Math Ed & Research 2019

No Power Point Presentations

The same Mathematics Curriculum for Different Students
Addition-free STEM-based Math for Migrants
Math Dislike Cured with Inside-Outside Deconstruction
Developing the Child's Own Mastery of Many
What is Math - and Why Learn it?
Flexible BundleNumbers

Allan.Tarp@MATHeCADEMY.net

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Preface

● The texts 01 and 02 concern the ICMI Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, held in Tsukuba, Japan, 26-30 November 2018. My paper, ‘A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many’, is included in the 2018 Articles. The discussion document theme B, “Analysing school mathematics curriculum reforms for coherence and relevance” had five sub-questions, and I was asked to contribute to writing a chapter addressing key-question B2, “How are mathematics content and pedagogical approaches in reforms determined for different groups of students (for e.g. in different curriculum levels or tracks) and by whom? How do curriculum reforms establish new structures in content, stakeholders (e.g. students and teachers), and school organisations; and what are their effects?”, in short called mathematics for all.

The first outline was scheduled to February 15, 2019. Having almost finished the outline, on February 12 we received a mail saying “The deadline for the chapter outlines has been extended until at least the end of February (from 15 February). I would like to open discussion between us on several matters.” On February 20 I sent in my response (see 02).

I never got any reaction so on March 30 I sent a mail to the organizing committee saying “To stimulate our work, would it be an idea to send out a monthly or quarterly newsletter reporting on the progress and challenges being made and met?” No response came so I began writing a proposal for a contribution (se 01). Then on May 10 I got a mail saying “There will not be a separate chapter on key question 2”, to which I responded “I think that the question ‘math for all’ as focused on in the question B2 is so important that it deserves an answer. When mentioned at the conference that no paper addressed this I objected since my paper is addressing the question if it with a different way of organizing math education will be able to include all. Furthermore, I have written a first draft that I send on February 12. Moreover, I have collected a substantial amount of material to include, just waiting for an answer to my mail on February 20 and March 28. So I am going to write a chapter with the focus originally decided upon since I think the research question deserves an answer as mentioned above. I will send it to you as planned before at the end of June. You might then decide not to include it, it will be your choice, then I will publish it elsewhere since the question is very important and since the material, I have collected shows that it is indeed possible to have mathematics for all in different ways.”

I then sent in my proposal before July 1 but heard nothing then or after the time limit for a reaction on August 15. So apparently my contribution will be the chapter that was commissioned and rejected without even being read.

01. The same Mathematics curriculum for different students.

The paper has the following chapters: 01. A need for curricula for all students 02. Addressing the need 03. Coherence and relevance 04. Parallel tracks to the main curriculum, examples 05. Pre-calculus, typically the last mandatory curriculum 06. Precalculus in the Danish parallel high school, a case study 07. A refugee camp curriculum 08. Do we really need parallel curricula 09. Conclusion.

02. Comments to a discussion paper

● At the Genoa University April 8-11, Paolo Boero held an international workshop called ‘Habermas’ elaboration on rationality and mathematics education’ over the Habermas text ‘Some further clarifications of the concept of communicative rationality’. I was allowed to give a short presentation.

03. A Mathematics Teacher Using Communicative Rationality Towards Children

● At Beijing Normal University June 28-30 2019, the 2019 Classroom Teaching Research for All Students Conference (CTRAS) took place with the conference theme ‘innovative practices and research initiatives in STEM integration that supports all students’ mathematics learning (..) The conference will provide participants from around the world with the opportunity to share: 1) best practices of STEM integration; 2) the most contemporary STEM research initiatives; 3) innovative curriculum of STEM integration; and 4) professional development approaches for STEM educators.' I contributed with a paper with a power point presentation, a proposal for a short presentation, a poster and a workshop, 04-08

● At Paderborn University September 16-19 the third international conference on mathematics textbooks research and development, ICMT3, will take place. Invitations were sent out to contribute with oral communications, workshops, posters, papers and symposia. I sent in one oral presentation (09), one paper (14), one workshop (11) and three posters (10, 12, 13). All were rejected except for the paper that was accepted for a ten minutes oral presentation.

09. Developing the Child's Own Mastery of Many, oral presentation

10. Math Dislike Cured with Inside-Outside Deconstruction, poster

11. Learning from The Child's Own Mathematics, workshop

12. Five Alternative Ways to Teach Proportionality, poster

13. New Textbooks, but for Which of the 3x2 Kinds of Mathematics Education, poster

14. Developing the Child's Own Mastery of Many, paper. The abstract says: Sociological imagination sees continuing educational problems as possibly caused by a goal displacement making mathematics see itself as the goal instead of its outside root, mastery of Many. Typically, the number-language is taught inside-inside as examples of its meta-language. However, as the word-language, it can also be taught inside-outside, thus bridging it to the outside world it describes. So, textbooks should not reject, but further guide the mastery of Many that children bring to school.

The chapters are called 'is one curriculum and textbook for all students possible, meeting many, children bundle to count and share, textbooks for a question guided counting curriculum, textbook for a question guided adding curriculum, discussion and future research.'

15. The PowerPointPresentation is called 'The Child's Own Mastery of Many, Count & ReCount & DoubleCount, before Adding NextTo & OnTop' and contains 43 slides.

● At Freiburg Pädagogische Hochschule October 7-8 the third Educating the Educators International Conference on approaches to scaling-up professional development in maths and science education will take place. Invitations were sent out to contribute with oral presentation sessions in the three dimensions (personal, material and structural) to report on projects, approaches and research, workshop sessions actively involving all participants, discussion group sessions also actively involving all participants, poster sessions and materials market, allowing participants to exhibit interesting professional development materials (including classroom materials) and learn about other materials.

The conference focused on three topics wanting to 'serve as a lever and platform for international exchange about concepts and experiences. The aim is to present and discuss different approaches which ensure a high quality of the education of educators:

* Personal dimension: Which roles, contents and activities have to be considered in the professional development courses for PD course leaders and facilitators in professional learning?

* Material dimension: Which role can materials play in professional development for maths and science teachers (classroom materials, face-to-face PD materials and e-learning PD materials)?

* Structural dimension: How can projects or initiatives for scaling up professional development look like and how can they be evaluated?

I sent in four proposals. One was rejected (16, sent as a poster for topic 3), two were accepted as posters (17 sent as a presentation for topic 1, 18 sent as a workshop for topic 2), one was accepted for presentation (19 sent as a discussion group 3). The proposal for a material market (21) was accepted.

16. Addition-Free Math Make Migrants and Refugees Stem Educators

17. Recounting Before Adding Makes Teachers Course Leaders and Facilitators

18. Self-explanatory Learning Material to Improve your Mastery of Many

19. Can Grounded Math and Education and Research Become Relevant to Learners

20. The PowerPointPresentation is called 'Can Grounded Mathematics & Education & Research become Relevant to Learners?' and contains 54 slides.

21. Recounting in Icon-Units and in Tens Before Adding Totals Next-To and On-Top, together with the posters presented at the stand.

● The following note is handed out to students and to teacher to have a basic discussion of the need and form of mathematics education.

22. What is Math - and Why Learn it?

● This material is meant for high school to illustrate how algebra and geometry should be always together and never apart.

23. Mathematics with Playing Cards

● This math compendium is meant for a high school pre-calculus course to illustrate the point made in (01) that it is possible to start all over from the bottom in a pre-calculus course, and also to give an introduction to calculus presenting integral calculus before differential calculus. The compendium also includes several projects modeling real world problems.

24. Mathematics Predicts, PreCalculus

● In Växjö January 14-15, 2020, the Swedish Society for Research in Mathematics Education welcome to Madif 12, its twelfth research seminar in connection with the Matematikbiennalen 2020. The theme of the seminar is 'Sustainable mathematics education in a digitalized world'. I sent in three papers inspired by (01), one on early childhood education (25), and one on middle school (26), and one on precalculus (27) as well as two proposals for a workshop (28, 29). All were rejected.

25. Sustainable Adaption to Quantity: From Number Sense to Many Sense

The abstract says: Their biological capacity to adapt to their environment make children develop a number-language based upon two-dimensional block- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language. And math education must accept that its goal is not to mediate the truth regime of setcentric university math, but to develop the child's already existing adaption to Many.

The chapters are called: Decreased PISA Performance Despite Increased Research Mathematics and its Education, Biology Looks at Education, Philosophy Looks at Education, Psychology Looks at Education, Sociology Looks at Education, Meeting Many, Children Bundle to Count and Share, A Contemporary Mathematics Curriculum, The Difference to a Typical Contemporary Mathematics Curriculum, Mathematics as a Number-Language, Discussing Number Sense and Number Nonsense, Conclusion and Recommendation.

26. Per-numbers connect Fractions and Proportionality and Calculus and Equations Sustainable Adaption to Quantity: From Number Sense to Many Sense

The abstract says: In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word ‘percent’ to also talk about other ‘per-numbers’ as e.g. ‘per-five’ thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

The chapters are called: Mathematics is Hard, or is it, The ICMT3 Conference, Different Ways of Seeing Fractions, Ratios and Rates, Per-numbers Occur when Double-counting a Total in two Units, Fractions as Per-numbers, Expanding and Shortening Fractions, Taking Fractions of Fractions, the Per-number Way, Direct and Inverse Proportionality, Adding Fractions, the Per-number Way, Solving Proportionality Equations by Recounting , Seven Ways to Solve the two Proportionality Questions, A Case: Peter, about to Peter Out of Teaching, Discussion and Recommendation

27. Sustainable Adaption to Double-Quantity: From Pre-Calculus to Per-Number Calculations

The abstract says: Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

The chapters are called: A need for curricula for all students, A Traditional Precalculus Curriculum, A Different Precalculus Curriculum, Precalculus, building on or rebuilding?, Using Sociological Imagination to Create a Paradigm Shift, A Grounded Outside-Inside Fresh-start Precalculus from Scratch, Solving Equations by Moving to Opposite Side with Opposite Sign, Recounting Grounds Proportionality, Double-counting Grounds Per-numbers and Fractions, The Change Formulas, Precalculus Deals with Uniting Constant Per-Numbers as Factors, Calculus Deals with Uniting Changing Per-Numbers as Areas, Statistics Deals with Unpredictable Change, Modeling in Precalculus Exemplifies Quantitative Literature, A Literature Based Compendium, An Example of a Fresh/start Precalculus Curriculum, An Example of an Exam Question, Discussion and Conclusion.

28. A Lyotardian dissension to the early childhood consensus on numbers and operations. The chapter are called: Can sociological imagination improve mathematics education? Consensus and Dissension on Early Childhood Numbers & Operations. Time Table for the Workshop.

29. Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis? The chapter are called: Does Mathematics Education Research have an Irrelevance Paradox? The Replication Crisis in Science. Time Table for the Workshop.

30. Bundle Counting Table. A guide to bundle-counting in pre-school. Written for the stand at the Matematikbiennale.

31. Proposals for the 2020 Swedish Math Biennale. All were rejected.

● At the Ho Chi Minh City University of Education on December 7, a conference was held called ‘Psychology and Mathematics education’. I was invited to give the plenary talk Saturday, which I named after the paper I send in (32), together with a Power Point Presentation (33). Sunday, I gave a talk on modeling to a group of master students. Monday, I gave a talk to a class of senior students on a poster presentation from the 'Educating the Educators' conference in Freiburg, Germany, in October, and handed out the notes 'What is Math - and Why Learn it?' and 'Bundle Counting Table'. Tuesday, I gave a talk to the staff on research in mathematics education and networks to join and design research as a methodology to use when researching the implementation of the new activity-based curriculum inspired by Kolb's experimental learning theory.

32. De-Modeling Numbers, Operations and Equations: From Inside-Inside to Outside-Inside Understanding

The abstract says: Adapting to the outside fact Many, children internalize social number-names, but how do they externalize them when communicating about outside numerosity? Mastering Many, children use bundle-numbers with units; and flexibly use fractions and decimals and negative numbers to account for the unbundled singles. This suggests designing a curriculum that by replacing abstract-based with concrete-based psychology mediates understanding through de-modeling core mathematics, thus allowing children to expand the number-language they bring to school.

The chapters are: 1. Introduction, 2. Materials/ Subjects and Methods, 2.1. Reflections on Different forms of Mathematics, 2.2. Reflections on Different forms of Psychology, 2.3. Merging Mathematics and Psychology, 2.4. De-modelling Digits, 2.4.1. Designing and Implementing a micro-curriculum, 2.5. Reflections on how to De-model Bundle-counting Sequences, 2.5.1. Designing and Implementing a micro-curriculum, 2.6. Reflections on how to De-model Operations, 2.7. Reflections on how to Recount into Tens, 2.7.1. Designing and Implementing a micro-curriculum, 2.8. Reflections on how to Model Double-counting with Per-numbers and Fractions, 2.9. Reflections on how to De-model Trigonometry, 3. Results and Discussion, 4. Conclusion.

33. De-Model Numbers, Operations and Equations, PPP.

34. Visit to Ho Chi Minh City University of Education December 7-13 2019.

● The ICMT3 and Educating Educators conferences used peer-reviews, and in the first you were allowed to comment on the reviews

35. Review 01 ICMT3

36. Review 02 ICMT3

37. Comments to ICMT3 Reviewers

38. Educating Educators Reviews

Aarhus, December 2019, Allan Tarp

01. The Same Mathematics Curriculum for Different Students

To offer mathematics to all students, parallel tracks often occur from the middle of secondary school. The main track continues with a full curriculum, while parallel tracks might use a reduced curriculum leaving out e.g. calculus; or they might contain a different kind of mathematics meant to be more relevant to students by including more applications. However, an opportunity here presents itself for designing the same curriculum for all students no matter which track they may choose. As number-language, why not let mathematics follow the communicative turn that took place in language education in the 1970s by prioritizing its connection to the outside world higher than its inside connection to its grammar? We will consider examples of all three curricula options.

01. A Need for Curricula for all Students

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing the Organisation for Economic Co-operation and Development (OECD, 2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)

Other countries also experience declining PISA results; and in high performing countries not all students are doing well.

02. Addressing the Need

By saying "All students should study mathematics in each of the four years that they are enrolled in high school." the US National Council of Teachers of Mathematics (2000, p. 18) has addressed the need for curricula for all students in their publication 'Principles and Standards for School Mathematics'. In the overview the Council writes

> We live in a mathematical world. Whenever we decide on a purchase, choose an insurance or health plan, or use a spreadsheet, we rely on mathematical understanding (..) In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors. (..) everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics. Students need to learn a new set of mathematics basics that enable them to compute fluently and to solve problems creatively and resourcefully. (p. 1)

In this way the Council points out that it is important to master 'mathematical competence', i.e. to understand and do mathematics to solve problems creatively and to compute fluently. This will benefit the personal life, the workplace, as well as further study leading to productive futures.

Consequently, the Council has included in the publication a curriculum that "is mathematically rich providing students with opportunities to learn important mathematical concepts and procedures with understanding". This in order to "provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals."

The publication includes a set of standards: "The Standards for school mathematics describe the mathematical understanding, knowledge, and skills that students should acquire from prekindergarten to grade 12." The five standards

> present goals in the mathematical content areas of number and operations, algebra , geometry, measurement and data analysis and probability. (..) Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century" (p. 2)

In the chapter 'Number and operations', the Council writes

> Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense. Students with number sense naturally decompose numbers (..) For example, children in the lower elementary grades can learn that numbers can be decomposed and thought about in many different ways - that 24 is 2 tens and 4 ones and also two sets of 12. (p. 7)

In the chapter 'The Curriculum Principle', the Council writes

> A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades (..) for teachers at each level to know what mathematics their students have already studied and will study in future grades. (p. 3, 4)

All in all, the Council points to the necessity of designing a curriculum that is relevant in students' 'personal life, in the workplace and in further study' and that is coherent at the same time to allow teachers to know 'what mathematics their students have already studied and will study in future grades'.

03. Coherence and Relevance

So, in their publication, the National Council of Teachers of Mathematics stresses the importance of coherence and relevance. To allow teachers follow a prescribed curriculum effectively, and to allow students build upon what they already know, it must be 'well articulated across the grades'. And, to have importance for students a curriculum must be relevant by supplying them with 'the basic skills and understandings that students will need to function effectively in the twenty-first century'.

With 'cohere' as a verb and 'relevant' as a predicate we can ask: "to what does this curriculum cohere, and to what is it relevant?" As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.'

We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that 'to cohere' means 'to hold together firmly as parts of the same mass'. As to synonyms for cohere, it lists: 'accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.' And as to antonyms, it lists: 'differ (from), disagree (with).'

In the same dictionary, the word 'relevant' means 'having significant and demonstrable bearing on the matter at hand'. As to synonyms for relevant, it lists: 'applicable, apposite, apropos, germane, material, pertinent, pointed, relative.' And as to antonyms, it lists: 'extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.'

If we accept the verb 'apply' as having a meaning close to the predicate 'relevant', we can rephrase the above analysis question using verbs only: "to what does this curriculum cohere and apply?"

Metaphorically, we may see education as increasing skills and knowledge by bridging individual start levels to a common end level described by institutional goals. So, we may now give a first definition of an ideal curriculum: "To apply to a learning process as relevant, a curriculum coheres to the individual start levels and to the end goal, which again coheres with the need expressed by the society funding the educational institution."

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?
- How to make the end level cohere to goals expressed by the society?
- How to make the end level cohere to goals expressed by the learners?
- How to make the bridge cohere to previous and following bridges?
- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

04. Parallel Tracks to the Main Curriculum, Examples

In their publication chapter Grades 9 through 12, the National Council of Teachers of Mathematics discusses to the possibility to introduce parallel courses in the high school.

> In secondary school, all students should learn an ambitious common foundation of mathematical ideas and applications. This shared mathematical understanding is as important for students who will enter the workplace as it is for those who will pursue further study in mathematics and science. All students should study mathematics in each of the four years that they are enrolled in high school.
>
> Because students' interests and inspirations may change during and after high school, their mathematics education should guarantee access to a broad spectrum of careers and educational options. They should experience the interplay of algebra, geometry, statistics, probability and discrete mathematics.
>
> High school mathematics builds on the skills and understandings developed in the lower grades. (..) High school students can study mathematics that extends beyond the material expected of all students in at least three ways. One is to include in the curriculum material that extends the foundational material in depth or sophistication. Two other approaches make use of supplementary courses. In the first students enroll in additional courses concurrent with those expected of all students. In the second, students complete a three-year version of the shared material and take other mathematics courses. In both situations, students can choose from such courses as computer science, technical mathematics, statistics, and calculus. Each of these approaches has the essential property that all students learn the same foundation of mathematics but some, if they wish, can study additional mathematics. (p. 18-19)

The Council thus emphasizes the importance of studying 'mathematics in each of the four years that they are enrolled in high school'. This the council sees as feasible if implementing one or more of three options allowing students to 'study mathematics that extends beyond the material expected of all students'. Some students may want to study 'material that extends the foundational material in depth or sophistication'. Others may want to take additional courses cohering to the college level, especially calculus. Others may want to take additional courses relevant to their daily life or a workplace. We will now look at two examples of that both including examples of finite mathematics, a subject that is normally outside a standard high school curriculum.

For all Practical Purposes, Introduction to Contemporary Mathematics

In the US, the Consortium for Mathematics and its Applications (COMAP) has worked out a material called ' For all practical purposes' (COMAP, 1988). In its preface, the material presents itself as

> (..) an introductory mathematics course for students in the liberal arts or other nontechnical curricula. The course consists of twenty-six half-hour television shows, the textbook, and this Telecourse guide. This series shows mathematics at work in today's world. (..) For all practical purposes aims to develop conceptual understanding of the tools and language of mathematics and the ability to reason using them. We expect most students will have completed elementary algebra and some geometry in high school. We do not assume students will be pursuing additional courses in mathematics, at least none beyond the introductory level. (p. iii)

As to content, the material has five parts (p. v - vi)

Part one focuses on graph theory and linear programming illustrated with network as scheduling and planning. It includes an overview show and four additional shows called street smarts: street networks; trains, planes and critical paths; juggling machines: scheduling problems; juicy problems: linear programming.

Part two deals with statistics and probability illustrated with collecting and deducing from data. It includes an overview show and four additional shows called behind the headlines: collecting data; picture this: organizing data; place your bets: probability; confident conclusions: statistical inference.

Part three focuses on social choice, fair division and game theory illustrated by different voting systems and conflict handling. It includes an overview show and four additional shows called the impossible dream: election theory; more equal than others: weighted voting; zero-sum games: games of conflict; prisoner's dilemma: games of partial conflict.

Part four focuses on using geometry, the classical conic sections, shapes for tiling a surface, geometric growth in finance in and in population, and measurement. It includes an overview show and four additional shows called how big is too big: scale and form; it grows and grows: populations; stand up conic: conic sections; it started in Greece: measurement.

Part five focuses on computer algorithms. It includes an overview show and four additional shows called rules of the games: algorithms; counting by two's: numerical representation; creating a cde: encoding information; moving picture show: computer graphics.

The video sections are available on YouTube.

A Portuguese Parallel High School Curriculum

Portugal followed up on the COMAP initiative. In his paper called "Secondary mathematics for the social sciences" (Silva, 2018), Jaime Silva describes how the initiative inspired an innovative two-year curriculum for the Portuguese upper secondary school.

As to the background, Silva writes

> There are two recurring debates about the mathematics curriculum in secondary schools, especially in countries like Portugal where compulsory education goes till the 12th grade. First, should all students study mathematics (not necessarily the same) or should the curriculum leave a part of the students "happy" without the mathematics "torture"? Second, should all students study the same "classic" mathematics, around ideas from differential and integral calculus with some Geometry and some Statistics?
>
> When the 2001 revision (in great part in application today) of the Portuguese Secondary School curriculum was made (involving the 10th, 11th and 12th grades) different kinds of courses were introduced for the different tracks (but not for all of them) that traditionally existed. Mathematics A is for the Science and Technology track and for the Economics track and is a compulsory course. Mathematics B is for the Arts track and is an optional course. Mathematics Applied to the Social Sciences (MACS) is for the Social Sciences track and is an optional course. The Languages track was left without mathematics or science. Later the last two tracks were merged and the MACS course remained optional for the new merged track. The technological or professional tracks could have Mathematics B, Mathematics for the Arts or Modules of Mathematics (3 to 10 to be chosen from 16 different modules, depending on the professions). (p. 309)

As to the result of debating a reform in Portugal, Silva writes

> When, in 2001, there was a possibility to introduce a new Mathematics course for the "Social Sciences" track, for the 10th and 11th grade students, there were some discussions of what could be offered. The model chosen was inspired in the course "For All Practical Purposes" (COMAP, 2000) developed by COMAP because it "uses both contemporary and classic examples to help students appreciate the use of math in their everyday lives". As a consequence, a set of independent chapters, each one with some specific purpose, was chosen for this syllabus, that included 2 years of study, with 4.5 hours of classes per week (normally 3 classes of 90 minutes each). The topics chosen were: 10th grade Decision Methods:

> Election Methods, Apportionment, Fair Division; Mathematical Models: Financial models, Population models Statistics (regression); 11th grade Graph models, Probability models, Statistics (inference). (p. 310)

As to the goal of the curriculum, Silva writes

> The stated goal of this course is for the students to have *"significant mathematical experiences that allow them to appreciate adequately the importance of the mathematical approaches in their future activities"*. This means that the main goal is not to master specific mathematical concepts, but it is really to give students a new perspective on the real world with mathematics, and to change the students view of the importance that mathematical tools will have in their future life. In this course it is expected that the students study simple real situations in a form as complete as possible, and *"develop the skills to formulate and solve mathematically problems and develop the skill to communicate mathematical ideas (students should be able to write and read texts with mathematical content describing concrete situations)".* (p. 310)

As to the reception of the curriculum, Silva writes

> This was a huge challenge for the Portuguese educational system because most of these topics had never been covered before, and most teachers did not even study Graph Theory at University. Election Methods, Apportionment and Fair Division were of course completely new to everybody. The reception was good from the part of the Portuguese Math Teacher Association APM, as it considered that *"the methodologies and activities suggested in the MACS program promote the development of the skills of social intervention, of citizenship and others"*. The reception from the scientific society SPM was rather negative because they considered the syllabus did not have enough mathematical content. (p. 310-311)

As to the present state of the curriculum, Silva writes

> After 15 years there is no thorough evaluation of how the course is run in practice in the schools, or which is the real impact on the further education or activities of the students that studied "Mathematics Applied to the Social Sciences". In Portugal there is no institution in charge of this type of work and evaluations are done on a case by case basis. All Secondary Schools need to do selfevaluations but normally just compare internal statistics to national ones to see where they are in the national scene. In the reports consulted there was no special mention to the MACS course and so we have the impression that the MACS course entered the normal Portuguese routine in Secondary School. (p. 315)

So as to a parallel track to the traditional curriculum, the National Council of Teachers of Mathematics suggests that including a different kind of mathematics might be an option, e.g. finite mathematics. In the US this idea was taken up by the Consortium for Mathematics and its Applications (COMAP) working out a material including a textbook and a series of television shows to show 'mathematics at work in today's world'. Part of this material was also included in a parallel curriculum in Portugal called 'Mathematics Applied to the Social Sciences' (MACS) offering to Portuguese students also to study mathematics in each of their high school years, as the National Council of Teachers of Mathematics recommends.

05. Precalculus, Typically the last Mandatory Curriculum

This chapter looks at the part of a mathematics curriculum called precalculus, typically being the first part that is described in a parallel curriculum since it contains operations as root and logarithm that is not considered part of a basic mathematics algebra curriculum. First, we look at an example of a traditional precalculus curriculum. The we ask what could be an ideal precalculus curriculum, and illustrates it with two examples. In the next chapter, we look at a special case, a Danish precalculus curriculum that has served both as a parallel and a serial curriculum during the last 50 years.

A Traditional Precalculus Course

An example of a traditional precalculus course is found in the Research and Education Association book precalculus (Woodward, 2010). The book has ten chapters. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

An Ideal Precalculus Curriculum

In their publication, the National Council of Teachers of Mathematics writes "High school mathematics builds on the skills and understandings developed in the lower grades. (p. 19)"

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they have from the lower grades?

Mathe-matics, Meta-matics, and Mathe-matism

Furthermore, what kind of mathematics has been taught? Was it 'grounded mathematics' abstracted bottom-up from its outside roots, or 'ungrounded mathematics' or 'meta-matics' exemplified top-down from inside abstractions, maybe becoming 'meta-matism' if mixed with 'mathe-matism' (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept 'function', Euler saw it as a bottom-up abstracted name for 'standby calculations' containing specified and unspecified numbers. Later meta-matics defined a function top-down as an example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may be seen as a number-language sentence containing a subject, a verb and a predicate allowing its value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group concepts applying the associative and commutative law as well as the neutral element and inverse elements thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2.

2*u = 6, so (2*u)*½ = 6*½, so (u*2)*½ = 3, so u*(2*½) = 3, so u*1 = 3, so u = 3.

However the equation 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula 'T = (T/B)*B' present all over mathematics as the proportionality formula thus solved in one step:

2*u = 6 = (6/2)*2, giving u = 6/2 = 3.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology. Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction about an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

Restarting from Scratch with Grounded Outside-Inside Mathematics

Let students see how outside degrees of Many are iconized by inside digits with as many strokes as it represents, five strokes in the 5-icon etc. Let students see that after nine we count by bundling creating icons for the counting operations as well, where division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see that an outside block of 2 3s becomes an inside calculation 2*3 and vice versa. Let students see the commutative law by turning and a*b block, and see the distributive law by splitting a into c and d, and see the associative law by turning an a*b*c box.

Let students see that both the word- and the number-language use full sentences with a subject, a verb, and an object or predicate, abbreviating ‘the total is 2 3s’ to ‘T = 2*3’

Let students enjoy flexible bundle-numbers where decimals and fractions negative and numbers are created to describe the unbundle ones placed next-to or on-top of the block, thus allowing 5 to be recounted in 3s as T = 5 = 1B2 = 1.2 B = 1 2/3 B = 2B-1.

Let student see, that recounting in other units may be predicted by the recount-formula ‘T = (T/B)*B’ saying “From the total T, T/B times, B may be pushed away”. Let students see that where the recount-formula in primary school recounts 6 in 2s as 6 = (6/2)*2 = 3*2, in secondary school the same task is formulated as solving the equation u*2 = 6 as u*2 = 6 = (6/2)*2 giving u = 6/2, thus moving 2 to the opposite side with the opposite calculation sign.

Let students see the power of ‘flexible bundle-numbers’ when the inside multiplication 7*8 = (B-3)*(B-2) = BB-2B-3B+6 = 5B6 = 56 may be illustrated on an outside ten by ten block, thus showing that of course minus times minus must give plus since the 2*3 corner has been subtracted twice.

Let students see that double-counting in two units create per-numbers as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and asking ‘9kg = ?$’ we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

And, that double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Thus finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

And, that double-counting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA * c; b = (b/c)*c = cosA * c; a = (a/b)*b = tanA * b. With a circle filled from the inside by right triangles, this also allows phi to be found from a formula: circumference/diameter = $\pi \approx$ n*tan(180/n) for n large.

And, how recounting and double-counting physical units create per-numbers and proportionality all over STEM, Science, Technology, Engineering and mathematics: kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter.

Also, let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation formula. Writing 'y = f(x)' means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Algebra and Geometry, Always Together, Never Apart

Let students enjoy the power and beauty of integrating algebra and geometry.

First, let students enjoy seeing that multiplication creates blocks with areas where 3*7 is 3 7s that may be algebraically recounted in tens as 2.1 tens. Or, that may be geometrically transformed to a square u^2 giving the algebraic equation u^2 = 21, creating root as the reverse calculation to power, u = √21. Which may be found approximately by locating the nearest number p below u, here p = 4, so that u^2 = (4+t)^2 = 4^2 + 2*4*t + t^2 = 21.

Neglecting t^2 since t is less than 1, we get 4^2 + 2*4*t = 21, which gives t = $\frac{21 - 4\text{^}2}{4*2}$, or t = $\frac{N - p\text{^}2}{p*2}$, if p is the nearest number below u, where u^2 = N.

As an approximation, we then get √N ≈ p + t = p + $\frac{N - p\text{^}2}{p*2}$; or √N ≈ $\frac{N + p\text{^}2}{p*2}$, if p^2 < N < (p+1)^2

Then let students enjoy the power and beauty of predicting where a line geometrically intersects lines or circles or parabolas by algebraically solving two equations with two unknowns, also predicted by a computer software.

A Number Seen as a multiple Numbering

Let students see the number 456 as what it really is, not three ordered digits obeying a place-value system, but three numberings of bundles-of-bundles, bundles, and unbundled ones as expressed in the number-formula, formally called a polynomial: T = 456 = 4*B^2 + 5*B + 6*1, with B = ten.

Let students see that a number-formula contains the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, students see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Figure 01. *The 'algebra-square' shows the four ways to unite or split numbers.*

Let students see calculations as predictions, where 2+3 predicts what happens when counting on 3 times from 2; where 2*5 predicts what happens when adding 2$ 5 times; where 1.02^5 predicts what happens when adding 2% 5 times; and where adding the areas 2*3 + 4*5 predicts how to add the per-numbers when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Solving Equations by Reversed Calculation Moving Numbers to Opposite Side

Let students see the subtraction '5-3' as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers 'to opposite sign with opposite calculation sign'; a rule that applies also to the other reversed operations:

- the division u = 5/3 is the number u that multiplied with 3 gives 5, u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, u^3 = 5, making root a 'factor-finder'
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, 3^u = 5, making logarithm a 'factor-counter'.

Let students see multiple calculations reduce to single calculations by unhiding 'hidden bracket' where 2+3*4 = 2+(3*4) since with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s. This will prevent solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5, by allowing the hidden bracket to be shown: 2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4 to be verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy singing a 'Hymn to Equations': "Equations are the best we know, they're solved by isolation. But first the bracket must be placed, around multiplication. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don't want to stop!"

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let student place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, the first two terms disappear by adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = 0 + b^2/4 – c = b^2/4 – c.

Now, moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c

u + b/2 = $\pm\sqrt{b^2/4 - c}$

u = -b/2 $\pm\sqrt{b^2/4 - c}$

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, containing also the formulas for constant change: T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change: constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to 'post-dict' what we cannot 'pre-dict'.

Combining linear and exponential change by n times depositing a$ to an interest rate r%, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest rate R is predicted by the formula 1+R = (1+r)^n. Such a saving may be used to neutralize a debt Do, that in the same period has changed to D = Do*(1+R).

The formula and the proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Constant Change

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers creating the operations integration and its reverse, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power and its two inverse operations, root and logarithm.

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$ changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, leading to the formula for change with a constant change-factor, also called exponential change, T = b*a^n. Reversing the question, this formula entails two equations.

The first equation asks about an unknown change-percent. Thus, we might want to find which percent that added ten times will give a total change-percent 70%, or, formulated with change-factors, what is the change-factor, a, that applied ten times gives the change-factor 1.70. So here the job is ‘factor-finding’ which leads to defining the tenth root of 1.70, i.e. 10√1.70, as predicting the factor, a, that applied 10 times gives 1.70: If a^10 = 1.70 then a = 10√1.70 = 1.054 = 105.4%. This is verified by testing: 1.054^10 = 1.692. Thus, the answer is “5.4% is the percent that added ten times will give a total change-percent 70%.”

We notice that 5.4% added ten times gives 54% only, so the 16% remaining to 70% is the effect of compound interest adding 5.4% also to the previous changes.

Here we solve the equation a^10 = 1.70 by moving the exponent to the opposite side with the opposite calculation sign, the tenth root, a = 10√1.70. This resonates with the ‘to opposite side with opposite calculation sign’ method that also solved the equations a+3 = 7 by a = 7-3, and a*3 = 20 by a = 20/3.

The second equation asks about a time-period. Thus, we might want to find how many times 7% must be added to give 70%, 1.07^n = 1.70. So here the job is factor-counting which leads to defining the logarithm log1.07(1.70) as the number of factors 1.07 that will give a total factor at 1.70: If 1.07^n = 1.70 then n = log1.07(1.70) = 7,84 verified by testing: 1.07^7.84 = 1.700.

We notice that simple addition of 7% ten times gives 70%, but with compound interest it gives a total change-factor 1.07^10 = 1.967, i.e. an additional change at 96.7%-70% = 26.7%, explaining why only 7.84 periods are needed instead of ten.

Here we solve the equation 1.07^n = 1.70 by moving the base to the opposite side with the opposite calculation sign, the base logarithm, n = log1.07(1.70). Again, this resonates with the ‘to opposite side with opposite calculation sign’ method.

An ideal precalculus curriculum could ‘de-model’ the constant percent change exponential formula T = b*a^n to outside real-world problems as a capital in a bank, or as a population increasing or radioactive atoms decreasing by a constant change-percent per year.

De-modeling may also lead to situations where the change-elasticity is constant as in science multiplication formulas wanting to relate a percent change in T with a percent change in n.

An example is the area of a square T = s^2 where a 1% change in the side s will give a 2% change in the square, approximately: With To = s^2, T1 = (s*1.01)^2 = s^2*1.01^2 = s^2* 1.0201 = To*1.0201.

Once mastery of constant change-percent is established, it is possible to look at time series in statistical tables asking e.g. “How has the unemployment changed over a ten-year period?” Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b+a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n. These average numbers would allow setting up a forecast predicting the yearly numbers in the ten-year period, if the numbers were predictable. However, they are not, so instead of predicting the number with a formula, we might 'post-dict' the numbers using statistics dealing with unpredictable numbers, but still trying to predict a plausible interval by describing the unpredictable random change by nonfictional numbers, median and quartiles, or by fictional numbers, mean and standard deviation.

Calculus Deals with Adding Per-Numbers by Their Areas

Likewise, real-world phenomena as unemployment occur in both time and space, so unemployment may also change in space, e.g. from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status, also called contingency tables or crosstabs. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add, they must be multiplied back to unit-numbers to find the total percent. However, once you multiply you create an area, and adding per-numbers by areas is what calculus is about, thus here introduced in a natural way through double-tables from statistical materials.

An example: in one region 10 persons have 50% unemployment, in another, 90 persons have 5% unemployment. To find the total, the unit-numbers can be added directly to 100 persons, but the percent-numbers must be multiplied back to numbers: 10 persons have 10*0,5 = 5 unemployed; and 90 persons have 90*0,05 = 4.5 unemployed, a total of 5+4.5 unemployed = 9.5 unemployed among 100 persons, i.e. a total of 9.5% unemployment, also called the weighted average. Later, this may be renamed to Bayes formula for conditional probability.

Calculus as adding per-numbers by their areas may also be introduced through mixture problems asking e.g. '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?' Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before being added, thus adding by their areas.

Modeling in Precalculus

Furthermore, the entry of graphing calculators allows authentic modeling to be included in a pre-calculus curriculum thus giving a natural introduction to the following calculus curriculum as well.

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A' = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A' = x' = 1; and that if A = ½*x^2, then y = A' = (½*x^2)' = x; and that if A = 1/3*x^3, then y = A' = (1/3*x^3)' = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to x = 3, instead of adding small strips we just calculate the change in the area over this distance.

This makes sense since adding many small strips means adding many small changes, which gives just one final change since all the in-between end- and start-values cancel out:

$\int_1^3 y * dx = \int_1^3 dA = \Delta_1^3 A = \Delta_1^3 (\frac{1}{3} * x^3)$ = end – start = $\frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$

On the calculus course we just leave out the area by renaming it to a ‘primitive’ or an ‘antiderivative’ when writing

$\int_1^3 y * dx = \int_1^3 x^2 * dx = \Delta_1^3 (\frac{1}{3} * x^3)$ = end – start = $\frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$

A graphing calculator show that this suggestion holds. So, finding areas under per-number graphs not only allows adding per-numbers, it also gives a grounded and natural introduction to integral and differential calculus where integration precedes differentiation just as additions precedes subtraction.

From the outside, regression allows giving a practical introduction to calculus by analysing a road trip where the per-number speed is measured in five second intervals to respectively 10 m/s, 30 m/s, 20 m/s 40 m/s and 15 m/s. With a five data-set table we can choose to model with a degree4 polynomial found by regression. Within this model we can predict when the driving began and ended, what the speed and the acceleration was after 12 seconds, when the speed was 25m/s, when acceleration and braking took place, what the maximum speed was, and what distance is covered in total and in the different intervals.

Another example of regression is the project ‘Population versus food’ looking at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2- line table regression can produce two formulas: with x counting years after 1850, the population is modeled by Y1= 815*1.013^x and the food by Y2= 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

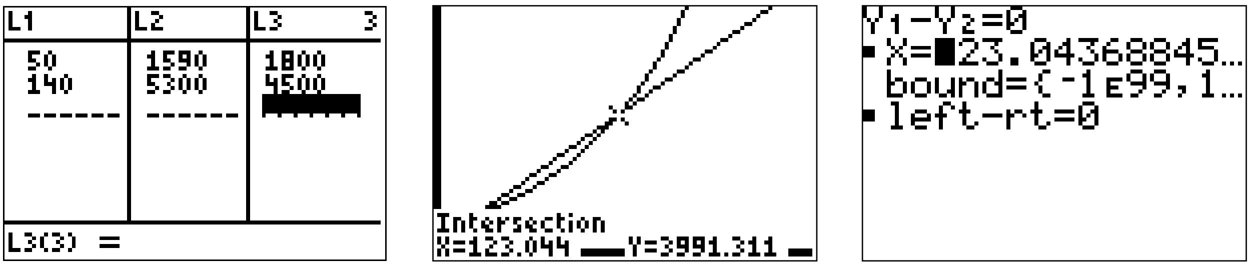

Figure 02. A Malthusian model of population and food levels

An example of an ideal precalculus curriculum is described in a paper called ‘Saving Dropout Ryan With a Ti-82’ (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren’t even able to use a TI-30. A compendium called ‘Formula Predict’ (Tarp, 2009) replaced the textbook. A formula’s left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by ‘solve Y1-Y2 = 0’. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

With the increased educational interest in STEM, modeling also allows including science-problems as e.g. the transfer of heat taking place when placing an ice cube in water or in a mixture of water and alcohol, or the transfer of energy taking place when connecting an energy source with energy consuming bulbs in series or parallel; as well as technology problems as how to send of a golf ball to hit a desired hole, or when to jump from a swing to maximize the jumping length; as well as engineering problems as how to build a road inclining 5% on a hillside inclining 10%.

Furthermore, precalculus allows students to play with change-equations, later called differential equations since change is calculated as a difference, $\Delta T = T2 – T1$. Using a spreadsheet, it is fun to see the behavior of a total that changes with a constant number or a constant percent, as well as with a decreasing number or a decreasing percent, as well as with half the distance to a maximum value or with a percent decreasing until disappearing at a maximum value. And to see the behavior of a total accelerating with a constant number as in the case of gravity, or with a number proportional to its distance to an equilibrium point as in the case of a spring.

So, by focusing on uniting and splitting into constant per-numbers, the ideal precalculus curriculum has constant change-percent as its core. This will cohere with a previous curriculum on constant change-number or linearity; as well as with the following curriculum on calculus focusing on uniting and splitting into locally constant per-numbers, thus dealing with local linearity. Likewise, such a precalculus curriculum is relevant to the workplace where forecasts based upon assumptions of a constant change-number or change-percent are frequent. This curriculum is also relevant to the students' daily life as participants in civil society where tables presented in the media are frequent.

Two Curriculum Examples Inspired by an Ideal Precalculus Curriculum

An example of a curriculum inspired by the above outline was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

Later, around year 2000, another version was designed but not tested. The curriculum goal was stated as: 'the students develop their number-language so they can participate in social practices involving quantitative descriptions of change and shape.' The curriculum means included

1. Numbers and calculations. Quantities and qualities. Number-language, word-language, meta-language. Unit-numbers and per-numbers, and how to calculate their totals. Equations as predicting statements. Forwards and reverse calculations.

2. Change calculations. Change measuring change with change-number and change-percent and index-number. Calculation rules for the change of a sum, a product and a ratio.

3. Constant change. Change with a constant change-number. Change with a constant change-percent. Change with both.

4. Unpredictable change. Fractals, mean and deviation, 95% confidence interval. Binomial distribution approximated by a normal distribution.

Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

5. Trigonometry. Dividing and measuring earth. Calculation the sides and angles in a triangle.

06. PRECALCULUS IN THE DANISH PARALLEL HIGH SCHOOL, A CASE STUDY

In the post-war era, the Organization for Economic Co-operation and Development (OECD) called for increasing the population knowledge level, e.g. by offering a second chance to take a high school degree giving entrance to tertiary education. In Denmark in 1966, this resulted in creating a two-year education called ‘Higher preparation exam’ as a parallel to the traditional high school. Two levels of two-years mathematics courses were included, a basic precalculus course for those who did not choose the calculus course.

The 1966 Curriculum

The precalculus curriculum came from leaving out small parts of the calculus curriculum, thus being an example of a reduced curriculum.

The goal of the calculus course stated it should ‘supply students with knowledge about basic mathematical thinking and about applications in other subject areas, thus providing them with prerequisites for carrying through tertiary education needing mathematics.’

The goal of the precalculus course was reduced to ‘supplying students with an impression of mathematical thinking and method and to mediate mathematical knowledge useful also to other subject areas.’

So, where the calculus curriculum has to cohere and be relevant to tertiary education needing mathematics, the precalculus course is a parallel curriculum meant to be relevant to the students themselves and to other high school subjects.

The content of the precalculus curriculum had five sections.

The first section contained basic concepts from set theory as sets, subsets, complementary set, union, intersection, product, difference. The function concept. Mapping into an on a different set, one-to one mapping, inverse mapping (inverse function), composite mappings. The calculus curriculum added nothing here.

Section two contained concepts from abstract algebra: Composition rules. The associative law. The commutative law. Neutral element. Inverse element. The group concept with examples. Rules for operations on real numbers. Numeric value. Here the calculus curriculum added the distributive law, the concept of a ring and a field, the ring of whole numbers as well as quotient classes. The calculus curriculum added nothing here.

Section three contained equations and inequalities. Examples on open statements in one or two variables. Equations and inequalities of degree one and two with one unknown. Equations and inequalities with the unknown placed inside a square root or a numeric sign. Simple examples of Equations and inequalities of degree one and two with two unknowns. Graphical illustration. The calculus curriculum added nothing here.

Section four contained basic functions. The linear function in one variable. A piecewise linear function. The second-degree polynomial. The logarithm function with base ten, the logarithmic scale, the calculator stick, the use of logarithm tables. Trigonometric functions, tables with functions values. Calculations on a right-angled triangle using trigonometric functions. Here the calculus curriculum added rational functions in one variable, exponential functions, and the addition formulas and logarithmic formulas in trigonometry.

Section five contained combinatorics. The multiplication principle. Permutations and combinations. Here the calculus curriculum added probability theory, probability field, and examples of probability based upon combinatorics.

Finally, the calculus curriculum added a section about calculus.

The new set-based mathematics coming into education around 1960 inspired the 1966 precalculus curriculum thus cohering with the university mathematics at that time, but it was not especially relevant to the students. Many had difficulties understanding it and they often complained about seeing no reason for learning it or why it was taught.

In my own class, I presented it as a legal game where we were educating us to become lawyers that could convince a jury that we were using lawful methods to solving equations in one of two different methods by referring to the relevant paragraphs in the law. The first method was the traditional one used at that time way by moving numbers to the opposite side with opposite calculation sign, now legitimized by the theorem that in a group the equation a*u = b has as a solution a^-1*b. The second method was a new way with many small steps where, for each step, you have to refer to laws for associativity, and commutativity etc.; and, where a group contained exactly the paragraphs needed to use this method. Once seen that way, the students found it easy but boring. However, they accepted since they needed the exam to go on, and we typically finished the course in half time allowing time for writing a script for a movie to be presented at the annual gala party.

So, all in all, the 1963 curriculum was coherent with the next step, calculus, and with the university math view at that time, set-based; but it was mostly irrelevant to the students.

The 1974 Curriculum

The student rebellion in 1968 asked for relevance in education, which led to a second precalculus in 1974 revision. Here the goal was stated as ‘giving the students a mathematical knowledge that could be useful to other subjects and to their daily life, as well as an impression of mathematical methods thinking’. Now the curriculum structure was changed from a parallel one to a serial one where all students took the precalculus course and some chose to continue with the calculus course afterwards just specifying in its curriculum what was needed to be added.

The 1974 precalculus curriculum now had four sections.

The first section contained concepts from set theory and logic and combinatorics. Set, subset; solution set to an open statement, examples on solving simple equations and inequalities in one variable; the multiplication principle, combinations.

Section two contained the function concepts: Domain, function value, range; injective function; monotony intervals; inverse function, composite function.

Section three contained special functions; graphical illustration. A linear function, a piecewise linear function, an exponential function; examples of functions defined by tables; coordinate system, logarithmic paper.

Section four contained descriptive statistics. Observations described by numbers; frequency and their distribution and cumulated distribution; graphical illustration; statistical descriptors.

Section five described probability and statistics. A random experiment, outcome space, probability function, probability field; sampling; binomial distribution; binomial testing with zero hypothesis, critical set, significance level, single and double-sided test, failure of first degree.

Section six was called ‘Free lessons’. 20m lessons are to be used for studying details in one of the above sections, or together with one or more other school subjects to work with an area applying mathematics.

The second 1974 curriculum thus maintains a basis of set-theory but leaves out the abstract algebra. As to functions, it replaces the second-degree polynomial with the exponential function. Here trigonometry is excluded to be included in the calculus curriculum.

The combinatorics section is to great extent replaced by descriptive statics.

Finally, the section has been added with quite detailed probability theory and testing theory within statistics.

All in all, the coherence with the university set-based mathematics has been softened by leaving out abstract algebra and second-degree polynomial. Instead of introducing a first-degree polynomial together with a second-degree polynomial, the former now is introduced as a linear function together with the exponential function allowing modelling outside change with both a constant change-number and a constant change-percent. This makes the curriculum more relevant to the students individually as well as to other high school subjects as required by the goal statement.

The quite detailed section on testing theory was supposed to make the curriculum more relevant to students but the degree of detail make it fail to do so by drowning in quite abstract concepts.

The 1990 Curriculum

As the years passed on it was observed that the free hours were used on trigonometry, and on savings and instalments, the first cohering with the following calculus course, the latter highly relevant to many students, and at the same time combining linear and exponential change, the core of the curriculum. This let to designing an alternative curriculum around 1990 to choose instead of the standard curriculum if wanted.

The 1990 curriculum did not change the goal but included the following subjects

1) Numbers, integers, rational and real numbers together with their calculation rules. Number sets. Calculations with power and root.

2) Calculations including percent and interest rates: Average percent, index number, weighed average. Simple and compound interest, saving and installments.

3) Geometry and trigonometry. Similar triangles. Right triangles. Calculations on sides and angles.

4) Functions. The function concept, domain, functional values, range, monotony. Various ways to define a function. Elementary functions as linear, piecewise linear and exponential growth and decay. Coordinate system. Examples of simple equations and inequalities including the functions mentioned above.

5) Probability and statistics. A stochastic experiment. Discrete stochastic variables, probability distribution, mean value, binomial distribution, observation sets described graphically, representation by statistical descriptors, examples of a normal distribution, normal distribution paper.

6) Calculation aids. Pocket calculator, formulas, tables, semi logarithmic paper, normal distribution paper.

The 2005 Curriculum

Then a major reform of the Danish upper secondary high school was planned for 2005. As to precalculus, it was inspired by the entry of graphing calculators and computer assisted systems allowing regression to transform tables into formulas, thus allowing realistic modeling to be included.

Now the goal defined the competences students should acquire:

The students can

- handle simple formulas and translate between symbolic and natural language and use symbolic language to solve simple problems with a mathematical content.
- apply simple statistical models for describing a given data set, pose questions based upon the model and sense what kind of answers are to be expected and knows how to formulate conclusion in a clear language.
- apply relations between variables to model a given data set, can make forecasts, and can reflect on them and their domain of relevance
- describe geometrical models and solve geometrical problems
- produce simple mathematical reasoning
- demonstrate knowledge about mathematical methods, applications of mathematics, and examples of cooperation between mathematics and other sciences, as well as its cultural and historical development
- apply information technology for solving mathematical problems

The means include

- The hierarchy of operations, solving equations graphically and with simple analytical methods, calculating percent and interest rates, absolute and relative change
- Formulas describing direct and inverse proportionality as well as linear, exponential ad power relations between variables
- Simple statistical methods for handling data sets, graphical representation of statistical materials, simple statistical descriptors
- Ratios in similar triangles and trigonometry used for calculations in arbitrary triangles.
- xy-plot of data sets together with characteristics of linear, exponential and power relations, the use of regression.
- Additional activities for 25 lessons are examples of mathematical reasoning and proofs, modeling authentic data sets, examples of historical mathematics.

The 2017 Curriculum

Then in 2017 e new reform was made to inspire more students to continue with the calculus level by moving some subjects to the precalculus level:

- interpreting the slope of a tangent as a growth rate in a mathematical model
- combinatorics, basic probability theory and symmetrical probability space
- the function concept and characteristics of linear, exponential and power functions and their graphs
- graphical handling of a quadratic function, and the logarithm functions and their characteristics
- graphical determination of a tangent, and monotony intervals, as well as finding extrema values in a closed interval
- prime characteristics at mathematical models and simple modelling using the functions above alone or in combination.

Relevance and Coherence

The 1966 had internal coherence with the previous and following curriculum, but with the emphasis on abstract algebra, there was little external coherence. It was indirectly relevant to students wanting later to take a calculus course but only little relevant to the daily life of students

The 1972 curriculum took the consequence and changed from a parallel curriculum to a serial curriculum so that it had internal coherence to the calculus curriculum, and by replacing quadratics with exponential functions, it obtained an external relevance to change calculations with a constant change-number or a constant change-percent. Also, including a considerable amount of probability gave coherence to eternal testing situations, however these were not part of student daily life, so they didn't add to the relevance for students. However, including the free lessons allowed the students to choose areas that they found relevant, in this case interest rates and saving and installment calculations as well as trigonometry.

The 1990 curriculum was inspired by this and re-included trigonometry and interest rates while at the same time reducing probability a little.

The 2005 reform was informed by the occurrence of competence concept as well as the advances in calculation technology. Her the function concept was replaced by variables to make it cohere more with external applications in science and economics and daily life. Now the probability was gone, so this curriculum showed coherence and relevance to external appliers and to the student's daily life as well for other school subjects. It was close to the ideal precalculus curriculum.

The 2017 reform was inspired by the wish to motive more to continue with a calculus course, so part of this was moved down to the precalculus level, making the two levels cohere better, however the things imported had little relevance to the students' daily life.

07. A Refugee Camp Curriculum

The name 'refugee camp curriculum' is a metaphor for a situation where mathematics is taught from the beginning and with simple manipulatives. Thus, it is also a proposal for a curriculum for early childhood education, for adult education, for educating immigrants and for learning mathematics outside institutionalized education. It considers mathematics a number-language parallel to our word-language, both describing the outside world in full sentences, typically containing a subject and a verb and a predicate. The task of the number-language is to describe the natural fact Many in space and time, first by counting and recounting and double-counting to transform outside examples of Many to inside sentences about the total; then by adding to unite (or split) inside totals in different ways depending on their units and on them being constant or changing. This allows designing a curriculum for all students inspired by Tarp (2018) that focuses on proportionality, solving equations and calculus from the beginning, since proportionality occurs when recounting in a different unit, equations occur when recounting from tens to icons, and calculus occurs when adding block-numbers next-to and when adding per-numbers coming from double-counting in two units.

Talking about 'refugee camp mathematics' thus allows locating a setting where children do not have access to normal education, thus raising the question 'What kind and how much mathematics can children learn outside normal education especially when residing outside normal housing conditions and without access to traditional leaning materials?'. This motivates another question 'How much mathematics can be learned as 'finger-math' using the examples of Many coming from the body as fingers, arms, toes and legs?'

So the goal of 'refugee camp mathematics' is to learn core mathematics through 'Finger-math' disclosing how much math comes from counting the fingers.

Focus 01. Digits as Icons with as Many Outside Sticks and Inside Strokes as They Present

Activity 01. With outside things (sticks, cars, dolls, animals), many ones are rearranged into one many-icon with as many things as it represents. Inside, we write the icon with as many strokes as it represents. Observe that the actual digits from 1 to 9 are icons with as many strokes as they represent if written less sloppy. A discovery glass showing nothing is an icon for zero. When counting by bundling in tens, ten become '1 Bundle, 0 unbundled' or 1B0 or just 10, thus needing no icon since after nine, a double-counting takes place of bundles and unbundled.

Focus 02. Counting Ten Fingers in Various Ways

Activity 01. Double-count ten fingers in bundles of 5s and in singles

● Outside, lift the finger to be counted; inside say "0 bundle 1, 0B2, 0B3, 0B4, 0B5 or 1B0. Then continue with saying "1B1, …, 1B5 or 2B". ● Outside, look at the fingers not yet counted; inside say "1 bundle less4, 1B-3, 1B-2, 1B-1, 1B or 1B0. Then continue with saying "2B-4, …, 2B or

2B0”. ● Outside, show the fingers as ten ones. ● Outside, show ten fingers as 1 5s and 5 1s; inside say “The total is 1Bundle5 5s” and write ‘T = 1B5 5s’. ● Outside, show ten fingers as 2 5s; inside say “The total is 2Bundle0 5s” and write ‘T = 2B0 5s’.

Activity 02. Double-count ten fingers in bundles of tens and in singles

● Outside, lift the finger to be counted; inside say “0 bundle 1, 0B2, 0B3, …, 0B9, 0Bten, or 1B0”. ● Outside, look at the fingers not yet counted; inside say “1 bundle less9, 1B-8, …, 1B-2, 1B-1, 1B or 1B0.

Activity 03. Counting ten fingers in bundles of 4s using ‘flexible bundle-numbers’.

● Outside, show the fingers as ten ones, then as one tens. ● Outside, show ten fingers as 1 4s and 6 1s; inside say “The total is 1Bundle6 4s, an overload” and write ‘T = 1B6 4s’ ● Outside, show ten fingers as 2 4s and 2 1s; inside say “The total is 2Bundle2 4s, a standard form” and write ‘T = 2B2 4s’. ● Outside, show ten fingers as 3 4s less 2; inside say “The total is 3Bundle, less2, 4s, an underload” and write ‘T = 3B-2 4s’.

Activity 04. Counting ten fingers in bundles of 3s using ‘flexible bundle-numbers’.

● Outside, show ten fingers as 1 3s and 7 1s; inside say “The total is 1Bundle7 3s, an overload” and write ‘T = 1B7 3s’. ● Outside, show ten fingers as 2 3s and 4 1s; inside say “The total is 2Bundle4 3s, an overload” and write ‘T = 2B4 3s’. ● Outside, show ten fingers as 3 3s and 1 1s; inside say “The total is 3Bundle1 3s, a standard form” and write ‘T = 3B1 3s’. ● Outside, show ten fingers as 4 3s less 2; inside say “The total is 4Bundle, less2, 3s, an underload” and write ‘T = 4B-2 3s’.

Activity 05. Counting ten fingers in bundles of 3s, now also using bundles of bundles.

● Outside, show ten fingers as 3 3s (a bundle of bundles) and 1 1s; inside say “The total is 1BundleBundle1 3s” and write ‘T = 1BB1 3s’. Now, inside say “The total is 1BundleBundle 0 Bundle 1 3s” and write ‘T = 1BB 0B 1 3s’. Now, inside say “The total is 1BundleBundle 1 Bundle, less2, 3s” and write ‘T = 1BB 1B -2 3s’.

