we have an equation to be solved, e.g. $3 + x = 5$, if not already solved, $3 + 5 = x$. With two unknown we have a formula as in $3 + x = y$, or a relation as in $x + y = 3$ that can be changed into the formula $y = 3 – x$. So, the natural relation between two unknown variables seems to be a formula.

As to solving exponential equations, is presenting solution formulas nature or choice presented as nature? Solving basic equations is just another way of defining inverse operations, so it is a natural thing to define a root and a logarithm as solutions to the basic equations involving power.

## Desiging a Grounded Core Course

So, a traditional core course seems to be filled with examples of choices presented as nature. This leads to the question: is it possible to design a different core course based upon nature instead of choices presented as nature? In other words, what would be the content of a core course in pre-calculus grounded in the root of mathematics, the natural fact Many?

### *Mathematics as a Number-language*

As to the nature of the subject itself, mathematics is a number-language that together with the word-language allows users to describe quantities and qualities in everyday life. Thus, a calculator is a typewriter using numbers instead of letters. A typewriter combines letters to words and sentences. A calculator combines figures to numbers that combined with operations becomes formulas. Thus, formulas are the sentences of the number-language.

### *The Number Formula shows the four Ways to Unite*

The four Algebra ways to unite is seen when writing out fully the total $T = 542 = 5*B$^2 + 4*$B$ + 2*1, i.e. as three blocks: 5 bundles of bundles and 4 bundles and 2 unbundled singles. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reverse splitting operation: subtraction, division, root and logarithm and differentiation.

Furthermore, we see that the number-formula, has as special cases the formulas for constant linear, exponential, elastic, and accelerated change:

*T* = *b**x+c*, *T* = *a**n^x*, *T* = *a**x^n*, and *T* = *a**x^2 + *b**x* + *c*.

### *Formulas Predict*

One difference between the word- and the number-language is that sentences describe whereas formulas predict the four different ways of uniting numbers:

Addition predicts the result of uniting unlike unit-numbers: uniting 2$ and 3$ gives a total that is predicted by the formula *T* = *a*+*b* = 2+3 = 5

Multiplication predicts the result of uniting like unit-numbers: uniting 2$ 5 times gives a total that is predicted by the formula *T* = *a***b* = 5*2 = 10.

Power predicts the result of uniting like per-numbers: uniting 2% 5 times gives a total that is predicted by the formula 1+*T* = *a*^*b* = 1.02^5, i.e. *T* = 0.104 = 10.4%.

Integration predicts the result of uniting unlike per-numbers: uniting 2kg at 7$/kg and 3kg at 8$/kg gives 5 kg at *T*$/5kg where *T* = 7*2 + 8*3 is the area under the per-number graph, *T* = ∫ *p***dx*.

### *Solving Equations with Solver*

Thus, inverse operations solve equations; as do the TI-82 using a solver. An equation as $2 + x = 6$ always has a left-hand side and a right-hand side that can be entered on the calculator’s Y-list as *Y1* and *Y2*. So, any equation has the form *Y1* = *Y2*, or *Y1* – *Y2* = 0 that only has to be entered to the solver once. After that, solving equations just means entering its two sides as *Y1* and *Y2*. Using graphs, *Y1* and *Y2* have the intersection points as solutions to the equation *Y1* = *Y2*.

If one of the numbers in a calculation is unknown, then so is the result. A formula with two unknowns can be described by a table answering the question ‘if *x* is this, then what is *y*?’ Graphing a table allows the inverse question to be addressed by reading from the *y*-axis.

***Producing Formulas with Regression***

Once a formula is known, it produces answers by being solved or graphed. Real world data often come as tables, so to model real world problems we need to be able to set up formulas from tables. Simple formulas describe levels as e.g. cost = price*volume. Calculus formulas describe change and pre-calculus describes constant change.

If a variable y begins with the value b and changes by a number a x times, the $y = b + a*x$. This is called linear change and occurs in everyday trade and in interest-free saving.

If a variable y begins with the value b and changes by a percentage r x times, the $y = b*(1+r)\text{^}x$ since adding 5% means multiplying with 105% = 1 + 5%. This is called exponential change and occurs when saving money in a bank and when populations grow or decay.

Combining linear and exponential change by depositing *a*$ *n* times to an interest rate *r*%, we get a saving *A*$ predicted by the formula $A/a = R/r$ where the total interest rate *R* is predicted by the formula $1+R = (1+r)\text{^}n$.

The proof: from an account with *a/r*$ the interest is moved to another account together with its interest, thus containing $a/r*R$ as a saving *A, which* gives $A/a = R/r$.

Thus, instalments can be studied as a race between a debt *D* growing with an interest rate *r*%, $T = D*(1+r)\text{^}n$, and a saving *A* growing from fixed deposits and interest rates, $A = a/r*R$. From this, the debt to be found be the formula $D*(1+r)\text{^}n = a/r*R$.

In the case of linear and exponential and power change, a two-line table allows us to find the two constants *b* and *a* using regression on a TI-82.

Multi-line tables can be modelled with polynomials. Thus, a three-line table might be modelled with a quadratic formula $y = b + a*x + c*x\text{^}2$ including also a bending-number *c*; and a four-line table by a cubic formula $y = b + a*x + c*x\text{^}2 + d*x\text{^}3$ including also a counter-bending-number *d*, etc.

Graphically, a second-degree polynomial is a bending ling, a parabola; and a third- degree polynomial is a double parabola. The top and the bottom of a bending curve as well as its zeros can be found directly by graphing methods on a TI-82.

***Fractions as Per-numbers***

Fractions are rooted in per-numbers: 3$ per 5 kg = 3$/5kg = 3/5 $/kg. To add per-numbers they first must be changed to unit numbers by being multiplied with their units: 3 kg at 4 $/kg + 5 kg at 6 $/kg = 8 kg at (3*4 + 5*6)/8 $/kg

Geometrically this means that the areas under its graph add per-numbers. Here again TI-82 comes in handy when calculating areas under graphs; also in the case where the graph is not horizontal but a bending line, representing the case when the per-number is changing continuously as e.g. in a falling body: 3 seconds at 4 m/s increasing to 6 m/s totals 15 m in the case of a constant acceleration.

***Models as Quantitative Literature***

With the ability to use TI-82 as a quantitative typewriter able to set up formulas from tables and to answer both x- and y-questions, it becomes possible to include models as quantitative literature.

All models share the same structure: A real-world problem is translated into a mathematical problem that is solved and translated back into a real-world solution.

| | Problem | | Solution |
|---|---|---|---|
| Mathematics | | → | |
| | ↑ | | ↓ |
| Real world | | ← | |

*Table 2: The four steps in modelling real-word situations*

The project ‘Population versus food’ looks at the Malthusian warning: If population changes in a linear and food in an exponential way, hunger will eventually occur.

The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period.

From this two-line table regression can produce two formulas: with x counting years after 1850, the population is modeled by *Y1*= 815*1.013^*x* and the food by *Y2*= 300 + 30**x*. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

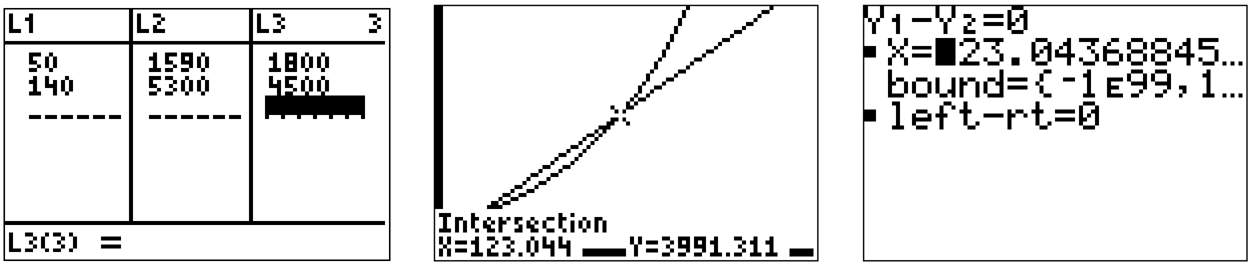


*Table 3: Two two-line table regressions allow graphical and algebraic solutions*

The project ‘Fundraising’ finds the revenue of a fundraising show assuming that all students will accept a free ticket, that 100 students will buy a 20$ ticket and that no one will buy a 40$ ticket.

From this three-line table the demand can be modeled by the quadratic formula *Y1* = .375**x*^2 – 27.5**x* + 500. Thus, the revenue formula is the product of the price and the demand, *Y2* = *x***Y1*. Graphing methods shows that the maximum revenue will be 2688 $ at a ticket price of 12$.

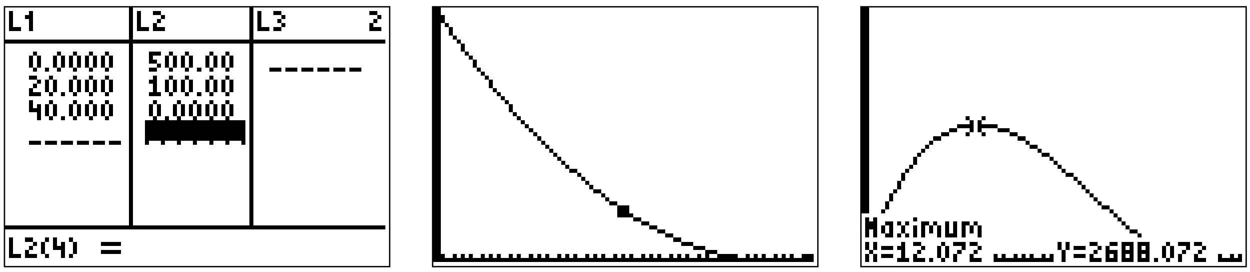


*Table 4: A three-line table regression offers a parabola to be studied graphically*

In the project ‘Driving with Peter’ his velocity is measured five times. A model answers two questions: When is Peter accelerating? And what distance did Peter travel in a given time interval?

From a five-line table the speed can be modeled by the 4th degree polynomial *Y1* = -0.009**x*^4 + 0.53**x*^3 - 10.875**x*^2 + 91.25**x* - 235. Visually, the triple parabola fits the data points. Graphing methods shows that a minimum speed is attained after 14.2 seconds; and that Peter traveled 115 meters from the 10th to the 15th second.

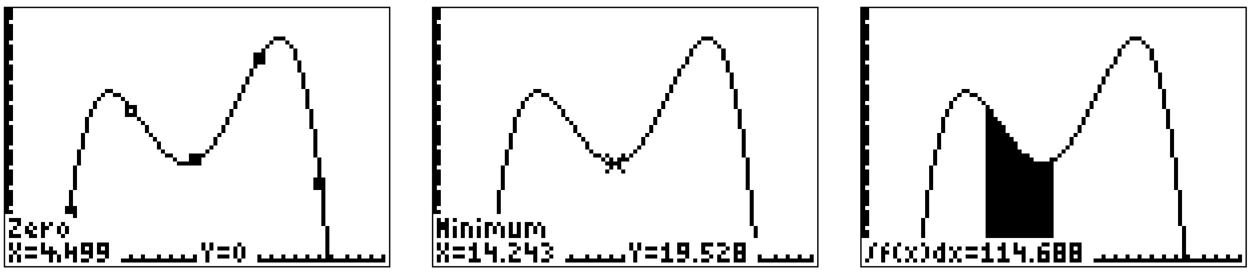


*Table 5: A five-line table regression offers a triple-parabola to be studied graphically*

Six other projects were included in the course.

The project ‘Forecasts’ modeled a capital growing constantly in three different ways: linear, exponential and potential.  The project ‘Determining a Distance’ uses trigonometry to predict the distance to an inaccessible point on the other side of a river. The project ‘The Bridge’ uses trigonometry to predict the dimensions of a simple expansion bridge over a canyon.

The project ‘Playing Golf’ asks to predict the formula for the orbit of a ball that has to pass three given points: a starting point, an ending point and the top of a hedge. The project ‘Saving and Pension’ asks about the size of a ten years monthly pension created by a thirty years monthly payment of 1000$ at an interest rate of 0.4% per month. And the project ‘The Takeover Try’ asks how much company A has spent buying stocks in company B given a course described by a four-line table.

***Testing the Core Course***

The students expressed surprise and content with the course. Their hand-in was delivered on time. And they course finished before time allowing the inclusion of additional models from classical physics: vertical falling balls, projectile orbits, colliding balls, circular motion, pendulums, gravity points, drying wasted whisky with ice cubes, and supplying bulbs with energy.

At the written and the oral exam, for the first time at the school, all the students passed. Some student wanted to move on to a calculus class, other were reluctant arguing that they had already learned the core of calculus.

***Reporting Back to the Headmaster***

The headmaster expressed satisfaction, but the teachers didn’t like the textbook and its traditional mathematics to be set aside. To encourage the teachers, the headmaster ordered the TI-82 to be bought to all pre-calculus classes.

## Conclusion: Make Losers Users

Using difference-research, this action research project showed that dropout students get an extra chance by boiling mathematics down to its core grounded in its roots, the natural fact Many. Here numbers are polynomials showing the operations that allows totals to be united from or split into constant or variable unit- or per-numbers according to Algebra’s reuniting project. In this way the core of algebra is solving equations with the ‘opposite side & sign’ method or with the solver on a graphing calculator. And the core of pre-calculus is using regression to translate tables to formulas that can be processed both geometrically and algebraically when entered into the Y-list of the TI-82. Thus, grounding mathematics in its root Many will allow all students to use a graphing calculator to predict the behavior of real-world quantities, thus reconquering the number-language taken from them by meta-matism.

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## 18. Conflicting Theories Help Teachers Improve Mathematics Education

*Traditionally, education is seen as teachers transferring institutionalized knowledge to individual learners. As such, education involves several choices. Shall teachers teach or guide? Is mathematics an eternal truth or a social construction? Is it knowledge about, or knowing how to? How to motivate learning? Should a class be optional or mandatory? To answer, teacher education refers to theory from philosophy, psychology and sociology. Including the existence of conflicting theories will allow teachers try out alternatives if wanting to improve mathematics education.*

*Keywords: teacher education, mathematics, philosophy, psychology, sociology.*

### Philosophical Controversies

Ancient Greece saw two forms of knowledge, called 'sophy'. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. The Christian Church eagerly took over a metaphysical patronage and changed the academies into monasteries, until the Reformation changed some back again.

By letting the laboratory precede the library, natural science reinvented scepticism. Newton discovered that falling objects obey their own will instead of that of a patronizor. This inspired the Enlightenment Century and it two republics, the American and the French, transforming scepticism into pragmatism and post-structuralism, based upon existentialism defined by Sartre as 'Existence preceding essence'; and with the Heidegger warning: in a sentence, respect the subject, but question the predicate since it might be gossip. Thus, post-structuralism deconstructs ungrounded diagnoses forcing humans to accept unfounded patronization.

### Psychological Controversies

As to how learners acquire knowledge, several constructivist theories exist among which are Vygotskian and Piagetian social and radical constructivism disagreeing by recommending teaching as much and as little as possible.

Vygotsky sees knowledge as true sentences to be transferred by good teaching. However, a learner can only take in unknown sentences about subjects already known, so the teacher must know the individual 'zone of proximal development' in order to successfully connect it to the institutionalized knowledge by scaffolding.

Whereas Piaget recommends meeting the new subjects directly to allow learners form individual concepts and sentences to be negotiated and accommodated socially.

### Sociological Controversies

As a social institution, education can be seen from a structure or an actor viewpoint, reflecting societies with high or low degree of institutionalization. Being highly institutionalized, continental Europa has developed a structure-based sociology seeing humans as bound by social structures. Thus, Foucault sees knowledge as socially constructed discourses; and describes a school as a 'prispital' mixing the power techniques of a prison and a hospital: the learners are fixed in classrooms and diagnosed as ignorant to be cured by discourses institutionalized as truth but instead exerting 'pastoral power'. Whereas their escape from Europe made Americans actors developing grounded theory, lifting Piagetian accommodation to a social level.

### Mathematical Controversies

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about Many by itself, in space, in time, and in space and time. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric. With astronomy and music gone, today mathematics should be a common label for geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about fifty years ago when the Greek ‘many-matics’ rooted in Many was replaced by the ‘New Mathematics’. Here the invention of the concept Set created a ‘meta-matics’ defining concepts as examples of abstractions instead of as abstractions from examples. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

So today two mathematics discourses exist. One is the institutionalized self-referring ‘meta-matics’ presenting digits and fractions as numbers that can be added without units, and still producing poor PISA results despite fifty years of research. The other is a natural science about Many, a ‘many-matics’ presenting digits and fractions as operators or factors needing a number to become a number, and where adding number-blocks on-top and next-to leads directly to core mathematics as linearity and calculus. And where calculators predict the result of re-counting, preceding addition.

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## 19. Addition-free Core STEM Curriculum for Late Learners along the Silk Road

*Its many applications make mathematics useful. But to solve core STEM tasks you need no addition, thus calling for an addition-free curriculum. Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount-formula being a core in all STEM subjects.*

### Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015a) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change'.

To find an unorthodox solution we pretend that a university in China arranges a curriculum architect competition: 'Theorize the low success of 50 years of mathematics education research; and derive from this theory a STEM-based core curriculum that can be used for late learners along the coming new silk road, One Belt and One Road (OBOR).'

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to improve schools in Sweden and elsewhere.

### Social theory looking at mathematics education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET changed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to ‘2x3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

### Difference research looking at mathematics education

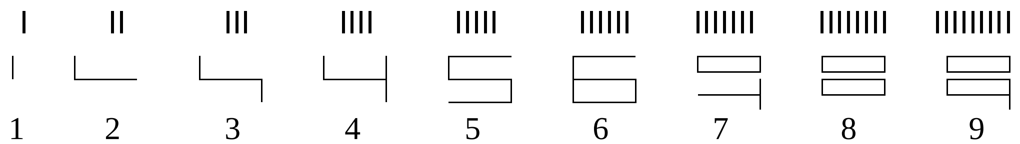
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ (Tarp, 2017) is searching for hidden differences making a difference. So, to avoid a goal displacement in math education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014). So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

### Meeting Many creates a ‘count-before-adding’ curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence (Tarp, 2018b).

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:




Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, a number of bundles that may or may not be recounted as ones.

This inspires ‘bundle-counting’, recounting a total in icon-bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted before being processed by on-top and next-to addition, direct or reversed.

Thus, a total $T$ of 5 1s is recounted in 2s as $T$ = 2 2s & 1; and is described by ‘bundle-writing’, $T$ = 2$B$1 2s, or ‘decimal-writing’, $T$ = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

So, to count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away).

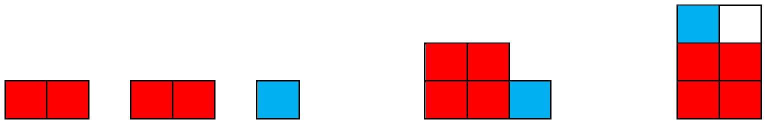
Entering ‘5/2’, we ask a calculator ‘from 5 we take away 2s’. The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking ‘5 – 2x2’. The answer, 1, predicts that 5 = 2*B*1 2s = 2.1 2s as indirectly predicted on the bottom line.

| | |
|---|---|
| 5 / 2 | 2.some |
| 5 – 2 x 2 | 1 |

A calculator thus predicts the result by a recount-formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: 5/2 gives 2.some, and 5 – 2x2 gives 1, so T = 7 = 2B1 3s.

This recount-formula occurs all over mathematics: when relating proportional quantities: $y = c*x$; in trigonometry as sine, cosine and tangent: $a = (a/c)*c = \sin A*c$ *and* $b = (b/c)*c = \cos A*c$ *and* $a = (a/b)*b = \tan A*b$; in coordinate geometry as line gradients: $\Delta y = \Delta y/\Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In economics, the recount-formula becomes a price-formula: $ = ($/kg)*kg, $ = ($/day)*day, etc.

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 5 = 2 2s & 1 = 2B1 2s = 2.1 2s = 2 1/2 2s

***Recounting in the same unit and in a different unit***

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2*B*1 2s as 1*B*3 2s with an outside ‘overload’; or as 3*B*-1 2s with an outside ‘underload’ thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48.

Recounting in a different unit means changing unit, also called proportionality or linearity. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2*B*2 5s.

Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’ The answer, ‘2.some’, predicts that the singles come by taking away 2 5s, thus asking ‘3*4 – 2*5’. The answer, ‘2’, predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

***Recounting to and from tens***

Asking ‘3 4s = ? tens’ is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $x*7 = 35$. It is easily solved by recounting 35 in 7s: $x*7 = 35 = (35/7)*7$. So $x = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

***Double-counting creates proportionality as per-numbers***

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘*T* = 6$ = ?kg’, we recount 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

### A short curriculum in addition-free mathematics

01. To stress the importance of bundling, the counting sequence can be: 01, 02, …, 09, 10, 11 etc. And 01, 02, 03, 04, 05, Ten less 4, T-3, T-2, T-1, Ten, Ten and 1, T and 2, etc.

02. Ten fingers can be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers can be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables can be formulated as re-counting from icon-bundles to tens and use underload counting after 5: T = 4*7 = 4 7s = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 can be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

06. Solving proportional equations as 3*x = 12 can be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks can be done by re-counting in the per-number: With 3$/4kg, 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages can be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$ = (60/3)*3$ giving (60/3)*2$ = 40$

09. Trigonometry can precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sinA*c.

10. Counting by stacking bundles into adjacent blocks leads to the number-formula or bundle-formula called a polynomial: T = 456 = 4*BundleBundle + 5*Bundle + 6*single = 4*B^2 + 5*B + 6*1. In its general form, the number-formula T = a*x^2 + b*x + c contains the different formulas for constant change: T = a*x (proportionality), T = a*x^2 (acceleration), T = a*x^c (elasticity) and T = a*c^x (interest rate); as well as T = a*x+b (linearity).

11. Predictable change roots pre-calculus (if constant) and calculus (if changing). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

12. Integral can precede differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers.  So both per-numbers and fractions are added with units as the area under the per-number graph.

### Meeting Many in a STEM context

Having met Many by itself, we now meet Many in time and space in the present culture based upon STEM, described by OECD (2015b) as follows: 'In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.'

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we must build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

Nature consists of things in motion, combined in the momentum = mass*velocity. Things contain mass and molecules and electric charge. Thus, nature is counted in meter and second and kilogram and mole and coulomb. STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), $/kg (price), $/hour (wages), etc.

A list of core formulas shows that nature is predictable by recounting & per-numbers.:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/litre) * litre = molarity * litre
- momentum = (Δ momentum/second) * second = force * seconds
- Δ energy= (Δ energy/meter) * meter = force * meter = work
- Δ energy = (energy/kg/degree) * kg * degree = heat * kg * degree

Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

***SCIENCE: counting and double-counting time, space, matter, force and energy***

Counting time, the unit is seconds. A bundle of 60 seconds is called a minute; a bundle of 60 minutes is called an hour, and a bundle of 24 hours is called a day, of which a bundle of 7 is called a week. A year contains 365 or 366 days, and a month from 28 to 31 days.

Counting space, the international unit is meter, of which a bundle of 1000 is called a kilometer; and if split becomes a bundle of 1000 millimeters, 100 centimeters and 10 decimeters. Counting squares, the unit is 1 square-meter. Counting cubes, the unit is 1 cubic-meter, that is a bundle of 1000 cubic-decimeters, also called liters, that split up as a bundle of 1000 milliliters.

Counting matter, the international unit is gram that splits up into a bundle of 1000 milligrams and that unites in a bundle of 1000 to 1 kilogram, of which a bundle of 1000 is called 1 tons.

Counting force and energy, a force of 9.8 Newton will lift 1 kilogram, that will release an energy of 9.8 Joule when falling 1 meter.

Cutting up a stick in unequal lengths allows the pieces to be double-counted in liters and in kilograms giving a per-number around 0.7 kg/liter, also called the density.

A walk can be double-counted in meters and seconds giving a per-number at e.g. 3 meter/second, called the speed. When running, the speed might be around 10 meter/second. Since an hour is a bundle of 60 bundles of 60 seconds this would be 60*60 meters per hour or 3.6 kilometers per hour, or 3.6 km/h.

A pressure from a force applied to a surface can be double-counted in Newton and in square meters giving a per-number Newton per square-meter, also called Pascal.

Motion can be double-counted in Joules and seconds producing the per-number Joule/second called Watt. To run properly, a bulb needs 60 Watt, a human needs 110 Watt, and a kettle needs 2000 Watt, or 2 kiloWatt. From the Sun the Earth receives 1370 Watt per square meter.

***Warming and boiling water***

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

Heating 1000 gram water 80 degrees in 167 seconds in a 2000 Watt kettle, the per-number will be 2000*167/80 Joule/degree, creating a double per-number 2000*167/80/ 1000 Joule/degree/gram or 4.18 Joule/degree/gram, called the specific heat of water.

Producing 100 gram steam in 113 seconds, the per-number is 2000*113/100 Joule/gram or 2260 J/g, called the heat of evaporation for water.

***Dissolving material in water***

In the sea, salt is dissolved in water. The tradition describes the solution as the number of moles per liter. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59gram = (100/59)*1mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

***An electrical circuit***

To work properly, a 2000Watt water kettle needs 2000Joules per second. The socket delivers 220Volts, a per-number double-counting the number of Joules per charge-unit.

Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm.

Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So, the 120Watt bulb only receives half of the energy of the 60Watt bulb.

***How high up and how far out***

A ping-pong ball is sent upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second.

To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the speed is split into a vertical and a horizontal part, both reducing it with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed by the formula: Horizontal distance to the top = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

The vertical distance is halved, but the vertical speed changes from 9.92 to 9.92*0.707 = 7.01 only. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the distance four times.

***TECHNOLOGY: letting steam work***

A water molecule contains two Hydrogen and one Oxygen atom weighing 2*1+16 units. A collection of a million billion billion molecules is called a mole; a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. The volume should increase accordingly. But, if kept constant, instead the inside pressure will increase.

Inside a cylinder, the ideal gas law, p*V = n*R*T, combines the pressure, p, and the volume, V, with the number of moles, n, and the absolute temperature, T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

So, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn.

In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus decreasing and increasing the pressure by letting steam in and out of the two holes. The leaving steam the is visible on steam locomotives.

In the third generation used in power plants, two cylinders, a hot and a cold, connect with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back to the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical power to industries and homes.

***ENGINEERING: how many turns on a steep hill***

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2.

In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2.

In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

***MATHEMATICS: the simplicity of counting before adding next-to and on-top***

Meeting Many, we ask ‘How many in total?’ To answer, we count and add. To count means to use division, multiplication and subtraction as icons for bundling, stacking and removing stacks to predict unit-numbers as blocks of stacked bundles; but also, to recount to change unit, and to double-count to get per-numbers bridging the units, both rooting proportionality.

Once counted and recounted and double-counted, totals can be added next-to or on-top, rooting integral calculus and proportionality; and that, if reversed, roots differential calculus and solving equations. Adding thus means uniting unit-numbers and per-numbers, but both can be constant or changing, so to predict, we need four uniting operations: addition and multiplication unite changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. As well as four splitting operations: subtraction and division split into changing and constant unit-numbers; and differentiation and root/logarithm split into changing and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite. And it appears in Arabic numbers written out fully as T = 456 = 4 bundles-of-bundles & 5 bundles & 6 unbundled, showing all four uniting operations, addition and multiplication and power and next-to addition of stacks; and showing that the word-language and the number-language share the same sentence form with a subject and a verb and a predicate or object. Finally, shapes can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

So, by its simplicity, mathematics is easy and quick to learn if education wants to do so.

### Adding addition to the curriculum

Once counted, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being re-counted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

***Uniting constant and changing unit-numbers and per-numbers***

The number-formula also shows the four ways to unite numbers offered by algebra meaning ‘reuniting’ in Arabic: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers:

| Uniting/*splitting into* | Changing | Constant |
|---|---|---|
| Unit-numbers<br>m, s, kg, $ | T = a + n<br>*T – a = n* | T = a * n<br>*T/n = a* |
| Per-numbers<br>m/s, $/kg, $/100$ = % | T = ∫ a dn<br>*dT/dn = a* | T = a^n<br>*loga(T) = n    n√T = a* |

**Conclusion and recommendation**

This paper argues that the low success of 50 years of mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views offer different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying 'To master Many, counting and recounting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating.' A proposal for two separate a twin-curricula in counting and adding is found in Tarp (2018a).

Thus, this simplicity of mathematics as expressed in a Count-before-Adding curriculum allows bundle-numbers to replace line-numbers, and to learn core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM-teachers or STEM-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

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## 20. Good, Bad & Evil Mathematics - Sociological Imagination in Math Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking “renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.” (p. 16). As to organizations, he warns against a so-called goal displacement: “The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right.” (p. 84)

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference. So, using sociological imagination we can ask the ‘Cinderella question’: ‘as an alternative to the tradition, is there is a different way to the goal of mathematics education, mastery of Many?’ In short, could there be different kinds of mathematics? And, could it be that among them, one is good, and one is bad, and one is evil?

Meaning ‘Earth measuring’ in Greek and ‘Reuniting numbers’ in Arabic, geometry and algebra clearly was grounded in the physical fact Many. But, around 1900 the set-concept transformed grounded mathematics into today’s self-referring ‘meta-matism’. Which is a mixture of ‘meta-matics’, defining its concepts by self-reference, i.e. top-down as examples of abstractions instead of bottom-up as abstractions from examples; and of ‘mathe-matism’ true inside but seldom outside classrooms. Here adding numbers without units as ‘2 + 3 IS 5’ meets counter-examples as 2weeks + 3days is 17 days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

The existence of three different versions of mathematics, ‘many-math’ and ‘meta-matics’ and ‘mathe-matism’, allows formulating the following definitions:

Good mathematics is absolute truths about things rooted in the outside world, e.g. $T = 2*3 = 6$. So good mathematics is tales about how totals are counted, united and changed; described by a number-language sentence with a subject, T, and a verb, is, and a predicate, 2*3.

Bad mathematics is relative truths about things rooted in the outside world. An example is claiming unconditionally that 2+3 = 5. So bad mathematics is tales about numbers without units.

Evil mathematics talks about something existing only inside classrooms. An example is claiming that fractions are numbers, and that they can be added without units as e.g. 1/2 + 2/3 = 7/6 even if 1 red of 2 apples plus 2 reds of 3 apples total 3 reds of 5 apples, and certainly not 7 reds of 6 apples. So bad mathematics is tales about fractions as numbers.

On this background, the paper will outline a grounded curriculum in Good Mathematics, free of self-reference and goal-displacement (Tarp, 2018).

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## 21. Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum

*Its many applications make mathematics useful; and of course, it must be learned before applied. Or, can it be learned through its original roots? Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. Asking 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a recount-formula occurring all over mathematics. Once counted, totals can be added next-to or on-top rooting calculus and proportionality. From this 'Count-before-Adding' curriculum, Many-matics offers remedial micro-curricula for classes stuck in a traditional curriculum.*

### Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change'. Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

### Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

### Mathematics, before and now

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET changed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to ‘2x3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

### Difference Research Looking at Mathematics Education

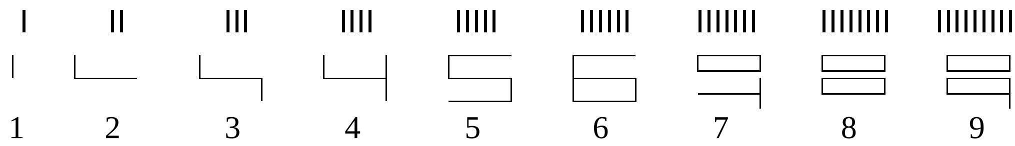
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ (Tarp, 2017) is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a list of remedial micro-curricula for classes stuck in a traditional mathematics curriculum. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

### Meeting Many Creates a ‘Count-before-Adding’ Curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence (Tarp, 2018b).

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:

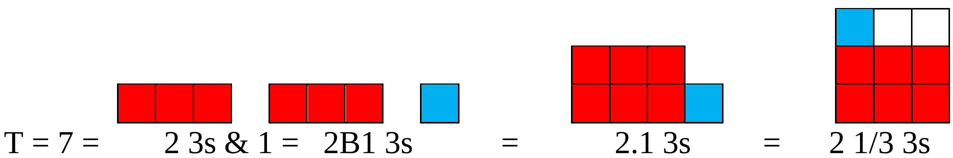



We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T, we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: 7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

| | |
|---|---|
| 7 / 3 | 2.some |
| 7 – 2 x 3 | 1 |

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations and differential calculus:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = $\Delta$T/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

### A typical mathematics curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed. Adding without units always precede bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, let us use difference research and imagination to uncover or invent remedial micro-curricula for classes stuck in the tradition.

### Remedial micro-curricula for classes stuck in the tradition

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to use two-dimensional block-numbers and bundle-counting, recounting in the same and in a different unit, and calculator prediction before next-to and on-top addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to Here a difference is to talk about 8/2 as ‘8 counted in 2s’ instead of as ‘8 divided between 2’; and to rewrite the number as ‘10 or 5 times less something’ and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of ‘mathe-matism’ true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes ‘middle school integration’ later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of

reverse operations to establish the basic rule for solving equations ‘move to the opposite side with the opposite sign’: In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn’t have a reverse sign. This gives 3*u = 14 – 2. Since u doesn’t have a reverse sign, 3 moves to the other side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. ‘transform the formula T = a/(b-c) so that all letters become subjects.’ A hymn can be created: “Equations are the best we know / they’re solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don’t want to stop.”

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to ‘pre-dict’ a number, instead statistics can ‘post-dict’ its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains

several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the $5^{th}$ root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children's own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Inspirational purposes have led to the creation of several DrAlTarp YouKu.com, SoKu.com videos, and MrAlTarp YouTube videso: Deconstructing Fractions, Deconstructing Calculus, Deconstructing PreCalculus Mathematics, Missing Links in Primary Mathematics, Missing Links in Secondary Mathematics, Postmodern Mathematics, PreSchool Math.

## Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, one alternative presents itself directly for future studies creating a paradigm shift (Kuhn, 1962): to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining the misery coming from teaching meta-matism. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research should search for differences and test if they make a difference. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a is-sentence, trust the subject since it exists, but doubt the predicate, it is a verdict that might be gossip. So, maybe we should stop teaching essence and instead start letting learners meet and experience existence.

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## 22. Mastering Many by Counting, Re-counting and Double-counting before Adding On-top and Next-to

*Observing the quantitative competence children bring to school, and by using difference-research searching for differences making a difference, we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.*

*Keywords:* numbers, operations, proportionality, calculus, early childhood

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed e.g. by the International Congress on Mathematics Education taking place each four year since 1969. However, despite 50 years of research, many countries still experience poor results in the Programme for International Student Assessment (PISA). In the former model country Sweden this caused the Organisation for Economic Co-operation and Development (OECD) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change' since 'more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.' (OECD, 2015, p. 3).

Mathematics thus seems to be hard by nature. But, with mathematics and education as social institutions, a different answer, by choice, may come from sociology, having imagination as a core part as pointed out by Mills (1959). Bauman (1990) agrees when talking about organizations:

> Sociological thinking is, one may say, a power in its own right, an *anti-fixating* power. It renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. (p.16) **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (p.79) Last but not least, the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right: the new end against which the organization tends to measure the rationality of its performance (p.84).

It is a general opinion that the goal of mathematics education is to learn mathematics. However, this goal is self-referring. So maybe traditional mathematics has a goal displacement hiding a different more fruitful way to the outside goal, to master Many as it occurs in space and time?