Focus 03. Counting Ten Sticks in Various Ways

The same as Focus 02, but now with sticks instead of fingers.

Focus 04. Counting Ten Cubes in Various Ways

The same as Focus 02, but now with cubes, e.g. centi-cubes or Lego Bricks, instead of fingers. When possible, transform multiple bundles into 1 block, e.g. 2 4s = 1 2x4 block; inside say “The total is 1 2x4 block” and write ‘T = 2B0 4s.’

Focus 05. Counting a Dozen Finger-parts in Various Ways

Except for the thumps, our fingers all have three parts. So, four fingers have three parts four times, i.e. a total of T = 4 3s = 1 dozen finger-parts.

Focus 05 is the same as focus 02, but now with a dozen finger-parts instead of ten fingers.

Focus 06. Counting a Dozen Sticks in Various Ways

Focus 06 is the same as focus 03, but now with a dozen sticks instead of ten.

Focus 07. Counting a Dozen Cubes in Various Ways

Focus 07 is the same as focus 04, but now with a dozen cubes instead of ten.

Focus 08. Counting Numbers with Underloads and Overloads.

Activity 01. Totals counted in tens may also be recounted in under- or overloads.

● Inside, rewrite T = 23 as T = 2B3 tens, then as 1B13 tens, then as 3B-7tens. ● Try other two-digit numbers as well. ● Inside, rewrite T = 234 as T = 2BB3B4 tens, then as T = 2BB 2B14, then as T =

2BB 4B-6. Now rewrite T = 234 as T = 23B4, then as 22B14, then as 24B-6. Now rewrite T = 234 as T = 3BB-7B4, then as 3BB-6B-6. ● Try other three-digit numbers as well.

Focus 09. Operations as Icons Showing Pushing, Lifting and Pulling

Activity 01. Transform the three outside counting operations (push, lift and pull) into three inside operation-icons: division, multiplication and subtraction.

● Outside, place five sticks as 5 1s. ● Outside, push away 2s with a hand or a sheet; inside say “The total 5 is counted in 2s by pushing away 2s with a broom iconized as an uphill stroke” and write ‘T = 5 = 5/2 2s’. ● Outside, rearrange the 2 2s into 1 2x2 block by lifting up the bundles into a stack; inside say “The bundles are stacked into a 2x2 block by lifting up bundles iconized as a lift” and write ‘T = 2 2s = 2x2’. ● Outside, pull away the 2x2 block to locate unbundled 1s; inside say “The 2x2 block is pulled away, iconized as a rope” and write ‘T = 5 – 2x2 = 1’.

Five counted in 2s:

I I I I I (push away 2s) II II I (lift to stack) II
II I (pull to find unbundles ones) II
II I

Focus 10. The Inside Recount-Formula T = (T/B)xB Predicts Outside Bundlecounting Results

Activity 01. Use a calculator to predict a bundle-counting result by a recount-formula T = (T/B)xB, saying “from T, T/B times, B is pushed away”, thus using a full number-language sentence with a subject, a verb and a predicate.

● Outside, place five cubes as 5 1s. ● Outside, push away 2s with a ‘broom’; inside say “Asked ‘5/2’, a calculator answers ‘2.some’, meaning that 2 times we can push ways bundles of 2s. ● Outside , stack the 2s into one 2x2 stack by lifting; inside say “We lift the 2 bundles into one 2x2 stack, and we write T = 2 2s = 2x2 ● Outside, we locate the unbundled by, from 5 pulling away the 2x2 block; inside we say “Asked ‘5-2x2’, a calculator answers ‘1’. We write T = 2B1 2s and say “The recount-formula predicts that 5 recounts in 2s as T = 2B1 2s, which is tested by recounting five sticks manually outside.”

Activity 02. The same as activity 01, but now with 4 3s counted in 5s, 4s and 3s.

Focus 11. Discovering Decimals, Fractions and Negative Numbers.

Activity 01. When bundle-counting a total, the unbundled can be placed next-to or on-top.

● Outside, chose seven cubes to be counted in 3s. ● Outside, push away 3s to be lifted into a 2x3 stack to be pulled away to locate one unbundled single. Inside use the recount-formula to predict the result, and say “seven ones recounts as 2B1 3s” and write T = 2B1 3s. ● Outside, place the single next-to the stack. Inside say “Placed next-to the stack the single becomes a decimal-fraction ‘.1’ so now seven recounts as 2.1 3s” and write T = 2.1 3s. ● Outside, place the single on-top of the stack. Inside say “Placed on-top of the stack the single becomes a fraction-part 1 of 3, so now seven recounts as 2 1/3 3s” and write T = 2 1/3 3s. Now, inside say “Placed on-top of the stack the single becomes a full bundle less 2, so now seven recounts as 3.-2 3s” and write T = 3.-2 3s. Finally, inside say “With 3 3s as 1 bundle-bundle of 3s, seven recounts as 1BB-2 3s.”

Activity 02. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 3s.

Activity 03. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 4s.

Activity 04. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 5s.

Focus 12. Recount in a New Unit to Change Units, Predicted by the Recount-Formula

Activity 01. When bundle-counting, all numbers have units that may be changed into a new unit by recounting predicted by the recount-formula.

● Outside, chose 3 4s to be recounted in 5s. ● Outside, rearrange the block in 5s to find the answer T = 3 4s = 2B2 5s. Inside use the recount-formula to predict the result, and say “three fours recounts as 2B2 5s" and write T = 3 4s = 2B2 5s = 3B-3 5s = 2 2/5 5s. Repeat with other examples as e.g. 4 5s recounted in 6s.

Focus 13. Recount from Tens to Icons

Activity 01. A total counted in tens may be recounted in icons, traditionally called division.

● Outside, chose 29 or 2B9 tens to be recounted in 8s. ● Outside, rearrange the block in 8s to find the answer T = 29 = 3B5 8s and notice that a block that decreases its base must increase its height to keep the total the same. Inside use the recount-formula to predict the result, and say "With the recount-formula, a calculator predicts that 2 bundle 9 tens recounts as 3B5 8s" and write T = 29 = 2B9 tens = 3B 5 8s = 4B-3 8s = 3 5/8 8s. Repeat with other examples as e.g. 27 recounted in 6s.

* Now, inside reformulate the outside question 'T = 29 = ? 8s' as an equation using the letter u for the unknown number, u*8 = 24, to be solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign. Use an outside ten-by-ten abacus to see that when a block decreases its base from ten to 8, it must increase its height from 2.4 to 3. Repeat with other examples as e.g. 17 = ? 3s.

Focus 14. Recount from Icons to Tens

Activity 01. Oops, without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. Use an outside ten-by-ten abacus to see that when a block increases its base from 7 to ten, it must decrease its height from 3 to 2.1.

Activity 02. Use 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently 'less less 1' means adding 1.

Focus 15. Double-Counting in Two Physical Units

Activity 01. We observe that double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$. Repeat with other examples as e.g. 4$ per 5days.

Focus 16. Double-Counting in the Same Unit Creates Fractions

Activity 01. Double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%. We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Repeat with other examples as e.g. 2$/5$.

Focus 17. Mutually Double-Counting the Sides in a Block Halved by its Diagonal

Activity 01. Recount sides in a block halved by its diagonal? Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. Use these formulas to predict the sides in a half-block with base 6 and angle 30 degrees. Use these formulas to predict the angles and side in a half-block with base 6 and height 4.

Focus 18. Adding Next-to

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Since 3*5 is an area, adding next-to in 8s means adding areas, called integral calculus. Asking a calculator, the two answers, '2.some' and '5', predict the result as 2B5 8s.

Focus 19. Reversed Adding Next-to

Activity 01. With T1 = 2 3s and T2 adding next-to as T = 4 7s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Asking '3 5s and how many 3s total 2B6 8s?', using sticks will give the answer 2B1 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula

y' = Δy/t = (y2 – y1)/t.

Focus 20. Adding On-top

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Adding on-top in 5s, '3 5s + 2 3s = ? 5s?', re-counting must make the units the same. Asking a calculator, the two answers, '4.some' and '1', predict the result as 4B1 5s.

Focus 21. Reversed Adding On-top

Activity 01. With T1 = 2 3s and T2 as some 5s adding to T = 4 5s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting. Reversed addition is called backward calculation or solving equations.

Focus 22. Adding Tens

Activity 01. With T1 = 23 and T2 = 48, what is T1+T2 id added as tens?" Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Focus 23. Subtracting Tens

Activity 01. "If T1 = 23 and T2 add to T = 71, what is T2?" Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Focus 24. Multiplying Tens

Activity 01. "What is 7 43s recounted in tens?" Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Focus 25. Dividing Tens

Activity 01. "What is 348 recounted in 6s?" Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Focus 26. Adding Per-Numbers

Activity 01. "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

Activity 02. Two groups of voters have a different positive attitude to a proposal. How to find the total positive attitude?

● Asking “20 voters with 30% positive + 60 voters with 10% positive = 80 voters with ? positive.” Here we see that the unit-numbers 20 and 40 add directly whereas the per-numbers 30% and 10% add by areas since they must first transform to unit-numbers by multiplication, creating the areas.

Focus 27. Subtracting Per-Numbers

Activity 01. “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Focus 28. Adding Differences

Activity 01. Adding many numbers is time-consuming, but not if the numbers are changes, then the sum is simply calculated as the change from the start to the end-number.

● Write down ten numbers vertically. The first number must be 3 and the last 5, the rest can be any numbers between 1 and 9. In the next column write down the individual changes ‘end-start’. In the third column add up the individual changes along the way. Try to explain why the result must be 5-3 regardless of the in-between numbers.

● Draw a square with side n. Let n have a small positive change t. Show that the square will change with two next blocks when disregarding the small txt square. This shows that the change in an n*n square is 2*n*t, so if we want to add arears under a y = 2*n curve we must add very many small areas y*t = 2*n*t. However, since each may be written as a change in a square, we just have to find the change of the square from the start-point to the end-point. That is how integral calculus works.

Focus 29. Finding Common Units

Activity 01. “Only add with like units, so how add T = 4ab^2 + 6abc?”. Here units come from factorizing: T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b).

Focus 30. Finding Square Roots

Activity 01. A 7x7 square can be recounted in tens as 4.9 tens. The inverse question is how to transform a 6x7 block into a square, or in other words, to find the square root of 4.2 tens. A quick way to approach a relevant number is to first find two consecutive numbers, p and p+1, that squared are too low and too high. Then the an approximate value for the square root may be calculated by using that if p^2 < N < (p+1)^2, then $\sqrt{N} \approx \frac{\text{N+ p\^{}2}}{p*2}$.

Conclusion

A curriculum for a refugee camp assumes that the learners have only the knowledge they acquire outside traditional education. The same is the case for street children living outside traditional homes; and for nomadic people always moving around.

However, a refugee camp curriculum might also be applied in a traditional school setting allowing the children to keep on to the two-dimensional block numbers they bring to school allowing them to learn core mathematics as proportionality, equations, functions and calculus in the first grade, thus not needing parallel curricula later on.

So, the need for parallel curricula after grade 9 is not there by nature, but by choice. It is the result of disrespecting the mastery of many children bring to school and force them to adopts numbers as names along a number line, and force them to add numbers that are given to them without allowing them to find them themselves by counting, recounting and double-counting.

08. Do We Really Need Parallel Curricula?

Why do we need different curricula for different groups of students? Why can’t all students have the same curriculum? After all, the word-language does not need different curricula for different groups, so why does the number-language?

Both languages have two levels, a language level describing the 'outside' world, and a grammar level describing the 'inside' language. In the word-language, the language level is for all students and includes many examples of real-world descriptions, both fact and fiction. Whereas grammar level details are reserved for special students. Could it be the same with the number-language, teaching the language level to all students including many examples of fact and fiction? And reserving grammar level details to special students?

Before 1970, schools taught language as an example of its grammar (Chomsky, 1965). Then a reaction emerged in the so-called 'communicative turn' in language education. In his book 'Explorations in the function of language' Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a "communicative approach to the teaching of language (p. ix)" allowing more students to learn a less correct language to be used for communication about outside things and actions.

Thus, in language teaching the communicative turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language. So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use
- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

After all, the word language seems more voluminous with its many letters, words and sentence rules. In contrast, a pocket calculator shows that the number language contains ten digits together with a minor number of operations and an equal sign. And, where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. (Tarp, 2018)

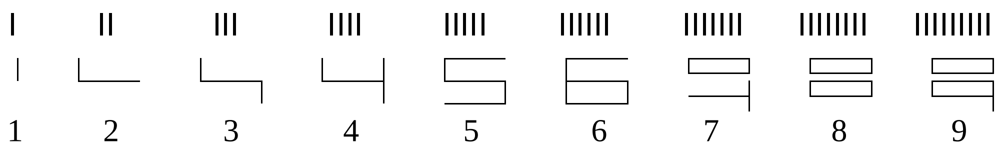

Figure 03. Digits as icons with as many sticks as they represent.

Furthermore, also the operations are icons describing how we total by counting unbundled, bundles, bundles of bundles etc. Here division iconizes pushing away bundles to be stacked, iconized by a multiplication lift, again to be pulled away, iconized by a subtraction rope, to identify unbundled singles that are placed next-to the stack iconized by an addition cross, or by a decimal point; or on-top iconized by a fraction or a negative number.

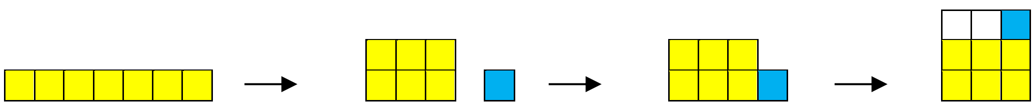

Figure 04. Seven counted as 2 3s & 1 or 2*B*1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

The operations allow predicting counting by a recount-sentence or formula ‘T = (T/B)*B’ saying that ‘from T, T/B times, B can be taken away’, making natural numbers as bundle- or block numbers as e.g. T = 3B2 4s or T = 3*4+2. And, using proportionality to change the unit when two blocks need the same unit to be added on-top, or next-to in a combined unit called integral calculus.

So, it seems as if early childhood education may introduce core mathematics as proportionality, solving equations, and integral calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction. Thus, there is indeed an opportunity to design a core curriculum in mathematics for all students without splitting it up in tracks. But, only if the word- and the number-language are taught and learned in the same way by describing outside things and actions in words and in numbers coming from counting and adding.

So, why not introduce a paradigm shift by teaching the number-language and the word-language in the same way through its use, and not before, thus allowing both languages being taught in the space between the inside language and the outside world.

Why keep on teaching the number-language in the space between the language and its meta-language or grammar, which makes the number-language more abstract, leaving many educational challenges unsolved despite close to half a century of mathematics education research.

Why not begin teaching children how to number, and stop teaching children about numbers and operation to be explained and learned before they can be applied to the outside world.

Why not accept and develop children’s already existing ‘many-sense’, instead of teaching them the eight different aspects of what is called ‘number-sense’ described by Sayers and Andrews (2015) that after reviewing research in the Whole Number Arithmetic domain created a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

And, why not simply let children talk about counting and adding constant and changing unit-numbers and per-numbers using full sentences with a subject, a verb, and a predicate; instead of teaching them the eight different components of what is called ‘mathematical competencies’ (Niss, 2003), thus reducing their numbers from eight to two: count and add (Tarp, 2002)?

So maybe we should go back to the mother Humboldt university in Berlin and reflect on Karl Marx thesis 11 written on the staircase: ”Die Philosophen haben die Welt nur verschieden interpretiert; es kömmt drauf an, sie zu verändern.” (The philosophers have only interpreted the world, in various ways. The point, however, is to change it.)

09. Conclusion

Let us return to the dream of the National Council of Teachers of Mathematics, to “provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.” Consequently, “everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics.” Furthermore, let us also accept what the council write about numbers: “Number pervades all areas of mathematics.”

So let us look for a curriculum that allows the students to understand and use and numbers, and see how far such a curriculum can carry all students without splitting into parallel tracks.

Now, what does it mean to understand a number like 456?

Is the ability to say that the three digits obey a place-value system where, from right to left, the first digits is ones, then tens, then hundred, then thousands, then, oops no-name unless we use the Chinese name wan, then no-name, then million, then no-name, then no-name, then billions or milliards, etc. Names and lack of names that give little meaning to children where only few understand why ten has its own name but not its own icon but has two digits as 10.

On the other hand, is it the ability to understand that of course ten becomes 10 since it is short for '1 bundle and no singles'? And, that it would have been 20 had we counted in bundles of 5s instead as they do on an eastern abacus, where the two digits 10 then would be used for the bundle size 5.

And that ten is just another word for bundle, and hundred for bundle-bundle, i.e. 2 times bundling; and thousand for bundle-bundle-bundle, i.e. or bundling 3 times, etc. where we never end in a situation with no name. Isn't it both power and beauty to transform an unorganized total into a repeated bundling with the ability that only the decimal point moves if you change the number of bundling, T = 32.1 tens = 3.21 tentens, which is not the case with romans bundling where 3 tens is 6 fives. The romans didn't stick to bundling bundles since they bundled in both fives and tens and fifties but not in 5 5s, i.e. in 25s. Power and beauty comes from bundle bundles only.

Consequently, to understand the number 456 is to see it, not as one number, but as three numberings of a total that has been bundled 0 times, bundled 1 times, bundled 2 times, etc. And to read the total as 4 bundled 2 times and 5 bundled once and 6 not bundled, or as 4 bundle-bundles and 5 bundles and 6 unbundles singles. And to write the total as T = 4BB 5B 6. And to allow the same total to be recounted with an underload as T = 4BB 6B -4, or with an overload as T = 45B 6 = 4BB 56; or as T = 45B -4 if combining overload and underload.

This understanding allows an existing unorganized total become a number-language sentence connecting the outside subject T to an inside calculation, T = 4*B^2 + 5*B^1 + 6*B^0. Which again is an example, or specification, of an unspecified number-formula or polynomial T = a*x^2 + 5*x + 6.

The power and beauty of the number-formula is manifold. It shows four ways to unite: power, multiplication, addition and next-to block addition also called integration. By including the units, we realize that there are only four types of numbers in the world as shown in the algebra-square above, constant and changing unit-numbers and per-numbers, united by precisely these four ways: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers.

Furthermore, we observe that splitting a total into parts will reverse uniting parts into a total, meaning that all uniting operations have reverse operations: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers. This makes root a factor-finder, and logarithm a factor-counter, and differentiation a finder of per-numbers.

And, if we use the word 'equation' for the need to split instead of unite, we observe that solving an equation means isolating the unknown by moving numbers to the opposite side with opposite calculation sign. Furthermore, using variables instead of digits we observe that the number-formula contains the different formulas for constant change as shown above.

As to a non-constant change, there are two kinds. Predictable change roots calculus as shown by the algebra-square; and unpredictable change roots statistics to instead 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for unpredictable numbers.

Thus the 'power and beauty' of mathematics resides in the number-formula, as does the ability 'to use mathematics in students' personal life, in the workplace and in further study'. So, designing a curriculum based upon the number-formula will 'provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.'

Furthermore, a number-formula based curriculum need not split into parallel curricula until after calculus, i.e. until after secondary education.

So, one number-language curriculum for all is possible, as it is for the word-language. Thus, it is possible to allow all students to learn about the four ways to unite and the five ways to split a total.

The most effective way to design a curriculum for all students is to adopt the curriculum designed refugee camp from the beginning since it accepts and develops the number-language children bring to school. Presenting figures and operations as icons, it bridges outside existence with inside essence. All four uniting methods occur in grade one when counting and recounting in different units, and when adding totals next-to and on-top. It respects the natural order of operations by letting division precede multiplication and subtraction, thus postponing addition until after counting, recounting and double-counting have taken place. It introduces the core recounting-formula expressing proportionality when changing units from the beginning, which allows a calculator to predict inside an outside recounting result. By connecting outside blocks with inside bundle-writing, geometry and algebra are introduced as Siamese twins never to part. Using flexible bundle-numbers connects inside decimals, fractions and negative numbers to unbundled leftovers placed next-to or on-top the outside block. It introduces solving equations when recounting from tens to icons. It introduces per-numbers and fractions when double counting in units that may be the same or different. And, it introduces trigonometry before geometry when double-counting sides in a block halved by its diagonal.

Another option is to integrate calculus in a precalculus course by presenting integral calculus before differential calculus, which makes sense since until now inverse operations are always taught after the operation, subtraction after addition etc. Consequently, differential calculus should wait until after it has been motivated by integral calculus that is motivated by adding changing per-numbers in trade and physics, and by adding percent in statistical double-tables.

In their publication, the National Council of Teachers of Mathematics writes "High school mathematics builds on the skills and understandings developed in the lower grades. (p. 19)" If this has to be like that then high school education will suffer from lack of student skills and misunderstandings; and often teachers say that precalculus is the hardest course to teach because of a poor student knowledge background.

So, we have to ask: Can we design a fresh-start curriculum for high school that integrates precalculus and calculus? And indeed, it is possible to go back to the power and beauty of the number-formula as described above, and build a curriculum based upon the algebra-square. It gives an overview of the four kinds of numbers that exist in the outside world, and how to unite or split them. It shows a direct way to solve equations based upon the definitions of the reverse operations: move to opposite side with opposite calculation sign.

Furthermore, it provides 2x2 guiding questions: how to unite or split into constant per-numbers, as needed outside when facing change with a constant change-factor? And how to unite or split into changing per-numbers that are piecewise or locally constant, as needed outside when describing e.g. the motion with a changing velocity of a falling object.

As a reverse operation, differential calculus is a quick way to deliver the change-formula that solve the integration problem of adding the many area-strips coming from transforming locally constant per-numbers to unit-numbers by multiplication. Also, by providing change-formulas, differential calculus can extend the formulas for constant change coming from the number-formula. An additional extension comes from combining constant change-number and change-percent to one of the most beautiful formulas in mathematics that is too often ignored, the saving-formula, $A/a = R/r$, a formula that is highly applicable in individual and social financial decisions.

Working with constant and changing change also raises the question what to do about unpredictable change, which leads directly into statistics and probability.

So designing and implementing a fresh-start integrated precalculus and calculus curriculum will allow the National Council of Teachers of Mathematics to have their dream come through, so that in the future high schools can provide all students "with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals."

As a number-language, mathematics is placed between its outside roots and its inside meta-language or grammar. So, institutionalized education must make a choice: should the number-language be learned through its grammar before being applied to outside descriptions; or should it as the word-language be learned through its use to describe the outside world? I short, shall mathematics education tach about numbers and operations and postpone applications till after this has been taught? Or shall mathematics education teach how to number and how to use operations to predict a numbering result thus teaching rooting instead of applications?

Choosing the first 'inside-inside' option means connecting mathematics to its grammar as a 'meta-matics' defining concepts 'from above' as top-down examples from abstractions instead of 'from below' as bottom-up abstractions from examples. This is illustrated by the function concept that can be defined from above as an example of a set-product relation where first component identity implies second-component identity, or from below as a common name for 'stand-by' calculations containing unspecified numbers.

Choosing the inside-inside 'mathematics-as-metamatics' option means teaching about numbers and operations before applying them. Here numbers never carry units but become names on a number-line; here numbers are added by counting on; and the other operations are presented as inside means to inside tasks: multiplication as repeated addition, power as repeated multiplication, subtraction as inverse addition, and division as inverse multiplication. Here fractions are numbers instead of operators needing numbers to become numbers. Here adding numbers and fractions without units leads to 'mathe-matism', true inside classrooms where 2+3 is 5 unconditionally, but seldom outside classrooms where counterexamples exist as e.g. 2weeks + 3days is 17days or 2 3/7 weeks. Here geometry and algebra occur independently and before trigonometry. Here primary and lower secondary school focus on addition, subtraction, multiplication and division with power and root present as squaring and square roots, thus leaving general roots and logarithm and trigonometry to the different tracks in upper secondary school where differential calculus is introduced before integral calculus, if at all.

Choosing the inside-outside 'mathematics-as-manymath' option means to teach digits as icons with as many strokes as they represent. And to also teach operations as icons, rooted in the counting process where division wipes away bundles to be stacked by multiplication, again to be removed by subtraction to identify unbundled singles. This will allow giving a final description of the total using a full sentence with a subject, a verb and a predicate predicted by the recount-formula T = (T/B)*B, e.g. T = 2Bundle 1 3s = 2.1 3s = 2 1/3 3s thus including decimal numbers and fractions in a natural number. Here a double description of Many as an outside block and an inside bundle-number allows outside geometry and inside algebra to be united from the start. Once counted, totals can be recounted. First in the same unit to create overloads and underloads introducing negative numbers. Then between icon- and ten-bundles introducing the multiplication table and solving equations. Then double-counting in two units creates per-numbers becoming fractions with like units. Finally, recounting the sides in a block halved by its diagonal will root trigonometry before geometry, that integrated with algebra can predict intersection points. Then follows addition and reversed addition in its two versions, on-top or next-to. On-top addition calls for recounting the totals in the same unit, thus rooting proportionality. And next-to addition means adding blocks as areas, thus rooting integral calculus. Reversed addition roots equations and differential calculus. Per-numbers are added as operators including the units, thus rooting integral calculus, later defined as adding locally constant per-numbers. Thus, this option means that the core of mathematics is learned in primary school allowing ample of time in secondary school to enjoy the number-language literature by examining existing models or producing models yourself. And it means that only one curriculum is needed for all students as in the word-language.

Furthermore, the root and use of calculus to add changing per-numbers is easily introduced at the precalculus level when adding ingredients with different per-numbers and when adding categories in statistics with different percent.

And, the fact that the difficulty by adding many numbers disappears when the numbers can be written as change-numbers since adding up any number of small changes total just one change from the start- to the end-number. Which of course motivates differential calculus.

Consequently, there is no need for a parallel curriculum to the traditional, since everybody can learn calculus in a communicative way. Of course, one additional optional course may be given to look at all the theoretical footnotes.

To offer a completely different kind of mathematics as graph theory and game theory and voting theory risks depriving the students of the understanding that mathematics is put in the world as a number-language that use operations to predict the result of counting, recounting and double-counting. A language that only needs four operations to unite parts into a total, and only five operations to split a total into parts.

Without calculus in the final high school curriculum, students may not understand how to add per-numbers and might add them as unit-numbers instead of as areas; and this will close many 'doors to productive futures' as the US National Council of Teachers of Mathematics talks about.

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02. COMMENTS TO A DISCUSSION PAPER

Dear Colleagues. First some information. The deadline for the chapter outlines has been extended until at least the end of February (from 15 February). I would like to open discussion between us on several matters. Please "Reply All" if you have any thoughts to add.

1. Recap conversations and conclusions from the conference.

Since neither Jennie nor I were with you two at the conference, it would be helpful if Allan and Jaime could summarise their work at the conference for us. This will provide a common starting point for us. Can you please outline (summary dot points will be fine) the key matters you discussed, and any conclusions you reached.

Comments to question one

The last day at the conference was for the most part used to discuss the meaning of the two terms 'coherence and relevance'. That left only very little time for us to discuss more than a general structure of our contribution.

We decided to use a format focussing on case studies of curricula from upper secondary school since this is where the side-curricula to the main curriculum occur; and to include our own contributions as exemplary cases.

Silva: a curriculum that introduces untraditional mathematics more relevant to students

Tarp: an untraditional curriculum for starters as e.g. children or migrants

- ("But we should be thinking outside the box, and beginning with the child, rather than the discipline", Discussion summary Session 1 Group B ICMI Study 24 Conference, Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Working\Records of conversations.)
- ("(..) it is appropriate that curricula should continue to evolve, perhaps in radical ways.", Taguma, p. 10)
- ("Coherence with what children bring to school. We colonize the "mastery of many" that children have when they bring to school. What kind of mathematics would we have if we built on it? In Europe we are still stuck with set-based mathematics. Be careful about adding without units. This leads to incoherence with what is inside the classroom and what is outside.", Wednesday session notes McCallum, Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Working\Records of conversations)

As well as a Danish end of high school pre-calculus curriculum that I had described in my working group

- ("Denmark: they tried to make a different sort of coherence between mathematics and the outside world with precalculus course. The used different names for linear and exponential functions; growth by adding and growth by percentage. Each was connected with a context from the real world (buying, bank accounts). Again, this is an example of coherence through relevance.", Session_2_McCallum, Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Working\Records of conversations)

The February draft follows these lines and includes also a section for contributions from other papers and sources.

2. Definitions of coherence and relevance

These two terms will need to be defined and used consistently in all the writing in this Theme. The discussion of the terms and their definitions will be an important part of the introduction to the section that contains the chapters for the Key Questions.

I would like to make the following distinction between the two terms as they apply to the intended curriculum:

coherence is 'internal' – the alignment (or otherwise) within and between various aspects of the curriculum

relevance is 'external' – the interaction between the curriculum and the needs and aspirations (of students/young people, the workplace, universities, society) that often drive reforms

Is this distinction sound? Please add your thoughts below.

Comments to question two

Thank you very much indeed for making this very clear, and for choosing definitions cohering with the dictionary definitions of the two terms. This is really very helpful.

At my short initial presentation at the conference, I included these two definitions in my presentation as seen in the Dropbox version (Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Presentations). On slide 11 I wrote “MerriamWebster: History and Etymology for cohere, borrowed from Latin cohaerēre "to stick together, be in contact with, be connected," from co- CO- + haerēre "to be closely attached, stick,". And on slide 12 I wrote about relevance “Oxford Dict.: Closely connected or appropriate to what is being done or considered.”

When we began discussing the meaning of the two terms the last day at the conference, I suggested that instead of trying to find a common understanding in the group we should accept and respect the general meaning of the two terms as described in dictionaries. However, the discussion between a minority of the persons continued almost all of the time where many were silently waiting for the work in the five writing groups to begin. Of course, it is a little difficult to have a common discussion among peoples with so many different cultural backgrounds and with so different access to the common language used. But the question could be raised if the time could have been spent in a different more productive way by accepting that day what now has been accepted, the dictionary meanings of the two words.

Also, in my post-conference essay “A New Curriculum - But For Which Of The 3x2 Kinds Of Mathematics Education” that I have sent to the members of theme B, I devoted the first chapter “Coherence And Relevance In The School Mathematics Curriculum” to discuss the meaning of these two terms, again arguing that we should stay with the dictionary definitions since they seem to agree.

So again, thank you very much for allowing us to do that.

3. The focus for our work

If we broadly accept the distinction above, it seems that the focus for the work on KQ2 will be mostly on ‘relevance’ of the reforms to the groups of students for whom it is intended. The coherence of the reforms may also need to be analysed, but my feeling is that the emphasis must be on relevance.

Do you agree? Please add your thoughts below.

Comments to question three

Again, thank you very much for making this clear. I could not agree more. As a social institution, education also has a social responsibility to educate all children and teenagers. So in the case where a structure has been chosen that keeps children and teenagers together in multi-year classes and tracks, the question about curricula for the side-track to the main track becomes an important one, that need to be addressed and that should be the main if not the full focus in this key question. The alternative is to allow teenagers to choose among a list of half-year or full-year blocks as in North America, practically absent at the conference, which seems to indicate that the curriculum reform problems reside with multi-year classes and tracks.

4. A lack of papers directly relating to KQ2

This is a known and difficult matter:

I have attached the spreadsheet of the other links for KQ2 from the survey

KQ1 and KQ3 (especially) would seem to have papers about reforms that are directed towards particular needs, you may be aware of other research (not part of Study 24) that is particularly relevant to our work

Please share your responses to these suggestions and any cases that you think will be worth analysing from these other sources.

Comments to question four

Again, thank you very much for giving us the opportunity to share inputs with the other groups, and to receive input from them.

Once the draft structure is discussed and adapted and accepted, contributors will be asked to elaborate on their suggestions as to specific questions asked, and to identify relevant references.

The questions, of course, will be inspired by the title of the ICMI study 24: "School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities." Auguste Comte invented the term 'sociology' and developed 'positivism' as the idea that the dynamic laws of physics also applies to social conditions. Although questioned, inspiration may still be found in this idea, also in the case of social curricula.

The fundamental law of physical dynamics is Newton's second law originally saying, "The rate of change of the momentum of a body is directly proportional to the net force acting on it." Today it is formulated as "A force's impulse gives a change in momentum", or as an equation "the product of the force and its period is proportional to the product of the inertial mass and the change affected."

This means that a change depends upon three factors: The change increases with the net force applied and with the period it is applied, and it decreases with the size of the inertia of the system. And here, the net force is the combination of all the forces applied.

Based upon this, two questions will be answered in the cases and asked specifically to contributors:

> In the latest reform for the side-tracks in upper secondary school:
>
> 01) Which outside and inside forces were acting and with what strength and over which period?
>
> 02) What was the resistance against the change exerted by systemic inertia?

5. A policy dimension

In my experience reforms can often be initiated in response to policy directions outside of mathematics education. For example, in my country policies relating to gender and mathematics, or improving educational outcomes of students from particular ethnic backgrounds (including indigenous students) tend to drive the goals and some of the substance of the intended mathematics curriculum.

Is this common? Do you think it is important to discuss this policy dimension in relation to any reforms we analyse for relevance (and coherence)?

<u>Comments to question five</u>

Thank you very much for asking this question that coheres with what is called the outside force in the above description of Newton's law exported to social dynamics. Interesting examples hopefully will occur describing what happens when outside forces pointing to increased relevance meet inside forces pointing to the importance of keeping inside coherence unchanged.

03. A Mathematics Teacher Using Communicative Rationality Towards Children

Defining, as Habermas, communicative rationality as ‘wanting to reach understanding to secure the participant speakers an intersubjectively shared lifeworld, thereby securing the horizon within which everyone can refer to one and the same objective world’; and defining the objective world as ‘the totality of entities concerning which true propositions are possible’ (thus, to avoid self-reference, not seeing propositions as part of the objective world); and seeing a speech act as ‘a speaker pursuing the aim of reaching understanding with a hearer about something’, we might ask:

How can a math teacher use communicative rationality to establish a non-patronizing power-free rational dialogue with grade one children about the objective fact Many, present in both the children and the teacher’s life-world; thus accepting four fingers held together two by two being rationalized as (as do children) ‘the total is two twos’ and not just as ‘four’?

It turns out, that accepting the children’s 2dimensional block-numbers instead of letting the system-world colonize their lifeworld by enforcing upon them 1dimensional line-numbers, will allow co-creating and co-developing a mastery of Many (a post-setcentric ‘ManyMath’) where digits are icons with as many strokes as they represent (5 strokes in the 5-icon); and where also operations are icons for the counting process (division is a broom sweeping away bundles, multiplication is a lift stacking bundles into a block, subtraction is a rope drawing away the block to look for unbundle singles, placed next to the block as decimals or on-top of the block counted in bundles as fractions or negative numbers).

Once counted, a total can be recounted in the same unit to create underload and overload (T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s); or in another unit predicted by a calculator with the recount formula ‘T = (T/B)*B’ saying ‘From T, T/B times, B can be taken away’; or from tens to icons rooting equations solved by recounting (? 7s = u*7 = 42 = (42/7)*7, so ? = u = 42/7); or from icons to tens rooting multiplication tables (T = 7 8s = ?tens); or in a different units creating per-numbers used to bridge the unites by recounting (with T = 2kg = 3$ we have the per-number 2kg/3$ = 2/3 kg/$, and T = 6kg = (6/2)*2kg = (6/2)*3$ = 9$).

Once counted and recounted, totals can be added on-top needing recounting (proportionality) to make the units like, or next-to that by adding areas is integral calculus, that leads to differential calculus when reversed.

In short, having as a dream to establish third generation Enlightenment republics in Europe, Habermas uses Weber’s warning against rationalization taken too far to become an iron cage to, in Habermas’ version, warn against a colonization of the lifeworld by systems.

Thus, in the case of mathematics education, the institutionalized system wants to colonize the children’s own Many-math by forcing upon them, not mathematics, but ‘meta-matism’, a mixture of ‘meta-matics’ defining concepts as examples of abstractions instead of as abstractions from examples; and ‘mathe-matism’ true inside itself where 2+3 IS 5 unconditionally, but seldom outside in the objective world where adding numbers without units creates counter-examples as for example 2weeks + 3days = 17 days = 2 3/7 weeks.

Maybe Marx has a point in his Feuerbach Thesis 11: “Philosophers have hitherto only interpreted the world in various ways; the point is to change it.”

Tarp, A. (2018). Mastering Many. *Journal of Mathematics Education 11*(1), 103-117.

04. ADDITION-FREE STEM-BASED MATH FOR MIGRANTS

A curriculum architect is asked to avoid traditional mistakes when designing a curriculum for young migrants that will allow them to soon become STEM pre-teachers and pre-engineers. Multiplication formulas expressing recounting in different units suggest an addition-free curriculum. To answer the question 'How many in total?' we count and recount totals by bundling in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. A recount formula that expresses proportionality when changing units is a core prediction formula in all STEM subjects.

DECREASED PISA PERFORMANCE DESPITE INCREASED RESEARCH

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise has funding, see e.g. the Swedish National Centre for Mathematics Education. Yet, despite extra research and funding, and despite being warned against the possible irrelevance of a growing research industry (Tarp, 2004), decreasing Swedish PISA results caused OECD to write the report "Improving Schools in Sweden" (2015a) describing its school system as "in need of urgent change" since "more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life (p. 3)."

To find an unorthodox solution to poor PISA performance we pretend that a university in southern Sweden, challenged by numerous young male migrants, arranges a curriculum architect competition: "Theorize the low success of 50 years of mathematics education research; and derive from this a STEM based core curriculum allowing young migrants to soon become STEM pre-teachers and pre-engineers."

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to improve schools in Sweden and elsewhere.

SOCIAL THEORY LOOKING AT MATHEMATICS EDUCATION

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now (p. 16)."

As to institutions, of which mathematics education is an example, Bauman talks about rational action "in which the *end* is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical (p. 79)". He then points out that "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement** (p. 84)." Of which one example is saying that the goal of mathematics education is to learn mathematics since such a goal statement is obviously made meaningless by its self-reference.

The link between a goal and its means is also present in existentialist philosophy described by Sartre (2007) as holding that "Existences precedes essence (p. 20)". Likewise, Arendt (1963) points out that practicing a means blindly without reflecting on its goal might lead to practicing "the banality of evil". Which makes Bourdieu (1977) says that "All pedagogic action is, objectively, symbolic violence insofar as it is the imposition of a cultural arbitrary by an arbitrary power (p. 5)". This raises the question if mathematics and education is universal or chosen, more or less arbitrarily.

DIFFERENT KINDS OF EDUCATION

The International Commission on Mathematical Instruction, ICMI, named its 24th study "School mathematics Curriculum Reforms: Challenges, Changes and Opportunities". At its conference in Tsukuba, Japan, in November 2018 it became clear during plenary discussions that internationally there is little awareness of two different kinds of educational systems practiced from secondary school.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

Moreover, as a social institution involving monopolizing and individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlightens teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching only one subject, and in their own classrooms.

To protect its republic against attack from its German speaking neighbors, France created elite schools with multi-year forced classes, called 'pris-pitals' by Foucault (1995) pointing out that it mixes power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

DIFFERENT KINDS OF MATHEMATICS

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, rhetoric and logic (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both being action-words rooted in the physical fact Many through their original meanings. This resonates with the primary goal of knowledge seeking and education, to be able to master the outside world through proper actions. And in Europe, Germanic countries taught counting and reckoning in primary school and algebra and geometry in the lower secondary school until about 50 years ago when they all were replaced by the setbased 'New Math' even if mathematics is a mere label and not an action-word. But the point was that by being setbased mathematics could become a self-referential 'meta-matics' needing no outside root. Instead it could define concepts top-down as examples of inside abstractions instead of bottom-up as abstractions from outside examples.

Russell objected by pointing to the set of sets not belonging to itself. Here a set belongs only if it does not: if $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. In this way Russell shows that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false. Instead Russell proposed a type theory banning self-reference. However, mathematics ignored Russell's paradox and his type theory since it prevented fraction from being numbers by being defined from numbers.

Instead, setbased mathematics changed classical grounded mathe-matics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be recounted as 6 1s.

Although spreading around the world, the United States rejected the New Math by going 'back to basics'. So today three kinds of mathematics may be taught: a pre-setbased, a present setbased and a post-setbased version (Tarp, 2017).

THE TRADITION OF MATHEMATICS EDUCATION

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5. This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations having addition as the base with subtraction as reversed operation, where multiplication is repeated addition with division as reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction creates negative numbers, where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2. Then halving a block by its diagonal creates a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

Then calculations with unspecified numbers leads to creating additional education about the results of operating on such numbers, e.g. that (*a*+*b*) * (*a*-*b*) = *a*^2 - *b*^2.

In a calculation, changing the input will change the output. Relating the changes creates an operation on the calculation called differentiation, also creating additional education about the results of operating on calculations, e.g. that (*f**g*)'/(*f**g*) = *f'*/*f* + *g'*/*g*. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6**x*^2d*x* = 2**x*^3.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modeling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

THEORIZING THE SUCCESS OF MATH EDUCATION RESEARCH

When trying to theorize the low success of 50 years of mathematics education research, the first question must be what we mean by mathematics and education and research.

As to education, who needs it if they already know? So, we must ask: what is it that students do not know and must be educated in? Or in other words: what is the goal of mathematics education? Two answers present themselves, one pointing to on the outside existence rooting mathematics, the other to its inside institutionalized essence.

Giving precedence to inside essence over outside existence the answer is: of course, the goal of mathematics education is to teach mathematics as defined by mathematicians at the universities. Modern societies institutionalize the creation and mediation of knowledge as universities and schools. Here priority should be given to useful knowledge as mathematics; and of course, mathematics must be taught before it can be applied, else there is nothing to apply! However, although very useful, mathematics is at the same time very hard to learn as witnessed again and again by research, carefully and in detail describing students' learning problems. So, 50 years of mathematics education research has not been unsuccessful, on the contrary, it has been extremely successful in proving that, by its very nature, mathematics is indeed difficult. The 'essence precedes existence' stance is typically argued by university scholars as e.g. Bruner (1962), Skemp (1971), Freudenthal (1973), and Niss (1994).

Giving precedence to outside existence over inside essence the answer is: It is correct that research has demonstrated many learning difficulties. However, what has been taught is not an outside rooted mathe-matics, but an inside self-referring meta-matism as defined above. And, until now research has primarily studied the two contemporary versions of mathematics, the pre-setbased and the present setbased version whereas very little if any research has studied the post-setbased mathematics that gives precedence to existence over essence by accepting and developing the mastery of Many in the number-language that children develop before school.

Giving precedence to essence or existence makes a difference to math education.

In its pre-setbased version, mathematics presents digits as symbols, and numbers as a sequence of digits obeying a place value system. Once a counting sequence is established, addition is defined as counting on, after which the other operations are defined from addition. Fractions are seen as numbers.

In its present setbased version, mathematics uses the inside concept set for deriving other concepts. Here numbers describe the cardinality of a set, and an operation is a function from a set product into a set. Again, addition is taught as the first operation.

In its post-setbased version, mathematics presents digits as icons with as many sticks as they represent; and numbers always carry units as part of number-language sentences bridging the outside existence with inside essence, thus connecting outside blocks with inside bundles, T = 2 3s = 2B0 3s. Here operations are icons also, and here counting comes before adding to respect that counting involves taking away bundles by division to be stacked by multiplication, to be pulled away by subtraction to find unbundled ones. And here counting and recounting and double-counting precedes the two forms of addition, on-top and next-to. And here fractions are per-numbers, both being operators needing numbers to become numbers.

Likewise, the core concept 'function' is treated differently. Pre-setbased mathematics sees a function as a calculation containing specified and unspecified numbers. Present setbased mathematics sees a function as a subset of set product where first-component identity implies second-component identity. Post-setbased mathematics sees a function as a number-language sentence T = 2*3 relating an outside existing total with an inside chosen essence.

Choosing an 'inside-outside' view will make mathematics self-referring and difficult by its missing link to its outside roots. Whereas choosing an 'outside-inside' view will allow mathematics develop the language children use to assign numbers to outside things and actions, i.e. a number-language similar to the word-language.

Mathematics as the Grammar of the Number-Language

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: ”This is a chair”. As does the number-language assigning numbers instead: ”The 3 chairs each have 4 legs”, abbreviated to ”The total is 3 fours”, or ”T = 3 4s” or ”T = 3*4”. Unfortunately, the tradition hides the similarity between word- and number-sentences by leaving out the subject and the verb and just saying ”3*4 = 12”.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ”This is a chair” leads to a meta-sentence ”The word 'is' is an auxiliary verb”. Likewise, the sentence ”T = 3*4” leads to a meta-sentence ”The operation '*' is commutative”.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead its self-referring setbased form has turned mathematics into a grammar labeling its outside roots as 'applications', used as means to dim the impeding consequences of teaching a grammar before its language.

Before 1970, language was taught as an example of its grammar (Chomsky, 1965). Then a reaction emerged. In his book 'Explorations in the function of language' Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to

explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a "communicative approach to the teaching of language (p. ix)" allowing more students to learn a less correct language to be used for communication about outside things and actions.

Time for a Linguistic Turn in the Number-Language also

Thus, in language teaching a new version of the linguistic turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language.

So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use
- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

Maybe the time has come to realize that the two statements '2+3 = 5' and '2*3 = 6' have a different truth status.

The former is a conditional truth depending on the units. But, with 3 as the unit, the latter is an unconditional truth since 2 3s may always be recounted as 6 1s.

In short, maybe it is time to look for a different outside-inside mathematics to replace the present tradition, inside-outside meta-matism? And to ask what kind of math grows from the mastery of Many that children develop through use and before school?

DIFFERENCE RESEARCH LOOKS AT MATHEMATICS EDUCATION

To answer, we let Many open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a mathematics core curriculum based upon examples of Many in a STEM context (Lawrenz et al, 2017). Using 'Difference-research' (Tarp, 2017) searching for hidden differences making a difference, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

MEETING MANY CREATES A 'COUNT-BEFORE-ADD' CURRICULUM

Meeting Many, we ask "How many in Total?" To answer, we count by bundling to create a number-language sentence as e.g. T = 2 3s that contains a subject and a verb and a predicate as in a word-language sentence; and that connects the outside total T with its inside predicate 2 3s (Tarp, 2018b). Rearranging many 1s into one symbol with as many sticks or strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting by bundling and stacking:

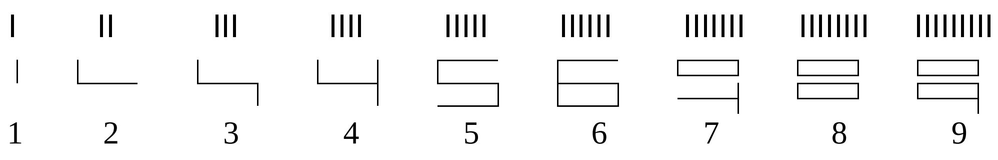

Figure 1: Digits as icons with as many sticks as they represent.

Holding 4 fingers together 2 by 2, a 3year-old will say 'This is not 4, this is 2 2s', thus describing what exists, bundles of 2s and 2 of them. This inspires 'bundle-counting', counting a total in icon-bundles to be stacked as bundle- or block-numbers, which can be recounted and double-counted before being processed by next-to and on-top addition, direct or reversed. Thus, a total T of 5 1s is recounted in 2s as T = 2 2s & 1; described by ‘bundle-writing’ as T = 2B1 2s; or by ‘decimal-writing’, T = 2.1 2s, where, with a bundle-cup, a decimal point separates the bundles inside from the outside unbundled singles; or by ‘deficit-writing’, T = 3B-1 2s = 3.-1 2s = 3 bundles less 1 2s.

To bundle-count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a rope pulling the block away).

A calculator thus predicts the result by a ‘recount formula’ $T = (T/B)*B$ saying that ‘from T, T/B times, Bs can be taken away’: entering ‘5/2’ on a calculator gives ‘2.some’, and ‘5 – 2x2’ gives ‘1’, so T = 5 = 2B1 2s. The unbundled can be placed next-to the stack as .1 or on-top as ½ counted in 2s, thus rooting decimals and fractions.

The recount formula occurs all over science. With proportionality: $y = c*x$; in trigonometry as sine, cosine and tangent: $a = (a/c)*c = \sin A*c$ and $b = (b/c)*c = \cos A*c$ and $a = (a/b)*b = \tan A*b$; in coordinate geometry as line gradients: $\Delta y = \Delta y/\Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In economics, the recount formula is a price formula: $ = ($/kg)*kg = price*kg, $ = ($/day)*day = price*day, etc.

Recounting in the Same Unit or in a Different Unit

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2B1 2s as 1B3 2s with an outside ‘overload’; or as 3B-1 2s with an outside ‘underload’ thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48; or 336 = 35B-14, so 336/7 = 5B-2 = 48.

Recounting in a different unit means changing unit, also called proportionality. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2B2 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’. The answer, ‘2.some’, predicts that the unbundled singles come from taking away 2 5s, now asking ‘3*4 – 2*5’. The answer, ‘2’, predicts that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s or 2 2/5 5s.

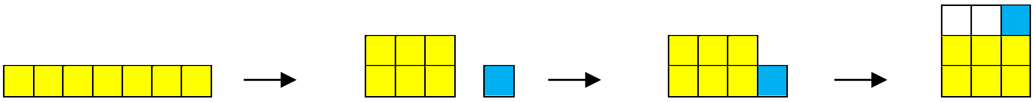

Figure 2: Seven counted as 2 3s & 1 or 2B1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

Recounting from Icons to and from Tens

Recounting from icons to tens by asking e.g. ‘2 7s = ? tens’ is eased by using underloads: T = 2 7s = 2*7 = 2*(B-3) = 20-6 = 14; and T = 6 8s = 6*8 = (B-4)*(B-2) = BB - 4B - 2B -- 4*2 = 10B – 4B – 2B + 8 = 4B8 = 48. This makes sense since widening the base form t7 to ten will shorten the height from 6 to 4.8.

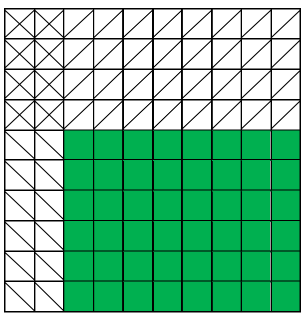

Figure 3: On an abacus 6 8s = 6*8 = (B-4)*(B-2) = 10B - 4B - 2B + 4 2s = 4B8 = 48.

Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer directly as 2*7 = 14, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation u*7 = 35. It is easily solved by recounting 35 in 7s: u*7 = 35 = (35/7)*7. So u = 35/7, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-Counting Creates Proportionality as Per-Numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in the per-number 2 and use the per-number to bridge 2$ and 3kg: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and vice versa: T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Double-counting in the same unit creates fractions: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%. So 2/3 of 60 = 2$/3$ of 60$, where 60$ = (60/3)*3$ then gives (60/3)*2$ = 40$.

Double-Counting the Sides in a Block Creates Trigonometry

Halving a block by its diagonal allows mutual recounting of the sides, which creates trigonometry to precede plane and coordinate geometry: $a = (a/c)*c = \sin A*c$, and $a = (a/b)*b = \tan A*b$. Filling a circle with blocks shows that $\pi = n*\tan(180/n)$ for n large.

A SHORT CURRICULUM IN ADDITION-FREE MATHEMATICS

00. Playing with '1digit math' (Zybartas et al, 2005): Rearranging 3 cars into one 3-icon, etc. Recounting a total of ten fingers in bundles of e.g. 3s: T = 1Bundle7 = 2B4 = 3B1 = 4Bundle less 2 or 4B-2, and using both fingers and sticks and centi-cubes or LEGO bricks to experience algebra and geometry as always together, never apart. Recounting in a different unit when asking e.g. T = 2 3s = ?4s. Recounting to and from tens when asking e.g. T = 5 6s = ? tens, and T = 4B2 tens = ? 7s. Uniting blocks next-to and on-top when asking e.g. T = 2 3s & 4 5s = ? 8s; and T = 2 3s & 4 5s = ? 3s; and T = 2 3s & 4 5s = ? 5s. Splitting blocks next-to and on-top when asking e.g. T = 2 3s & ? 5s = 3 8s; and T = 2 3s & ? 5s = 7 3s; and T = 2 3s & ? 5s = 4 5s.

01. Until nine, many ones may be rearranged into one icon with as many sticks or strokes as it represents. As one bundle, ten needs no icon. So, a total typically consists of several countings: of unbundled ones, of bundles, of bundles of bundles, etc.

02. Parallel counting sequence stress the importance of bundling: 0Bundle1, 0*B*2, …, 0*B*9, 1*B*0, 1*B*1 etc.; or 0*B*1, 0*B*2, 0*B*3, 0*B*4, 0*B*5 or half Bundle, Bundle less 4, *B*-3, *B*-2, *B*-1, Bundle or 1*B*0, Bundle and 1 or 1*B*1, Bundle and 2 or 1*B*2, etc., thus rooting negative numbers. Here we mention that the Vikings used the words 'eleven' and twelve' as short for 'one-left' and 'two-left'. Using other bundles as units, ten fingers may be counted as 1*B*3 7s, 2*B*0 5s, 2*B*2 4s, 3*B*1 3s, 1*BB*0*B*1 3s, 5*B*0 2s, and 1*BBB*0*BB*1*B*0 2s. A Total of five fingers can be recounted in 2s in three ways, standard or with overload or underload: T = 2*B*1 2s = 1*B*3 2s = 3*B*-1 2s = 3 bundles less 1 2s.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. For prediction, a calculator uses a 'recount formula', $T = (T/B)*B$, saying that 'from T, T/B times, Bs can be taken away'.

04. Recounting in a different unit is called proportionality. Asking '3 4s = ? 5s', we enter '3*4/5' to ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted in 5s as 2*B*2 5s. The unbundled can be placed next-to the bundles separated by a decimal point, or on-top counted in bundles, thus rooting decimals and fractions, T = 3 4s = 2*B*2 5s = 2.2 5s = 2 2/5 5s.

05. Recounting from tens to icons by asking '35 = ? 7s' is called an equation u*7 = 35, solved by recounting 35 in 7s: u*7 = 35 = (35/7)*7. So u = 35/7, showing that equations are solved by moving

to opposite side with opposite calculation sign. Division is eased by using overloads or underloads: $T = 336 = 33B6 = 28B56 = 35B\text{-}14$, so $336/7 = 4B8 = 5B\text{-}2 = 48$. As is multiplication: $T = 4*78 = 4*7B8 = 28B32 = 31B2 = 312$.

06. Recounting from icons to tens by asking '2 7s = ? tens' is eased by underloads: $T = 2*7 = 2*(B\text{-}3) = 20\text{-}6 = 14$; $6*8 = (B\text{-}4)*(B\text{-}2) = BB\text{-}4B\text{-}2B\text{-}\text{-}8 = 100 - 60 + 8 = 48$.

07. Double-counting a quantity in two units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in the per-number: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Trigonometry can precede plane and coordinate geometry to show how, in a block halved by its diagonal, the sides can be mutually recounted as e.g. $a = (a/c)*c = \sin A*c$.

MEETING MANY IN A STEM CONTEXT

OECD (2015b) says: "In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth." STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature's behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

Nature as Things in Motion

To meet, we must specify space and time in a nature consisting of things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature's three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

In nature, we count matter in kilograms, space in meters and time in seconds. Things in motion have a momentum = mass * velocity, a multiplication formula as most STEM formulas expressing recounting by per-numbers:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/liter) * liter = molarity * liter
- energy = (energy/kg/degree) * kg * degree = heat * kg * degree
- Δ momentum = (Δ momentum/second) * second = force * seconds
- Δ energy = (Δ energy/meter) * meter = force * meter = work
- energy/sec = (energy/charge) * (charge/sec) or Watt = Volt * Amp.

Thus, STEM-subjects swarm with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Warming and Boiling Water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number 410/70/1.4 Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is 316/0.14 kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving Material in Water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 mole/liter, or 0.676 mole/liter.

Building Batteries with Water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun’s energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H2O, and carbon dioxide CO2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH4, the energy comes from using 4 oxygen atoms to burn it.

Technology & Engineering: Steam and Electrons Produce and Distribute Energy

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units making a mole of water weigh 18 gram. Since the density of water is roughly 1 kilogram/liter, the volume of 1000 moles is 18 liters. With about 22.4 liter per mole, its volume increases to about 22.4*1000 liters if transformed into steam, which is an increase factor of 22,400 liters per 18 liters = 1,244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, *p*V = n*R*T*, combining the pressure *p*, and the volume *V*, with the number of moles *n*, and the absolute temperature *T*, which adds 273 degrees to the Celsius temperature. *R* is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g. steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to consumers.

An Electrical Circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per 'carrier' (charge-unit). Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 carriers per second, which is called the electrical current counted in Ampere, a per-number double-counting the number of carriers per second. To create this current, the kettle must have a resistance R according to a circuit law 'Volt = Resistance*Ampere', i.e., 220 = Resistance*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per carrier)*(carrier per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120 Watt bulb, the latter needs twice the energy and current, and consequently has half the resistance of the former, making the latter receive half the energy if connected in series.

How High Up and How Far Out

A spring sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, e.g. 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the spring 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, e.g. 2.5 meters, 5 meters and 0,71 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin45 = cos45 = 0,707. The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app. Compared with the horizontal distance, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2. So horizontally, the distance and the speed are proportional. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

ADDING ADDITION TO THE CURRICULUM

Once counted as block-numbers, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being recounted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations where the question 2 3s + ? 4s = 5 7s leads to differentiation: ? = (5*7 – 2*3)/4 = ΔT/4.

Integral calculus thus precedes differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers. So, both per-numbers and fractions must be multiplied by the units before being added as the area under the per-number graph.

Using overloads and underloads eases addition and subtraction: $T = 23 + 49 = 2B3 + 4B9 = 6B12 = 7B2 = 72$; and T = 56 – 27 = 5$B$6 – 2$B$7 = 3$B$-1 = 2$B$9 = 29.

Moving in a coordinate system, distances add directly when parallel; and by squares when perpendicular. Re-counting the y-change in the x-change creates a linear change formula $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$, algebraically predicting geometrical intersection points, thus observing a 'geometry & algebra, always together, never apart' principle.

The number-formula $T = 456 = 4*B$^2 + 5*B + 6*1 shows the four ways to unite numbers offered by algebra meaning 'reuniting' in Arabic: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers (Tarp, 2018b):

Uniting/splitting into	Changing	Constant
Unit-numbers m, s, kg, $	***T = a + n*** *T – a = n*	***T = a * n*** *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	***T = ∫ a dn*** *dT/dn = a*	***T = a^n*** *loga(T) = n* *n√T = a*

Figure 4: An 'Algebra-Square' with the 4 and 5 ways to unite and split totals.

In its general form, the number formula $T = a*x$^2 + $b*x$ + c contains the different formulas for constant change: $T = a*x$ (proportionality), $T = a*x$^2 (acceleration), $T = a*x$^c (elasticity) and $T = a*c$^x (interest rate); as well as $T = a*x+b$ (linearity, or affinity, strictly).

As constant/changing, predictable change roots pre-calculus/calculus. Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Engineering: How Many Turns on a Steep Hill

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance $BC = u$.

In the triangle BCD, the angle B is 30 degrees, and $BD = u$*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2. In the triangle ACD, the angle A is 10 degrees, and $AD = AC$*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2. In the triangle ACB, AB = 1 and $BC = u$, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or $u = AC$*sin(10)/sin(30) = $AC*r$,
or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance $BC = u$ = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

CONCLUSION AND RECOMMENDATION

This paper argues that 50 years of unsuccessful mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views lead to different kinds of mathematics: a setbased top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying "To master Many, counting and recounting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating." A proposal for two separate twin-curricula in counting and adding is found in Tarp (2018a).

Thus, the simplicity of mathematics as expressed in a 'count-before-adding' curriculum allows replacing line-numbers with block-numbers. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM pre-teachers or pre-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

Thus, it is possible to solve core STEM problems without learning addition, that later should be introduced in both versions since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

So, as with another foreign language, why not learn the number-language through its use. And celebrate that core mathematics as proportionality, equations, per-numbers and calculus grow directly from the mastery of Many that children develop through use and before school? Let us see math, not as a goal in itself, but as an inside means to an outside goal that is reached better and by more with quantitative communication.

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05. BUNDLE-COUNTING PREVENTS & CURES MATH DISLIKE

Inside-Outside Mathematics

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5. This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations having addition as the base with subtraction as reversed operation, where multiplication is repeated addition with division as reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction creates negative numbers, where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2. Then halving a block by its diagonal creates a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2, and where π = n*sin(180/n) for n large.

Then calculations with unspecified numbers leads to creating additional education about the results of operating on such numbers, e.g. that (a+b) * (a-b) = a^2 - b^2.

In a calculation, changing the input will change the output. Relating the changes creates an operation on the calculation called differentiation, also creating additional education about the results of operating on calculations, e.g. that (f*g)/(f*g) = f/f + g/g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6*x^2dx = 2*x^3.

Having taught inside how to operating on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modeling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature.

Outside-Inside Mathematics

But, as with another foreign language, why not learn the number-language through its use? Is the goal of mathematics education to learn mathematics – or to learn how to master Many? Is math a goal in itself, or an inside means to an outside goal, that may be reached better and by more through quantitative communication? What math grows from the mastery of Many that children develop through use and before school?

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total *T* by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. *T* = 2 5s.

02. Icons thus inspires 'bundle-counting' and 'bundle-writing' where a total *T* of 5 1s is recounted in 2s as *T* = 1*B*3 2s = 2*B*1 2s = 3*B*-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, *T* = 5 = 2*B*1 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a 'recount formula', *T* = (*T*/*B*)**B*, to predict that 'from *T*, *T*/*B* times, *B*s can be taken away'. This recount

formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

04. Recounting in a different unit is called proportionality. Asking '3 4s = ? 5s', sticks say 2*B*2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

05. Recounting from tens to icons by asking '35 = ? 7s' is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking '2 7s = ? tens' is eased by using underloads: $T = 2*7 = 2*(B\text{-}3) = 20\text{-}6 = 14$; and $6*8 = (B\text{-}4)*(B\text{-}2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$.

07. Double-counting a quantity in units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question '*T* = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

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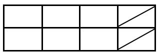
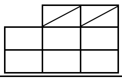
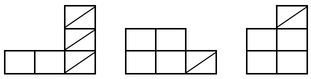
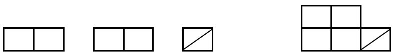
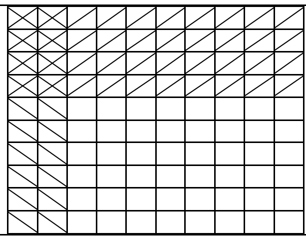
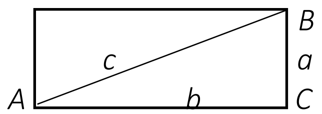
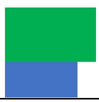
06. Flexible Bundle-Numbers

respect & develop **Kids Own Math** *Outside & Inside Math*

Digits as ICONS III IIII IIIII	ㄣ 4 5	3 4 5
Operations as ICONS	Push Lift Pull	/ X –
Count Fingers in 5s using BundleCounting & BundleNumbers	I●●●● II●●● III●● IIII● IIIII IIIII I●●●●	T = 0*B*1 = 1*B*-4 5s T = 0*B*2 = 1*B*-3 5s T = 0*B*3 = 1*B*-2 5s T = 0*B*4 = 1*B*-1 5s T = 1*B*0 = 5 T = 1*B*1 = 2*B*-4 5s
Unbundled creates Decimals & Fractions & Negative Numbers IIIIIIII ➔ IIII IIII II		T = 2*B*2 **3s** = 2.2 **3s** T = 2 2/3 **3s** T = 3*B*-1 3s = 3.-1 **3s** T = 1*BB* 0*B* -1 (T = *p***x*^2 + *q***x* + *r*)
ReCount in Same Unit creates Flexible Numbers ▌▌▌▌▌ III ➔ 53	5: IIII I I I IIII IIII I IIII IIII IIII 	T = 1*B*3 *Overload* T = 2*B*1 *Standard* T = 3*B*-1 *Underload* T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 **tens**
Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7* 48 = ? = 336 /7 = ? =	6*B*5 + 2*B*7 = 8*B*12 = 9*B*2 = 92 6*B*5 – 2*B*7 = 4*B*-2 = 3*B*8 = 38 7* 4*B*8 = 28*B*56 = 33*B*6 = 336 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48
ReCount in New Unit *ReCount-Formula:*	5 = ? **2s**  T = (T/*B*) * *B*	T = 5 = (5/2)*2 = ? = 2*B*1 **2s** 5/2 2.some 5 - 2*2 1
ReCount: Tens to Icons ▌▌▌ IIIII = ? **7s**	3*B*5 **tens** = *u**7	*u**7 = 35 = (35/7)*7 so *u* = 35/7
ReCount: Icons to Tens 6 **8s** = ? **tens** 		T = 6 **8s** = 6*8 = (*B*-4)*(*B*-2) = *BB* - 4*B* - 2*B* - - 8 = 10*B* - 6*B* + 8 = 4*B*8 = 4.8 **tens** = 48
DoubleCount gives PerNumbers	2$ per 3kg = 2$/3kg	T = 6$ = (6/2) * 2$ = (6/2) * 3kg = 9kg
Like Units: Fractions 5% of 40	5$/100$ of 40$	T = 40$ = (40/100)*100$ gives (40/100)*5$ = 2$
DoubleCount a Block halved by its Diagonal		*a* = (*a*/*c*)**c* = sin *A***c* *a* = (*a*/*b*)**b* = tan *A***b* π= *n**tan(180/*n*) *for n large*

Allan.Tarp@MATHeCADEMY.net MrAlTarp YouTube:

Add NextTo OnTop	T = 2 **3s** + 4 **5s** = 3*B*2 **8s** *Integration*  T = 2 **3s** + 4 **5s** = 1*B*1 **5s** + 4 **5s** = 5*B*1 **5s** *Proportionality* 
MatheMatism	ADDING WITHOUT UNITS Digits or Fractions or Per-numbers

Flexible Bundle-Numbers Respect & Develop Kids Own Math

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total T by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. $T = 2$ 5s.

02. Icons thus inspires 'bundle-counting' and 'bundle-writing' where a total T of 5 1s is recounted in 2s as T = 1B3 2s = 2B1 2s = 3B-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, T = 5 = 2B1 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: T = 336 = 33B6 = 28B56 = 35B-14, so 336/7 = 4B8 = 5B-2 = 48.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a 'recount formula', $T = (T/B)*B$, to predict that 'from T, T/B times, Bs can be taken away'. This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

04. Recounting in a different unit is called proportionality. Asking '3 4s = ? 5s', sticks say 2B2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

05. Recounting from tens to icons by asking '35 = ? 7s' is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking '2 7s = ? tens' is eased by using underloads: $T = 2*7 = 2*(B\text{-}3)$ = 20-6 = 14; and 6*8 = (B-4)*(B-2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48.

07. Double-counting a quantity in units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Next-to addition geometrically means adding by areas, so multiplication precedes addition. Next-to addition is also called integral calculus, or differential if reversed.

09. On-top addition means using the recount-formula to get like units. Changing units is also called proportionality, or solving equations if reversed.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education*, *11*(1), 103-117.

07. WORKSHOP IN ADDITION-FREE STEM-BASED MATH

Nature as Heavy Things in Motion in Time and Space

A falling ball introduces nature's three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

Matter and force and motion all represent different degrees of Many, thus calling for a science about Many. This is how mathematics arose in ancient Greece, so it should respect its root as a natural science by letting multiplication precede addition since the basic science formulas are multiplication formulas expressing 'per-numbers' coming from double-counting: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter)*sq.-meter = pressure*sq.-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg)*kg = heat*kg; Δ momentum = (Δ momentum/sec)*sec = force*sec = impulse; Δ energy = (Δ energy/meter)*meter = force*meter = work; gram = (gram/mole)*mole = molar mass*mole; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp.

Counting in Icon-Bundles Allows Recounting in the Same and in a Different Unit

Meeting many, we observe that five ones may be recounted as one five-icon. Likewise, with the other digits; thus being, not symbols, but icons with as many strokes or sticks as they represent. 'Bundle-counting' in icon-bundles allows 'bundle-writing' where a total *T* of 5 1s is recounted in 2s as *T* = 1*B*3 2s = 2*B*1 2s = 3*B*-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, *T* = 5 = 2*B*1 2s = 2.1 2s = 2 ½ 2s.

Recounting in the same unit to create or remove over- or underloads eases operations. Example: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48.

Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross.

This creates a 'recount formula', *T* = (*T/B*)**B*, saying that 'from *T*, *T/B* times, *B*s can be taken away'. This formula predicts the result of recounting in another unit, called proportionality: Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2*B*2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the unbundled singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s recount in 5s as 2*B*2 5s or 2.2 5s or 2 2/5 5s.

This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

Recounting to and from Tens

Times tables ask ‘2 7s = ? tens’, eased by using underloads: $T = 2*7 = 2*(B\text{-}3) = 20\text{-}6 = 14$; and $6*8 = (B\text{-}4)*(B\text{-}2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-counting Creates Proportionality as Per-Numbers

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘T = 6$ = ?kg’, we recount 6 in 2s since the per-number is 2$/3kg: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Workshop exercises in addition-free STEM-based math

STEM: Mathematics as one of the natural Sciences, applied in Technology and Engineering

Using ‘Outside-Inside Math’ allows outside science problems to be solved by inside math formulas.

Outside degrees of Many create inside number-icons with the number of strokes they represent. Outside counting-operations, occurring when bundles are pushed away, lifted and pulled away to find unbundled ones, create the operation-icons division, /, and multiplication, x, and subtraction -.

Once bundle-counted, recounting in different units (called proportionality) create a ‘recount-formula‘, $T = (T/B)*B$, saying that ‘from T, T/B times, Bs can be taken away’; occurring all over math and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$; in science as speed: $\Delta s = (\Delta s/\Delta t)*\Delta t = v*\Delta t$.

Asking ‘3 **4s** is how many **5s**?’, outside sticks show that 3 **4s** becomes 2*B*2 **5s**: IIII IIII IIII -> VV II.

To predict inside, we enter ‘3*4/5’ to ask a calculator ‘from 3 **4s** we take away **5s’**. The answer ‘2.some’ predicts that the unbundled ones come by taking away 2 **5s**. Now, asking ‘3*4 – 2*5’ gives ‘2’. So, 3 **4s** = 2*B*2 **5s = 2.2 5s**.

3*4/5	2.some
3*4 – 2*5	2

Recounting a quantity in two different physical units gives a ‘per-number’ as e.g. 2m per 3sec, or 2m/3sec. To answer the question ‘T = 6m = ?sec’, we recount 6 in **2s** since the per-number is 2m/3sec: T = 6m = (6/2)*2m = (6/2)*3sec = 9sec. Double-counting in the same unit creates fractions and %: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%. 5% of 40 = ?; T = 40 = (40/100)*100 gives (40/100)*5 = 2.

kg = (kg/cubic-meter)*cubic-meter = density*cub.-meter force = (force/square-meter)*sq.-meter = press.*sq.-meter meter = (meter/sec)*sec = velocity*sec energy = (energy/sec)*sec = Watt*sec energy = (energy/kg)*kg = heat*kg	Δ momentum = (Δ mom./sec)*sec = force*sec = impulse Δ energy = (Δ energy/meter)*meter = force*meter = work gram = (gram/mole)*mole = molar mass*mole energy/sec = (energy/charge)*(charge/sec), or Watt = Volt*Amp.

Science multiplication formulas expressing ‘per-numbers’ coming from double-counting

Five Ways to Solve Proportionality Questions

Inside recounting solves outside questions as “If 2m need 5sec, then 7m need ?sec; and 12sec gives ?m”

● Europe used the ‘Regula de Tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: ‘2m takes 5s, 7m takes ?s’ to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: ‘5s gives 2m, 12s gives ?m’ to get to the answer (12*2)/5s = 4.8m.

Then, two new methods appeared, ‘find the unit’, and cross multiplication in an equation expressing like proportions or ratios:

● Q1: 1m takes 5/2s, so 7m takes 7*(5/2) = 17.5s. Q2: 1s gives 2/5m, so 12s gives 12*(2/5) = 4.8m.

● Q1: 2/5 = 7/x, so 2*x = 7*5, x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, so 5*x = 12*2, x = (12*2)/5 = 4.8.

● Alternatively, we may recount in the ‘per-number’ 2m/5s coming from ‘double-counting’ the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● SET introduced modeling with linear functions to show the relevance of abstract algebra’s group theory: Let us define a linear function $f(x) = c*x$ from the set of m-numbers to the set of s-numbers, having as domain $DM = \{x \in R \mid x>0\}$. Knowing that $f(2) = 5$, we set up the equation $f(2) = c*2 = 5$ to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: $c*2 = 5$, $(c*2)*½ = 5*½$, $c*(2*½) = 5/2$, $c*1 = 5/2$, $c = 5/2$. With $f(x) = 5/2*x$, the inverse function is f-1(x) = 2/5*x. So with 7m, $f(7) = 5/2*7 = 17.5$s; and with 12s, f-1(12) = 2/5*12 = 4.8m.

Three different kinds of mathematics answering the question: What is a function?

pre-setcentric:	*a function is a calculation with specified and unspecified numbers.*
present setcentric:	*a function is a subset of a set-product where component identity transfers.*
post-setcentric:	*a function is a number-language sentence with a subject, a verb and a predicate.*

EXERCICES

E01. With sticks, transform many OUTSIDE ones into one INSIDE many-icon with as many strokes as it represents.

E02. Name fingers as 5s using BundleCounting & BundleNumbers: 0**B**1 = 1**B**-4, 0**B**2 = 1**B**-3, …**5s**

E03. Count 5 fingers in 2s using flexible bundle-numbers: T = 5 = 1**B**3 = 2**B**1 = 3**B**-1 **2s** (overload, standard, underload)

E04. Recount ten fingers in **4s**, **3s** and **2s**: T = ten = 1**B**6 = 2**B**2 = 3**B**-2 **4s**; T = 3**B**1 = 1**BB**1 = 1**BB** 0**B** 1 = 10.1 **3s**; T = 5**B**0 = 4**B**2 = 2**BB** 1**B** 0 = 1**BBB** 0**BB** 1**B** 0 **2s**. ReCount 7 fingers in **3s**: T = 7 = 2B1= 1BB-2 = 2.1 = 3.-2 = 2 1/3.

E05. Write traditional numbers as flexible BundleNumbers: T = 53 = 5**B**3 = 4**B**13 = 6**B**-7 tens

E06. Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7* 48 = ? = 336 /7 = ? =	6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92 6**B**5 – 2**B**7 = 4**B**-2 = 3**B**8 = 38 7* 4**B**8 = 28**B**56 = 33**B**6 = 336 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

E07. With cubes, transform the three OUTSIDE parts of a counting process, PUSH & LIFT & PULL, into three INSIDE operation-icons: division & multiplication & subtraction.

Five counted in **2s**: I I I I I (push away **2s**) II II I (lift to stack) II II I (pull to find unbundles ones) II II I.

E08. OUTSIDE BundleCounting with icons as units may be predicted INSIDE by a **recount-formula T = (T/B)*B**, (from T, T/B times, take Bs away) using a full number-language sentence with a subject, a verb and a predicate.

OUTSIDE: T = I I I I I; T counted in **2s**: II II I; T – 2x2: = II II I ; INSIDE: $\boxed{\begin{array}{ll} 5/2 & 2.\text{some} \\ 5-2\text{x}2 & 1 \end{array}}$

E09. Recount in a new unit to change units, predicted by the recount-formula

OUTSIDE, use sticks or cubes to answer 3 **4s** = ? **5s**. INSIDE, the recount-formula predicts 3x4/5

E10. Recount from tens to icons

OUTSIDE, to answer the question '40 = ? **5s'**, on squared paper transform the block 4.0 **tens** to **5s**.

INSIDE, formulate an equation to be solved by recounting 40 in **5s**:

*u**5 = 40 = (40/5)*5, so *u* = 40/5.

Notice that recounting gives the solution rule 'move to opposite side with opposite calculation sign'.

E11. Recount from icons to tens

OUTSIDE, to answer '3 **7s** = ? **tens'** on squared paper transform the block 3 **7s** to **tens**.

INSIDE: oops, with no ten-button on a calculator we can't use the recount-formula? Oh, we just multiply!

E12. ReCounting in two physical units

Recounting in two physical units gives a 'per-number' as e.g. 2m per 3sec, or 2m/3sec.

To answer the question 'T = 6m = ?sec', we just recount 6 in the per-number **2s**: T = 6m = (6/2)*2m = (6/2)*3sec = 9sec.

E13. Solving STEM proportionality heating problems with recounting

With a heater giving 20 J in 30 sec, what does 40 sec give, and how many seconds is needed for 50J?

With 40 Joules melting 5kg, what will 60 Joules melt and what will 7 kg need?

With 3 degrees needs 50 Joules, what does 7 degrees need; and what does 70 Joules give?

With 4 deg. in 20kg needing 50 Joules, what does 9 deg. in 30 kg need? What does 70 Joules give in 40 kg?

E14. Mutual ReCounting the sides in a block halved by its diagonal creates trigonometry:

a = (a/b)*b = tanA*b

Draw a vertical tangent to a circle with radius r. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/radius with tangent of the angle on a calculator.

E15. Engineering

A 12x12 square ABCD has AB on the ground and is inclined 20 degrees. From B, a straight road is to be constructed intersecting the borderline AD in the point E, inclined 5 degrees. Find the length DE.

Hint: Show that if DE = 2, then the incline of the road is 3.2 degrees.

E16. Traveling

With 4 meters taking 5 seconds, what does 6 meters take; and what does 7 seconds give?

With distance d and speed v and time t related as d = v*t, what time is needed to go 20m with velocity 4m/s?

With distance d and time related as d = 5*t^2, what time is needed to go 30m?

Hint: Use that if p^2 < N < $(p+1)^2$, then √N ≈ $\frac{N+p^2}{2p}$

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**10~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	~~10**B**10~~
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

09. DEVELOPING THE CHILD'S OWN MASTERY OF MANY

Oral Presentation Outline

Present setcentric and pre-setcentric math are challenged by post-setcentric math seeing math, not as a goal, but as a means to develop the mastery of Many children bring to school.

Asked "How old next time?", a 3year old says "Four" showing 4 fingers; but protests if held together 2 by 2: "That is not four, that is two 2s", thus describing what exists, and with units: the total is bundles of 2s, and 2 of them. Children thus develop both word- and number-sentences with a subject, a verb and a predicate. The outside total exists as a natural fact, but the inside predication it chosen and can be changed: T = 4 1s = 2 2s = 1Bundle 1 3s = 0Bundle less 1 5s, etc.

Post-setcentric textbooks allow children to develop further their mastery of Many by counting and recounting totals before adding them; and to number instead of being taught about numbers. A textbook thus can be based upon the following 'research' questions.

- The digit 5 is an icon with five sticks. Does this apply to all digits?
- How to count fingers in different sequences and bundles?
- How can a calculator predict a recounting result?
- What to do with the unbundled singles?
- How to recount in the same or in another unit?
- How to recount between tens and icons?
- How to recount the sides in a block halved by its diagonal?
- How to perform and reverse next-to and on-top addition?
- How to perform and reverse adding per-numbers and fractions?

Tarp, A. (2018). Mastering Many. *Journal of Mathematics Education 11*(1), 103-117.

10. MATH DISLIKE CURED WITH INSIDE-OUTSIDE DECONSTRUCTION

Poster

In math, division often creates dislike. Sociological imagination asks: Could math teach how to number instead of about numbers? Here existentialism and deconstruction ask: can the outside existence be predicated differently?

Outside totals are predicated more naturally by digits as icons with as many strokes as they represent.

Outside operations are predicated more naturally by division wiping away bundles to be stacked by multiplication, to be taken away to look for singles by subtraction.

Once reported by a full number-sentence with an outside subject and an inside predicate, a total can be recounted in the same unit with overloads or underloads:

T = 47 = 4Bundle7 tens = 3B17 tens = 5B-3 tens.

Such flexible numbers ease operations, e.g.:

T = 245 /7 = 24B5 /7 = 21B35 /7 = 3B5 = 35.

Recounting in another unit uses a 'recount formula', T = (T/B)xB, saying 'from T, T/B times, Bs can be taken away'. (4x5/6)x6 thus predicts that 4 5s recounts as 3B2 6s. Recounting 3 in 5s as T = 3 = (3/5) 5s creates fractions. Recounting a total in 2kg and 3L gives a 'per-number' 2kg per 3L = 2/3 kg/L, bridging the units by recounting in the per-number: T = 6kg = (6/2)x2kg = (6/2)x3L = 9L.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation ux7 = 35 (= (35/7)x7), giving u = 35/7 by recounting.

Recounting from icons to tens by asking '3 4s = ? tens' is called times tables eased by flexible numbers.

Tarp, A. (2018). Mastering Many. *Journal of Mathematics Education 11*(1), 103-117.

11. LEARNING FROM THE CHILD'S OWN MATHEMATICS

Workshop

Post-setcentric math (Tarp, 2018) allows children to develop their mastery of Many by counting and re-counting totals before adding them; and to number instead of being taught about numbers. A textbook is based upon the following 'research' questions.

"The digit 5 is an icon with five sticks, does this apply to all digits?" Many ones change to one icon with as many sticks as it represents, shown on a folding ruler.

"How to number fingers in different sequences?" Five fingers: '01, 02, 03, 04, Hand' to include the bundle; or '01, 02, Hand less2, H-1, Hand. Ten fingers: …, H&1, H&2, (ten) less 2, (ten) less one, ten, one left, two left (en levnet, to levnet in Viking counting).

"How to number fingers by bundle-counting?" Ten fingers count in 5s as T = ten = 2Bundle0 5s = 2B0 5s (= 2.0 5s later), using a number-sentence with a subject, a verb and a predicate, and called an inside flexible 'bundle-number' describing the outside Lego 'block-number'. In 4s as T = 2B2 4s; in 3s as T = 3B1 3s = 1BB1 3s = 1BB0B1 3s, or T = 1xB^2 + 0xB + 1, showing the four ways to unite: on-top addition, multiplication, power and next-to block addition called calculus.

"How to number fingers with overloads or underloads?" Ten fingers bundle-count in 3s as T = ten = 1B7 3s = 2B4 3s = 3B1 3s = 4B-2 3s. Over- and underloads ease operations: T = 225 /3 = 22B5 /3 = 21B15 /3 = 7B5 = 75.

"How can a calculator predict a recounting result?" Division iconizes wiping away bundles. Multiplication iconizes stacking bundles into a block. Subtraction iconizes dragging it away to look for unbundled singles. Showing '7/2' as '3.some', and '7-3x2' as '1' predicts that 7 recounts in 2s as 3B1 2s, using a 'recount-formula' T = (T/B)xB, saying "From T, T/B times B can be taken away", occurring all over math and science.

"How to recount in another unit?" Asking T = 4 5s = ? 6s, the recount-formula says T = 4 5s = (4x5/6) 6s. The answer '3.some' suggests entering '4x5–3x6'. The answer '2' predicts that 4 5s recount in 6s as 3B2 6s. Asking T = 3 = ? 5s, the recount-formula says T = 3 = (3/5) 5s thus creating fractions. Recounting a total in 2kg and 3L gives a 'per-number' 2kg per 3L = 2/3 kg/L, bridging the units by recounting in the per-number: T = 6kg = (6/2)x2kg = (6/2)x3L = 9L; and T = 12L = (12/3)x3L = (12/3)x2kg = 8kg. Likewise, a fraction of a total is found by recounting in the per-number.

"How to recount tens into icons?" Asking 'T = 2.4 tens = 24 = ? 8s' is an equation, ux8 = 24 (= (24/8)x8), giving u = 24/8 by recounting.

"How to recount icons into tens?" Asking 'T = 3 8s = ? tens creates the multiplication tables eased by flexible numbers: 3 8s = 3x(B less 2) = 3B-6 = 2B4 = 24.

"How to recount the sides in a block halved by its diagonal?" With base b, height a, and diagonal c, mutual recounting gives the trigonometric per-numbers: a = (a/c)xc = sinAx c, etc.

Tarp, A. (2018). Mastering Many by Counting. *Journal of Mathematics Education 11*(1), 103-117.

12. FIVE ALTERNATIVE WAYS TO TEACH PROPORTIONALITY

Poster

Proportionality is rooted in questions as “If 2kg costs 5$, what does 7kg cost; and what does 12$ buy?”

A1) Europe used the ‘Regula de Tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide.

So first we ask, Q1: ‘2kg cost 5$, 7kg cost ?$’ to get to the answer (7*5/2)$ = 17.5$.

Then we ask, Q2: ‘5$ buys 2kg, 12$ buys ?kg’ to get to the answer (12*2)/5$ = 4.8kg.

A2) Then, two new methods appeared, ‘find the unit’:

Q1: 1kg costs 5/2$, so 7kg cost 7*(5/2) = 17.5$.

Q2: 1$ buys 2/5kg, so 12$ buys 12*(2/5) = 4.8kg.

A3) And cross multiplication in an equation e*pressing like proportions or ratios:

Q1: 2/5 = 7/u, so 2*u = 7*5, u = (7*5)/2 = 17.5.

Q2: 2/5 = u/12, so 5*u = 12*2, u = (12*2)/5 = 4.8.

A4) Set-based New Math chose modelling with linear functions to show the relevance of abstract algebra’s group theory: Let us define a linear function f(u) = c*u. Knowing that f(2) = 5, we set up the equation f(2) = c*2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law:

c*2 = 5, (c*2)*½ = 5*½, c*(2*½) = 5/2, c*1 = 5/2, c = 5/2.

With f(u) = 5/2*u, the inverse function is f-1(u) = 2/5*u.

So with 7kg, f(7) = 5/2*7 = 17.5$

And with 12$, f-1(12) = 2/5*12 = 4.8kg.

A5) Recounting in two units we get a ‘per-number’ 2kg/5$ to bridge the units.

Q1: T = 7kg = (7/2)*2kg = (7/2)*5$ = 17.5$;

Q2: T = 12$ = (12/5)*5$ = (12/5)*2kg = 4.8kg.

13. NEW TEXTBOOKS, BUT FOR WHICH OF THE 3X2 KINDS OF MAT EDUCATION

Poster

A curriculum must choose between a pre-, a present, and a post-setcentric mathematics, differing in answers to the question: "What is the difference between 5/3 and 5÷3".

Pre-setcentric math sees 5/3 as a number on the number-line reached by taking 5 steps of the length coming from dividing one in 3; and sees 5÷3 as 5 shared between 3.

Present setcentric mathematics sees 5/3 as an equivalence class a/b created by cross-multiplication, a x d = b x c. With 1/3 as the inverse element to 3 under multiplication, 5÷3 should be written as 5 x 1/3, i.e. the solution to the equation 3 x u = 5, found by applying and thus legitimizing abstract algebra and group theory.

Post-setcentric mathematics sees 5/3 as a per-number coming from recounting the same total in different units (5£/3kg), becoming a fraction with like units (5£/3£ = 5/3), used to bridge the two units: T = 20£ = (20/5)x5£ = (20/5)x3kg = 12 kg. And sees 5÷3 as 5 counted in 3s occurring in the 'recount-formula' recounting a total T in bundles of 3s as T = (T/3)x3, saying 'from T, T/3 times, 3s can be taken away'. This gives flexible numbers: T = 5 = 1B2 3s = 1.2 3s = 1 2/3 3s = 2B-1 3s = 2.-1 3s

Unitary states typically have one multi-year curriculum for primary and lower secondary education, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even self-chosen half-year curricula from secondary school as in North America.

14. DEVELOPING THE CHILD'S OWN MASTERY OF MANY

Sociological imagination sees continuing educational problems as possibly caused by a goal displacement making mathematics see itself as the goal instead of its outside root, mastery of Many. Typically, the number-language is taught inside-inside as examples of its meta-language. However, as the word-language, it can also be taught inside-outside, thus bridging it to the outside world it describes. So, textbooks should not reject, but further guide the mastery of Many that children bring to school.

IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA results made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life' (p. 3). Research thus still leaves many issues unsolved after half a century. Inspired by Sartre (2007, p. 20) saying that in existentialism 'existence precedes essence' and by Bauman's (1990, p. 84) sociological imagination, we can ask if mathematics education has a 'goal displacement' seeing its present essence as a goal instead of as an inside means to its outside existing root and goal, mastery of Many?

Mathematics education is based upon textbooks that again are based upon a curriculum for primary and lower secondary school supplemented with side-curricula for upper secondary school. But why can't all students have the same curriculum? After all, the word-language does not need different curricula for different groups of students, so why does the number-language?

Both languages have two levels, a language level describing the outside world, and a grammar level describing the inside language. In the word-language, the language level is for all students and includes examples of real-world descriptions, both fact and fiction, whereas grammar level details are reserved for special students. Could it be the same with the number-language, learned by all students through describing fact and fiction? And where grammar level details are reserved to special students?

Also, in contrast to the many letters, words and sentence rules in word-language, a pocket calculator shows that the number-language contains only ten digits and a few operations. And where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. And so are the operations describing counting unbundled, bundles, bundles of bundles where division iconizes wiping away bundles to be stacked, iconized by a multiplication lift, again to be drawn away, iconized by a subtraction rope, to identify unbundled singles that may be placed next-to the stack iconized by an addition cross.

Could it be that the numbering competence children bring to school contain core mathematics as proportionality and calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction? This would allow designing a curriculum for all students without splitting it up into tracks. And allow the word-language and the number-language to be taught and learned in the same way by describing outside things and actions with inside words and numbers coming from counting and adding.

However, instead of teaching children how to number, the tradition teaches children about numbers, and about operations, both to be learned before being applied to the outside world (Bussi and Sun, 2018). Thus, where word-language is taught in the space between the inside language and the outside world, the number-language is taught in the inside space between the language and its meta-language or grammar, which makes the number-language more abstract and difficult to learn and to apply.

So maybe research should go back to the mother Humboldt university in Berlin and reflect on the Karl Marx thesis 11 written on the staircase: “The philosophers have only interpreted the world, in various ways. The point, however, is to change it.”

MEETING MANY, CHILDREN BUNDLE TO COUNT AND SHARE

How to master Many can be observed in a power-free dialogue (Habermas, 1981) with preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly if held together 2 by 2: ‘That is not four, that is two twos’, thus insisting that the outside existing bundles should inside be predicated by a ‘bundle-number’ including the unit. Children also use bundle-numbers when talking about Lego blocks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, integrating them next-to each other, they typically say ‘2 7s and 4’.

Children have fun ‘bundle-counting’ their fingers in 3s in various ways: as 1Bundle7 3s, ‘bundle-written’ as T = 1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Sharing 9 cakes, 4 children take one by turn, and they smile when seeing that ‘9/4’ predicts that they can take a cake twice, thus seeing division by 4 as taking away 4s.

Children thus master numbering and sharing before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the core mastery of Many that children bring to school?

Numeracy as ‘the ability to understand and work with numbers’ (Oxford Dictionary) thus has an outside interpretation by the child’s own mastery of Many that contrasts the inside interpretations seeing numeracy as applying institutionalized mathematics.

TEXTBOOKS FOR A QUESTION GUIDED COUNTING CURRICULUM

Typically, a mediating curriculum sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a developing curriculum sees mathematics as an exoteric means to develop the children’s existing ability to master Many by numbering outside totals and blocks with inside two-dimensional bundle-numbers. This calls for different textbooks from grade 1 that don’t mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Tarp (2018) holding that only two competences are needed to master Many, counting and adding. The corresponding pre-service and in-service teacher education may be found at the MATHeCADEMY.net.

Q01, icon-making: “The digit 5 seems to be an icon with five sticks. Does this apply to all digits?” Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons; and by comparing with Roman numbers.

Q02, counting sequences: “How to count fingers?” Here the learning opportunity is that five fingers can also be counted “01, 02, 03, 04, Hand” to include the bundle; and ten fingers as “01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H”.

Q03, icon-counting: “How to count fingers by bundling?” Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate: T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s, called an ‘inside bundle-number’ describing the ‘outside block-number’. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, unbundled as decimals, fractions or negative numbers: “Where to put the unbundled singles?” Here the learning opportunity is to see that the unbundled occur in three ways: Next-to the block as a block of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q05, calculator-prediction: “How can a calculator predict a counting result?” Here the learning opportunity is to see the division sign as an icon for a broom wiping away bundles: 7/2 means ‘from 7, wipe away bundles of 2s’. The calculator says ‘3.some’, thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes a rope dragging away the block to look for unbundled singles. By showing ‘7-3*2 = 1’ the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional learning opportunity is to write and use the ‘recount-formula’ T = (T/B)*B, saying “From T, T/B times B can be taken away”, to predict counting and recounting examples.

Q06, recounting in another unit: “How to change a unit?” Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives fractions: T = 3 = (3/5)*5. Another learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Q07, prime or foldable units: “Which blocks can be folded?” Here the learning opportunity is to examine the symmetry of a block. The block T = 2 4s = 2*4 has 4 as the unit. Turning over gives T = 4*2, now with 2 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a ‘prime unit’ and 4 a ‘foldable unit’, 4 = 2 2s. A number is called even or symmetrical if it can be written with 2 as the unit, else odd.

Q08, finding units: “What are possible units in T = 12?”. Here the learning opportunity is that units come from factoring in prime units, 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

Q09, recounting from tens to icons: “How to change unit from tens to icons?” Here the learning opportunity is that asking ‘T = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

Q10, recounting from icons to tens: “How to change unit from icons to tens?” Here the learning opportunity is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking ‘T = 3 7s = ? tens’. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently ‘less less 1’ means adding 1.

Q11, recounting block-sides. “How to recount sides in a block halved by its diagonal?” Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

Q12, navigating. “Avoid the rocks on a squared paper”. Rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock?

TEXTBOOK FOR A QUESTION GUIDED ADDING CURRICULUM

Counting ten fingers in 3s gives T = 1BundleBundle1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the algebra square’.

Q13, next-to addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?” Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

Q14, reversed next-to addition: “If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

Q15, on-top addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?” Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

Q16, reversed on-top addition: “If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

Q17, adding tens: “With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?” Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Q18, subtracting tens: “If T1 = 23 and T2 add to T = 71, what is T2?” Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Q19, multiplying tens: “What is 7 43s recounted in tens?” Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Q20, dividing tens: “What is 348 recounted in 6s?” Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Q21, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

Q22, subtracting per-numbers: "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

Q23, finding common units: "Only add with like units, so how add T = 4ab^2 + 6abc?". Here units come from factorizing: T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b).

DISCUSSION AND FUTURE RESEARCH

So yes, a curriculum for all students is possible without splitting it up into tracks. For the mastery of Many that children bring to school contains core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many inspired by, and not repressed by, the present of many historically versions of mathematics.

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16. ADDITION-FREE MATH MAKE MIGRANTS AND REFUGEES STEM EDUCATORS

TOPIC 03, research-based poster

Abstract

This poster presents a design for migrants and refugees as STEM educators, relating to the topic 3 question 1 about projects aiming at a widespread implementation of innovative teaching.

As an integrated subject in a STEM package, mathematics should respect its historic Greek roots as a common name for four studies of Many in time and space, arithmetic and geometry and music and astronomy. And it should respect the mastery of Many, children bring to school which includes using full number-language sentences with a subject and a verb and a predicate as in the word-language, and using numbers with units as in T = 3*4 = 3 4s, coming from taking away bundles by division to be stacked by multiplication. With units as multipliers, the recount-formula T = (T/B)*B, saying 'From T, T/B times, B can be taken away', has the same proportionality form as most STEM formulas. So, a different Many-based mathematics should be designed to practice recounting before adding. This ManyMath presents a shortcut to the core of mathematics, proportionality and equations and calculus, that allow migrants and refugees to return to help develop or rebuild their countries as STEM educators.

The future need for education as described by the UN and OECD

Among the 17 'UN Sustainable Development Goals', goal 4, quality education, states that 'obtaining a quality education is the foundation to improving people's lives and sustainable development. (..) The reasons for lack of quality education are due to lack of adequately trained teachers (..) For quality education to be provided to the children of impoverished families, investment is needed in educational scholarships, teacher training workshops (..) More than half of children that have not enrolled in school live in sub-Saharan Africa. An estimated 50 per cent of out-of-school children of primary school age live in conflict-affected areas. 617 million youth worldwide lack basic mathematics and literacy skills.'

The UN states that by 2030 the goal 4 targets, will 'substantially increase the number of youth and adults who have relevant skills, including technical and vocational skills, for employment, decent jobs and entrepreneurship (..) ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy (..) substantially increase the supply of qualified teachers, including through international cooperation for teacher training in developing countries'.

The 'OECD 2030 Learning Framework', states that 'The members of the OECD Education 2030 Working Group are committed to helping every learner develop as a whole person, fulfil his or her potential and help shape a shared future built on the well-being of individuals, communities and the planet. (..) Future-ready students need to exercise agency, in their own education and throughout life. (..) To help enable agency, educators must not only recognise learners' individuality, (..) Two factors, in particular, help learners enable agency. The first is a personalised learning environment that supports and motivates each student to nurture his or her passions, make connections between different learning experiences and opportunities, and design their own learning projects and processes in collaboration with others. The second is building a solid foundation: literacy and numeracy remain crucial.'

An answer: addition-free STEM-based recounting math for migrants and refugees

A01. Counting fingers shows the existence of a number-language with sentences containing a subject and a verb and a predicate as in the word-language: the total is two fives, or T = 2 5s = 2*5. Here the outside total T exists unchanged while its inside predication changes with how it is bundled. Five fingers thus may recount in 2s as T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s, using an overload or a normal or an underload form, rooting negative numbers. The unbundled singles can be placed next-to the stack as decimals, or on-top counted as a fraction: T = 5 = 2.1 2s = 2 1/2 2s. Counting ten fingers in 3s show that also bundles can be bundled: T = ten = 1B7 = 2B4 = 3B1 =

4B-2 = 1BB1 = 1BB0B1 3s, or T = 1*B^2 + 0*B + 1 showing directly the four ways to unite numbers, on-top addition and multiplication and power and next-to block-addition called calculus, corresponding to the Arabic meaning of the word algebra, to reunite. Other units are tens, dozens, scores, meters, seconds, dollars, etc. Bundling bundles create names as hundred for ten tens, gros for a dozen dozens, a minute for 60 seconds, wan for BBBB tens, etc.

A02. With bundles we only need ten digits, each an inside icon bridging to the outside quantity it represents if written less sloppy, and with zero as a looking glass finding nothing.

A03. Likewise, operators are inside icons reflecting outside actions: a division broom wipes away bundles, to be stacked by a multiplication lift, to be removed by a subtraction rope, to identify unbundles singles, to be placed next-to the stack as decimals or on-top as fractions.

A04. Recounting in another unit, we ask ‘T = 3 4s = ? 5s’. The recount-formula allows a calculator predict the answer. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s recounts in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives a fraction: T = 3 = (3/5)*5. Recounting in physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg, bridging the units by recounting in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and vice versa.

A05. Recounting from tens to icons, we ask ‘T = 2.4 tens = 24 = ? 8s’. This is an equation, u*8 = 24, that is easily solved by recounting 24 in 8s as 24 = (24/8)*8. Thus, the unknown number u = 24/8 is found by moving 8 to the opposite side with the opposite calculation sign.

A06. Recounting from icons to tens, we ask ‘T = 3 7s = ? tens’. With no ten-button, a calculator cannot use the recount-formula to predict the answer. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. The multiplication tables may use ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten, less 3) = 3 * ten, less 3*3 = 3ten, less 9 = 3ten, less (ten less1) = 2ten, less less 1 = 2ten & 1 = 21. And, 7*9 = (ten, less 3)*(ten, less 1) = tenten, less 3ten, less 1ten, lessless3 = 6ten & 3 = 63.

A07. Recounting an axb block halved by its diagonal c, will create per-numbers:

a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b; and $\pi \approx$ n*sin(180/n).

A08. Trying to stay inside a squared paper when navigating from the middle, and with two dices telling the horizontal and vertical change where odd numbers are negative, roots statistics.

A09. STEM contains multiplication formulas with per-numbers: meter = (meter/sec)*sec = velocity*sec, kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter)*square-meter = pressure*square-meter; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg)*kg = heat * kg. Lego-bricks: number = (number/meter)*meter = density*meter.

A10. Adding 2 3s and 4 5s on-top, the units must be harmonized by recounting. Adding next-to means adding areas, called integral calculus, as when adding per-numbers and fractions that must change to unit-numbers by multiplication, thus creating areas to be added.

A11. Reversing addition, asking ‘2 3s and ? 5s total 4 5s or 2 8s’ will become equations, 2*3 + u*5 = 4*5 and 2*3 + u*5 = 2*8, solved by moving to opposite side with opposite sign.

References

Tarp, A, 2018, “Mastering Many”, *Journal of Mathematics Education, vol 11*(1), pp. 103-117.

17. RECOUNTING BEFORE ADDING MAKES TEACHERS COURSE LEADERS AND FACILITATORS

TOPIC 01, Research-based poster

Abstract

This Poster presents innovative perspectives in educating math educators from the child's perspective, relating to the topic 1 question 2 about successful programs and essential contents, and to question 6 how primary teachers can become learning community facilitators.

Recounting in bundle-numbers allows teachers to secure that no students are left behind.

The three learning levels at the MATHeCADEMY.net allows a teacher to become both a facilitator for a professional learning community and a course leader initiating pyramid-organized professional development locally or globally on the internet. The open source inquiry-based material is organized as individual inspiration, group reflection and school development, thus creating self-sustaining learning communities that ensures sustainability. The learning levels are research based; and by seeing mathematics as a natural science about the physical fact Many, they develop the quantitative competence children bring to school, thus including all students despite diversity as to gender or ethnicity or social or cultural background.

Peter, stuck in division, until learning about recounting in flexible bundle-numbers

Being a mathematics teacher in an ordinary class and in an adult class, both showing severe dislike towards division and fractions, Peter is about to give up teaching when he hears about a one-day workshop on curing math dislike by recounting totals in flexible bundle-numbers.

Here 5 sticks are recounted in 2s in three different ways, overload and standard and underload, occurring as outside blocks, and inside bundle-formulas: T = 5 = I I I I I = II I I I = 1*B*3 2s = II II I = 2*B*1 2s = II II I~~I~~ = 3*B*-1 2s. Likewise, if using ten-bundling: T = 57 = 5*B*7 tens = 4*B*17 tens = 6*B*-3 tens; or T = 567 = 56*B*7 = 50*B*67 = 60*B*-33 = 5*BB*6*B*7 tens.

Operations are eased by recounting in over- or underloads:

When dividing 336/7, 336 is bundle-written as 33*B*6. This is recounted as 28*B*56 that divided by 7 gives 4*B*8 or 48; or as 35*B*-14 that divided by 7 gives 5*B*-2 or 4*B*8 or 48.

Likewise, with subtraction: T = 65 – 48 = 6*B*5 – 4*B*8 = 2*B*-3 = 1*B*7 = 17; or T = 65 – 48 = 6*B*5 – 4*B*8 = 5*B*15 – 4*B*8 = 1*B*7 = 17.

Likewise, with multiplication: T = 7 x 48 = 7 x 4*B*8 = 28*B*56 = 33*B*6 = 336.

Likewise, with addition: T = 17 + 48 = 1*B*7 + 4*B*8 = 5*B*15 = 6*B*5 = 65.

A chatroom recommends watching the video 'CupCount and ReCount before you Add' (https://goo.gl/eBRFTy), and to download a 'CupCount & ReCount Booklet' for self-testing. Realizing its innovative potentials, he gives a copy to his colleagues, and they ask the school to arrange a free 1day Skype seminar in curing math dislike by recounting in bundle-numbers.

In the morning they watch the PowerPoint presentation 'Curing Math Dislike' confronting the three forms of mathematics, a pre- and a present and a post-setcentric version.

Present setcentric mathematics is called 'MetaMatism' as a mixture of 'MatheMatism', true inside a classroom but rarely outside where '2+3 = 5' is contradicted by e.g. 2weeks+3days = 17days, and 'MetaMatics', presenting a concept top-down as an example of an abstraction instead of bottom-up as an abstraction from many examples: 'A function IS an example of a set-product', instead of 'a function is a name for a formula with some unspecified numbers.'

The post-setcentric version is called 'ManyMath' by seeing mathematics as a natural science about the physical fact Many, to be counted and recounted in bundle-units before being added (or split) next-to or on-top. Here digits are icons with as many sticks as they represent. Likewise, operations iconize bundle-counting: a division broom pushes away bundles, to be stacked by a multiplication lift, to be pulled away by a subtraction rope, to look for unbundled singles, to be placed in a separate stack as decimals, or on-top counted as a fraction of a bundle. A 'recount-formula', T = (T/B)xB, allowing a calculator predict that 'from T, T/B times, B can be taken away', occurs as proportionality all over mathematics and science.

The recounting seminar includes two Skype sessions with an external course leader.

Observing ManyMath curing math dislike, the school asks Peter to take in 1year e-learning course at the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMath. Peter here experiences PYRAMIDeDUCATION where 8 are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. An external coach assists the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays by coaching a new group of 8 teachers.

The four e-learning courses for primary and for secondary school are called CATS, inspired by the fact that, to deal with Many, we Count & Add in Time & Space.

Primary school mathematics is learned through educational sentence-free meetings with the sentence subject, thus developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic 'grasp-to-grasp' learning. Thus, learning means asking, not the instructor but the subject talked about. Using full number-language sentences with a subject and a verb and a predicate as in the word-language allows modelling from the beginning by recounting both bundles, distances, time periods, money etc.

Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic 'gossip-learning': Thank you for telling me something new about something I already knew.

The material is inquiry-based with guiding questions. In primary school, the four sets of questions are as follows. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives 14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

In secondary school, the four sets of questions are as follows. COUNT: How to count possibilities? How to predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? QUANTITATIVE LITERATURE, what is that? Does it also have the 3 different genres: fact, fiction and fiddle?

The three MATHeCADEMY.net learning level thus allows Peter to become a facilitator for a local learning community and a course leader initiating pyramid-organized professional development locally or globally on the internet. After coaching a learning pyramid at the school to allow eight other teachers to be trained as facilitators and course leaders, the school may ask Peter to take the secondary school course also so the school can become as a local center for curing math dislike, thus allowing students to excel in both primary and secondary mathematics.

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Tarp, A, 2018, "Mastering Many". *Journal of Mathematics Education, vol 11*(1), pp. 103-117.

18. SELF-EXPLANATORY LEARNING MATERIAL TO IMPROVE YOUR MASTERY OF MANY

TOPIC 02, Research-based poster or oral or workshop

Abstract

This poster illustrates a hands-on experience with educating math educators from the child's perspective, relating to the topic 2 question 4 about designing innovative self-explanatory material that has large potentials for scaling-up.

It is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but will protest when held together two by two by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. Meeting Many, children develop a number-language with full sentences including a subject and a verb and a predicate as in the word-language, as well as 2-dimensional block-numbers with units, neglected by the school's 1-dimensional line-names, making some children count-over by saying 'twenty-ten'. So, the goal of the workshop is to inquire into the mastery of Many children bring to school to see what kind of mathematics occur if allowing the children to develop their already existing quantitative competence under proper guidance.

Digits and operations as icons bridging inside signs and outside existence

Matches and a folding ruler show that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent: five sticks in the 5-icon, etc.

Operations also are inside icons reflecting outside actions: a division broom pushes away bundles, to be stacked by a multiplication lift, to be pulled by a subtraction rope to identify unbundles ones, to be placed next-to the stack as decimals, or on-top as fractions or negatives, predicted by a 'recount-formula', *T* = (*T/B)*B*, saying 'from T, T/B times, B is taken away'.

Bundle-counting fingers roots negative numbers and polynomials

To emphasize bundles, the fingers may be bundle-counted as: 0Bundle1, 0*B*2, 0*B*3, 0*B*4, ½*B*, 0*B*6, 0*B*7, and then 0*B*8, 0*B*9, 1*B*0; or 1*B* less2, 1*B*-1, 1*B*0, continuing with 'Viking-counting' one-left (eleven), two left (twelve), and finally BundleBundle as 100. Two-digit numbers are named by their two neighbours: *T* = 68 = 6*B*8 tens = 7*B*-2 tens = 6ten8 = 7ten-2.

Counting ten fingers in 3s introduces bundles of bundles: *T* = ten = 3*B*1 3s = 1*BB*1 3s, leading on to the general number-formula or polynomial *T* = ten = 1*B^2 + 0*B + 1*1 3s. Likewise counting in tens, *T* = 345 = 3*BB + 4*B + 5*1 = 3*B^2 + 4*B + 5*1, showing the four ways to unite numbers (the Arabic meaning of Algebra): on-top addition, multiplication, power and next-to block-addition called integration, all with reverse splitting operations: subtraction, division, factor-finding (root), factor-counting (logarithm), and differentiation.

Block-counting cubes roots decimals, fractions and negative numbers

Block-counting 8 cubes in 5s gives 1 5s and 3 unbundled 1s as predicted: *T* = 8 = (8/5)*5 = 1*5 & 3. Placing the 3 1s after the 1 5s roots decimal-writing, *T* = 1.3 5s = 2.-2 5s. Placing the unbundled instead on-top of the block of bundles roots fractions and decimal numbers, *T* = 8 = 1 3/5 5s = 2 - 2/5 5s = 2 5s less 2. Counting in tens, *T* = 68 = 6 8/10 tens = 6.8 tens = 7.-2 tens.

Recounting roots flexible numbers and proportionality and per-numbers

Recounting in the same unit creates flexible numbers: *T* = 68 = 6.8 tens = 7.-2 tens

Recounting in another unit by asking e.g. '*T* = 3 4s = ? 5s', the recount-formula allows a calculator to predict the answer. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s recounts in 5s as 2*B*2 5s or 2.2 5s. Counting 3 in 5s gives a fraction: *T* = 3 = (3/5)*5. Recounting in physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg, bridging the units by recounting in the per-number: Asking '6$ = ?kg', we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and vice versa.

Recounting from tens and to tens roots equations and multiplication tables

Recounting from tens to icons by asking e.g. 'T = 2.4 tens = 24 = ? 8s' becomes an equation, u*8 = 24, that is easily solved by recounting 24 in 8s as 24 = (24/8)*8 so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite calculation sign.

Recounting from icons to tens by asking e.g. 'T = 3 7s = ? tens' we notice that with no ten-button on a calculator, the recount-formula cannot predict the answer. But, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. The multiplication tables may use 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten, less 3) = 3 * ten, less 3*3 = 3ten, less 9 = 3ten, less (ten less1) = 2ten, less less 1 = 2ten & 1 = 21. And, 7*9 = (ten, less 3)*(ten, less 1) = ten ten, less 3ten, less 1ten, lessless3 = 6ten & 3 = 63.

Recounting is exemplified in STEM-formulas

STEM contains multiplication formulas with per-numbers: meter = (meter/sec)*sec = velocity*sec, kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter)*square-meter = pressure*square-meter; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg)*kg = heat * kg. Lego-bricks: number = (number/meter)*meter = density*meter.

Recounting sides in a block halved by its diagonal roots angles, trigonometry and pi

Recounting a block with base b and height a, halved by its diagonal c, creates per-numbers:

$a = (a/c)*c = \sin A*c$; $b = (b/c)*c = \cos A*c$; $a = (a/b)*b = \tan A*b$; and $\pi \approx n*\sin(180/n)$.

Adding totals on-top and its reverse roots proportionality and differential calculus

Adding 2 3s and 4 5s on-top, the units must be harmonized by recounting. Adding next-to means adding areas, called integral calculus; as when adding per-numbers and fraction that must change to unit-numbers by multiplication, thus creating areas to be added.

Reversing addition by asking e.g. '2 3s and ? 5s total 4 5s or 2 8s' will become equations, 2*3 + u*5 = 4*5 or 2*3 + u*5 = 2*8, solved by moving to opposite side with opposite sign.

Grand theory holds conflicting conceptions on concepts

Within philosophy, Platonism and Existentialism argue if concepts are examples of abstractions or abstractions from examples. Within psychology, Vygotsky and Piaget argue if concepts are constructions mediated socially or experienced individually. Within sociology, the agent-structure debate is about establishing inclusion by accepting the agent's own concepts or establishing exclusion by insisting on teaching and learning institutionalized concepts.

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19. CAN GROUNDED MATH AND EDUCATION AND RESEARCH BECOME RELEVANT TO LEARNERS

TOPIC 03, presentation

Abstract

This presentation relates to the topic 3 question 3 about challenges to be overcome even if innovative teaching has been designed and initiated. The main question about mathematics education and its research is: 'If 50 years of research fails to solve the problems of math education, then what can?' The presentation allows the audience to give comments to the five section questions that are inspired by the Chomsky-Foucault debate on Human Nature.

Humans communicate in languages, a word-language and a number-language. We learn to speak the word-language in the family, and we are taught to read and write in institutionalized education, also mediating the number-language under the name mathematics, thus emphasizing the three r's: Reading, Writing and Arithmetic. Despite intensive research, international tests show that the learning of the number-language is deteriorating in many countries.

This raises two questions: May a change in mathematics, education and research make more learners reach the goal of math education? Is the goal of mathematics education to echo an inside university truth regime labelled mathematics, or to master the outside fact Many?

Education in general

On our planet, life takes the form of single black cells, or green or grey cells combined as plants or animals. Humans only need a few children in their lifetime, since transforming the forelegs to hands and fingers allows humans to grasp the food, and to share information through communication and education by developing a language when associating sounds to what they grasp. Where food must be split in portions, information can be shared through education.

Education takes place in the family and in the workplace; and in institutions with primary, secondary and tertiary education for children, for teenagers and for adults. English language does not have continental Europe's words for education using Plato's cave to picture learners as unformed and living below: Bildung, Unterricht, Erziehung, Didactics, etc. Likewise, Europe still holds on to the multi-year line-organized office preparing education that was created by the German autocracy shortly after 1800 to mobilize the population against the French democracy, whereas the North American republics use self-chosen half-year block-organized talent developing education from secondary school. So, how well-defined is 'education'?

Mathematics and its education

The Pythagoreans used the word 'mathematics' as a common label for their knowledge about Many by itself and in space and time, arithmetic and geometry and music and astronomy. Without the two latter, mathematics later became a common label for arithmetic, algebra and geometry, which may be called pre-setcentric math, challenged by the present setcentric 'New Math' appearing in the 1960s, again challenged by a post-setcentric math seeing math as a natural science about its outside roots, Many, since setcentric mathematics never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we master the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon. Continental Europe taught reckoning, called 'Rechnung' in German, until the arrival of the New Math. When opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only offers classes in mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. So, how well-defined is mathematics and its education?

The teacher and the learner

It seems natural to say that the job of a teacher is to teach learners so that learning takes place, checked by written tests. However, continental Europe calls a teacher a 'Lehrer' thus using the same word as for learning. In addition, a Lehrer is supposed to facilitate Bildung, Unterricht and Erziehung and to foster competences. In teacher education, the subject didactics, meant to determine the content of Bildung, is unknown outside the continent. In the American high school, teachers have their own classroom to teach one subject; outside teachers must teach several subjects to students forced to stay in the same class for several years.