### Difference-research

To find differences we use 'Difference-research' (Tarp, 2018a) searching for differences making a difference, thus containing two parts: finding a difference, and testing it to see if it makes a difference. This paper focuses on the first part in order to find differences that can be tested to create a background for a possible paradigm shift (Kuhn, 1959).

Difference-research builds on sociological imagination; and on the skeptical thinking of the ancient Greek sophists warning against choice presented as nature. Thus disagreeing with Plato seeing choice as an illusion since the physical is but examples of meta-physical forms visible only to philosophers educated at his academy, later by Christianity turned into monasteries before being changed back again by the Reformation. In the Renaissance, this created the skeptical thinking of natural science, which rooted the Enlightenment century with its two republics, the American and the French (Russell, 1945).

Where France now has its fifth republic, the USA still has its first with skepticism as pragmatism and symbolic interactionism and grounded theory. To protect its republic, France has developed a skepticism inspired by the German thinker Heidegger, seen by Bauman as starting ‘the second Copernican revolution’ by asking: What is ‘is’? (Bauman, 1992, p. ix).

Heidegger (1962) sees three of our seven basic is-statements as describing the core of Being: ‘I am’ and ‘it is’ and ‘they are’; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the ‘I’ must create an authentic relationship to the ‘It’. However, this is made difficult by the ‘dictatorship’ of the ‘They’, shutting the ‘It’ up in a predicate-prison of idle talk, gossip.

Heidegger thus uses existentialist thinking, described by Sartre (Marino, 2004) as holding that ‘existence precedes essence’ (p. 22). In France, Heidegger inspired the poststructuralist thinking pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Foucault, 1995; Lyotard, 1984; Tarp, 2016).

Difference-research tells what can be different from what cannot. From a Heidegger view, an is-sentence contains two things: a subject that exists and cannot be different, and a predicate that can and that may be gossip masked as essence, provoking ‘the banality of Evil’ (Arendt, 1963) if institutionalized. So, to discover its true nature, we need to meet the subject, Many, outside the predicate-prison of traditional mathematics. We will use Grounded Theory (Glaser and Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1970) from a personal to a social level, to allow Many create its own categories and properties. In this way, we can see if our observations can be assimilated to traditional mathematics or will suggest it be accommodated.

### Our Two Languages with Word- and Number-Sentences

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: ‘This is a chair’. As does the number-language assigning numbers instead: ‘the 3 chairs each have 4 legs’, abbreviated to ‘the total is 3 fours’, or ‘$T$ = 3 4s’ or ‘$T$ = 3*4’. Unfortunately, the tradition hides the similarity between word- and number-sentences by leaving out the subject and the verb by just saying ‘3*4 = 12’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is an auxiliary verb’. Likewise, the sentence ‘$T$ = 3*4’ leads to a meta-sentence ‘’*’ is a commutative operation’.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead its self-referring set-based form has turned mathematics into a grammar labeling its outside roots as ‘applications’, used as means to dim the impeding consequences of teaching a grammar before its language.

So, using full sentences including the subject and the verb in number-language sentences is a difference to the tradition; as is teaching language before grammar.

### Mathematics, Rooted in Many, or in Itself

The Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. Together they formed the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic.

However, 50 years ago the set-concept created a self-referring ‘New Math’ or ‘meta-matics’ with concepts defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. And neglecting that Russell, by looking at the set of sets not belonging to itself, showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

So, to find a difference we now return to the Greek origin to meet Many openly to uncover a ‘Many-matics’ as a natural science about Many.

### Meeting Many, Children use Block-numbers to Count and Share

How to master Many can be observed from preschool children. Asked ‘How old next time?’, a 3year-old child will say ‘Four’ and show 4 fingers; but will react strongly if held together 2 by 2, ‘That is not 4, that is 2 2s.’

Children thus describes what exists in the world: bundles of 2s, and 2 of them. So, what children bring to school is 2-dimensional block-numbers, illustrated geometrically by LEGO blocks, together with some quantitative competence. Children thus love re-counting 5 sticks in 2s in various ways as 1 2s & 3, as 2 2s & 1, and as 3 2s less 1.

Sharing nine cakes, four children take one by turn saying ‘I take 1 of each 4’. With 1 left they might say ‘let’s count it as 4’. Thus, children share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts.

Children quickly observe the difference between a ‘stack-number’ as 6 = 3 2s or 2 3s, and a prime number as 3, serving only as a bundle-number by always leaving singles if stacked.

Finally, by turning and splitting 2-dimensional or 3-dimensional blocks, children see their commutative, distributive and associative properties as self-evident: of course, 2 3s is the same as 3 2s; and 6 3s can be split in 4 3s and 2 3s; and 2 3*4s is the same as 2*3 4s.

### Meeting Many Openly

Many exists in space and time as multiplicity and repetition. Meeting Many we ask: ‘how many in total?’ To answer, we count and add. We count by bundling and stacking as seen when writing out fully the total $T$ = 456 = 4*$B$^2 + 5*$B$ + 6*1 showing three stacks or blocks added next-to each other: one with 4 bundles of bundles, one with 5 bundles, and one with 6 unbundled singles. Typically, we use ten as the bundle-size, formally called a base.

Digits occur by uniting e.g. five ones to one fives, rearranged as an icon with five strokes if written less sloppy. As the bundle-size, ten needs no icon when counted as 10, one bundle and no unbundled. Then follow eleven and twelve coming from Danish Vikings counting ‘one left’ and ‘two left’.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| I | II | III | IIII | IIIII | IIIIII | IIIIIII | IIIIIIII | IIIIIIIII |

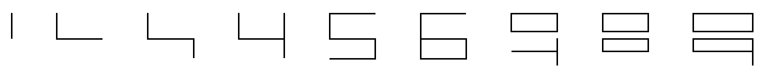

*Figure 1. Digits as icons with as many sticks as they represent.*

#### *Counting by Bundling*

We count in several ways. Some gather-hunter cultures count ‘one, two, many’. Agriculture needs to differentiate degrees of Many and typically bundles in tens. To include the bundle, we can count ‘0Bundle1, 0*B*2, 0*B*3,…, 1*B*, 1*B*1, 1*B*2’, etc.; or ‘0.1 tens, 0.2 tens’, etc., using a decimal point to separate the bundles from the unbundled singles. To signal nearness to the bundle we can count '1, 2, …, 7, bundle less 2, bundle less 1, bundle', etc. Thus a number always contains three numbers: a number of bundles, a number of singles, and a number for the bundle-size.

Bundle-counting, we ask e.g. 'A total of 7 is how many 3s?' Using blocks, we stack the 3-bundles on-top of each other. The single can be placed next-to, or on-top counted in 3s. Thus, the result of counting 7 in 3s, $T = 2$ 3s & 1, can be written as $T = 2B1$ 3s using 'bundle-writing', and as $T = 2.1$ 3s using 'decimal-writing', and as $T = 2\ 1/3$ 3s using 'fraction-writing'.

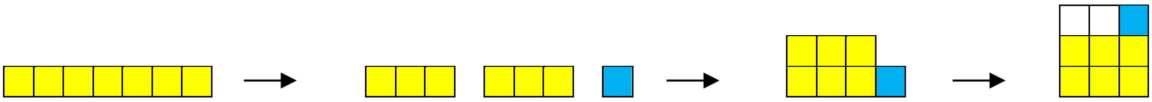

*Figure 2. Seven bundle-counted as 2Bundle1 3s, as 2.1 3s, and as 2 1/3 3s.*

***Bundle-counting in Space and Time***

We include space and time by using 'geometry-counting' in space, and 'algebra-counting' in time. Counting in space, we stack the bundles and report the result on an abacus in 'geometry-mode'. Here the total 7 is on the below bar with 1 unbundled and a block with 2 bundles on the bars above.

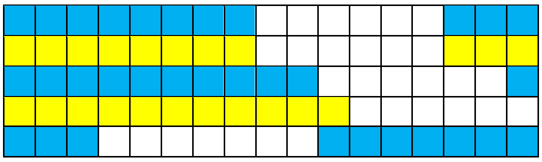

*Figure 3. Seven bundle-counted as 2B1 3s on an abacus in geometry-mode.*

Counting in time, we count the bundles and report the result on an abacus in 'algebra-mode'. Here the total 7 is on the below bar with 1 unbundled and the number of bundles on the bars above.

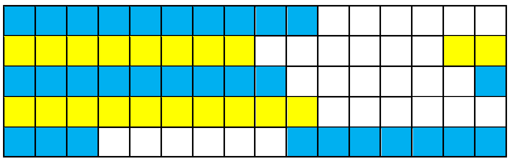

*Figure 4. Seven bundle-counted as 2.1 3s on an abacus in algebra-mode.*

***A Calculator Predicts Counting-results***

Iconizing the counting processes also, a calculator can predict a counting-result. A stack of 2 3s is iconized as 2x3 (or 2*3) showing a lift used 2 times to stack the 3s. As for taking away, subtraction shows the trace left when taking away just once, and division shows the broom wiping away several times.

So, entering '7/3' we ask the calculator 'from 7, 3s can be taken away how many times?' The answer is '2. some'. To find the leftover singles we take away the stack of 2 3s by asking '7 – 2*3'. From the answer '1' we conclude that 7 = 2$B$1 3s. Showing '7 – 2*3 = 1', a display indirectly predicts that 7 can be re-counted as 2 3s and 1, or as 2$B$1 3s or 2.1 3s.

| | |
|---|---|
| **7 / 3** | 2.some |
| **7 – 2 * 3** | 1 |

*Figure 5. A calculator predicts how 7 re-counts in 3s as 2.1 3s.*

A calculator thus uses a 're-count formula', $T = (T/B)*B$, saying that 'from $T$, $T/B$ times, $B$s can be taken away'; and a 're-stack formula', $T = (T–B)+B$, saying that 'from $T$, $T–B$ is left, if $B$ is taken away and placed next-to'. The formulas may be illustrated by LEGO blocks. The re-count formula introduces early algebra (Kieran, Pang, Schifter and Ng, 2016) from grade one; and it occurs all over mathematics and science as proportionality formulas. Likewise, the early use of a calculator shows the importance of mathematics as a language for prediction.

### *Cup-Counting Allows Re-Counting in the Same Unit*

Cup-counting uses a cup when bundle-counting e.g. 7 in 3s. For each bundle we place a stick inside the cup, leaving the unbundled singles outside.

T = 7 = I I I I I I I → III III I → [ ▌▌] I → 2B1 3s = 2.1 3s

One stick moves outside the cup as a bundle of 1s, that moves back inside as 1 bundle. This will change the 'normal' form to an 'overload', or to an 'underload' leading to negative numbers that may be used freely in childhood even if adults abstain from doing so:

*T* = 7 = I I I I I I I → III I I I I → [ ▌] I I I I → 1*B*4 3s = 1.4 3s

*T* = 7 = I I I I I I I → III III I-II II → [ ▌▌▌] II → 3*B*-2 3s = 3.-2 3s

### *Re-Counting in a Different Unit*

Re-counting in a different unit means changing units, also called proportionality. Re-counting 3 4s in 5s, the re-count formula and a calculator predict the result 2 5s & 2 by entering '3*4/5' and taking away the 2 5s.

| | |
|---|---|
| **3 * 4 / 5** | 2.some |
| **3 * 4 – 2 * 5** | 2 |

*Figure 6. A calculator predicts how 3 4s re-counts in 5s as 2.2 5s.*

### *Re-Counting from Icons to Tens*

A calculator has no ten-button. Instead, to re-count an icon-number as 3 4s in tens, it gives the result 1.2 tens directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

| | |
|---|---|
| **3 * 4** | 12 |

*Figure 7. A calculator predicts how 3 4s re-counts in tens as 1.2 tens.*

Re-counting from icons to tens, 3 4s is a geometrical block that increases its base. Therefore, it must decrease its height to keep the total unchanged.

Re-counting in tens is called multiplication tables to be learned by heart. However, the ten-by-ten table can be reduced to a 4-by-4 table since 5 is half of ten and 6 is ten less 4, and 7 is ten less 3 etc. Thus *T* = 4*7 = 4 7s that re-counts in bundles of tens as

*T* = 4*7 = 4*1*B*-3 tens = 4*B*-12 tens = 3*B*-2 tens = 2*B*8 tens = 28

Such results generalize to algebraic formulas as *a*(b – c) = a*b – a*c*.

### *Re-Counting from Tens to Icons*

Re-counting from tens to icons will decrease the base and increase the height. The question '38 is ? 7s' is called an equation '38 = *u**7', using the letter *u* for the unknown number. An equation is easily solved by recounting 38 in 7s, thus providing a natural 'to opposite side with opposite sign' method as a difference to the traditional 'do the same to both sides' method.

*u**7 = 38 = (38/7)*7 so *u* = 38/7 = 5 3/7

*Figure 8. An equation solved by re-counting, the OppositeSide&Sign method.*

### *Once Counted, Totals Can be Added On-Top or Next-To*

To add on-top by asking '3 5s and 2 3s total how many 5s?', the units must be the same. So, 2 3s must be re-counted in 5s as 1*B*1 5s that added to the 3 5s gives 4*B*1 5s.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking '(3*5 + 2*3)/5', the answer is '4. Some'. Taking away 4 5s leaves 1. So again, we get the result 4*B*1 5s.

| | |
|---|---|
| **(3 * 5 + 2 * 3)/ 5** | 4.some |
| **(3 * 5 + 2 * 3) – 4 * 5** | 1 |

*Figure 9. A calculator predicts how 3 5s and 2 3s re-counts in 5s as 4.1 5s.*

To add next-to by asking '3 5s and 2 3s total how many 8s?', we add by areas, called integral calculus. With blocks we get the answer 2*B*5 8s.

Using a calculator to predict the result, we use a bracket before counting in 8s: Asking '(3*5 + 2*3)/8', the answer is '2. Some'. Taking away 2 8s leaves 5. So again, we get the result 2*B*5 8s.

| | |
|---|---|
| **(3 * 5 + 2 * 3)/ 8** | 2.some |
| **(4 * 5 + 2 * 3) – 2 * 8** | 5 |

*Figure 10. A calculator predicts how 3 5s and 2 3s re-counts in 8s as 2.5 8s.*

***Reversing Adding On-Top and Next-To***

Reversed addition may be called backward calculation or solving equations. Reversing next-to addition may be called reversed integration or differentiation. Asking '3 5s and how many 3s total 2*B*6 8s?', using blocks gives the answer 2*B*1 3s.

Using a calculator to predict the result, the remaining is bracketed before counting in 3s.

| | |
|---|---|
| **(2 * 8 + 6 – 3 * 5)/ 3** | 2 |
| **(2 * 8 + 6 – 3 * 5) – 2 * 3** | 1 |

*Figure 11. A calculator predicts how 2.6 8s re-counts in 3 5s and 2.1 3s.*

Adding or integrating two areas next-to each other means multiplying before adding. Reversed integration, i.e. differentiation, then means subtracting before dividing, as shown by the gradient formula $y' = \Delta y/t = (y2 - y1)/t$.

***Double-Counting in Two Units Creates Per-Numbers and Proportionality***

Double-counting the same total in two units is called proportionality, which produces 'per-numbers' as e.g. 2$ per 5kg, or 2$/5kg, or 2/5 $/kg.

To answer the question '*T* = 6$ = ?kg' we use the per-number to re-count 6 in 2s, that many times we have 5kg: *T* = 6$ = (6/2)*2$ = (6/2)*5kg = 3*5kg = 15kg. And vice versa: Asking '*T* = 20kg = ?$', the answer is *T* = 20kg = (20/5)*5kg = (20/5)*2$ = 4*2$ = 8$.

A total can be double-counted in colored blocks of different values, e.g. 1 red per 3 blues. Here, a total of 10 blues re-counts as *T* = 7*b* & 1*r* = 4*b* & 2*r* = 1*b* & 3*r*. Likewise, a total of 3 reds re-counts as *T* = 3*b* & 2*r* = 6*b* & 1*r* = 9*b*. Placed next to each other, this introduces a primitive coordinate system.

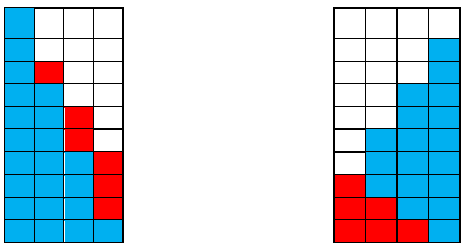

*Figure 12. 10 blues left, and 3 reds right, re-counted in combinations.*

***Double-Counting in the Same Unit Creates Fractions as Per-Numbers***

Double-counting a total in the same unit, per-numbers take the form of fractions, e.g. as 3$ per 5$ = 3/5; or percentages as 3$ per 100$ = 3/100 = 3%.

Thus, to find 3$ per 5$ of 20$, or 3/5 of 20, we re-count 20 in 5s as 20 = (20/5)*5 = 4*5. Now we have two options. Seeing 20 as 4 5s, 4 times we get 3, i.e. 4*3 = 12; and seeing 20 as 5 4s, we get 3 4s, i.e. 3*4 = 12.

Likewise, to find what 3$ per 5$ is in percent, i.e. per 100, we re-count 100 in 5s as 100 = (100/5)*5 = 20*5. Again, we have two options. Seeing 100 as 20 5s, 20 times we get 3, i.e. 20*3 = 60; and seeing 100 as 5 20s, we get 3 20s, i.e. 3*20 = 60. So, 3 per 5 gives 60 per 100 or 60%.

Including or removing units will enlarge or reduce fractions:

4/6 = 4 3s/6 3s = 4*3/6*3 = 12/18

4/6 = 2*2/3*2 = 2 2s/3 2s = 2/3

***Adding Per-numbers Roots Integral Calculus before Differential Calculus***

Adding 2kg at 3$/kg and 4kg at 5$/kg, the 'unit-numbers' 2 and 4 add directly, but the per-numbers 3 and 5 must be multiplied first, thus creating areas. So per-numbers and fractions are not numbers, but operators needing numbers to become numbers. Per-numbers thus add by the areas under the per-number graph, here being 'piecewise constant'.

Asking '3 seconds at 4m/s increasing steadily to 5m/s', the per-number is 'locally constant'. This concept is formalized by an '*epsilon-delta* criterion' seeing three forms of constancy: *y* is 'globally constant' *c* if, for any positive number *epsilon*, the difference between *y* and *c* is less than *epsilon.* And *y* is 'piecewise constant' *c* if an interval-width *delta* exists such that, for any positive number *epsilon*, the difference between *y* and *c* is less than *epsilon* in this interval. Interchanging *epsilon* and *delta* makes *y* 'locally constant' or continuous. Likewise, the change ratio □*y*/□*x* can be globally, piecewise or locally constant, in the latter case written as *dy/dx = y'*.

With locally constant per-numbers, the area under the graph splits up into countless strips that add easily if written as differences since the middle terms then will disappear, leaving just the difference between the end- and start-values. Thus, adding areas precedes and motivates differential calculus.

***Using Letters and Functions for Unspecified Numbers and Calculations***

At the language level we can set up a calculation with an unspecified number *u*, e.g. *T* = 2 + ? = 2 + *u*. Also, at the meta-language level we can set up an unspecified formula with an unspecified number *u*, written as *T* = *f(u).*

With one unspecified number, a formula becomes an equation as 8 = 2**u*; with two, a formula becomes a function as T = 2**u*; and with three, a formula becomes a surface as T = 2**u* + 2**w*.

Although we can write it, *T* = *f*(2) is meaningless since 2 is not an unspecified number. When specified, a function can be linear or exponential, but it cannot be a number or increase. A total can increase, but the way it does so cannot. Mixing language and meta-language creates meaningless sentences as 'the predicate ate the apple'.

A general number-formula as e.g. *T = a*x^2 + b*x + c* is called a polynomial. It shows the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

| Operations **unite**/ *split* Totals in | Changing | Constant |
|---|---|---|
| Unit-numbers<br>m, s, kg, $ | $\boldsymbol{T = a + n}$<br>$T - n = a$ | $\boldsymbol{T = a*n}$<br>$T/n = a$ |
| Per-numbers<br>m/s, $/kg, $/100$ = % | $\boldsymbol{T = \int a*dn}$<br>$dT/dn = a$ | $\boldsymbol{T = a^{\wedge}n}$<br>$n\sqrt{T} = a \quad log_a T = n$ |

*Figure 13. The 'algebra-square' shows the four ways to unite or split numbers.*

The number-formula contains the formulas for constant change:

$T = b*x$ (proportional)

$T = b*x + c$ (linear)

$T = a * x^{\wedge}n$ (elastic)

$T = a * n^{\wedge}x$ (exponential)

$T = a*x^{\wedge}2 + b*x + c$ (accelerated)

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to 'post-dict' what we cannot 'pre-dict'.

Combining linear and exponential change by *n* times depositing *a*$ to an interest rate *r*%, we get a saving *A*$ predicted by a simple formula, $A/a = R/r$, where the total interest rate *R* is predicted by the formula $1+R = (1+r)^{\wedge}n$.

The formula and the proof are both elegant: in a bank, an account contains the amount *a/r*. A second account receives the interest amount from the first account, $r*a/r = a$, and its own interest amount, thus containing a saving *A* that is the total interest amount *R*a/r*, which gives $A/a = R/r$.

***Trigonometry before Geometry***

The tradition introduces plane geometry before coordinate geometry and trigonometry. A difference is the opposite order with trigonometry first since halving a block by its diagonal allows the base and the height to be re-counted in the diagonal or in each other to create the per-numbers sine, cosine, tangent and gradient:

height = (height/base)*base = tangent*base = gradient*base.

This allows a calculator to find *pi* from a formula: $\pi = n*\tan(180/n)$ for *n* sufficiently large; and it allows to predict an angle *A* form its base *b* and height *a* by reversing the formula tan $A = a/b$.

Integrating plane and coordinate geometry allows geometry and algebra to always go hand in hand. In this way solving algebraic equations predicts intersection points in geometrical constructions, and vice versa.

***Testing a Many-matics Micro-curriculum***

A '1 cup and 5 sticks' micro-curriculum can be designed to help a class stuck in division. The intervention begins by bundle-counting 5 sticks in 2s, using the cup for the bundles. The results, 1*B*3 2s and 2*B*1 2s and 3*B*-1 2s, show that a total can be counted as an inside number of bundles, and an outside number of singles; and written in three ways: overload and normal and underload.

So, to divide 336 by 7, we move 5 bundles outside as 50 singles to re-count 336 with an overload: 336 = 33*B*6 = 28*B*56, which divided by 7 gives 4*B*8 = 48. With multiplication, singles move inside as bundles: 7* 4*B*8 = 28*B*56 = 33*B*6 = 336. 'Is it that easy?' is a typical reaction.

***Algebra before Arithmetic may now be Possible***

Introducing algebra before arithmetic was central to the New Math idea and to the work of Davidov (Schmittau, 2004). Introducing algebra as generalized arithmetic, the book 'Early Algebra' describes how 'a fourth-grade USA class is investigating what happens to the product of a multiplication expression when one factor is increased by a certain amount.' (Kieran et al, 2016, p.17). The investigation begins with an example showing that 7*3 = 21, and 7*5 = 35, and 9*3 = 27.

In a first-grade class working with block-numbers with the bundle as the unit, the answer would be: 7*3 is 7 3s, and 7*5 is 7 5s, and 9*3 is 9 3s. So 7*5 means that 7 2s is added next-to 7 3s. Re-counted in tens this will increase the 2*B*1 tens with 1*B*4 tens to 3*B*5 tens. Likewise, 9*3 means that 2 3s is added on-top of 7 3s. Re-counted in tens this will increase the 2*B*1 tens with 0*B*6 tens to 2*B*7 tens.

Adding 2 to both numbers means adding additional 2 2s. Re-counted in tens this will increase the 2*B*1 tens with 1*B*4 tens and 0*B*6 tens and additional 0*B*4 tens to 4*B*5 tens.

Counting 7 as 9 less 2, and 3 as 5 less 2, will decrease the 9 5s with 2 5s and 2 9s. Only now we must add the 2 2s that was removed twice, so (9-2)*(5-2) = 9*5 - 9*2 - 2*5 + 2*2 as shown on a western ten by ten abacus as a 9 by 5 block. This roots the algebraic formula $(a – b)*(c – d) = a*c – a*d – b*c + b*d$.

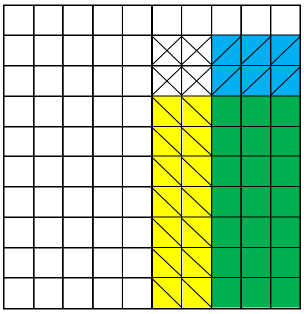

*Figure 14. An abacus shows that 7*3 = (9-2)*(5-2) = 9*5 - 9*2 - 2*5 + 2*2.*

Later follows a discussion on solving equations (pp. 25-29). In a first-grade class working with block-numbers with the bundle as the unit, solving the equation 3*x* + 9 = 5*x* + 1, the geometrical answer would be: to the left we have a block of 3*B*9 *x*s, and to the right we have a block of 5*B*1 *x*s. Removing 3 bundles and 1 single from both, we get 8 = 2*x*. Re-counting 8 in 2s we get 2**x* = 8 = (8/2)*2, so *x* = 8/2 = 4.

The algebraic answer would be similar: to the left we have 3 bundles inside and 9 singles outside the bundle-cup, and to the right we have 5 bundles inside and 1 single outside. Removing 3 bundles from the inside and 1 single from the outside, we get 8 = 2*x*. Re-counting 8 in 2s we get 2**x* = 8 = (8/2)*2, so *x* = 8/2 = 4.

Using block-numbers instead of line-numbers thus allows introducing algebra before arithmetic since with the re-count formula, counting and re-counting and double-counting precede addition.

***Conclusion and Recommendation***

Among the many research articles on counting and arithmetic, only few deal with block-numbers (Zybartas and Tarp, 2005). Dienes (2002), the inventor of Multi-base blocks, has similar ideas when saying (p. 1):

> The position of the written digits in a written number tells us whether they are counting singles or tens or hundreds or higher powers. (..) My contention has been, that in order to fully understand how the system works, we have to understand the concept of power. (..) In school, when young children learn how to write numbers, they use the base ten exclusively and they only use the exponents zero and one (namely denoting units and tens), since for

> some time they do not go beyond two digit numbers. So neither the base nor the exponent are varied, and it is a small wonder that children have trouble in understanding the place value convention.

Instead of talking about bases and higher powers, working with icon-bundles and bundles of bundles will avoid that ‘neither the base nor the exponent are varied’. By seeing bundles as existence and bases as essence, block-numbers differ from Dienes’ multi-base blocks that seem to have set-based mathematics as the goal, and blocks as a means.

Set, however, changed mathematics from a bottom-up Greek ‘Many-matics’ into today’s self-referring top-down ‘meta-matism’, a mixture of ‘meta-matics’ with concepts defined top-down instead of bottom-up, and ‘mathe-matism’ with statements true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meets counter-examples as 2weeks + 3days is 17 days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

So, mathematics is not hard by nature but by choice. And yes, a different way exists to its outside goal, mastery of Many. Still, it teaches line-numbers as essence to be added without units and without being first bundle-counted and re-counted and double-counted. By neglecting the existence of block-numbers and re-counting, it misses the golden learning opportunities from introducing formulas, proportionality, calculus and equations in early childhood education through its grounded alternative, Many-matics.

Consequently, let us welcome ‘good’ 2-dimensional block-numbers and drop ‘bad’ 1-dimensional line-numbers and ‘evil’ fractions (Tarp, 2018b). Let us bundle-count and re-count and double-count before adding on-top and next-to. Let us use full sentences about how to count and (re)unite totals. And, let difference-research use sociological imagination to design a diversity of micro-curricula (Tarp, 2017) to test if Many-matics makes a difference by fulfilling the ‘Mathematics for All’ dream.

Let existence precede essence in mathematics education also. So, think things.

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## 23. Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions

### Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.' (p. 3).

This may prove that, by its very nature, mathematics is indeed hard to learn. On the other hand, since mathematics education is a social institution, social theory may provide a different reason.

### Social Theory Looking at Mathematics Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference.

So, inspired by sociology we can ask the 'Cinderella question': 'as an alternative to the tradition, is there is a different way to the goal of mathematics education, mastery of Many?'

In short, could there be different kinds of mathematics? And could it be that among them, one is good, and one is bad, and one is evil? In other words, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics'.

Here the invention of the concept Set created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

### Good and Bad and Evil Mathematics

The existence of three different versions of mathematics, many-matics and meta-matics and mathe-matism, allows formulating the following definitions:

Good mathematics is absolute truths about things rooted in the outside world. An example is $T$ = 2*3 = 6 stating that a total of 2 3s can be re-counted as 6 1s. So good mathematics is tales about totals, and how to count and unite them.

Bad mathematics is relative truths about things rooted in the outside world. An example is claiming that 2+3 = 5, only valid if the units are the same, else meeting contradictions as 2weeks + 3days = 17days. So bad mathematics is tales about numbers without units.

Evil mathematics talks about something existing only inside classrooms. An example is claiming that fractions are numbers, and that they can be added without units as claiming that 1/2 + 2/3 = 7/6 even if 1 red of 2 apples plus 2 reds of 3apples total 3reds of 5 apples and not 7reds of 6apples. So bad mathematics is tales about fractions as numbers.

### Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks the grounded theory question: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

### Meeting Many Creates a 'Count-before-Adding' Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create number-language sentences, $T$ = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to use as units when counting:

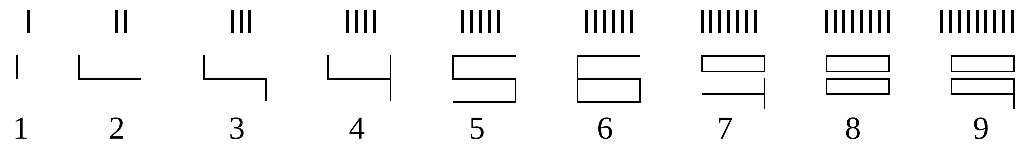


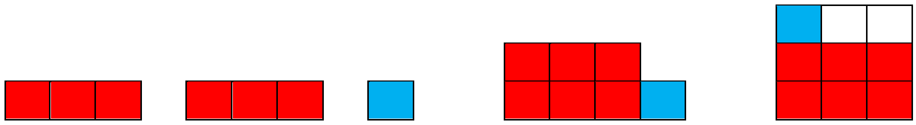

*Figure 1. Digits as icons containing as many sticks as they represent*

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total $T$ we take away bundles $B$ thus rooting and iconizing division as a broom wiping away the bundles. Stacking the bundles roots and iconizes multiplication as a lift stacking the bundles into a block. Moving the stack away to look for unbundled singles roots and iconizes subtraction as a trace left when dragging the block away. A calculator predicts the counting result by a 're-count formula' $T = (T/B)*B$ saying that 'from $T$, $T/B$ times, $B$ can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so $T$ = 7 = 2$B$1 3s.

Placing the unbundled singles next-to or on-top of the stack of 3s roots decimals and fractions:



*T* = 7 = 2 3s & 1 = 2*B*1 3s = 2.1 3s = 2 1/3 3s

*Figure 2. Re-counting a total of 7 in 3s, the unbundled single can be placed in three different ways*

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, *T* = 42 = ? 7s = *u**7, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations, and differential calculus:

*T* = 2 3s + ? 4s = 5 7s gives differentiation: ? = (5*7 – 2*3)/4 = $\Delta T/4$

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers *sine*, *cosine* and *tangent*. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the *y*-change in the *x*-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

**A Short Version of a Curriculum in Good Mathematics, Grounded Many-matics**

01. To stress the importance of bundling, the counting sequence should be: 01, 02, …, 09, 10,11 etc.

02. The ten fingers should be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers should be re-counted in three ways (standard, and with over- and underload): *T* = 2*B*1 5s = 1*B*3 5s = 3*B*-1 5s = 3 bundles less 1 5s.

04. Multiplication tables should be formulated as re-counting from icon-bundles to tens and use overload counting after 5: *T* = 4 7s = 4*7 = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 should be formulated as re-counting from tens to 7s and use overload counting: *T* = 336 /7 = 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48

06. Solving proportional equations as 3**x* = 12 should be formulated as re-counting from tens to 3s: 3**x* = 12 = (12/3)*3 giving *x* = 12/3 illustrating the relevance of the ‘opposite side & sign’ method.

07. Proportional tasks should be done by re-counting in the per-number: With 3$/4kg, *T* = 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and *T* = 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages should be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$, so T = 60$ = (60/3)*3$ gives (60/3)*2$ = 40$

09. Integral should precede differential calculus and include adding both piecewise and locally constant per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at (2*3+4*5)$/(2+4)kg thus showing that per-numbers and fractions are added with their units as the area under the per-number graph.

10. Trigonometry should precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. *a* = (*a/c*)**c* = sin*A***c*.

## Good and Bad Mathematics

Today's tradition begins with arithmetic telling about line-numbers, processed by four basic operations, later extended with negative numbers and rational numbers and reel numbers. Algebra then repeats it all with letters instead. Geometry begins with plane geometry followed by coordinate geometry and trigonometry later. Functions are special set-products, and differential calculus precedes integral calculus.

In general, we see mathematics as truths about well-defined concepts. So we begin by discussing what can be meant by good and bad concepts.

***Good and Bad Concepts***

As an example, let us look at a core concept in mathematics, a calculation. To differentiate between $y = 2*3$ and $y = 2*x$, around 1750 Euler defined the concept 'function' as a calculation containing unspecified numbers. Later, around 1900, set-based mathematics defined a function as an example of a set-product where first component identity implies second component identity.

So where the former is a bottom-up definition of a concept as an abstraction from examples, the latter is a top-down definition of a concept as an example of an abstraction.

Since examples are in the world and since Russell warned that by its self-reference the set-concept is meaningless, we can label bottom-up and top-down definitions good and bad concepts respectively.

***Good and Bad Numbers***

Good numbers should reflect that our number-language describes a total as counted in bundles and expressing the result in a full sentence with subject and verb and predicate as in the word-language, as e.g. $T$ = 2 3s. These are the numbers that children bring to school, two-dimensional block-numbers that contain three different number-types: a 'unit-number' for the size, a 'bundle-number' and a 'single-number' for the number of bundles and unbundled singles. Totals then are written in bundle- form or in decimal-form with a unit where a bundle-B or a decimal point separates the inside bundles from the outside singles, as e.g. T = 3*B*2 tens = 3.2 tens.

Good numbers are flexible to allow a total to be re-counted in a different unit; or in the same unit to create an overload or underload to make calculations easier, as e.g. T = 3*B*2 tens = 2*B*12 tens = 4*B*-8 tens. Good numbers are shown in two ways: an algebraic with bundles, and a geometrical with blocks. Good numbers also tell that eleven and twelve come from the Vikings saying 'one left' and 'two left'.

Bad numbers do not respect the children's own two-dimensional block-numbers by insisting on one-dimensional line-numbers be introduced as names along a line without practicing bundling. Numbers follow a place value system with different places for the ones, tens, hundreds, and thousands; but seldom renaming them as bundles, bundle of bundles, and bundles of bundles of bundles.

***Good and Bad Counting***

A good counting sequence includes bundles in the names, as e.g. 01, 02, …, Bundle, 1*B*1, etc.; or 0Bundle1, 0*B*2, etc. Another sequence respects the nearness of a bundle by saying 0*B*6, 1*B*less3, 1*B*-2, etc.

Good counting lets counting and re-counting and double-counting precede addition; and allows the re-count formula to predict the counting-result; and it presents the symbols for division, multiplication and subtraction as icons coming from the counting process, thus introducing the operations in the opposite order.

Bad counting neglects the different forms of counting by going directly to adding, thus not respecting that totals must be counted before they can be added.