As to learners, the tradition sees learning taking place when learners follow external instructions from the teacher in class and from the textbook at home. Then constructivism came along suggesting that instead learning mostly takes place through internal construction when working with peers or with manipulatives. So how well-defined is a 'teacher' and a 'learner'?

Research and conflicting theories

Typically, research is seen as a search for laws predicating essence to an existent subject. But, is the subject the root or an example of its predication? Holding that existence precedes essence, Existentialism has no doubt, but what about other philosophical observations?

Using the word sophy for knowledge, the ancient Greek sophists warned against choice masked as nature whereas the philosophers saw choice as an illusion since the physical is but examples of metaphysical forms only visible to them when educated at the Plato academy as scholastic 'late opponents' defending their comments to an already defended comment against three opponents. Newton's natural science installed validation by unfalsified predictions instead, which inspired the 18th century Enlightenment period, which again created counter-enlightenment, so today research still uses Plato scholasticism outside the natural sciences.

Using classrooms to gather data, math education research could be a grounded natural science, but seems to prefer scholastics by researching, not math education itself, but theories on math education instead. But this raises questions about what to do with conflicting theories:

Within philosophy the Greek controversy between sophists and philosophers is revived today between structuralism on one side and French post-structuralism and American pragmatism on the other side. Within Psychology, Vygotsky sees education as building ladders from the present theory regime to the learner's learning zone, where Piaget replaces this top-down view with a bottom-up view inspired by American Grounded Theory allowing inside categories to grow from concrete outside experiences and observations. And Sociology fiercely discusses who constructs who in the relation between individual agency and social structure.

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21. RECOUNTING IN ICON-UNITS AND IN TENS BEFORE ADDING TOTALS NEXT-TO AND ON-TOP

Material presentation

At the conference the plan is to exhibit the MATHeCADEMY.net's three levels for developing mastery of Many individually, school- and nation-wise, and globally. The focus is on a teacher being emotionally touched by the student's learning problems and wanting to cure math dislike. The background is the math dislike often occurs when teaching division.

The aim is to, by seeing how the CATS approach cures math dislike, a teacher is mobilized to be a facilitator in a one-day Skype seminar as well as a course leader by taking a one-year online training in the CATS approach to math education: to master Many, we Count and Add in Time and Space on a primary and on a secondary level. Accepting and developing the mastery of Many children bring to school, the material emphasizes bundle- and block-counting and recounting totals in icon-units and in tens before adding them next-to and on-top.

The school subjects concerned are mathematics and STEM subjects. The language available is English. The material was designed by Allan Tarp as part of a phd project and was applied and developed when hired as a web-based distance education educator at pre-service education in Denmark, and draws upon experiences from a period as a visiting professor at an in-service and pre-service teacher training academy in South Africa.

The material as well as the CATS teacher training program for primary and for secondary teachers is available at the MATHeCADEMY.net website which may be franchised freely by any university, together with MrAlTarp YouTube videos.

Bedsides allowing educators to be educated as facilitators and course leaders, the material also allows migrants and refugees to be educated as STEM educators able to return to help develop or rebuild their countries.

References

Tarp, A, 2018, "Mastering Many", *Journal of Mathematics Education, vol 11*(1), pp. 103-117.

Wrong Numbers

~~LineNumbers~~
~~with place values~~ ☹

IconNumbers
BundleNumbers
PerNumbers ☺

Respect & Develop
Kids' own Flexible BundleNumbers with Units

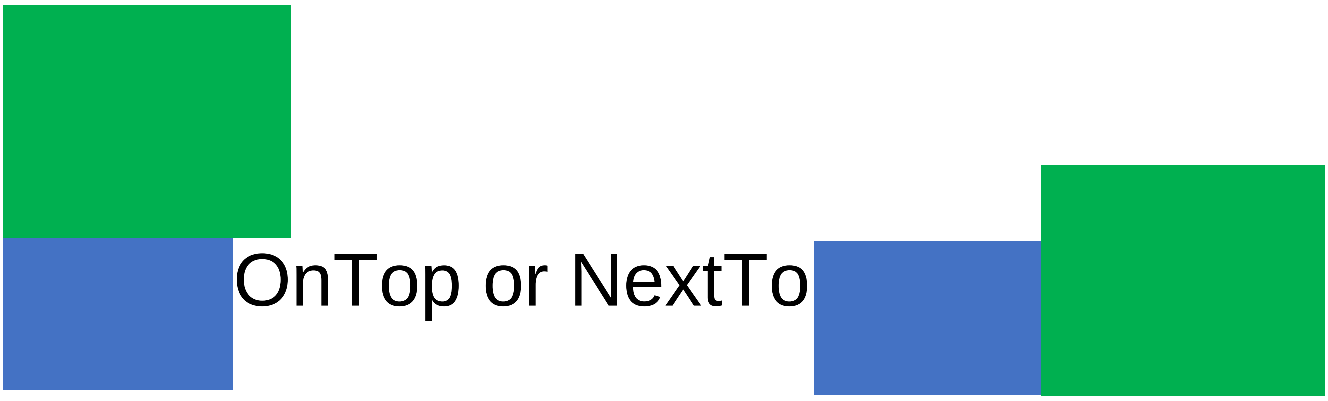
~~T is 48~~ No:
T is **4*B*8 = 3*B*18 = 5*B*-2**

Wrong Operations

~~8/2 is 8 split by 2~~ NO:
8/2 is 8 counted in **2s**

~~5x8 is 40~~ NO:
5x8 is 5 **8s**

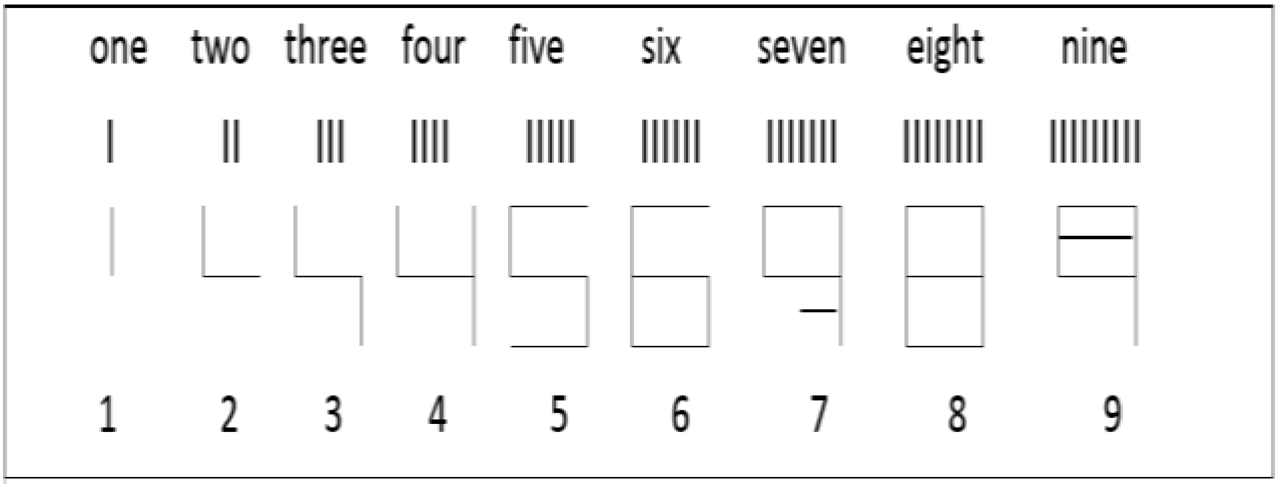
2 **3s** + 4 **5s** = ???



Wrong Math = **Dislike**

Numbers are Icons

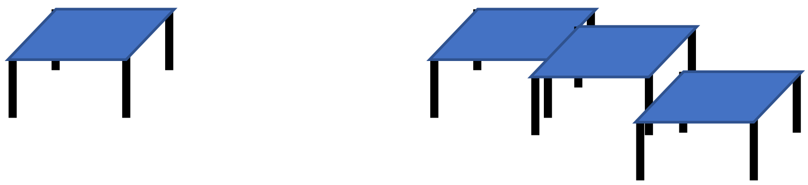
5 sticks in the 5-icon etc.



Our two Language Houses have two Floors

The WORD-language assigns words in sentences with a subject, a verb & a predicate.
The NUMBER-language assigns numbers instead.
Both languages have a META-language, a grammar, describing the language, that is learned before the grammar in the word-language, but not in the number-language.

	WORD language	NUMBER language
META-language grammar	***'is' is a verb***	***'x' is an operation***
Language	**This is a table**	**T = 3x4**



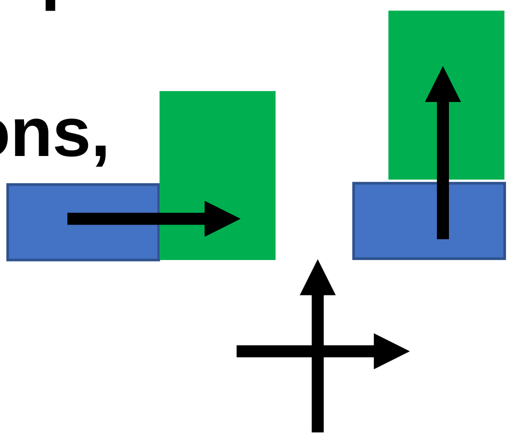
Operations are Icons

From 9 PUSH away 2s we write <u>9/2</u> iconized by a broom, called *division*.

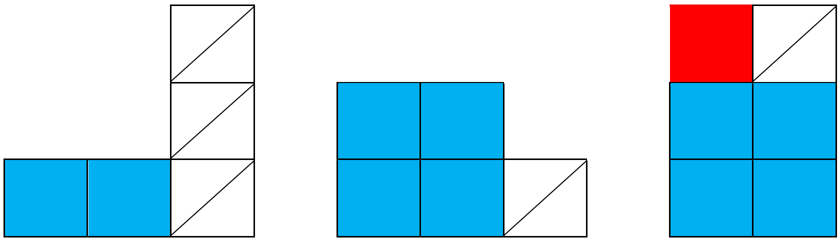
4 times LIFTING 2s to a stack we write <u>4x2</u> iconized by a lift called *multiplication*.

From 9 PULL away 4 2s' to find un-bundled we write <u>9 – 4x2</u> iconized by a rope, called *subtraction*.

UNITING next-to or on-top we write A+C <u>iconized</u> by two directions, called *addition.*



Flexible Bundle-Numbers



Overload *Standard* *Underload*

I I I I I = ╫ I I I = ╫ ╫ I = ╫ ╫ ╫

5 = 1*B*3 = 2*B*1 = 3*B*-1 **2s**

5 = 1.3 = 2.1 = 3.-1 **2s**

= 2 ½ **2s**

48 = 4*B*8 = 3*B*18 = 5*B*-2

T = 65 + 27 = ? = 6***B***5 + 2***B***7 = 8***B***12 = 9***B***2 = 92

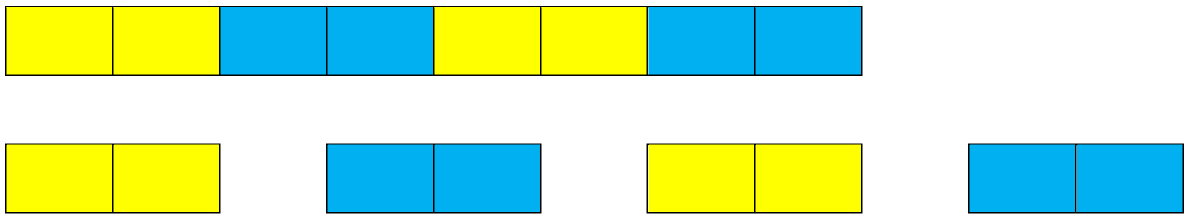
T = 65 – 27 = ? = 6***B***5 – 2***B***7 = 4***B***-2 = 3***B***8 = 38

T = 7* 48 = ? = 7* 4***B***8 = 28***B***56 = 33***B***6 = 336

T = 336 /7 = ? = 33***B***6 /7 = 28***B***56 /7 = 4***B***8 = 48

The RecountFormula

Recounting a total T in B-bundles



8 = (8/2)*2   = 4*2

T = (T/B)*B

From T, T/B times, push B away

Solves equations:

u*2 = 8   *= (8/2)*2*

u   = 8/2 (opposite side & sign)

$u + 2 = 8$	$u*2 = 8$	u^$8 = 2$	2^$u = 8$
$u = 8 – 2$	$u = 8/2$	$u = \sqrt[8]{2}$	$u = \log_2(8)$

Root: factor-finder & log: factor-counter

Used in STEM-formulas

m = (m/sec)*sec = speed*sec

$ = ($/hour)*hour = rate*hour

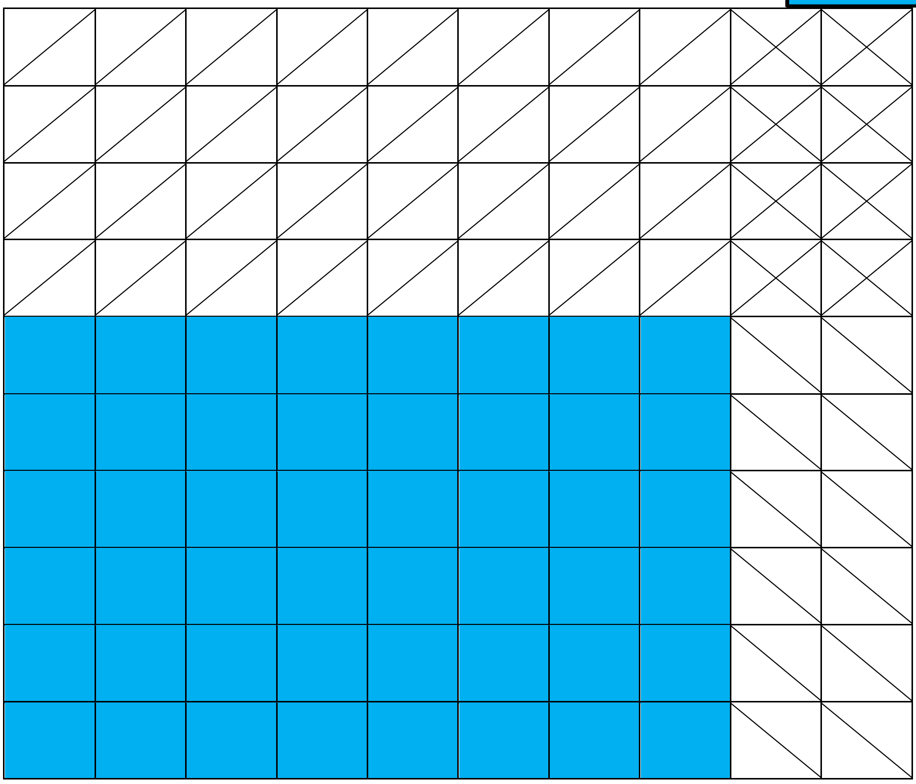
Joule = Joule/sec)*sec = Watt*sec

Ten-numbers

Tables: recount to tens

6 **8s** = ? **tens**

longer base - shorter height:



T = 6 **8s** = 6*8

= (***B***-4)*(***B***-2)

= ***BB*** - 4***B*** - 2***B*** - - 8

= 10***B*** - 6***B*** + 8

= 4***B***8 = 4.8 **tens** = 48

Per-numbers

DoubleCounting in kg & $ gives a **Per-number 2$/3kg**

<u>8$ = ?kg</u>

8$ = (8/2) x 2$

= (8/2) x 3kg = 12kg

<u>9kg = ?$</u>

9kg = (9/3) x 3kg

= (9/3) x 2$ = 6$

With like units, per-numbers are fractions: **2$/3$ = 2/3**

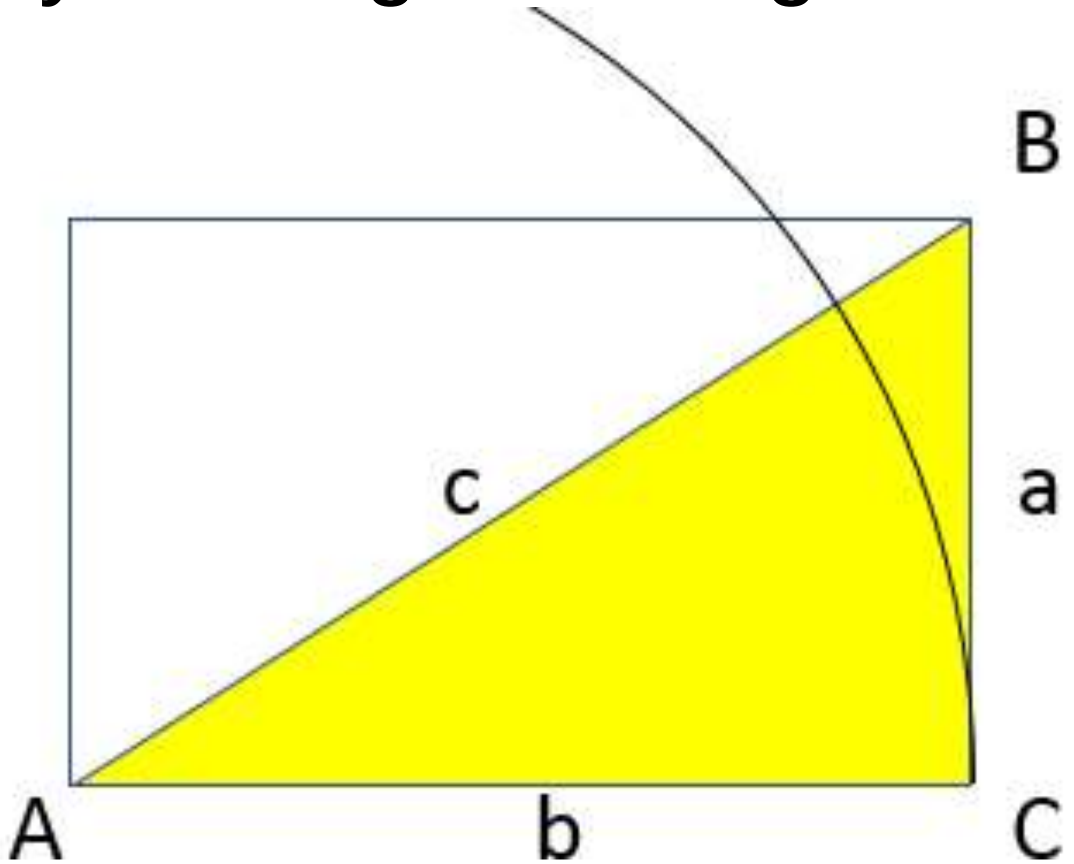
STEM-formulas contain per-numbers coming from double-counting:

m = (m/sec) * sec = speed * sec

kg = (kg/m^3) * m^3 = density * m^3

Side-numbers

Recount sides in a box halved by its diagonal: Trigonometry



T = (T/B)*B

a = (a/c)*c = **sinA***c

a = (a/b)*b = **tanA***b

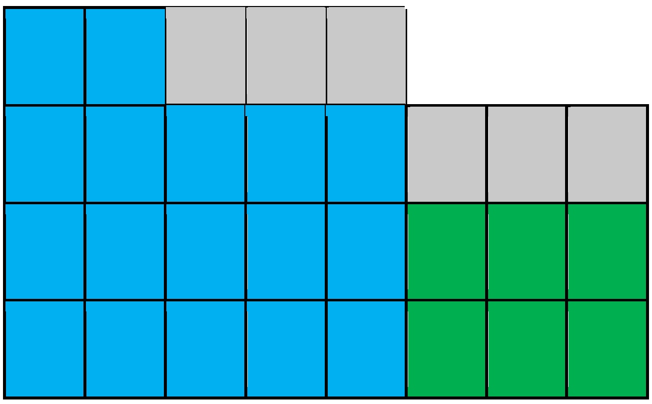
π = n*sin(180/n) for n large

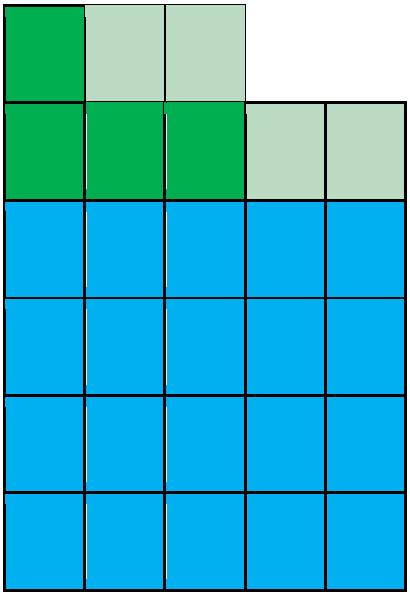
c*c = a*a + b*b

Addition is not Well Defined

Counted & Recounted, Totals may Add

BUT: NextTo	or OnTop
4 **5s** + 2 **3s** = 3**B**2 **8s**	4 **5s** + 2 **3s** = 5**B**1 **5s**
The areas are integrated *Adding areas = Integration*	Units changed to the same *Change unit = Proportionality*

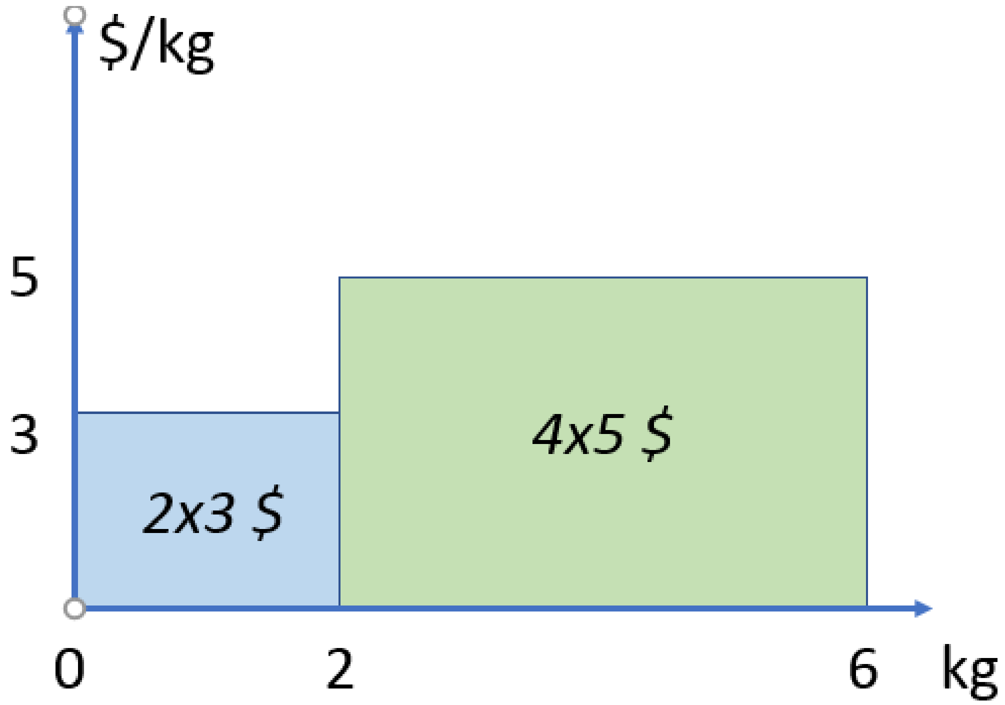




Adding fractions and per-numbers: Calculus

2 kg	at	3 $/kg
+ 4 kg	at	5 $/kg
(2+4) kg	at	? $/kg

Unit-numbers add on-top.
Per-numbers add next-to as areas under the per-number graph:



4 Ways to Unite & Split

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2+4***B**+5 (a polynomial) shows the four ways to add:

+, *, ^, next-to block-addition (integration)

<u>Add</u> & <u>multiply</u> add changing and constant unit-numbers.
<u>Integrate</u> & <u>power</u> add changing and constant per-numbers.

The 4 uniting operations have a reverse splitting operation:

Add has <u>subtract</u> (–), and multiply has <u>divide</u> (/).

Power has factor-find (<u>root</u>, √) and factor-count (<u>logarithm</u>, log).

Integrate has per-number find (<u>differentiate</u> dT/dn = T‘).

Reversing operations solve equations by moving to **opposite side** with **opposite sign**.

Operations unite/ *split into*	**changing**	**constant**
Unit-numbers *m, s, \$, kg*	**T = a + n** *T – a = n*	**T = a * n** *T/n = a*
Per-numbers *m/s, \$/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $\log_a T = n$, $\sqrt[n]{T} = a$

We call this beautiful simplicity the ‘**Algebra Square**’ since in Arabic, algebra means to reunite.

Solving Equations

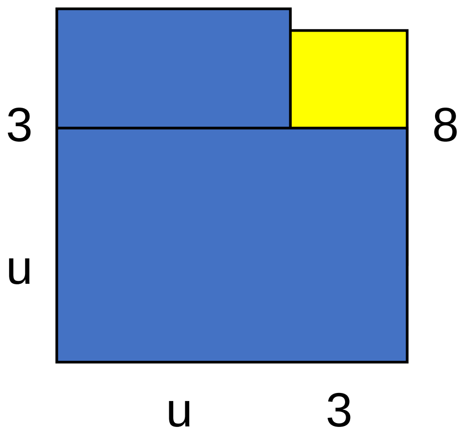
ManyMath: Recount

2 x u = 6 = (6/2) x 2	Solved by recounting 6
u = 6/2 **= 3**	Test: 2 x 3 = 6 OK

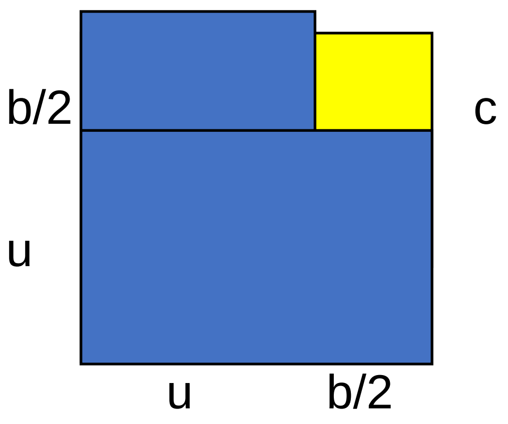
MatheMatics: Neutralize with Abstract Algebra

2 x u = 6	Multiply has 1 as neutral element, and 2 has ½ as inverse element
↕ (2 x u)x½ = 6x½	Multiply 2's inverse element to both number-names
↕ (u x 2)x½ = 3	Apply the commutative law to ux2, 3 is the short number-name for 6x½
↕ u x (2x½) = 3	Apply the associative law
↕ u x 1 = 3	Apply the definition of an inverse element
↕ **u = 3**	Apply definition of a neutral element *With arrows, a test is not needed*

Quadratic Equations with 3 Cards

<table>
<tr>
<td>Solve the quadratic equation
</td>
<td>$u^2 + 6u + 8 = 0$
$(u+3)^2 = u^2 + 6u + 8 + 1$
$(u+3)^2 = \quad 0 \quad + 1$
$u+3 = \pm 1$
$u = -3 \pm 1$
Solution: $u = -4, u = -2$</td>
</tr>
</table>

With unspecified numbers:

<table>
<tr>
<td>Solve the quadratic equation
</td>
<td>$u^2 + b*u + c = 0$
$(u+b/2)^2 = u^2 + b*u + c + (b/2)^2-c$
$(u+b/2)^2 = \quad 0 \quad + D/4$
$u+b/2 = \pm \sqrt{D}/2$, $D = b^2-4c$
$u = -b/2 \pm \sqrt{D}/2$
Solution: $u = (-b \pm \sqrt{D})/2$</td>
</tr>
</table>

MATHeCADEMY.net

- Teaches Teachers to Teach MatheMatics as **ManyMath**, a natural science about **Many**
- Cures **Math Dislike** when counting fingers in flexible bundle-numbers
- YouTube videos
- Free 1day Skype Seminars

IconNumbers • ReCounting 7 in **5s** & **3s** & **2s**

22. WHAT IS MATH - AND WHY LEARN IT?

"What is math - and why learn it?" Two questions you want me to answer, my dear nephew.

0. What does the word mathematics mean?

In Greek, 'mathematics' means 'knowledge'. The Pythagoreans used it as a common label for their four knowledge areas: Stars, music, forms and numbers. Later stars and music left, so today it only includes the study of forms, in Greek called geometry meaning earth-measuring; and the study of numbers, in Arabic called algebra, meaning to reunite. With a coordinate-system coordinating the two, algebra is now the important part giving us a number-language, which together with our word-language allows us to assign numbers and words to things and actions by using sentences with a subject, a verb and a predicate or object:

"The table is green" and "The total is 3 4s" or "T = 3*4". Our number-language thus describes Many by numbers and operations.

1. Numbers and operations are icons picturing how we transform Many into symbols

The first ten degrees of Many we unite: five sticks into one 5-icon, etc. The icons become units when counting Many by uniting unbundles singles, bundles, bundles of bundles. Operations are icons also:

Counting 8 in **2s** can be predicted by <u>division</u>, iconized by a broom pushing away **2s**: 8/2 = 4, so 8 = 4 **2s**.

Stacking the **2s** into a block can be predicted by <u>multiplication</u>, iconized by a lift pushing up the **2s**: 8 = 4x2.

Looking for unbundled can be predicted by <u>subtraction</u>, iconized by a rope pulling away the 4 **2s**: 8 – 4x2.

Uniting bundles and singles is predicted by <u>addition</u>, iconized by a cross, +, placing blocks next-to or on-top.

Recounting a total T in B-bundles is predicted by a 'recount-formula':	**T = (T/B)*B**
saying 'From T, T/B times, B can be pushed away'.	9/2 4.some 9 – 4x2
Recounting 9 in **2s**, the calculator predicts the result	1
9 = 4B1= 4.1 **2s** = 4 **2s** + 1	

Now, let us write out the total 345 as we say it when bundling in ones, tens, and ten-tens, or hundreds, we get T = 3*B^2 + 4*B + 5*1.

This shows that uniting takes place with four operations: number-addition unite unlike numbers, multiplication unite like numbers, power unite like factors, and block-addition (integration) unite unlike areas. So, one number is really many numberings united by calculations.

Thus, mathematics may also be called calculation on specified and unspecified numbers and formulas.

2. Placeholders

A letter like x is a placeholder for an unspecified number. A letter like f is a placeholder for an unspecified calculation formula. Writing 'y = f(x)' means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Writing y = f(2) is meaningless, since 2 is not an unspecified number. The first letters of the alphabet are used for unspecified numbers that do not vary.

3. Calculation formula predict

The addition calculation T = 5+3 predicts the result without having to count on. So, instead of adding 5 and 3 by 3 times counting on from 5, we can predict the result by the calculation 5+3 = 8.

Likewise, with the other calculations:

- The multiplication calculation 5*3 predicts the result of 3 times adding 5 to itself.
- The power calculation 5^3 predicts the result of 3 times multiplying 5 with itself.

4. Reverse calculations may also be predicted

'5 + 3 = ?' is an example of a forward calculation. '5 + ? = 8' is an example of a reversed calculation, often written as 5 + x = 8, called an equation that asks: which is the number that added to 5 gives 8? An equation may be solved by guessing, or by inventing a reverse operation called subtraction, x = 8 – 5; so, by definition, 8-5 is the number x that added to 5 gives 8. The calculator says that 8-5 is 3. We now test to see if this is the solution by calculating separately the left and right side of the equation. The left side gives 5 + x = 5 + 3 = 8. The right side is already calculated as 8. When the left side is equal to the right side, the test shows that x = 3 is indeed a solution to the equation.

Likewise, with the other examples of reverse calculations:

- $\frac{8}{5}$ is the number x, that multiplied with 5 gives 8. So, it solves the equation 5*x = 8.
- $\sqrt[5]{8}$ is the number x, that multiplied with itself 5 times gives 8. So, it solves the equation x^5 = 8.
- $\log_5(8)$ is the number x of times to multiply 5 with itself to give 8. So, it solves the equation 5^x = 8.

Thus, where the root is a factor-finder, the logarithm is a factor-counter.

Together we see that an equation is solved by 'moving to opposite side with opposite sign'

5 + x = 8	5*x = 8	x^5 = 8	5^x = 8
x = 8 – 5	x = $\frac{8}{5}$	x = $\sqrt[5]{8}$	x = $\log_5(8)$

5. Double-counting creates per-numbers and fractions

Double-counting in two units creates per-numbers as e.g. 3$ per 4kg or 3$/4kg or ¾ $/kg.

To bridge the units, we just recount the per-number: 15$ = (15/3)*3$ = (15/3)*4kg = 20kg.

With the same unit, a per-number becomes a fractions or percent: 3$/4$ = ¾, 3$/100$ = 3%.

Again, the per-number bridges: To find ¾ of 20, we recount 20 in 4s. 20 = (20/4)*4 gives (20/4)*3 = 15.

6. Change formulas

The unspecified number-formula T = a*x^2 + c*x + d contains basic change-formulas:

- T = c*x; proportionality, linearity
- T = c*x+d; linear formula, change by adding, constant change-number, degree1 polynomial
- T = a*x^2 + c*x + d; parabola-formula, change by acceleration, constant changing change-number, degree2 polynomial
- T = a*b^x; exponential formula, change by multiplying, constant change-percent
- T = a*x^b; power formula, percent-percent change, constant elasticity

7. Use

- Asking '3kg at 5$ per kg gives what?', the answer can be predicted by T = 3*5 = 15$.

- Asking '10 years at 5% per year gives what?', the answer can be predicted by the formula T = 105%^10 – 100% = 62.9% = 50% in plain interest plus 12.9% in compound interest.
- Asking 'If an x-change of 1% gives a y-change of 3%, what will an x-change of 7% give?', the answer can be predicted by the approximate formula T = 1.07^3 – 100% = 22.5% = 21% plus 1.5% extra elasticity.
- Asking 'Will 2kg at 3$/kg plus 4kg at 5$/kg total (2+4)kg at (3+5)$/kg?', the answer is 'yes and no'.

The unit-numbers 2 and 4 can be added directly, whereas the per-numbers 3 and 5 must first be multiplied to unit-numbers 2*3 and 4*5 before they can be added as areas.

Thus, geometrically per-numbers add by the area below the per-number curve, also called by integral calculus.

A piecewise (or local) constant p-curve means adding many area strips, each seen as the change of the area, p*Δx = ΔA, which allows the area to be found from the equation A = Δp/Δx, or A = dp/dx in case of local constancy, called a differential equation since changes are found as differences. We therefore invent d/dx-calculation also called differential calculus.

Geometrically, dy/dx is the local slope of a locally linear y-curve. It can be used to calculate a curve's geometric top or bottom points where the curve and its tangent are horizontal with a zero slope.

8. Conclusion.

So, my dear Nephew, Mathematics is a foreign word for calculation, called algebra in Arabic. It allows us to unite and split totals into constant and changing unit- and per-numbers.

Love, your uncle Allan.

Algebra **unites**/*splits into*	**Changing**	**Constant**
Unit-numbers (meter, second, dollar)	**T = a + b** $T - b = a$	**T = a*b** $\frac{T}{b} = a$
Per-numbers (m/sec, m/100m = %)	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a \qquad \log_a(T) = b$

23. Mathematics with Playing Cards

This booklet contains short articles, most of which have been printed in the LMFK member magazine for Danish upper secondary school math teachers. Thus, (2013.6) indicates that the article has been published in magazine nr. 6 from 2013. The goal is to show how mathematics formulas may be discovered by working with ordinary playing cards. Some formulas are limited by the fact that cards only have positive numbers, so the question if the formulas also apply to negative numbers may be partly answered by testing.

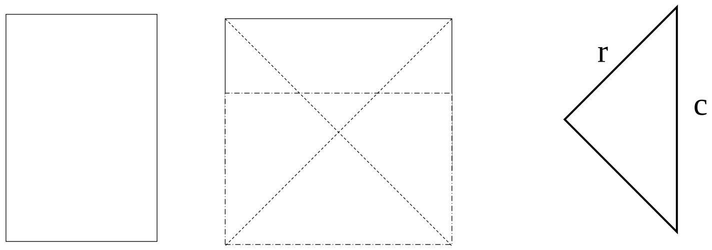
The article on Heron's formula is the only one not using playing cards.

Content

1. The Little, Medium and Large Pythagoras with 3, 4 and 5 Playing Cards

I. The Little Pythagoras

A third playing card can show how much two others should be moved to form a square with the side length c. The two diagonals, each with length $2*r$, form four like isosceles right-angled triangles, each with the area $\frac{1}{2}*r^2$. So, $c^2 = 4*\frac{1}{2}*r^2 = r^2 + r^2$.

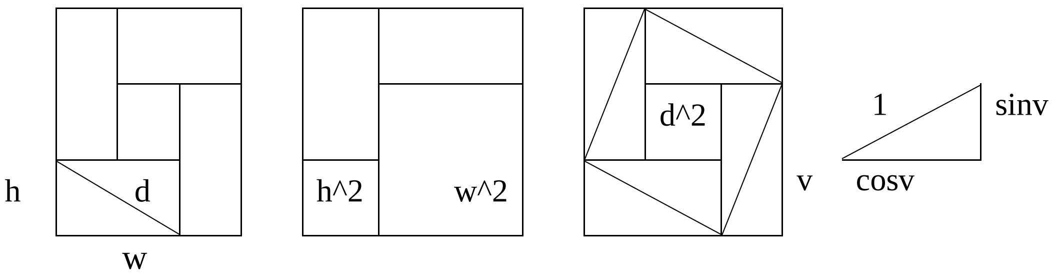


II. The Medium Pythagoras

Four playing cards have the width w and height h. The deck is rotated a quarter turn and placed to the right of the lower card, which remains unturned. This is repeated 3 times thus forming a shape that covers the area $h^2 + w^2$ + two cards.

During this process, also the diagonals with length d also make a quarter turn, and they now form the area d^2, which covers the shape above together with four half cards. Since four half-cards is the same as two whole cards, $d^2 = h^2 + w^2$.

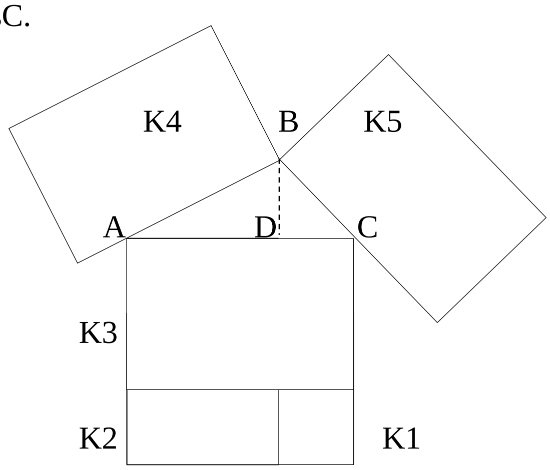
In particular, $\sin v^2 + \cos v^2 = 1$ in a right triangle with diagonal 1 and sides sin v and cos v.



III. The Large Pythagoras

The card K1 is placed horizontally. K2 is placed on-top and makes a quarter turn so the lower left corners are congruent. K3 is placed on-top so that K1 and K3 form a square. The cards K4 and K5 are used to generate the triangle ABC.

Here, the height BD is an extension of K2 's right side. Also, BD divides ABC into two right-angled triangles, BDA and BDC. In the right triangle BDC we see that $DB = a*\sin C$ and $DC = a*\cos C$.



AC's outer square consists of two squares formed by AD and DC, as well as two small strips. AD's square will be AC's square minus the two large strips, plus DC's square, which is deducted twice:

AD ^ 2 = AC^2 + DC ^ 2 – 2*DC*AC = b ^ 2 + (a*cosC) ^ 2 – 2*a*b*cosC

Using that

a^2*sinC^2 + a^2*cosC^2 = a^2*(sinC^2 + cosC^2) = a^2 *1 = a^2

We see that, in the left triangle ABD

AB^2 = DB^2 + AD^2.

= a^2*sinC^2 + a^2*cosC^2 + b^2 – 2*a*b*cos C

= a^2 + b^2 – 2*a*b*cos C

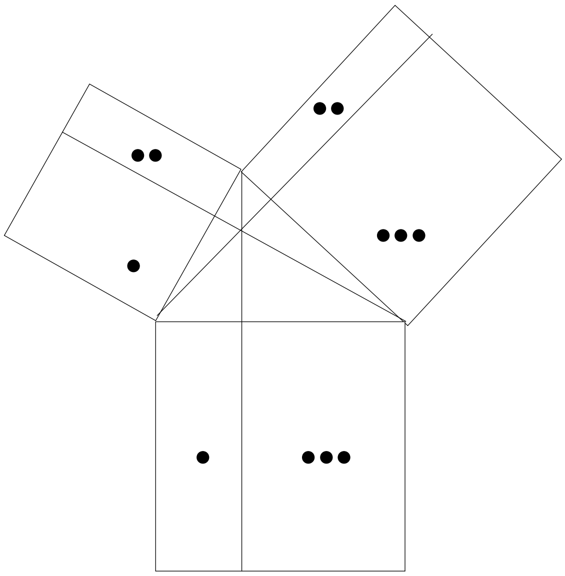
This implies that c^2 = a^2 + b^2 – 2*a*b*cosC (the large Pythagoras).

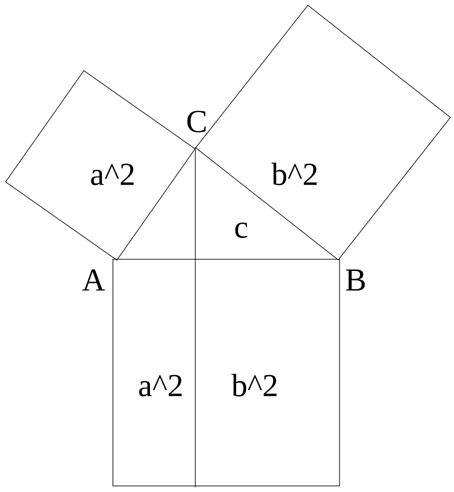
At the same time, we see that the height from B divides b's outer square in two parts with the areas b*c*cosA to the A side, and a*b*cosC to the C side. Likewise, we see that the height from A divides a's outer square in two parts with the areas a*c*cosB to B's side, and a*b*cosC to C's side. As well as to height from C divides c's outer square in two parts with the areas a*c*cosB to B's side and area b*c*cosA to a's side.

Consequently, the following two rules apply:

In a triangle without obtuse angles, the heights divide the opposite side’s outer squares in parts that are pairwise identical at the three angles, with a*b*cosC at angle C, and so on.

In a right triangle, the height divides the diagonal’s outer square into parts corresponding to the short side’s squares.





02. Pi with three playing cards

Three playing cards placed horizontally have the length b. The two top cards are rotated 90 degrees and placed so the three cards lower left corners are congruent. The top card is shifted to the right until it covers the lower card. The 2 top cards now form a square with side length b and diagonal d.

This square can be inscribed in a circle with the center at the intersection of diagonals and with the diagonal as diameter. Divided into 4 identical (red) isosceles triangles, their outer sides can be considered as a first approximation A1 to the circle circumference.

The outsides may be calculated by halving the center angles

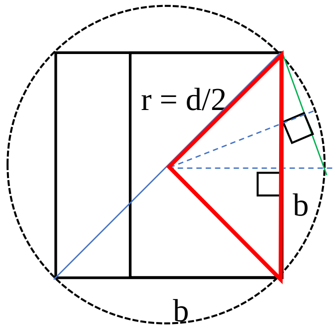
A1 = 4*2*d/2*sin (360/4/2) = d*4*sin (180/4).

The next approximation A2 is available as the outsides of the 8 (green) isosceles triangles, which comes when halving the center angles and keeping the radius r as inside:

A2 = 8*2*(d/2)*sin (180/4/2) = d*8*sin (180/8).

Continuing in this way, the approximation An = d*n*sin(180/n) will approach more and more to the circle circumference, which then can be written as d*pi where pi = n*sin(180/n) for n sufficiently large:

n	100	1000	10000	table value
n*sin(180/n) ≈ π	3.141076	3.141587	3.141593	3.141593…

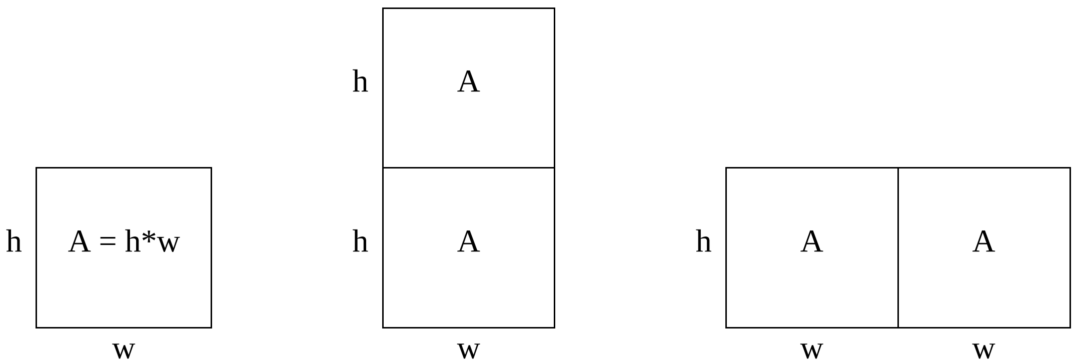


03. Proportionality with 2 playing cards

The product formulas A = h*w is very common, often called the proportionality formula or a bridging formula with the per-number h connecting the two unit-numbers A and w. The per-numbers may be $/kg, meter/second, or mol/liter in economics, physics or chemistry.

A product formula is illustrated by a playing card where the height h and width w will give the area A = h*w. Placed after the other vertically, the width w is constant, and doubling the height h implies doubling the area A. Thus, the height and the area are proportional. Likewise, we see that the width and the area are proportional when placing the cards after each other horizontally.

Moving the top card from vertical to horizontal position keeps the surface area A constant whereas the width w and height h will be doubled and halved respectively, thus being inverse proportional.



04. The Product Rules with 2-4 playing cards

A playing card with width a and height b has the area a*b.

A. Card 1 is placed horizontally with card 2 turned and placed on top so the bottom left corners are congruent. In this way, the upper-right corner becomes a square with the area (a-b) ^ 2, obtained by removing two cards from the area a ^ 2 and add b ^ 2, since this area is removed twice:

$(a – b)^2 = a^2 – 2*a*b + b^2$

Card 1 divides card 2 in an upper part with the area (a-b)*a, and a lower part with the area b^2:

$(a – b)*b = a*b – b^2$

B. Card 2 moves vertically up until leaving card 1. Card 2 will then split card 1 into two parts, where the left part with the area b ^ 2 together with card 2 covers the area (a + b)*b:

$(a + b)*b = a*b + b^2$

C. Card 3 is placed horizontally on top of card 2 so the upper left corners are congruent. Card 2 divides card 1 and card 3 into two parts, where the right parts are included in a rectangle with area (a+b)*(a-b). And where card 1 's share and the visible part of card 2 both have the area (a-b)*b. The bent area then is the difference between a^2 and b^2:

$(a + b)*(a – b) = a^2 – b^2$

D. Card 3 is shifted to the right. A vertical card 4 is added below so that the four cards form a square with the side length a + b, combined to the left by the area a ^ 2 at the bottom, and b^2 at the top, together with the two cards C3 and C4 to the right.

$(a + b)^2 = a^2 + b^2 + 2*a*b$

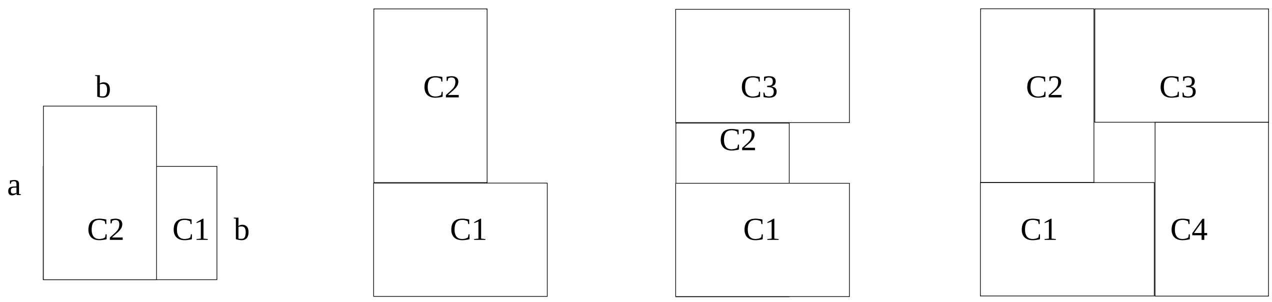


figure A figure B figure C figure D

05. A quadratic equation solved with two playing cards

The equation x + 2 = 8 asks: what is the number x that added to 2 gives 8? To answer, we invent an opposite operation to addition, called subtraction, where the number x = 8 – 2 by definition is the number x that added to 2 gives 8. Thus, we see that an equation is solved when the unknown number is isolated by moving a number to the opposite side with the opposite calculation sign.

Likewise, by definition, the number x = 8/2 is the solution to the equation x*2 = 8 asking for the number x that multiplied with 2 gives 8. Likewise, the number x = ± √ 8 is the solution to the equation x ^ 2 = 8 asking for the number x that multiplied with itself gives 8. Again, we see that an equation is solved by moving a number to the opposite side with the opposite calculation sign.

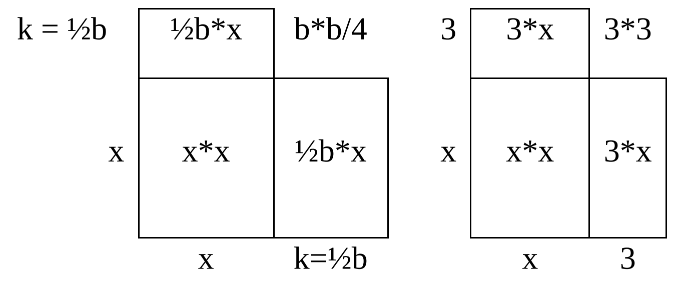
In the quadratic equation x ^ 2 + 6*x + 8 = 0 there are two unknown x's so it needs to be rewritten, so there is only one x.

Two playing cards has the width k and the height x + k. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that

(x+k)^2 = x^2 + 2*k*x + k^2, or, have the unknown x only once on the right side:

(x+k)^2 – k^2 = x^2 + 2*k*x, or ‘x plus k squared, minus k squared gives x squared + double-k x

We can now rewrite the equation x^2 + 6x + 8 = 0 first to (x^2 + 2*3*x) + 8 = 0, then to (x+3)^2 – 3^2 + 8 = 0, and finally to (x+3)^2 – 1 = 0 that is solved by three times moving to the opposite side:



x^2 + 6x + 8 = 0

(x+6/2)^2 – (6/2)^2 + 8 = 0

(x+3)^2 – 1 = 0;

Now 3 times, we move to opposite side:

(x+3)^2 = 1

x+3 = ±√1

x = -3+1 = -2, and x = -3-1 = -4

With unspecified letter-numbers, the quadratic equation is solved in the same way:

x^2 + b*x + c = 0, giving

(x^2 + 2*b/2*x) + c = 0, giving

(x + b/2)^2 – (b/2)^2 + c = 0, giving

(x + b/2)^2 = b^2/4 – c = D/4 = 0, where D = b^2 – 4*c is called the discriminant.

Thus, the quadratic equation x^2 + b*x + c = 0 has two solutions, or one, or none, depending on if the value of the discriminant D is positive, zero, or negative.

$(x + \frac{b}{2})^2 = \frac{D}{4}$; so $x + \frac{b}{2} = \pm \frac{\sqrt{D}}{2}$; so $x = \frac{-b}{2} \pm \frac{\sqrt{D}}{2}$; so $x = \frac{-b \pm \sqrt{D}}{2}$

06. Change by adding or multiplying with playing cards

Change by adding or multiplying occurs when x times an initial value b will be respectively added or multiplied by the same number a. This gives the terminal values respectively y = b + a*x and y = b*a ^ x, also called linear and exponential growth. The two change forms may be illustrated with two decks of playing cards.

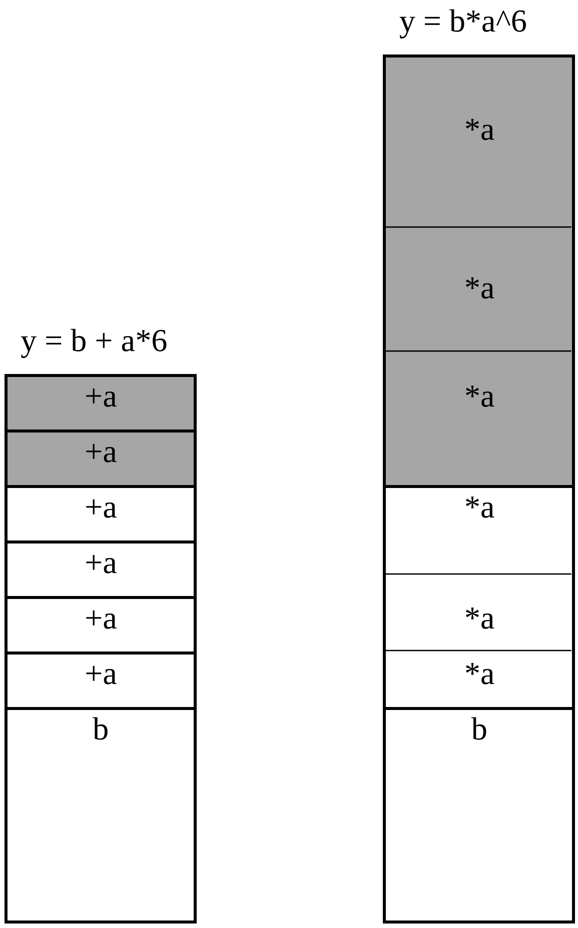
To the left is placed a deck of 7 cards. The top card stays. The next six cards are shifted upwards with a quarter of the length of the first card to show the terminal number after six times adding with the same change-number a.

Next to we place a deck of 6 cards. The top card stays.

The next three cards are shifted upwards with a quarter of the length of the previous stack to show that the terminal number after 3 changes is roughly the same as with four changes when changing by adding since as 125%*125*125%% = 195.3%, i.e. almost 200%. Thus, a change-percent at 25% means doubling after three changes.

So, after 6 changes, the initial number has been doubled twice, thus having 4 as the total change-factor or change-multiplier.

Linear change with b = 100$, and a = 25$. Exponential change with b = 100$, and a = 125%.



07. The saving formula with nine playing cards

With a monthly deposit a $ and interest percent, a saving combines change by adding and by multiplying. A saving is also called an annuity.

A saving occurs if a bank creates two accounts, K1 and K2. K2 receives the one-time deposit a/r $. Each month, K1 receives first the monthly interest percent r of its own amount, and then the monthly interest amount of the amount in K2, i.e. a fixed deposit of a/r $ * r = a $.

After n months, K1 will contain a saving A growing monthly from a deposit of a$ and an interest percent r. But at the same time K1 will contain the total interest percent R of the initial amount a/r $ in K2, so A = a/r*R or A/R = a/r, where 1+R = (1 + r) ^ n.

This can be illustrated with nine playing cards placed on a A4 paper divided into two with K1 to the left, and K2 to the right.

After the first month, K1 receives the interest percent r of its deposit, 0$; as well as a$ from K2, shown with a playing card placed horizontally with the backside up.

After the second month, K1 receives the interest percent r of its deposit, r*a $, shown with a vertical card push up a little; as well as a $ from K2, shown with a playing card placed horizontally with the backside up.

We will continue until the end of the fifth month. We gradually increase the pushing up of the vertical interest card because of the growing deposit in K1. Finally, we cover the right part of the two last cards with a white paper strip.

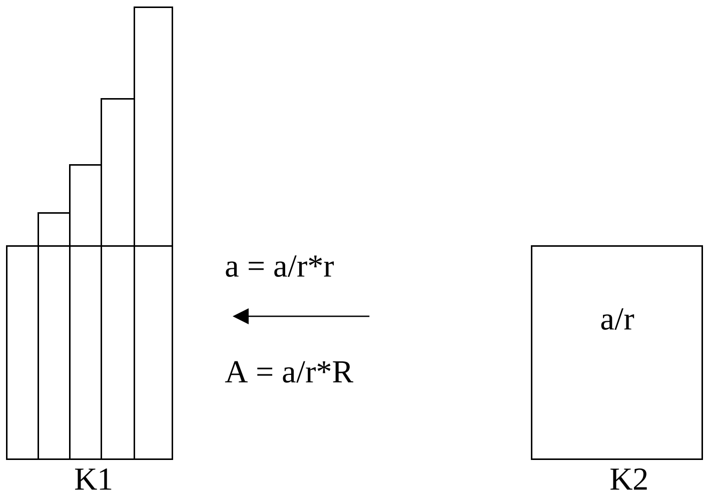
K1 now contains a saving, A, consisting of a series of constant $-deposits, the horizontal cards, and the interest on these amounts, the vertical cards.

At the same time, the horizontal cards represent the simple interest of K2, whereas the vertical cards represent the compound interest of K2.

So, again we have that the saving is A = a/r*R, giving A/R = a/r.

As to the relation between the total interest rate R, single interest rate r, and compound-interest rate RR, the cards show that R = n*r + RR or RR = R – n*r.

Thus, the simple and the compound interest might be taxed differently.



08. Product change with three playing cards

In geometry, the product of two numbers h and w occurs as the area A of a rectangle with height h and width w, A = h*w.

In economics the product occurs as the total $-number for w kg at h $/kg, T = h*w, or, more generally, each time a per-number is multiplied up to a quantity.