Bad counting treats numbers as names thus hiding their bundle nature by a place value system. This leads some to count ‘twenty-ten’ instead of ‘thirty’, and to confuse 23 and 32.

***Good and Bad Addition***

Good addition waits until after totals have been counted and re-counted in the same and in a different unit, to and from tens, and double-counted in two units to create per-numbers bridging the units. Likewise, good addition respects its two forms: on-top rooting proportionality since changing the units might be need; and next-to rooting integral calculus by being added by the areas.

Bad addition claims it priority as the fundamental operation defining the others: multiplication as repeated addition, and subtraction and division as reversed addition and multiplication. It insists on being the first operation being taught. Numbers must be counted in tens. Therefore there is no need to change or mention the unit; nor is there a need to add next-to as twenties.

Bad addition does not respect that in block-numbers as T = 2*B*3 4s, the three digits add differently. Unit-numbers, as 4, only add if adding next-to. Bundle-numbers, as 2, only add if the units are the same; else re-counting must make them so. Single-numbers, as 3, always add, but might be re-counted because of an overload.

***Good and Bad Subtraction***

Good subtraction sees its sign as iconizing the trace left when dragging away a stack to look for unbundled singles, thus leading on to division as repeated subtraction moving bundles away. It does not mind taking too much away and leaving an underload, as in 3*B*2 - 1*B*5 = 2*B*-3.

Bad subtraction sees its sign as a mere symbol; and sees itself as reversed addition; and doesn’t mind subtracting numbers without units.

***Good and Bad Multiplication***

Good multiplication sees its sign as iconizing a lift stacking bundles. It sees 5*7 as a block of 5 7s that may or may not be re-counted in tens as 3.5 tens or 35; and that has the width 7 and the height 5 that, if recounted in tens, must widen it width and consequently shorten its height. Thus, it always sees the last factor as the unit.

Good multiplication uses flexible numbers when re-counting in tens by multiplying, as e.g. *T* = 6*8 = 6*(ten-2) = (ten-4)*8 = (ten-4)*(ten-2). This allows reducing the ten by ten multiplication table to a five by five table.

Bad multiplication sees its sign as a mere symbol; and insists that all blocks must be re-counted in tens by saying that 5*7 IS 35. It insists that multiplication tables must be learned by heart.

***Good and Bad Division***

Good division sees its sign as iconizing a broom wiping away the 2s in *T* = 8/2. It sees 8/2 as 8 counted in 2s; and it finds it natural to be the first operation since when counting, bundling by division comes before stacking by multiplication and removing stacks by subtraction to look for unbundled singles.

Bad division sees its sign as a mere symbol; and teaches that 8/2 means 8 split between 2 instead of 8 counted in 2s. Bad division accepts to be last by saying that division is reversed multiplication; and insists that fractions cannot be introduced until after division.

***Good and Bad Calculations***

Good calculations use the re-count formula to allow a calculator to predict counting-results.

Bad calculations insist on using carrying so that the result comes out without overloads or underloads.

***Good and Bad Proportionality***

Good proportionality is introduced in grade 1 as re-counting in another unit predicted by the re-count formula. It is re-introduced when adding blocks on-top; and when double-counting in two units to create a per-number bridging the units by becoming a proportionality factor.

Bad proportionality is introduced in secondary school as an example of multiplicative thinking or of a linear function.

***Good and Bad Equations***

Good equations see equations as reversed calculations applying the opposite operations on the opposite side thus using the 'opposite side and sign' method in accordance with the definitions of opposite operations: 8-3 is the number $x$ that added to 3 gives 8; thus if $x+3 = 8$ then $x = 8-3$. Likewise with the other operations.

Good equations sees equations as rooted in re-counting from tens to icons, as e.g. 40 = ? 8s, leading to an equation solved by re-counting 40 in 8s: $x*8 = 40 = (40/8)*8$, thus $x = 40/8 = 5$.

Bad equations insist that the group definition of abstract algebra be used fully or partwise when solving an equation. It thus sees an equation as an open statement expressing identity between two number-names. The statements are transformed by identical operations aiming at neutralizing the numbers next to the unknown by applying commutative and associative laws.

| | |
|---|---|
| $2*x = 8$ | an open statement about the identity of two number-names |
| $(2*x)*(1/2) = 8*(1/2)$ | ½, the inverse element of 2, is multiplied to both names |
| $(x*2)*(1/2) = 4$ | since multiplication is commutative |
| $x*(2*(1/2)) = 4$ | since multiplication is associative |
| $x*1 = 4$ | by definition of an inverse element |
| $x = 4$ | by definition of a neutral element |

*Figure 3. Solving an equation using the formal group definition from abstract algebra*

***Good and Bad Pre-calculus***

Good pre-calculus shows that the number-formula, $T = 345 = 3*BB + 4*B + 5*1 = 3*x^2 + 4*x + 5$, has as special cases the formulas for constant linear, exponential, elastic, or accelerated change: $T = b*x+c$, $T = a*n^x$, $T = a*x^n$, and $T = a*x^2 + b*x + c$. It uses 'parallel wording' by calling root and logarithm 'factor-finder' and 'factor-counter' also. It introduces integral calculus with blending problems adding piecewise constant per-numbers, as e.g. 2kg at 3 $/kg plus 4kg at 5$/kg. It includes modeling examples from STEM areas (Science, Technology, Engineering, Mathematics)

Bad pre-calculus introduces linear and exponential functions as examples of a homomorphism satisfying the condition f($x$#$y$) = f($x$)$f($y$). It includes modeling from classical word problems only.

***Good and Bad Calculus***

Good calculus begins with primary school calculus, adding two blocks next-to each other. It also includes middle school calculus adding piecewise constant per-numbers, to be carried on as high school calculus adding locally constant per-numbers.

It motivates the epsilon-delta definition of constancy as a way to formalize the three forms of constancy: global, piecewise and locally. It shows series with single changes and total changes calculated to realize that many single changes sum up as one single change, calculated as the difference between the end- and start-values since all the middle terms disappear.

This motivates the introduction of differential calculus as the ability to rewrite a block $h*dx$ as a difference $dy$, $dy/dx = h$; and where the changes of block with sides f and g leads on to the fundamental formula of differential calculus, (f*g)’/(f*g)= f’/f + g’/g, giving (x^n)’/x^n = n*1/x, or (x^n)’ = n*x^(n-1).

Bad calculus introduces differential calculus before integral calculus that is defined as anti-differentiation where the area under *h* is a primitive to *h*; and it introduces the epsilon-delta criterion without grounding it in different kinds of constancy.

***Good and Bad Modeling***

Good modeling is quantitative literature or number-stories coming in three genres as in word stories: Fact, fiction and fiddle. Fact and fiction are stories about factual and fictional things and actions. Fiddle is nonsense like ‘This sentence is false’ that is true if false, and vice versa.

Fact models, also called ‘since-then’ or ‘room’ models, quantify quantities and predict predictable quantities: “What is the area of the walls in this room?”. Since the prediction is what is observed, fact models can be trusted. Fiction models, also called ‘if-then’ or ‘rate’ models, quantify quantities but predict unpredictable quantities: “My debt is gone in 5 years at this rate!”. Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based on alternative assumptions. Fiddle models, also called ‘then-what’ or ‘risk’ models, quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. (Tarp, 2017).

Bad modeling does not distinguish between the three genres but sees all models as approximations.

***Good and Bad Geometry***

Good geometry lets trigonometry precede plane geometry that is integrated with coordinate geometry to let algebra and geometry go hand in hand to allow formulas predict geometrical intersection points.

Bad geometry lets plane geometry precede coordinate geometry that precedes trigonometry.

## Evil Mathematics

Evil mathematics talks about something existing only inside classrooms. Fractions as numbers and adding fractions without units are two examples. The tradition presents fractions as rational numbers, defined as equivalence classes in a set product created by the equivalence relation R, where $(a,b)$ R $(c,d)$ if $a*d = b*c$.

Grounded in double-counting in two units, fractions are per-numbers double-counted in the same unit, as e.g. 3$ per 5$ or 3 per 5 or 3/5. Both are operators needing a number to become a number. Both must be multiplied to unit-numbers before adding, i.e. they add by their areas as in integral calculus.

Shortening or enlarging fractions is not evil mathematics. They could be called ‘footnote mathematics’ since they deal with operator algebra seldom appearing outside classrooms. They deal with re-counting numbers by adding or removing common units: to shorten, 4/6 it is re-counted as 2 2s over 3 2s giving 2/3. To be enlarged, both take on the same unit so that 2/3 = 2 4s over 3 4s = 8/12.

Educating teachers, it is evil to silence the choices made in mathematics education. Instead, teachers should be informed about the available alternatives without hiding them in an orthodox tradition. Especially the difference between good and bad mathematics should be part of a teacher education.

## Good and Bad Education

When children become teenagers, their identity work begins: ‘Who am I; and what can I do?’ So good education sees its goal as allowing teenagers to uncover end develop their personal talent through daily lessons in self-chosen practical or theoretical half-year blocks with teachers having only one subject; and praising the students for their talent or for their courage to try out something unknown.

Bad education sees its goal as selecting the best students for offices in the private or public sector. It uses fixed classes forcing teenagers to follow their age-group despite the biological fact that girls are two years ahead in mental development.

### Good and Bad Research

Good research searches for truth about things that exist. It poses a question and chooses a methodology to transform reliable data into valid statements. Or it uses methodical skepticism to unmask choice masked as nature.

Bad research is e.g. master level work applying instead of questioning existing research. Or journalism describing something without being guided by a question.

With these three research genres, peer-review only works inside the same genre.

### Conclusion and Recommendation

This paper used difference-research to look for different ways to the outside goal of mathematics education, mastery of Many. By meeting Many outside the present self-referring set-based tradition three ways were found, a good, and a bad, and an evil. Good mathematics respects the original tasks in Algebra and Geometry, to reunite Many and to measure earth. By identifying a hidden alternative, good mathematics creates a paradigm shift (Kuhn, 1962) that opens up a vast field for new research seeing mathematics as a many-matics, i.e. as a natural science about Many (cf. Tarp, 2018).

In short, we need to examine what happens if we allow children to keep and develop the quantitative competence they bring to school, two-dimensional block-numbers to be recounted and double-counted before being added on-top or next-to; and reported with full number-language sentences including both a subject that exists, and a verb, and a predicate that may be different.

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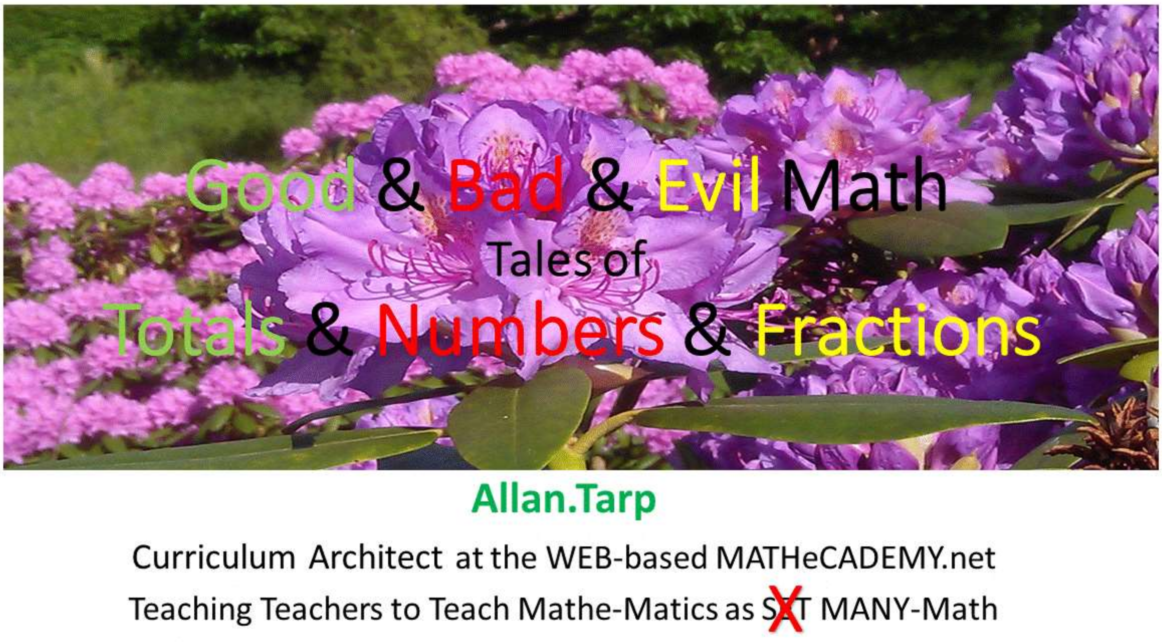
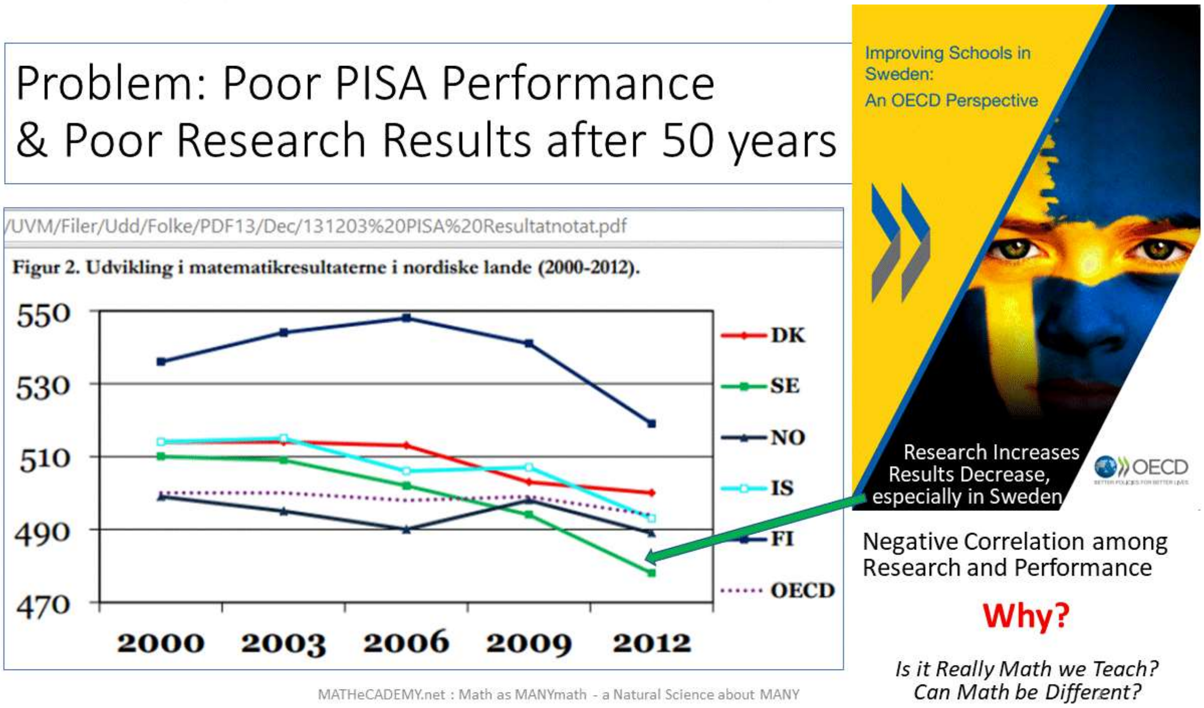
## 24. Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions, PPP



## Solution in a Nutshell: From BAD to GOOD Math

1) All teach numbers. Don't. Tell tales about how Totals unite and change
2) All use 1D line-numbers. Don't. Use 2D block-numbers
3) All begin with addition. Don't. Begin with counting and division, multiplication and subtraction before adding next-to and on-top
4) All add fractions without units. Don't. Use units as in integral calculus
5) All include only the predicate (3*5). Don't. Use full language sentences with a subject, a verb and a predicate (T = 3*5)
6) All call it MatheMatics. Don't. It is MetaMatism, derived from SET, and falsified by e.g. 2+3 is 17 and not 5 in the case of weeks and days. Real MatheMatics is rooted in MANY.

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY

## One Definition of Mathematics


Pythagoras: mathematics, meaning knowledge, is a common label for 4 areas describing Many by itself and in space & time.

Together they formed the '**quadrivium**' recommended by Plato as a general curriculum after the '**trivium**' consisting of grammar & logic & rhetoric.

| *Grounded in Many as shown by names:* | **Geometry** means to measure earth in Greek<br>**Algebra** means to reunite numbers in Arabic |
|---|---|

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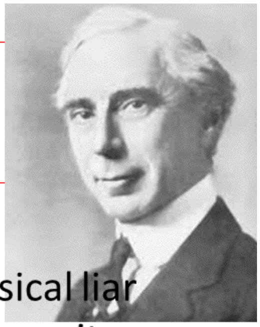

# Another Definition of Mathematics

Around 1900, **SET** made mathematics self-referring.

However, Russell said: Self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true & opposite.

Let M be the set of sets not belonging to itself, $M = \{ A \mid A \notin A \}$.

Then $M \in M \Leftrightarrow M \notin M$. Forget about sets. Use type theory instead. So, by self-reference, fractions cannot be numbers.

**Mathematics**: Forget about Russell, he is not a mathematician.

Of course fractions are numbers, they are rational numbers.



# Two Different Mathematics


The Ruling Set-based Top-Down <u>Meta-matics</u> from above

- Mathematics exists by itself as a collection of well-proven statements about well-defined concepts
- Concepts are defined from above as **examples from abstractions**
- Mathematics has many applications; and of course it must be taught and learned before it can be applied

*a FUNCTION is <u>an example of</u> a set relation where component1-identity implies component2-identity*

The Silenced Many-based Bottom-Up <u>Many-matics</u> from below

- Many exists all over the outside world, that schools prepare children and teenagers and adults for
- Concepts are defined from below as **abstractions from examples**
- Mathematics has many roots; but teaching it before applied is like teaching a grammar before its language

*a FUNCTION is <u>for example</u> 2+x, but not 2+3; i.e. a name for a calculation with an unspecified number*

## How to Define Good & Bad & Evil Math:
## Four Questions to Answer (please discuss)

| ***This is true*** | always | never | sometimes |
|---|---|---|---|
| 2 + 3 = 5 | | | |
| 2 x 3 = 6 | | | |
| $\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$ | | | |
| $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$ | | | |



## Four Questions Answered

| ***This is true*** | always | never | sometimes |
|---|---|---|---|
| 2 + 3 = 5 | | | X<br>Only with the same unit: 2weeks + 3days = 17days |
| 2 x 3 = 6 | X<br>2x3 is 2 **3s** III III that exist and may be recounted as 6 **1s** I I I I I I | | |
| $\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$ |  <br>1 red of 2 apples + 2 of 3 apples is 3 of 5 apples, and not 7 of 6 | | X<br>Depends on the units |
| $\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$ | | | X<br>Only if taken of the same total |

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# Defining Good & Bad & Evil Mathematics

**Good mathematics** is absolute truths about outside existing things

- T = 2*3 = 6 stating that a total of 2 **3s** can be re-counted as 6 **1s**: III III = I I I I I I
- So good mathematics is tales about how to count and unite and change totals

**Bad mathematics** ('mathe-matism') is relative truths about outside existing things

- 2+3 = 5, valid with like units, else falsified by e.g. 2weeks + 3days = 17days
- So bad mathematics is tales about numbers without units

**Evil mathematics** is about what exists only inside classrooms

- 1/2 + 2/3 = 7/6, but 1red of 2 + 2reds of 3 = 3reds of 5, and not 7reds of 6
- So bad mathematics is tales about fractions as numbers.
  Fractions are not numbers, but operators, needing numbers to become numbers.

Today's **BAD** MatheMatics = **MetaMatism** = MetaMatics + MatheMatism

*What is **GOOD** MatheMatics =* **ManyMatics***?*

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# Difference-Research finds Differences making a Difference, inspired by

**Philosopy**

- The ancient Greek sophists: To unmask choice masked as nature, find a difference
- In existentialism, Sartre said: EXISTENCE precedes ESSENCE
- Heidegger said: In a sentence, the SUBJECT exists, the PREDICATE is essence that can be different

**Sociology (Bauman)**

- Sociological imagination *"renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. "*
- Goal Displacements: *"The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right."*

**Psychology**

- Don't teach about subjects, bring them to class to allow 'greifen vor begreifen' (Piaget, not Vygotsky)

So let us meet the existing subject **MANY** directly & outside its 'essence-prison'
so **MANY** can create its own categories using Grounded Theory

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# Our Two Language Houses

The **WORD language** assigns words in sentences with <u>a subject, a verb, and a predicate</u>.
The **NUMBER language** assigns numbers instead with <u>a subject, a verb, and a predicate</u>.
Both languages have a meta-language, a grammar, describing the language, describing the world.

The meta-language is about the language, so we should teach and learn language before grammar.
This is the case with the word-language only, since SET-math is a grammar of the number-language.
Mixing language levels creates nonsense: 'The verb smiles' & 'The function increases'.

| | WORD language | NUMBER language |
|---|---|---|
| Meta-language, grammar | 'is' is a verb | '*' is an operation |
| Language | This is a chair | T = 3*4 |

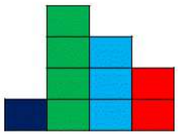
*WORLD*



# Children see Many as Bundles <u>with Units</u>

Asked 'How old next time?', a 3year-old says
4, but reacts when held together 2 by 2:
'<u>That is not 4, that is 2 **2s**</u>'.

Seeing bundles as units, children use 2D LEGO-like **block-numbers**, not 1D **line-numbers**, taught in school, even if 2D Arabic block-numbers replaced 1D Roman line-numbers centuries ago.



T = 1 4 3 2

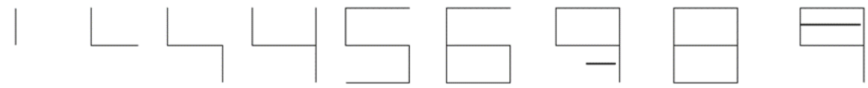
T = MCCCCXXXII

# Many as Icons: I I I I → IIII → 4

Meeting Many, we ask: "**How Many in Total**?"
To answer, we Math … oops sorry, it's a label, not an action word.
To answer, first we count, then we add. We name and iconize the degrees of Many until ten, that as 1 bundle has no icon or digit itself.

- Thus there are four sticks in a 4-icon, five in a 5-icon, etc.

| one | two | three | four | five | six | seven | eight | nine |
|---|---|---|---|---|---|---|---|---|
| I | II | III | IIII | IIIII | IIIIII | IIIIIII | IIIIIIII | IIIIIIIII |



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|

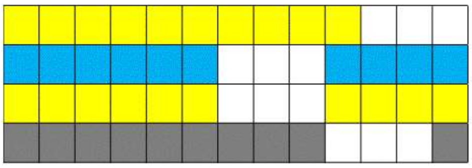


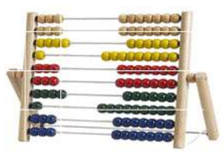
# Cup- or BundleCounting in Icons: 9 = ? 4s

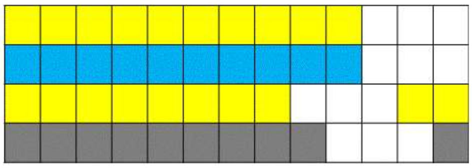
9 = I I I I I I I I I = IIII IIII I = [IIII IIII] I = 2B1 4s = 2.1 4s

To count, we bundle & use a bundle-cup with 1 stick per bundle.
We report with **bundle-writing** or **decimal-writing** where the decimal point separates inside bundles from outside single leftovers.
Shown on a western IKEA **ABACUS,** letting geometry & algebra go together.

**G**eometry/space mode or **A**lgebra/time mode



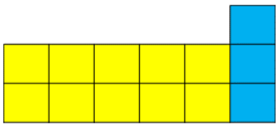




# The UnBundled become Decimals or Fractions
## 0.3 **5s** or 3/5

When counting by bundling and stacking,
the unbundled single leftovers can be placed

**NextTo** the stack
counted as a stack of **1s**



T = 2**B**3 **5s** = 2.3 **5s**
*A decimal number*

**OnTop** of the stack
counted as a bundle

T = 2 3/5 **5s**
*A fraction*

15

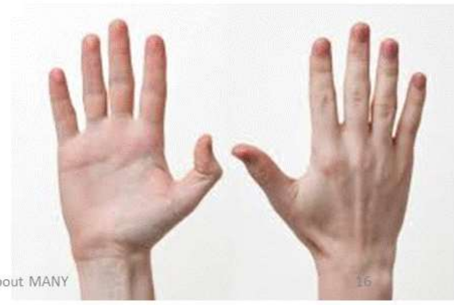
# Counting Sequences

We may include bundling if saying '0**Bundle**3' or '03' instead of plain '3'

- '0**Bundle**1, 0**B**2, 0**B**3, …, 0**B**8, 0**B**9, 1**B**0, 1**B**1, 1**B**2, … **tens**, or
- '01, 02, …, 1**Bundle** less 2, 1**B**-1, 1**B**0, 1**B**1(1left), 1**B**2, … **tens**

Counting fingers gives 1**B**0 **tens**, or

- 2**B**0 **5s** IIII IIII
- 2**B**2 **4s** IIII IIII I I
- 3**B**1 **3s** III III III I or 1**BB**1 **3s** IIIIIIIII I



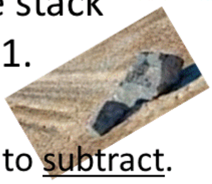
## Operations as Icons

- To count 7 in **3s** we take away 3 many times, iconized by an uphill stroke, 7/3, showing the broom wiping away the **3s**.

| 7/3 | 2.some |
| --- | --- |
| 7 – 2x3 | 1 |

- A calculator predicts: 3 can be taken away 2 times. Stacking the bundles is iconized as a lift, 2x3.

- To look for unbundled singles, we drag away the stack of 2 **3s,** iconized by a horizontal trace: 7 – 2x3 = 1.

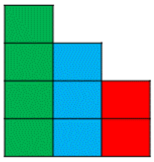


Counting creates 3 operations: to divide & to multiply & to subtract.



## More Operations as Icons

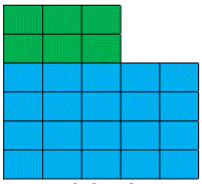
- To bundle bundles also, **power** is iconized as a cap, 5^2, showing the number of times bundles are bundled.



- Counting a Total gives a **BundleFormula,** a polynomial:

T = 432 = 4***BundleBundle** + 3***Bundle** + 2*1 = 4***B**^2 + 3***B**^1 + 2***B**^0

- **Addition** is a cross + showing blocks placed on-top of or next-to each other.



4 **5s** & 2 **3s** added OnTop

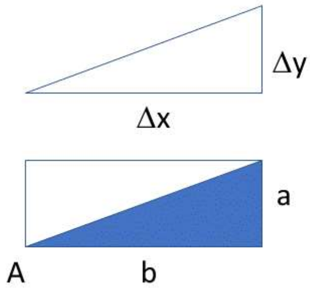
4 **5s** & 2 **3s** added NextTo

# The ReCount Formula

| | |
|---|---|
| 7/3 | 2.some |
| 7 – 2 * 3 | 1 |

Predicting T = 7 = 2.1 **3s**, the **ReCount formula T = (T/B)*B**
saying 'from T, T/B times, B can be taken away', is all over:

| | |
|---|---|
| Proportionality | y = k * x |
| Linearity | Δy = (**Δy/Δx**) * Δx = **m** * Δx |
| Local linearity | dy = (**dy/dx**) * dx = **y'** * dx |
| Trigonometry | a = (**a/b**) * b = **tanA** * b |
| Trade | \$ = (**\$/kg**) * kg = **price** * kg |
| Science | meter = (**meter/second**) * second = **velocity** * second |

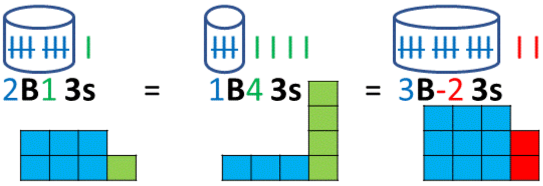





# ReCounting in the Same Unit gives Flexible Totals

A total can be counted in 3 ways: Normal Overload Underload

T = 7 = I I I I I I I = 2B1 3s = 1B4 3s = 3B-2 3s



Or, when counting in tens
T = 37 = 3**B**7 **tens** = 2**B**17 **tens** = 4**B**-3 **tens**

**BundleWriting** and flexible totals may cure Math Dislike in classes stuck in Division:

☹☹☹ T = 336 /7 = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48 ☺☺☺

Likewise with

| | |
|---|---|
| Multiplication | T = 7* 48 = 7* 4**B**8 = 28**B**56 = 33**B**6 = 336 |
| Subtraction | T = 53 – 29 = 5**B**3 – 2**B**9 = 3**B**-6 = 2**B**4 = 24 |
| Addition | T = 53 + 29 = 5**B**3 + 2**B**9 = 7**B**12 = 8**B**2 = 82 |

## ReCounting in a Different Unit creates Proportionality & Multiplication & Equations

| | |
|---|---|
| 4*5/6 | 3.some |
| 4*5 – 3*6 | 2 |

ReCounting in different units changes units (**Proportionality**)

- T = 4 **5s** = ? **6s**. A calculator predicts with ReCount-formula: T = 3.2 **6s**

ReCounting from icons to tens gives **Multiplication**

- T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens**, predicted by multiplication

*So multiplication is a special form of division*

ReCounting from tens to icons creates **Equations** solved by recounting

- T = ? **7s** = 42 = (42/7)*7 with the solution ? = 42/7 = 6.

*An equation is solved by moving to Opposite Side with Opposite Sign*

**u*7 = 42 = (42/7)*7**
**u = 42/7 = 6**



## Solving Equations by ReCounting, we may bracket Group Theory from Abstract Algebra

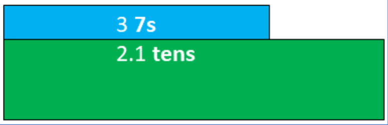
ManyMath

| | |
|---|---|
| 2 x *u* = 8 = (8/2) x 2 | Solved by re-counting 8 in 2s |
| *u* = 8/2 = 4 | Move: Opposite Side & Sign |

SetMath (Don't test, but do remember bi-implication arrows)

| | | |
|---|---|---|
| | 2 x *u* = 8 | Multiplication has 1 as its **neutral element**, and 2 has ½as its **inverse element** |
| ↕ | (2 x *u*) x (½) = 8 x (½) | Multiplying 2's inverse element ½ to both number-names |
| ↕ | (*u* x 2) x (½) = 4 | Applying the **commutative** law to u x 2; 4 is the short number-name for 8 x ½ |
| ↕ | *u* x (2 x (½)) = 4 | Applying the **associative** law |
| ↕ | *u* x 1 = 4 | Applying the definition of an inverse element |
| ↕ | *u* = 4 | Applying the definition of a neutral element. *With arrows a test is not needed.* |

# ReCounting Simplifies Multiplication Tables




Geometry: Multiplication means that, recounted in tens, a block increases its width and therefore must decrease its height to keep the total unchanged.

Thus T = 3*7 means 3 **7s** that may be recounted in tens as T = 2.1 **tens** = 21.

Algebra: The full ten-by-ten table can be reduced using that 6 is Bundle less 4, 7 is Bundle less 3, etc. This roots Early Algebra.

T = 2 **6s** = 2*6 = 2*(**B**-4) = 2**B**-8 = 2**B**-(1**B**-2) = 1**B**--2 = 1**B**+2 = 1**B**2 = 12

T = 4 **7s** = 4*7 = 4*(**B**-3) = 4**B** – 1**B**2 = 3**B**-2 = 2**B**8 = 28

T = 8 **7s** = 8*7 = (**B**-2)*(**B**-3) = **BB** – 2**B** – 3**B** + 6 = 10**B** – 2**B** – 3**B** + 6 = 5**B**6 = 56



# DoubleCounting in 2 Units creates PerNumbers

Apples are double-counted in kg and in $.

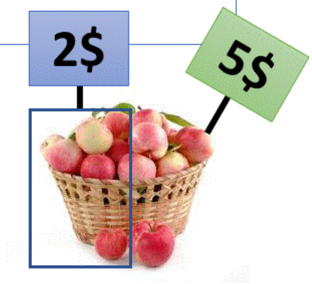
With **4kg = 5$** we have the **per-number** 4kg/5$ = 4/5 kg/$

*Questions:*

| **12kg = ?$** | **20$ = ?kg** |
|---|---|
| 12kg = (12/4)*4kg<br>= (12/4)*5$<br>= 15$ | 20$ = (20/5)*5$<br>= (20/5)*4kg<br>= 16kg |


*Answer:* Recount in the per-number

## DoubleCounting in the Same Unit creates Fractions




The same unit: 2$ per 5$ = 2$/5$ = 2/5

• *Question: 2/5 = ? per 100; or 2$/5$ is ? per 100$*

Answer: recount 100 in 5s!

100$ = (100/5)*5$ gives (100/5)*2$ = 40$, so 2/5 = 40/100 = 40%

• *Question: 2/5 of 40 = ?; or with units: 2$ per 5$ of 40 $.*

Answer: recount 40 in 5s!

40$ = (40/5)*5$ gives (40/5)*2$ = 16$, so 2/5 of 40 = 16

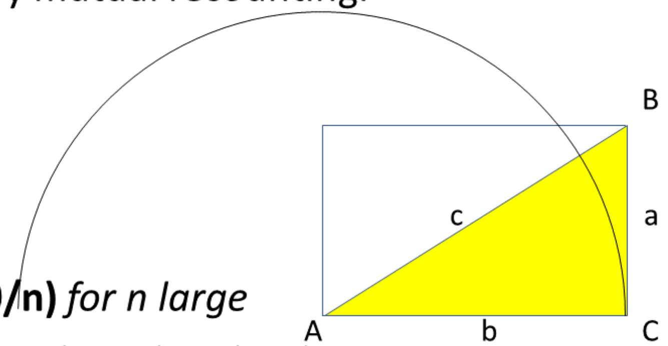


## Trigonometry ReCounts Sides in a HalfBlock

Halved by its diagonal, a block becomes a right angled triangle with three sides: the base b & the height a & the diagonal c, creating trigonometry by mutual recounting.

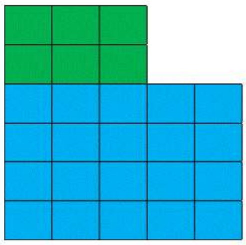
a = (**a/c**) * c = **sinA** * c
b = (**b/c**) * c = **cosA** * c
a = (**a/b**) * b = **tanA** * b

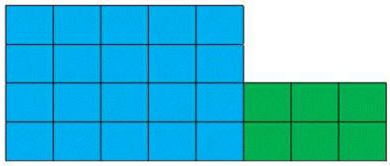
½Circle = π = **n*tan(180/n)** *for n large*



## Once Counted & ReCounted, Totals can be Added

| **OnTop** | **NextTo** |
|---|---|
| 4 **5s** **+** 2 **3s** = 4 **5s** **+** 1**B**1 **5s** = 5**B**1 **5s** | 4 **5s** + 2 **3s** = 3**B**2 **8s** |
| The units are changed to be the same<br>*Change unit = Proportionality* | The areas are added<br>*Adding areas = Integration* |







## The 3 Numbers in a Total add Differently

From totals as T1 = 2.3 **4s** and T2 = 3.4 **5s** we see that a Total has 3 numbers that add differently:

*The bundle-size, the bundle-number, the single-number.*

- Bundle-sizes stay unchanged unless the blocks are added next-to each other as in integration
- Bundle-numbers only add with like bundle-sizes.
- Singles always add.