The question now is how changes in h and w will change the product.

Three playing cards has the height h and width w. The top card stays, the middle card is shifted off a piece Δw to the right, and the bottom card is shifted off a piece Δh upwards.

We see that the change in area, ΔA, consists of three parts, Δh*w and h*Δw and Δh*Δw.

With small changes, the last corner part may be neglected since the product of two small numbers gives a very small number: Assume that in the product 2*3 both numbers are changed with 0.01 to

(2+0.01)*(3+0.01) = 2*3 + 2*0.01 + 0.01*3 + 0.01*0.01
= 6 + 0.02 + 0.03 + 0.0001
= 6. 0501
= 6.05 with three significant figures.

Furthermore, the corner part has less and less influence, the smaller the change is.

Change t	0.1	0.01	0.001
h*w = (2+t)*(3+t)	6.51	6.0501	6.005001

Writing a small change as d instead of Δ will give the following rule for how a product is changed by small changes in its factors:

dA = d(h*w) = dh*w + h*dw, or as percentages:

dA/A = d(h*w)/(h*w) = dh/h + dw/w

Thus, with products, the change-percentages almost just add: Changing a kg-number with 3% and a $/kg-number with 5% will make the $-number change with approximately 3% + 5% = 8%. This rule applies to changes less than 10% with decreasing precision.

Since A = h*w, the per-number h = A/w.

Moving to opposite side with opposite calculation sign, we get

dh/h*w = dA/A – dw/w, or d(A/w)/(A/w) = dA/A – dw/w

Thus, with ratios, the change-percentages almost just subtract: Changing a $-number with 7% and a kg-number with 3% will make the $/kg-number change with approximately 7% – 3% = 4%. Again, this rule applies to changes less than 10% with decreasing precision.

So, with y = x^n we get that dy/y = n*dx/x, or dy/dx = n*y/x = n*x^n/x = n*x^(n-1)

09. Integral- and differential calculus with 2 playing cards

Where unit-numbers add directly, per-numbers add by their area: 2 kg at 6$/kg plus 3 kg at 4$/kg gives a total (2 + 3) kg at (6*2 + 4*3)/(2 + 3) $/kg.

This can be shown with two cards placed side by side, card1 placed vertically, and card2 placed horizontally. Card1 has the per-number 6$/kg vertically and the unit-number 2 kg horizontally, giving the area 12 $.

The unit-numbers add directly, the dollar-numbers to 12+12 = 24, and the kilo-numbers to 2+3 = 5.

The per-numbers, the $/kg-numbers 6 and 4, add by their area 24/5 = 4.8.

So Σ ($/kg) = Σ $ / Σ kg.

Graphing the cards in a coordinate system provides the rule: per-numbers add by the area under the per-number graph, i.e. by integration.

Integration uses multiplication before addition. In contrast, subtraction comes before division when reversing integration, also called differentiation:

the question "2 kg at 6$/kg plus 3 kg at u $/kg gives a total of 24$" is answered by first removing card1, and then recount the card2 area in 3s, thus applying subtraction before division:

u = (24 – C1)/3 = ΔC/3

Integration may be seen as a process of change: Card1 specifies a starting area. Placed next-to, card2 makes the area change to a new width w and the new area A. But now the variable height means a variable per-number, that might be found with differential calculus:

Card1 contains the initial numbers: the height ho, the width wo and area number Ao. Card2 contains the change-numbers

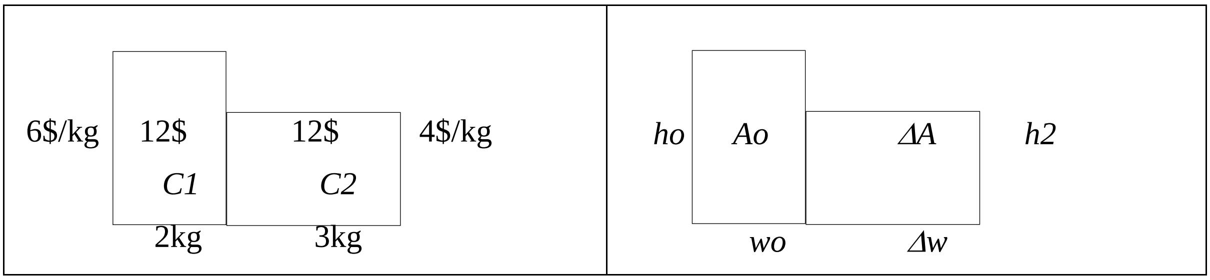
Δw = w – wo, and

$\Delta A = A - Ao$, which is also $h2 * \Delta w$.

We then find the per-number for card2 as

$h2 = (A - Ao)/(w - wo) = \Delta A/\Delta w = \Delta y/\Delta x = \Delta f/\Delta x$

Here the width-number and the area-number are graphed as a line in an x-y coordinate system, where the per-number will be the slope of the area or unit-number curve y, typically given as A = y = f (x), i.e. as a formula with a variable number x.



10. How to differentiate sine and cosine with three playing cards

Three playing cards have the height h and width w. Card 1 rotates so w forms the angle v with the horizontal direction. Card 2 rotates 90 degrees and is placed at the end of card 1, so w here forms the angle v with the vertical direction. Card 3 remains horizontal and kicks under card 2 and over card 1 until forming a triangle with h as the long side.

As known, sine and cosine may be read as the first and second coordinate in a unit circle.

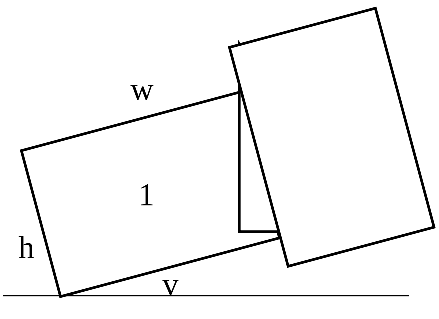
If the angle v gets a very small change dv, the circle (left w on card 2) is approximately linear.

Furthermore, the two left legs from v and v + dv (lower and upper w on card 1) are approximately parallel.

If v is measured in radian, the triangle's vertical, horizontal and long side will be three increment sides d(sin v) and -d (cos v) and dv. The long side forms the angle v with vertical, so

$\cos v = \frac{d(\sin v)}{dv}$, and $\sin v = -\frac{d(\cos v)}{dv}$,

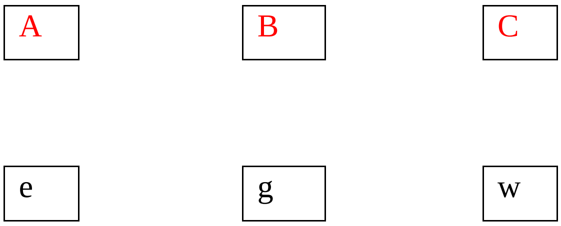
or, $\frac{d(\sin v)}{dv} = (\sin v)' = \cos v$, and $\frac{d(\cos v)}{dv} = (\cos v)' = -\sin v$



11. Topology with six playing cards

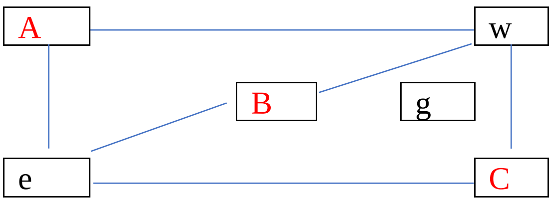
Six playing cards may illustrate a supply problem, a classic problem in topology, i.e. geometry, where neither distances nor angles, but only the relative positions between the points matter.

Problem: How can three houses A, B and C be provided with electricity, gas and water without the wires crossing?



We notice that the connection from house A to electricity to house C to water forms a closed ring that splits the plane in two areas, inside and outside. To be connected, house B and gas must be on the same side.

Suppose House B and gas is inside. Then the connection from electricity to house B to water will be splitting the inside in two closed areas with houses A and C in different areas. Located in one area, gas cannot be connected to the house located in the other area.



Now, suppose that house B and gas is outside. Then the connection from electricity to house B to water to another house back to electricity will enclose the third House, and the argument above can now be repeated.

Conclusion: the task cannot be solved unless we add a bridge whereby the plan changes its topology to a torus which is a plane with a handle.

Gluing a strip together at the ends, you can get from the outside to the inside in two ways: You can turn the strip a half turn (a Möbius strip), or you can punch a 'wormhole' (crosscap) in the strip.

In network analysis, topology is used to describe the number of bridges or handles in a given network.

12. Heron’s formula, a triangle’s circles, and Pythagoras on factor form

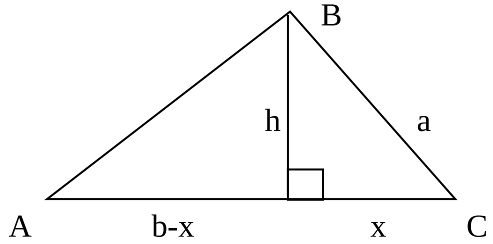
Heron’s area formula T = √(s*(s-a)*(s-b)*(s-c)) comes from repeating the factor formula

p^2 – q^2 = (p+q)*(p-q).

If 2*s = a+b+c, then 2*s – 2*a = -a+b+c, 2*s – 2*b = a-b+c and 2*s – 2*c = a+b-c

Proof.

Let B be the largest angle in the triangle ABC. Then the height h from B will divide the triangle into two right triangles, and it will divide the side b into two parts, x towards C and b – x towards A.



The lengths h and x are found from the two right triangles: x^2 + h^2 = a^2, so h^2 = a^2 – x^2.

Inserting this in (b-x)^2 + h^2 = c^2 gives c^2 = (b-x)^2 + a^2 – x^2.

Moving to opposite side with opposite sign gives

c^2 – a^2 = (b-x)^2 – x^2 = (b-x+x)*(b-x-x) = b*(b-2*x) = b^2 – 2*b*x.

Consequently, 2*b*x = a^2 + b^2 – c^2.

Now T = ½*h*b, so 16*T^2 = 4*h^2*b^2 = 4*(a^2 – x^2)*b^2 = 4*a^2*b^2 – 4*x^2*b^2 = (2*a*b)^2 – (2*x*b)^2 = (2*a*b + 2*b*x)* (2*a*b – 2*b*x)

Now we insert that 2*b*x = a^2 + b^2 – c^2:

16*T^2 = (2*a*b + a^2 + b^2 – c^2)* (2*a*b – a^2 – b^2 + c^2) = ((a+b)^2 - c^2)*(-(a-b)^2 + c^2) = ((a+b+c)*(a+b-c))*((a-b+c)*(-a+b+c)) = 2*s*(2s-2c)*(2s-2b)*(2s-2a) = 16*s*(s-a)*(s-b)*(s-c).

Consequently T^2 = s*(s-a)*(s-b)*(s-c).

Et viola: T = √(s*(s-a)*(s-b)*(s-c)).

Let the inscribed circle have the radius r. Then

T = ½*r*a + ½*r*b + ½*r*c = ½*r*2s = r*s, which gives

r = √((s-a)*(s-b)*(s-c)/s).

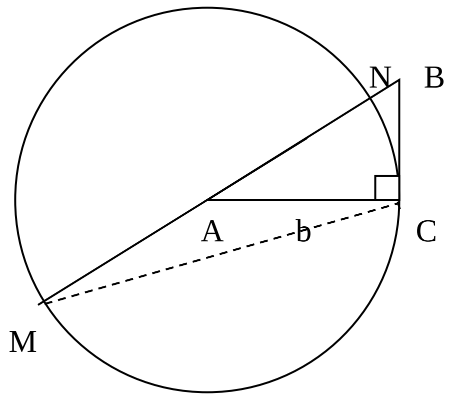
Let the circumscribed circle have the radius R. this allows extending the sinus relations

2*R = a/sinA.

Inserting this into the area formula T = ½*b*c*sinA gives the formula 4*R*T = a*b*c, “4 Round Turns teaches you abc”

The Pythagorean theorem can be found both on term form, a^2 + b^2 = c^2, and on factor form, a^2 = (c + b)*(c-b).

The factor form is found by drawing a circle with center A and radius b. Let M and N be the points where c intersect the circle. Then BM = c + b and BN = c-b. The factor form follows from of the two similar triangles BCN and BMC, where the angles BCN and BMC are like, as they span the same arc; or from calculating the point B's ‘power-of-a-point theorem’.



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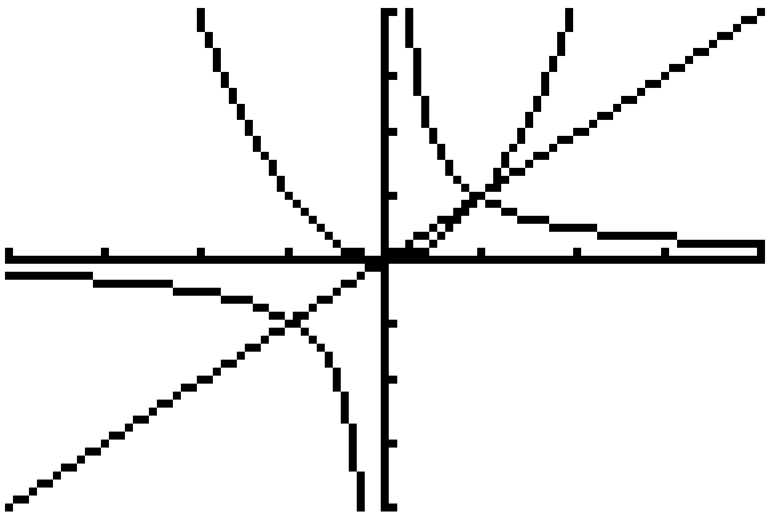
24. Mathematics Predicts
PreCalculus
How to add constant PerNumbers



y = 1*x , y = x*x , y = 1/x

CONTENTS

Compendium & Projects

by Allan.Tarp@MATHeCADEMY.net

Version 0719

Mathematics Predicts

Mathematics Algebra Uniting and Splitting Numbers Geometry Measuring Earth Statistics Accounting Many	Mathematics contains Algebra, Geometry and Statistics Algebra (calculation) can predict counting processes, both the end result and the parts. Geometry (earth measuring) can be used to calculate plane figures and spatial forms. Statistics (counting) is used for counting the actual size of different quantities.

Mathematics has two main fields, Algebra and Geometry, as well as Statistics. Geometry means 'earth-measuring' in Greek. Algebra means 'reuniting' in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus, together algebra and geometry give an answer to the fundamental human question: how do we divide the earth and what it produces?
Originally human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus, trade was only possible with those highlanders that had silver in their mountains.
The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire. The dark Middle Ages came when the Greek silver mines were empty, and the Arabs conquered the Spanish mines. German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number-system and algebra.
The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: A+B+C = 180 and a side law: a^2+b^2=c^2. Pythagoras generalized these findings by claiming: All is numbers.
This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don't know Geometry. However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation.
The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has an acceleration g; and the distance s and the time t are connected by the formula: s = ½*g*t^2. However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middlemen. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine.
The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move? The church said 'among the stars'. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth.
But why do things fall? The church said: everything follows the unpredictable will of our metaphysical lord only attainable through belief prayers and church attendance. Newton objected: It follows its own will, a force called gravity that can be predicted by a formula telling how a force changes the motion, which made Newton develop change-calculations, calculus. So instead of obeying the church, people should enlighten themselves by knowledge, calculations and school attendance.
Brahe used his life to write down the positions of the planets among the stars. Kepler used these data to suggest that the sun is the center of the solar system but could not prove it without sending up new planets. Newton, however, could validate his theory by different examples of falling and swinging bodies.
Newton's discoveries laid the foundation of the Enlightenment period realizing that when an apple follows its own will and not that of a metaphysical patronizor, humans could do the same. Thus, by enlightening themselves people could replace the double patronization of the church and the prince with democracy, which lead to two democracies, one in The US and one in France. Also formulas could be used to predict and therefore gain control over nature, using this knowledge to build an industrial welfare society based upon a silver-free economy emerging when the English replaced the import silk and pepper from the Far East with production of cotton in the US creating the triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton.

Calculations Predict

<table>
<tr><td>Calculations predict the total T. 2*4 calculation types are used to unite and split into four different types of numbers:
<table>
<tr><td>Uniting or splitting</td><td>Variable</td><td>Constant</td></tr>
<tr><td>Unit-numbers
$, kg, s</td><td>Plus +
Minus -</td><td>Multiplication *
Division /</td></tr>
<tr><td>Per-numbers
$/kg, $/100$, %</td><td>Integration ∑ ∫
Differentiation Δ</td><td>Power ^
Log or root √</td></tr>
</table></td>
<td>a $ and n $ total T $: a+n = T
a $ n times total T $: n*a = T
r % n times total T%: (1+r)^n = 1+T
a1 kg at p1 $/kg +
a2 kg at p2 $/kg total T $:
p1*a1+ p2*a2 = T: ∑ p*a = T</td></tr>
</table>

Algebra means re-uniting in Arabic and can be translated to predictions. Algebra thus predicts the result of uniting singles into totals or splitting totals into singles.

There are four ways of uniting numbers: addition (+), multiplication (*), power (^) and integration (∑ or ∫).

Addition + predicts the result of uniting variable singles:
2$ and 3 $ and 4 $ total T $: 2+3+4 = T
Multiplication * predicts the result of uniting constant singles:
2$ + 2$ + 2$ + 2$ + 2$ = 5 times 2$ = T, 5*2 = T

Job	Count	Predict
32$ and 63 $	1,2,…,95	T = 32+63
2$ 36 times	2,4, …,72	T = 72*2
20% 5 times	120, 144...	T = 120%^5

Power ^ predicts the result of uniting constant percentages: 5 times 2% totals T%, 102%^5 = 1+T
Integration ∑ or ∫ predicts the result of uniting constant per-numbers:
2kg at 7$/kg + 3kg at 8$/kg totals T $: 7*2 + 8*3 = T , ∑ $/kg * kg = T , ∫ p*dx = T
Inverse or backward calculations predicts the result of splitting a Total into singles.

x+3 = 15	Question: Which number added to 3 gives 15?
x = 15-3	Prediction: 15-3 is the number that added to 3 gives 15. Test: 3+(15-3) = 15
Rule	**Plus-numbers move across as minus-numbers, and vice versa**

x*3 = 15	Question: Which number multiplied with 3 gives 15?
$x = \frac{15}{3}$	Prediction: $\frac{15}{3}$ is the number that added to 3 gives 15. Test: $3*\frac{15}{3} = 15$
Rule	**Multiplication-numbers move across as minus-numbers, and vice versa**

x^3 = 125	Question: Which number raised to power 3 gives 125?
$x = \sqrt[3]{125}$	Prediction: $\sqrt[3]{125}$ is the number that raised to power 3 gives 125. Test: $(\sqrt[3]{125})$^3= 125
Rule	**Exponent-numbers move across as reciprocal exponent-numbers, and vice versa**

3^x = 243	Question: 3 raised to which power gives 243?
$x = \frac{\log 243}{\log 3}$	Prediction: 3 raised to power $\frac{\log 243}{\log 3}$ gives 243. Test: 3^$\frac{\log 243}{\log 3}$ = 243
Rule	**Base-numbers move across as logarithm-numbers, and vice versa**

A multi-calculation containing more calculations reduce to a single calculation by bracketing the stronger one.
T = 2+3*4 = 2+(3*4), T = 2+3^4 = 2+(3^4) , T = 2*3^4 = 2*(3^4) <u>Priority: 1. (), 2.^, 3. *, 4. +</u>
A formula-table can be used to document the solving of an equation.

The unknown number	c = ?	T = a+b*c	*The formula*
The known numbers	a = 2 b = 3 T = 14	14 = 2+(3*c) $\frac{(14-2)}{3} = c$ **4 = c**	*From a mixed to a single calculation by bracketing the stronger* *+ moves across as the opposite -, and * moves across as /* *Bracket the calculation already present* *Perform the calculation*
Tests	Test	14 = 2+3*4 14 = 14 ☺	'MATHSolver 0 = -14 + 2+3*x' gives 'x = 4'

Exercises

Find the unknown number in the formula. Make more with randM (3,1)

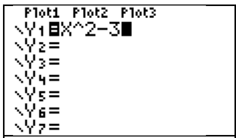
1. T = a+b*c
2. T = a+b/c
3. T = a*b^c
4. T = a+b^c
5. T = a-b*c
6. T = a-b/c
7. T = a/b^c
8. T = a-b^c

	T	b	a	c
I	60		12	20
II	60	1.5		20
III	60	1.5	12	

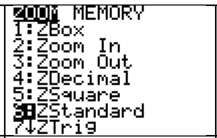
Formulas Predict

A formula contains a quantity y and its calculation f, y = f(x,z,t) **An equation** is a formula with 1 unknown. An equation can be calculated or solved by finding the unknown. **A function** is a formula with 2 unknowns. A function can be tabled or graphed showing different scenarios: If x = a then y = f(a).	Purchase-formula: b \$ + x kg at a \$/kg totals y \$: b + x * a = y Sharing-formula: b \$ + a \$ shared between x persons totals y \$: b + a/x = y

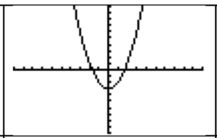
A formula contains a quantity y and a its calculation f, y = f(x,z,t). Thus a formula might contain 2, 3, 4 or more variables. If the variables are replaced by fixed numbers, a formula is transformed into an equation or a function.
An equation is a formula with 1 unknown: y = 10 + 2*3, or 16 = b + 2*3, or 16 = 10 + a*3, or 16 = 10 + 2*x
An equation can be solved manually or by a calculator using MATH-Solver. After using 'solve' the solution is tested by inserting all known numbers: 16 = 10 + 2*3 gives 16 = 16
A function is a formula with 2 (or more) unknowns: y = b + 2*3, or y = 10 + 2*x, or 16 = b + 2*x, or 16 = 10 + a*x.
In a function one of the unknowns is isolated and entered on the calculators y-list. Thus x^2-y+3=0 gives y=x^2-3.

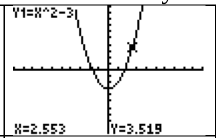


Formulas are put on the y-list

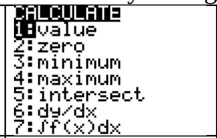


Always start with Standard Zoom

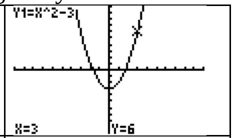

Choose Graph to graph



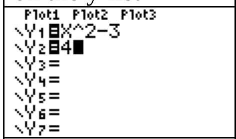
Choose Trace to see scenarios



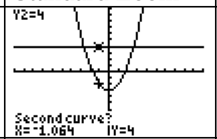
Calc Value gives specific values



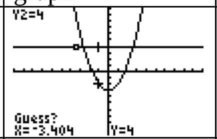
And is used for knownx/unknowny



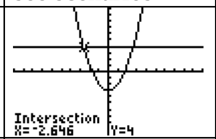
Knowny/unknownx y is on the y-list



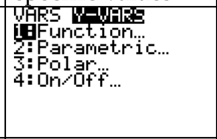
The intersecting curves marked



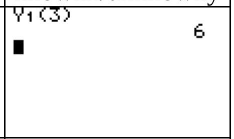
The cursor is close to the sol.



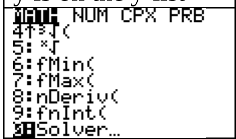
The procedure is repeated



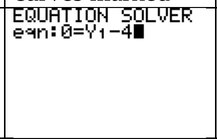
VARS gives access to the Y-s



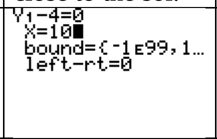
The known x is put after the Y



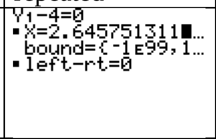
MATHSolver is used to find y's



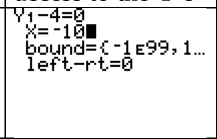
CLEAR old and enter new



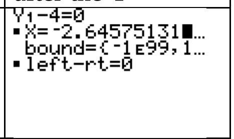
Enter a guess



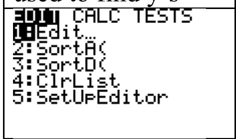
Read the solution close to guess



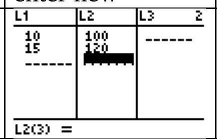
Enter a new guess



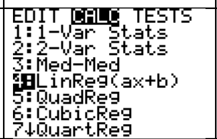
Read the solution close to guess



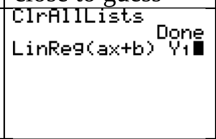
From table to formula use STAT



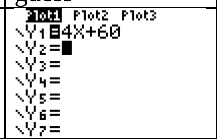
Enter the table as lists



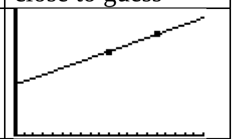
Choose a formula type



Add Y1 to bring formulas to y-list



Add Plot for visual control


Adjust window before graphing

Exercises: Find the question marks in 3 different ways: manually in a formula table, using graphs and using calculation.

1 x	y=3+2*x	2 x	y=3-2*x	3 x	y=x^2-4	4 x	y=-x^2+5
-3.7	?	-3.7	?	-3.7	?	-3.7	?
-2.4	?	-2.4	?	-2.4	?	-2.4	?
3.1	?	3.1	?	3.1	?	3.1	?
4.5	?	4.5	?	4.5	?	4.5	?
?	-3.7	?	-3.6	?	-3.8	?	-3.2
?	-2.4	?	-2.5	?	-2.2	?	-2.6
?	3.1	?	3.2	?	3.7	?	3.3
?	4.5	?	4.6	?	4.7	?	4.3

						Ans:	a	b	Formula	y	x	T
5	x	10	20	30		lin	2	10	y = 10+2*x	70	35	
	y	30	50		80	exp	1,052	18	y = 18*1,052^x	83,33	29,2	13,6
						pow	0,737	5,5	y = 5,5*x^0,737	67,41	37,84	
6	x	10	15	25		lin	6	40	y = 40+6*x	190	23,33	
	y	100	130		180	exp	1,054	59,17	y = 59,17*1,054^x	219,7	21,2	13,2
						pow	0,647	22,54	y = 22,54*x^0,647	180,92	24,8	
7	x	10	20	40		lin	-3	130	y = 130+-3*x	10	40	
	y	100	70		10	exp	0,965	142,86	y = 142,86*0,965^x	34,3	74,56	-19,4
						pow	-0,515	327,02	y = 327,02*x^-0,515	49	877,72	

PerNumbers and Fractions

Unit-numbers have 1 unit, per-numbers have 2 or %. Per-numbers and fractions come from double-counting.	Per-numbers must convert to unit-numbers before being added (weighted average, conditional probability).

The ReCount-formula T = (T/B)*B predicts a counting result: 8 recounted in 2s gives: 8 = (8/2)*2 = 4*2.
The ReCount-formula solves equations: <u>u*5 = 40</u> = (40/5)*5, so u = 40/5 (move to opposite side with opposite sign).
Double-counting (proportionality) occurs in many places. A commodity may be double-counted as 4 kg and $5kr thus giving the per-number 4 kg per 5, or 4 kg/5 $, or 4/5 kg/$. Typical questions are: 60 kg = ? $ and 60$ =? kg

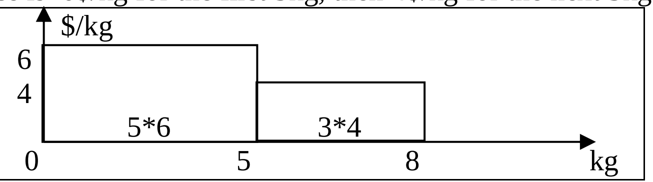
Method 1, recounting numbers	60 kg = (60/4) *4 kg = (60/4) *5 $ = 75 $	60$ = $(60/5)*5$r = 60/5*4kg = 48kg
Method 2, recounting units	$ = ($/kg)*kg = 5/4 *60 = 75	kg = kg/$*$ = 4/5*60 = 48

Per-numbers as fractions and percent
With like units, per-numbers become fractions or percentages: 4kg/5kg = 4/5, 4kg/100kg = 4/100 = 4%

40% of 30$ = u$	30$ is u% of 75$	30$ is 40% of u$
30 = (30/100)*100 giving (30/100)*40 = 12	100 = (100/75)*75 giving (100/75)*30 = 40	100% = (100/40)*40% giving (100/40)*30$ = 75$

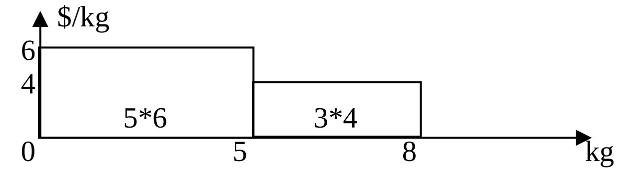
Adding per-numbers and fractions
Example 1. Big volumes may give a discount: The price is 6$/kg for the first 5kg, then 4$/kg for the next 3kg.

T1 = 5kg at 6 $/kg = 5*6 $ = 30 $
T2 = 3kg at 4 $/kg = 3*4 $ = 12 $
T = 8kg at u $/kg = 8*u $ = 42 $
8*u = 42 = (42/8)*8, so u = 42/8 = 5.25 $\frac{kr}{kg}$



Example 2. Fractions are operators needing a number to become a number: 3/5 of 5kg added to 2/3 of 3 kg totals what?

T1 = 3/5 of 5 kg = 5*3/5*5 kg = 3 kg
T2 = 2/3 of 3 kg = 3*2/3*3 kg = 2 kg
T = u of 8 kg = u*8 kg = 5 kg
u*8 = 5 = (5/8)*8, so u = $\frac{5}{8}$ $\boxed{\text{so, } \frac{3}{5} + \frac{2}{3} = \frac{5}{8}}$



Notice: Per-numbers (and fractions) add as areas under the per-number graph, also called integral calculus.

Interest rate
250$ + 8% = ?$. $ + % doesn't work. But 100%+8% = 108%, and 108% of 250$ = 1.08*250$ = (1+0.08)*250$ = 270$.
Rate-formula: T = b*(1+r) , T: terminal number, b: beginning number, r: rent, 1+r: rate-factor = rate-multiplier.
The rate r% added n times: T = b*(1+r)*(1+r) *(1+r)* … = b*(1+r)^n = b*(1+R), R: total rate, 1+R = (1+r)^n
8% added 4 times gives R = 1.08^4 – 1 = 0.360 = 36.0% = 8%*4 + 4.0% = simple rate + compound rate (CR)
Continuous rate: 8% added continuously gives the rate-factor e^0,08 = 1.0833, i.e. a rate of 8.33% including 0.33% as compound rate. e = (1+1/n)^n for n very large.

Exercises

1. Density = 1.23 kg/l. 7.5 kg = ? l, ? kg = 34 l
2. Molar mass = 16 g/mol. 234 g = ? mol. ? g = 34.5 mol
3. Concentration = 2.4 mol/l. 34 mol = ? l. ? mol = 3.5 l
4. Price = 3.6 $/kg. 346 $ = ? kg. ? $ = 234 kg
5. Speed = 23.4 m/s. 34 m = ? s. ? m = 56 s
6. 20 is ?% of 30. ? is 20% of 30. 20 is 30% of ?.

7: Rent = 80 $/day. 200 $ = ? days. ? $ = 40 days
8: 40 kg = 60 l. 75 kg = ? l. ? kg = 200 l.
9: 40 $ = 82 days. 75 kg = ? days. ? kg = 200 days.
10: 30 m = 20 s. 65 m = ? s. ? m = 800 s.
11: 60 J = 90 s. 55 J = ? s. ? J = 400 s.

12. Calculate the empty fields when T = b*(1+r)^n

	T	b	r	n	R	n*r	CR
1		10	3.2%	20			
2	34		3.2%	20			
3	34	10		20			
4	34	10	3.2%				
5	34		3.2%		72%		
6	34			20		50%	
7		10	3.2%		68%		
8		10		20			12%
9	34		3.2%				23%

PerNumber Problems

In a per-number problem, the per-number must be multiplied to a unit-number to set up an equation.

Type2.1 Traveling

Problem21: Train1 travels from A to B with the speed 40 km/t. Two hours later train2 leaves A for B with the speed 60 km/t. When and where will they meet?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*	
Hours		x = ?	*4*	40*(x+2)	= 60*x
Speed1	40 km/h			40*x + 80	= 60*x
Speed2	60 km/h			80	= 60*x – 40*x = 20*x
Km1		40*(x+2) km	*240*	80/20	= x
Km2		60*x km	*240*	4	= x

Problem22: Train1 travels from A to B with the speed 40 km/t. At the same time train2 travels from B to A with the speed 60 km/t. The distance from A to B is 300 km? When and where do they meet?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*	
Hours		x = ?	*4*	40*x + 60*x	= 300
Speed1	40 km/h			100*x	= 300*x
Speed2	60 km/h			x	= 300/100
Km1		40*x km	*120*	x	= 3
Km2		60*x km	*180*		

Problem23: A motorboat travels the same distance in 3 hours upstream as 2 hours downstream. The stream has the speed 5 km/t . What is the speed of the boat?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*	
Speed	x = ? km/h		*25*	km = km/h*h = (x-5)*3	= (x+5)*2
Speed up	x – 5 km/h		*20*	3*x - 15	= 2*x+10
Speed down	x + 5 km/h		*30*	3*x - 2*x	= 10+15
Hours		3 hours		x	= 25

Type2.2 Mixture problems

? Liter 40% alcohol + 3 liter 20% alcohol gives ? liter 32% alcohol

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*	
Liter-number		x = ? liter	*4.5*	0.4*x + 0.2*3	= 0.32*(x+3)
Liter-number		x+3 liter	*7.5*	0.4*x + 0.6	= 0.32*x + 0.96
Alcohol1	40%	0.4*x liter		0.4*x - 0.32*x	= 0.96- 0.6
Alcohol2	20%	0.2*3 liter		0.08*x	= 0.36
Alcohol3	32%	0.32*(x+3)	liter	x	= 0.36/0.08
				x	= 4.5

Type2.3 Finance

400.000$. giving a yearly yield at 20.000$ is invested in the following way: One part goes to a bank paying an interest at 3% p.a., the rest goes to 8% bonds. How much goes to the bank?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*	
Bank in thousands		x = ? $	*240*	3%*x + 8%*(400-x)	= 20
Bonds in thousands		x+3 $	*160*	0.03*x + 32 – 0.08*x	= 20
Interest rate in bank	3%			32 - 20	= 0.08*x – 0.03*x
Interest rate on bonds	8%			12	= 0.05*x
Bank part		3%*x $		12/0.05	= x
Bond part		8%*(400-x)$		240	= x

Type2.4 Work problems

A can dig a ditch in 4 hours. B can dig the same ditch in 3 hours. How many hours if working together?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*	
Hours		x = ? hours	*12/7*	¼*x + 1/3*x	= 1
A’s speed	1/4 ditch/t			(¼ + 1/3)*x	= 1
B’s speed	1/3 ditch/t			7/12*x	= 1
A’s part		¼*x		x	= 12/7
B’s part		1/3*x			

Constant Change

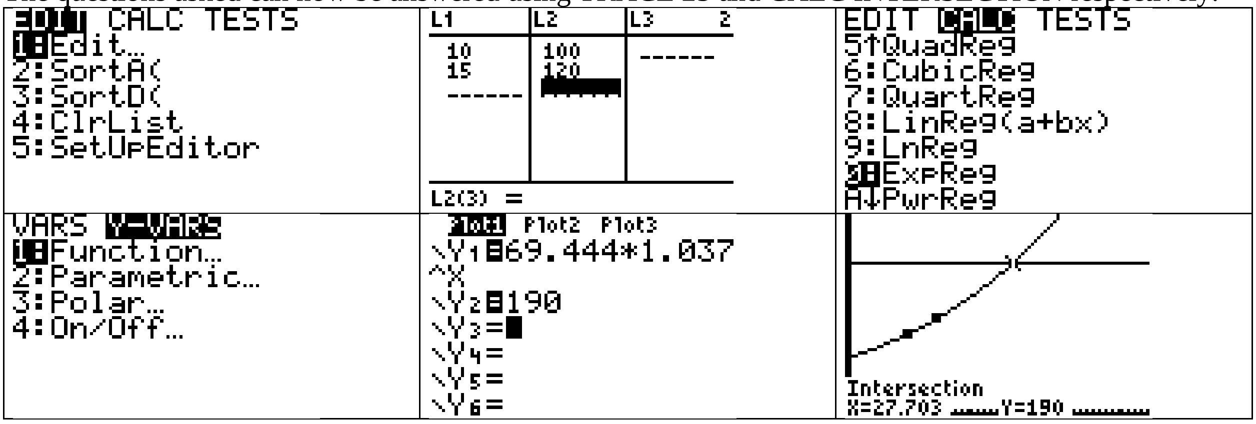
Table

x	y
10	100
15	120
25	?
?	190

Change forms

Linear ++ change	Exponential +* change	Power ** change
y = b+a*x	y = b*a^x	y = b*x^a
x: +1 day y: +a $ (change-number)	x: +1 day y: +r % (change-percent) x: + T, y: + 100% (or -50%) T = log2/loga = ln2/lna	x: +1 % y: +a % (elasticity)

To model and forecast we transform a table into a formula by using regression to find the numbers a and b. To use regression, first we select STATE EDIT. Then we enter the table. Now we select STATE CALC and e.g. ExpReg. Under the VARS Y-VARS we add Y1 to have the regression formula placed directly on the y-list. On the Y-list we add Y2 = 190 and turn on PLOT1 to see if the curve goes through the table points. The questions asked can now be answered using TRACE 25 and CALC INTERSECTION respectively.



The questions asked can also be answered using formula schemes:

y = ?	y = 60+4·x
x = 25	y = 60+4·25
	y = 160
x = ?	y = 60+(4·x)
y = 180	y–60 = 4·x
	$\frac{y-60}{4} = x$
	$\frac{180-60}{4} = x$
	30 = x

y = ?	y = 69.44·1.037^x
x = 25	y = 69.44·1.037^25
	y = 172.80
x = ?	y = 69.44·(1.037^x)
y = 180	$\frac{y}{69.44}$ = 1.037^x
	$\frac{\log\left(\frac{180}{69.44}\right)}{\log(1.037)} = x$
	26.21 = x
Test	180 = 69.44*1.037^26.21
	180 = 179.958 ☺

y = ?	y = 35.48·x^0.450
x = 25	y = 35.48·25^0.450
	y = 151.03
x = ?	y = 35.48·x^0.450
y = 180	$\frac{y}{35.48}$ = x^0.450
	$\sqrt[0.450]{\frac{180}{35.48}} = x$
	36.92 = x
Test	180 = 35.48*36.92^0.450
	180 = 179.991 ☺

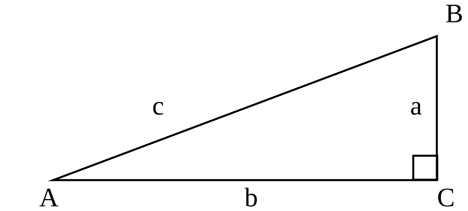
Exercises

						Ans:	a	b	Formula	y	x	Doubl. T
1	x	10	20	30		lin	2	10	y = 10+2*x	70	35	
	y	30	50		80	exp	1,052	18	y = 18*1,052^x	83,33	29,2	13,6
						pot	0,737	5,5	y = 5,5*x^0,737	67,41	37,84	
2	x	10	15	25		lin	6	40	y = 40+6*x	190	23,33	
	y	100	130		180	exp	1,054	59,17	y = 59,17*1,054^x	219,7	21,2	13,2
						pot	0,647	22,54	y = 22,54*x^0,647	180,92	24,8	
3	x	10	20	35		lin	-2	80	y = 80+-2*x	10	35	
	y	60	40		10	exp	0,96	90	y = 90*0,96^x	21,77	54,19	-17,1
						pot	-0,585	230,74	y = 230,74*x^-0,585	28,83	213,92	
4	x	10	20	40		lin	-3	130	y = 130+-3*x	10	40	
	y	100	70		10	exp	0,965	142,86	y = 142,86*0,965^x	34,3	74,56	-19,4
						pot	-0,515	327,02	y = 327,02*x^-0,515	49	877,72	

Trigonometry

Any land can be divided in triangles Any triangle can be divided into right-angled triangles	Two Greek Formula: A+B+C = 180 a^2 + b^2 = c^2 Three Arabic Formula: $\sin A = \frac{a}{c}$ $\cos A = \frac{b}{c}$ $\tan A = \frac{a}{b}$

A triangle is defined by 3 pieces. The rest can be predicted by formulas. The Greeks only found two formulas, so trigonometry first was developed when the Arabs added three extra formulas.

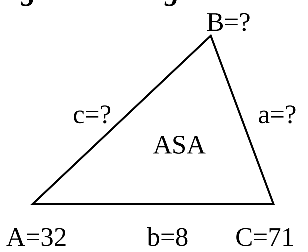


Greek formulas
A+B+C = 180 a^2 + b^2 = c^2 (Pythagoras)
Arabic formulas:
$\sin A = \frac{a}{c}$ (height in % of c) $\cos A = \frac{b}{c}$ (base in % of c) $\tan A = \frac{a}{b}$
A right triangle can be seen as a rectangle halved by a diagonal.

In a non right-angled triangle, the sine and cosine formulas have to be extended:

The Sine Rule	**The Cosine Rule (The Extended Pythagoras)**
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	a^2 = b^2 + c^2 - 2*b*c*cosA b^2 = a^2 + c^2 - 2*a*c*cosB c^2 = a^2 + b^2 - 2*a*b*cosC

Angle-Side-Angle

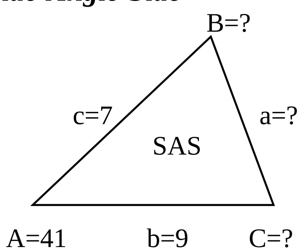


B = ?	A+B+C=180
A=32 C=71	B=180-A-C B=180-32-71 B=77

a = ?	$\frac{a}{\sin A} = \frac{b}{\sin B}$
b=8 A=32 B=77	$a = \frac{b*\sin A}{\sin B}$ $a = \frac{8*\sin 32}{\sin 77}$ a = 4.351

c = ?	$\frac{c}{\sin C} = \frac{b}{\sin B}$
b=8 C=71 B=77	$c = \frac{b*\sin C}{\sin B}$ $c = \frac{8*\sin 71}{\sin 77}$ c = 7.763

Side-Angle-Side



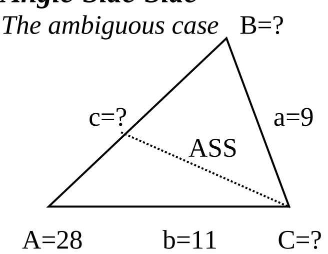
a = ?	a^2=b^2+c^2 -2*b*c*cosA
b=9 c=7 A=41	a^2=9^2+7^2 -2*9*7*cos41 a = √34.907 a = 5.908

B = ?	b^2=a^2+c^2 -2*a*c*cosB
b=9 c=7 a=5.9	$\cos B = \frac{a^2+c^2-b^2}{2*a*c}$ $\cos B = \frac{5.9^2+7^2-9^2}{2*5.9*7}$ B=cos-1(0.035)=88.0

C = ?	A+B+C=180
A=41 B=88	C=180-A-B B=180-41-88 B=51

Angle-Side-Side

The ambiguous case

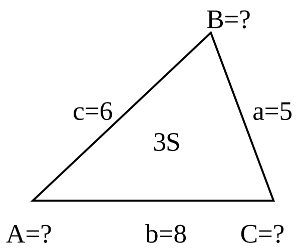


B = ?	$\frac{\sin B}{b} = \frac{\sin A}{a}$
b=11 A=28 a=9	$\sin B = \frac{b*\sin A}{a}$ $\sin B = \frac{11*\sin 28}{9}$ B=sin-1(0.574) B = [35, 145

C = ?	A+B+C=180
A=28 B=35 or B=145	C=180-A-B C=180-28-35 C=117 or C=180-28-145 C=7

c = ?	$\frac{c}{\sin C} = \frac{a}{\sin A}$
a=9 A=28 C=117 or C=7	$c = \frac{a*\sin C}{\sin A}$ $c = \frac{9*\sin 117}{\sin 28}$ c = 17.081 or c = 2.336

Side-Side-Side



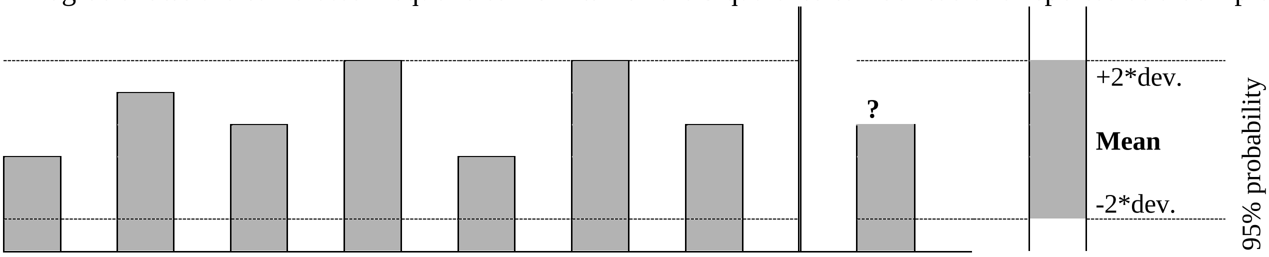
a=?	a^2=b^2+c^2 - 2*b*c*cosA
a=5 b=8 c=6	$\cos A = \frac{b^2+c^2-a^2}{2*b*c}$ $\cos A = \frac{8^2+6^2-5^2}{2*8*6}$ A = cos-1(0.781) A = 38.6

b=?	b^2=a^2+c^2 - 2*a*c*cosB
a=5 b=8 c=6	$\cos B = \frac{a^2+c^2-b^2}{2*a*c}$ $\cos B = \frac{5^2+6^2-8^2}{2*5*6}$ B = cos-1(-0.05) B = 92.9

C = ?	A+B+C=180
A=38.6 B=92.9	C=180-A-B C=180-38.8-92.9 C=48.5

Statistics, Stochastic Variation

Numbers may be predictable or unpredictable. Unpredictable numbers are also called random or stochastic numbers. Numbers that cannot be pre-dicted can often be post-dicted by setting up a statistics on their former behavior. A statistical table contains two columns, one with the numbers and one with their frequencies.
If arranged in increasing order:
The median = the middle observation, 1. (3.) quartile = the middle observation in the 1. (2.) half.
A histogram shows the frequencies
An ogive shows the cumulated frequencies from which the 3 quartiles can be read and reported as a box-plot.



Statistics — Post-dicts mean and deviation

Probability — pre-dicts x = Mean ± 2* deviation

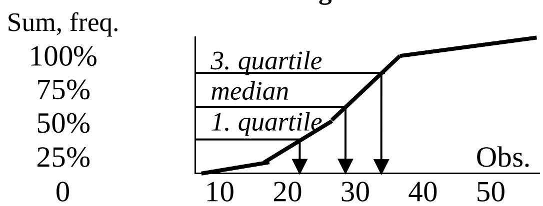
1. Observations

x: 10, 12, 22, 12, 15, ...

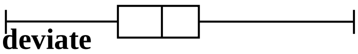
2. Grouping and counting frequencies

Observations	Frequency	Rel. Freq.	Sum. freq.
x	h	p	∑p
0-10	3	3/40=0.075	0.075
10-20	12	0.300	0.375
20-30	18	0.450	0.825
30-50	7	0.175	1.000
Total	40	1.000	

Ogive



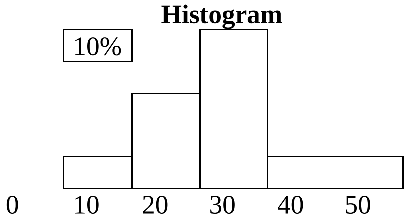
A Boxplot contains the median and two quartiles and the least and greatest observation



3. Mean or average: IF all the observations were the same ... however, the deviate

Observations	Frequency	Rel. Freq.	Sum. freq.	Mean
x	h	p	∑p	μ = ∑xi*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375
10-20	12	0.300	0.375	4.5
20-30	18	0.450	0.825	11.25
30-50	7	0.175	1.000	7
Total	40	1.000		23.1

Histogram



4. Variance, deviation: IF all the deviations were the same ...

Observations	Frequency	Rel. Freq.	Sum. freq.	Mean	Distance	Variance
x	h	p	∑p	μ = ∑xi*pi	|xi - μ|	v = ∑(xi-μ)^2*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375	|5-23.1|=18.13	18.13^2*0.075=24.64
10-20	12	0.300	0.375	4.5	8.13	19.80
20-30	18	0.450	0.825	11.25	1.88	1.58
30-50	7	0.175	1.000	7	16.88	49.83
Total	40	1.000		23.1		1 s^2 = 95.86

Deviation s = √95.86 = 9.8

5. Prediction: x = Mean ± 2*deviation =μ ± 2*s = 23.1 ± 19.6 *Confidence-interval* = [3.5 ; 42.7]

6. Using technology

On a GDC the interval midpoints are entered under STAT. Rel. frequency = freq/sum(freq). CumFreq = cumsum(freq).

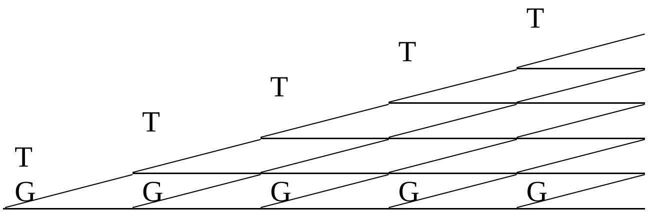
Obs.	Freq	Rel.freq	CumFreq
0	2	.05	.050
1	5	.125	.175
2	9	.225	.400
3	12	.300	.700
4	8	.200	.900
5	4	.100	1.000

The different numbers can be calculated using 1-var statistics:
Mean m = 2.8
Standard deviation, s = 1.3
Confidence-interval = m ± 2*s = [0.2;5.4]
1. quartile = 2
Median = 3
3. quartile = 4

Probability

A game has a 25% winning chance. Repeated 100 times, winning 25 times has the highest probability, which is less than not winning 25 times.	Probability predicts the unpredictable.

Example 1. A game with two results, Gain and Loss, is repeated 5 times. This implies 6 possible outcomes since we can win 0, 1, 2, 3, 4, 5 times. There is only 1 way to win 5 times: GGGGG, there are 5 ways to win 4 times because we can lose the 1st , 2nd , 3rd , 4th and 5th time: TGGGG, GTGGG, GGTGG, GGGTG, GGGGT. How many ways are there to win 3 times? We can count or predict the answer: the number of roads with 3 G out of 5 = nCr (5.3) = 10.
The 6 outcomes are equally possible, but not equally likely. The distribution obtained by repeating a 2-result experiment many times is called a **binomial distribution**. If the gain probability p is 50% = 0.5, then:
The probability of winning just 3 times of 5: p (x = 3) = BinomCdf (5, 0.5, 3, 3) = 0313 = 31.3%.
The probability of winning 2, 3 or 4 times of 5: p (2 ≤ x ≤ 4) = BinomCdf (5, 0.5, 2.4) 0718 = 71.8%
Setting up a statistics of the different outcome, we can calculate the mean and deviation. However, these numbers may be predicted by the formulas m = n*p and s = √ (n*p*(1-p)), where n is the number of repetitions and p is the winning chance.



x	nCr	Cdf
0 G	1	0.031
1 G	5	0.156
2 G	10	0.313
3 G	10	0.313
4 G	5	0.156
5 G	1	0.031

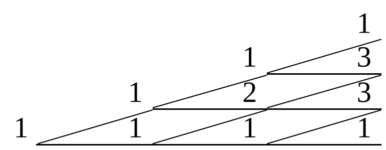
Example 2. Repeating a game with a 2/3 winning chance 30 times will give the mean value m = n*p = 30*2/3 = 20. The deviation is s = √ (n*p*(1-p)) = √ (20*1/3) = 2.6. The confidence interval is 20 ± 2 * 2.6 = [14.8; 25.2].
So there is approximately 95% chance that the next repetition will result in winning between 15 and 25 times.
If we only win 12 times, we must consider rejecting the hypothesis about a 2/3 winning chance.

Normal distribution

With many repetitions, a binomial distribution will approach the normal distribution, which often occurs in nature where there usually will be some variation of, for example, animal height. In such cases, the random variables x may assume decimal values or negative values.
Example 3. Repeating a game with a 60% winning chance 20,000 times will give the mean value m = n*p = 20000*0.60 = 12000, and deviation s = √ (n*p*(1-p)) = √ (12000*0.40) = 69.3.
Binomial distribution: P (0 < x < 12123) = BinomCdf (20000, 0.60, 0,12122) = 0962 = 96.2%
Normal distribution: P (0 < x < 12123) = NormCdf (0, 12123, 12000, 69.3) = 0962 = 96.2%
A normal distribution will have a straight sum-curve on a normal paper.

Exercises

1. Describe example 1 with p = 30%?
2. Describe example 1 with 6 repetitions
3. Describe the example 1 with 8 repetitions.
4. 'when (RAND() < 0.7, 1.0)' is an game with p = 0.7. Combined with itself 5 times is equivalent to performing the experiment in example 1. Perform the game 32 times and make a statistics on the outcome.
5. Describe example 1 with 200 repetitions and different p-values. Compare responses from the binomial and the normal distribution.
6. In a population, weight is normal distributed with mean value 13.2 and deviation 2.4. What is the probability of not exceeding 12.6? at least 13.5? between 13.0 and 14.0?
7. In a population height is normal distributed with mean value 132 and spread 24. What is the probability of a maximum of 126? at least 135? between 130 and 140?
8. Arrange the nCr numbers systematically in the triangle below to create a 'Pascal's triangle'. Which features have this triangle? Who was Pascal?

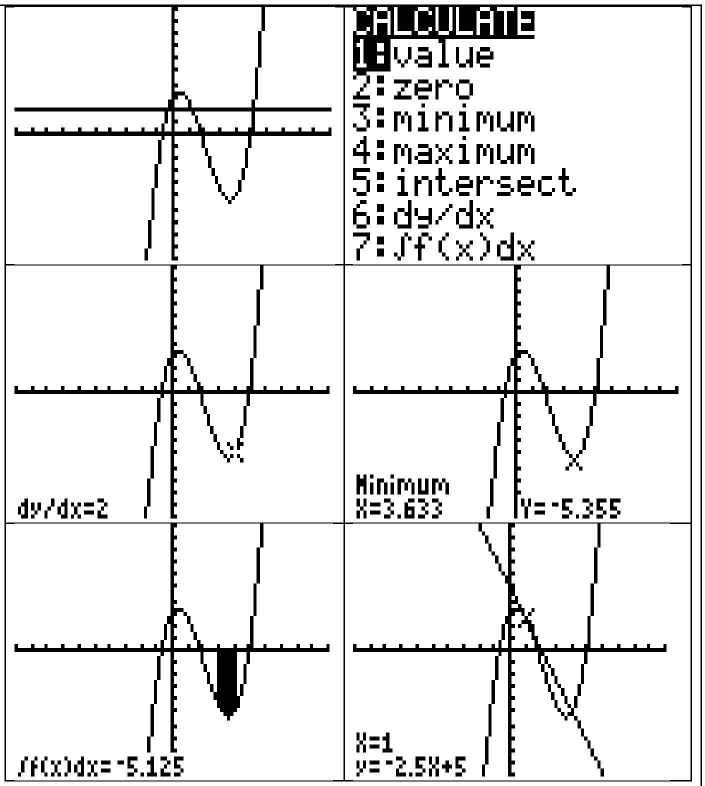


Polynomials and Calculus

0. degree polynomial tells the (initial) point	y = 5
1. degree polynomial tells the (initial) gradient or steepness	y = 5 + 2*x
2. degree polynomial tells the (initial) bending	y = 5 + 2*x + 0.3*x^2
3. degree polynomial tells the (initial) counter-bending	y = 5 + 2*x + 0.7*x^2 – 0.2*x^3
4. degree polynomial tells the (initial) counter-counter-bending	y = 5 + 2*x + 0.7*x^2 – 0.2*x^3 + 0.3*x^4

Arabic numbers are polynomials: 4352 = 4*10^3 + 3*10^2 + 5*10 + 2. General form: y = 4*x^3 + 3*x^2 + 5*x + 2
Polynomials with bending graphs (degree over 1) have some interesting points:
Turning points, either top-points (maximum) or bottom-points (minimum).
Intersections with the x-axis (zeros), with the y-axis (y-intercept), or with other graphs.
Intersecting other graphs (equations graphically), Intersecting vertical lines (tracing values).
Shifting bending or curvature, where the bending changes its sign.
Tangent-point. A tangent is a straight line practically coinciding with the graph around the contact point, thus showing a scenario: this is how the graph would look like if the steepness stays constant.
If the curve graphs per-numbers, the total is found as the area under the per-number graph, i.e. by integration
If the curve graphs a Total, the per-numbers are found as the steepness of the total graph, i.e. by differentiation
Differentiation twice gives the bending, being positive when bending upwards and negative when bending downwards.
Finding the steepness (gradient, slope) formula is called differentiation. Finding the area under a curve is called integration. Together differentiation and integration are inverse operations called Calculus
Example: y = 0.5x^3-3x^2+2x+3

	Graphics	Formula
Intersecting the y-axis CALC value	y = 3	y1(0)
Intersecting the x-axis CALC zero	x = -0.694 x = 1.748 x = 4.946	Solve(0=Y1)
Intersecting y = 2 CALC intersection	x = -0.329 x = 1.181 x = 5.147	Solve(0=Y1-2)
Top CALC Maximum	x = 0.367 y = 3.355	MATH fMax(Y1,x,0,7)
Bottom CALC Minimum	x = 3.633 y = -5.355	MATH fMin(Y1,x,0,7)
Steepness in x = 4 CALC dy/dx	2	MATH nDeriv(Y1,x,4)
Area from 3 to 4 CALC Sf(x)dx	-5.125	MATH fnInt(Y1,x,3,4)
Tangent in x = 1 DRAW tangent x = 1	y=-2.5x+5	



Exercises

1. Repeat as above with y = 0.7x^3-4x^2+3x+4. 2. Repeat as above with y = -0.4x^3+2x^2-0.5x-3. 3. Produce your own polynomials using randM(4,1). 4. Produce your own polynomials using regression 5. Find the cheapest box without lid containing 1 liter. 6. Find the cheapest pipe without lid containing 1 liter. 7. Find the cheapest box with lid containing 1 liter. 8. Find the cheapest pipe with lid containing 1 liter. 9. Find the cheapest box with double lid containing 1 liter. 10. Find the cheapest pipe with double lid contain. 1 liter.	11. Find the cheapest cone without lid containing 1 liter. 12. Find the cheapest cone with lid containing 1 liter. 13. Find the cheapest cone with double lid contain. 1 liter. 14. y1 is a polynomial of degree 0. If y1 is a Total, what is its per-number? If y1 is a per-number, what is its total? 15. As 14 with polynomials of degree 1. 16. As 14 with polynomials of degree 2. 17. As 14 with polynomials of degree 3.