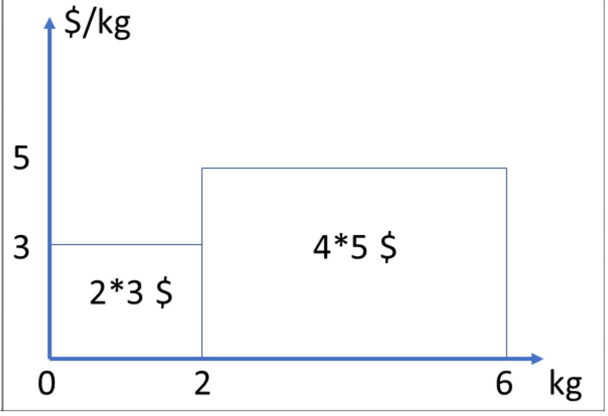
Never add without units: Mars Climate Orbiter, planes?

# Adding PerNumbers as Areas (Integration)

|  |  |  |  |
|---|---|---|---|
|  | 2 kg | at | 3 $/kg |
| + | 4 kg | at | 5 $/kg |
| (2+4) | kg | at | ? $/kg |

Unit-numbers add on-top.

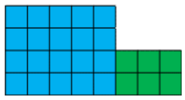
Per-numbers add next-to as **areas** under the per-number graph.



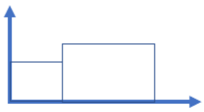



# Primary & Middle & High School Calculus

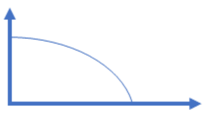
<u>Primary calculus</u>: Next-to addition of block-numbers



<u>Middle calculus</u>: Add piecewise constant per-numbers

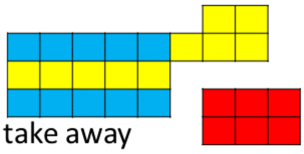


<u>High school calculus</u>: Add locally constant (continuous) per-numbers



# Reversed Addition = Solving Equations

| OppoSite Side with OppoSite Sign | | NextTo |
|---|---|---|
| **2 x ? = 8** = (8/2) x 2 | **2 + ? = 8** = (8-2) + 2 | **2 3s + ? 5s = 3.2 8s** |
| ? = 8/2<br>*Solved by ReCounting* | ? = 8-2<br>*Solved by ReStacking* | ? = (3.2 8s – 2 3s)/5<br>*Solved by differentiation: (T-T1)/5 = ΔT/5* |



**Hymn to Equations**

Equations are the best we know,
they are solved by isolation.
But first, the bracket must be placed
around multiplication.

We change the sign and take away
and only x itself will stay.
We just keep on moving, we never give up.
So feed us equations, we don't want to stop!



The Algebra project: How to re-unite

# Concrete Algebra: 4 ways we Unite, + * ^ ∫ as shown by the Bundle Formula

**T = 456 = 4*B^2 + 5*B^1 + 6*B^0**

Totals exist as changing or constant **unit-numbers** or **per-numbers**

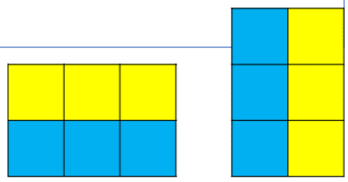
- Addition & Multiplication unite changing & constant unit-numbers
  - Subtraction & division split into changing & constant unit-numbers
- Integration & Power unite changing & constant per-numbers
  - Differentiation & root/logarithm (factor finder/counter) split into changing & constant per-numbers

| Operations **unite** / *split into* | Changing | Constant |
|---|---|---|
| Unit-numbers<br>*m, s, \$, kg* | **T = a + n**<br>*T – a = n* | **T = a*n**<br>*T/n = a* |
| Per-numbers<br>*m/s, \$/kg, m/(100m) = %* | **T = ∫ a dn**<br>*dT/dn = a* | **T = a^n**<br>$log_a T = n$, $\sqrt[n]{T} = a$ |

The Simplicity of Mathematics

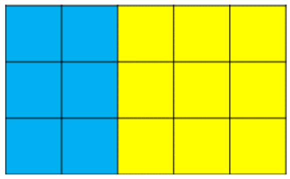
The Algebra project: How to re-unite

# Abstract Algebra: (re)Uniting Units



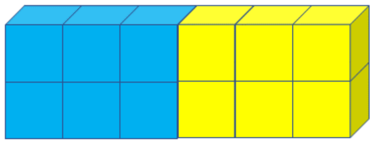
- Turning a block will change the unit
  T = 2 3s = 2*3 -> T = 3 2s = 3*2, so T = 2*3 = 3*2

*(The Commutative law)*



- A block may be split in two parts
  T = 3 5s = 3 2s + 3 3s or
  T = 3*5 = 3*(2+3) = 3*2 + 3*3

*(The Distributive Law)*



- A united unit as 6 that can be folded and fully stacked
  - a prime unit as 3 cannot.
  T = 2 6s = 2*(2*3) = (2*2)*3

*(The Associative law)*



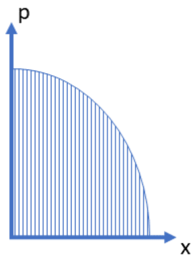
**T = 456 = 4*B^2 + 5*B + 6*1**

# Bundle Formula: 5 ways of Constant Change

The number-formula contains formulas for constant change:

| | | |
|---|---|---|
| • T = b*x | (proportional) | *trade* |
| • T = b*x + c | (linear) | *trends* |
| • T = a * x^n | (elastic) | *science* |
| • T = a * n^x | (exponential) | *economy* |
| • T = a*x^2 + b*x + c | (accelerated) | *physics* |

# Two forms of NonConstant Change




Adding locally constant per-numbers means finding the area under the per-number graph as a sum of a large number of thin area-strips. But, if written as changes, this reduces to finding one total change since the middle terms cancel out. Writing p*dx = dF, or p = dF/dx motivates differential calculus, also useful to describe non-constant **predictable change**.

**Unpredictable change** roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot predict.



# Three forms of Constancy

A class is stuck in the **epsilon-delta** definition of continuity and differentiability. Here a difference is to rename them to 'local constancy' and 'local linearity'. As to constancy:

- y is globally constant c if for all positive numbers **epsilon**, the difference between y and c is less than epsilon.
- y is piecewise constant c if an interval-width **delta** exists such that for all positive numbers **epsilon**, the difference between y and c is less than epsilon, in this interval.
- y is locally constant c if for all positive numbers **epsilon**, an interval-width **delta** exists such that the difference between y and c is less than epsilon, in this interval.

Likewise, the change per-number **Δy/Δx** can be globally, piecewise or locally constant.
If locally constant , it is written as **dy/dx,** and y is called 'locally linear'.

## Quantitative Literature or Modeling comes in 3 Genres also: Fact & Fiction & Fiddle

- Fact models or 'since-then' calculations use numbers and formulas to quantify and to predict predictable quantities as e.g. 'since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20'. Fact models can be trusted once the numbers and the formulas and the calculation has been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars Climate Orbiter.
- Fiction models or 'if-then' calculations use numbers and formulas to quantify and to predict unpredictable quantities as e.g. 'if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20'. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made.
- Fiddle models or 'what-then' models use numbers and formulas to quantify and to predict unpredictable qualities as e.g. 'since a graveyard is cheaper than a hospital, then a bridge across the highway is too costly.' Fiddle models should be rejected and relegated to a qualitative description.

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## How Different is the Difference? I
## SET Math versus MANY Math

| | **SET Math** | **Many Math** |
|---|---|---|
| Goal/ Means | Learn Mathematics / Teach Mathematics | Learn to master Many / Tales of Many as counted, united, changed |
| Digits | Symbols like letters | Icons with as many sticks as they represent |
| Numbers | Line-numbers with place-value system Never with units | Block-numbers, stacking singles, bundles, bundle-bundles etc. Always with units |
| Number-types | Four types: Natural, Integers, Rational, Real | Positive & negative decimal numbers with units |
| Operations | Mapping from a set-product to the set. Order: Add, subtract, multiply, divide | Counting-icons: bundle /, stack x, remove -, unite on-top & next-to +). Opposite order |
| Division | 8/2 means 8 split in 2 | 8/2 means 8 split in (counted in) 2s |
| ReCount PerNumber | Do not exist | Core concepts |

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## How Different is the Difference? II
## SET Math versus MANY Math

| | SET Math | Many Math |
|---|---|---|
| Fractions | Rational numbers without units, and adding without units | Per-numbers, not numbers but operators needing a number to become a number, so added by integration |
| Equation | Statement about equivalent number-names | A recounting from tens to icons. Reversed operations |
| Function | A set relation where component1-identity implies comp.2-identity | A number-language sentence about the Total with a subject & a verb & a predicate |
| Propotio-nality | A linear function | A name for double-counting in two units |
| Calculus | Differentiation before integration (anti-differentiation) | Integration adds locally constant per-numbers. Integration before differentiation |
| Geometry | Plane before Coordinate before Trig. | Trigonometry before Coordinate Geometry |



## Difference-Research, Main Warning:
## The 3x3 Goal Displacements in Math Education

| | | |
|---|---|---|
| Primary | Numbers | Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1<br>Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure |
| | Operations | Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times<br>Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +). |
| | Addition | Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication<br>Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based |
| Middle | Fractions | Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration)<br>Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms) |
| | Equations | Could: be introduced in primary as recounting from ten-bundles to icon-bundles; and as reversed on-top and next-to addition<br>Instead: Defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra |
| | Proportionality | Could: be introduced in primary as recounting in another unit when adding on-top; be double-counting producing per-numbers<br>Instead: defined as linear functions, or as multiplicative thinking supporting the claim that fractions and ratios are rational numbers |
| High | Trigonometry | Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal<br>Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra. |
| | Functions | Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate<br>Instead: are introduced as set-relations where first-component identity implies second-component identity |
| | Calculus | Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers<br>Instead: differential calculus precedes integral calculus, presented as anti-differentiation |

# ReCounting looks like Dienes MultiBase Blocks

- "Dienes' name is synonymous with the Multi-base blocks (also known as Dienes blocks) which he invented for the teaching of place value.
- Dienes' place is unique in the field of mathematics education because of his theories on how mathematical structures can be taught from the early grades onwards using multiple embodiments through manipulatives, games, stories and dance."

(http://www.zoltandienes.com/about/)



# Dienes on Numbers and MultiBase Blocks

"The position of the written digits in a written number tells us whether they are counting singles or tens or hundreds or higher powers. This is why our system of numbering, introduced in the middle ages by Arabs, is called the place value system. My contention has been, that in order to fully understand how the system works, we have to understand the concept of power. (..)

In school, when young children learn how to write numbers, they use the base ten exclusively and they only use the exponents zero and one (namely denoting units and tens) , since for some time they do not go beyond two digit numbers. So neither the base nor the exponent are varied, and it is a small wonder that children have trouble in understanding the place value convention. (..)

Educators today use the "multibase blocks", but most of them only use the base ten, yet they call the set "multibase". These educators miss the point of the material entirely."

(What is a base?, http://www.zoltandienes.com/academic-articles/)

## Power & Base from Above, or Bundles from Below

Dienes teaches the 1D place value line-numbers with 2D & 3D blocks to show the importance of the power concept.

- ManyMatics teaches 2D block-numbers with units to show the importance of bundling singles, bundles & bundle-bundles.

Dienes sees numbers as examples of the abstract label **base**

- ManyMatics sees counting as an action with a concrete verb **bundle**

base the base
bundle the bundle

Dienes teaches top-down ‘MetaMatics‘ derived from the concept Set

- ManyMatics teaches a bottom-up natural science about the fact Many; and sees Set as meaningless because of Russell’s set-paradox.



Different Education

## EU: Line-organized & Office-directed Schools

From secondary school, continental Europe uses **line-organized** education with forced classes and forced schedules, making teenagers stay together in age groups - even if boys are two years behind in mental development.

The classroom belongs to the class. This forces teachers to change room and (in lower secondary school) to teach several subjects outside their training.

Tertiary education is also **line-organized** preparing for offices in the public or private sector. This makes it difficult to change line in the case of unemployment.

This makes reproduction fall to 1.5 child/family, causing the European population to be halved each two generations since per female, (1.5/2)*(1.5/2) = .75*.75 ≈ 0.5.

Different Education

# US: Block-organized & Talent-directed Schools

Alternatively, North America uses **block-organized** education saying to teenagers:

"Welcome, inside you carry a **talent**! Together we will uncover and develop your personal talent through self-chosen daily half-year blocks, academical or practical, together with 1subject teachers. If successful the school will say '**good job**, you have a **talent**, you need some more'. If not, the school will say '**good try**, you have **courage** to try out the unknown, now try something new'".

The classroom belongs to the teacher teaching one subject only.

Likewise, college is **block-organized** easy to supplement with additional blocks in the case of unemployment.

At the age of 25, most students have an education, a job and a family with three children, 1 for mother, 1 for father, and 1 for the state to secure reproduction.



# Good & Bad Research

- Good research searches for truth about things that exist. It poses a question, and chooses a methodology to transform reliable data into valid statements. Or it uses methodic skepticism to unmask choice masked as nature.
- Bad research is e.g. master level work applying instead of questioning existing research. Or journalism describing something without being guided by a question.
- With these three research genres, peer-review only works inside the same genre.
- All conferences should have a **'salon des refusé'** to foster and boost new paradigms (Kuhn), as it does in art.

## More Conflicting Theory in Math Ed Research

**Philosophy**

- Sophists: Unmask choice masked as nature by finding hidden differences
- Philosophy: All is nature and examples of meta-physical forms only visible to us

**Sociology**

- Structure: Institutions are good if rational and democratic
- Agent: Goal displacements in institutions lead to 'the banality of evil' (Arendt)

**Psychology**

- Piaget: Teach little, but allow the learner to meet the **existing** subject directly
- Vygotsky: We need good teaching to mediate institutionalized **essence**



## More Enlightenment Sociology in Math Ed Research

Sociology can question institutions by asking: Offering education as a cure for the diagnose 'uneducated' is a self-referring irrationality. A power agenda behind?

Thus, inspired by Heidegger's: 'In sentences, trust the subject & doubt the predicate', and wanting to protect its Enlightenment republic, French post-structuralism says:

- Derrida: Words can be fake, and install instead of label (DeConstruction)
- Lyotard: Truth can be fake (PostModern skepticism towards meta-narratives)
- Foucault: Diagnoses and discourses can be fake, still allowing curing institutions to expand (a school is really a 'pris-pital' mixing power techniques from a prison and a hospital, and with learners as 'patien-mates')
- Bourdieu: Education is fake by using symbolic violence (and mathematics especially) to create a new knowledge-nobility

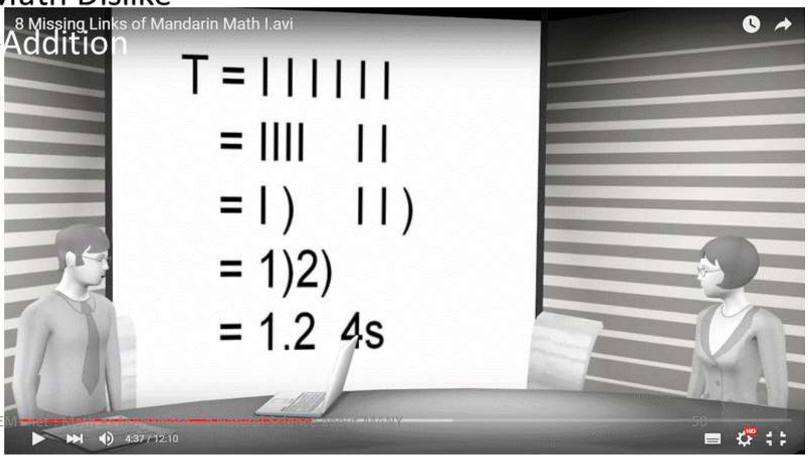
## ManyMath is Different
## But does it make a Difference? Try it out

- Watch some YouTube or YouKu videos (MrAlTarp/DrAlTarp)
- Try the **CupCount before you Add** Booklet
- Try a 1day free Skype seminar **How to Cure Math Dislike**
- Try Action Learning and Action Research, e.g. **1Cup & 5Sticks**
- Collect data and Report on 8 **MicroCurricula**, M1-M8
- Try a 1year online InService TeacherTraining at the MATHeCADEMY.net using PYRAMIDeDUCATION to teach teachers to teach MatheMatics as **ManyMatics**, a Natural Science about the root of mathematics, **Many**



## Some MrAlTarp YouTube Videos
*Screens & Scripts on MATHeCADEMY.net*

- Postmodern Mathematics Debate
- CupCounting removes Math Dislike
- IconCounting & NextTo-Addition
- PreSchool Mathematics
- Fractions
- PreCalculus
- Calculus
- Mandarin Mathematics
- World History



## Action Learning & Action Research

Imagine a difference

Design → Report

Teach ↓ ↑ Share

Observe

Lyotard dissenting Paralogy
Quality indicator:
Ungrounded rejection

**Example**
Calculators in PreSchool
and Special Needs education
*Paper rejected at MADIF10*



## Numbers as Icons & ReCounting 7 in 5s & 3s & 2s

# MatheMatics: Unmask Yourself, Please

- In Greek you mean ‘knowledge’. You were chosen as a common label for 4 activities: Music, Astronomy, Geometry & Arithmetic. Later only 2 activities remained: Geometry and Algebra
- Then Set transformed you from a natural science about the physical fact Many to a metaphysical subject, MetaMatism, combining MetaMatics and MatheMatism
- So please, unmask your true identity, and tell us how you would like to be presented in education:
- MetaMatism for the few - or ManyMatics for the many.

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 53

# From Bad & Evil Math to Good Math:

1) Respect the Child’s own 2D Block
2) Count, ReCount & DoubleCount before Adding OnTop & NextTo
3) Let Existence precede Essence: **Think Things**

*Slides on MATHeCADEMY.net*
*Details in **Journal of Mathematics Education***

**Thank You for Listening**

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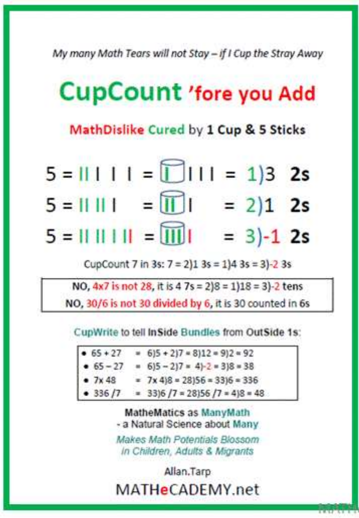
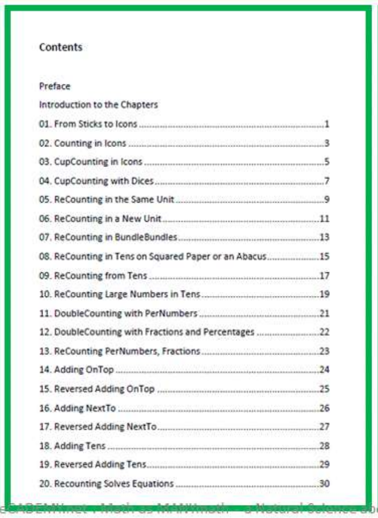
# CupCount 'fore you Add Booklet, free to Download



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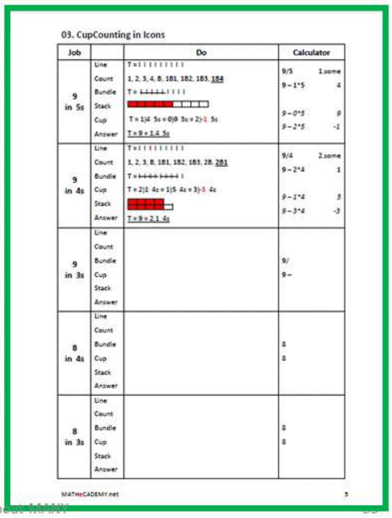



# 1day free Skype Seminar:
# To Cure Math Dislike, CupCount before you Add

*Action Learning based on the Child's own 2D NumberLanguage*

**09-11. Listen and Discuss the PowerPointPresentation**
<u>To Cure MathDislike, replace MetaMatism with ManyMath</u>

- **MetaMatism** = MetaMatics + MatheMatism
- **MetaMatics** presents a concept TopDown as an example instead of BottomUp as an abstraction
- **MatheMatism** is true inside but rarely outside classrooms
- **ManyMath**, a natural science about Many mastering Many by BundleCounting & Adding NextTo and OnTop.

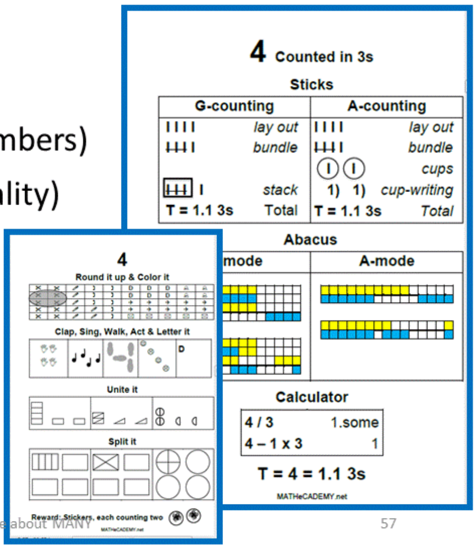
**11-13. Skype Conference. Lunch.**

**13-15. Do: Try out the <u>CupCount before you Add</u>** booklet to experience proportionality & calculus & solving equations as golden LearningOpportunities in BundleCounting & NextTo Addition.

**15-16. Coffee. Skype Conference.**

# 8 MicroCurricula for Action Learning & Research

C1. Create Icons
C2. Count in Icons
C3. ReCount in the Same Icon (Negative Numbers)
C4. ReCount in a Different Icon (Proportionality)
A1. Add OnTop (Proportionality)
A2. Add NextTo (Integrate)
A3. Reverse Adding OnTop (Solve Equations)
A4. Reverse Adding NextTo (Differentiate)

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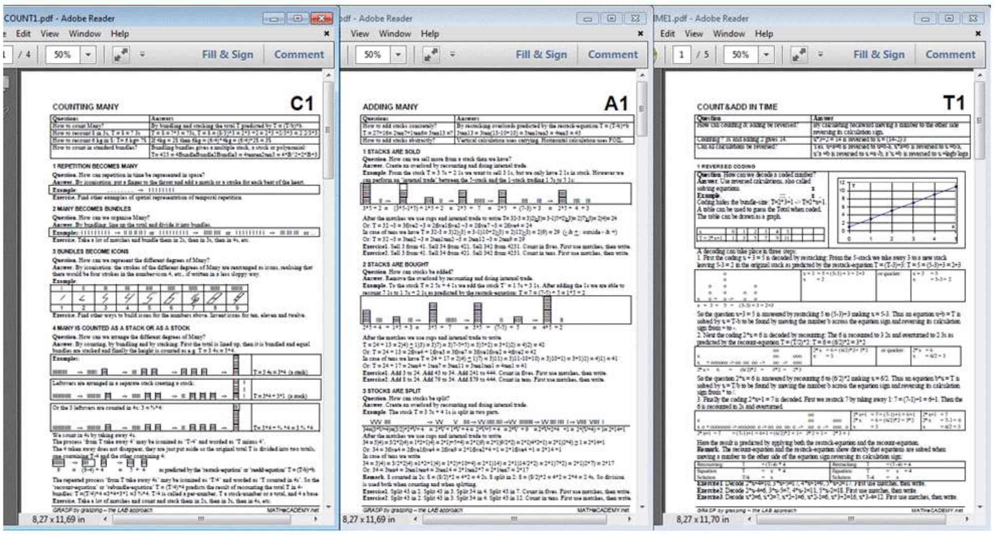
# Teacher Training in CATS ManyMath
# Count & Add in Time & Space

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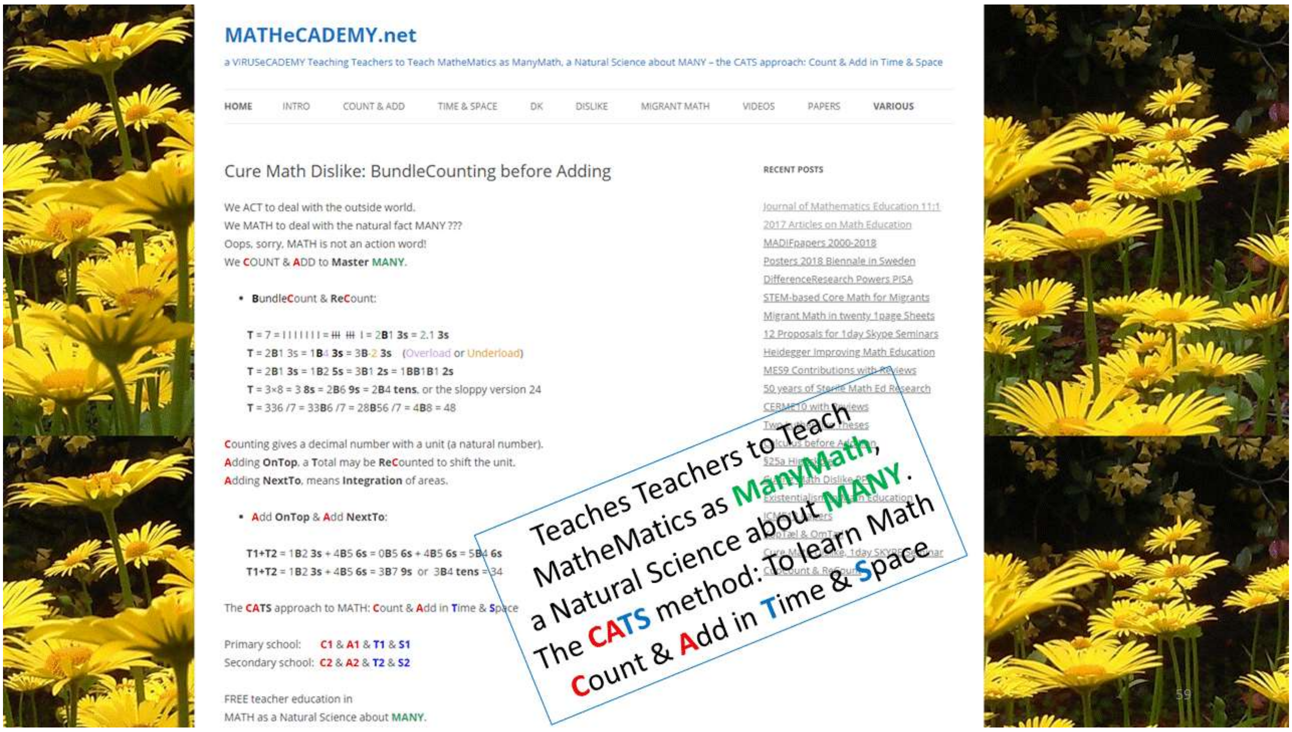


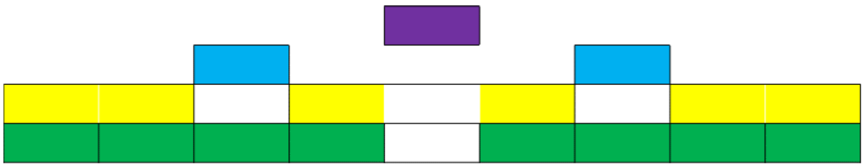

PYRAMIDeDUCATION

To learn MATH: Count&Add MANY
*Always ask Many, not the Instructor*
MATHeCADEMY.net - a VIRUSeCADEMY

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

- Each pair works together to solve Count&Add problems.
- The coach assists the instructors when instructing their team and when correcting the Count&Add assignments.
- Each teacher pays by coaching a new group of 8 teachers.

1 Coach
2 Instructors
3 Pairs
2 Teams



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28b. Different Mathematics

## Main Parts of a ManyMath Curriculum

**Primary School – respecting and developing the Child's own 2D NumberLanguage**

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- BundleCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: divide, multiply, subtract, add on-top & next-to

**Middle school – integrating algebra and geometry, the content of the label math**

- DoubleCounting produces PerNumbers as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always, so length becomes change and vv.

**High School – integrating algebra and geometry to master CHANGE**

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

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MATHeCDMY

## Quadratic Equations with 3 Cards

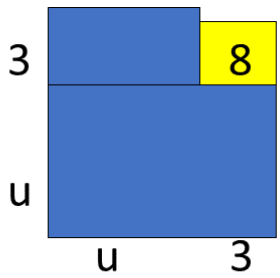
Solve the quadratic equation **u^2 + 6u + 8 = 0**

(u+3)^2 = u^2 + 6u + 8 + 1

(u+3)^2 = 0 + 1

u+3 = ± 1

u = -3 ± 1




Solution: **u = -4, u = -2**

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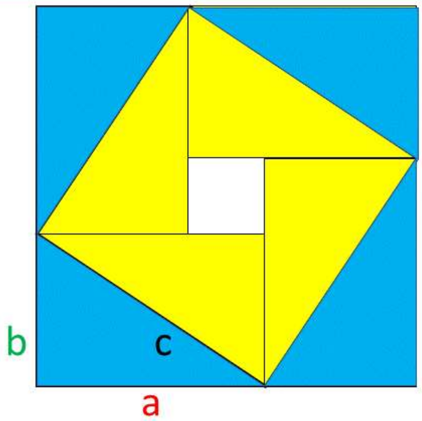
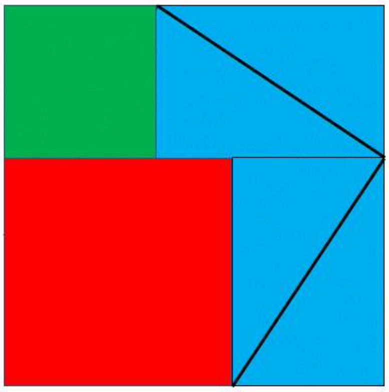
# Pythagoras shown by 4 Cards with Diagonals




**c^2** + 4 ½cards



**a^2** **+** **b^2** + 2 cards

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## 25. The Simplicity of Math reveals a Core Curriculum

### Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015a) to write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change’.

To find an unorthodox solution we pretend that a university in southern Sweden arranges a curriculum architect competition: ‘Theorize the low success of 50 years of mathematics education research; and derive a STEM-based core curriculum from this theory.’

Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

### Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking ‘renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now’ (p. 16).

Mathematics education is an example of ‘rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)’. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn’t the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here the invention of the concept SET created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2x3=6' stating that 2 3s can always be re-counted as 6 1s.

### Difference Research Looking at Mathematics Education

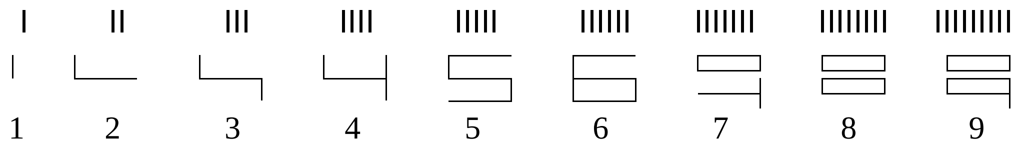
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014). So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

### Meeting Many Creates a 'Count-before-Adding' Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:

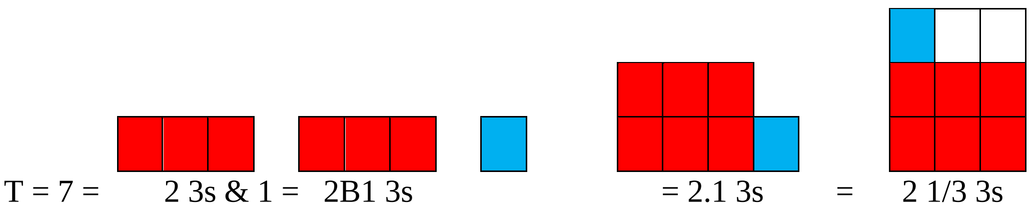



We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)*B saying that 'from T, T/B times, B can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s




A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations and differential calculus:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

**Alternative Versions of Standard Mathematics**

01. To stress the importance of bundling, the counting sequence should be: 01, 02, …, 09, 10,11 etc.

02. The ten fingers should be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers should be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables should be formulated as re-counting from icon-bundles to tens and use underload counting after 5: T = 4*7 = 4 7s = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 should be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

06. Solving proportional equations as 3*x = 12 should be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks should be done by re-counting in the per-number: With 3$/4kg, 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages should be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$ = (60/3)*3$ giving (60/3)*2$ = 40$

09. Integral should precede differential calculus and include adding both piecewise and locally constant per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at (2*3+4*5)$/(2+4)kg thus showing that per-numbers and fractions are added with their units as the area under the per-number graph.

10. Trigonometry should precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sinA*c.

**Level & Change Formulas**

Re-counting and double-counting leads to the recount-formula T = (T/B)*B occurring all over mathematics: when re-counting or double-counting to change unit in proportional quantities; when re-counting to solve equations; in trigonometry to mutually re-count the sides in a right triangle; and in calculus to mutually re-count the changes as dy = (dy/dx)*dx = y'*dx. In economics, the recount-formula becomes a price-formula: $ = ($/kg)*kg, $ = ($/day)*day, etc.

Counting by stacking bundles into adjacent blocks leads to the number-formula called a polynomial:

T = 456 = 4*BundleBundle + 5*Bundle + 6*single = 4*B^2 + 5*B + 6*1.

In its general form, the number-formula T = a*x^2 + b*x + c contains the different formulas for constant change: T = a*x (proportionality), T = a*x+b (linearity), T = a*x^2 (acceleration), T = a*x^c (elasticity) and T = a*c^x (interest rate).

The number-formula also shows the four ways to unite numbers offered by algebra meaning 'reuniting' in Arabic: addition and multiplication add variable and constant unit-numbers; and integration and power unite variable and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into variable and constant unit-numbers; and differentiation and root & logarithm split a total in variable and constant per-numbers:

| Uniting/*splitting into* | Variable | Constant |
|---|---|---|
| Unit-numbers<br>m, s, kg, $ | T = a + n<br>*T – a = n* | T = a * n<br>*T/n = a* |
| Per-numbers<br>m/s, $/kg, $/100$ = % | T = ∫ a dn<br>*dT/dn = a* | T = a^n,<br>*loga(T) = n    n√T = a* |

**Meeting Many in a STEM Context**

Having met Many by itself, we now meet Many in time and space in the present culture based upon STEM, described by OECD (2015b) as follows: 'In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.'

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we must build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), $/kg (price), $/hour (wages), etc.

***An Electrical Circuit***

To work properly, a 2000Watt water kettle needs 2000Joules per second. The socket delivers 220Volts, a per-number double-counting the number of Joules per charge-unit.

Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm.

Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So, the 120Watt bulb only receives half of the energy of the 60Watt bulb.

***Warming and Boiling Water***

In a water kettle, a double-counting can take place between the time and the energy used to warm the water to boiling, and to transform the water to steam.

Heating 1000gram water 80degrees in 167seconds in a 2000Watt kettle, the per-number will be 2000*167/80Joule/degree, creating a double per-number 2000*167/80/1000Joule/degree/gram or 4.18Joule/degree/gram, called the specific heat of water.

Producing 100gram steam in 113seconds, the per-number is 2000*113/100Joule/gram or 2260Joule/g, called the heat of evaporation for water.

## Conclusion and Recommendation

This paper argues that the low success of 50 years of mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views offer different kinds of mathematics: a set-based top-down ‘meta-matics’ that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based ‘Many-matics’ simply saying ‘To master Many, counting produces constant or variable unit-or per-numbers, uniting by adding or multiplying or powering or integrating.’

Thus, this simplicity of mathematics as expressed in a Count-before-Adding curriculum allows bundle-numbers to replace line-numbers, and to learn core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young male migrants learn core STEM subjects at the same time, thus allowing them to become STEM-teachers or STEM-engineers to return help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

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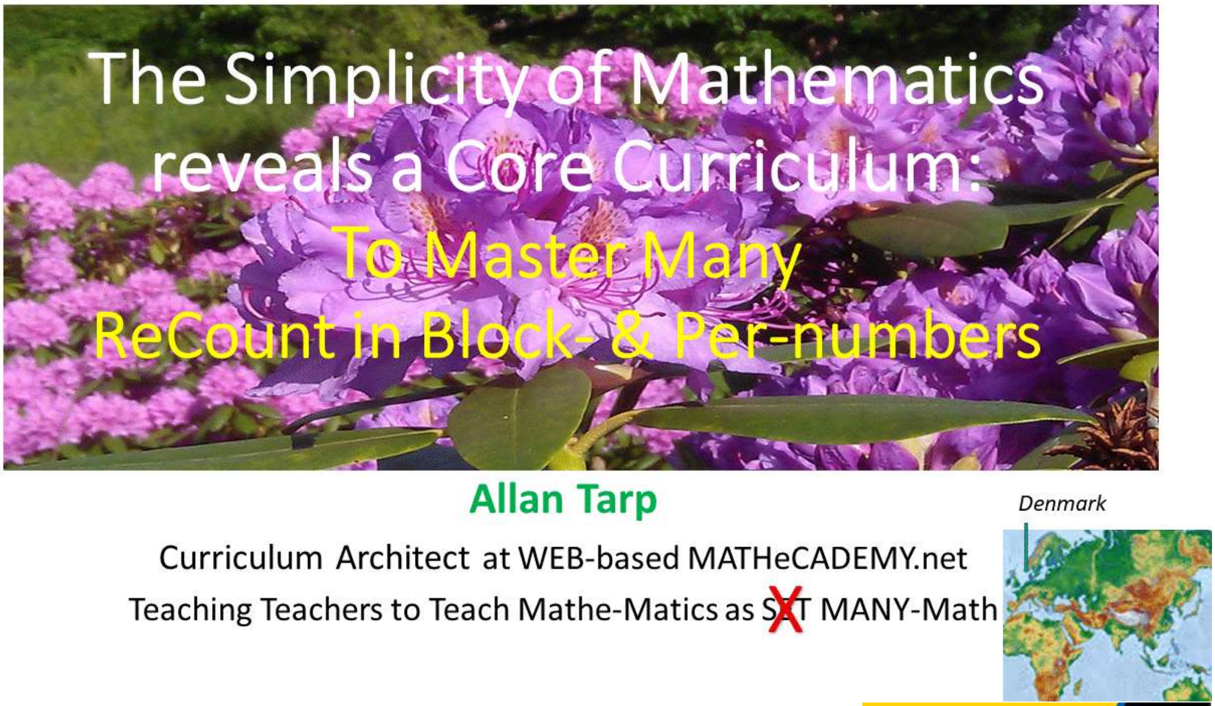
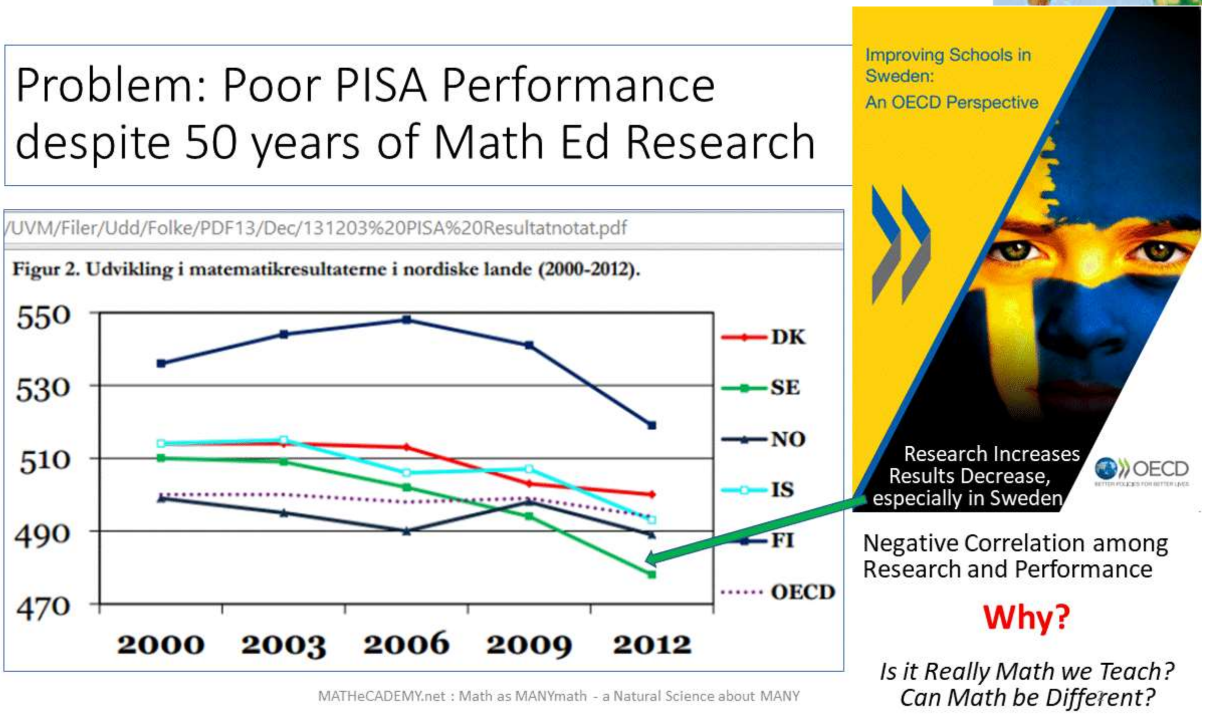
**26. The Simplicity of Math reveals a Core Curriculum, PPP**



## Solution in a Nutshell
## From BAD to GOOD Math

1) All teach numbers. Don't. Tell tales about how Totals unite and change
2) All use 1D line-numbers. Don't. Use 2D block-numbers
3) All begin with addition. Don't. Begin with counting and division, multiplication and subtraction before adding next-to and on-top
4) All add fractions without units. Don't. Use units as in integral calculus
5) All include only the predicate (3*5). Don't. Use full language sentences with a subject, a verb and a predicate (T = 3*5)
6) All call it MatheMatics. Don't. It is MetaMatism, derived from SET SET, and falsified by e.g. 2+3 is 17 and not 5 in the case of weeks and days. MatheMatics is rooted in MANY MANY.

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## A Call for Curriculum Architects

With many young male migrants in Sweden, a university may write out a competition:

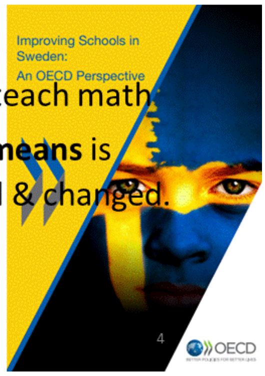
**'Theorize the poor PISA performance; and derive from this a STEM-based core curriculum for young male migrants.'**

**Didactics**: Define one **goal** and several **means**.

**The Tradition:** The **goal** is to learn mathematics. The **means** is to teach math.

**A Difference**: The **goal** is to master Many in space and time. The **means** is number-language sentences about how Many is counted & added & changed.

**Prerequisites**: None, start from scratch.

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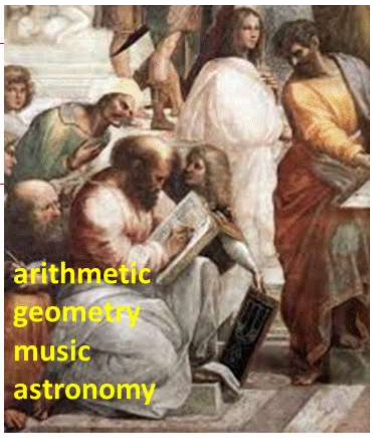
# Definitions of MatheMatics




Pythagoras: mathematics, meaning knowledge, is a common label for 4 areas describing Many by itself and in space & time:

- **Geometry** means to measure earth in Greek
- **Algebra** means to reunite numbers in Arabic

Around 1900, **SET** made mathematics self-referring. However, Russell said:

Self-reference leads to the classical liar paradox 'this sentence is false', being false if true & opp.

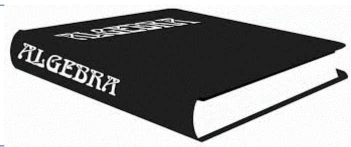
Just look at the set of sets, not belonging to itself. If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

So, forget about sets, and forget about fractions as numbers, by self-reference they cannot be so.

**Mathematics**: Forget about Russell, he is not a mathematician. Of course fractions are numbers.



# Two Different Mathematics




The ruling Set-based Top-Down **Meta-matics**

- Concepts are defined from above as **examples from abstractions**

*a FUNCTION is an example of a set relation with component-1 identity implying component-2 identity*

The silenced Many-based Bottom-Up **Many-math**

- Concepts are defined from below as **abstractions from examples**

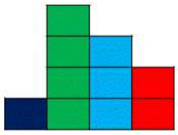
*a FUNCTION is for example 2+x, but not 2+3;*
*i.e. a name for a calculation with an unspecified number*

# Children see Many as Bundles with Units

Asked ‘How old next time?’, a 3year-old says 4, but reacts when held together 2 by 2:

‘<u>That is not 4, that is 2 **2s**</u>’.

Seeing bundles as units, children use 2D LEGO-like **block-numbers**, not 1D **line-numbers**, taught in school, even if 2D Arabic block-numbers replaced 1D Roman line-numbers centuries ago.



T = 1 4 3 2

T = MCCCCXXXII

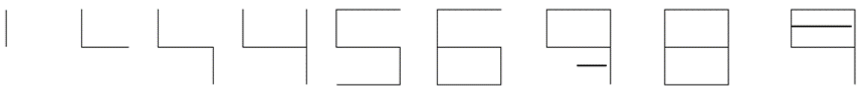


# Many as Icons: I I I I → IIII → 4

Meeting Many, we ask: “**How Many in Total**?”
To answer, we Math … oops sorry, it’s a label, not an action word.
To answer, first we count, then we add. We name and iconize the degrees of Many until ten, that as 1 bundle has no icon or digit itself.

- Thus there are four sticks in a 4-icon, five in a 5-icon, etc.

| one | two | three | four | five | six | seven | eight | nine |
|---|---|---|---|---|---|---|---|---|
| I | II | III | IIII | IIIII | IIIIII | IIIIIII | IIIIIIII | IIIIIIIII |



| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|

# Operations as Icons also

A decimal point
parts inside bundles
from outside singles

We count by bundling & stacking:

T = 7 = I I I I I I I = III III I = [III / III] I = 2**B**1 **3s** = 2.1 **3s**

- Thus, to count 7 in **3s** we take away 3 many times, iconized by an uphill stroke, 7/3, showing the broom wiping away the **3s**.

| | |
|---|---|
| 7/3 | 2.some |
| 7 – 2x3 | 1 |

- A calculator predicts: 3 can be taken away 2 times. Stacking the bundles is iconized as a lift, 2x3.

- To look for unbundled singles, we drag away the stack of 2 **3s,** iconized by a horizontal trace: 7 – 2x3 = 1.

Counting creates 3 operations: to divide & to multiply & to subtract.

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# Totals as a Bundle Formula

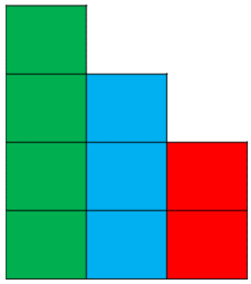
- To bundle bundles also, **power** is iconized as a cap, 5^2, showing the number of times bundles have been bundled.
- **Addition** is a cross + showing blocks juxtaposed next-to or on-top of each other.

Counting gives a Total as a **BundleFormula** called a polynomial.

Here all numbers have units:

T = 432 = 4***BundleBundle** + 3***Bundle** + 2*1

= 4***B^2** + 3***B +** 2*1



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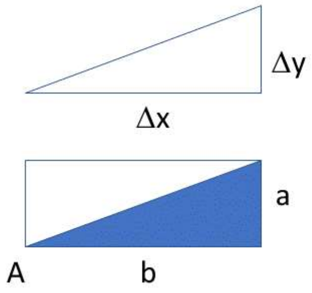
# The ReCount Formula

| | |
|---|---|
| 7/3 | 2.some |
| 7 – 2 * 3 | 1 |

Predicting T = 7 = 2.1 **3s**, the saying 'from T, T/B times, B can be taken away', is all over:

**ReCount formula T = (T/B)*B**

| | |
|---|---|
| Proportionality | y = k * x |
| Linearity | Δy = (**Δy/Δx**) * Δx = **m** * Δx |
| Local linearity | dy = (**dy/dx**) * dx = **y'** * dx |
| Trigonometry | a = (**a/b**) * b = **tanA** * b |
| Trade | $ = (**$/kg**) * kg = **price** * kg |
| Science | meter = (**meter/second**) * second = **velocity** * second |



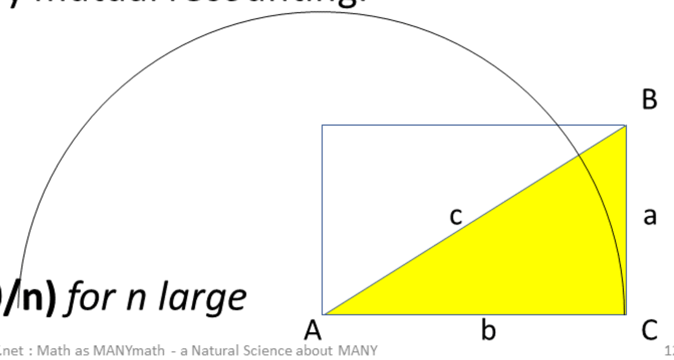



# Trigonometry ReCounts Sides in a HalfBlock

Halved by its diagonal, a block becomes a right angled triangle with three sides: the base b & the height a & the diagonal c, creating trigonometry by mutual recounting.

a = (**a/c**) * c = **sinA** * c
b = (**b/c**) * c = **cosA** * c
a = (**a/b**) * b = **tanA** * b

½Circle = π = **n*tan(180/n)** *for n large*



# ReCounting creates Proportionality and Overloads & Underloads

ReCounting in a **new unit** changes units (**proportionality**)
T = 4 **5s** = ? **6s**. The ReCount-formula predicts T = 3.2 **6s**

| | |
|---|---|
| 4*5/6 | 3.some |
| 4*5 – 3*6 | 2 |

ReCounting in the **same unit** creates overloads & underloads
T = 7 = I I I I I I I = IIIII IIIII I = IIIII I I I I = IIIII IIIII IIIII II
T = 7 = 2**B**1 **3s** = 1**B**4 **3s** = 3**B**-2 **3s**

**BundleWriting** may cure Math Dislike in classes stuck in Division:
T = 336 /7 = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

Likewise:

| | |
|---|---|
| Multiplication | T = 7* 48 = 7* 4**B**8 = 28**B**56 = 33**B**6 = 336 |
| Subtraction | T = 53 – 28 = 5**B**3 – 2**B**8 = 3**B**-5 = 2**B**5 = 25 |
| Addition | T = 53 + 28 = 5**B**3 + 2**B**8 = 7**B**11 = 8**B**1 = 81 |



# ReCounting creates Multiplication & Equations

ReCounting from icons to tens is predicted by **Multiplication**
T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens**

ReCounting from tens to icons is predicted by **Equations**
T = ? **7s** = 42 = (42/7)*7 recounting 42 in 7s, so ? = 42/7

| | |
|---|---|
| u*7 = 42 = (42/7)*7 | |
| u = 42/7 | = 6 |

*An equation is solved by moving to Opposite Side with opposite Sign*

| | | |
|---|---|---|
| | 7 x *u* = 42 | Multiplication has 1 as its **neutral element**, and 7 has 1/7 as its **inverse element** |
| ↕ | (7 x *u*) x (1/7) = 42x(1/7) | Multiplying 7's inverse element 1/7 to both …ber-names |
| ↕ | (*u* x 7) x (1/7) = 6 | Applying the **commutative law** to u x 7; 6 is the s… umb… ne for 42 x 1/7 |
| ↕ | *u* x (7 x (1/7)) = 6 | Applying the **associative law** |
| ↕ | *u* x 1 = 6 | Applying the definition of an inverse element |
| ↕ | *u* = 6 | Applying the definition of a neutral element… *n arrows a test is … needed.* |

## DoubleCounting in 2 units creates PerNumbers

Apples are double-counted in **kg** and in **$**.
With 4**kg = **5**$** we have the **PerNumber** 4kg/5$ = 4/5 kg/$
*Questions:*

| 12**kg** = ?**$** | 20**$** = ?**kg** |
|---|---|
| 12kg = (12/4)*4kg<br>= (12/4)*5$<br>= 15$ | 20$ = (20/5)*5$<br>= (20/5)*4kg<br>= 16kg |


*Answer:* Recount in the per-number

- With like units, per-numbers become fractions: 2$ per 5$ = 2$/5$ = 2/5



## The BundleFormula T = 432 = 4*B^2 + 3*B + 2*1 shows the 4 ways, Many Unite (*the Simplicity of Math*)

Many exists as **changing** & **constant block-numbers** & **per-numbers**

- Addition & Multiplication unite changing & constant block-numbers
  *Subtraction & Division split into changing & constant block-numbers*
- Integration & Power unite changing & constant per-numbers
  *Differentiation & Root/Logarithm split into changing & constant per-numbers*

| Operations **unite** / *split into* | **Changing** | **Constant** |
|---|---|---|
| **Block-numbers**<br>*m, s, $, kg* | T = a + n<br>*T – a = n* | T = a*n<br>*T/n = a* |
| **Per-numbers**<br>*m/s, $/kg, m/(100m) = %* | T = ∫ a dn<br>*dT/dn = a* | T = a^n<br>*$log_a T = n$, $\sqrt[n]{T} = a$* |

# Theorizing Poor PISA Performance

Poor PISA performance is caused by 4 blind spots:

- Mathematics should respect its nature as a NumberLanguage with 3part sentences (subject-verb-predicate) and a grammar, as in the WordLanguage.
- Seen as a goal in itself, math hides its outside goal, to master Many, so we teach TopDown MetaMatics instead of BottomUp ManyMath
- We use 1D line-numbers instead of 2D block-numbers with 3 numbers: the size of the bundle & the number of bundles & the number of unbundled – and they add differently
- By this complexity, addition OnTop and NextTo should be postponed to after BundleCounting & ReCounting & DoubleCounting in STEM-tasks

1D: + - x /

2D: / x - +



# STEM (ScienceTechEngMath) based Core Curriculum for Migrants

**Nature** consists of things in motion, combined in momentum = mass*velocity

Things contain **mass** & **molecules** & **eclectric charge**.

Nature is counted in **meter** & **second** & **kilogram** & **mole** & **coulomb**.

***<u>Nature is predictable by ReCounting & PerNumbers:</u>***

kilogram = (**kilogram/cubic-meter**) * cubic-meter = **density** * cubic-meter

meter = (**meter/second**) * second = **velocity** * second

Δ momentum = (Δ **momentum/second**) * second = **force** * seconds

Δ energy= (Δ **energy/meter**) * meter = **force** * meter = work

*Energy = ½*mass*velocity squared*

# PerNumbers/ReCounting in ScienceTechEngMath II

Energy = (**energy/kg**) * kg = **melting/evaporation heat** * kg

Energy = (**energy/kg/degree**) * kg * degree = **heat** * kg * degree

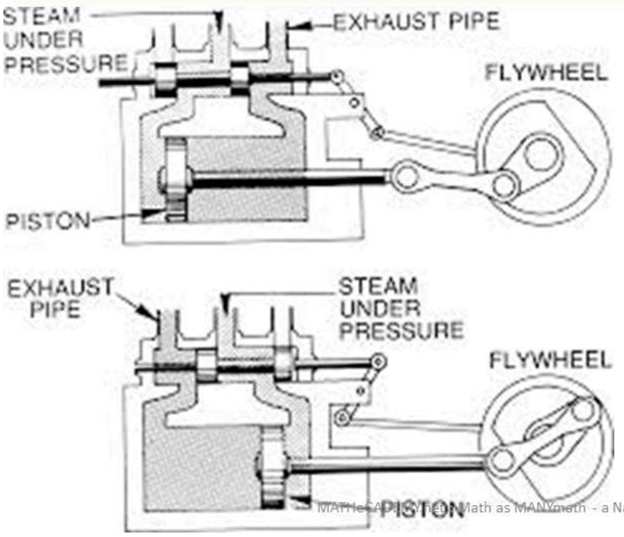
force = (**force/square-meter**) * square-meter = **pressure** * square-meter

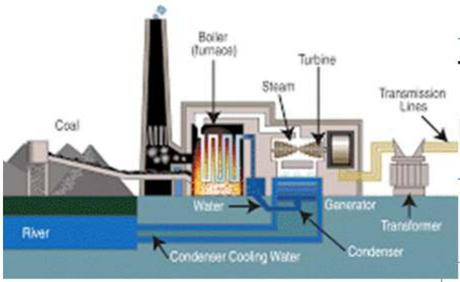
gram = (**gram/mole**) * mole = **molar mass** * mole

mole = (**mole/liter**) * liter = **molarity** * liter

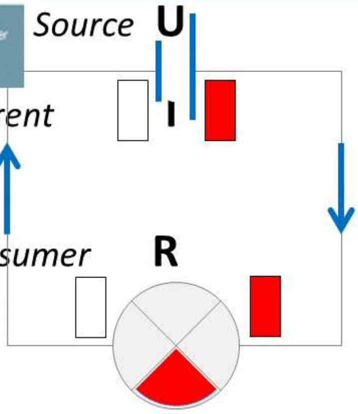


# Technology I
Steam at work: $p*V = n*R*T$




# Technology II
## Electrons at work: P= U*I & U = R*I



*Source* **U** — **Volt** = Energy/Coulomb

*Current* **I** — **Ampere** = Coulomb/second

*Consumer* **R** — **Resistance** in Ohm

*Consumption* **P** — **Watt** = Energy/second



# Engineering
## How many turns on a steep hill

On a 30 degree hillside, a 10 degree road is to be constructed. How many turns will there be on a 1 x 1 km hillside?

- We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.
- In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2.
- In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2.
- In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).
- Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

*Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.*

The Simplicity of Mathematics reveals a Core Curriculum
*To Master Many: ReCount in Block- & Per-numbers*

**Thank You for Listening**

Slides & full paper on
MATH**e**CADEMY.net

Details in
**Journal of Mathematics Education**
**vol. 11 #1**

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DifferenceResearch finds Differences making a Difference, inspired by

- The ancient Greek sophists:

*Differences unmask choice masked as nature*

- In existentialism, Sartre: *EXISTENCE precedes ESSENCE.* Heidegger: *In sentences, the SUBJECT exists, but the PREDICATE is essence that often can be different.*

Let's meet the subject, **MANY,** directly & outside its 'essence-prison'

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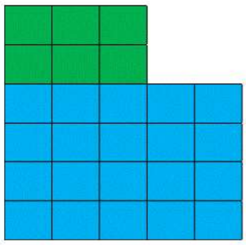
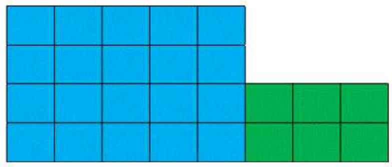
# Totals Add OnTop & NextTo

| OnTop | NextTo |
|---|---|
| 4 5s + 2 3s = 4 5s + 1B1 5s = 5B1 5s | 4 5s + 2 3s = 3B2 8s |
| The units are changed to be the same<br>*Change unit = Proportionality* | The areas are added<br>*Adding areas = Integration* |







# PerNumbers & Fractions add as **Integral Calculus**

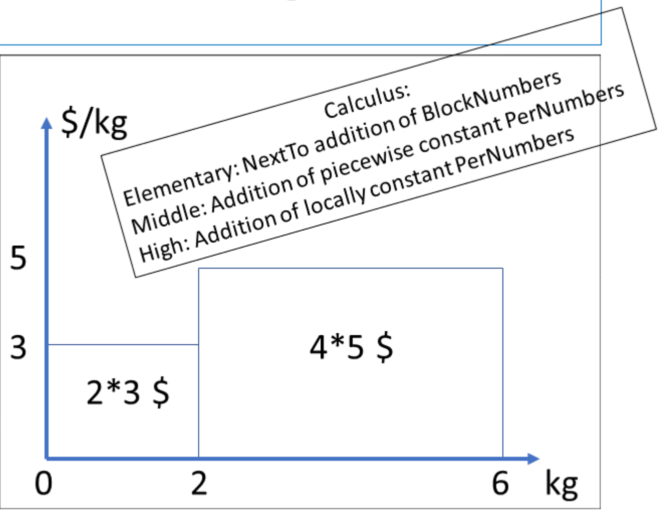
2 kg at 3 $/kg

+ 4 kg at 5 $/kg

**(2+4) kg at ? $/kg**

Unit-numbers add on-top.

Per-numbers add next-to as **areas** under the per-number graph, i.e. as **integral calculus**



## 27. Addition-Free Migrant-Math Rooted in STEM Re-Counting Formulas

*STEM typically contains multiplication formulas expressing re-counting in different units, thus calling for an addition-free curriculum. The mastery of Many children bring to school uncovers a Many-based 'Many-matics' as an alternative to the present self-referring set-based mathe-matics. To answer the question 'How many in total?' we count and re-count totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a re-count formula as a core formula in all STEM subjects.*

### Decreased PISA performance despite increased research

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish Centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD to write the report "Improving Schools in Sweden" (2015a) describing its school system as "in need of urgent change".

To find an unorthodox solution we pretend that a university in southern Sweden, challenged by numerous young male migrants, arranges a curriculum architect competition: "Theorize the low success of 50 years of mathematics education research; and derive from this theory a STEM based core curriculum allowing young migrants to return as STEM pre-teachers and pre-engineers."

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to help migrants and how to improve schools in Sweden and elsewhere.

### Social theory looking at mathematics education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16).

As a institutions, of which mathematics education is an example, he talks about of rational action "in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)". He then points out that "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement (p. 84)."

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since such a goal statement is meaningless by its self-reference. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'. Here the invention of the concept Set created a Set-based 'meta-matics', self-referential defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. But, then Russell looked at the set of sets not belonging to itself. Here a set belongs only if it does not: if $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. Thus pointing out that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false.

In this way, Set changed grounded classical mathe-matics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

**Difference research looks at mathematics education**

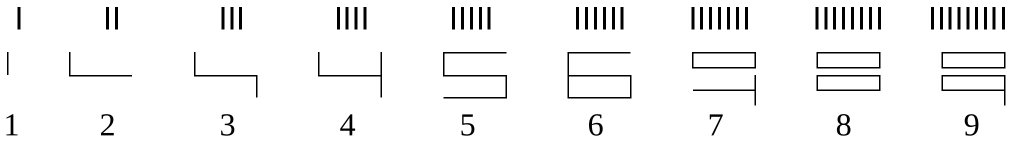
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ (Tarp, 2017) is searching for hidden differences making a difference. An additional inspiration comes from existentialist philosophy described by Sartre (2007, p. 20) as holding that “Existences precedes essence”. So, to avoid a goal displacement in math education, difference-research asks: How will math look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a mathematics core curriculum based upon examples of Many in a STEM context (Lawrenz et al, 2017). So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

**Meeting Many creates a ‘count-before-adding’ curriculum**

Meeting Many, we ask “How many in Total?” To answer, we total by counting to create number-language sentences as e.g. T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence; and connecting the outside total T with its inside predicate 2 3s (Tarp, 2018b).

Rearranging many 1s into one symbol with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:




Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, bundles of 2s and 2 of them. This inspires ‘bundle-counting’, re-counting a total in icon-bundles to be stacked as bundle- or block-numbers, which can be re-counted and double-counted before being processed by on-top and next-to addition, direct or reversed. Thus, a total T of 5 1s is re-counted in 2s as T = 2 2s & 1; described by ‘bundle-writing’, T = 2B1 2s; or by ‘decimal-writing’, T = 2.1 2s, where, with a bundle-cup, a decimal point separates the bundles inside from the outside unbundled singles; or by ‘deficit-writing’, T = 3B-1 2s = 3.-1 2s = 3 bundles less 1 2s.

So, to count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator thus predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: entering ‘5/2’ on a calculator gives ‘2.some’, and ‘5 – 2x2’ gives ‘1’, so T = 7 = 2B1 3s. The unbundled can be placed next-to or on-top the stack thus rooting decimals, fractions and negative numbers.

The re-count formula occurs all over. With proportionality: y = c*x; in trigonometry as sine, cosine and tangent: a = (a/c)*c = sinA*c and b = (b/c)*c = cosA*c and a = (a/b)*b = tanA*b; in coordinate geometry as line gradients: Δy = Δy/Δx = c* Δx; and in calculus as the derivative, dy = (dy/dx)*dx = y'*dx. In economics, the re-count formula is a price formula: $ = ($/kg)*kg, $ = ($/day)*day, etc.

***Re-counting in the same unit or in a different unit***
Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 2B1 2s as 1B3 2s with an outside 'overload'; or as 3B-1 2s with an outside 'underload' thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48; or 336 = 35B-14, so 336/7 = 5B-2 = 48. Re-counting in a different unit means changing unit, also called proportionality. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2B2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s' The answer, '2.some', predicts that the singles come from taking away 2 5s, now asking '3*4 – 2*5'. The answer, '2', predicts that 3 4s can be re-counted in 5s as 2B2 5s or 2.2 5s.

***Re-counting to and from tens***
Asking '3 4s = ? tens' is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right. Re-counting from tens to icons by asking '35 = ? 7s' is called an equation x*7 = 35. It is easily solved by re-counting 35 in 7s: x*7 = 35 = (35/7)*7. So x = 35/7, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

***Double-counting creates proportionality as per-numbers***
Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we re-count 6 in the per-number 2s: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

**A short curriculum in addition-free mathematics**

01. To stress the importance of bundling, the counting sequence can be: 01, 02, …, 09, 10, 11 etc.; or 01, 02, 03, 04, 05, Ten less 4, T-3, T-2, T-1, Ten, Ten and 1, T and 2, etc.

02. Ten fingers can be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers can be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables can be formulated as re-counting from icon-bundles to tens and use underload counting after 5: T = 4*7 = 4 7s = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 can be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

06. Solving proportional equations as 3*x = 12 can be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks can be done by re-counting in the per-number: With 3$/4kg, 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg.

08. Fractions and percentages are per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$ = (60/3)*3$ giving (60/3)*2$ = 40$

09. Trigonometry can precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sinA*c.

10. Counting by stacking bundles into adjacent blocks leads to the number formula or bundle formula called a polynomial: T = 456 = 4*BundleBundle + 5*Bundle + 6*single = 4*B^2 + 5*B + 6*1. In its general form, the number formula T = a*x^2 + b*x + c contains the different formulas for constant change: T = a*x (proportionality), T = a*x^2 (acceleration), T = a*x^c (elasticity) and T = a*c^x (interest rate); as well as T = a*x+b (linearity or affinity, strictly).

11. Predictable change roots pre-calculus (if constant) and calculus (if changing). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

12. Integral can precede differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers. So, both per-numbers and fractions are added with units as the area under the per-number graph.

**Meeting Many in a STEM context**

OECD (2015b) says: 'In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.' STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature's behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

***Nature as heavy things in motion***

To meet we must specify place and time in a nature consisting of heavy things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature's three main factors, matter and force and motion, like the three social factors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water into electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to dissolving matter in water; to the trajectory of a ball pulled down by gravity; to put steam and electrons to work in a power plant creating an electrical circuit transporting energy from a source to many consumers.

Heavy things in motion are combined by the momentum = mass*velocity, a multiplication formula as most STEM formulas expressing re-counting by per-numbers: kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter, where force is the per-number change in momentum per second. Thus, STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

***Warming and boiling water***

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 2000 Joule per second in 167 seconds will heat 1000 gram water 80 degrees we get a double per-number 2000*167/80/1000 Joule/degree/gram or 4.18 Joule/degree/gram, called the specific heat capacity of water. Producing 100 gram steam in 113 seconds, the per-number is 2000*113/100 Joule/gram or 2260 J/g, called the heat of evaporation for water.

***Dissolving material in water***

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so re-counting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

***Building batteries with water***

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun’s energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H2O, and carbon dioxide CO2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH4, the energy comes from using 4 Os to burn it.

***Technology and engineering: letting steam and electrons produce and distribute energy***

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units. Thus a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, p*V = n*R*T, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g. steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to industries and homes.

***An electrical circuit***

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per charge-unit. Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere. To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120 Watt bulb, because of proportionality the latter needs twice the current, and consequently half the resistance of the former.

***How high up and how far out***

An inclined gun sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

Compared with the horizontal, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

**Adding addition to the curriculum**

Once counted as block-numbers, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being re-counted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations where the question 2 3s + ? 4s = 5 7s leads to differentiation: ? = (5*7 – 2*3)/4 = $\Delta$T/4. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular.

The number formula T = 456 = 4*B^2 + 5*B + 6*1 shows there are four ways to unite numbers: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers (Tarp, 2018b).

## Conclusion and recommendation

This paper argues that 50 years of unsuccessful mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views lead to different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying "To master Many, counting and re-counting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating." A proposal for two separate twin-curricula in counting and adding is found in Tarp (2018a). Thus, the simplicity of mathematics as expressed in a 'count-before-adding' curriculum allows replacing block-numbers with line-numbers, and learning core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM pre-teachers or pre-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017). Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

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## 28. A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many

*Mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory, we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum.*

### POOR PISA PERFORMANCE DESPITE FIFTY YEARS OF RESEARCH

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in math education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 years since 1969. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to significantly below the OECD average in 2012, causing OECD (2015) to write the report 'Improving Schools in Sweden'. Likewise, math dislike seems to be widespread in high performing countries also. With mathematics and education as social institutions, grand theory may explain this 'irrelevance paradox', the apparent negative correlation between research and performance.

### GRAND THEORY

Ancient Greece saw two forms of knowledge, 'sophy'. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Christianity eagerly took over metaphysical patronage and changed the academies into monasteries. The sophist skepticism was revived by Brahe and Newton, insisting that knowledge about nature comes from laboratory observations, not from library books (Russell, 1945).

Newton's discovery of a non-metaphysical changing will spurred the Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. Two republics arose, in the United States and in France. The US still has its first Republic, France its fifth, since its German-speaking neighbors tried to overthrow the French Republic again and again.

In North America, the sophist warning against hidden patronization lives on in American pragmatism and symbolic interactionism; and in Grounded Theory, the method of natural research resonating with Piaget's principles of natural learning. In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructural thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnosed cures, and education; all may hide patronizing choices presented as nature (Lyotard, 1984).

Within philosophy itself, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that 'existence precedes essence', exemplified by the Heidegger-warning: In a sentence, trust the subject, it exists; doubt the predicate, it is essence coming from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber was the first to theorize the increasing goal-oriented rationalization that de-enchant the world and create an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16). But he also formulates a warning (p. 84): "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right". Which may lead to 'the banality of evil' (Arendt, 1963).

As to what we say about the world, Foucault (1995) focuses on discourses about humans that, if labeled scientific, establish a 'truth regime'. In the first part of his work, he shows how a discourse disciplines itself by only accepting comments to already accepted comments. In the second part he shows how a discourse disciplines also its subject by locking humans up in a predicate prison of abnormalities from which they can only escape by accepting the diagnose and cure offered by the 'pastoral power' of the truth regime. Foucault thus sees a school as a 'pris-pital' mixing the power techniques of a prison and a hospital: the 'pati-mates' must return to their cell daily and accept the diagnose 'un-educated' to be cured by, of course, education as defined by the ruling truth regime.