Two equations With Two Unknowns; and Three

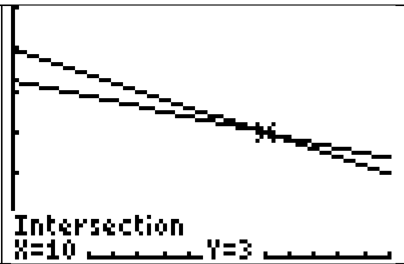
Two equations with two unknowns are solved manually, by intersection or by matrices	b \$ + 5kg at a \$/kg = 25 \$ b \$ + 8kg at a \$/kg = 34 \$	x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1 & 5\\1 & 8\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$

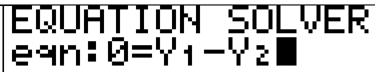
2 equations with 2 unknowns: The formula b \$ + 5kg at a \$/kg = 25\$ contains 2 unknowns and cannot be solved, unless we know another example of the same formula as e.g. b \$ + 8kg at a \$/kg = 34 \$.

Written as an equation system	Written as a matrix equation
x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1 & 5\\1 & 8\end{pmatrix} * \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$

Manually one variable is isolated in the first and inserted in the second equation: x=25-5*y, 25-5*y+8*y=34, y=3 & x=10.

Using graphs, the y's are isolated and inserted into the y- editor.
x + 5*y = 25 gives y = (25-x)/5 ,
x + 8*y = 34 gives y = (34-x)/8
The intersection point is found by 'Calc Intersection' to x = 10 and y = 3.
Also we can use Math Solver 0 = Y1 –Y2.





Matrix-solutions is found by entering the matrices into the matrix-editor as ml and mr (matrix-left & -right):

$\underline{V} = \begin{pmatrix}x\\y\end{pmatrix} = ?$	$\underline{\underline{ml}} * \underline{V} = \underline{mr}$
$\underline{\underline{ml}} = \begin{pmatrix}1 & 5\\1 & 8\end{pmatrix}$ $\underline{mr} = \begin{pmatrix}25\\34\end{pmatrix}$	$\underline{V} = \underline{\underline{ml}}$^-1 * $\underline{mr}$ $\underline{V} = \begin{pmatrix}1 & 5\\1 & 8\end{pmatrix}$ ^-1 * $\begin{pmatrix}25\\34\end{pmatrix}$ $\underline{V} = \begin{pmatrix}10\\3\end{pmatrix}$
Test	$\begin{pmatrix}1 & 5\\1 & 8\end{pmatrix} * \begin{pmatrix}10\\3\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$ $\begin{pmatrix}25\\34\end{pmatrix} = \begin{pmatrix}25\\34\end{pmatrix}$

$\underline{V} = \begin{pmatrix}x\\y\\z\end{pmatrix} = ?$	$\underline{\underline{ml}} * \underline{V} = \underline{mr}$
$\underline{\underline{ml}} = \begin{pmatrix}2 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix}$ $\underline{mr} = \begin{pmatrix}18\\-2\\16\end{pmatrix}$	$\underline{V} = \underline{\underline{ml}}$^-1 * $\underline{mr}$ $\underline{V} = \begin{pmatrix}2 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix}$ ^-1 * $\begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\underline{V} = \begin{pmatrix}1\\2\\3\end{pmatrix}$
Test	$\begin{pmatrix}2 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix} * \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\begin{pmatrix}18\\-2\\16\end{pmatrix} = \begin{pmatrix}18\\-2\\16\end{pmatrix}$

3 equations with 3 unknowns cannot be solved graphically, but manually and by using matrices:

Written as an equation system	Written as a matrix equation
3*x + 5*y + 2*z = 19 x – z = -2 4*x – 3*y + 6*z = 16	$\begin{pmatrix}3 & 5 & 2\\1 & 0 & -1\\4 & -3 & 6\end{pmatrix} * \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}19\\-2\\16\end{pmatrix}$

A matrix-solution is found by entering the matrices into the matrix-editor as ml and mr.

4 equations with 4 unknowns, 5 equations with 5 unknowns etc. Like 3 equations with 3 unknowns.

Equation systems for skill building can be generated by 'randM(3,3)' and 'randM(3,1).

Exercises. Solve the equation systems

1. 4x - 1*y = -9 4x - 4*y = 0 2. 4x + 2*y = 16 5x - 3*y = -2 3. 7x + 4*y = -1 -3x + 2*y = 19 4. 2x - 5*y = 16 3x - 4*y = 17	5. -7*x - 3*y - 7*z = 3 -1*x - 5*y + 1*z = -13 9*y - 5*z = 36 6. 4*x + 3*y + 7*z = 81 5*x + 3*y + 1*z = 54 2*x + 9*y + 5*z = 57 7. 2*x + 3*y - 1*z = - 6 5*x + 3*y - 4*z = -15 2*x – 2*y + 5*z = 40	8. 2*x + 5*y - 1*z + 9t = 118 1*x + 1*y - 9*z - 5t = -88 - 3*y + 7*z + 5t = -51 -3*x + 5*y + 2*z - 5t = -10 9. -6*x - 1*y + 8*z + 8t = 129 -2*x + 2*y - 5*z + 7t = 60 8*x + 6*y + 3*z + 3t = -40 -7*x - 4*y - 8*z - 4t = 12

The Quantitative Literature: The Three Kinds of Models

Geometry and algebra are the classical quantitative literature meaning ‘measuring earth’ and ‘uniting number’s in Greek and Arabic respectively. The modern quantitative literature is mainly created by questions coming from the production: How to extract silver and coal from mines? How to navigate at sea? How to predict the orbit of a falling body? How to build machines? How to optimize a production? How to optimize profits? Etc.
Calculations took place to lift silver and water up from the mines, and how to transform crude silver into silver of varying degree of purity. Silver now travelled through Germany to Italy, and from there to the east to be exchanged with pepper and silk, either via the expensive way by land transported by caravans, or via the cheap way over sea transported by Arab merchants. Helped by replacing their own Roman numbers with Arabic numbers allowing multiplication and division to take place, Italian wealth created banks to lend out money for an interest in return. This implies a need for adding interest rates leading to the discovery of power calculations and compound interest: 7 years at 6% yearly = 42% interest rate + 8% compound interest rate = 50% since 106% ^ 7 = 150%. Part of the profit goes to consumption when building magnificent palaces everywhere in Renaissance Italy, and when hiring artists.
Then Portugal took over by decreasing the price on pepper to 1/3 when skipping the Arabic middlemen by sailing silk and pepper home around Africa on their own ships. Spain then tried to find another way to India by sailing west. But in West India there was no spice or silk, but plenty of gold and silver, e.g. in the land of silver, Argentine. The Pope divided the new world between Spain and Portugal. Portugal gets everything east of the 60^{th} longitude, Spain everything west of it. In Spain and Portugal profits went to consumption through the building of churches and monasteries and mansions. The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move? The church said ’among the stars’. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth pulled down by a gravitational force, that changes motion thus leading to the creation of change-calculations, calculus.
Once in India, England exchanged silver for cotton to be planted in their North American colonies bought form Spain who had no interest in land without silver. By replacing silk and pepper trade from the Far East with production of cotton in the US, England created a triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton. The profit was used for investment buying stock and establishing an industrial production. Fighting for colonies led to the second world war, leading to the creation of the computer and operation research.

The three genres: facts, fiction and fiddle

Qualitative and quantitative literature divide into three genres: facts, fiction and fiddle. In qualitative literature, the three qualitative of genres are
Fact: ‘SINCE Copenhagen is located on Zealand, THEN Copenhagen is close to water level.’
Fiction: ‘IF Copenhagen was located in the Alps, THEN Copenhagen is far from water level.’
Fiddle: ‘IF Copenhagen is located first in a sentence, THEN Copenhagen is close to water level.’

Fact

Fact models are ‘SINCE-THEN’ calculations, quantifying quantities, and calculating the calculable: ‘SINCE the price is 4 $/kg, THEN 6 kg costs 6 * 4 = 24$.
‘SINCE-THEN’ calculations may also be called ‘room-calculations: ‘SINCE the room has dimensions 3x4x5, THEN the volume is 3*4*5 = 60’
Fact models are re-calculated for testing: T = 3 kg at 4$/kg. = 3*4 $ = 15 $, oops, calculation error, T = 12 $.
Fact models may be wrong if omitting the units. The Mars Climate Orbiter thus failed when mixing cm and inch.

Fiction

Fiction models are ‘IF-THEN’ calculations, quantifying quantities, and calculating the incalculable: ‘IF my daily income is 4$, THEN 6 days will give the income be 6*4 = 24$’. But my income may vary?
‘IF-THEN’ calculations may also be called ‘rate-calculations’: ‘IF the growth rate is 3% per year, THEN the total growth rate after 5 years be 15.9%, since 103% ^ 5 = 115.9%.’ But the growth rate may vary?
Fiction models must be supplemented by parallel scenarios: The daily income is estimated to be between 4$ and 5$, so in 3 days, ‘The expected income will be between 3*4$ = 12$ and 3*5$ =15$.

Fiddle

Fiddle models are ‘SO-WHAT’ calculations, quantifying qualities, and calculate the incalculable: ’If the consequence C= broken bone is set to 2 million $, and if the probability P set to 30%, then will the risk be R = C*P = 2 * 0.3 = 0.6 million $.’ But SO-WHAT? Who says that a broken leg costs 2 million $? And who says that a general probability of breaking a leg can be measured?
Another example: ‘If the cost of a burial ground is 10 $/day, and the cost of a hospital bed is 10,000 $/day, then it is cheaper to have people lying in the cemetery than at the hospital. But SO-WHAT? Should we increase the speed limit to 200 km/hour to save money?’
‘SO-WHAT’ calculations may also be called ‘risk-calculations’: ‘IF we can increase the probability of death and reduce the probability of injury, then will the risk of crossing a school road could be reduced.’ But SO-WHAT? Should we then dismantle the zebra crossing? Fiddle models must be rejected and referred to a qualitative treatment in the word-language instead of a quantitative one in the number-language.

Letter Calculation, Transposing Formulas

Change the T-formulas to a-formulas, b-formulas and c-formulas, and vice versa.

	T		a	b	c
1	$T = a + b\cdot c$		$a = T - b\cdot c$	$b = \frac{T-a}{c}$	$c = \frac{T-a}{b}$
2	$T = a - b\cdot c$		$a = T + b\cdot c$	$b = \frac{a-T}{c}$	$c = \frac{a-T}{b}$
3	$T = a + \frac{b}{c}$		$a = T - \frac{b}{c}$	$b = (T-a)\cdot c$	$c = \frac{b}{T-a}$
4	$T = a - \frac{b}{c}$		$a = T + \frac{b}{c}$	$b = (a-T)\cdot c$	$c = \frac{b}{a-T}$
5	$T = (a + b)\cdot c$		$a = \frac{T}{c} - b$	$b = \frac{T}{c} - a$	$c = \frac{T}{a+b}$
6	$T = (a - b)\cdot c$		$a = \frac{T}{c} + b$	$b = a - \frac{T}{c}$	$c = \frac{T}{a-b}$
7	$T = \frac{a+b}{c}$		$a = T\cdot c - b$	$b = T\cdot c - a$	$c = \frac{a+b}{T}$
8	$T = \frac{a-b}{c}$		$a = T\cdot c + b$	$b = a - T\cdot c$	$c = \frac{a-b}{T}$
9	$T = \frac{a}{b+c}$		$a = T\cdot (b+c)$	$b = \frac{a}{T} - c$	$c = \frac{a}{T} - b$
10	$T = \frac{a}{b-c}$		$a = T\cdot (b-c)$	$b = \frac{a}{T} + c$	$c = b - \frac{a}{T}$
11	$T = \frac{a}{b} + c$		$a = (T-c)\cdot b$	$b = \frac{a}{T-c}$	$c = T - \frac{a}{b}$
12	$T = \frac{a}{b} - c$		$a = (T+c)\cdot b$	$b = \frac{a}{T+c}$	$c = \frac{a}{b} - T$
13	$T = a\cdot b^{c}$		$a = \frac{T}{b^{c}}$	$b = \sqrt[c]{\frac{T}{a}}$	$c = \frac{\log(\frac{T}{a})}{\log b}$
14	$T = \frac{a}{b^{c}}$		$a = T\cdot b^{c}$	$b = \sqrt[c]{\frac{a}{T}}$	$c = \frac{\log(\frac{a}{T})}{\log b}$
15	$T = (a\cdot b)^{c}$		$a = \frac{\sqrt[c]{T}}{b}$	$b = \frac{\sqrt[c]{T}}{a}$	$c = \frac{\log T}{\log(a\cdot b)}$
16	$T = (\frac{a}{b})^{c}$		$a = \sqrt[c]{T}\cdot b$	$b = \frac{a}{\sqrt[c]{T}}$	$c = \frac{\log T}{\log(\frac{a}{b})}$
17	$T = (a + b)^{c}$		$a = \sqrt[c]{T} - b$	$b = \sqrt[c]{T} - a$	$c = \frac{\log T}{\log(a+b)}$
18	$T = (a - b)^{c}$		$a = \sqrt[c]{T} + b$	$b = a - \sqrt[c]{T}$	$c = \frac{\log T}{\log(a-b)}$
19	$T = a + b^{c}$		$a = T - b^{c}$	$b = \sqrt[c]{T-a}$	$c = \frac{\log(T-a)}{\log b}$
20	$T = a - b^{c}$		$a = T + b^{c}$	$b = \sqrt[c]{a-T}$	$c = \frac{\log(a-T)}{\log b}$
21	$T = a^{(b+c)}$		$a = \sqrt[(b+c)]{T}$	$b = \frac{\log T}{\log a} - c$	$c = \frac{\log T}{\log a} - b$
22	$T = a^{(b-c)}$		$a = \sqrt[(b-c)]{T}$	$b = \frac{\log T}{\log a} + c$	$c = b - \frac{\log T}{\log a}$

Homework

1. In the triangle ABC, C is 90, A = 42, c = 5. Find the rest.
2. In the triangle ABC, C is 90, A = 34, a = 6. Find the rest.
3. In the triangle ABC, C is 90, A = 28, b = 7. Find the rest.
4. In the triangle ABC, C is 90, a = 5, c = 7. Find the rest.
5. In the triangle ABC, C is 90, b = 4, c = 7. Find the rest.
6. In the triangle ABC, C is 90, a = 4, b = 5. Find the rest.
7. In the triangle ABC, A is 32.6, b = 4.6, c = 5.2. Find the rest.
8. In the triangle ABC, A is 34.8, b = 5.6, a = 7.2. Find the rest.
8. In the triangle ABC, A is 42.6, B = 74.6, c = 6.2. Find the rest.
10. In the triangle ABC, A is 34.8, C = 54.6, a = 5.2. Find the rest.

11. (all lin, exp & pow)

x	y
2	10
7	15
9	?
?	30

12

x	y
3	8
7	12
9	?
?	28

13

x	y
1	20
5	30
9	?
?	80

14

x	y
10	80
20	62
30	?
?	30

15

x	y
12	64
18	42
25	?
?	24

16

x	y
3	50
12	28
20	?
?	10

17. In 1993 there was 420 $. In 1998 there was 630 $. In 2005 there was ? $. In ? there was 950 $. Linear and exponential and power change.
18. In 1994 there was 520 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 1250 $. Linear and exponential and power change.
19. In 1992 there was 920 $. In 1996 there was 730 $. In 2005 there was ? $. In ? there was 450 $. Linear and exponential and power change.
20. In 1994 there was 720 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 250 $. Linear and exponential and power change.
21. A capital had 753 $. increased with 20% 4 times and became ? $. What is the doubling-time?
22. A capital had 956 $. decreased with 25% 5 times and became ? $. What is the half-time?
23. A capital had 486 $. increased with 30% ? times and became 2345.83 $. What is the doubling-time?
24. A capital had 324 $. decreased with 35% ? times and became 25.88 $. What is the half-time?
25. A capital had 743 $. increased with ?% 4 times and became 2854.32 $. What is the doubling-time?
26. A capital had 896 $. decreased with ?% 5 times and became 45.09 $. What is the half-time?
27. A capital had ? $. increased with 50% 6 times and became 2423.83 $. What is the doubling-time?
28. A capital had ? $. decreased with 55% 7 times and became 2.45 $. What is the half-time?

31. (Polynomial regr.)

x	y
2	10
7	30
9	35
12	?
?	30
?	turn

32

x	y
3	8
7	5
11	12
9	?
?	28
?	turn

33

x	y
1	20
5	30
7	35
9	?
?	10
?	turn

34

x	y
10	60
20	120
30	30
40	70
50	?
?	80
?	turn

35

x	y
12	74
18	22
20	43
25	41
30	?
?	34
?	turn

36

x	y
3	9
12	28
15	8
17	14
20	?
?	10
?	turn

41. (Mean, ogive. & boxplot)

Obs	Freq
0-10	6
10-20	9
20-30	12
30-40	15
40-50	6

42

Obs	Freq
0-10	50
10-20	20
20-30	10
30-40	20
40-50	30

43

Obs	Freq
0-10	16
10-20	29
20-30	52
30-40	25
40-50	16

44

Obs	Freq
0-10	16
10-20	29
20-30	32
30-40	45
40-50	56
50-60	66

45

Obs	Freq
0-10	12
10-20	56
20-30	42
30-40	13
40-50	73
50-60	25

46

Obs	Freq
0-10	23
10-20	45
20-30	25
30-40	12
40-50	86
50-60	23
60-70	45

51. Solve the equation 2+3*(1+x)^4 = 20
52. Solve the equation 4+5*(1+x)^6 = 30
53. Solve the equation 40-3*(1-x)^4 = 20
54. Solve the equation 50-4*(1-x)^5 = 10
55. Transpose the equations $T = d-e$, $T = d - \frac{e}{f}$, $T = d - \frac{e-f}{g}$
56. Transpose the equations $T = \frac{d}{e}$, $T = \frac{d}{e} - f$, $T = \frac{d-e}{f} - g$

01. Project Forecasting

Problem: How to set up a forecast assuming constant growth?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The real-world problem

A capital is assumed to grow constantly. From two data sets we would like to establish a forecast predicting the capital at a certain time and when a certain level will be reached.

2. The mathematical problem

We set up a table showing the capital to two different times. x are years, y is 1000 $

x	y = ?		
2	10	1. Linear ++ growth: y = a*x + b	x: +1, y: +a (gradient, slope)
5	30	2. Exponential +* growth: y = a*b^x = a*(1+r)^x	x: +1, y: + r% (interest rate, b = 1+r)
8	?	3. Power ** growth: y = a*x^b	x: +1%, y: + r% (elasticity)
?	60		

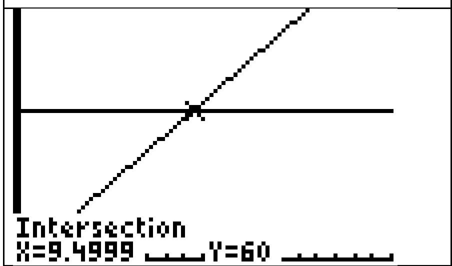
3. Solving the mathematical problem

First we find the y-formulas using regression. We enter the table as lists L1 and L2 und STAT.
'LinReg Y1' produces a linear model transferred to the y-list as Y1
'ExpReg Y1' produces an exponential model transferred to the y-list as Y1
'PowerReg Y1' produces a power model transferred to the y-list as Y1

Linear growth

y = ?	y = 6.667*x – 3.333
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 6.667*8 – 3.333 = 50
Test	Trace x = 8 gives y = 50

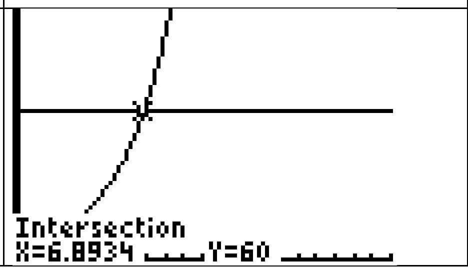
x = ?	y = 6.667*x – 3.333
y = 60	60 = (6.667*x)– 3.333 60 + 3.333 = 6.667*x 63.333/6.667 = x 9.5 = x
Test1	60 = 6.667*9.5– 3.333 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 9.5
Test3	CALC Intersection with y2=60 gives x = 9.5



Exponential growth

y = ?	y = 4.807 * 1.442^x
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.807 * 1.442^8 = 89.9
Test	Trace x = 8 gives y = 89.9

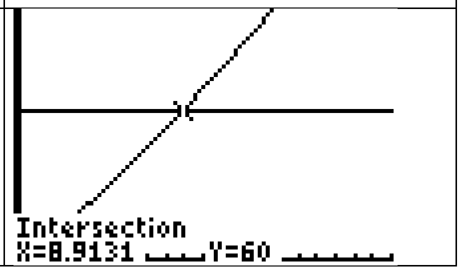
x = ?	y = 4.807 * 1.442^x
y = 60	60 = 4.807 * (1.442^x) 60/4.807 = 1.442^x log(60/4.807)/log1.142 = x 6.89 = x
Test1	60 = 4.807 * 1.442^6.89 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 6.89
Test3	CALC Intersection with y2=60 gives x = 6.89



Power growth

y = ?	y = 4.356*x^1.199
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.356*8^1.199 = 52.7
Test	Trace x = 8 gives y = 52.7

x = ?	y = 4.356*x^1.199
y = 60	60 = 4.356*(x^1.199) 60/4.356 = x^1.199 1.199√(60/4.356) = x 8.91 = x
Test1	60 = 4.356*8.91^1.199 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 8.91
Test3	CALC Intersection with y2=60 gives x = 8.91



4. Solving the real-world problem

We see that forecast can be made by using technology's regression lines. The forecasts give different answers to the same questions since different forms of growth is assumed.
Linear growth assumes that the gradient is constant.
Exponential growth assumes that the interest rate is constant.
Power growth assumes that the elasticity is constant.

02. Project Population and Food Growth

Problem: When will the population exceed food supply

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

Around the year 1800, the English economist Malthus (1766-1834) predicted a future food crisis, "since the world's population grows exponentially and food supply linearly, the population will one day overtake the food supply with famine to follow" (Malthus ' principle of population). Is Malthus right?

2. The mathematical problem

We set up a table of time x as the number of years after 1850; and the world's population, which is assumed to be 1.59 billion in 1900 and 5.3 billion in 1990; and world food production, which is assumed to be 1,800 billion daily rations in 1900 and 4.5 billion daily rations in 1990. The population is assumed to grow exponentially, and food quantity is assumed to grow linearly. The table scope is assumed to be 0 < x < 250.

x Years after 1850	Y1 World population in mio.	Y2 World food supply in mio. in daily rations
(1900) 50	1590	1800
(1990) 140	5300	4500

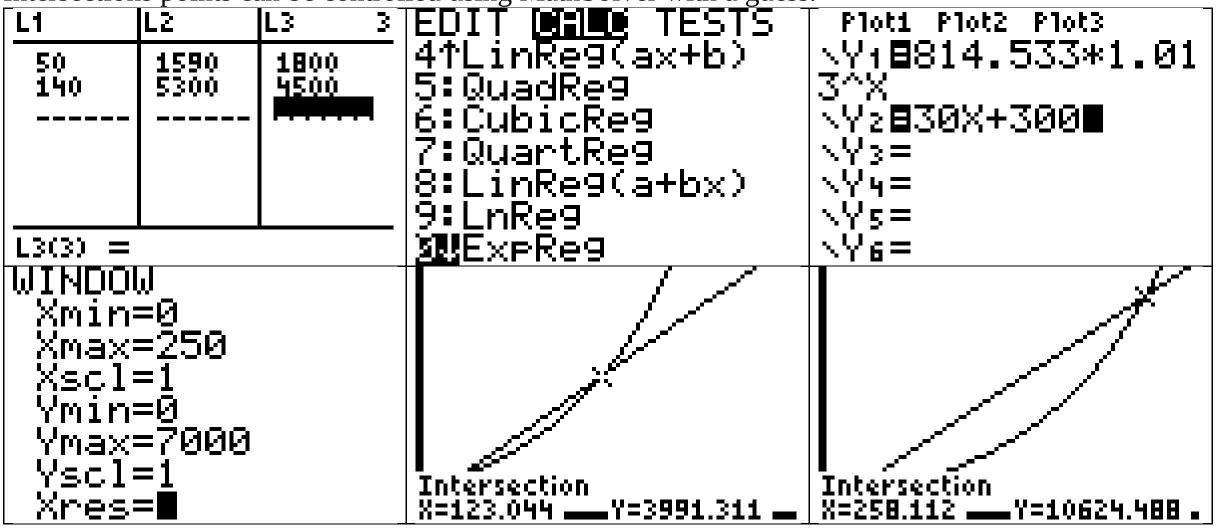
3. Solving the mathematical problem

On a TI-82, the x-numbers are on list L1, and the y-numbers are on lists L2 and L3. The formulas for the population are determined by L1, L2, ExpReg Y1. The result is y1 = 815 * 1,013 ^ x. So, when x is 0 in 1850, the population is y = 815; and when x increases by 1, the population increases by 1.3% 1. The formula for the volume of food is determined by LinReg L1, L3, Y2. The result is y1 = 300 + 30 x. So, when x is 0 in the 1850, the food amount is y = 300; and when x increases by 1, the food increases by 30.

Famine occurs where Y1 is greater than Y2. Y1 and Y2 are the intersection found with 'Calc Intersection ' to around x = 30 and 123. So, as to the model there was famine from 1850 to 1880, and again after 1973.

Assume instead that the world population is growing by 1% a year, and the amount of food with 40 per year. Then the formulas become Y3 = 815 * 1.01^ x, and Y4 = 300 + 40 x. They will intersect at approximately x = 16 and x = 258. I.e. in this case, there was famine from 1850 to 1866, and again after 2108.

Intersections points can be controlled using MathSolver with a guess.



4. Solving the real-world problem

Malthus is right in saying that there will be famine if the world's population continues to grow exponentially, and the quantity of food continues to grow linearly, for a curved path will always outpace a straight. If food grows by 30 million daily rations per year, then famine will occur years 1973 if the world's population is growing by 1.3% per year, and in the year 2108 if the world's population is growing by 1% per year and food production grows by 40/year. However, it will be earlier if a part of the food is used for fueling cars instead.

03. Project Saving and Pension

Problem: How much pension will a saving provide?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

A saving comes from sending in a fixed amount each month to a bank. In the end, a saving can be used to drawing out a fixed amount each month. What is the relationship between the monthly saving input and the pension output?

2. The mathematical problem

If paying $1000 monthly for 30 years, what will the monthly pension be for 10 years? The interest rate is 0.4% monthly. By saving two formulas apply, the first applies to a single deposit, the second for many monthly deposits:

1) K = Ko*(1+R), 1+R = (1+r)^n, K/Ko: terminal/initial capital, r: monthly rate, R: total interest, n number of months.
2) K/a = R/r, K: terminal capital, a. monthly deposit, r: monthly rate, R: total interest,

3. Solving the mathematical problem

First we find the total interest rate per year R: 1+R = (1+r)^n = (1+0.004)^12, so R = 1.049 –1 = 0.049 = 4.9% per year.
Then we find the total interest rate for 30 years R: 1+R = (1+r)^n = (1+0.004)^(30*12) = 4.209. So. R = 4.209-1 = 3.209 = 321%.
The simple interest rate is 30*12*0.4% = 144%. So the effect of compound interest is 321% - 144% = 177%.
With a = 1000, r = 0.4% the saving after x deposits will be K = a*R/r = 1000*(1.004^x –1) / 0.004.
We find the saving after 10, 20 and 30 years:

Months	120	240	360
Saving	153632	401675	802147

The total deposit after 30 years is 1000*360 = 360000. The effect of compound interest is 802147 –360000 = 442147.
We observe that the saving is 500000 after 275 deposits.

x = ?

$$\frac{1000*(1.004^x-1)}{0.004} = 500000$$

$$1.004^x - 1 = \frac{500000*0.004}{1000}$$

$$1.004^x = 2+1$$

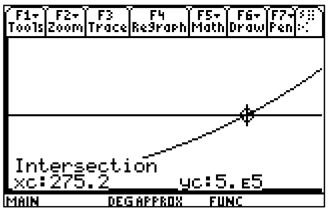
$$x = \frac{\ln(3)}{\ln(1.004)} = 275.2$$

Test1

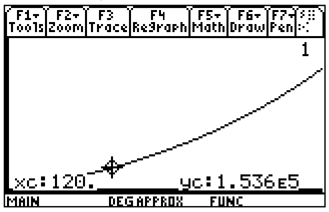
$$\frac{1000*(1.004^{275.2}-1)}{0.004} = 500000$$

499994 = 500000

Test 2



x = 120, K = ?



To use the saving for a 10 years pension we use two accounts.
On the first, the saving grows from 10 years of interest to K = Ko (1 + R) = 802147 * (1 + 0.004) ^ 120 = 1295089.
The second is used for a 'negative saving' with a monthly redraw, a, that makes the two accounts balance after 10 years: K = a*R/r = Ko*(1+R), giving the equation a*(1,004^120 – 1)/0.004 = 1295089 that is solved by a = 8430.
Thus the relationship between output and input is (10*12*8430)/(30*12*1000) = 2.8.
Repeating the calculations with a monthly interest rate of 0.3% and 0.5% gives a different relationship:

Monthly %	Yearly %	Saving	Monthly pension	Relationship between output and input
0.3%	3.7%	646640	6425	(10*12*6425)/(30*12*1000) = 2.1
0.4%	4.9%	802147	8430	(10*12*8430)/(30*12*1000) = 2.8
0.5%	6.2%	1004515	11152	(10*12*11152)/(30*12*1000) = 3.7

4. Solving the real-world problem

When saving, an account grows from three monthly sources: a deposit, and an interest rate of the total deposit as well as of the total interest amount. When terminated, a saving continues to grow, but, used as a pension fund, the monthly input is replaced by a monthly output, the pension. Depositing $1000 monthly for 30 years allows taking out monthly $8430 in 10 years. With a monthly interest rate of respectively 0.3%, 0.4% and 0.5% the output-input ratio is respectively. 2.1, 2.8 and 3.7. However, it should be remembered that 40 years of inflation will reduce this factor.

Proof of the saving-formula: Account 1 contains the amount a/r. Each month we transfer the interest amount, r*a/r = a, to account 2, also receiving the monthly interest of its own amount. Account 2 thus will contain a saving K, but at the same time it contains total interest amount R of account 1, i.e. R*a/r. Therefore K = R*a/r , or K/a = R/r.

04. Project Supply, Demand and Market Price

Problem: How does supply and demand determine the market price?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

We assume we know supply curve and the demand for a given commodity, e.g. apples. That is, we know the market price determines supply and demand. If supply is larger than demand, a decrease in price should increase demand and decrease supply. If supply is less than demand, an increase in price should decrease demand and increase supply. The equilibrium price therefore should occur where supply equals demand.

2. The mathematical problem

We set up a table showing the relationship between price and demand and supply. The table is supposed to be valid for prices between 0 and 10, 0 < x < 10. Regression allows finding the two formulas, becoming an equation when set to be equal, thus finding the point of intersection that determines the equilibrium price.

Linear graphs

Price x	Supply S	Demand D
2	40	80
4	60	50

Bending graphs

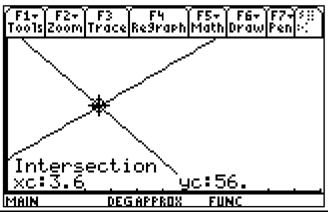
Price x	Supply S	Demand D
2	40	80
4	60	50
6	75	30

3. Solving the mathematical problem

Entering a table to the data/matrix-editor of a graphic display calculator allows finding regression formulas.
With 2 data-sets we choose LinREg, giving a degree 1 polynomial without bending.
With 3 data-sets we choose QuadREg, giving a degree 2 polynomial with bending.
The intersection point is found graphically, or algebraically by solving two equations with two unknowns.

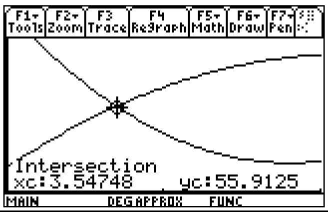
Degree one polynomial

x = ?	Supply = Demand
S = 10x+20 D = -15x+110	10x+20 = -15x+110 10x + 15x = 110-20 25x = 90 x = 90/25 = 3.6
Test1	y1(x) x=3.6 gives y=56 y2(x) x=3.6 gives y=56
Test2	Solve(y1(x)=y2(x),x) gives x=3.6
Test3	Graphical reading gives (x,y)=(3.6,56) (intersection)



Degree two polynomial

x = ?	Supply = Demand
S = -0.625x^2+13.75x+15 D = 1.25x^2-22.5x+120	-0.625x^2+13.75x+15 = 1.25x^2-22.5x+120 -1.875x^2+36.25x-105 = 0
Factorizing Zero rule	-1.875*(x-15.79)*(x-3.55) = 0 x = 15.79 and x = 3.55 15.79 is outside the validity area
Test1	y1(x) x=3.55 gives y=55.91 y2(x) x=3.55 gives y=55.91
Test2	Solve(y1(x)=y2(x),x) gives x=3.55
Test3	Graphical reading gives (x,y)=(3.55,55.91) (intersection)



4. Solving the real-world problem

We see that with linear supply and demand curves, the equilibrium price is $3.6 resulting in an equilibrium level at 56 units. And we see that with bending supply and demand curves, the equilibrium price is $3.55 resulting in an equilibrium level at 55.9 units. The solution assumes that the tables are unchanged. If changed, the regression formulas will change accordingly, and so will the solutions.

05.Project Collection, Laffer-Curve

Problem: How Which ticket price will optimize a collection income?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

We want to collect a charity fund among the school's 500 students by selling tickets at a fixed price. Which of the following three collection models will provide the highest contribution?

A) No marketing.	B) Marketing.	C) Marketing and lottery.

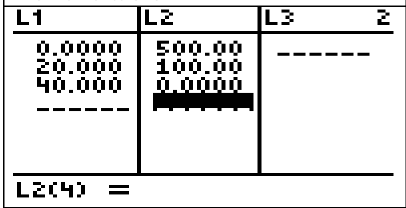
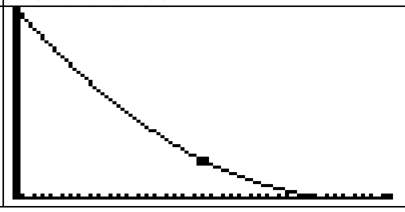
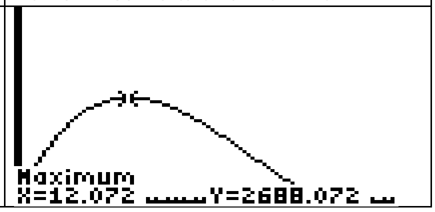
2. The mathematical problem

The demand Y1 will depend on the price x. The collected fund then will be Y2 = Y1*x.

3. Solving the mathematical problem

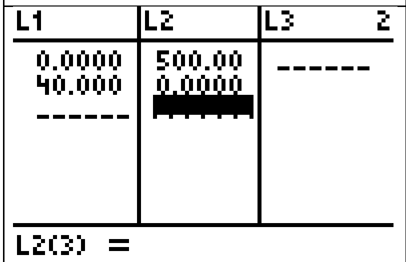
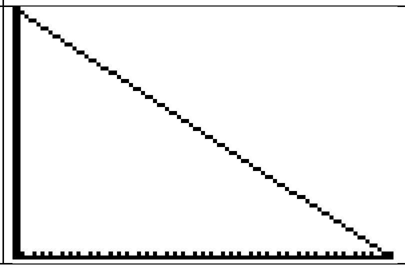
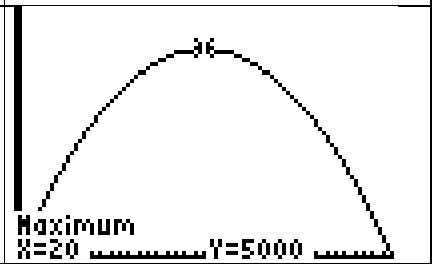
Model A. We assume that all 500 customers will buy a ticket at the price \$0, that no one will provide over \$40; and that demand is falling rapidly as only 100 customers will give \$20.

Demand: 3 data-sets gives a degree2 polynomial. 'QuadReg' gives Y1 = .375*x^2 – 27.5*x + 500

Demand table	Demand curve	Laffer-income with a maximum
		

Test: Calc dy/dx ≈ 0 in x = 12.07. dy/dx = gradient, slope

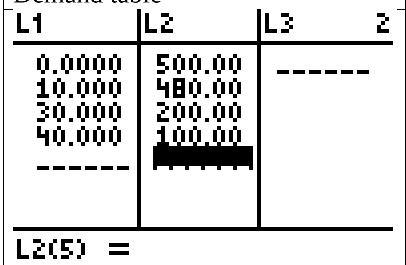
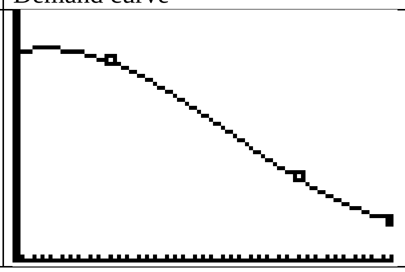
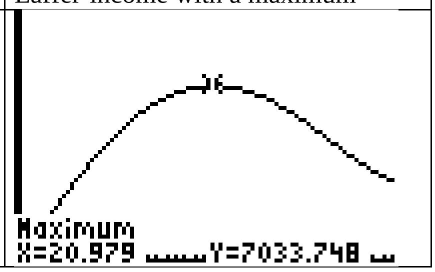
Model B. We assume that with marketing, 500 will buy a ticket at \$0, no one will give more than \$40, demand is supposed to fall evenly. Demand: 2 data-sets gives a degree1 polynomial. 'LinReg' gives Y1 = -12.5*x + 500

Demand table	Demand curve	Laffer-income with a maximum
		

Test: Calc dy/dx ≈ 0 in x = 20. dy/dx = gradient, slope

Model C. Here marketing includes a lottery with a Grand Prize \$500 and 3 extra prizes of \$200. We assume this will result in all 500 customers will buy a ticket at the price \$0, 480 customers will give \$10, 400 customers will give \$20, 200 customers will give \$30 and 100 customers will give \$40.

Demand: 4 data-sets gives a degree3 polynomial. 'CubicReg' gives Y1 = 0.013*x^3 – 0.933*x^2 + 6*x +500

Demand table	Demand curve	Laffer-income with a maximum
		

Test: Calc dy/dx ≈ 0 in x = 21.28. dy/dx = gradient, slope

4. Solving the real-world problem

Collection without marketing will provide an income of \$2688 at a ticket price of \$12.
Marketing without lottery will give an income of \$5000 at a price at of \$20.
Marketing with lottery will provide an income of 7034 – (500 + 3 * 200) = 5934 at a price at \$21.
The Laffer-curve is an argument for increasing income tax together with decreased tax percentage.
The demand curve tells then that with growing tax rate, undeclared work will also grow.

06. Project Linear Programming

Problem: How to optimize a product mix?

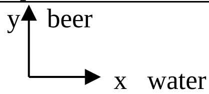
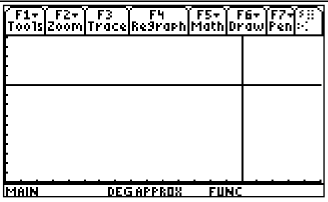
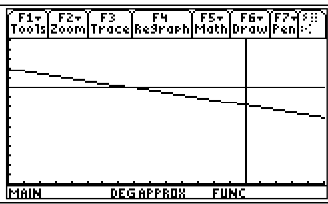
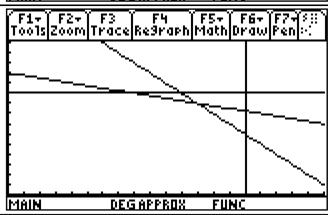
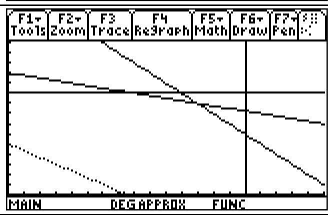
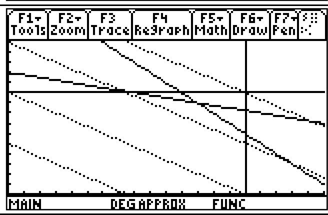
A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

A market booth sells water (max 15 boxes) and beer (max 10 boxes). The cost for water and beer is per box \$25 and \$100 respectively. A maximum of \$1200 DKK can be invested. At most 21 boxes can be sold in opening hours. Income is \$80/120 per box beer/water. How to optimize the income?

2. The mathematical problem

Outside world	Inside equations	Graphics
Boxes of water Boxes of beer	x y	
Restriction on goods: The booth has place for a maximum of 15 boxes of water 10 boxes of beer	$0 \le x \le 15$ $0 \le y \le 10$	
Restriction on capital \$1200, given the cost: \$25 per box of water \$100 per box of beer	$25*x + 100*y \le 1200$ $(100*y \le -25*x + 1200$ $y \le -\frac{1}{4}*x + 12)$	
Restriction on labor: At most 21 boxes can be sold during opening hours.	$x + y \le 21$ $(y \le -x + 21)$	
Total income T is \$80/120 per box water/beer.	$T = 80*x + 120*y$ $y = -\frac{2}{3}*x + \frac{T}{120}$ N0: T = 0: $y = -2/3*x$ N600: T = 600: $y = -2/3*x + 5$	
Solution Buying 12 boxes of water and 9 boxes of beer will maximize the total income to \$2040.	'Solve($-\frac{1}{4}$*x+12 = -x+21,x)' gives x = 12 'y =-x + 21|x=12' gives y = 9 'T = 80*x + 120*y | x=12 and y = 9' gives T = 2040	

3. Solving the mathematical problem

Graphically, the restrictions give a polygon. The level-lines are since the income T only influences the intersection with the y-axis. A parallel translation of the level-lines across the polygon thus will increase or decrease T. So, the optimal value comes where a level-line leaves or is a tangent to the polygon, which will always happen in a corner. We can therefore predict the optimal situation by calculating all corner points (n equations with n unknowns), as well as T's values in these points (simplex method). This method is used when the number of variables is greater than 2.

4. Solving the real-world problem

We see that the maximum income will be \$2040 when12 boxes of water and 9 boxes of beer are sold. Furthermore, we see that the effective restrictions are the opening hours and the invested capital. Linear programming is used to optimize a particular quantity (to maximize profit, to minimize cost, etc.) within a series of restrictions on other quantities.

07. Project Game Theory

Problem: Which strategy will maximize my outcome?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

Two players A and B choose between different strategies. The outcome table shows what B must pay to A. The 2person game is called a ZeroSum game, since what A wins, B loses, and vice versa.

2. The mathematical problems

We look at two different games:

I

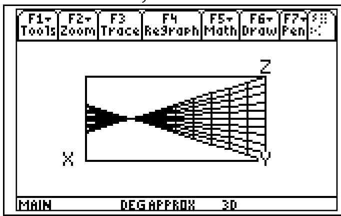
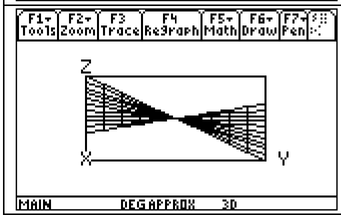
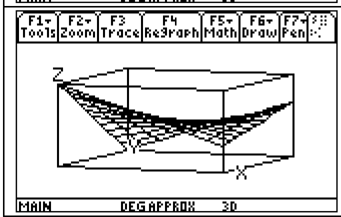
		B: b1	b2
A	a1	5	0
	a2	15	10

II

			y B: b1	1-y b2
A	x	a1	5	0
	1-x	a2	-5	10

3. Solving the mathematical problem

The two players analyze the game:
A: With a1 is I <u>risk</u> outcome 0, with a2 I risk 10. In order to maximize my minimum-number I should choose a2. And, if I was B, I would choose b2, so I choose a2.
B: With b1 is I <u>risk</u> paying 15, with b2 I risk 10. In order to minimize my maximum-pay I should choose a2. And, if I was A, I would choose a2, so I choose b2
The pair (a2, b2) is called the game **equilibrium** with the game value 10. No player will gain from not choosing the equilibrium strategy: A risks to get 0 instead of 10, so A sticks with the **maximin**-strategy. B risks paying 15 instead of 10, so B sticks with the **minimax**-strategy.

	'Solve (10*x-5 = -10*x+10, x)' gives x= 0.75
	'Solve (-15*y + 10 = 5*y, y)' gives y= 0.5
	Hold down the arrow for 1 second to watch the surface rotate

The two players analyze the game:
A: With a1 is I <u>risk</u> outcome 0, with a2 I risk -5. In order to maximize my minimum-number I should choose a1.
B: With b1 is I <u>risk</u> to pay 5, with b2 I risk 10. In order to minimize my maximum-pay I should choose b1.
But the maximin and minimax strategies a1 and b1 are not in equilibrium, since B gain by choosing b2, making A choose a2, making B choose b1, making A choose a1, etc.
Thus, with no equilibrium point, A should mix a1 and a2 in the ratio x to 1-x, randomly, and likewise B.
If B chooses b1 or b2, the outcomes will be respectively
P1 = 5*x – 5*(1-x) = 10*x – 5.
P2 = 0*x + 10*(1-x) = -10*x +10.
The two outcomes are like with 10*x – 5 = -10*x +10, or 20*x = 15, or x = 15/20 = 3/4 giving P = 10*3/4-5 = 2.5.
So by mixing the strategies a1 and a2 in the ration 3 to 1, player A will secure an average outcome 2.5 no matter what B chooses. So this game has the value 2.5.
Likewise we find that by mixing the strategies b1 and b2 in the ration 1 to 1, player B will secure an average loss 2.5 no matter what A chooses.
With A and B randomly mixing their strategies in the ratios x to 1-x, and y to 1-y respectively, the outcome will be
P = 5*x*y + 0*x*(1-y) – 5*(1-x)*y + 10*(1-x)*(1-y)
P = -10*x – 15*y + 20*x*y + 10
On a graphical display calculator P becomes a surface called a **saddle point**, going down one way, and up the other way.

4. Solving the real-world problem

In game I, A and B should choose strategy a2 and b2 respectively resulting B losing 10 per game. In game II, the solution x =. 75, y = .5, P = 2.5 means that with a deck of cards, A should choose a2 when dragging clubs, otherwise a1. And that B should choose b2 when dragging black, otherwise b1. B will have an average loss at 2.5 per game.

We see that a 2person ZeroSum-game always has an equilibrium point in a saddle point going up to one side and down to the other. With the saddle point in a corner, this corner is the solution to the problem. With an inside saddle point, the strategies must be mixed randomly in a ration found by looking at the saddle point from the side.

08. Project Distance to a Far-away Point

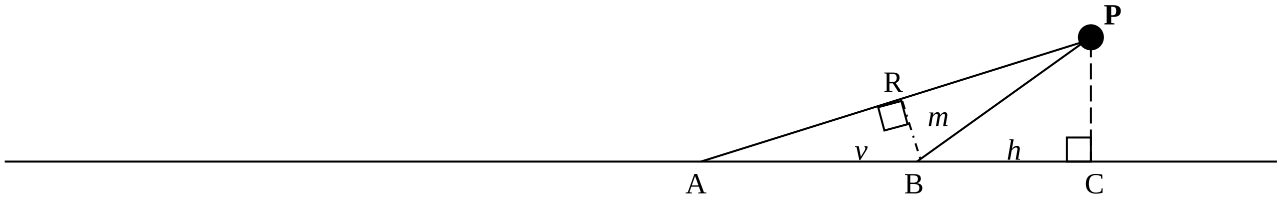
Problem: How to determine the distance to an inaccessible distant point?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

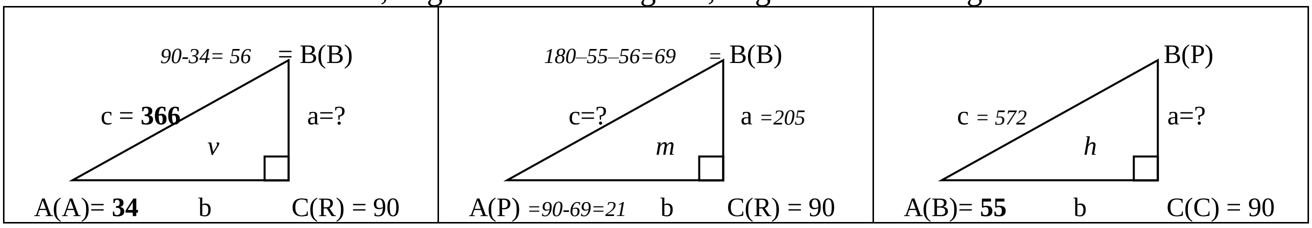
From a given baseline we want to determine the distance to a far-away inaccessible point P.



2. The mathematical problem

From a known baseline AB we measure the angles A and B to the inaccessible point P.
From the three right angled triangles ABR, BRP and BCP we calculate RB, BP as well as the distance PC.

Measurements: AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees



3. Solving the mathematical problem

We set up three formula tables

Triangle ABR

a = ?	$\sin A = \frac{a}{c}$
A = 34 c = 366	$\sin 34 = \frac{a}{366}$ sin34 * 366 = a 205 = a
Test1 ☺	$\sin 34 = \frac{205}{366}$ 0.559 = 0.560
Test2 ☺	Math Solver $0 = \frac{x}{366} - \sin 34$ gives x = 205

Triangle PBR

c = ?	$\sin A = \frac{a}{c}$
A = 21 a = 205	$\sin 21 = \frac{205}{c}$ c * sin21 = 205 $c = \frac{205}{\sin 21}$ c = 572
Test1 ☺	$\sin 21 = \frac{205}{572}$ 0.358 = 0.358
Test2 ☺	Math Solver $0 = \frac{205}{x} - \sin 21$ gives x = 572

Triangle PBC

a = ?	$\sin A = \frac{a}{c}$
A = 55 c = 572	$\sin 55 = \frac{a}{572}$ sin55 * 572 = a 469 = a
Test1 ☺	$\sin 55 = \frac{469}{572}$ 0.819 = 0.820
Test2 ☺	Math Solver $0 = \frac{x}{572} - \sin 55$ gives x = 469

4. Solving the real-world problem

Using trigonometry, we are able to determine the distance to the inaccessible point P to 469 cm.

09. Project Bridge

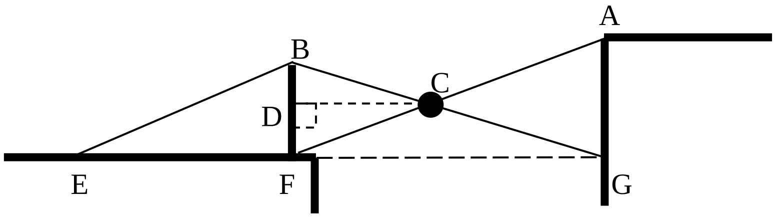
Problem: How to determine the dimensions of a bridge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

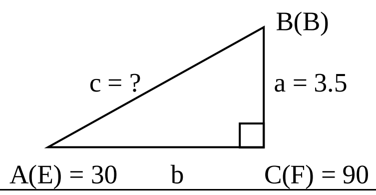
Over a canyon a suspension bridge made of steel i fastened to the cliff and to a vertical upright. We want to determine the length of the 3 beams as well as the welding point. The left fixing angle must be 30 degrees.

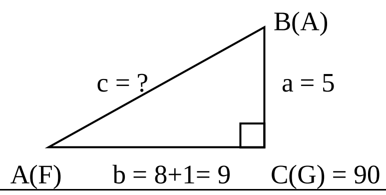


2. The mathematical problem

From the right-angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between the lines BG and FA.

Measurements: angle FEB = 30 degrees, FB = 3.5m, FG = 8m + 1m = 9m and AG = 5m.





I a coordinate system with F as zero the following coordinates emerge: F: (0,0) and A: (9,5), as well as B: (0,3.5) and G: (9,0). Using linear regression, we determine the equations for the lines FA and BG.

3. Solving the mathematical problem

We set up formula tables

Triangle EFB

c = ?	sinA = $\frac{a}{c}$
A = 30 a = 3.5	sin30 = $\frac{3.5}{c}$ sin30*c = 3.5 c = 3.5/sin30 c = 7.0
Test1 ☺	sin30 = $\frac{3.5}{7}$ 0.5 = 0.5
Test2 ☺	Math Solver 0 = $\frac{3.5}{x}$ - sin30 gives x = 7

Triangle FGA og GFB

c = ?	$a^2 + b^2 = c^2$
a = 5 b = 9	$5^2 + 9^2 = c^2$ √(106)= c 10.30 = c
Test1 & Test2	
c = ?	$a^2 + b^2 = c^2$
a = 3.5 b = 9	$3.5^2 + 9^2 = c^2$ √(93.25) = c 9.66 = c

Lines BG and FA

BG: ?	y = ax + b
Test	y = – 0.389x + 3.5 Found by LinReg L1, L2, Y1 Trace x=0 gives 3.5 Trace x=9 gives 0 StatPlot fits

Likewise we find
FA: ? y = 0.556x

Calc Intersection gives
x = 3.71 and y = 2.06
In the triangle FDC, DC = 3.71 and FD = 2.06
Pythagoras gives:
FC = √(3.71^2 + 2.06^2) = 4.24
In the triangle BDC, DC = 3.71 and
FD = 3.6-2.06 = 1.54
Pythagoras gives:
BC = √(3.71^2 + 1.54^2) = 4.02

4. Solving the real-world problem

Using trigonometry, we have found the lengths of the three steel beams as EB = 7.00 m, FA = 10.30 m and BG = 9.66

The welding point is determined by FC = 4.24 m and BC = 4.02 m.

As an extra control the bridge can be drawn and build by pipe cleaners in the ration 1:100.

10. Project Driving

Problem: How far and how did Peter drive?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

When driving, the velocity 100 km/t is 100*1000/(60*60) = 27.8 m/s. A camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s and 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

2. The mathematical problem

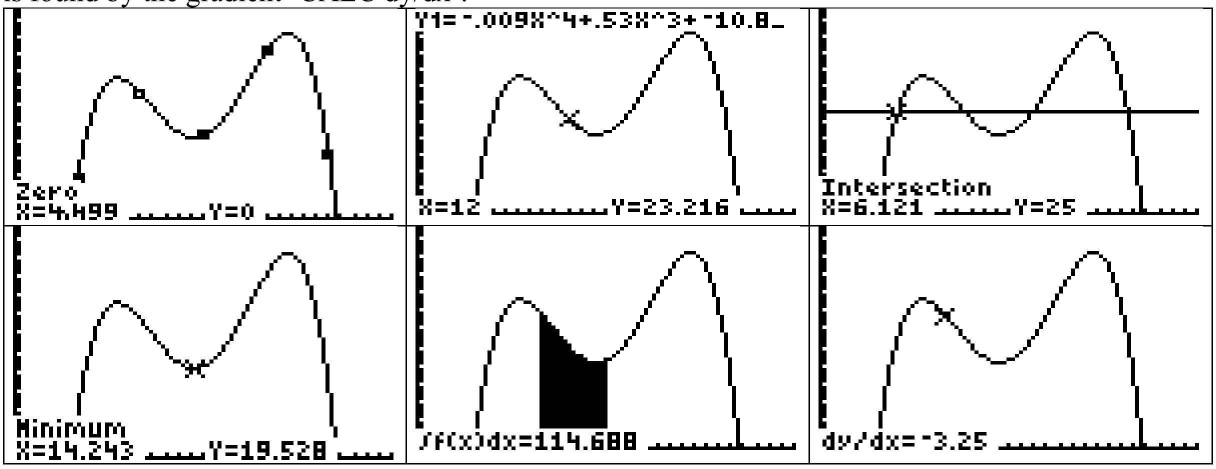
We set up a table showing time x and velocity y. The domain of the table is taken to be 0<x<30.

Time x sec	Velocity y m/s	Accel. dy/dx
5	10	?
10	30	?
15	20	?
20	40	?
25	15	?

3. Solving the mathematical problem

On TI-84 the table is entered as the lists L1 and L2. 5 data sets allow quartic regression (a 4. degree polynomial with a 3-fold parabola) providing the formula y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235 placed as Y1. No the question asked can be answered using formula tables, or using technology, i.e. graphical readings or calculations.