**Mathematics, stable until the arrival of SET**

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught 'reckoning' in primary school and 'arithmetic' and 'geometry' in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here a wish for exactness and unity created a SET-derived 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

SET thus transformed classical grounded 'many-matics' into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meets counter-examples as e.g. 1week + 2days is 9days.

**Proportionality illustrates the variety of mastery of Many and of quantitative competence**

Proportionality is rooted in questions as "2kg costs 5$, what does 7kg cost; and what does 12$ buy?"

Europe used the 'Regula de Tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, the divide. So first we ask, Q1: '2kg cost 5$, 7kg cost ?$' to get to the answer (7*5/2)$ = 17.5$. Then we ask, Q2: '5$ buys 2kg, 12$ buys ?kg' to get to the answer (12*2)/5$ = 4.8kg.

Then, two new methods appeared, 'find the unit', and cross multiplication in an equation expressing like proportions or ratios:

Q1: 1kg costs 5/2$, so 7kg cost 7*(5/2) = 17.5$. Q2: 1$ buys 2/5kg, so 12$ buys 12*(2/5) = 4.8kg.
Q1: $2/5 = 7/x$, so $2*x = 7*5$, $x = (7*5)/2 = 17.5$. Q2: $2/5 = x/12$, so $5*x = 12*2$, $x = (12*2)/5 = 4.8$.

SET chose modeling with linear functions to show the relevance of abstract algebra's group theory: Let us define a linear function $f(x) = c*x$ from the set of kg-numbers to the set of $-numbers, having as domain $DM = \{x \in R \mid x>0\}$. Knowing that $f(2) = 5$, we set up the equation $f(2) = c*2 = 5$ to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: $c*2 = 5$, $(c*2)*½ = 5*½$, $c*(2*½) = 5/2$, $c*1 = 5/2$, $c = 5/2$. With $f(x) = 5/2*x$, the inverse function is $f\text{-}1(x) = 2/5*x$. So with 7kg, $f(7) = 5/2*7 = 17.5$$; an with 12$, $f\text{-}1(12) = 2/5*12 = 4.8$kg.

In the future, we simply 're-count' in the 'per-number' 2kg/5$ coming from 'double-counting' the total *T*. Q1: *T* = 7kg = (7/2)*2kg = (7/2)*5$ = 17.5$; Q2: *T* = 12$ = (12/5)*5$ = (12/5)*2kg = 4.8kg.

**Grand theory looks at mathematics education**

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologically, how we perceive and verbalize it. University mathematics holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB+4*B+5*1, resonating with what children bring to school, e.g. T = 2 5s.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? University mathematics holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian psychology.

Sociologically, a Weberian viewpoint would ask if SET is a rationalization of Many gone too far leaving Many de-enchanted and the learners in an iron cage. A Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, contemporary university mathematics has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word 'mathematics' must go from goal descriptions. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a pris-pital, a Foucaultian viewpoint, would ask, first which structure to choose, European line-organization forcing a return to the same cell after each hour, day and month for several years; or the North American block-organization changing cell each hour, and changing the daily schedule twice a year? Next, as prisoners of a 'the goal of math education is to learn math' discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g. by examining and developing the existing mastery children bring to school?

**Meeting Many, children bundle in block-numbers to count and share**

How to master Many can be learned from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists, and with units: bundles of 2s, and 2 of them.

Children also use block-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 extra'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 4s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as *T* = 7 = 1*B*5 = 2*B*3 = 3*B*1 = 4*B*~~1~~; or even as 1*BB*3 or 1*BB*1*B*1. Also, children love to count in 3s, 4s, and in hands.

Sharing 9 cakes, 4 children take one by turn saying they take 1 of each 4. Taking away 4s roots division as counting in 4s; and with 1 left they often say "let's count it as 4". Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. And they smile when seeing that entering '9/4' allows a calculator to predict the sharing result as 2 1/4; and when seeing that entering '2*5/3' will predict the result of sharing 2 5s between 3 children.

Children thus master sharing, taking parts and splitting into parts before division and counting- and splitting-fractions is taught; which they may like to learn before being forced to add without units.

So why not develop instead of rejecting the core mastery of Many that children bring to school?

**A typical contemporary mathematics curriculum**

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, $T = 2$ 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1*B*14 tens is not allowed. Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

## A QUESTION GUIDED COUNTING CURRICULUM

The question guided re-enchantment curriculum in counting could be named 'Mastering Many by counting, recounting and double-counting'. The design is inspired by Tarp (2018). It accepts that while eight competencies might be needed to learn university mathematics (Niss, 2003), only two are needed to master Many (Tarp, 2002), counting and uniting, motivating a twin curriculum. The corresponding pre-service or in-service teacher education can be found at the MATHeCADEMY.net. Remedial curricula for classes stuck in contemporary mathematics can be found in Tarp (2017).

Q01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

Q02, counting sequences: "How to count fingers?" Here the learning opportunity is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H". Follow-up activities could be counting things.

Q03, icon-counting: "How to count fingers by bundling?" Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported in a number-language sentence with subject, verb and predicate: $T = 5 =$ 1Bundle3 2s = 2*B*1 2s = 3*B*-1 2s = 1*BB*1 2s, called an 'inside bundle-number' describing the 'outside block-number'. A western abacus shows this in 'outside geometry space-mode' with the 2 2s on the second and third bar and 1 on the first bar; or in 'inside algebra time-mode' with 2 on the second bar and 1 on the first bar. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, calculator-prediction: "How can a calculator predict a counting result?" Here the learning opportunity is to see the division sign as an icon for a broom wiping away bundles: 5/2 means 'from 5, wipe away bundles of 2s'. The calculator says '2.some', thus predicting it can be done 2 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes the trace left when dragging away the block to look for unbundled singles. By showing '5-2*2 = 1' the calculator indirectly predicts that a total of 5 can be recounted as 2B1 2s. An additional learning opportunity is to write and use the 'recount-formula' $T = (T/B)*B$ saying "From *T*, *T*/*B* times *B* can be taken away." This proportionality formula occurs all over mathematics and science. Follow-up activities could be counting cents: 7 2s is how many fives and tens? 8 5s is how many tens?

Q05, unbundled as decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the learning opportunity is to see that with blocks, the unbundled occur in three ways. Next-to the block as a block of its own, written as $T = 7 =$ 2.1 3s, where a decimal point separates the bundles from the singles. Or on-top as a part of the bundle, written as $T = 7 =$ 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q06, prime or foldable units: “Which blocks can be folded?” Here the learning opportunity is to examine the stability of a block. The block *T* = 2 4s = 2*4 has 4 as the unit. Turning over gives *T* = 4*2, now with 2 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a ‘prime unit’ and 4 a ‘foldable unit’, 4 = 2 2s. So, a block of 3 2s cannot be folded, whereas a block of 3 4s can: *T* = 3 4s = 3 * (2*2) = (3*2) * 2. A number is called even if it can be written with 2 as the unit, else odd.

Q07, finding units: “What are possible units in *T* = 12?”. Here the learning opportunity is that units come from factorizing in prime units, *T* = 12 = 2*2*3. Follow-up activities could be other examples.

Q08, recounting in another unit: “How to change a unit?” Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. *T* = 3 4s = ? 5s, the recount-formula will say *T* = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

Q09, recounting from tens to icons: “How to change unit from tens to icons?” Here the learning opportunity is that asking ‘*T* = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter *u* for the unknown number, *u**8 = 24. This is easily solved by recounting 24 in 8s as 24 = (24/8)*8 so that the unknown number is *u* = 24/8 attained by moving 8 to the opposite side with the opposite sign. Follow-up activities could be other examples of recounting from tens to icons.

Q10, recounting from icons to tens: “How to change unit from icons to tens?” Here the learning opportunity is that if asking ‘T = 3 7s = ? tens’, the recount-formula cannot be used since the calculator has no ten-button. However, it is programmed to give the answer directly by using multiplication alone: *T* = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: *T* = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Follow-up activities could be other examples of recounting from icons to tens.

Q11, double-counting in two units: “How to double-count in two different units?” Here the learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To answer questions we just recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa, asking ‘?$ = 12kg’, the answer is T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Follow-up activities could be numerous other examples of double-counting in two different units since per-numbers and proportionality are core concepts.

Q12, double-counting in the same unit: “How to double-count in the same unit?” Here the learning opportunity is that when double-counted in the same unit, per-numbers take the form of fractions, 3$ per 5$ = 3/5; or percentages, 3 per hundred = 3/100 = 3%. Thus, to find a fraction or a percentage of a total, again we just recount in the per-number. Also, we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Follow-up activities could be other examples of double-counting in the same unit since fractions and percentages are core concepts.

Q13, recounting the sides in a block. “How to recount the sides of a block halved by its diagonal?” Here, in a block with base *b*, height *a*, and diagonal *c*, mutual recounting creates the trigonometric per-numbers: *a* = (*a*/*c*)**c* = sin*A***c*; *b* = (*b*/*c*)**c* = cos*A***c*; *a* = (*a*/*b*)**b* = tan*A***b*. Thus, rotating a line can be described by a per-number *a*/*b*, or as tan*A* per 1, allowing angles to be found from per-numbers. Follow-up activities could be other blocks e.g. from a folding ruler.

Q14, double-counting in STEM (Science, Technology, Engineering, Math) multiplication formulas with per-numbers coming from double-counting. Examples: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter) * square-meter = pressure * square-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg) * kg

= heat * kg; gram = (gram/mole) * mole = molar mass * mole; Δ momentum = (Δ momentum/sec) * sec = force * sec; Δ energy = (Δ energy/ meter) * meter = force * meter = work; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp; dollar = (dollar/hour)*hour = wage*hour.

Q15, navigating. "Avoid the rocks on a squared paper". Four rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock? Predict and measure the angles on the journey.

**A QUESTION GUIDED UNITING CURRICULUM**

The question guided re-enchantment curriculum in uniting could be named 'Mastering Many by uniting and splitting constant and changing unit-numbers and per-numbers'.

A general bundle-formula *T* = *a***x*^2 + *b***x* + *c* is called a polynomial. It shows the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Q21, next-to addition: "With *T*1 = 2 3s and *T*2 = 4 5s, what is *T*1+*T*2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus. Follow-up activities could be other examples of next-to addition.

Q22, reversed next-to addition: "If *T*1 = 2 3s and *T*2 add next-to as *T* = 4 7s, what is *T*2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Follow-up activities could be other examples of reversed next-to addition.

Q23, on-top addition: "With *T*1 = 2 3s and *T*2 = 4 5s, what is *T*1+*T*2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Follow-up activities could be other examples of on-top addition.

Q24, reversed on-top addition: "If *T*1 = 2 3s and *T*2 as some 5s add to *T* = 4 5s, what is *T*2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again is called differential calculus. An underload is removed by recounting. Follow-up activities could be other examples of reversed on-top addition.

Q25, adding tens: "With *T*1 = 23 and *T*2 = 48, what is *T*1+*T*2 when added as tens?" Again, recounting removes an overload: *T*1+*T*2 = 23 + 48 = 2*B*3 + 4*B*8 = 6*B*11 = 7*B*1 = 71; or *T* = 236 + 487 = 2*BB*3*B*6 + 4*BB*8*B*7 = 6*BB*11*B*13 = 6*BB*12*B*3 = 7*BB*2*B*3 = 723.

Q26, subtracting tens: "If *T*1 = 23 and *T*2 add to *T* = 71, what is *T*2?" Again, recounting removes an underload: *T*2 = 71 – 23 = 7*B*1 – 2*B*3 = 5*B*-2 = 4*B*8 = 48; or *T*2 = 956 – 487 = 9*BB*5*B*6 – 4*BB*8*B*7 = 5*BB*-3*B*-1 = 4*BB*7*B*-1 = 4*BB*6*B*9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, showing that -(-1) = +1.

Q27, multiplying tens: "What is 7 43s recounted in tens?" Here the learning opportunity is that also multiplication may create overloads: *T* = 7*43 = 7*4*B*3 = 28*B*21 = 30*B*1 = 301; or 27*43 = 2*B*7*4*B*3 =8*BB*+6*B*+28*B*+21 =8*BB*34*B*21 =8*BB*36*B*1 = 11*BB*6*B*1 = 1161, solved geometrically in a 2x2 block.

Q28, dividing tens: “What is 348 recounted in 6s?” Here the learning opportunity is that recounting a total with overload often eases division: *T* = 348 /6 = 3*BB*4*B*8 /6 = 34*B*8 /6 = 30*B*48 /6 = 5*B*8 = 58.

Q29, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform into unit-number by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining constancy by epsilon and delta.

Q30, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Q31, finding common units: “Only add with like units, so how to add *T* = 4ab^2 + 6abc?”. Here units come from factorizing: *T* = 2*2*a*b*b* + 2*3*a*b*c* = 2*b**(2*a*b*) + 3*c**(2*a*b*) = 2*b*+3*c* 2*ab*s.

## CONCLUSION

A curriculum wants to develop brains, and colonizing wants to develop countries. De-colonizing accepts that maybe countries and brains can develop themselves if helped by options instead of directions from developed countries and brains. Some prefer a direction-giving multi-year macro-curriculum; others prefer option-giving half-year micro-curricula. Some prefer a curriculum to be a cure prescribing mathematics competencies and literacy; others prefer developing the existing quantitative competence and numeracy, defined by dictionaries as the ability to use numbers and operations in everyday life, thus silencing the word ‘mathematics’ to avoid a hidden continuing colonization. In the transition period between colonizing and decolonizing brains, grand theory has an advice to the ‘irrelevance paradox’ of mathematics education research: accept the brain’s own epistemology to avoid a goal displacement blocking the road to its educational goal, mastery of Many.

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**29. Counting Before Adding, a PPP for the Article on a Twin Curriculum**

# COUNTING before ADDING

*The Child's Own Twin Curriculum*

Count & ReCount & DoubleCount

before Adding NextTo & OnTop

Master Many with

ManyMath

Allan.Tarp@MATHeCADEMY.net, November 2018

MATHeCADEMY

## Background: Our two language houses

| | |
|---|---|
| The **WORD language** assigns words in sentences with<br>The **NUMBER language** assigns numbers instead with | • a subject<br>• a verb<br>• a predicate |

Both languages have a META-language, a grammar, describing the language, that is learned before the grammar. But does mathematics respect teaching language before grammar?

| | WORD language | NUMBER language |
|---|---|---|
| *META-language, grammar* | *'is' is a verb* | *'x' is an operation* |
| Language | This is a chair<br>Mother tongue | Mathematics<br>T = 3x4 |

WORLD

MATHeCADEMY.net | Math as MANYmath - a Natural Science about MANY 2

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# How well defined is mathematics?


In ancient Greece, Pythagoras used mathematics, meaning knowledge, as a common label for four descriptions of Many by itself & in space & time:

Together they formed the '**quadrivium**' recommended by Plato as a general curriculum after the '**trivium**' consisting of grammar & logic & rhetoric.

*Geometry & algebra are both grounded in Many as shown by names:*

**Geometry** means to measure earth in Greek

**Algebra** means to reunite numbers in Arabic



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# Modern mathematics, MetaMath

Around 1900, **SET** made math a self-referring **MetaMath**.

But Russell saw that self-reference leads to the classical liar paradox 'this sentence is false', being false if true & opposite:

"Let M be the set of sets not belonging to itself, $M = \{A \mid A \notin A\}$.

Then $M \in M \Leftrightarrow M \notin M$. Forget about sets. Use type theory instead.

So, by self-reference, fractions cannot be numbers."

**Mathematics**: "Forget about Russell, he is not a mathematician.

We just institutionalize fractions as so-called rational numbers."

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## Institutions as thorns protecting Sleeping Beauties

- Weber on institutionalization: Rationalized too far, mathematics may become an **iron cage** that **disenchants** its subject.
- Bauman on self-reference: „The ideal model of action subjected to rationality as the supreme criterion contains a danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right".
- Arendt: Just following orders may lead to '**the banality of evil**'.
- Sartre on existentialism: „**Existence precedes essence**".



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## Three curriculum choices to make

What is the goal of mathematics education?

1. To learn institutionalized mathematics, or
2. To learn to master what exists, Many


What is the core means?

1. To learn about numbers without units, or
2. To learn how to number with units

1, -2, 3/4, √5, π, e

T = 3B2 = 3.2 = 4.-2 4s

What are numbers?

1. One-dimensional line-numbers without units, or
2. Two-dimensional block-numbers with units

*Choosing 1 may have caused 50 years of less successful math education research.*

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# Different curricula: MetaMath or ManyMath

What is the goal of mathematics education?

1. To learn mathematics (self reference pointing up, Vygotsky theory)
2. To learn to Master Many (external reference pointing down, Piaget theory)

What is a core means?

1. To learn about numbers (operations on specified and unspecified numbers)
2. To learn how to number (number-language sentences about counting & adding totals)

What are numbers?

1. 1D line-numbers (integer, natural, rational, real, place value system)
2. 2D bundle-numbers (constant & changing unit-numbers & per-numbers)



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# Why teach children if they already know?

With education curing un-educatedness, we ask:

To CURE, be SURE

1. The diagnosed is not already cured
2. The diagnose is not self-referring: *teach math to learn math*

Core Questions:

- What Mastery of Many does the child have already?
- What could be a ChildCenteredCurriculum in Quantitative Competence?

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## Creating icons: IIII → IIII →  →

Children love making number-icons of cars, dolls, spoons, sticks.
Counting in ones means naming the different degrees of Many.
Changing four ones to one fours creates a 4-icon with four sticks.
An icon contains as many sticks as it represents, if written less sloppy.
Once created, icons become units to use when counting in bundles.

| one | two | three | four | five | six | seven | eight | nine |
|---|---|---|---|---|---|---|---|---|
| I | II | III | IIII | IIIII | IIIIII | IIIIIII | IIIIIIII | IIIIIIIII |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |



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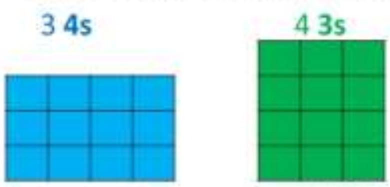
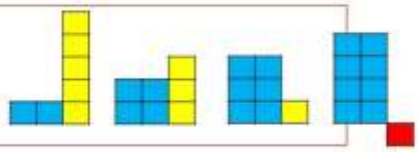
## Children see Many as bundles with units

- "How old next time?" A 3year old says "Four" showing 4 fingers: I I I I
- But, the child reacts strongly to 4 fingers held together 2 by 2: II II
- "That is not four, that is two twos"
- The child describes what exists, and with units: bundles of 2s, and 2 of them
- The block 3 **4s** has two numbers: 3 (the counting-number) and **4** (the unit-number)
- Children also use bundle-numbers with Lego blocks:




# To count Many, children bundle



- Children are flexible when re-counting a Total of 7 sticks in **2s**:

IIIIIII • ǁ IIIII • ǁ ǁ III • ǁ ǁ ǁ I • ǁ ǁ ǁ ǁ

T = 7 = 1 **2s** &5 = 2 **2s** &3 = 3 **2s** &1 = 4 **2s** less 1

- And children don't mind writing a total of 7 using 'bundle-writing':

T = 7 = 1B5 = 2B3 = 3B1 = 4B ~~1~~, or even as

T = 7 = 1BB3 = 1BB1B1 = 2BB ~~1~~

- Also, children love to count in **3s**, **4s**, and in **hands**:

Thus, a number is a multi-counting of bundles as units

*(..., bundles-of-bundles, bundles, unbundled)*

| IIII II |
|---|
| T = 7 = 1 **5s** & 2 |
| T = 7 = 1B2 **5s** |





# Counting bundles gives a number formula

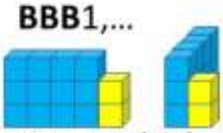
Children have fun when counting bundles, bundles of bundles, etc.:

With ten-bundles: 01, 02, ..., 09, **Bundle**,

**B**1, **B**2,..., 9**B**8, 9**B**9, **BundleBundle**,

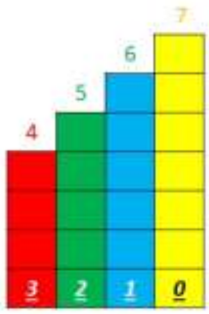
**BB**1,..., 2**BB**3**B**4,..., 9**BB**9**B**9, **BundleBundleBundle**,

**BBB**1,...



With blocks turned to hide the units behind:

**B** is marked with 1, **BB** with 2, **BBB** with 3, etc., singles with 0.




Later, this is a number formula T = 4567 = 4**BBB**5**BB**6**B**7 = 4x**B**^3 + 5x**B**^2 + 6x**B** + 7

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# Counting ten fingers & counting in tens

Children have fun when flexibly counting ten fingers in different ways:

- The Roman way: 01, 02, 03, HandLess1, **HAND**, Hand1, H2, H3, 2H-1, 2H, 2H1, 2H2
- The Viking way: 01, 02, 03, 04, HALF, 06, 07, less2, less1, **FULL**, 1left, 2left
- The modern way: 01, 02,..., 09, **ten**, ten1, ten2,..., 9ten8, 9ten9, **tenten**, tenten1,..., 2tenten3ten4,..., 9tenten9ten9, **tententen**, tententen1,...



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# Division, multiplication & subtraction as icons also

'From 9 take away **4s**' we write <u>9/4</u>
<u>iconizing</u> the sweeping away by a broom, called division.

'2 times stack **4s**' we write <u>2x4</u>
<u>iconizing</u> the stacking up by a lift called multiplication.

'From 9 take away 2 **4s**' to look for un-bundled we write <u>9 – 2x4</u>
<u>iconizing</u> the dragging away by a trace called subtraction.

So counting includes division and multiplication and subtraction:

Finding the bundles: 9 = 9/4 **4s**. Finding the un-bundled: 9 – 2x4 = 1.

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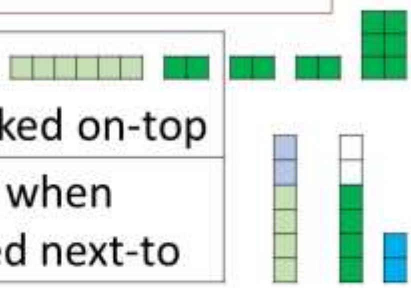
# Counting creates two counting formulas

| | |
|---|---|
| *ReCount*<br>**T = (T/B) x B** | from a total **T**, **T/B** times,<br>**Bs** is taken away and stacked on-top |
| *ReStack*<br>**T = (T–B) + B** | from a stack **T**, **T–B** is left when<br>**B** is taken away and placed next-to |



With formulas, a calculator can **predict** the counting-result 9 = 2**B**1 **4s**

| | |
|---|---|
| **9/4** | **2.some** |
| **9 – 2x4** | **1** |


*As sentences of the number language, **formulas predict***



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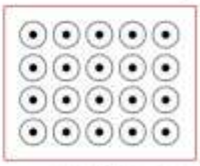
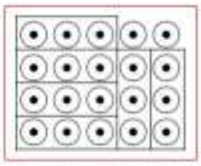
# To share Many, children take away bundles predicted by division, multiplication and subtraction

2 preschoolers share 6 cakes by taking away **2s** from 6, thus rooting division as counting in **2s**. ⊙⊙⊙⊙⊙⊙ → ⊙⊙⊙⊙ ⊙⊙ → ⊙⊙ ⊙⊙ ⊙⊙

- They smile when seeing that entering '6/2' allows a calculator to predict that they can take cakes 3 times.
- And when seeing that '4x5/3' predicts that 3 children can take cakes 6 times (or 6 cakes 1 time) when sharing 4 rows of 5 cakes.
- And when seeing that '4x5-6x3' predicts that 2 will be left.

| | |
|---|---|
| 6/2 | 3 |
| 4x5/3 | 6.some |
| 4x5-6x3 | 2 |





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# Question Guided Counting Curriculum

A question guided re-enchanting COUNTING curriculum could be named Mastering Many by counting, re-counting & double-counting.

- The design accepts that while 8 competences might be needed to learn university mathematics, only 2 are needed to Master Many: COUNTING & ADDING, motivating a twin curriculum.
- The corresponding pre-service or in-service question guided teacher education can be found at the MATHeCADEMY.net.
- Remedial micro-curricula for classes stuck in traditional mathematics can be found there also.



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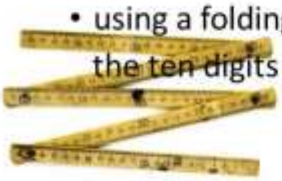
# Q01, icon making IIII → IIII → 4

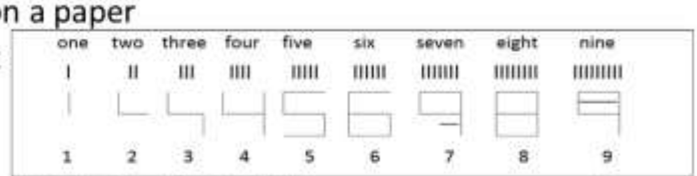
"The digit-icon 4 seems to be have four sticks. Does this apply to all digit-icons?"

We can change many ones to one icon with as many sticks or strokes as it represents, if written in a less sloppy way.

Follow-up activities could be:

- rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.
- rearranging sticks on a table or on a paper
- using a folding ruler to construct the ten digits as icons





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# Q02, counting sequences I

"How to count fingers?"

- Using **5s** as the bundle-size, five fingers can be counted as
  "01, 02, 03, 04, **Bundle**"
- And ten fingers can be counted as
  "01, 02, **Bundle** less2, **Bundle** -1, **Bundle**"
  "**Bundle**&1, **B**&2, 2**B** less2, 2**B**-1, 2**B**".

Follow-up activities could be counting the fingers in **3s** and **4s** and **7s**:

**T** = ten = 1**B**3 **7s** = 2**B**2 **4s** = 3**B**1 **3s** = 1**BB**1 **3s**.



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# Q02, counting sequences II

Counted as 1**B**, the bundle-number needs no icon. So counting a dozen cakes we say:

| *in* | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ | ⊙ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **4s** | 01 | 02 | 03 | **B** | 1**B**1 | 1**B**2 | 1**B**3 | 2**B** | 2**B**1 | 2**B**2 | 2**B**3 | 3**B** |
| **7s** | 01 | 02 | 03 | 04 | 05 | 06 | **B** | 1**B**1 | 1**B**2 | 1**B**3 | 1**B**4 | 1**B**5 |
| **tens** | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | **B** | 1**B**1 | 1**B**2 |

The number names, eleven and twelve come from 'one left' and 'two left' in Danish, (en / twe levnet), again showing that counting takes place by taking away bundles.

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# Q03, bundle-counting in icon-units I

## "How to count by bundling?"

Five fingers can be bundle-counted in pairs or triplets, allowing both an overload and an underload; and reported in a number-language sentence with a subject & a verb & a predicate as e.g. T = 2 **3s.**

IIIII • ǁ III • ǁ ǁ I • ǁ ǁ ǁ • ǁ ǁ I

T = 5 = 1Bundle3 **2s** = 2B1 **2s** = 3B-1 **2s** = 1BB1 **2s**

- Cup- & decimal-writing separates inside bundles from outside singles:

T = 5 = 1]3 **2s** = 2]1 **2s** = 3]-1 **2s** = 1]]0]1 **2s**

T = 5 = 1.3 **2s** = 2.1 **2s** = 3.-1 **2s** = 10.1 **2s**

Likewise, if counting in ten-bundles: T = 57 = 5**B**7 = 4**B**17 = 6**B**-3 **tens**



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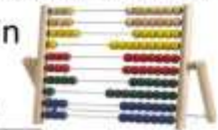
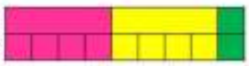
# Q03, bundle-counting in icon-units II

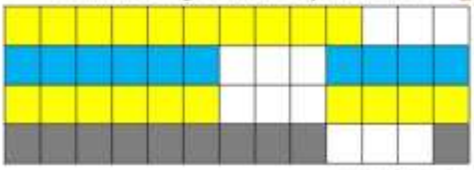
To count 9 in **4s**, we may bundle in a cup with 1 stick per bundle.
9 = IIIIIIIII = IIII IIII I = **II**]I = 2]1 **4s** = 2B1 **4s** = 2.1 **4s**

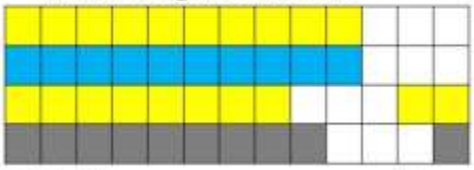
We may report with cup-, bundle- or decimal-writing,
or on a western **ABACUS** in






Outside geometry mode or Inside algebra mode





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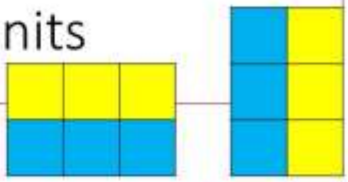
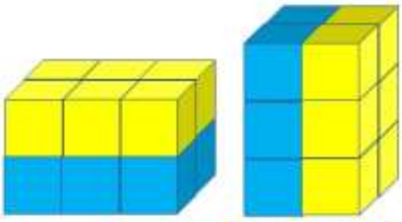
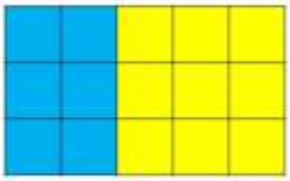
# Switching & uniting & splitting units



- Turning a 2D block will change the unit
  T = 2 **3s** = 2x3 → T = 3 **2s** = 3x2,
  So T = 2x3 = 3x2 *(The Commutative law)*



- Turning a 3D block will also change the unit
  So T = 2x(2x3) = (2x2)x3 *(The Associative law)*



- A block may split into two parts
  T = 3 **5s** = 3 **2s** + 3 **3s** or
  So T = 3x5 = 3x(2+3) = 3x2 + 3x3 *(The Distributive law)*



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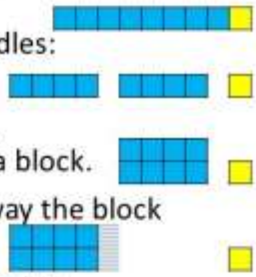
# Q04, calculators predict

## "Can a calculator predict a counting result?"



We may see the <u>division</u> sign as an icon for a broom wiping away bundles:
9/4 means 'from 9, wipe away bundles of 4s'.

- The calculator says '2.some', thus predicting it can be done 2 times.
  Now the <u>multiplication</u> sign iconizes a lift stacking the bundles into a block.
- Finally, the <u>subtraction</u> sign iconizes the trace left when dragging away the block to look for unbundled singles.
- With '9-2x4 = 1' the calculator predicts that 9 can be recounted as 2B1 **4s**.

| | |
|---|---|
| **9/4** | **2.some** |
| **9 – 2x4** | **1** |

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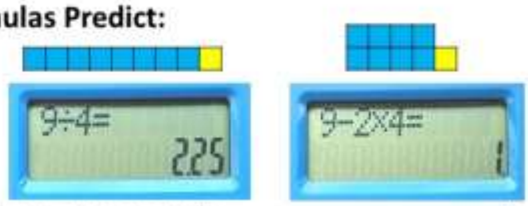
# Q04, counting creates 2 counting formulas

| | |
|---|---|
| *ReCount*<br>**T = (T/B) x B** | from a total **T**, **T/B** times,<br>**Bs** is taken away and stacked |
| *ReStack*<br>**T = (T–B) + B** | from a total **T**, **T–B** is left, when<br>**B** is taken away and placed next-to |

As sentences of the number language, **Formulas Predict:**
Predicting that **T** = 9 = 2.1 **4s:**

| | |
|---|---|
| **9/4** | **2.some** |
| **9 – 2x4** | **1** |



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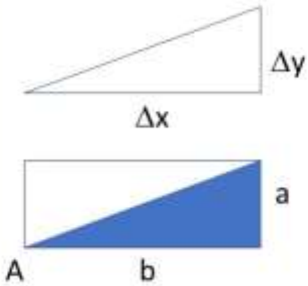
# Q04, the recounting formula is a core formula

**T = (T/B)*B** saying 'from T, T/B times, Bs can be taken away', is all over:

| | |
|---|---|
| Proportionality | y = k * x |
| Linearity | Δy = (**Δy/Δx**) * Δx = **m** * Δx |
| Local linearity | dy = (**dy/dx**) * dx = **y'** * dx |
| Trigonometry | a = (**a/b**) * b = **tanA** * b |
| Trade | $ = (**$/kg**) * kg = **price** * kg |
| Science | meter = (**meter/second**) * second<br>= **velocity** * second |



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## Q05, unbundled as decimals or negatives or fractions

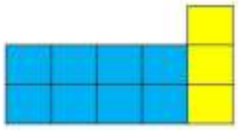
0.3 **4s** or 0.-1 **4s** or 3/4 **4s**

### "Where to put the unbundled singles?"

When counting by bundling, the unbundled singles can be placed

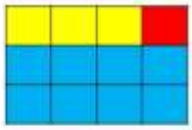
**NextTo** the block — counted as a block of **1s**

**OnTop** of the block — counted as a bundle — counted in bundles



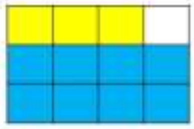
T = 2B3 **4s** = 2.3 **4s**

*A decimal number*



T = 3B-1 **4s** = 3.-1 **4s**

*A negative number*



T = 2 3/4 **4s**

*A fraction*



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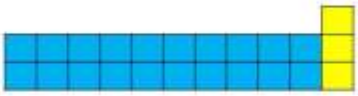
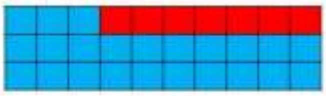
## Q05, counting in tens

### "Where to put the unbundled singles with tens?"

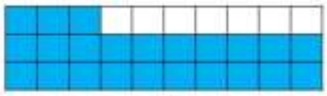
Counting in tens, an outside Total of 2 **tens** & 3 can be described inside as T = 23 if leaving out the unit, or as



T = 2.3 **tens**



T = 3.-7 **tens**



T = 2 3/10 **tens**

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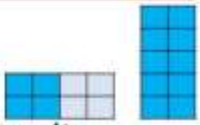
# Q06, prime & foldable bundle-units

## "When can blocks be folded in like bundles?"



The block T = 2 **4s** = 2 x 4 has 4 as the bundle-unit.

Turning over gives T = 4 **2s** = 4 x 2, now with 2 as the bundle-unit.

**4s** can be folded in another bundle as 2 **2s**, whereas 2s cannot.

(1 is not a bundle, nor a unit since a bundle-of-bundles stays as 1).

We call 2 a **prime bundle-unit** and 4 a **foldable bundle-unit**, 4 = 2 **2s**.

A block of 3 **2s** cannot be folded.

A block of 3 **4s** can be folded: T = 3 **4s** = 3 x (2 x 2) = (3 x 2) x 2 = 2 **3x2s**.

A number is called **even** if it can be written with 2 as the unit, else **odd**.



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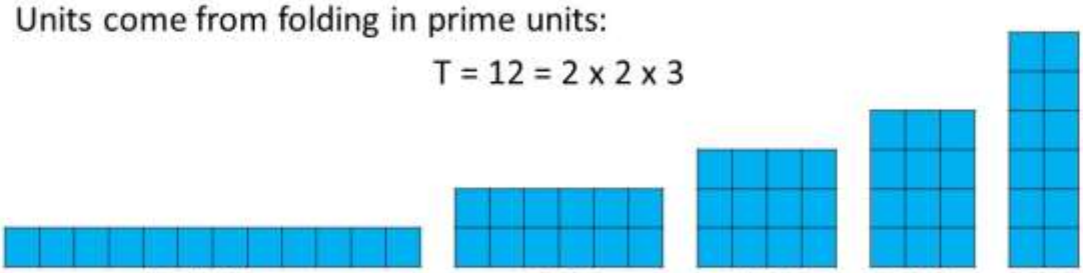
# Q07, finding possible units

## "What are possible units in T = 12?"

Units come from folding in prime units:

T = 12 = 2 x 2 x 3



T = 12 **1s** = 2 **2x3s** = 3 **2x2s** = 2x2 **3s** = 2x3 **2s**

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# Q08, recounting in a different unit

## "How to change a unit?"

The recount-formula allows changing the unit.