Starting and ending points are found using 'CALC Zero'. Y-numbers are found using 'TRACE'. X-numbers are found using 'CALC Intersection'. Maximum and minimum are found with 'CALC Maximum/Minimum'. The total meter-number is obtained by summing up the m/s*s = ∫Y1dx. Acceleration is found by the gradient 'CALC dy/dx'.



y = ?	y = y1
x=12	y = y1(12) = 3.667
Test	TRACE x = 12 gives y = 23.216

x = ?	y = y1
y = 25	MATH Solver 0 = y1 - 25 gives x =6.12 and …
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC intersection gives x = 6.12, 11.44, 16.86 and 24.47

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx = 0 at x = 5.5

4. Solving the real-world problem

The driving began after 4.50 sec. and ended after 25.62 sec. After 12 sec the velocity was 23.2 m/s. And it was 25m/s after 6.12 sec, 11.44 sec, 16.86 sec and 24.47 sec. Acceleration took place in the time-intervals (4.50; 8.19) and (14.24; 21.74). Deceleration in the intervals (8.19;14.24) and (21.74;25.62). Max-velocity was 44.28 m/s = 159 km/t. after 21.7 sec. In the time-intervals (5;10), (10;15), (15;20) and (20;25) the distance traveled was 142.8 m, 114.7 m, 142.8 m and 189.7 m. The acceleration in the beginning of these time-intervals were 17.75, -3.25, 1.25, 4.25, -21.25 m/s^2. The total distance traveled was 597.4 m.

11. Project Vine Box

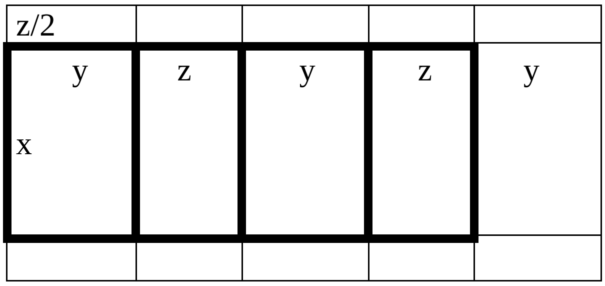
Problem: What are the dimensions of a 3 liters vine bag with the least surface area?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

Vine is sold in bottles or in boxes. A 3 liter bag will be constructed by cutting out a piece of cardboard.



2. The mathematical problem

The cardboard dimensions are called x, y & z all in dm. We express the volume V and the Surface S as formulas:

V = x*y*z = 3,   S = x*(3y+2z) + 2*z/2*(3y+2z)

3. Solving the mathematical problem

We expand the S-formula: S = x*(3y+2z) + 2*z/2*(3y+2z) = 3xy + 2xz + 3yz + 2z^2

We now insert z = 3/(x*y) so that S only depends on two variables x and y:

S = 3xy + 2xz + 3yz + 2z^2 and z = 3/(x*y) gives $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2}$

Scenario A. We assume that y should be half the length of x: y = 0.5*x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 1.5x^2 + \frac{21}{x} + \frac{72}{x^4}$, which gives $\frac{dS}{dx} = 3x - \frac{21}{x^2} - \frac{288}{x^5} = 0$ for x = 2.4

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

x = 2.4 and S = 19.56, so y = 0.5*x = 0.5*2.4 = 1.2, and z = 3/(2.4*1.2) = 1.0

Scenario B. We assume that y should be the same length of x: y = x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 3x^2 + \frac{15}{x} + \frac{18}{x^4}$, which gives $\frac{dS}{dx} = 6x - \frac{15}{x^2} - \frac{72}{x^5} = 0$ for x = 1.7

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

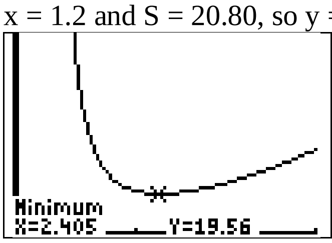
x = 1.7 and S = 19.65, so y = x = 1.7, and z = 3/(1.7*1.7) = 1.0

Scenario C. We assume that y should be double the length of x: y = 2*x. This restriction is inserted:

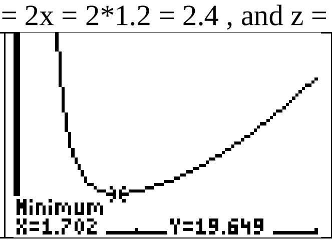
$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 6x^2 + \frac{12}{x} + \frac{4.5}{x^4}$, which gives $\frac{dS}{dx} = 12x - \frac{12}{x^2} - \frac{18}{x^5} = 0$ for x = 1.2

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

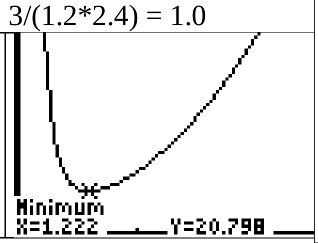
x = 1.2 and S = 20.80, so y = 2x = 2*1.2 = 2.4 , and z = 3/(1.2*2.4) = 1.0



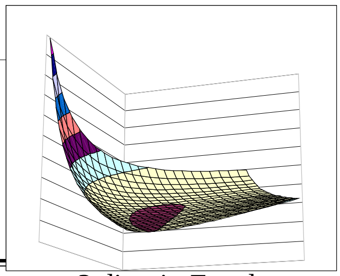
Scenario A



Scenario B



Scenario C


3 dim. in Excel

4. Solving the real-world problem

We see that the minimum surface area is a little above 19 dm^2. Using an Excel-spreadsheet we can find the optimal solution to be x = 2.1 and y = 1.4 and z = 1.0, giving a minimum surface area at 19.47 dm^3.

(Graphing $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2}$ does not give a curve but a surface as shown on the Excel-pict.)

12. Project Golf

Problem: How to hit a golf hole behind a hedge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

From a position on a 2 meter high flat hill we want to send a golf ball over a 3 meter hedge 2 meter away on the hill to hit a hole situated 12 meters away at level zero.
What is the orbit of the ball? How high is the ball at the distance 10 meters? When does the ball have a height of 6 meters? How high does the ball go? What is the direction of the ball in the beginning, at 10 meters distance and at the impact?

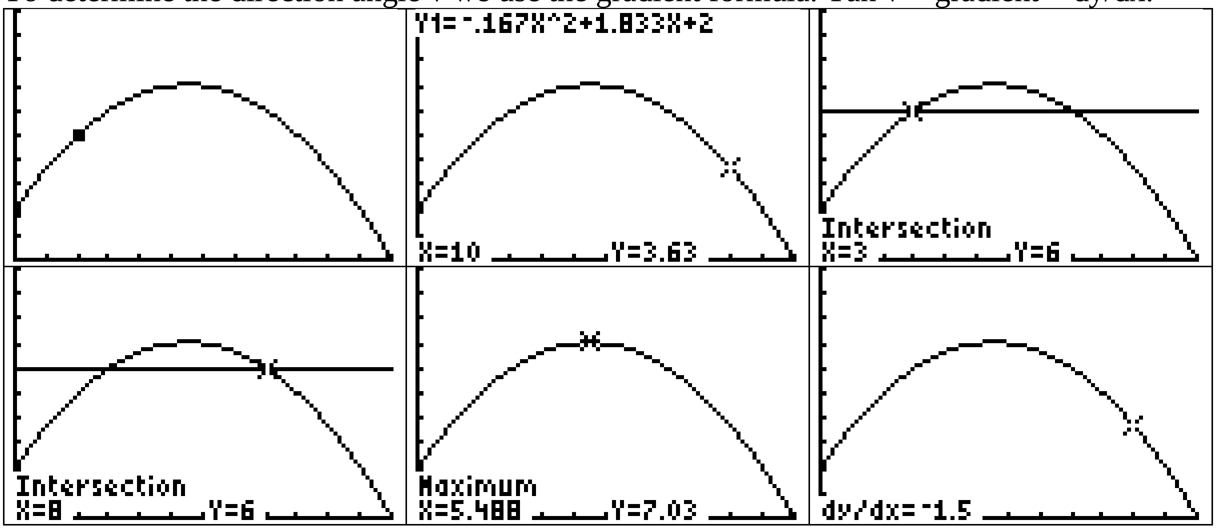
2. The mathematical problem

We set up a table with the length x and the height y having the domain 0<x<12.

Length x	Height y	Direction v
0	2	?
2	5	
12	0	?
10	?	?
?	6	

3. Solving the mathematical problem

We insert the table as lists L1 and L2. Three data-sets allows a 2nd degree polynomial, quadratic regression, which produces the formula y= -0.167x^2 + 1.833x + 2, which is transferred to Y1.
Now the questions asked can be answered using formula tables and a calculator for graphing or calculating. The Y-number can be found by CALC Value, the x-number by CALC Intersection, the maximum by CALC Maximum, and the gradient by CALC dy/dx.
To determine the direction angle v we use the gradient formula: Tan v = gradient = dy/dx.



y = ?	y = y1
x=10	y = y1(10) = 3.667
Test	Trace x=10 gives y=3.67

x = ?	y = y1
y = 6	Math solver 0 = y1 - 6 gives x =3 & x = 8
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC Intersection gives x = 3 and 8

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx ≈ 0 at x = 5.5

v = ?	tan v = dy/dx
x = 0	tan v = 1.833 v = tan-1(1.833) v = 61.4

v = ?	tan v = dy/dx
x = 12	tan v = -2.167 v=tan-1(-2.167) v =- 65.2

v = ?	tan v = dy/dx
x =10	tan v = -1.5 v = tan-1(-1.5) v = -56.3

4. Solving the real-world problem

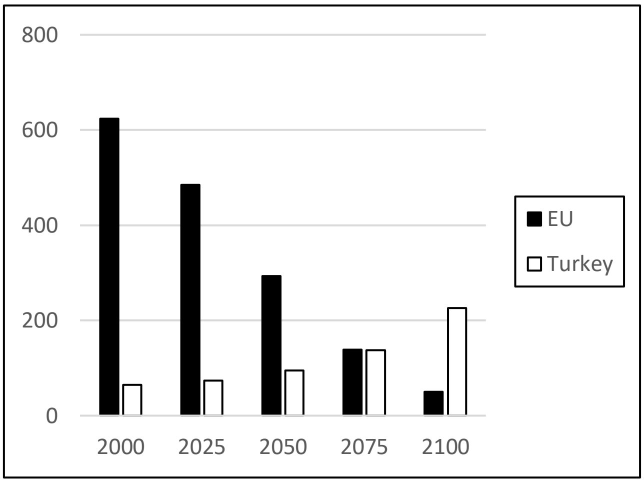
The orbit of the ball is a parabola. The height of the ball at the distance 10 meters is 3.67 meters? At the distances 3 meters and 8 meters the ball has a height of 6 meters. The ball goes to the maximum height 7.04 meters? The direction of the ball in the beginning, at 10 meters distance and at the impact are 61.4 grader, -65.2 grader and -56.3 grader.

13. Project Population Forecast

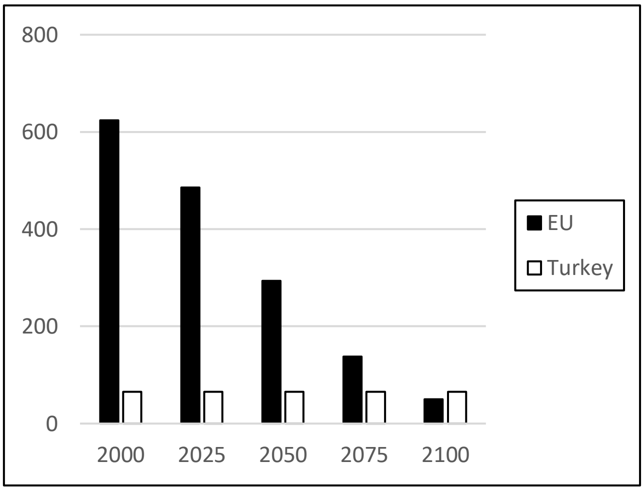
Problem:

From year 2000, find a 100year forecast for the populations in the EU and Turkey, building on the assumption that the yearly change percent in the EU is -1% if on average each woman gives birth to 1.5 child.
Please, construct three scenarios: ,
1. In Turkey, the yearly change percent in Turkey is 0,5% if on average each woman gives birth to 2.2 child
2. In Turkey, the yearly change percent in Turkey is 0% if production causes women to have fewer children
3. In Turkey, the yearly change percent in Turkey is 1% if religion causes women to give birth to 4.4 children

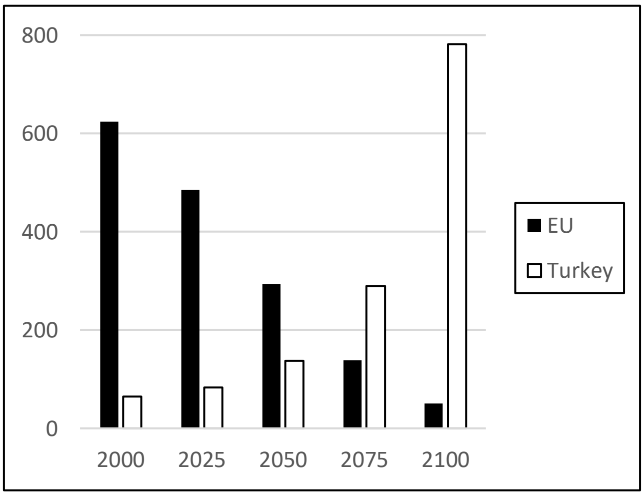
Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	74	94	137	226



Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	65	65	65	65



Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	83	137	289	782



Answer: In all three scenarios Turkey will exceed EU in year 2100

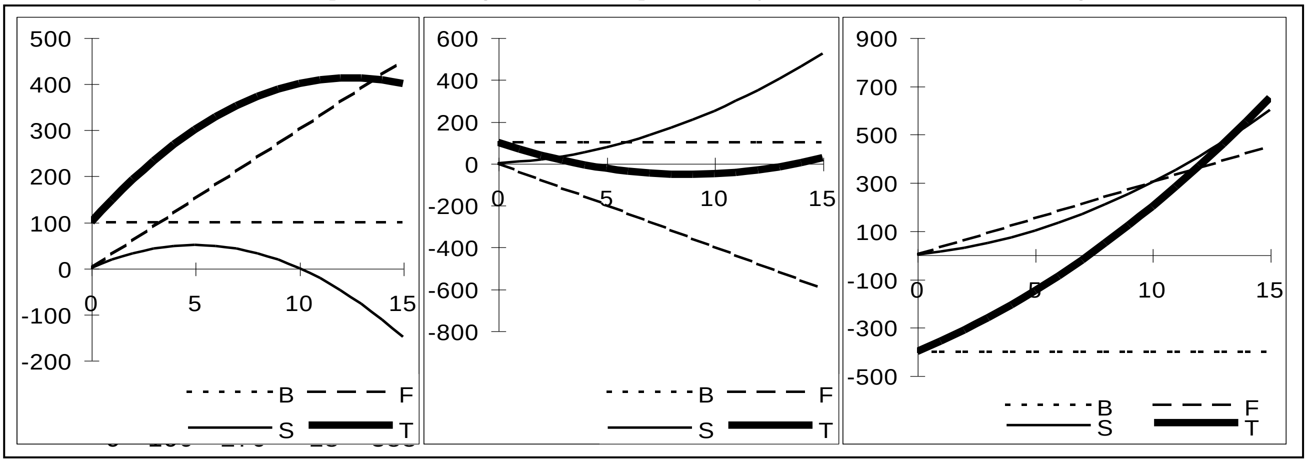
14. Project Family Firm

A family firm builds its total capital T through income from three generations. The grandfather has retired and has left the capital To $. The father has established a routine which earns b $/day. The son has just come back from the university, where he has learned a new technology enabling him to slowly raise the daily income d$/day to s = so + d*n. The total capital after n days, since the family is calculated as a sum of polynomials:

Grandfather B	To	degree 0 polynomial, a constant
Father F	b*n	degree 1 polynomial, a line
Son S	s*n = (so + ½*d*n)* n = so*n + ½*d*n^2	degree 2 polynomial, a bended line
Total T	T = To + (b+so)*n + d*n^2	degree 2 polynomial, a parabola

Gatherers and Spreaders

Some families have both a spreader and a gatherer. The spreader may be the son, the father or the grandfather



Spreader: Son Father Grandfather

Pricing the tea

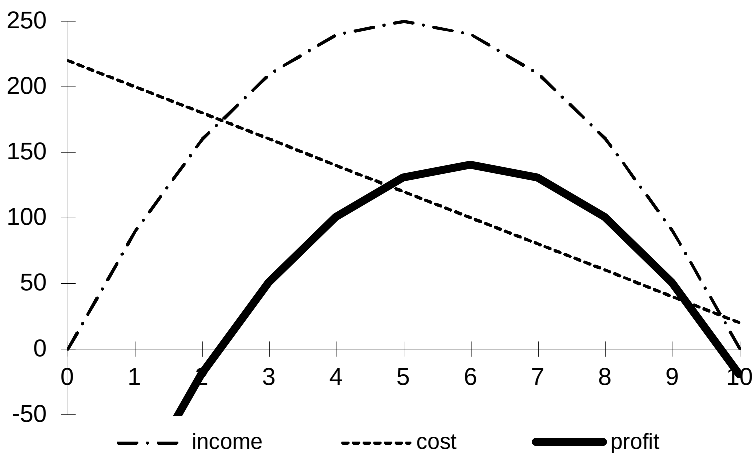
In the family firm they discuss if increasing the unit price by necessity will make the sale decrease.

Grandfather: The sale will decrease with an increasing unit price. I believe in a linear relation y = a + b*x found from the table	Father: The sale will decrease slower at high unit prices. I believe in a degree 2 polynomial y = a + b*x + c*x^2 found from the table	Son: The sale will decrease more at low and high unit prices. I believe in a degree 3 polynomial y = a + b*x + c*x^2+ d*x^3 found from the table
price x / sale y: 0 / 100 10 / 0	price x / sale y: 0 / 100 5 / 80 10 / 0	price x / sale y: 0 / 100 2 / 60 8 / 40 10 / 0

The grandfather scenario

The sale will be y = 100-10*x found by regression. The total income T is T = unit price*sale = x*y = x*(100-10 x) = 100*x-10*x ^ 2, a degree 2 polynomial. The cost C to produce y units consists of a fixed cost co = 20 and a variable unit-cost m = 2. So, C = co + m*y = = 20 + 2*y = 20 + 2*(100–10*x) = 20 + 200–20*x = 220–20*x. The profit P will be when P = T – C = (100*x–10*x^2) – (220–20*x) = -220 + 80x – 10*x^2, i.e. again a degree 2 polynomial.

price	sale	income	cost	profit
0	100	0	220	-220
1	90	90	200	-110
2	80	160	180	-20
3	70	210	160	50
4	60	240	140	100
5	50	250	120	130
6	40	240	100	140
7	30	210	80	130
8	20	160	60	100
9	10	90	40	50
10	0	0	20	-20

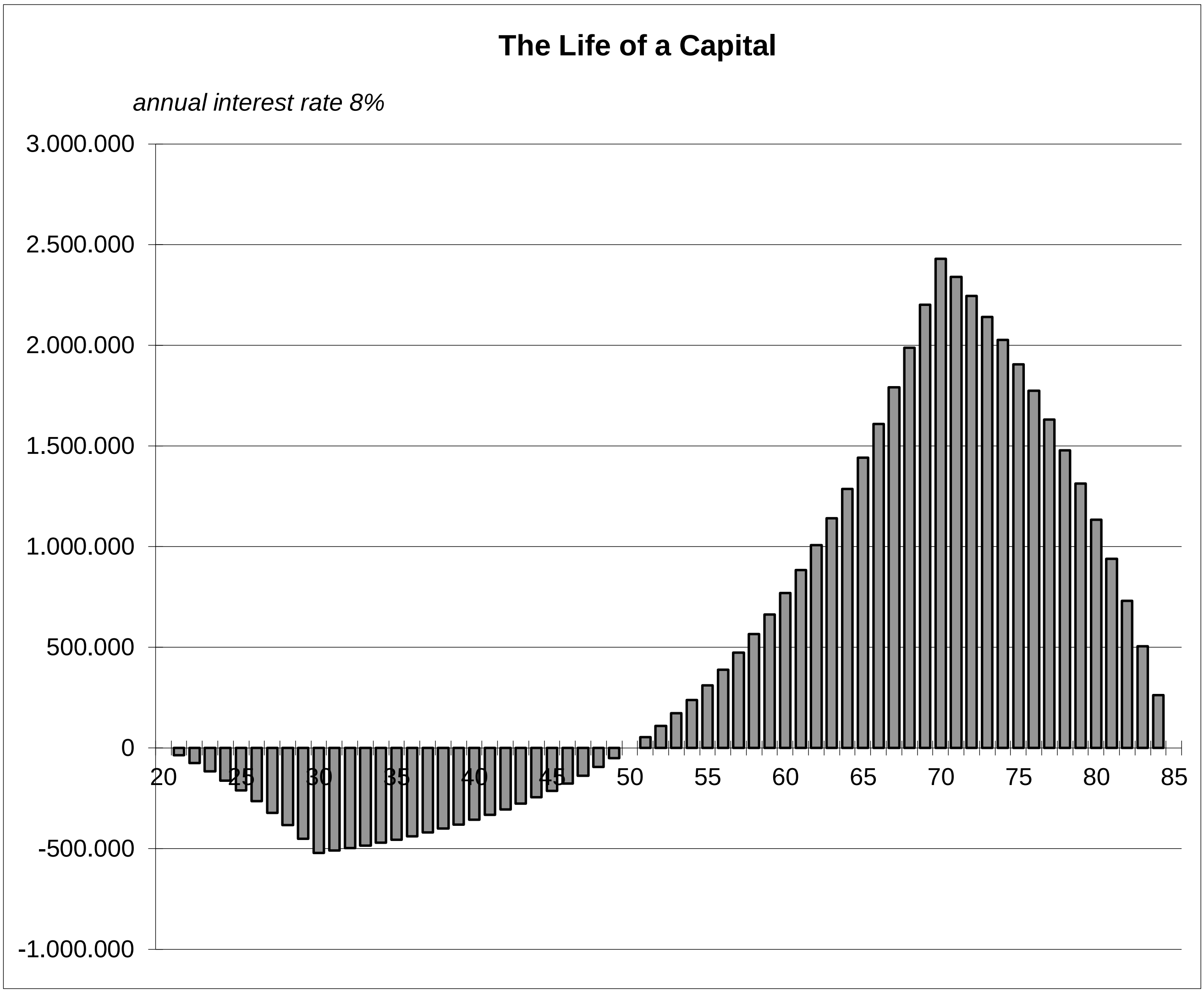


Exercises

1. Set up the father's scenario
2. Set up the son's scenario

Project 15. The Life of a Capital

During the lifetime, a person's capital will change. In the following example, life is divided into four periods:
1) 20-30 years: student loans at $36,000 per year for 10 years.
2) 30-50 years: debt settlement, where the student loan is settled by paying $53,000 per year for 20 years.
3) 50-70 years: building a capital by continuing depositing $53,000 per year for 20 years
4) 70-85 years: pension, where assets are settled by paying $284,000 per year for 15 years.
The example assumes an interest rate at 8% per year. Other examples may be set up with different interest rates.



Compound interest

A constant monthly interest rate will doubling-time on T months. Counting in doublings-times, we look at an account A0 receiving 1 unit at time 0. At time 1, the interest of A0 is transferred to account A1. At time 2, the interest of A0 is transferred to account A1, and the interest of A1 is transferred to account A2, etc. What do you observe?

A10											
A9											
A8											
A7											
A6											
A5											
A4					1						
A3				1	1+3 = 4						
A2			1	1+2 = 3	3+3 = 6						
A1		1	1+1 = 2	2+1 = 3	3+1 = 4						
A0	1	1	1	1	1						
Time	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Sum	1	2	4	8	16						

Revision Problems Using TI-84

<table>
<tr><td>1.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>?</td></tr><tr><td>?</td><td>40</td></tr></table></td><td>Answer the question marks in case of a linear model.
Answer the question marks in case of an exponential model. What is the doubling time?
Answer the question marks in case of a power model.</td></tr>
<tr><td>2.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>18</td></tr><tr><td>15</td><td>?</td></tr><tr><td>?</td><td>10</td></tr></table></td><td>Answer the question marks in case of a quadratic model.
Find maxima or minima.
Find the equation for the tangent line in x=2.
Find the gradient formula.
Find the gradient number in x = 5
Find the area formula
Find the area number from x= 1 to x = 6
Find the intersection points with the line y = 3 + 2x</td></tr>
<tr><td>3.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>14</td></tr><tr><td>12</td><td>18</td></tr><tr><td>15</td><td>?</td></tr><tr><td>?</td><td>30</td></tr></table></td><td>Answer the question marks in case of a cubic model.
Find maxima and minima.
Find the equation for the tangent line in x=2.
Find the gradient formula.
Find the gradient number in x = 5
Find the area formula
Find the area number from x= 1 to x = 6
Find the intersection points with the line y = 3 + 2x</td></tr>
<tr><td>4.</td><td>3x + 4y = 15 & 5x – 6y = 12</td><td>Solve the simultaneous equations</td></tr>
<tr><td>5.</td><td>Given two points in a coordinate system P(2,4) and Q(6,10)</td><td>Find the midpoint of the line PQ.
Find the equation for the line through P and Q
Find the equation for the normal line to PQ passing through P
Find the angle between PQ and the x-axis.
Find the distance between P and Q
Find the distance from the line PQ to the point S(8,1)
Find the equation for the circle through P and Q and with the midpoint of PQ as center.
Find the intersection point between the circle and the line y = 12-2x</td></tr>
<tr><td>6.</td><td>Let X be a normal random variable with mean m = 100 and standard deviation d = 12</td><td>P(X<89) = ?
P(X>108) = ?
P(93<X<109) = ?</td></tr>
<tr><td>7.</td><td>X counts the numbers of wins in 100 repetitions of a game with 65% winning chance.</td><td>P(X<70) = ? P(X≤60) = ?
P(X≥58) = ? P(63<X≤72) = ?</td></tr>
<tr><td>8.</td><td>sin(3x) = 0.4, 0≤x≤2π
con(½x) = -0.3, 0≤x≤2π
tan(2x) = 0.7, 0≤x≤2π</td><td>Find the solutions: Remember to adjust the window
Find the solutions:
Find the solutions:</td></tr>
<tr><td>9.</td><td>A = 40, b = 7, C = 90</td><td>Find a, B and c.</td></tr>
<tr><td>10.</td><td>a = 4, c = 7, C = 90</td><td>Find A, B and b.</td></tr>
<tr><td>11.</td><td>A = 40, b = 7, C = 68</td><td>Find a, B and c.</td></tr>
<tr><td>12.</td><td>A = 40, b = 7, c = 6.8</td><td>Find a, B and C.</td></tr>
<tr><td>13.</td><td>A = 40, b = 7, a = 6.2</td><td>Find c, B and C.</td></tr>
<tr><td>14.</td><td>a = 4, b = 7, c = 6.8</td><td>Find A, B and C.</td></tr>
<tr><td>15.</td><td>$T = \frac{d}{e\text{-}f} + g$</td><td>Transpose the T-formula to a d-, e-, f-, and g-formula</td></tr>
<tr><td>16.</td><td>The capital 785 increased with 2.7% 5 times and became ?</td><td>Find the answer
Find the corresponding doubling time.</td></tr>
<tr><td>17.</td><td>The capital 785 increased with 2.7% ? times and became 980</td><td>Find the answer
Find the corresponding doubling time.</td></tr>
<tr><td>18.</td><td>The capital 785 increased with ?% 5 times and became 980</td><td>Find the answer
Find the corresponding doubling time.</td></tr>
<tr><td>19.</td><td>-21</td><td>As 16-18, but with $ instead of %</td></tr>
</table>

***Problem 1.* Linear model**

Equation:	y=ax+b
	y=x+9, found by Stat, Calc, LinReg
Test	y1(3) = 12 ☺

y=?	y=x+9
x=10	y=19 found by y1(10)
Test	y=19 found by CalcValue ☺

x=?	y=x+9
y=40	x=31, found by Math, Solver 0=y1-40
Test	y1(31) = 40 ☺

Exponential model

Equation:	y=a*b^x
	y=9.671*1.075^x, found by Stat, Calc, ExpReg
Test	y1(3) = 12 ☺

y=?	y=9.671*1.075^x
x=10	y=19.853 found by y1(10)
Test	y=19.853 found by CalcValue ☺

x=?	y=9.671*1.075^x
y=40	x=19.740, found by Math, Solver 0=y1-40
Test	y1(19.740) = 40 ☺

Doubling time T = log2/logb = log2/log1.075 = 9.6

Power model

Equation:	y=a*x^b
	y=8.264*x^0.340 found by Stat, Calc, PwrReg
Test	y1(3) = 12 ☺

y=?	y=8.264*x^0.340
x=10	y=18.060 found by y1(10)
Test	y=18.060 found by CalcValue ☺

x=?	y=8.264*x^0.340
y=40	x=104.024 found by Math, Solver 0=y1-40
Test	y1(104.024) = 40 ☺

***Problem 2.* Quadratic model**

Equation:	y=a*x^2+b*x+c
	y=-0.048x^2+1.476x+8 found by Stat, Calc, QuadReg
Test	y1(3) = 12 ☺

y=?	y=-0.048x^2+1.476x+8
x=15	y=19.429 found by y1(15)
Test	y=19.429 found by Graph, Calc, Value ☺

x=?	y=-0.048x^2+1.476x+8
y=10	x=1.420 or 29.580 found by Math, Solver 0=y1-10
Test	y1(1.420) = 10 y1(29.580) = 10 ☺

Maximum:	y=-0.048x^2+1.476x+8
	(x,y) = (15.500,19.140) found by Graph, Calc, Maximum
Test	dy/dx = 0 for x = 15.5 y1(15.5)= 19.14 ☺

Tangent in x=2	y=-0.048x^2+1.476x+8
x=2	y=1.286x + 8.190 found by Graph, Draw, Tangent

Gradient formula	y=-0.048x^2+1.476x+8
	y’= -0.095*x + 1.476, found by TI89
Test	∫y’dx=-0.048x^2+1.476x found by TI89 ☺

Gradient number:	y=-0.048x^2+1.476x+8
x=5	dy/dx = 1 for x=5 found by Graph, Calc, dy/dx
Test	1, found by Math, nDeriv ☺

Area formula:	y=-0.048x^2+1.476x+8
x=2	∫ydx = -0.016*x^3 + 0.738*x^2 + 8.000*x found by TI89
Test	d(∫ydx)/dx = -0.048x^2+1.476x+8 found by TI89 ☺

Area number:	y=-0.048x^2+1.476x+8
	$\int_1^6 y\,dx$ = 62.421, found by Graph, Calc, ∫f(x)dx
Test	62.421, found by Math, fnInt ☺

Intersection points	y = -0.048x^2+1.476x+8 and y = 3+2x (y1 = y3)
	(x,y) = (-17.130,-31.260) and (x,y) = (6.130, 15. 260), found by Math, Solver 0=y1-y3 and y1(-17.130) = -31.260 etc.
Test	tested by Graph, Calc, Intersect ☺

***Problem 3.* Cubic model**

Equation:	y=a*x^3+b*x^2+c*x+d
	y=0.086x^3-1.952x^2 +13.752x-14, found by Stat, Calc, CubicReg
Test	y1(3) = 12 ☺

y=?	y=0.086x^3-1.952* x^2 +13.752x-14
x=15	y=42.286 found by y(15)
Test	y=42.286 found by Graph, Calc, Value ☺

x=?	y=0.086x^3-1.952x^2 +13.752x-14
y=30	x=13.885 found by Math, Solver 0=y1-30
Test	y1(13.885) = 30 ☺

Maximum Minimum:	y=0.086x^3-1.952x^2+13.752x-14
	Max: (x,y) = (5.552, 16.841) found by Graph, Calc, Maximum Min: (x,y) = (9.634, 13.925) found by Graph, Calc, Minimum
Test	dy/dx = 0 for x = 5.552 y1(5.552)= 16.841 dy/dx = 0 for x = 9.643 y1(9.643)= 13.925 ☺

Tangent in x=2	y=0.086x^3-1.952x^2+13.752x-14
x=2	y = 6.971x – 7.562 found by Graph, Draw, Tangent

Gradient formula	y=0.086x^3-1.952x^2+13.752x-14
	y’= 0.257*x^2 – 3.905*x + 13.752, found by TI89
Test	∫y’dx=0.086x^3-1.952x^2+13.752x found by TI89 ☺

Gradient number:	y=0.086x^3-1.952x^2+13.752x-14
x=5	y'(5) = 0.657 found by Graph, Calc, dy/dx
Test	0.657, 1 found by Math, nDeriv ☺

Area formula:	y=0.086x^3-1.952x^2+13.752x-14
x=2	∫ydx = 0.021*x^4 – 0.651*x^3+6.876*x^2+14*x found by TI89
Test	d(∫ydx)/dx = 0.086x^3-1.952x^2+13.752x-14 found by TI89 ☺

Area number:	y=0.086x^3-1.952x^2+13.752x-14
	$\int_1^6 ydx$ = 58.496, found by Graph, Calc, ∫f(x)dx
Test	58.496, 62.421 found by Math, fnInt ☺

Intersection points with y=3+2x: (x,y) = (2.129,-7.259) and (x,y) = (6.657, 16.315) and (x,y) = (13.991, 30.981) found by Math, Solver 0=y1-y3 , tested by Graph, Calc, Intersect.

Problem 4

Solutions: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix}$, found by A*B=C, B=A^-1*C, where A = $\begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix}$ and B = $\begin{pmatrix} x \\ y \end{pmatrix}$ and C = $\begin{pmatrix} 15 \\ 12 \end{pmatrix}$

Tested by A*B=C : A*B = $\begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix} * \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$ = C ☺

Problem 5

Midpoint:	$(x,y) = (\frac{x1+x2}{2}, \frac{y1+y2}{2})$
x1=2 x2=6 y1=4 y2=10	$(x,y) = (\frac{2+6}{2}, \frac{4+10}{2})$ (x,y) = (4,7)
Test	Tested geometrically ☺

Gradient PQ:	$a = \frac{y2-y1}{x2-x1}$
x1=2 x2=6 y1=4 y2=10	$a = \frac{10-4}{6-2}$ a= 3/2 a = 1.5
Test	Tested geometrically ☺

Line PQ:	y = y1 + a*(x – x1)
a =1.5 x1=2 y1=4	y = 4 + 1.5*(x – 2) y = 1.5*x + 1
Test	Tested geometrically ☺

Gradient perpend.:	c*a = -1
a=3/2	c = -2/3 found by Math, Solver 0 = c*3/2+1
Test	Tested geometrically ☺

Normal:	y = y1 + a*(x – x1)
a=-2/3 x1=2 y1=4	y = 4 + -2/3*(x – 2) y = -2/3*x + 5.333
Test	Tested geometrically ☺

Distance PQ	$d = \sqrt{(x2-x1)^2 + (y2-y1)^2}$
x1=2 x2=6 y1=4 y2=10	$d = \sqrt{(6-2)^2 + (10-4)^2}$ d = 7.21
Test	Tested geometrically ☺

Distance point-line	$d = \frac{\lvert y1 - a*x1 - b\rvert}{\sqrt{1 + a^2}}$
a=1.5 b=1 x1=8 y1=1	$d = \frac{\lvert 1 - 1.5*8 - 1\rvert}{\sqrt{1 + 1.5^2}}$ d = 6.66
Test	Tested geometrically ☺

Circle equation	$(x-c1)^2 + (y - c2)^2 = r^2$
r=½*7.21 r=3.61 c1=4 c2=7	$(x -4)^2+(y - 7)^2 = 3.61^2$ $(x -4)^2+(y - 7)^2 = 13.03$
Test	Tested geometrically ☺

Intersection	$(x-c1)^2 + (y - c2)^2 = r^2$ and y = 12-2x
r =½*7.21 =3.61 c1 =4 c2 =7	(x,y) = (1.30,9.40) and (4.30,3.40) found by Math, Solver $0 = (x-4)^2 + (12-2x-7)^2 – 3.61^2$
Test	Tested geometrically ☺

Angle*:* tan(v) = a, a=3/2 ; v = 56.31 found by Math, Solver 0 = tanv - 3/2, v>0 and v<90. Tested geometrically ☺

Problem 6	***Problem 7***
p(X<115) = 0.894, found by normalCdf(1EE-99,115,100,12) p(X<89) = 0.180, found by normalCdf(1EE-99,89,100,12) p(X>108) = 0.253, found by normalCdf(108,1EE99,100,12) p(93<X<109) = 0.494, found by normalCdf(93,109,100,12)	p(X<70) = 0.827, found by binomCdf(100,0.65,0,69) p(X≤60) = 0.172, found by binomCdf(100,0.65,0,60) p(X≥58) = 0.941, found by binomCdf(100,0.65,58,100) p(63<X≤72) = 0.571, found by binomCdf(100,0.65,64,72)

Problem 8

x=?	sin(3x) = 0.4
	x =0.137, or 0.910, or 2.232 or 3.004 or 4.326 or 5.099 found by Math, Solver 0=y1-0.4
Test	tested by Graph, Calc, Intersect ☺

x=?	cos(½x) = -0.3
	x =3.745 found by Math, Solver 0=y1+0.3
Test	tested by Graph, Calc, Intersect ☺

x=?	tan(2x) = 0.7
	x =0.305, or 1.876, or 3.447 or 5.018 found by Math, Solver 0=y1-0.7
Test	tested by Graph, Calc, Intersect ☺

Problem 9

a = ?	tan A = a/b
A =40 b=7	a = 5.874 found by Math, Solver 0=a/7-tan40
Test	tan40 = 5.874/7 0.839 = 0.839 ☺

c = ?	cos A = b/c
A =40 b=7	c = 9.138 found by Math, Solver 0=7/c-cos40
Test	cos40 = 7/9.138 0.766 = 0.766 ☺

B = ?	A + B = 90
A = 40	B = 50 found by Math, Solver 0=40+B-90
Test	50+40 = 90 90 = 90 ☺

Problem 10

b = ?	a^2 + b^2 = c^2
a =4 c=7	b = 5.745 found by Math, Solver 0=4^2+b^2-7^2
Test	4^2 + 5.745^2 = 7^2 49 = 49 ☺

A = ?	sinA = a/c
a =4 c=7	A =34.85 found by Math, Solver 0=4^2+b^2-7^2
Test	Sin34.85 = 4/7 0.571 = 0.571 ☺

B = ?	A + B = 90
A = 34.85	B = 55.15 found by Math, Solver 0=34.85+B-90
Test	34.85+55.15 = 90 90 = 90 ☺

Problem 11

B = ?	A+B+C=180
A =40 C =68	B = 72 found by Math, Solver 0=40+B+68-180
Test	40+72+68=180 180= 180 ☺

a = ?	a/sinA = b/sinB
A =40 B =72 b = 7	a =4.731 found by Math, Solver 0=a/sin40- 7/sin72
Test	4.731/sin40 = 7/sin72 7.360 = 7.360 ☺

c = ?	c/sinC = b/sinB
C =68 B =72 b = 7	c =6.824 Math, Solver 0=c/sin68 - 7/sin72
Test	6.824/sin68 = 7/sin72 7.360 = 7.360 ☺

Problem 12

a = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
A =40 c =6.8 b = 7	a =4.724 found by Math, Solver $0=a^2-6.8^2-7^2+2*6.8*7*\cos 40$
Test	$4.724^2=6.8^2+7^2–2*6.8*7*\cos 40$ 22.316= 22.316 ☺

B = ?	a/sinA = b/sinB
A =40 b = 7 a=4.724	B = 72.3 found by Math, Solver 0=4.724/sin40- 7/sinB
Test	4.724/sin40 = 7/sin72.3 7.348 = 7.348 ☺

C = ?	A+B+C=180
A =40 B =72.3	C = 67.7 found by Math, Solver 0 = 40+72.3+C-180
Test	40+72.3+67.7=180 180= 180 ☺

Problem 13

B = ?	a/sinA = b/sinB
A =40 a =6.2 b = 7	B = 46.53 or B = 133.47 found by Math, Solver 0 = 6.2/sin40 - 7/sinB
Test	6.2/sin40=7/sin46.53=7/sin133.47 9.645 = 9.645 = 9.645 ☺

C = ?	A+B+C=180
A =40 B =46.53 or B=133.47	C = 93.47or C = 6.53 found by Math, Solver 0 = 40+B+C-180
Test	40+46.53+93.47=180 180= 180 ☺

c = ?	a/sinA = c/sinC
A =40 a=6.2 C =93.47, C=6.53	c = 9.628 or C = 1.097 found by Math, Solver 0 = 6.2/sin40 - c/sinC
Test	6.2/sin40=9.628 /sin93.47=9.628 /sin6.53 9.645 = 9.645 = 9.645 ☺

Problem 14

A = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
a = 4 c =6.8 b = 7	A =33.66 found by Math, Solver $0 = 4^2-6.8^2-7^2+ 2*6.8*7*\cos A$
Test	$4^2 =6.8^2+7^2–2*6.8*7*\cos 33.66$ 16 = 16 ☺

B = ?	$b^2 = a^2+c^2–2*a*c*\cos B$
a = 4 c = 6.8 b = 7	B =75.91 found by Math, Solver $0 = 7^2-4^2-6.8^2+2*6.8*4*\cos B$
Test	$7^2=4^2+6.8^2–2*6.8*4*\cos 75.91$ 49 = 49 ☺

C = ?	A+B+C=180
A =33.66 B =75.91	C = 70.43 found by Math, Solver 0 = 33.66+75.91+C-180
Test	33.66+75.91+70.43=180 180= 180 ☺

Problem 15

d=?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$ d = (e-f)*(T-g)
Test	$T= \frac{(e-f)*(T-g)}{e-f} +g = T$

e=?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$ (T-g)(e-f) = d $e = \frac{d}{T-g} + f$
Test	$T= \frac{d}{\frac{d}{T-g} + f - f} +g =T$

f=?	$T = (\frac{d}{(e-f)}) + g$
	(T-g)(e-f) = d $e = \frac{d}{T-g} + f$ $e - \frac{d}{T-g} = f$
Test	$T= \frac{d}{e- e - \frac{d}{T-g}} +g =T$

g=?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$ $T - (\frac{d}{(e-f)}) = g$
Test	$T= \frac{d}{e-f} +T - (\frac{d}{(e-f)})=T$

Problems 16-18

y = ?	y = a*b^x
a=785 b=1.027 x=5	y = 785*1.027^5 y = 896.85

T = log(2)/log(1.027) = 26.0

x = ?	y = a*b^x
a=785 b=1.027 y=980	x = 8.3 found by Math, Solver 0=785*1.027^x - 980
Test	980 = 785*1.027^8.3 980 = 980 ☺

T = log(2)/log(1.027) = 26.0

b = ?	y = a*b^x
a=785 y=980 x=5	b = 1.045 = 1 + 4.5% found by Math, Solver 0=785*b^5 - 980
Test	980 = 785*1.045^5 980 = 980 ☺

T = log(2)/log(1.045) = 15.7

Problems 19-21

y = ?	y = a*x + b
b=785 a= 2.7 x=5	y = 2.7*5 + 785 y = 798.5

x = ?	y = a*x + b
b=785 a= 2.7 y=980	x = 72.2 found by Math, Solver 0 = 2.7*x+785 - 980
Test	980 = 2.7*72.2 + 785 = 980 ☺

a = ?	y = a*x + b
b=785 y=980 x=5	a = 39 found by Math, Solver 0 = a*5+785 - 980
Test	980 = 39*5 + 785 = 980 ☺

25. SUSTAINABLE ADAPTION TO QUANTITY: FROM NUMBER SENSE TO MANY SENSE

Their biological capacity to adapt to their environment make children develop a number-language based upon two-dimensional block- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language. And math education must accept that its goal is not to mediate the truth regime of setcentric university math, but to develop the child's already existing adaption to Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden.

However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

> The highest performing education systems across OECD countries are those that combine excellence with equity. A thriving education system will allow every student to attain high level skills and knowledge that depend on their ability and drive, rather than on their social background. Sweden is committed to a school system that promotes the development and learning of all its students, and nurtures within them a desire for lifelong learning. PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD 2015, p. 3).

Widespread innumeracy also resides in Denmark, where the use of multi-year office-directed lines with fixed classes from secondary school has lowered the exam passing limit at the end of lower and upper secondary school to about 15% and 20% compared to the North-American limit at 70%, using instead self-chosen half-year blocks to uncover and develop the student's individual talent.

Furthermore, two different forms of mathematics are taught, one accepting and one rejecting the 'New Math' occurring around 1960.

Mathematics and its Education

The Pythagoreans used the word 'mathematics' as a common label for their knowledge about Many by itself and in space and time: arithmetic, geometry, music and astronomy. (Freudenthal, 1973)

Without the two latter, mathematics later became a label for arithmetic, algebra and geometry, which may be called pre-setcentric math, replaced by the present setcentric 'New Math' in 1960 despite it never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we adapt to the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon.

Consequently, continental Europe taught reckoning, called 'Rechnung' in German, until the arrival of the New Math. And, when opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only teach set-centric mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. But also here precalculus is seen as a very difficult class to teach, discouraging many students from taking calculus classes.

However, in their 'Learning framework 2030', OECD (2018) points to the necessity of a solid background for all in literacy and numeracy, which raises the 'Cinderella question': with pre-setcentric and setcentric mathematics unable to 'make the prince dance', is there a third hidden post-setcentric alternative, that may prove sustainable so it will last?

The nature of education has been studied by different sciences. To discuss how to find a sustainable solution we should begin with biology, specializing in sustainability through adaption.

Biology Looks at Education

As a life science, biology sees life as built from green, grey and black cells.

Green cells form plants able to perform photosynthesis that store the energy form solar photons in carbon hydrate molecules by replacing oxygen with water in carbon dioxide molecules. To survive, plants must access light and water where they are situated since they are unable to move.

Grey cells form animals able to release the energy from plants or other animals by the replacing hydrogen with oxygen when inhaling oxygen and exhaling carbon dioxide through breathing. To survive, animals must move using muscles and limbs, as well as a brain to decide which way to move. Also, according to ethology (Darwin, 2003) they must adapt to the environment.

Black cells exist individually in oxygen depleted areas on the bottom of lakes or in the stomach of animals, surviving by removing oxygen from carbon hydrate molecules, thus being transformed to carbon or carbon hydrogen or oil allowing energy to be used by machines.

The holes in their head allow animals to satisfy their two basic needs for information and food. Animals come in three forms.

Reptiles have one brain allowing it to transform outside information into a choice between alternative actions.

Mammals also have a second brain for feelings binding them to a mate and to the offspring to allow it to gradually adapt to the environment through childhood before having offspring themselves.

Finally, humans also have a third brain to store and share information, made possible by transforming forelegs to arms with hands that can grasp food and things that are named by sounds, thus developing a language for mutual sharing information about what they observe and know about the six core ingredients of their life: I, you1, it, we, you2, and they; or in German: ich, du, es, wir, ihr, sie.

The combination of individual and collective adaption is so effective that to reproduce, humans only need two to three offspring in a lifetime, where other mammals need it per year.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured e.g. by a social institution called a school.

With life existing in space and time, institutional education has to answer two core questions: what things and events in the environment is important to address in education? And will learning take place through a meeting allowing individual representations to be created, or will it need to be mediated through the teaching of socially constructed representations.

To answer this, we now turn to three other sciences: philosophy, psychology and sociology.

Philosophy Looks at Education

Philosophy looks at the relation between outside existence, ontology, and inside representation, epistemology, or, in other words, the 'it-we' relation. Within philosophy, precedence is given to outside phenomena by empiricism, to inside rationality by rationalism, and to questioning ruling knowledge claims by skepticism.

A controversy within philosophy began in ancient Greece where the sophists pointed out that to practice democracy, a population must be enlightened, especially about the difference between nature and choice to avoid being patronized by political choice masked as unpolitical nature. In opposition to this, Plato saw choice as an illusion since all physical is but examples of metaphysical forms only visible to philosophers educated at the Plato academy. (Russell, 1945)

Later, the Christian church changed the academies into monasteries, where some changed back into universities after the reformation; and after Newton's discovery of physical laws controlling nature without being physical or metaphysical patronized. This inspired the Enlightenment Century installing two republics, one in North America taking over empiricism when developing 'it is right if it works' pragmatism, symbolic interactionism, grounded theory, and action research; and one in France taking over skepticism after seeing its republic turned over several times because of resistance from its German speaking neighbors.

Today, opposition against rationalism is seen within existentialism, claiming with Sartre (2007) that existence precedes essence. And explicated by Heidegger (1962) arguing that in defining verdict-sentences "subject is predicate", the subjects should be respected for naming what exist outside, whereas the predicates should be questioned and appealed since they represent an inside choice between alternatives in risk of being masked as nature.

In contrast to this, continental rationalism gives precedence to inside essence developed by rational universities through peer-reviewed research.

Psychology Looks at Education

Psychology looks at cognitive aspects of learning, or, in other words, the 'it-I' relation. Here, the philosophical controversy between outside existence and inside essence becomes a controversy between different forms of inside constructivism.

Supporting the philosophical existence stance, Piaget (1971) sees learning as a biological process of adapting inside to the outside environment through outside assimilation and inside accommodation, where assimilation makes the outside conform to inside schemata, whereas accommodation makes inside schemata conform to the outside resistance against assimilation.

Thus, to Piaget, learning takes place in the meeting between outside existence and inside schemata that accommodate through outside operations and inside peer communication. Here, teaching socially constructed schemata should be kept to a minimum to not influence the construction of individual schemata.

Siding with Piaget, Ausubel says that "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1968, p. vi).

Supporting the philosophical essence stance, Vygotsky (1986) sees learning as adapting to the socially institutionalized knowledge mediated through good teaching respecting that the knowledge taught must be attachable to what the learners already know in their zone of approximate development. Consequently, high quality must be given to teacher education and textbooks to provide good teaching. And teaching should be structured and well-organized aiming at students being able to reproduce what teachers teach.

Sociology Looks at Education

Sociology looks at the social aspects of human interaction, or, in other words, the 'they-I' relation. Here, the philosophical controversy between outside existence and inside essence carries on as a controversy between different forms of social theory emphasizing individual agency or social structure.

Individual agency is emphasized in the first Enlightenment republic in North America showing strong resistance against institutional answers since they may lead to a goal displacement (Baumann, 1990) becoming an inside goal itself instead of staying as an inside means to an outside goal, thus suppressing the 'sociological imagination' (Mills, 1959) that might keep the answer fluid instead of fossilizing into what Weber (1930) calls an 'disenchanting Iron Cage'.

Structuralism is preeminent in continental Europe seeing established science as being a true inside representation of the outside environment if accepted by the society's knowledge institutions, the universities.

In the second Enlightenment republic in France, institutional skepticism inspired by Heidegger led to French post-structuralism where Derrida, Lyotard, Foucault and Bourdieu warn against hidden patronization in our most basic institutions: words, correctness, diagnosing discourses, curing institutions, and education especially that might make mathematics so difficult it may serve as symbolic violence to establish a new knowledge nobility.

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006: 41)

As to education, Foucault (1995) sees schools as 'pris-pitals' mixing social power techniques from a prison and a hospital. Here students are 'pati-mates' forced to return to the same class, hour after hour, day after day, month after month for several years. Furthermore, self-reference is used to create diagnose 'ignorant' without specifying of what: 'you don't know math, so we must teach you math'. Consequently, humans are placed in a Kafkaesque (2015) situation where they, unable to have their diagnose defined, finally accept it and therefore also accept to be retained and treated to be cured.

In Germany, Habermas (1981) theorizes the possibility of creating a third Enlightenment republic in post-war Germany. Inspired by Weber's warning that rationalization carried too far might lead to an iron cage dis-enchanting the world, Habermas warns against system-worlds tending to colonize the life-world and recommends a power free communication rationality to prevent this from happening where "peers exchange views".

Likewise, Arendt (1963) points out that by definition, institutions lack competition, forcing employees to follow order, which might lead to the banality of evil, where ordinary citizens must act evilly to keep the job.

Agency-based education sees knowledge construction as best taking place in symbolic interaction between peers exchanging views about the sentence subject in order to negotiate a common view. This resonates with Piagetian constructivism, and with philosophical existentialism giving precedence to existence; and with the construction of social knowledge by using Grounded Theory (Glaser and Strauss, 1967). Secondary education should be block-organized to support the student's identity work through self-chosen half-year blocks with teachers teaching only one subject.

Structuralism sees democratically controlled educational institutions as the best way to mediate the university knowledge heritage. This resonates with Vygotskyan constructivism. Secondary education should be line-organized to supply the state with skilled academic and non-academic workers, thus forcing students to choose career line early, and to start all over if changing career.

Meeting Many, Children Bundle to Count and Share

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists: bundles of 2s, and 2 of them. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units.

Likewise, children use bundle-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 more'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 more'.

Children love placing four cars or dolls in patterns; and they smile when the items form a 4-icon. Likewise, they like to form number-icons with footprints in the sand, with body-parts etc.

Children love counting their fingers in 4s using a rubber band to hold the bundles together. They smile when seeing that the fingers can be counted in 4s as 1Bundle6, 2B2 or 3B less2. Or, if counting in 3s, as 1B7, 2B4, 3B1, or 4Bless2. Some even see that 3 bundles is the same as one bundle of bundles, 3B = 1BB.

Likewise, children love bundle-counting the fingers in e.g. 4s as 0Bundle1, 0B2, 0B3, 0B4 no 1B0, 1B1, 1B2, 1B3, 1B4 no 2B0, 2B1, 2B2.

A special case is counting in pairs or 2s. Here the fingers can be counted as 1B8, 2B6, 3B4, 4B2, 5B0. A different color for the rubber band used for the bundle of bundles will allow the fingers to be counted as 1BundleBundle6, 2BB2, 3BBless2. Some might suggest a new color for the bundles of bundles of bundles, thus counting the fingers as 1BBB2 or 1BBB1B0; or even 1BBB0BB1B0.

And children don't mind writing using 'bundle-writing' with a full sentence containing a subject, a verb and a predicate as in the word-language: T = 8 = 1B5 = 2B2 = 3B-1 3s. Some might even write T = 8 = 3B-1 = 1BB-1 3s.

Also, children smile when they see that, counting in hands, T = 5 = 1B0 5s, thus realizing that ten is written as 10 because ten becomes 1B0 if we count in tens.

Sharing 8 cakes, 2 children take away 2 to have one each; and smile when they see that entering '8/2', a calculator predicts they can have 4 each; thus seeing the division sign as an icon for a broom pushing away 2s. This motivates rooting division by 2 as counting in 2s.

Likewise, when counting 9 cubes in 2s they may stack the 2s on-top as a block of 4 2s, smiling when they see that entering '4x2', a calculator predicts they have a total of 8; thus seeing the multiplication sign as an icon for a lift pushing up 2s.

And again, they smile they see that entering '8 – 4*2', a calculator predicts that 1 is left when pulling away a stack of 4 2s from 8; thus seeing the subtraction sign as an icon for a rope pulling away the 4 2s.

Children thus see that counting involves three processes: pushing away, pushing up and pulling away, that can be performed by a broom, a lift and a rope; and that can be predicted on a calculator by using division, multiplication and subtraction. Some may even accept that the counting prescription 'From the total 8, 8/2 times, 2s can be pushed away' may be shortened to the calculation formula '8 = 8/2x2', later with unspecified numbers becoming a core formula expressing proportionality, the recount-formula 'T = (T/B)*B'.

Exposed to counting, children adapt in a natural way to the three basic operations division, multiplication and subtraction; and typically enjoys using a calculator, or even the recount-formula, to predict the counting result.

A Contemporary Mathematics Curriculum

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5.

This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about the results of operating on such numbers, e.g. that (a+b) * (a-b) = a^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles and areas. Later, geometry and algebra are coordinated in coordinate geometry.

To be followed by halving a rectangle by its diagonal to create a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

In a calculation, changing the input x will change the output y, making y a function of x, y = f(x), using f for an unspecified calculation. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about the results of operating on calculations, e.g. that (f*g)'/(f*g) = f'/f + g'/g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6*x^2dx = 2*x^3 + c, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

The Difference to a Typical Contemporary Mathematics Curriculum

Thus, typically the core of a curriculum is about how to operate on specified and unspecified numbers and calculations.

Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent.

Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles, bundles, and bundles of bundles.

Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Seldom children may say 'bundle' instead of ten, or to say 'ten-ten' or 'bundle-bundle' instead of hundred.

Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit.

Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit.

Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed.

Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction.

In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

Mathematics as a Number-Language

Wanting to respect the child's own number-language, Tarp (2018, p. 103) talks about word- and number-languages with word- and number-Sentences:

> Observing the quantitative competence children bring to school, (..) we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

Discussing Number Sense and Number Nonsense

The basic question in grade one mathematics is: shall education be about numbering or about numbers? Shall education guide and support the development of the children's already existing adaption to quantity, or shall education teach numbers? Shall the 'I' keep on adapting to the 'it' directly, or indirectly by having the adaption replaced by what is mediated by the 'they'?

Choosing numbers over numbering, the US National Council of Teachers of Mathematics, NCTM, in their publication 'Principles and Standards for School Mathematics' (2000) says: "Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense (p. 7)."