Asking T = 3 **4s** = ? **5s**, the recount-formula gives T = 3 **4s** = (3x4/5) **5s**.

Entering 3x4/5, the answer '2.some' shows that a block of 2 **5s** can be taken away.

With 3x4–2x5, the answer '2' shows that 3 **4s** can be recounted as 2B2 **5s** or 2.2 **5s**.

3 **4**s = IIII IIII IIII = IIII I III II II = IIIII IIIII II = 2B2 **5**s = 2.2 **5s**

| | |
|---|---|
| **3x4/5** | **2.**some |
| **3x4 – 2x5** | **2** |

**Change Unit = Proportionality**



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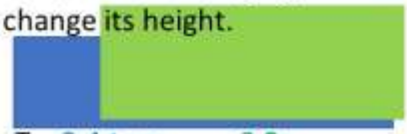
# Q09, recounting from tens to icons

## "How to change unit from tens to icons?"

Asking 'T = 2.4 **tens** = 24 = ? **8s**', we just recount 24 in **8s**:

T = 24 = (24/8)x8 = 3x8 = 3 **8s.**

| | |
|---|---|
| Formulated as an **equation** we use ***u*** for the unknown number, ***u***x8 = 24. Recounting 24 in 8s shows that ***u*** is 24/8 attained by moving 8 **to opposite side - with opposite sign** | To keep its size, a block changing its unit must also change its height.   |

***u*** x 8 = 24 = (24/8)x8

***u*** = 24/8 = 3

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## Q10, recounting from icons to tens (multiplication)   3 7s = ? tens

"How to change unit from icons to tens?"

Asking 'T = 3 **7s** = ? **tens**', the recount-formula cannot be used since the calculator has no ten-button. However, it gives the answer directly by using multiplication alone: T = 3 **7s** = 3x7 = 21 = 2.1 **tens**, only it leaves out the unit and the decimal point.

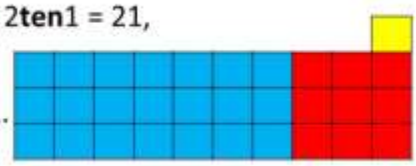
Alternatively, we may use 'less-numbers', so 7 = **ten** less 3

T = 3x7 = 3 x (**ten** less 3) = 3 x **ten** less 3x3 = 3**ten** less 9 = 2**ten**1 = 21,

or with 9 = **ten** less 1:

T = 3**ten** less (**ten** less1) = 2**ten** lessless 1 = 2**ten** & 1 = 21.

*showing that 'lessless' cancel out*





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## Recounting large numbers in or from tens: *same size, but new form*

Recounting 6 **47s** in **tens**   Recounting 476 in **7s**

*BundleWriting seprates **INSIDE** bundles from **OUTSIDE** singles*

T = 6 x 47 = 6 x 4**B**7
= 24**B**42
= 28**B**2
= 28.2
**tens**

T = 476 = 47.6 **tens**
= 47**B**6
= 42**B**56
= 6x7**B**8x7
= 68 x 7

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# Q11, double-counting in two units creates bridging **PerNumbers** & proportionality


## "How to double-count in two units?"

DoubleCounting in kg & $, we get **4kg = 5$** or
4kg **per** 5$ = 4kg/5$ = 4/5 kg/$ = a PerNumber.

With 4kg bridged to 5$ we answer questions by recounting in the per-number.

**Questions**:

| **7kg = ?$** | **8$ = ?kg** |
|---|---|
| 7kg = (7/4) x 4kg<br>= (7/4) x 5$ = 8.75$ | 8$ = (8/5) x 5$<br>= (8/5) x 4kg = 6.4kg |

**Answer**: *Recount in the PerNumber (Proportionality)*



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# Q12, double-counting in the same unit creates fractions


## "How to double-count in the same unit?"

Double-counted in the same unit, per-numbers are fractions, 2$ per 9$ = 2/9, or percentages, 2 per 100 = 2/100 = 2%.

To find a fraction or a percentage of a total, again we just recount in the per-number.

- **Taking 3 per 4 = taking ? per 100.** With 3 bridged to 4, we recount 100 in 4s:
100 = (100/4)*4 giving (100/4)*3 = 75, and 75 per 100 = 75%.
- **Taking 3 per 4 of 60 gives ?.** With 3 bridged to 4, we recount 60 in 4s:
60 = (60/4)*4 giving (60/4)*3 = 45.
- **Taking 20 per 100 of 60 gives ?.** With 20 bridged to 100, we recount 60 in 100s:
60 = (60/100)*100 giving (60/100)*20 = 12 .

We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number.

# Q12, enlarging or shortening units

## "How to enlarge or shorten units in fractions?"

Taking 2/3 of 12 means taking 2 per 3 of 12.

With 2 bridged to 3, we recount 12 in **3s**, 12 = (12/3)*3 = 4*3

So 4 times we can take 2, i.e. 8 of the 12. Thus 2 per 3 = 8 per 12.

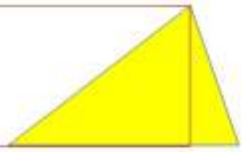
This may be used for enlarging or shortening fractions by inserting or removing the same unit above and below the fraction line:

$\frac{2}{3} = \frac{2\ \mathbf{4s}}{3\ \mathbf{4s}} = \frac{2*4}{3*4} = \frac{8}{12}$ ● $\frac{8}{12} = \frac{2*4}{3*4} = \frac{2\ \mathbf{4s}}{3\ \mathbf{4s}} = \frac{2}{3}$ ● $\frac{12abc}{8a} = \frac{3*4*a*b}{2*4*a} = \frac{3*b\ \mathbf{4as}}{2\ \mathbf{4as}} = \frac{3b}{2}$





# Q13, recounting the sides in a block



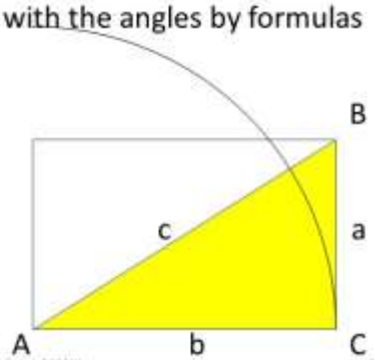
Geometry means to measure earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula. And connected with the angles by formulas recounting a side in the other side or in the diagonal:

A+B+C = 180

a*a + b*b = c*c (the Pythagoras formula)

sinA = a/c; cosA = b/c; tanA = a/b = Δy/Δx = gradient

Circle: circum./diam. = π = **n*tan(180/n)** for n large



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## Q14, double-counting gives per-numbers in STEM multiplication formulas I

STEM (Science, Technology, Engineering, Math) typically contains multiplication formulas with per-numbers coming from double-counting.

Examples:

- kg = (**kg/cubic-meter**) x cubic-meter = **density** x cubic-meter
- force = (**force/square-meter**) x square-meter = **pressure** x square-meter
- meter = (**meter/sec**) x sec = **velocity** x sec
- energy = (**energy/sec**) x sec = **Watt** x sec
- energy = (**energy/kg**) x kg = **heat** x kg



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## Q14, double-counting gives per-numbers in STEM multiplication formulas II

Extra STEM examples:

- gram = (**gram/mole**) x mole = **molar mass** x mole;
- $\Delta$ momentum = ($\Delta$ **momentum/sec**) x sec = **force** x sec;
- $\Delta$ energy = ($\Delta$ **energy/ meter**) x meter = **force** x meter = work;
- **energy/sec** = (energy/charge) x (charge/sec) or **Watt** = **Volt** x **Amp**;
- dollar = (**dollar/hour**) x hour = **wage** x hour;
- dollar = (**dollar/meter**) x meter = **rate** x meter
- dollar = (**dollar/kg**) x kg = **price** x kg.

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# Q15, navigating on a squared paper

First steps into coordinate geometry, to always keep algebra and geometry together.

| **"Collect treasures on the rocks "** | **"Plan a trip to treasure island"** |
|---|---|
| Three rocks are placed on a squared paper. | Departure point: 3cm out & 2cm up |
| The rocks have the values -1, 1, and 2. | Destination point: 7cm out & 4cm up. |
| A journey begins in the midpoint. | Plan a voyage with 1 out per day. |
| Two dices tell the out- and up- change, | How many days before reaching the island? |
| where odd numbers are negative. | What is your position after 2 days? |
| How many points before reaching the edge? | What is your position after n days? |
| Predict and measure angles on the journey. | What is the angle traveled? |



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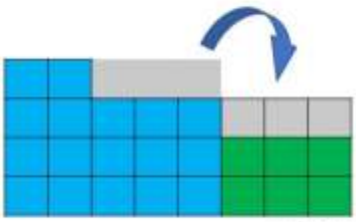
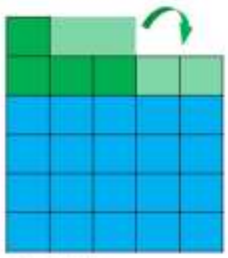
# Counted & recounted, Totals can be added

| **BUT: NextTo →** | **or OnTop ↑** |
|---|---|
| 4 5s + 2 3s = 3B2 8s | 4 5s + 2 3s = 4 5s + 1B1 5s = 5B1 5s |
| The areas are integrated<br>*Adding areas = Integration* | The units are changed to be the same<br>*Change unit = Proportionality* |





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# Four ways to unite into a Total

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2 + 4***B** + 5 (a polynomial) shows the four ways to add: +, *, ^, next-to block-addition (integration).
Addition and multiplication add changing and constant unit-numbers.
Integration and power add changing and constant per-numbers.
We might call this beautiful simplicity the 'Algebra Square'.

| **Operations unite** | **changing** | **constant** |
|---|---|---|
| **Unit-numbers** *m, s, $, kg* | **T = a + n** | **T = a * n** |
| **Per-numbers** *m/s, $/kg, m/(100m) = %* | **T = ∫ a dn** | **T = a^n** |



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# Five ways to split a Total

The 4 uniting operations (+, *, ^, ∫) each has a reverse splitting operation:
Addition has subtraction (–), and multiplication has division (/).
Power has factor-finding (root, √) and factor-counting (logarithm, log).
Integration has per-number finding (differentiation dT/dn = T').

Reversing operations is solving equations, done by moving to **opposite side** with **opposite sign**.

| **Operations unite** / *split into* | **changing** | **constant** |
|---|---|---|
| **Unit-numbers** *m, s, $, kg* | **T = a + n** *T – a = n* | **T = a * n** *T/n = a* |
| **Per-numbers** *m/s, $/kg, m/(100m) = %* | **T = ∫ a dn** *dT/dn = a* | **T = a^n** $log_a T = n$, $\sqrt[n]{T} = a$ |

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# Question Guided Adding Curriculum

A question guided re-enchanting ADDING curriculum could be named '<u>Mastering Many by uniting and splitting constant and changing unit-numbers and per-numbers</u>'.

- A corresponding pre-service and in-service question guided teacher education can be found at the MATHeCADEMY.net.
- Remedial curricula for classes stuck in traditional mathematics can be found there also.



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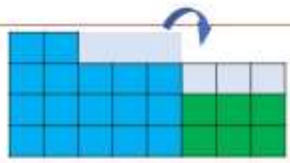
# Q21, next-to addition



"With T1 = 4 **5s** and T2 = 2 **3s**, what is T1+T2 when added next-to as **8s**?"

<u>Outside</u>, next-to addition geometrically means adding areas. Next-to addition is also called integral calculus.

<u>Inside</u>, the recount formula algebraically predicts the result. Here multiplication precedes addition.

T = (T/**B**) x **B**

= ( (4x5 + 2x3)/8) x 8 = 3.2 **8s**

| | |
|---|---|
| **(4x5 + 2x3)/8** | **3.some** |
| **(4x5 + 2x3) – 3x8** | **2** |

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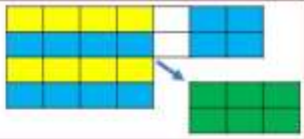
# Q22, reversed next-to addition



"If T1 = 2 **3s** and T2 add next-to as 4 **7s**, what is T2?"

<u>Outside</u>, we remove the initial block T1 and recount the rest in **4s**.

Thus reversed next-to addition geometrically means subtracting areas.

Reversed next-to addition is also called differential calculus.

<u>Inside</u>, the recount formula algebraically predicts the result.

Here subtraction precedes division; which is natural as reversed integration.

T2 = (T2/**B**) x **B**

= ( (4x7 – 2x3)/4) x 4 = 5.2 **4s**

| | |
|---|---|
| **(4x7 – 2x3)/4** | **5.some** |
| **(4x7 – 2x3) – 5x4** | **2** |



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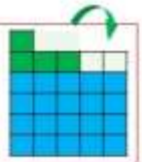
# Q23, on-top addition

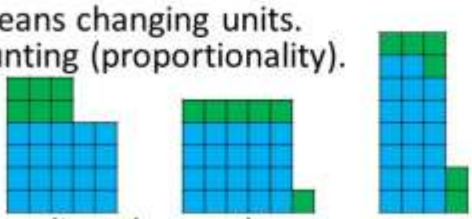


"With T1 = 4 **5s** and T2 = 2 **3s**, what is T1+T2 when added on-top?"

<u>Outside</u>, on-top addition geometrically means changing units.
On-top addition thus often involves recounting (proportionality).

T = 4 **5s** + 2 **3s** = 4 **5s** + 1.1 **5s** = 5.1 **5s**

T = 4 **5s** + 2 **3s** = 6.2 **3s** + 2 **3s** = 8.2 **3s**



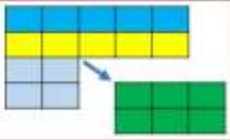
<u>Inside</u>, the recount formula algebraically predicts the result.
Here again, multiplication precedes addition.

T = (T/**B**) x **B**

= ( (4x5 + 2x3)/5) x 5 = 5.1 **5s**

| | |
|---|---|
| **(4x5 + 2x3)/5** | **5.some** |
| **(4x5 + 2x3) – 5x5** | **1** |

# Q24, reversed on-top addition



## "T1 = 2 3s and how many **5s** (T2) add on-top as 4 **5s**?"

Outside, we remove the initial block T1 and recount the rest in **5s**.

Thus reversed next-to addition geometrically means subtracting areas.

Reversed on-top addition is also called differential calculus.

Inside, the recount formula algebraically predicts the result.

Here again, subtraction precedes division.

T2 = (T2/**B**) x **B**

= ( (4x5 – 2x3)/5) x 5 = 2.4 **5s**

| | |
|---|---|
| **(4x5 – 2x3)/5** | **2.some** |
| **(4x5 – 2x3) – 2x5** | **4** |



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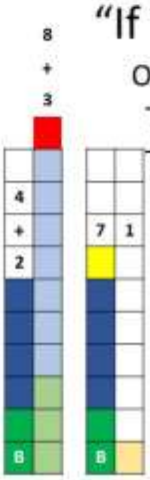
# Q25, adding tens on-top

## "If T1 = 23 and T2 = 48, what is T1+T2 as **tens**?"

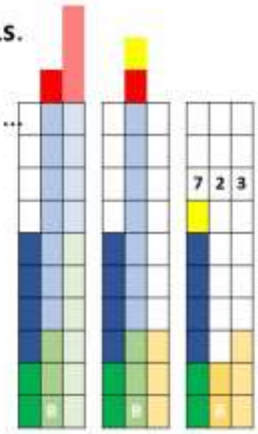



Outside and inside, we recount overloads by changing 1 **tens** to 10 **1s**.

T = 23 + 48 = 2**ten**3 + 4**ten**8 = 6**ten**11 = 6**ten**1**ten**1 = 7**ten**1 =71

T = 236 + 487 = 2**tenten**3**ten**6 + 4**tenten**8**ten**7 = 6**tenten**11**ten**13 = ...

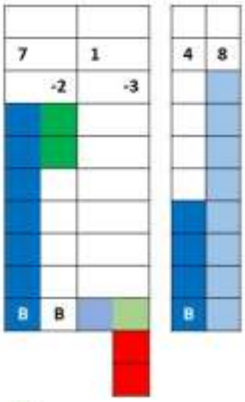
| | |
|---|---|
| T1+T2 = 23 + 48 | T = **2**36 + 487 |
| = 2**B**3 + 4**B**8 | = 2**BB**3**B**6 + 4**BB**8**B**7 |
| = 6**B**11 | = 6**BB**11**B**13 |
| = 7**B**1 | = 6**BB**12**B**3 |
| = 71 | = 7**BB**2**B**3 |
| | = 723 |



# Q26, subtracting tens on-top

## "If T1 = 23 and T2 add to T = 71, what is T2 as **tens**?"

Outside and inside, we recount underloads by changing 1 **tens** to 10 **1s.**




T = 71 – 23 = 7**ten**1 – 2**ten**3 = 5**ten**-2 = 4**ten**8 = 48

T = 956 – 487 = 9**tenten**5**ten**6 – 4**tenten**8**ten**7 = 5**tenten**-3**ten**-1 = …

| | |
|---|---|
| T2 = 71 – 23<br>= 7**B**1 – 2**B**3<br>= 5**B**-2<br>= 4**B**8<br>= 48 | T2 = 956 – 487<br>= 9**BB**5**B**6 – 4**BB**8**B**7<br>= 5**BB**-3**B**-1<br>= 4**BB**7**B**-1<br>= 4**BB**6**B**9<br>= 469 |



# Q27, from icons to tens, multiplication

A multiplication table recounts icon-blocks in ten-blocks: T = 7 **3s** = ? **Tens.**

To recount 7 **3s** in **tens** we can use that 7 is **ten** less3, and 3 is 5 less2:

From the 10 **5s** we remove 3 **5s** (/) and 2 **tens** (\).

But then we must add the 3 **2s** that was removed twice.

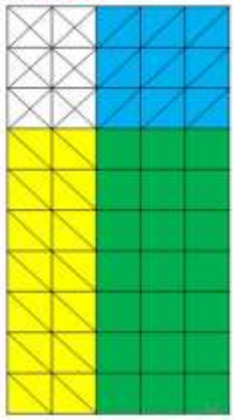
T = 7x3 = (ten – 3)x(5 – 2) = tenx5 – 3x5 – tenx2 + 3x2

= 50 – 15 – 20 + 6 = 21.

Shown on a western ten by ten abacus as a 10 by 5 block.

This roots the algebra formula showing that – x – is +

***(a – b)* x *(c – d)* = *a* x *c* – *a* x *d* – *b* x *c* + *b* x *d***



# Q27, multiplication tables

## "What is 7 **8s** recounted in **tens**?"

Using underload-numbers after 5, we recount to remove underloads:

| | |
|---|---|
| T = 7 x 8 = 7 x **B**-2 = 7**B**-14 = 7**B** – 1**B**4 = 6**B**-4 = 5**B**6 = 56 | T = 7 x 8 = **B**-3 x **B**-2 = 1**BB** – 3**B** – 2**B** + 6 = 10**B** – 3**B** – 2**B** + 6 = 5**B**6 = 56 |

| | 2 | 3 | 4 | 5 | B-4 | B-3 | B-2 | B-1 |
|---|---|---|---|---|---|---|---|---|
| **2** | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **3** | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| **4** | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| **5** | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
| **6** | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 |
| **7** | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 |
| **8** | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |
| **9** | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 |

| | 2 | 3 | 4 | 5 | B-4 | B-3 | B-2 | B-1 |
|---|---|---|---|---|---|---|---|---|
| **2** | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **3** | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| **4** | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| **5** | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
| **B-4** | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 |
| **B-3** | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 |
| **B-2** | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |
| **B-1** | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 |





# Q27, Recounting BundleBundles in tens (squares: ..., 4 **4s** = ? **tens,** 5 **5s** = ? **tens**, ...)

Using the multiplication table, we recount the different bundle-bundles (called squares) in **tens**:

S4 = 4 **4s** = 4x4 = 16

S5 = 5 **5s** = 5x5 = 25, etc.

We see that to get to the next square we add the sides twice, + 1:

5*5 = 4*4 + 2*4 + 1, or with 4 = n:

(n+1)*(n+1) = n*n + 2*n + 1, or

$(n+1)^2 = n^2 + 2*n + 1$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | | | | | | | | | |
| **2** | | 4 | | | | | | | | |
| **3** | | | 9 | | | | | | | |
| **4** | | | | 16 | | | | | | |
| **5** | | | | | 25 | | | | | |
| **6** | | | | | | 36 | | | | |
| **7** | | | | | | | 49 | | | |
| **8** | | | | | | | | 64 | | |
| **9** | | | | | | | | | 81 | |
| **10** | | | | | | | | | | 100 |

# Q27, recounting from icons to tens (multiplication)

Recount 43 **7s** in **tens:**

T = 43x7 = 301 = 30.1 tens

Horizontally we write 43 as 4ten3 or 4**B**3.
Vertically, we write 7.
Multiplying, we get 28**B** and 21.
So, T = 43x7 = 28**B**21 = 30**B**1 = 301.

With underload, 43 is 5ten-7 or 5**B**-7.
Vertically, we write 7.
Multiplying, we get 35**B** and -49.
So, T = 43x7 = 35**B**-49 = 30**B**1 = 301.

overload

| | | |
|---|---|---|
| **4B** | 3 | 43x |
| ? | ? | 7 |
| **4B** | 3 | 43x |
| 28**B** | 21 | 7 |
| 30**B** | 1 | 301 |

underload

| | | |
|---|---|---|
| **5B** | -7 | 43x |
| ? | ? | 7 |
| **5B** | -7 | 43x |
| 35**B** | -49 | 7 |
| 30**B** | 1 | 301 |





# Q27, recounting 27 43s in tens (multiplication)

Recounting 27 **43s** in **tens:** 27 x 43 = 1161 = 116.1 **tens**

overload

| | | | |
|---|---|---|---|
| | 2B | 7 | 27x43 |
| | ? | ? | 4B |
| | ? | ? | 3 |
| **?BB** | ?**B** | ? | |
| | 2B | 7 | 27x43 |
| | 8BB | 28B | 4B |
| | 6B | 21 | 3 |
| **8BB** | 34**B** | 21 | |
| **8BB** | 36**B** | 1 | |
| **11BB** | 6**B** | 1 | 1161 |

underload

| | | | |
|---|---|---|---|
| | 3B | -3 | 27x43 |
| | ? | ? | 4B |
| | ? | ? | 3 |
| **?BB** | ?**B** | ? | |
| | 3B | -3 | 27x43 |
| | 12BB | -12B | 4B |
| | 9B | -9 | 3 |
| **12BB** | -3**B** | -9 | |
| **12BB** | -4**B** | 1 | |
| **11BB** | 6**B** | 1 | 1161 |

underload

| | | | |
|---|---|---|---|
| | 3B | -3 | 27x43 |
| | ? | ? | 5B |
| | ? | ? | -7 |
| **?BB** | ?**B** | ? | |
| | 3B | -3 | 27x43 |
| | 15BB | -15B | 5B |
| | -21B | 21 | -7 |
| **15BB** | -36**B** | 21 | |
| **15BB** | -34**B** | 1 | |
| **11BB** | 6**B** | 1 | 1161 |

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# Q28, recounting from tens to icons (division)

Recount 30.1 **tens** in **7s**: 301/7 = 43

Recount 30.6 **tens** in **7s**: 306/7 = 43 5/7

overload

| | | |
|---|---|---|
| **4B** | 3 | 43x |
| **28B** | 21 | 7 |
| **30B** | 1 | 301 |
| ? | ? | ?x |
| ? | ? | 7 |
| **30B** | 1 | 301 |

underload

| | | |
|---|---|---|
| **5B** | -7 | 43x |
| **35B** | -49 | 7 |
| **30B** | 1 | 301 |
| ? | ? | ?x |
| ? | ? | 7 |
| **30B** | 1 | 301 |

Multiplying is top-down; division is bottom-up.

Below, we write 301 = 30**B**1. Above we recount 301 as 28**B**21 to count in **7s**. So, T = 301 = 43x7.

Below, we write 306 as 28**B**26 first, then as 28**B**21 + 5 to count in **7s**. So, T = 306 = 43x7 +5.

| | | |
|---|---|---|
| 4B | 3 +5/7 | 43 5/7x |
| 28B | 21+5 | 7 |
| 30B | 6 | 306 |
| ? | ? | ?x |
| ? | ? | 7 |
| 30B | 6 | 306 |



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# Q28, recounting 1161 in **43s** (division)

| | | | |
|---|---|---|---|
| | 2B | 7 | 27x |
| | 8BB | 28B | 4B |
| | 6B | 21 | 3 |
| **8BB** | **34B** | 21 | |
| **8BB** | **36B** | 1 | |
| **11BB** | **6B** | 1 | 1161 |
| | ? = 2B | ? | ?x |
| | 8BB | 30B | 4B |
| | 6B | 1 | 3 |
| ? | ? | ? | |
| **8BB** | **36B** | 1 | |
| **11BB** | **6B** | 1 | 1161 |

**Overload**

**Underload**

| | | | |
|---|---|---|---|
| | 3B | -3 | 27x |
| | 8BB | -12B | 4B |
| | 9B | -9 | 3 |
| **12BB** | **-3B** | -9 | |
| **12BB** | **-4B** | 1 | |
| **11BB** | **6B** | 1 | 1161 |
| | ? = 3B | ? | ?x |
| | 12BB | -13B | 4B |
| | 9B | 1 | 3 |
| ? | ? | ? | |
| **12BB** | **-4B** | 1 | |
| **11BB** | **6B** | 1 | 1161 |

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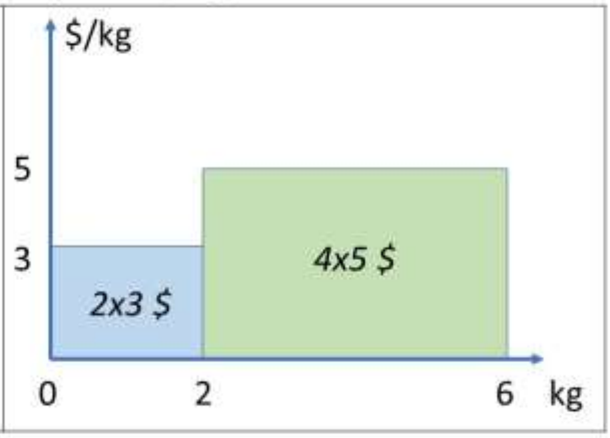
# Q29, adding PerNumbers as areas (integration)

"2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg?"

2 kg at 3 $/kg
+ 4 kg at 5 $/kg

(2+4) kg at ? $/kg

- Unit-numbers add on-top.
- Per-numbers add next-to as **areas** under the per-number graph. Here multiplication precedes addition.



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# Q30, subtracting PerNumbers (differentiation)

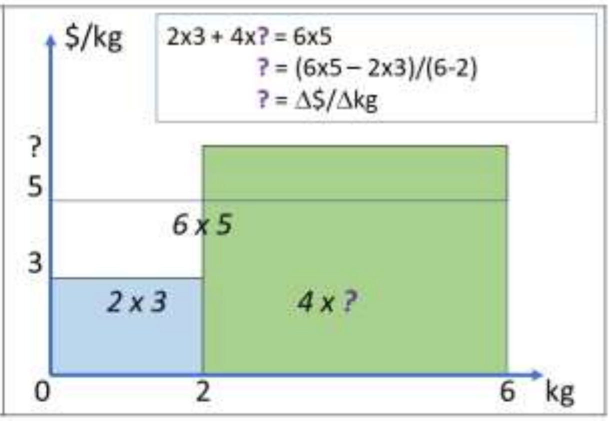
"2kg at 3$/kg + 4kg at **what** = 6kg at 5$/kg?"

2 kg at 3 $/kg
+ 4 kg at ? $/kg

6 kg at 5 $/kg

Outside, we remove the initial 2x3 block and recount the rest in 4s. Geometrically, reversed per-number addition means subtracting areas to be reshaped, called differential calculus.

Inside, the recount-formula algebraically predicts the result. Here subtraction (giving a change, Δ) precedes division.



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# Never add without units, the fraction paradox

| The Teacher | The Students |
|---|---|
| What is 1/2 + 2/3? | Well, 1/2 + 2/3 = (1+2)/(2+3) = 3/5 |
| No! 1/2 + 2/3<br>= 3/6 + 4/6<br>= 7/6 | But 1/2 of 2 cakes + 2/3 of 3 cakes<br>is 1+2 of 2+3 cakes, i.e. 3/5 of 5 cakes!<br>How can it be 7 cakes out of 6 cakes? |
| Inside this classroom<br>1/2 + 2/3 **IS** 7/6 ! |  |

Fractions are not numbers, but operators, needing numbers to become numbers.
**2+3 IS 5!** No, 2weeks + 3days is 17days; and 2m + 3cm = 203cm.
**2*3 IS 6!** Yes, since 3 is the unit, and 2 **3s** can be recounted to 6 1s.
*Adding without units: MatheMatism.*

*Mixing English and metric units made NASA's Mars Climate Orbiter fail in 1999.*



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# Q31, adding unspecified numbers

"Only add like units, so how to add $T = 4ab^2 + 6abc$?"

Here units come from folding (factoring):

$T = 4ab^2 + 6abc = T1 + T2$

$= 2 * 2 * a * b * b \quad + \quad 2 * 3 * a * b * c$

$= 2 * b * (2 * a * b) \quad + \quad 3 * c * (2 * a * b)$

$= (2b+3c) * \mathbf{2ab}$

$= 2b+3c$ **2abs**

*a factor-filter*

| | | | | | |
|---|---|---|---|---|---|
| $4ab^2$ | 2 | 2 | *a* | *b* | *b* |
| $6abc$ | 2 | 3 | *a* | *b* | *c* |
| **unit** | **2** | | ***a*** | ***b*** | |
| T1: | | 2 | | | *b* |
| T2: | | 3 | | | *c* |

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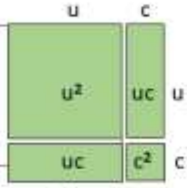
# Q31, multiplying unspecified numbers




## "How to multiply unspecified two-digit numbers T1 and T2?"

T1*T2 = (2u+3)*(4u-5)

| | 2u | +3 | T1*T2 |
|---|---|---|---|
| | ? | ? | 4u |
| | ? | ? | -5 |
| **?uu** | ?**u** | ? | |
| | 2u | +3 | T1*T2 |
| | 8uu | +12u | 4u |
| | -10u | -15 | -5 |
| 8u*u | +2u | -15 | |
| 8u² | +2u | -15 | |

T1*T2 = (u+c)*(u-c)

| | u | +c | T1*T2 |
|---|---|---|---|
| | ? | ? | u |
| | ? | ? | -c |
| **?uu** | ?**u** | ? | |
| | u | +c | T1*T2 |
| | uu | +cu | u |
| | -cu | -cc | -c |
| **uu** | | -cc | |
| **u²** | | **-c²** | |

T1*T2 = (u+c)*(u+c) = $(u+c)^2$

| | u | +c | T1*T2 |
|---|---|---|---|
| | ? | ? | u |
| | ? | ? | +c |
| **?uu** | ?**u** | ? | |
| | u | +c | T1*T2 |
| | uu | +cu | u |
| | +cu | +cc | +c |
| **uu** | **+2cu** | +cc | |
| **u²** | **+2cu** | **-c²** | |



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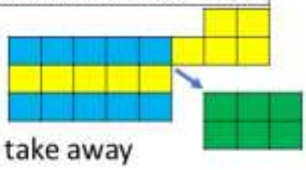
# Reversed Addition = Solving Equations

| OppoSite Side with OppoSite Sign | | NextTo |
|---|---|---|
| **2 x ? = 8** = (8/2) x 2 | **2 + ? = 8** = (8-2) + 2 | 2 **3s** + ? **5s** = 3.2 **8s** |
| ? = 8/2 | ? = 8-2 | ? = (3.2 **8s** – 2 **3s**)/5 |
| *Solved by ReCounting* | *Solved by ReStacking* | *Solved by differentiation: (T-T1)/5 = ΔT/5* |



**Hymn to Equations**
Equations are the best we know,
they are solved by isolation.
But first, the bracket must be placed
around multiplication.

We change the sign and take away
and only x itself will stay.
We just keep on moving, we never give up.
So feed us equations, we don't want to stop!

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## Solving equations by recounting, we may bracket Group Theory from Abstract Algebra

ManyMath

| | |
|---|---|
| 2 x *u* = 8 = (8/2) x 2 | Solved by re-counting 8 in 2s |
| *u* = 8/2 = 4 | Move: Opposite Side with OppoSite Sign |

MetaMath (Don't test, but DO remember the bi-implication arrows)

| | | |
|---|---|---|
| | 2 x *u* = 8 | Multiplication has 1 as its **neutral element**, and 2 has ½ as its **inverse element** |
| ⇕ | (2 x *u*) x (½) = 8 x (½) | Multiplying 2's inverse element ½ to both number-names |
| ⇕ | (*u* x 2) x (½) = 4 | Applying the **commutative** law to u x 2; 4 is the short number-name for 8 x ½ |
| ⇕ | *u* x (2 x (½)) = 4 | Applying the **associative** law |
| ⇕ | *u* x 1 = 4 | Applying the definition of an inverse element |
| ⇕ | *u* = 4 | Applying the definition of a neutral element. *With arrows a test is not needed.* |



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## Conclusions

*What Mastery of Many does the child have already?*

- Children typically see Many as blocks with a number af bundles, and use flexible numbers with units and with over- or underloads

*In ManyMath, BLOCKS are fundamental:*

- in numbers: 456 = three blocks

- in algebra: adding blocks next-to or on-top

- in geometry: recounting half-blocks

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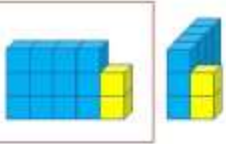
# The child's own twin math curriculum



1) Digits are (sloppy) icons, with as many sticks as they represent.

2) Totals are counted by bundling, giving <u>outside</u> geometrical multi-blocks, & (when turned to hide the units behind) <u>inside</u> algebraic bundle-numbers.

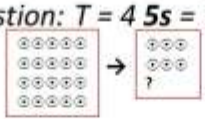
3) Operations are icons, showing the 3 counting steps: Removing bundles & stacking bundles & removing stacks to find the unbundled.

4) The operation order is division first, then multiplication, then subtraction. Addition next-to & on-top comes later after totals are counted & re-counted.

5) Counting & re-counting & double-counting is big fun, when predicted by a calculator with the recount formula: **T = (T/B)xB** (from T, T/B times, Bs can be taken away)

*Question: T = 4 **5s** = ? **3s*** • *Answer: T = 4 **5s** = 6**B**2 **3s*** • *Prediction:*

| | |
|---|---|
| 4x5/3 | 6.some |
| 4x5 – 6x3 | 2 |



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# Comparing with a traditional math curriculum I

A traditional curriculum: operations on specified and unspecified numbers.

- Digits are given directly as symbols, without letting children discover digits as icons with as many strokes or sticks as they represent.
- Numbers are one-dimensional line-numbers with digits respecting a place value system, without letting children discover the thrill of two-dimensional bundling and stacking counting both <u>singles</u> and <u>bundles</u> and <u>bundles-of-bundles</u> etc., and that includes the unit.
- Seldom, if ever, 0 is included as '01, 02, 03' in the counting sequence to show the importance of bundling.

## Comparing with a traditional math curriculum II

- Never children are told that eleven and twelve comes from the Vikings, counting '(ten and) 1 left', '(ten and) 2 left'.
- Never children use full number-language sentences, T = 2 **5s**, including both a subject & a verb & a predicate with a unit.
- Seldom children are asked to describe numbers after ten as 1**B**4 **tens** or 1**ten**4 or 1.4 **tens** with a unit and with a decimal point separating bundles and unbundled singles.
- Seldom 17 is recounted as 2**B**-3 or 2.-3 **tens**. Nor is 24 recounted as 1**B**14 **tens** or 3**B**-6 **tens**.





## Comparing with a traditional math curriculum III

- Never it respects the natural order of operations. Instead it turns the order around by giving addition without units priority over subtraction & multiplication & division.
- In short, children never experience the enchantment of counting, re-counting and double-counting Many before being forced to add on-top only, thus neglecting next-to addition.

So, re-enchanting Many is the goal of the twin curriculum in Mastery of Many through respecting and developing the children's existing mastery and quantitative competence.

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## Proportionality shows the variety of mastery of Many I

Proportionality, **Q1**: "2kg costs 5$, what does 7kg cost"; **Q2**: "What does 12$ buy?"

1) Regula de Tri (rule of three)

Re-phrase with shifting units, the unknown at last. From behind, first multiply then divide.

**Q1**: '2kg cost 5$, 7kg cost ?$'. Multiply-then-divide gives the $-number 7x5/2 = 17.5.

**Q2**: '5$ buys 2kg, 12$ buys ?kg'. Multiply-then-divide gives the kg-number 12x2/5 = 4.8.

2) Find the unit

**Q1**: 1kg costs 5/2$, so 7kg cost 7x(5/2) = 17.5$. **Q2**: 1$ buys 2/5kg, so 12$ buys 12x(2/5) = 4.8kg

3) Cross multiplication

**Q1**: 2/5 = 7/*u*, so 2**u* = 7*5, *u* = (7*5)/2 = 17.5. **Q2**: 2/5 = *u*/12, so 5**u* = 12*2, *u* = (12*2)/5 = 4.8

4) 'Re-counting' in the 'per-number' 2kg/5$ coming from 'double-counting' the total T.

**Q1**: T = 7kg = (7/2)x2kg = (7/2)x5$ = 17.5$; **Q2**: T = 12$ = (12/5)x5$ = (12/5)x2kg = 4.8kg.



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## Proportionality shows the variety of mastery of Many II

5) Modeling with linear functions using group theory from abstract algebra.

- A linear function $f(x) = c*x$ from the set of positive kg-numbers to the set of positive $-numbers, has the domain $DM = \{x \in R \mid x>0\}$.
- Knowing that $f(2) = c*2 = 5$, this equation is solved by multiplying with the inverse element to 2 on both sides, and applying the associative law, and the definition of an inverse element, and of the neutral element under multiplication:
  $c*2 = 5$ • $(c*2)*½ = 5*½$ • $c*(2*½) = 5/2$ • $c*1 = 5/2$ • $c = 5/2$.
- With $f(x) = 5/2*x$, the inverse function is $f^{-1}(x) = 2/5*x$.
- With 7kg, the answer is $f(7) = 5/2*7 = 17.5$$.
- With 12$, the answer is $f^{-1}(12) = 2/5*12 = 4.8$kg.

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# Main parts of a ManyMath curriculum

**Primary School – respecting and developing the Child's own 2D NumberLanguage**

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- BundleCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: divide, multiply, subtract, add on-top & next-to

**Middle school – integrating algebra and geometry, the content of the label 'math'**

- DoubleCounting produces PerNumbers and fractions as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always, so length becomes change and vv.

**High School – integrating algebra and geometry to master CHANGE**

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus



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# Question guided teacher education

## MATHeCADEMY.net

Teaches Teachers to Teach MatheMatics as ManyMath, a Natural Science about MANY.

To learn Math, Count & Add MANY, using the CATS method:

Count & Add in Time & Space

- Primary: C1 & A1 & T1 & S1
- Secondary: C2 & A2 & T2 & S2

MATHeCADEMY.net
a VIRUSeCADEMY:
*ask Many, not the Instructor*

**SUMMARY**

| | QUESTIONS | ANSWERS |
|---|---|---|
| C1 COUNT | How to count Many?<br>How to recount 8 in 3s: T= 8 = ? 3s<br>How to recount 6kg in $: T=6kg=?$<br>How to count in standard bundles? | By bundling and stacking the total T predicted by T = (T/b)*b<br>T= 8= 7*3= 7 3s, T= 8=(8/3)*3 = 2*3 +2 = 2*3 +2/3*3 = 2 2/3*3<br>If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$<br>Bundling bundles gives a multiple stack, a stock or polynomial:<br>T = 423 = 4BundleBundle+2Bundle+3 = 4tenten2ten3 = 4*B^2+2*B+3 |
| C2 COUNT | How can we count possibilities?<br>How can we predict unpredictable numbers? | By using the numbers in Pascal's triangle<br>We 'post-dict' that the average number is 8.2 with the deviation 2.3.<br>We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation) |
| A1 ADD | How to add stacks concretely?<br>T=27+16= 2ten7+1ten6= 3ten13=?<br>How to add stacks abstractly? | By restacking overloads predicted by the restack-equation T= (T-b)+b<br>T = 27+16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43<br>Vertical calculation uses carrying. Horizontal calculation uses FOIL |
| A2 ADD | What is a prime number?<br>What is a per-number?<br>How to add per-numbers? | Fold-numbers can be folded: 10=2fold5. Prime-numbers cannot: 5=1fold5<br>Per-numbers occur when counting, when pricing and when splitting<br>The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b |
| T1 TIME | How can counting & adding be reversed?<br>Counting ? 3s and adding 2 gave 14.<br>Can all calculations be reversed? | By calculating backward, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign.<br>x*3+2=14 is reversed to x = (14-2)/3<br>Yes. x+a=b is reversed to x=b-a, x*a=b is reversed to x=b/a, x^a=b is reversed to x=a√b, a^x=b is reversed to x=logb/loga |
| T2 TIME | How to predict the terminal number when the change is constant?<br>How to predict the terminal number when the change is variable, but predictable? | By using constant change-equations:<br>If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44<br>If Ko = 30 and ΔK/K = r = 2%, then K7= Ko*(1+r)^n= 30*1.02^7= 34.46<br>By solving a variable change-equation:<br>If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx |
| S1 SPACE | How to count plane and spatial properties of stacks and boxes and round objects? | By using a ruler, a protractor and a triangular shape.<br>By the 3 Greek Pythagoras', mini, midi & maxi<br>By the 3 Arabic recount-equations: sinA=a/c, cosA=b/c, tanA=a/b |
| S2 SPACE | How to predict the position of points and lines?<br>How to use the new calculation technology? | By using a coordinate-system. If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14)<br>Computers can calculate a set of numbers (vectors) and a set of vectors (matrices) |
| QL | What is quantitative literature?<br>Does quantitative literature also have the 3 different genres: fact, fiction and fiddle? | Quantitative literature tells about Many in time and space<br>The word and the number language share genres:<br>Fact is a since-so calculation or a room-calculation<br>Fiction is an if-then calculation or a rate-calculation<br>Fiddle is a so-what calculation or a risk-calculation |

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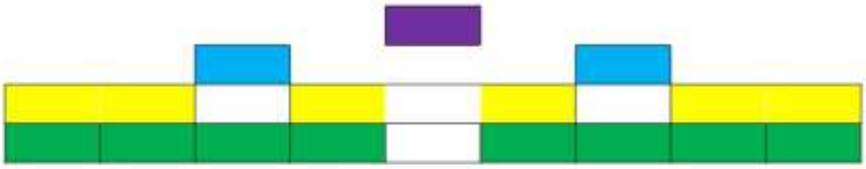
# PYRAMIDeDUCATION

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

- Each pair works together to solve Count&Add problems.
- The coach assists the instructors when instructing their team and when correcting their Count&Add assignments.
- Each teacher pays by coaching a new group of 8 teachers.

1 Coach
2 Instructors
3 Pairs
2 Teams





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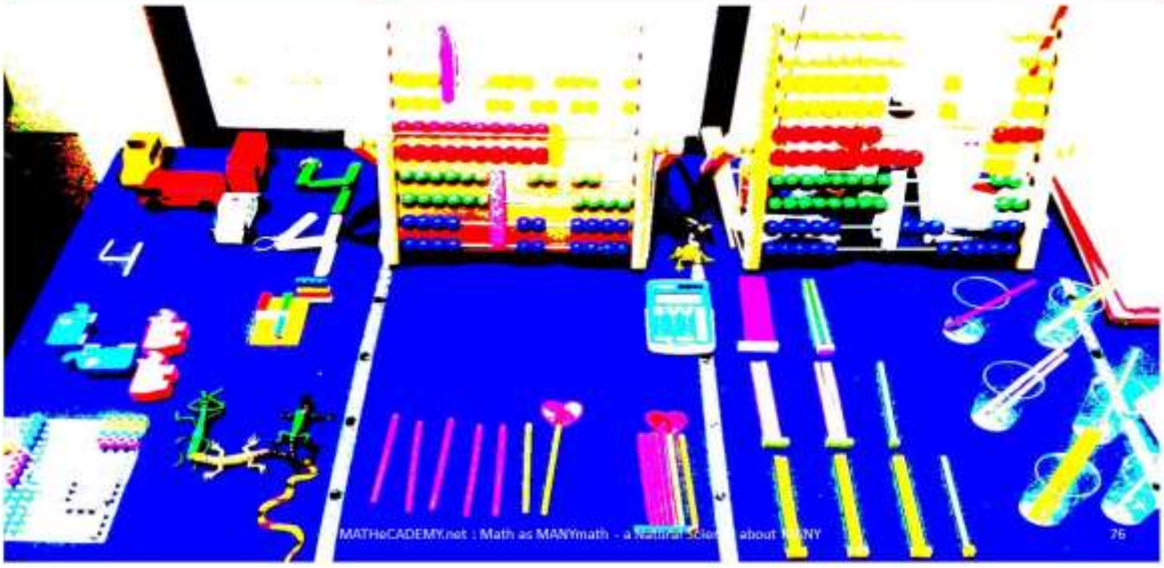
# Number Icons

# ReCounting 7 in 5s & 3s & 2s



# Theoretical background

Tarp, A. (2018). Mastering Many by counting, recounting and double-counting before adding on-top and next-to.
*Journal of Mathematics Education, March 2018, 11*(1), 103-117.

## COUNTING before ADDING

***The Child's Own Twin Curriculum***
**Count & ReCount & DoubleCount**
**before Adding NextTo & OnTop**

**master many**
**manymath**

## 30. A New Curriculum - But for Which of the 3x2 Kinds of Mathematics Education

An essay on observations and reflections at the ICMI study 24 curriculum conference

*As part of institutionalized education, mathematics needs a curriculum describing goals and means. There are however three kinds of mathematics: pre-, present and post-'setcentric' mathematics; and there are two kinds of education: multi-year lines and half-year blocks. Thus, there are six kinds of mathematics education to choose from before deciding on a specific curriculum; and if changing, shall the curriculum stay within the actual kind or change to a different kind? The absence of federal states from the conference suggests that curricula should change from national multi-year macro-curricula to local half-year micro-curricula; and maybe change to post-setcentric mathematics.*

### COHERENCE AND RELEVANCE IN THE SCHOOL MATHEMATICS CURRICULUM

The International Commission on Mathematical Instruction, ICMI, has named its 24th study "School mathematics Curriculum Reforms: Challenges, Changes and Opportunities". Its discussion document has 5 themes among which theme B, "Analysing school mathematics curriculum for coherence and relevance" says that "All mathematics curricula set out the goals expected to be achieved in learning through the teaching of mathematics; and embed particular values, which may be explicit or implicit."

So, to analyze we use the verb 'cohere' and the predicate 'relevant' when asking: "to what does this curriculum cohere and to what is it relevant?" As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.'

We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that 'to cohere' means 'to hold together firmly as parts of the same mass'. As to synonyms for cohere, it lists: 'accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.' And as to antonyms, it lists: 'differ (from), disagree (with).'

In the same dictionary, the word 'relevant' means 'having significant and demonstrable bearing on the matter at hand'. As to synonyms for relevant, it lists: 'applicable, apposite, apropos, germane, material, pertinent, pointed, relative.' And as to antonyms, it lists: 'extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.'

If we accept the verb 'apply' as having a meaning close to the predicate 'relevant', we can rephrase the above analysis question using verbs only: "to what does this curriculum cohere and apply?"

Seeing education metaphorically as bridging an individual start level for skills and knowledge to a common end level described by goals and values, we may now give a first definition of an ideal curriculum: "To apply to a learning process as relevant and useable, a curriculum coheres to the start and end levels for skills and knowledge."

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?
- How to make the end level cohere to goals and values expressed by the society?
- How to make the end level cohere to goals and values expressed by the learners?
- How to make the bridge cohere to previous and following bridges?

- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

**GOALS AND VALUES EXPRESSED BY THE SOCIETY**

In her plenary address about the 'OECD 2030 Learning Framework', Taguma shared a vision:

> The members of the OECD Education 2030 Working Group are committed to helping every learner develop as a whole person, fulfil his or her potential and help shape a shared future built on the well-being of individuals, communities and the planet. (..) And in an era characterised by a new explosion of scientific knowledge and a growing array of complex societal problems, it is appropriate that curricula should continue to evolve, perhaps in radical ways (p. 10).

Talking about learner agency, Taguma said:

> Future-ready students need to exercise agency, in their own education and throughout life. (..) To help enable agency, educators must not only recognise learners' individuality, (..) Two factors, in particular, help learners enable agency. The first is a personalised learning environment that supports and motivates each student to nurture his or her passions, make connections between different learning experiences and opportunities, and design their own learning projects and processes in collaboration with others. The second is building a solid foundation: literacy and numeracy remain crucial. (p. 11)

By emphasizing learner's individual potentials, personalised learning environment and own learning projects and processes, Taguma seems to indicate that flexible half-year micro-curricula may cohere better with learners' future needs than rigid multi-year macro-curricula. As to specifics, numeracy is mentioned as one of the two parts of a solid foundation helping learners enable agency.

**DIFFERENT KINDS OF NUMERACY**

Numeracy, however, is not that well defined. Oxford Dictionaries and Merriam-Webster agree on saying 'ability to understand and work with numbers'; whereas the private organization National Numeracy (nationalnumeracy.org.uk) says 'By numeracy we mean the ability to use mathematics in everyday life'.

The wish to show usage was also part of the Kilpatrick address, describing mathematics as bipolar:

> I want to stress that bipolarity because I think that's an important quality of the school curriculum and every teacher and every country has to deal with: how much attention do we give to the purer side of mathematics. The New Math thought that it should be entire but that didn't work really as well as people thought. So how much attention do we give to the pure part of mathematics and how much to the applications and how much do we engage together. Because it turns out if the applications are well-chosen and can be understood by the children then that helps them move toward the purer parts of the field. (p. 20)

After discussing some problems caused by applications in the curriculum, Kilpatrick concludes:

> If we stick with pure mathematics, with no application, what students cannot see, "when will I ever use this?", it's not surprising that they don't go onto take more mathematics. So, I think for self-preservation, mathematicians and mathematics educators should work on the question of: how do we orchestrate the curriculum so that applications play a good role? There is even is even a problem with the word applications, because it implies first you do the mathematics, then you apply it. And actually, it can go the other way. (p. 22)

So, discussing what came first, the hen or the egg, applications or mathematics, makes it problematic to define numeracy as the ability to apply mathematics since it gives mathematics a primacy and a monopoly as a prerequisite for numeracy. At the plenary afterwards discussion, I suggested using the word 're-rooting' instead of 'applying' to indicate that from the beginning, mathematics was rooted in the outside world as shown by the original meanings of geometry and algebra: 'to measure earth' in Greek and 'to reunite' in Arabic.

## MATHEMATICS THROUHG HISTORY

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy, seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra. And in Europe, Germanic countries taught 'reckoning' in primary school and 'arithmetic' and 'geometry' in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here a wish for exactness and unity created a 'setcentric' 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

Setcentrism thus changed classical grounded 'many-matics' into a self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meets counter-examples as e.g. 1week + 2days is 9days.

The introduction of the setcentric New Mathematics created different reactions. Inside the United States it was quickly abandoned with a 'back-to-basics' movement. Outside it was implemented at teacher education, and in schools where it gradually softened. However, it never retook its original form or name, despite, in contrast to 'mathematics', 'reckon' is an action-word better suited to the general aim of education, to teach humans to master the outside world through appropriate actions.

## DIFFERENT KINDS OF MATHEMATICS

So, a curriculum must choose between a pre-, a present, and a post-setcentric mathematics as illustrated by an example from McCallum's plenary talk. After noting that "a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships" (p. 4), McCallum asked the audience: "What is the difference between 5/3 and 5÷3".

Pre-setcentric mathematics will say that 5/3 is a number on the number-line reached by taking 5 steps of the length coming from dividing the unit in 3 parts; and that 5÷3 means 5 items shared between 3.

Present setcentric mathematics will say that 5/3 is a rational number defined as an equivalence class in the product set of integers, created by the equivalence relation $(a,b)$ eq. $(c,d)$ if cross-multiplication holds, $a$x$d$ = $b$x$c$; and, with 1/3 as the inverse element to 3 under multiplication, 5÷3 should be written as 5x1/3, i.e. the as the solution to the equation 3x$u$ = 5, found by applying and thus legitimizing abstract algebra and group theory; thus finally saying goodbye to the Renaissance use of a vertical line to separate addends from subtrahends, and a horizontal line to separate multipliers from divisors.

Post-setcentric mathematics (Tarp, 2018) sees setcentric mathematics as meta-matism hiding the original Greek meaning og mathematics as a science about Many. In this 'Many-math', 5/3 is a per-number coming from double-counting in different units (5$/3kg), becoming a fraction with like units (5$/3$ = 5/3). Here per-numbers and fractions are not numbers but operators needing a number to become a number (5/3 of 3 is 5, 5/3 of 6 is 10); and 5÷3 means 5 counted in 3s occurring in the 'recount-formula' recounting a total $T$ in bundles of 3s as $T$ = ($T$/3)x3, saying 'from $T$, $T$/3 times, 3 can be taken away'. This gives flexible numbers: $T$ = 5 = 1B2 3s = 1.2 3s = 1 2/3 3s = 2B-1 3s = 2.-1 3s, introduced in grade one where bundle-counting and re-counting in another unit precedes adding, and where recounting from tens to icons, $T$ = 2.4 tens = ? 6s, leads to the equation $T$ = $u$x6 = 24 = (24/6)x6 solved by recounting. In post-setcentric mathematics, per-numbers, fractions, ratios and proportionality melt together since double-counting in two units gives per-numbers as ratios, becoming fractions with like units. And here proportionality means changing units using the recount-formula to recount in the per-number: With 5$/3kg, "how much for 20$?" is found by re-counting 20 in 5s: $T$ = 20$ = (20/5)x5$ = (20/5)x3kg = 12 kg. Likewise if asking "how much for 15 kg?"

## DIFFERENT KINDS OF EDUCATION

As to education, from secondary school there is a choice between multi-year lines and half-year blocks. At the discussion after the Kilpatrick plenary session I made a comment about these two educational systems, which mas a lady from the United States say I was misinforming since in the states Calculus required a full year block. Together with other comments in the break, this made me realize that internationally there is little awareness of these two different kinds of educational systems. So here is another example of what the Greek sophists warned against, choice masked as nature.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

At the conference, the almost total absence of federal states as Germany, Canada, the United States and Russia seems to indicate that the problems reside with multi-year national curricula, becoming rigid traditions difficult to change. While federal competition or half-year blocks creates flexibility through an opportunity to try out different curricula.

Moreover, as a social institution involving individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlightens teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching only one subject in their own classrooms.

To protect its republic against its German speaking neighbors, France created elite schools, criticized today for exerting hidden patronization. Bourdieu thus calls education 'symbolic violence', and Foucault points out that a school is really a 'pris-pital' mixing power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings.

## DIFFERENT KINDS OF COMPETENCES

As to competences, new to many curricula, there are at least three alternatives to choose among. The European Union recommends two basic competences, acquiring and applying, when saying that "Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge."

At the conference two alternatives notions of competences were presented. In his plenary address, Niss recommended a matrix with 8 competences per concept (p. 73). In his paper, Tarp (pp. 317-324) acknowledged that 8 competences may be needed if the goal of mathematics education is to learn present setcentric university mathematics; but if the goal is to learn to master Many with post-setcentric mathematics, then only two competences are needed: counting and adding, rooting a twin curriculum teaching counting, recounting in different units and double-counting before adding.

**MAKING THE LEARNING ROAD MORE PASSABLE**

Once a curriculum is chosen, the next question is to make its bridge between the start and end levels for skills and knowledge more passable. Here didactics and pedagogy come in; didactics as the captain choosing the way from the start to the end, typically presented as a textbook leaving it to pedagogy, the lieutenants, to take the learners through the different stages.

The didactical choices must answer general questions from grand theory. Thus, philosophy will ask: shall the curriculum follow the existentialist recommendation, that existence precedes essence? And psychology will ask: shall the curriculum follow Vygotsky mediating institutionalized essence, or Piaget arranging learning meetings with what exists in the outside world? And sociology will ask: on which ethical grounds are children and teenagers retained to be cured by institutionalized education?

**COLONIZING OR DECOLONIZING CURRICULA**

The conference contained two plenary panels, the first with contributors from France, China, The Philippines and Denmark, almost all from the northern hemisphere; the second with contributors from Chile, Australia, Lebanon and South Africa, almost all from the southern hemisphere. Where the first panel talked more about solutions, the second panel talked more about problems.

In the first panel, France and Denmark represented some of the world's most centralized states with war-time educational systems dating back to the Napoleon era, which in France created elite-schools to protect the young republic from the Germans, and in Germany created the Humboldt Bildung schools to end the French occupation by mediating nationalism, and to sort out the population elite for jobs as civil servants in the new central administration; both just replacing the blood-nobility with a knowledge-nobility as noted by Bourdieu. The Bildung system latter spread to most of Europe.

Not surprisingly, both countries see university mathematics as the goal of mathematics education ('mathematics is what mathematicians do'), despite the obvious self-reference avoided by instead formulating the goal as e.g. learning numerical competence, mastery of Many or number-language. Seeing mathematics as the goal, makes mathematics education an example of a goal displacement (Bauman) where a monopoly transforms a means into a goal. A monopoly that makes setcentric mathematics an example of what Habermas and Derrida would call a 'center-periphery colonization', to be decentered and decolonized by deconstruction.

Artigue from France thus advocated an anthropological theory of the didactic, ATD, (p. 43-44), with a 'didactic transposition process' containing four parts: scholarly knowledge (institutions producing and using the knowledge), knowledge to be taught (educational system, 'noosphere'), taught knowledge (classroom), and learned available knowledge (community of study).

> The theory of didactic transposition developed in the early 1980s to overcome the limitation of the prevalent vision at the time, seeing in the development of taught knowledge a simple process of elementarization of scholarly knowledge (Chevallard 1985). Beyond the well-known succession offered by this theory, which goes from the reference knowledge to the knowledge actually taught in classrooms (..), ecological concepts such as those of niche, habitat and trophic chain (Artaud 1997) are also essential in it.

Niss from Denmark described the Danish 'KOM Project' leading to eight mathematical competencies per mathematical topic (pp. 71-72).

> The KOM Project took its point of departure in the need for creating and adopting a general conceptualisation of mathematics that goes across and beyond educational levels and institutions. (..) We therefore decided to base our work on an attempt to define and characterise mathematical competence in an overarching sense that would pertain to and make sense in any mathematical context. Focusing - as a consequence of this approach - first and foremost on the *enactment* of mathematics means attributing, at first, a secondary role to mathematical content. We then came up with the following definition of

> mathematical competence: Possessing *mathematical competence* – mastering mathematics – is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve actual or potential mathematical challenges of any kind. In order to identify and characterise the fundamental constituents in mathematical competence, we introduced the notion of mathematical competencies: A *mathematical competency* is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve a certain kind of mathematical challenge.

Some of the consequences by being colonized by setcentrism was described in the second panel.

In his paper 'School Mathematics Reform in South Africa: A Curriculum for All and by All?' Volmink from South Africa Volmink writes (pp. 106-107):

> At the same time the educational measurement industry both locally and internationally has, with its narrow focus, taken the attention away from the things that matter and has led to a traditional approach of raising the knowledge level. South Africa performs very poorly on the TIMSS study. In the 2015 study South Africa was ranked 38th out of 39 countries at Grade 9 level for mathematics and 47th out of 48 countries for Grade 5 level numeracy. Also in the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ), South Africa was placed 9th out of the 15 countries participating in Mathematics and Science – and these are countries which spend less on education and are not as wealthy as we are. South Africa has now developed its own Annual National Assessment (ANA) tests for Grades 3, 6 and 9. In the ANA of 2011 Grade 3 learners scored an average of 35% for literacy and 28% for numeracy while Grade 6 learners averaged 28% for literacy and 30% for numeracy.

After thanking for the opportunity to participate in a cooperative effort on the search of better education for boys, girls and young people around the world, Oteiza from Chile talked about 'The Gap Factor' creating social and economic differences. A slide with the distribution of raw scores at PSU mathematics by type of school roughly showed that out of 80 points, the median scores were 40 and 20 for private and public schools respectively. In his paper, Oteiza writes (pp. 81-83):

> Results, in national tests, show that students attending public schools, close to de 85% of school population, are not fulfilling those standards. How does mathematical school curriculum contribute to this gap? How might mathematical curriculum be a factor in the reduction of these differences? (..) There is tremendous and extremely valuable talent diversity. Can we justify the existence of only one curriculum and only one way to evaluate it through standardized tests? (..) There is a fundamental role played by researchers, and research and development centers and institutions. (..) How do the questions that originate in the classroom reach a research center or a graduate program? "*Publish or perish*" has led our researchers to publish in prestigious international journals, but, are the problems and local questions addressed by those publications?"

The Gap Factor is also addressed in a paper by Hoyos from Mexico (pp. 258-259):

> The PISA 2009 had 6 performance levels (from level 1 to level 6). In the global mathematics scale, level 6 is the highest and level 1 is the lowest. (..) It is to notice that, in PISA 2009, 21.8% of Mexican students do not reach level 1, and, in PISA 2015, the percentage of the same level is a little bit higher (25.6%). In other words, the percentage of Mexican students that in PISA 2009 are below level 2 (i.e., attaining the level 1 or zero) was 51%, and this percentage is 57% in PISA 2015, evidencing then an increment of Mexican students in the poor levels of performance. According to the INEE, students at levels 1 or cero are susceptible to experiment serious difficulties in using mathematics and benefiting from new educational opportunities throughout its life. Therefore, the challenges of an adequate educational attention to this population are huge, even more if it is also considered that approximately another fourth of the total Mexican population (33.3 million) are children under 15 years of age, a population in priority of attention".

As a comment to Volminks remark "Another reason for its lack of efficacy was the sense of scepticism and even distrust about the notion of People's Mathematics as a poor substitute for the "real mathematics"" (p. 104), and inspired by the sociological Centre–Periphery Model for colonizing, by post-colonial studies, and by Habermas' notion of rationalization and colonization of the lifeworld by the instrumental rationality of bureaucracies, I formulated the following question in the afterwards discussion: “As former colonies you might ask: Has colonizing stopped, or is it still taking place? Is there an outside central mathematics that is still colonizing the mind? What happens to what could be called local math, street math, ethno-math or the child’s own math?”

## CONCLUSION AND RECOMMENDATIONS

Designing a curriculum for mathematics education involves several choices. First pre-, present and post-setcentric mathematics together with multi-year lines and half-year blocks constitute 3x2 different kinds of mathematics education. Combined with three different ways of seeing competences, this offers a total of 18 different ways in which to perform mathematics education at each of the three educational levels, primary and secondary and tertiary, which may even be divided into parts.

Once chosen, institutional rigidity may hinder curriculum changes. So, to avoid the ethical issues of forcing cures from self-referring diagnoses upon children and teenagers in need of guidance instead of cures, the absence of participants from federal states might be taken as an advice to replace the national multi-year macro-curriculum with regional half-year micro-curricula. At the same time, adopting the post version of setcentric mathematics will make the curriculum coherent with the mastery of Many that children bring to school, and relevant to learning the quantitative competence and numeracy desired by society.

And, as Derrida says in an essay called ‘Ellipsis’ in ‘Writing and Difference’: “Why would one mourn for the centre? Is not the centre, the absence of play and difference, another name for death?”

## POSTSCRIPT: MANY-MATH, A POST-SETCENTRIC MATHEMATICS FOR ALL

As post-setcentric mathematics, Many-math, can provide numeracy for all by celebrating the simplicity of mathematics occurring when recounting the ten fingers in bundles of 3s:

T = ten = 1B7 3s = 2B4 3s = 3B1 3s = 4B-2 3s. Or, if seeing 3 bundles of 3s as 1 bundle of bundles,

T = ten = 1BB0B1 3s = 1*B^2 + 0*B + 1 3s, or T = ten = 1BB1B-2 3s = 1*B^2 + 1*B – 2 3s.

This number-formula shows that a number is really a multi-numbering of singles, bundles, bundles of bundles etc. represented geometrically by parallel block-numbers with units. Also, it shows the four ways to unite: on-top addition, multiplication, power and next-to addition, also called integration. Which are precisely the four ways to unite constant and changing unit- and per-numbers numbers into totals as seen by including the units; each with a reverse way to split totals. Thus, addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the Algebra Square’, also showing that equations are solved by moving to the opposite side with opposite signs.

| Operations **unite**/<br>*split* Totals in | Changing | Constant |
|---|---|---|
| Unit-numbers<br>m, s, kg, $ | ***T = a + n***<br>*T – n = a* | ***T = a*n***<br>*T/n = a* |
| Per-numbers<br>m/s, $/kg, $/100$ = % | ***T = ∫a*dn***<br>*dT/dn = a* | ***T = a^n***<br>*n√T = a    logaT = n* |

An unbundled single can be placed on-top of the block counted in 3s as T = 1 = 1/3 3s, or next-to the block as a block of its own written as T = 1 = .1 3s Writing T = ten = 3 1/3 3s = 3.1 3s = 4.-2 3s thus introduces fractions and decimals and negative numbers together with counting.

The importance of bundling as the unit is emphasized by counting: 1, 2, 3, 4, 5, 6 or bundle less 4, 7 or B-3, 8 or B-2, 9 or B-1, ten or 1 bundle naught, 1B1, …, 1B5, 2B-4, 2B-3, 2B-2, 2B-1, 2B naught.

This resonates with ‘Viking-counting’: 1, 2, 3, 4, hand, and1, and2, and3, less2, less1, half, 1left, 2left. Here ‘1left’ and ‘2left’ still exist as ‘eleven’ and ‘twelve’, and ‘half’ when saying ‘half-tree’, ‘half-four’ and ‘half-five’ instead of 50, 70 and 90 in Danish, counting in scores; as did Lincoln in his Gettysburg address: “Four scores and seven years ago …”

Counting means wiping away bundles (called division iconized as a broom) to be stacked (called multiplication iconized as a lift) to be removed to find unbundled singles (called subtraction iconized as a horizontal trace). Thus, counting means postponing adding and introducing the operations in the opposite order of the tradition, and with new meanings: 7/3 means 7 counted in 3s, 2x3 means stacking 3s 2 times. Addition has two forms, on-top needing recounting to make the units like, and next-to adding areas, i.e. integral calculus. Reversed they create equations and differential calculus.

The recount-formula, T = (T/B)*B, appears all over mathematics and science as proportionality or linearity formula:

- Change unit, T = (T/B)*B, e.g. T = 8 = (8/2)*2 = 4*2 = 4 2s
- Proportionality, $ = ($/kg)*kg = price*kg
- Trigonometry, a = (a/c)*c = sinA*c, a = (a/b)*b = tanA*b, b = (b/c)*c = cosA*c
- STEM-formulas, meter = (meter/sec)*sec = speed*sec, kg = (kg/m^3)*m^3 = density*m^3
- Coordinate geometry, $\Delta y = (\Delta y/\Delta x)*\Delta x = m*\Delta x$
- Differential calculus, dy = (dy/dx)*dx = y’ * dx

The number-formula also contains the formulas for constant change: *T = b*x* (proportional), *T = b*x + c* (linear), *T = a*x^n* (elastic), *T = a*n^x* (exponential), *T = a*x^2 + b*x + c* (accelerated).

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics ‘post-dicting’ what we cannot be ‘pre-dicted’.

## THE GENERAL CURRICULUM CHOICES OF POST-SETCENTRIC MATHEMATICS

Making the curriculum bridge cohere with the individual start levels in a class is obtained by always beginning with the number-formula, and with recounting tens in icons less than ten, e.g. T = 4.2 tens = ? 7s, or u*7 = 42 = (42/7)*7, thus solving equations by moving to opposite side with opposite sign. And by always using full number-language sentences with a subject, a verb and a predicate as in the word language, e.g. T = 2*3. This also makes the bridge cohere to previous and following bridges.

Making the end level cohere to goals and values expressed by the society and by the learners is obtained by choosing mastery as the end goal, not of the inside self-referring setcentric construction of contemporary university mathematics, but of the outside universal physical reality, Many.

Making the bridge passable is obtained by choosing Piagetian psychology instead of Vygotskyan.

## FLEXIBLE NUMBERS MAKE TEACHERS FOLLOW

Changing a curriculum raises the question: will the teachers follow? Here, seeing the advantage of flexible numbers makes teachers interested in learning more about post-setcentric mathematics:

Typically, division creates problems to students, e.g. 336/7. With flexible numbers a total of 336 can be recounted with an overload as T = 336 = 33B6 = 28B56, so 336/7 = 28B56 /7 = 4B8 = 48; or with an underload as T = 336 = 33B6 = 35B-14, so 336/7 = 35B-14 /7 = 5B-2 = 48.

Flexible numbers ease all operations:

T = 48*7 = 4B8*7 = 28B56 = 33B6 = 336

T = 92 – 28 = 9B2 – 2B8 = 7B-6 = 6B4 = 64

T = 54 + 28 = 5B4 + 2B8 = 7B12 = 8B2 = 82

To learn more about flexible numbers, a group of teachers can go to the MATHeCADEMY.net designed to teach teachers to teach MatheMatics as ManyMatics, a natural science about Many, to watch some of its YouTube videos. Next, the group can try out the “Free 1day Skype Teacher Seminar: Cure Math Dislike by ReCounting” where, in the morning, a power point presentation ‘Curing Math Dislike’ is watched and discussed locally, and at a Skype conference with an instructor. After lunch the group tries out a ‘BundleCount before you Add booklet’ to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows after the coffee break.

To learn more, a group of eight teachers can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for secondary school. For modelling, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where the 8 teachers form 2 teams of 4, choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers. The material mediates learning by experimenting with the subject in number-language sentences, i.e. the total T. Thus, the material is self-instructing, saying “When in doubt, ask the subject, not the instructor”.

The material for primary and secondary school has a short question-and-answer format. The question could be: “How to count Many? How to recount 8 in 3s? How to count in standard bundles?” The corresponding answers would be: “By bundling and stacking the total T, predicted by T = (T/B)*B. So, T = 8 = (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s = 3.-1 3s. Bundling bundles gives multiple blocks, a polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4*B^2 + 5*B + 6*1.”

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