Likewise choosing numbers over numbering, ICMI study 23 creates a WNA-discourse (Whole Number Arithmetic) asking:

> To what extent is basic number sense inborn and to what extent is it affected by socio-cultural and educational influences? How is the relationship between these precursors/foundations of WNA, on the one hand, and children's whole number arithmetic development?" (Bussi and Sun, 2018, pp 500-501)

Thus, both to the NCTM and in the WHA discourse, the concept 'number sense' is central, although not being that well defined (Griffin, 2004). In the ICMI study there are several references to Sayers and Andrews (2015) that based upon reviewing research in the WHA domain create a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

However, several questions may be raised to this FoNS framework.

In his book, Dantzig (2007) uses the term 'number sense' for a natural property shared by humans and animals. However, from a biological view it is sensing the environment that is fundamental to all grey cells. And as human constructs, numbers are not part of the environment, in contrast to what they number and what is embedded in human language as the singular in plural forms, the physical fact many or 'more-ness'. Using the term 'cardinality' just adds a religious power aspect demanding respect for the Cardinal.

Thus, the term ‘many sense’ is more precise than the term ‘number sense’. Especially since, with its reference to numbers, ‘number sense’ becomes a self-reference that removes meaning from four of the eight categories.

Furthermore, using the word ‘understanding’ makes three categories dubious since there are many different understandings of the word understanding.

What is left is category seven, simple arithmetic competence, which is about adding and subtraction, thus neglecting that division and multiplication come first when counting in bundles.

Thus, it seems difficult to define number-sense without self-reference and without referring to a tradition giving priority to addition and subtraction.

A grounded definition of number-sense or many-sense should come from how numbers emerge in the numbering process counting and recounting a total in bundles, to allow seeing the link between the number and what it numbers by including the ‘missing link’, the bundle and the unit, absent in everyday use: T = 6B7 tens = 67.

Therefore, a short definition could be: Having number-sense or many-sense means including the word ‘bundle’ as a unit for the numbers. That is:

To bridge the outside total with an inside numbering by bundling creating flexible bundle-numbers expressed in a full number-language sentence with an outside subject, a verb and an inside predicate, e.g. T = 2 3s.

To count 5 fingers in fives as 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0; and as 1Bundle less 4, 1B-3, 1B-2, 1B-1, 1B0; and to recount five fingers with ‘flexible bundle-numbers’ with overload, underload or fraction, i.e. as 1B3 2s, 2B1 2s or 3B-1 2s or 2 ½B 2s, and later as 1BB 0B1 2s or 1BB1B-1 2s.

And to recount ten fingers in 3s as 1B7, 2B4, 3B1, 4B-2, 31/3, 1BB0B1, or 1BB1B-2.

And to let 67 = 6B7 = 5B17 = 7B-3 = 6.7 tens = 7.-3 tens.

And 678 = 67B8 = 6BB7B8. (Tarp, 2019)

To see the digits as icons with as many sticks or strokes as they represent if written less sloppy; and with ten needing no icon when used as bundle-size.

To see the operations as icons coming from the counting process, where division iconizes a broom pushing away bundles, where multiplication iconizes a lift pushing up bundles into a block, where subtraction iconizes a rope pulling away the block to find unbundles singles, and where addition iconizes placing blocks next-to or on-top.

To see the counting process predicted by the recount-formula T = T/(B)*B, saying ‘From the total T, T/B times, B-bundles can be pushed way’; and to use a calculator to enter ‘9/4’ giving ‘2’, and ‘9-2*4’ giving ‘1’ to predict that from 9, 4s can be pushed away 2 times, and that pulling away the 2 4s from 9 leaves 1, thus predicting that 9 may be recounted as 2B1 4s.

To see totals as double described both as outside blocks and as inside bundles.

To see 678 as a numbering containing four numbers counting unbundled, bundles, bundles of bundles and specifying the bundle-size.

To see a multiplication task as recounting from icons to tens, facilitated by using flexible block&bundle numbers so that 6*8 = 1B-4 * 1B-2 = 1BB – 4B – 2B + 4*2 = 4B8 = 48, thus realizing that -*- is + since the corner was pulled away twice. And to see that 4*67 may be calculated as 4*6B7 giving 24B28, which may be recounted without an overload as 26B8 or 268.

To see a multiplication equation 4*x = 20 as recounting from tens to icons, solved by the recount-formula.

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**1~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	10**B**10
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

Figure 1. A counting table that includes the bundles in the number names

The WHA discourse defines numbers by internal reference as a set of whole numbers included in the set of integers, included in etc. All created to describe what is called cardinality which is claimed to be linear and represented by a number-line.

The WHA discourse thus presents 678 as one number, or if asked to be more precise, as 6 numbers: 6, 7, 8, ones, tens and hundreds, even if the correct answer is fou4r numbers: 6, 7, 8 and bundles, which typically is ten where it is twenty when the French and the Danes count four twenties instead of eight tens.

Furthermore, 67 is not even a whole number but decimal number that might include a negative number as well:

67 = 6ten7 = 6B7 tens = 7B-3 tens, or 6ten7 = 6.7 tens = 7.-3 tens.

The WNA discourse subscribes to setcentric mathematics. Even if Russell proved that self-reference leads to the nonsense of the classical liar paradox, ‘this sentence is false’, since the set of sets not belonging to itself will belong if and only if it will not.

Russell’s point is that it is OK to talk about elements and sets since that is how a language is organized, but when you talk about sets of sets you talk from a meta-level that should not be mixed with the language level, even if this was precisely what Zermelo and Fraenkel did when trying to save the set theory by disregarding the difference between a set and its elements, thus disregarding the difference between examples and abstractions that is the basis in any language.

Grounded in outside observations, the numbers zero, one and two are rooted in fingers on a hand. Defined inside the WNA discourse, zero is defined as the empty set Ø = {}. With 0 = Ø, 1 is defined as the set containing the set Ø, 1 = {Ø}, but as a set of sets, this places 1 on a different language level where it cannot be added to 0. Then 2 is defined as the set that contains a set, and a set of sets, Ø and 1, 2 = {Ø, {Ø}} thus placing 0, 1 and 2 on three different language levels. Which is nonsense according to Russell.

As to the sociological effect of creating an educational concept ‘number-sense’ we should remember that sociologically, a school is a pris-pital. So, the moment you introduce a new construct you may also introduce a new diagnose: this child lacks number sense, so it must be treated. Especially since it is claimed that children who start with a poorly developed understanding of numbers remain low achievers throughout school (Geary, 2013). And with eight diagnose components, you need eight cures. This might be good news for universities selling teacher education courses, but bad news for the curers, the teachers, now having three times eight additional tasks forced upon them: How to understand the diagnoses, how to find material to use in the cure, and how to evaluate if the cure works.

Introducing diagnoses is an example of what Foucault calls pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1982: 213, 215)

In this way Foucault describes the salvation promise of the generalized church: 'You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and do exactly what we tell you. If not, you only have yourself to than for your decline'.

Introducing diagnoses may also be seen as an example of 'symbolic violence' used as an exclusion technique to keep today's knowledge nobility in power (Bourdieu 1977).

To master Many, humans invented numbers as a means, typically rooted in the hands as the Roman numbers bundling fingers in hands and double hands (Dantzig, 2007). But numbers may lose their outside link and become examples of inside abstractions instead of abstractions from the outside. Likewise, outside quantity may become an example of inside cardinality. In that moment numbers undertake what Baumann calls a goal displacement, where inside derived setcentric numbers become the goal instead with outside quantity as a means thus leaving Many as what Weber calls disenchanted.

The situation with eight components in number sense reminds of the claimed eight 'mathematical competencies' (Niss, 2003) also made meaningless by self-reference, but meaningfully reduced to two competences, count and add (Tarp, 2002). Likewise, both situations remind of the eight sacraments in the catholic church, challenged by the two sacraments of the protestant church.

To look for meaningful diagnoses in a sustainable mathematics education adapted to quantity we must ask: What is it in the outside world that the children are not adapted to? Will bringing this inside the classroom allow children to extend their existing adaption?

So, instead of using the eight number sense components as diagnoses, we may use the alternative definition of number sense given above as diagnoses to be cured by guiding questions to outside subjects brought inside to receive common predicates, thus reifying the subject in the number language sentences.

Conclusion and Recommendation

This paper asked if there is a third hidden post-setcentric alternative, that may prove sustainable so it will last? The answer is yes, and maybe, since testing for sustainability has to be carried out on what may be called post-setcentric mathematics respecting instead of colonizing the way children adapt to quantity by using two-dimensional bundle-numbers with units instead of the one-dimensional line-numbers forced upon them by setcentric education. Thus, mathematics education should see itself as a language education allowing children develop their quantitative number-language like their qualitative word-language, both using sentences typically with a subject, a verb and a predicate.

A core question in language education is the following: should education develop further the children's own language, or should education colonize it by replacing their native language with a foreign language. And should language be taught before, together with or after its grammar?

Word-language education chose to respect the children's native language and to develop it before introducing a grammar. Likewise, with foreign language after the language revolution in the 1970s made language be taught before grammar (Widdowson, 1978, and Halliday, 1973).

Number-language education chose to disrespect the children's native language. Furthermore, its revolution in the 1970s made language be taught after its grammar, that was introduced not through bottom-up reference to examples, but as top-down examples of the abstraction Set.

So, to establish as sustainable tradition that will allow all to learn and practice a number-language, mathematics education must stop using a setcentric grammar-based foreign language to colonize the children's own native language.

The consequences of not decolonizing is seen in the OECD-report on the Swedish school system as well as in the widespread innumeracy documented by various PISA studies. The time has come for a paradigm shift (Kuhn, 1962) in early childhood education in adaption to quantity by developing the children's already existing many-sense.

Therefore, if the goal is a sustainable mathematics education it might be a good idea to respect and develop the natives' own natural number-language; and to say: 'only cure the diagnosed'.

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26. Per-numbers connect Fractions and Proportionality and Calculus and Equations

In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word ‘percent’ to also talk about other ‘per-numbers’ as e.g. ‘per-five’ thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

Mathematics is Hard, or is it

“Is mathematics hard by nature or by choice?” is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature.

That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969.

Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report ‘Improving Schools in Sweden’ describing the Swedish school system as being ‘in need of urgent change’.

Witnessing poor PISA performance, Denmark has lowered the passing limit at the final exam is to around 15% and 20 % in lower and upper secondary school.

Other countries also witness poor PISA performance. And high-ranking countries admit they have a high percentage of low scoring students.

As to finding the cause, Kilpatrick, Swafford, and Findell (2001, p. 36) points out that “what is actually taught in classrooms is strongly influenced by the available textbooks”. Personally, working ethnographically in schools in Denmark and abroad, listening to teachers and students confirms the picture that textbooks are followed quite strictly.

So, it seems only natural to look at what is currently being discussed in textbook research e.g. by looking at the Third International Conference on Mathematics Textbook Research and Development, ICMT3, in Germany.

The ICMT3 Conference

The September 2019 ICMT3 conference consisted of 4 keynote addresses, 15 symposium papers, 2 workshops, 40 oral presentations and 13 posters.

The name ‘fraction’ occurred 212 times in the proceedings, and one of the keynotes addressed the problems students have when asked to find 3/5 of 2/4.

As to fractions, Ripoll and Garcia de Souza writes that “The integer numbers structure and the idea of equivalence are elementary in the mathematical construction of the ordered field of the rational numbers. Hence, the concept of equivalence should not be absent in the Elementary School’s classrooms and textbooks.” (Rezat et al, 2019, p. 131). Looking at 13 Brazilian textbooks from 4th to 7th grade they conclude that

> The conclusion, with respect to equivalence, was that no (complete) characterization of equivalent fractions is present in the moment the content fractions is carried on in the 6th grade Brazilian textbooks, like “Two given fractions a/b and c/d are equivalent if and only if ad = bc.” In most cases only a partial equivalence criterion is presented, like “Two fractions are equivalent if one can transform one into the other by multiplying (or dividing) the numerator and the denominator by the same natural number.”

The authors thus take it that fractions should obey the New Math 'set-centrism' (Derrida, 1991) by saying: in a set-product of integers, a fraction is an equivalence class created by the equivalence relation statin that a/b ~c/d if a*d = b*c; and thus neglect the pre-setcentric version mentioned above where a fraction keeps its value by being expanded or shortened; as well as the post-setcentric version seeing a fractions as an example of a per-number, described later in this paper.

Confirming in the afterwards discussion that fractions are introduced by the part-whole model, an argument was made that if a fraction is defined as a part of a whole then a fraction must always be a fraction of something; thus being an operator needing a number to become a number, and not a number in itself.

Of course, in a 30 minutes presentation there is little time to discuss the nature of fractions thoroughly, so this question needs to be addressed in more details.

Also addressing middle school problems, Watanabe writes that "Ratio, rate and proportional relationships are arguably the most important topics in middle grades mathematics curriculum before algebra. However, many teachers find these topics challenging to teach while students find them difficult to learn." (p. 353)

And, talking about proportionality, Memis and Yanik writes that "Proportional reasoning is an important skill that requires a long process of development and is a cornerstone at middle school level. One of the reasons why students cannot demonstrate this skill at the desired level is the learning opportunities provided by textbooks." (p. 245)

Textbooks must follow curricula, and middle school problems were also mentioned at the International Commission on Mathematical Instruction Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, in Japan November 2018. Here in his plenary talk, McCallum after noting that "a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships" challenged the audience by asking "What is the difference between 5/3 and 5÷3?" (ICMI, 2018, p. 4).

So, this paper will focus on these challenges by asking: "Is there a hidden different way to see and teach core middle school concepts as fractions, quotients, ratios, rates and proportionality?" A question that might be answered answer by Difference-research (Tarp, 2018) using sociological imagination (Mills, 1959) to search for differences making a difference by asking two questions: 'Can this be different – and will the difference make a difference?'

Different Ways of Seeing Fractions

In a typical curriculum using a 'part-whole' approach, fractions are introduced after division has been taught as sharing a whole in equal parts: 8/4 is 8 split in 4 parts or 8 split by 4.

Representing the whole geometrically as a bar or a circle, dividing in 4 parts creates 4 pieces each called ¼ of the total. Assigning numbers to the whole allows finding ¼ of e.g. 8 by the division 8/2. Then the fraction ¾ means taking ¼ three times, so that taking ¾ of 8 involves two calculations, first 8/4 as 2, then 3*2 as 6, so that ¾ of 8 is 8/4*3, later reformulated to one calculation, 8*¾, multiplying the integer 8 with the rational number ¾.

However, in the 'part-whole' approach a fraction is a fraction of something, thus introducing a fraction as an operator needing a number to become a number.

This becomes problematic when the fraction later is claimed to be a point on a number line, i.e. a number in its own right, a rational number, defined by set-centrism as an equivalence class in a set-product as described above.

Furthermore, set-centrism is problematic in itself by making mathematics a self-referring 'Meta-matics', defined from above as examples from abstractions instead of from below as abstractions from examples.

And, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If M = $\{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

To avoid self-reference Russell introduced a type theory allowing reference only to lower degree types. Consequently, fractions could not be numbers since they refer to numbers in their setcentric definition.

Neglecting the Russell paradox by defining fractions as rational numbers leads to additional educational questions: When are two fractions equal? How to shorten or expand a fraction? What is a fraction of a fraction? Which of two fractions is the bigger? How to add fractions? Etc.

Fraction later leads on to percentages, the special fractions having 100 as the denominator; which leads to the three percentage questions coming from the part-whole formula defining a fraction, fraction = part/whole.

Seeing fractions as, not numbers, but operators still raises the first three questions whereas the two latter are meaningless since the answer depends on what whole they are taken of as seen by ‘the fraction paradox’ where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes, and never 7 cokes of 6 bottles.

Adding numbers without units may be called ‘mathe-matism’, true inside but seldom outside the classroom. And strangely enough the two latter questions are only asked with fractions and seldom with percentages.

Ratios and Rates

When introduced, ratios are often connected to fractions by saying that splitting a total in the ratio 2:3 means splitting it in 2/5 and 3/5.

Where fractions and ratios typically are introduced without units, rates include units when talking e.g. about speed as the ratio between the meter-number and the second-number, speed = 2m/3s.

Per-numbers Occur when Double-counting a Total in two Units

The question “What is 2/3 of 12?” is typically rephrased as “What is 2 of 3 taken from 12?” Seldom it is rephrased as “What is 2 per 3 of 12?”. Even if the word ‘per’ occurs in many connections, meter per second, per hundred, etc.

When we rephrase “taking 30% of 400” as “taking 30 per 100 of 400”, why don’t we rephrase “taking 3/5 of 400” as “taking 3 per 5 of 400” ?

In short, why don’t we rephrase 3/5 both as ‘3 of 5’ and as ‘3 per 5’?

In his conference paper, Tarp (p. 332) introduces per-numbers and recounting:

> An additional learning opportunity is to write and use the ‘recount-formula’ T = (T/B)*B, saying “From T,T/B times B can be taken away”, to predict counting and recounting examples. (..) Another learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Of course, you might argue that we cannot write ‘6$ = 9kg’ since the units are not the same. But then again, we write ‘2 meter = 200 centimeter’ even if the units are different, and we are allowed to do so since the bridge between the two units is the per-number 1m/100cm. Likewise, we should be allowed to write ‘6$ = 9kg’ since the bridge between the two units for now is the per-number 2$/3kg.

The difference is that the per-number between meter and centimeter is globally valid, whereas the per-number between kilogram and dollar is only locally valid. Still, it has validity as long as you are talking about the same outside total.

The interesting thing is that by including units, per-numbers connects fractions and proportionality. And that by including units, the recount-formula gives an introduction to fractions saying that 1/3 is '1 counted in 3s': 1 = (1/3)*3 = 1/3 3s.

Fractions as Per-numbers

With per-numbers coming from double-counting the same total in two units, we see that when double-counting in the same unit, the unit cancels out and we get a ratio between two numbers without units, a fraction as e.g. 3$/8$ = 3/8.

Reversely, inside fractions without units may be 'de-modeled' outside by adding new units, e.g. 'good' and 'total' transforming 3/8 to 3g/8t. This allows per-numbers and recounting to be used when solving the three fraction questions:

"What is 3/4 of 60?", and "20 is what of 60?", and "20 is 2/3 of what?"

Asking "What is ¾ of 60" means asking "What is 3 per 4 of 60", or de-modeled with units, "What is 3g per 4t of 60t",

Of course, 60t is not 4t, but 60 can be recounted in 4s by the recount-formula, 60t = (60/4)*4t = (60/4)*3g = 45g, giving the inside answer "¾ of 60 is 45".

Asking "20 is which fraction of 60" means asking "What fraction is 20 per 60", or with units, "Which per-number is 20g per 60t", giving the answer directly as 20g/60t or 20/60 g/t. Here we might look for a common unit in 20 and 60 to cancel out, e.g. 20, giving 20/ 60 = 1 20s/3 20s = 1/3. This allows transforming the outside answer "20 per 60 is 1 per 3" to the inside answer "20 is 1/3 of 60".

Asking "20 is 2/3 of what" means asking "20 is 2 per 3 of what", or with units, "20g is 2g per 3t of which total". Of course, 20g is not 2g, but 20 can be recounted in 2s by the recount-formula, 20g = (20/2)*2g = (20/2)*3t = 30t. This allows transforming the outside answer "20 is 2 per 3 of 30" to "20 is 2/3 of 30."

Expanding and Shortening Fractions

With fractions as per-numbers coming from double counting in the same unit that has cancelled out, we are always free to add a common unit to both numbers.

Using numbers as units will expand the fraction:

2/3 = 2 7s/3 7s = 2*7/3*7 = 14/21

Reversely, if both numbers contain a common unit, this will cancel out:

14/21= 2*7/3*7 =2 7s/ 3 7s = 2/3

Taking Fractions of Fractions, the Per-number Way

One of the keynotes pointed out that to understand that 6/20 is the answer to the question "What is 3/5 of 2/4?" we must draw a rectangle with 4 columns if which 2 are yellow, and with 5 rows of which 3 are blue. Then 6 double-colored squares out of a total of 20 squares gives an understanding that 3/5 of 2/4 is 6/20, which also comes from multiplying the numerators and the denominators.

Seeing fractions as per-numbers the question "What is 3/5 of 2/4?" translates into "What is 3 per 5 of 2 per 4". Knowing that using per-numbers to bridge two units involves recounting them in the per-number which again involves division, we might begin with a number that is easily recounted in 4s and 5s, e.g. 4*5 = 20, and reformulate the question to "3 per 5 of 2 per 4 is what per 20?".

To find 2 per 4 of 20 means finding 2g per 4t of 20t, so we recount 20 in 4s:

20t = (20/4)*4t = (20/4)*2g = 10g, so 2 per 4 of 20 is 10.

To find 3 per 5 of 10 means finding 3g per 5t of 10t, so we recount 10 in 5s:

10t = (10/5)*5t = (10/5)*3g = 6g, so 3 per 5 of 10 is 6

Thus, we can conclude that 3 per 5 of 2 per 4 is the same as 6 per 20, or, with fractions, that 3/5 of 2/4 is 6/20, again coming from multiplying the numerators and the denominators.

Of course, we could discuss, which method gives a better understanding, but we might never reach an answer, given the many different understandings of the word 'understanding'

Direct and Inverse Proportionality

Using a coordinate system with decimal numbers comes natural if bundle-writing totals in tens so e.g. T = 26 becomes T = 2.6 tens. This allows fixing a 3x5 box in the corner with the base and the height on the x- and y-axes. The recount-formula T = (T/B)*B then shows a total T as a box with base x = B and height y = T/B.

To keep the total unchanged, increasing the base will decrease the height (and vice versa) making the upper right corner create a curve called a hyperbola with the formula height = T/base, or y = T/x, showing inverse proportionality.

In a 3x5 box, the raise of the diagonal is the per-number 3/5. Expanding or shortening the per-number by adding or removing extra units will make the diagonal longer or shorter without changing direction. This will make the upper right corner move along a line with the formula 3/5 = height/base = y/x, or y = 3/5*x, showing direct proportionality.

Adding Fractions, the Per-number Way

Adding per-numbers occurs in mixture problems asking e.g. "What is 2kg at 3$/kg plus 4kg at 5$/kg?". We see that the unit-numbers 2 and 4 add directly, whereas the per-numbers cannot add before multiplication changes them to unit-numbers. However, multiplication creates the areas 2*3 and 4*5, which gives the answer: 2kg at 3$/kg + 4kg at 5$/kg gives (2+4)kg at (2*3+4*5)/(2+4)$/kg.

So we see that per-numbers add by the areas under the per-number graph in a coordinate system with the kg-numbers and the per-numbers on the axes.

But adding area under a graph is what integral calculus is all about. Only here, the per-number graph is piecewise constant, where the velocity graph in a free fall, is not piecewise, but locally constant, which means that the total area comes from adding up very many small area-strips.

This may be done by observing that the total area always changes with the last area-strip thus creating a change equation $\Delta A = p*\Delta x$, which motivates differential calculus to answer questions as dA/dx = p, thus finding the area formula that differentiated gives the give per-number formula p, e.g. d/dx (x^2) = 2*x.

Interchanging epsilon and delta to change piecewise constancy to local may be postponed to high school, that would benefit considerably by a middle school introduction of integral calculus as adding locally constant per-numbers by the area under the per-number graph, using differential calculus to find the area in a quicker way than asking a computer to add numerous small area-strips.

Solving Proportionality Equations by Recounting

Reformulating the recount-formula from T = (T/B)*B to T = c*B shows that with an unknown number u it may turn into an equation as 8 = u*2 asking how to recount 8 in 2s, which of course is found by the recount-formula, u*2 = 8 = (8/2)*2, thus providing the equation u*2 = 8 with the solution u = 8/2 obtained by isolating the unknown by moving a number to the opposite side with the opposite sign.

This resonates with the formal definition of division saying that 8/2 is the number u that multiplied by 2 gives 8: if u*2 = 8 then u = 8/2.

Set-centrism of course prefers applying and legitimizing all concepts from abstract algebra's group theory (commutativity, associativity, neutral element and inverse element) to perform a series of reformulations of the original equation: 2*u = 8, so (2*u)*½ = 8*½, so (u*2)*½ = 4, so u*(2*½) = 4, so u*1 = 4, so u = 4.

Seven Ways to Solve the two Proportionality Questions

The need to change units has mad the two proportionality questions the most frequently asked questions in the outside world, thus calling for multiple solutions.

With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g. Q1: "7 meters need how many seconds?", and Q2: "How many meters is covered in 12 seconds?"

● Europe used 'Regula-de-tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.

● Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter.

● Equating the rates. The velocity rate is constantly 2meter/5second. So we can set up an equation equating the rates. Q1: 2/5 = 7/x, where cross-multiplication gives 2*x = 7*5, which gives x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, where cross-multiplication gives 5*x = 12*2, which gives x = (12*2)/5 = 4.8.

● Recount in the per-number. Double-counting produces the per-number 2m/5s used to recount the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.

● Multiply with the per-number. Using the fact that T = 2m, and T = 5s, division gives T/T = 2m/5s = 1, and T/T = 5s/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.

● Modeling a linear function f(x) = c*x, with f(2) = 5, f(7) = ?, and f(x) =12.

A Case: Peter, about to Peter Out of Teaching

As a new middle school teacher, Peter is looking forward to introducing fractions to his first-year class coming directly from primary school where the four basic operations have been taught so that Peter can build upon division when introducing fractions in the traditional way. However, Peter is shocked when seeing many students with low division performance, and some even showing dislike when division is mentioned. So, Peter soon is faced with a class divided in two, a part that follows his introduction of fractions, and a part that transfers their low performance or dislike from divisions to fractions.

The following year seeing his new class behaving in the same way, Peter is about to give up teaching when a colleague introduces him to a different approach where division is used for bundle-counting instead of sharing called 'Recounting fingers with flexible bundle-numbers'. The colleague also recommends some YouTube videos to watch and some material to download from the MATHeCADEMY.net to try it yourself.

Inspired by this, Peter designs a micro-curriculum for his class aiming at introducing the class to bundle-counting leading to the recount-formula leading to double-counting in two units leading to per-numbers having fractions as the special case with like units.

“Welcome class, this week we will not talk about fractions!” “?? Well, thank you Mr. teacher, then what will we do?” “We will count our five fingers.” “Ah, Mr. teacher we did that in preschool!” “Correct, in preschool we counted our fingers in ones, now we will bundle-count them in 2s and 3s and 4s using bundle-writing. In this way we will see that a total can be counted in three different ways: overload, standard and underload. Look here:

Outside we have IIIII = ǂIII = ǂǂI = ǂǂǂ

Inside we write: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s

We will call this to recount 5 with flexible bundle-numbers. Now count the five fingers in 3s and 4s in the same way. Later, we will count all ten fingers.”

The following class, Peter began by rehearsing.

“Welcome class. Yesterday we saw that an outside total can be recounted in different units, and that the result inside can be bundle-written in three ways, with overload, standard and underload. Today we will begin by recounting twenty in hands, in six-packs and in weeks. Why twenty? Because counting in twenties was used by the Vikings who also gave us the words eleven and twelve, meaning one-left and two-left in Viking language.”

Later, Peter introduced the recount-formula:

“Here we have 6 cubes that we will count in 2s. We do that by pushing away 2-bundles, and write the result as T = 6 = 3B 2s. We see that the inside division stroke looks like an outside broom pushing away the bundles. And asking the calculator, 6/2, and we get the answer 3 predicting it can be done 3 times. We can illustrate this prediction with a recount formula ‘T = (T/B)xB’ saying that ‘from the total T, T/B times, B can be pushed away’. So, from now on, 6/2 means 6 recounted in 2s; and 3x2 means 3 bundles of 2s. And since it is counted in tens, 42 is seen as 4B2 or 3B12 or 5B-8 using flexible bundle-numbers.

Now let us read 42/3 as 4bundle2 tens recounted in 3s; and let us use flexible bundle-numbers to rewrite 4B2 with an overload as 3B12. Then we have T = 42 /3 = 4B2 /3 = 3B12 / 3 = 1B4 = 14. We notice that squeezing a box from base 10 to base 3 will increase the height, here from 4.2 to 14.

And, by the way, flexible bundle-numbers also come in handy when multiplying: Here 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 bundles and 56 unbundled singles, which can be recounted to remove the overload:

T = 7 x 4B8 = 28B56 = 33B6 = 336.”

The third day Peter repeated the lesson with 7 cubes counted in 3s to show that where the unbundled single was placed would decide if the total should be written using a decimal number when placed next-to as separate box of ones, T = 2B1 3s = 2.1 3s. Placed on-top means missing 2 to form a bundle, thus written as T = 3B-2 3s = 3.-2 3s. Or it means recounting 1 in 3s as 1 = (1/3)x3 = 1/3 3s, a fraction.

Later, Peter introduced per-numbers and fractions as described above, which allowed Peter to work with fractions and ratios and proportionality at the same time; and later to introduce calculus as adding fractions and per-numbers by areas.

Observing the increase of performance and the disappearance of dislike, the headmaster suggested to the headmaster of the nearby primary school that Peter be used as a facilitator for in-service teacher training. This would allow primary school children to meet fractions and negative numbers and proportionality when recounting and double-counting a total in a new bundle-unit.

Discussion and Recommendation

This paper asked “Is there a hidden different way to see and teach core middle school concepts as fractions, quotients ratios, rates and proportionality?“ The answer is yes: per-numbers includes them all as examples, as well as integral calculus and equations.

So introducing per-numbers through double-counting the same total in two units makes a difference by allowing fractions, quotients, rates and ratios to be seen and taught as examples of per-numbers, and by allowing integral calculus to be introduced in middle school, and by allowing a more natural way to solve multiplication equations, and by allowing STEM examples in the classroom since most STEM formulas are proportional formulas.

Furthermore, introducing recounting with flexible bundle-numbers allows math dislike to be cured by taking the hardness out of division, seen traditionally as the basis for fractions but becoming a tumbling stone instead if not learned well.

Consequently, it is recommended that primary school accepts and develops the double-numbers children bring to school. And that middle school introduces students to recounting in flexible bundle-numbers from the start to provide a strong division foundation for fractions that becomes connected with quotients, ratios, rates, proportionality, equations and calculus if introduced as per-numbers coming from double-counting in two units that may be the same.

So yes, mathematics is hard, not by nature, but by a choice replacing it with a mixture of top-down meta-matics and mathe-matism seldom true outside the class.

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27. SUSTAINABLE ADAPTION TO DOUBLE-QUANTITY: FROM PRE-CALCULUS TO PER-NUMBER CALCULATIONS

Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

A need for curricula for all students

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

Traditionally, a school system is divided into a primary school for children and a secondary school for adolescents, typically divided into a compulsory lower part, and an elective upper part having precalculus as its only compulsory math course. So, looking for a change we ask: how can precalculus be sustainably changed?

A Traditional Precalculus Curriculum

Typically, basic math is seen as dealing with numbers and shapes; and with operations transforming numbers into new numbers through calculations or functions. Later, calculus introduces two additional operations now transforming functions into new functions through differentiation and integration as described e.g. in the ICME-13 Topical Survey aiming to "give a view of some of the main evolutions of the research in the field of learning and teaching Calculus, with a particular focus on established research topics associated to limit, derivative and integral." (Bressoud et al, 2016)

Consequently, precalculus focuses on introducing the different functions: polynomials, exponential functions, power functions, logarithmic functions, trigonometric functions, as well as the algebra of functions with sum, difference, product, quotient, inverse and composite functions.

Woodward (2010) is an example of a traditional precalculus course. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

A Different Precalculus Curriculum

Inspired by difference research (Tarp, 2018) we can ask: Can this be different; is it so by nature or by choice?

In their 'Principles and Standards for School Mathematics' (2000), the US National Council of Teachers of Mathematics, NCTM, identifies five standards: number and operations, algebra, geometry, measurement and data analysis and probability, saying that "Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century (p. 2)." In the chapter 'Number and operations', the Council writes: "Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number (p. 7)."

Their biological capacity to adapt to their environment make children develop a number-language allowing them to describe quantity with two-dimensional block- and bundle-numbers. Education could profit from this to teach children primary school calculus that adds blocks (Tarp, 2018). Instead, it imposes upon children one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must be learned before it can be applied to operate on the functions introduced at the precalculus level.

However, listening to the Ausubel (1968) advice "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly (p. vi).", we might want to return to the two-dimensional block-numbers that children brought to school.

So, let us face a number as 456 as what it really is, not a one-dimensional linear sequence of three digits obeying a place-value principle, but three two-dimensional blocks numbering unbundled singles, bundles, bundles-of-bundles, etc., as expressed in the number-formula, formally called a polynomial:

T = 456 = 4*B^2 + 5*B + 6*1, with ten as the international bundle-size, B.

This number-formula contains the four different ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant 'double-numbers' or 'per-numbers'. We might call this beautiful simplicity 'the algebra square' inspired by the Arabic meaning of the word algebra, to re-unite.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Figure 01. The 'algebra-square' has 4 ways to unite, and 5 to split totals

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers, creating the operations integration and its reverse operation, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power, and its two reverse operations, root and logarithm.

Precalculus, building on or rebuilding?

In their publication, the NCTM writes "High school mathematics builds on the skills and understandings developed in the lower grades (p. 19)."

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they may have from the lower grades?

Furthermore, what kind of mathematics has been taught? Was it 'grounded mathematics' abstracted 'bottom-up' from its outside roots as reflected by the original meaning of 'geometry' and 'algebra' meaning 'earth-measuring' in Greek and 're-uniting' in Arabic? Or was it 'ungrounded mathematics' or 'meta-matics' exemplified 'top-down' from inside abstractions, and becoming 'meta-matism' if mixed with 'mathe-matism' (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept 'function', Euler saw it as a bottom-up name abstracted from 'standby calculations' containing specified and unspecified numbers. Later meta-matics defined a function as an inside-inside top-down example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may also be seen as an outside-inside bottom-up number-language sentence containing a subject, a verb and a predicate allowing a value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group operation applying the associative and commutative law as well as the neutral element and inverse elements, thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2:

2*u =6, so (2*u)*½ =6*½, so (u*2)*½ =3, so u*(2*½) =3, so u*1 =3, so u =3.

However, 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula 'T = (T/B)*B' (Tarp, 2018), present all over mathematics as a proportionality formula, thus solved in one step:

2*u = 6 = (6/2)*2, giving u = 6/2 = 3.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology.

Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as a number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar.

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction for an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

A Grounded Outside-Inside Fresh-start Precalculus from Scratch

Let students see that both the word-language and the number-language provide 'inside' descriptions of 'outside' things and actions by using full sentences with a subject, a verb, and an object or predicate, where a number-language sentence is called a formula connecting an outside total with an inside number or calculation, shortening 'the total is 2 3s' to 'T = 2*3';

Let students see how an outside degree of Many at first is iconized by an inside digit with as many strokes as it represents, five strokes in the 5-icon etc. Later the icons are reused when counting by bundling, which creates icons for the bundling operations as well. Here division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation. Writing 'y = f(x)' means that the y-number is found by specifying the x-number and the f-calculation. Thus, with x = 3, and with f(x) = 2+x, we get y = 2+3 = 5.

Let students see how calculations predict: how 2+3 predicts what happens when counting on 3 times from 2; how 2*5 predicts what happens when adding 2$ 5 times; how 1.02^5 predicts what happens when adding 2% 5 times; and how adding the areas 2*3 + 4*5 predicts adding the 'per-numbers' when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Solving Equations by Moving to Opposite Side with Opposite Sign

Let students see the subtraction 'u = 5-3' as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers 'to opposite side with opposite calculation sign'; a rule that applies also to the other reversed operations:

- the division u = 5/3 is the number u that multiplied with 3 gives 5, thus solving the equation u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, thus solving the equation u^3 = 5, and making root a 'factor-finder'
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, thus solving the equation 3^u = 5, and making logarithm a 'factor-counter'.

Let students see multiple calculations reduce to a single calculation by unhiding 'hidden brackets' where 2+3*4 = 2+(3*4) since, with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s.

This prevents solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5. Allowing to unhide the hidden bracket we get:

2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4

This solution is verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy a 'Hymn to Equations': "Equations are the best we know, they're solved by isolation. But first the bracket must be placed, around multipli-cation. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don't want to stop!"

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let students place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, three terms disappear if adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = b^2/4 – c.

Moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c, so u + b/2 = $\pm\sqrt{b^2/4 - c}$, so u = -b/2 $\pm\sqrt{b^2/4 - c}$

Recounting Grounds Proportionality

Let students see how recounting in another unit may be predicted by a recount-formula 'T = (T/B)*B' saying "From the total T, T/B times, B may be pushed away" (Tarp, 2018). In primary school this formula recounts 6 in 2s as 6 = (6/2)*2 = 3*. In secondary school the task is formulated as an equation u*2 = 6 solved by recounting 6 in 2s as u*2 = 6 = (6/2)*2 giving u = 6/2, thus again moving 2 'to opposite side with opposite calculation sign'.

Thus an inside equation u*b = c can be 'demodeled' to the outside question 'recount c from ten to bs', and solved inside by the recount-formula: u*b = c = (c/b)*b giving u = c/b.

Let students see how recounting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. And see how filling a circle with right triangles from the inside allows phi to be found from a formula: circumference/diameter = □ ≈ n*tan(180/n) for n large.

Double-counting Grounds Per-numbers and Fractions

Let students see how double-counting in two units create 'double-numbers' or 'per-numbers' as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number:

- Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg.
- Asking '9kg = ?$' we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Let students see how double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 40%.

To find 40% of 20$ means finding 40$ per 100$, so we re-count 20 in 100s:

T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$:

T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

Let students see how double-counting physical units create per-numbers all over STEM (Science, Technology, Engineering and mathematics):

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter;
- meter = (meter/second) * second = velocity * second;
- joule = (joule/second) * second = watt * second

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, T = 456 = 4*B^2 + 5*B + 6*1, containing the formulas for constant change:

T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change. So where constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics to 'post-dict' what we can't 'pre-dict'; and using confidence for predicting intervals.

Combining linear and exponential change by n times depositing a$ to an interest percent rate r, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest percent rate R is predicted by the formula 1+R = (1+r)^n. This saving may be used to neutralize a debt Do, that in the same period changes to D = Do*(1+R).

This formula and its proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Uniting Constant Per-Numbers as Factors

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$, changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, the formula for change with a constant change-factor, also called exponential change.

Reversing the question, this formula entails two equations. Asking 600 = 300*a^5, we look for an unknown change-factor. So here the job is ‘factor-finding’ which leads to defining the fifth root of 2, i.e. 5√2, found by moving the exponent 5 to opposite side with opposite calculation sign, root.

Asking instead 600 = 300*1.2^n, we now look for an unknown time period. So here the job is ‘factor-counting’ which leads to defining the 1.2 logarithm of 2, i.e. log1.2(2), found by moving the base 1.2 to opposite side with opposite calculation sign, logarithm.

Calculus Deals with Uniting Changing Per-Numbers as Areas

In mixture problems we ask e.g. ‘2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?’ Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before added, thus adding by areas. So here multiplication precedes addition.

Asking inversely ‘2kg at 3$/kg + 4kg at how many $/kg gives 6kg at 5 $/kg?’, we first subtract the areas 6*5 – 2*3 before dividing by 4, a combination called differentiation, ΔT/4, thus meaningfully postponed to after integration.

Statistics Deals with Unpredictable Change

Once mastery of constant change is established, it is possible to look at time series in statistical tables asking e.g. “How has the unemployment changed over a ten-year period?” Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b+a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n.

The average numbers allow calculating all totals in the period, assuming the numbers are predictable. However, they are not, so instead of predicting the number with a formula, we might ‘post-dict’ the numbers using statistics dealing with unpredictable numbers. This, in turn, offers a likely prediction interval by describing the unpredictable random change with nonfictional numbers, median and quartiles, or with fictional numbers, mean and standard deviation.

Calculus as adding per-numbers by their areas may also be introduced through cross-tables showing real-world phenomena as unemployment changing in time and space, e.g. from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add. First, they must multiply back to unit-numbers to find the total percent. However, multiplying creates areas, so per-numbers add by areas, which is what calculus is about.

Modeling in Precalculus Exemplifies Quantitative Literature

Furthermore, graphing calculators allows authentic modeling to be included in a precalculus curriculum thus giving a natural introduction to the following calculus curriculum, as well as introducing ‘quantitative literature’ having the same genres as qualitative literature: fact, fiction and fiddle (Tarp, 2001).

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator thus using a natural science proof, valid until finding counterexamples.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A' = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A' = x' = 1; and that if A = ½*x^2, then y = A' = (½*x^2)' = x; and that if A = 1/3*x^3, then y = A' = (1/3*x^3)' = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to 3, instead of adding small strips we just calculate the change in the area over this distance, later called the fundamental theorem of calculus.

A Literature Based Compendium

An example of an ideal precalculus curriculum is described in 'Saving Dropout Ryan With a Ti-82' (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' (Tarp, 2009) replaced the textbook. A formula's left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

An Example of a Fresh-start Precalculus Curriculum

This example was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

An Example of an Exam Question

Authentic material was used both at the written and oral exam. The first had specific, the second had open questions as the following asking 'What does the table tell?

Agriculture: Number of agricultural farms allocated over agricultural area

	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977
Farms in total	**161142**	**154 694**	**148 512**	**144 070**	**143093**	**141 137**	**137712**	**13424S**	**130 7S3**	**127117**
0,0- 4,9 ha	25 285	23 493	21 533	21623	22123	21872	21093	19915	18 852	17 833
5.0- 9.9-	34 644	32129	30 235	28 404	27693	26 926	26109	25072	24066	23152
10,0-19.9-	48 997	46482	43 971	41910	40850	39501	38261	36 702	35 301	34 343
20.0-29.9-	25670	25 402	25181	24 472	24 195	23 759	23 506	23134	22737	22376
30,0-49.9-	18 505	18 779	18 923	18 705	18 968	18 330	19 095	19 304	10 305	19 408
50,0-99.9-	6 552	6 852	7 076	7 275	7 549	7956	7 847	8247	8 556	8723
100.0 ha and over	1489	1 557	1611	1681	1 715	1791	1801	1871	1934	1882

Figure 02. A table found in a statistical survey used at an oral exam.

Discussion and Conclusion

Asking "how can precalculus be sustainably changed?" an inside answer would be: "By its nature, precalculus must prepare the ground for calculus by making all function types available to operate on. How can this be different?"

An outside answer could be to see precalculus, not as a goal but as a means, an extension to the number-language allowing us to talk about how to unite and split into changing and constant per-numbers. This could motivate renaming precalculus to per-numbers calculations.

In this way, precalculus becomes sustainable by dealing with adding, finding and counting change-factors using power, roots and logarithm. Furthermore, by including adding piecewise constant per-numbers by their areas, precalculus gives a natural introduction to calculus by letting integral calculus precede and motivate differential calculus since an area changes with the last strip, thus connecting the unit number, the area, with the per-number, the height.

Finally, graphing calculators allows authentic modeling to take place so that precalculus may be learned through its use, and through its outside literature.

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28. A Lyotardian Dissension to the Early Childhood Consensus on Numbers and Operations

Can Sociological Imagination Improve Mathematics Education?

Decreasing Swedish PISA results made OECD (2015) write the report ‘Improving Schools in Sweden’ describing its school system as “in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)”

As a social institution, mathematics education might improve by inspiration form sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology; and also emphasized in Lyotard’s report on knowledge in a postmodern digitalized condition (1984):

> “We no longer have recourse to grand narratives (..) But as we have seen, the little narrative remains the quintessential form of imaginative invention most particularly in science. In addition, the principle of consensus as a criterion of validation seems to be inadequate. (..) consensus is a component of the system, which manipulates it in order to maintain and improve its performance. It is the object of administrative procedures (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. The problem is therefore to determine whether it is possible to have a form of legitimation based solely on paralogy. Paralogy must be distinguished from innovation: the latter is under the command of the system, or at least used by it to improve its efficiency; the former is a move (the importance of which is often not recognized until later) played in the pragmatics of knowledge. (..) It is necessary to posit the existence of a power that destabilizes the capacity for explanation, manifested in the promulgation of new norms for understanding (p. 60-61).”

As a number-language, mathematics would create a paradigm shift (Kuhn, 1962) if copying the communicative turn in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by connecting to its outside world before its inside grammar,

In the workshop we focus on early childhood mathematics education as described in the ICME study 23 (Sun et al, 2015); and with a dissension by Tarp (2018).

Consensus and Dissension on Early Childhood Numbers & Operations

Question 01: There seems to be a consensus saying ‘Of course numbers must be learned before being applied in numbering. And as one-dimensional, numbers are names for points along a number line obeying a place value principle when containing more digits’. Thus, a dissension may ask: ‘From the age around four, children seem to distinguish between four ones and two twos thus developing double-numbers with units when adapting to outside quantity. So, why not develop the double-numbers with units children bring to school?’

Question 02: There seems to be a consensus saying ‘Of course addition must be learned before subtraction, multiplication and division since they are all defined from addition’. Thus, a dissension may ask: “Counting an outside total in bundle-counted by a broom pushing away the bundles, iconized as division, to be stacked by a lift iconized as multiplication, to be pulled away by a rope iconized as subtraction, thus finding unbundled singles that placed next-to or on-top the block roots decimals, fractions and negative numbers. This creates a ‘recount-formula’ T = (T/B)xB saying ‘From T, T/B times, B is pushed away’, present all over mathematics and science. Once counted, blocks may be added, but on-top needing units to be changed by recounting, or next-to as areas as in integral calculus? This ambiguity leaves addition not that well defined. So, why not accept the opposite order of the operations as the natural?’

Question 03: There seems to be a consensus saying ‘Of course functions are postponed to secondary school since their algebra builds upon the algebra of letter expressions.’ Thus, a dissension may ask: ‘The word- and the number-language both offer an inside description of an outside object or action by using sentences with a subject, a verb and a predicate, abbreviating ‘the total is 2 3s’ to ‘T = 2x3’. So, why not use functions as number-language sentences from the start?

Time Table for the Workshop

A 20minutes introduction will focus on the core question: As to the goal of mathematics education, is it to master inside mathematics as the means to later master outside Many; or is it to master outside Many by choosing among its three inside versions; the present setcentric Skemp-based 'meta-matics' defining concepts as examples of abstractions instead of as abstractions from examples, the pre setcentric Skinner-based 'mathe-matism' true inside but seldom outside classrooms by adding numbers and fractions without units; and the post setcentric Lyotard-based 'many-math', accepting the number-language children develop when adapting to Many before school.

A 30minutes group discussion on the three questions below is followed by 20 minutes in exchange-groups, and a 20minutes plenum for summing up.

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29. Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis?

Does Mathematics Education Research have an Irrelevance Paradox?

The Swedish Centre for Mathematics Education is meant to mediate research findings and facilitate their implementation. Still, decreasing Swedish PISA results made OECD (2015) write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life’ (p. 3).

Increasing research together with decreasing student performance points to an ‘irrelevance paradox’ in mathematics education research, possibly caused by peer reviewing failing to assure research quality. The so-called ‘replication crisis’ suggests that this might indeed be the case. First noticed in medical science, the crisis may also occur in schools seen by Foucault (1995) as ‘pris-pitals’, i.e. prison-like hospitals using education to cure humans from the diagnose ‘uneducated’.

Consequently, there is a need for a workshop discussing this hypothesis, as well as ways to make peer reviewed conferences produce more quality. We may ask: When mathematics itself has abandoned peer review, why shouldn’t also mathematics education?

The Replication Crisis in Science

In the article “How Science goes Wrong”, The Economist writes:

> A rule of thumb among biotechnology venture-capitalists is that half of published research cannot be replicated. Even that may be optimistic. Last year researchers at one biotech firm, Amgen, found they could reproduce just six of 53 "landmark" studies in cancer research. (..) The most enlightened journals are already becoming less averse to humdrum papers. (..) But these trends need to go much further. Journals should allocate space for "uninteresting" work, and grant-givers should set aside money to pay for it. Peer review should be tightened - or perhaps dispensed with altogether, in favour of post-publication evaluation in the form of appended comments. That system has worked well in recent years in physics and mathematics (The Economist, 19 Oct. 2013).

The replication crisis thus comes from the ‘metascience’ observation that many research studies are difficult or impossible to replicate or reproduce. It applies to different fields, e.g. psychology where Pashler and Wagenmakers (2012) writes:

> Is there currently a crisis of confidence in psychological science reflecting an unprecedented level of doubt among practitioners about the reliability of research findings in the field? It would certainly appear that there is (p. 528).

The authors refer among others to Ioannidis (2005) who writes:

> Scientists in a given field may be prejudiced purely because of their belief in a scientific theory or commitment to their own findings. Many otherwise seemingly independent, university-based studies may be conducted for no other reason than to give physicians and researchers qualifications for promotion or tenure. (..) Prestigious investigators may suppress via the peer review process the appearance and dissemination of findings that refute their findings, thus condemning their field to perpetuate false dogma (p. 0698).

As to the peer review process, LeBel (2015) writes:

> In recent years, there has been a growing concern regarding the replicability of findings in psychology (..) I propose a new replication norm that aims to further boost the dependability of findings in psychology (p. 1).

Addressing case series studies, Horton (1996) writes:

> The importance of the case series in surgical research is beyond doubt. Therefore, it seems reasonable to ask whether we can trust this study method to yield a valid result. According to conventional epidemiological wisdom, the answer is no (p. 984).

The quality of research was also questioned by Lyotard (1984) distinguishing between consensus and dissension:

> Consensus is a component of the system, which manipulates it (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. (..) Returning to the description of scientific pragmatics, it is now dissension that must be emphasized (p. 60-61).

Time Table for the Workshop

A 20minute introduction to the replication crisis and to conflicting theories within sociology, psychology and philosophy also includes examples on peer-reviews from MADIF 10, CERME 11, ICMT 3, and a journal (Tarp, 2018); and a proposal for a 'Salon des Refusés' created in France in 1863 to display rejected paintings later inspiring important innovation.

Then a 30minutes group discussion will use a short reader with excerpts of the authors cited above to discuss questions as: What kind of dissension risks being silenced by a peer review consensus? Will master-level papers applying existing theory oust research-level papers questioning or expanding it? Also, the groups are invited to exchange experiences on peer reviews; and to exchange opinions on how to increase the quality of the peer review process.

The next 20minutes, the groups split up to join the other groups to exchange views. Finally, a 20minutes plenum will sum up and formulate recommendations as to how to add quality to the coming MADIF sessions.

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30. A BUNDLE COUNTING TABLE

A guide to bundle-counting in pre-school.

Bundle-counting clarifies that we count by bundling, typically in tens

Example 01. Counting Mikado Sticks

The Mikado sticks are positioned next to each other to the right. Counting is done by taking one stick at a time to the left and assembling them in a bundle with an elastic band when we reach ten.

When counting, we say: "0 Bundle 1, 0 bundle 2,. .. "

"Why 0 bundle?" "Because we don't have a bundle yet, before we'll reach ten."

"..., 0 bundle 8, 0 bundle 9, 0 bundle ten, well no, 1 bundle 0".

Example 02. Counting matches

The box says 39, which we read as '3 bundles 9'. We bundle-count as with Mikado sticks.

Extra-option

Some children may find it fun later to count ' 1 bundle less 2, 1 bundle less 1, 1 bundle and 0, 1 bundle and 1 ' as a new way to count ' 0 bundle 8, 0 bundle 9, 1 bundle 0, 1 bundle 1 '. Later again, some children may find It fun to say ' 1 bundle-bundle 0 ' instead of ' ten bundles 0 ' or ' hundred '.

Example 03. Counting ten fingers or ten matches

The ten fingers (or ten matches) bundle are counted in 4s and in 3s while saying "The total is..." and possibly writing "T =..."

Ten counted in 4s	Ten counted in 3s
T = I I I I I I I I I I = ten 1s	T = III I I I I I I I = 1B7 3s
T = IIIIIIIIII = 1 tens = 1B0 tens	T = III III I I I I = 2B4 3s
T = IIII I I I I I I = 1B6 4s	T = III III III I = 3B1 4s
T = IIII IIII I I = 2B2 4s	T = III III III III = 4B-2 3s
T = IIII IIII IIII = 3B-2 4s	T = III III III I = 1BB 0B 1 3s

Table for counting ten tens, or 1 bundle bundles, or 1 hundred:

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~

31. PROPOSALS FOR THE 2020 SWEDISH MATH BIENNALE

Start-math for children and migrants: bundle-count and recount before adding

Assembling 4 fingers 2 and 2, a 3-year-old will protest: "It is not 4, but two 2s". The child counts in bundle- and block-numbers just like we: 456 = 4 Bundle Bundles + 5 bundles + 6 unbundled. And recounts 3 4s to 5s, which leads to proportionality. And recounts 42 to 7s, which leads to equations. And adds 2 3s and 4 5s to 3Bundle2 8s that leads to calculus. The child is directed directly to core mathematics if allowed to keep its 2D bundle- and block-numbers, and to count and recount before adding.

Start-matte for børn og migranter: Bundt-tæl og om-tæl før addition

Samles 4 fingre 2 og 2, vil en 3årig protestere: "Det er ikke 4, men to 2ere". Barnet tæller i bundt- og bloktal ligesom vi: 456 = 4 bundtbundter + 5 bundter + 6 ubundtede. Og om-tæller 3 4ere til 5ere, der fører til proportionalitet. Og om-tæller 42 til 7ere, der fører til ekvationer. Og adderer 2 3ere og 4 5ere til 3Bundt2 8ere, der fører til calculus. Barnet føres direkte til kernematematikken hvis det må beholde sine 2D bundt- og bloktal, samt må tælle og om-tælle før addition.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

Forskningen i matematikuddannelse er vokset siden dens første internationale kongres ICME1 i 1969. Ligeledes har finansiering, se fx 'National Center for Matematik'. På trods af ekstra forskning og finansiering og til trods for at være blevet advaret mod den mulige irrelevans af en voksende forskningsindustri (Tarp, 2004) har faldende svenske PISA-resultater forårsaget OECD til at skrive rapporten "Improving Schools in Sweden" (2015), der beskriver den svenske skolens som "havende brug for akut ændring", da "mere end en ud af fire studerende ikke engang opnår basisniveauet 2 i matematik, hvor eleverne begynder at demonstrere kompetencer for aktivt at deltage i livet (s. 3)."

Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

Fantasi som kernen i sociologi er beskrevet af Mills (1959). Bauman (1990) er enig ved at sige, at sociologisk tænkning "genindfører fleksibilitet til en verden, der er fastfrosset i rutiner ved at vise en alternativ verden, som den kunne være forskellig fra hvad den er nu (s. 16). "

Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

Et sådant eksempel er at sige, at formålet med matematikuddannelse er at lære matematik, da en sådan målsætning er åbenbart meningsløs ved sin selvreferencemåde.

Forbindelsen mellem et mål og dets midler er også til stede i den eksistentialistiske filosofi, der er beskrevet af Sartre (2007) som at fastholde, at "Eksistens går forud for essens (s. 20)". På samme måde påpeger Arendt (1963) at det at praktisere et middel blindt uden at reflektere over sit mål kan føre til at praktisere "ondskabets banalitet". Tilsvarende siger Bourdieu (1977) at "Alle pædagogiske handlinger er objektivt symbolsk vold, for så vidt som det er påtvingelse af en kulturel vilkårlighed ved en vilkårlig magt (s. 5)". Dette rejser spørgsmålet om matematik og uddannelse er universelt eller valgt, mere eller mindre vilkårligt.

Inspireret af de gamle græske sofister, der ønsker at undgå at blive patroniseret af valg præsenteret som natur, søger 'differensforskning' efter skjulte forskelle, der gør en forskel (Tarp, 2017). For at undgå en målforskydning i matematikuddannelsen spørger differensforskning:

Hvordan ville matematikken se ud, hvis den grundfæstes i sin udvendige rod, den fysiske faktum Mange?

For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Bundt-tæl og om-tæl før addition.

Cifre samler mange streger til ét ikon: Fem streger i 5tallet, osv. indtil ti = 1bundt0, 10.

Med en kop til bundter kan en total T på 7 pinde bundt-tælles i ikon-bundter, fx T = 7 = 2 3ere & 1 = 2B1 3ere. Herefter kan totalen om-tælles i samme enhed og skabe overlæs og underlæs: T = 7 = 2B1 3ere = 1B4 3ere = 3B-2 3ere.

En total kan også om-tælles i en ny enhed (proportionalitet), fx 2 4ere = ? 5ere, forudsagt af en regner som 2*4/5 = 1, og 2*4-1*5 = 3, altså T = 2 4ere = 1B3 5ere.

Vi tæller ved at bundte og stakke forudsagt med regnearter, som også er ikoner: Ved op-tælling af en total T i B-bundter, T/B, viser division den kost, der fra T fejer Bere væk. Multiplikation er den kran der løfter bundter op i en stak, og subtraktion er den snor, der trækker stakken væk for at finde de ubundtede. Resultatet kan derfor forudsiges af en 'omtællings-formel' T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere'.

Om-tælling fra ikon-bundter til 10ere fører til multiplikationstabellen: T = 3 4ere = 3*4 = 12 = 1ti2 = 1B2 10ere.

Tilbage-tælling fra 10ere til ikon-bundter bliver til ligninger, som løses ved at bruge om-tællingsformlen: 'Hvor mange 5ere giver 40' fører til ligningen: x*5 = 40, der løses ved at om-tælle 40 til 5ere: 40 = (40/5)*5, så x = 40/5. Så en ligning løses ved at flytte til modsat side med modsat regnetegn. For flere detaljer, se det web-baserede lærerakademi MATHeCADEMY.net og MrAlTarp YouTube videoer.

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Counting before adding will strengthen the number sense by children and migrants

We master many with a number-language sentences, formulas, e.g. T = 4 5ere = 4*5. Which shows that we enumerate totals T by bundling and stacking. So, 4*5 is 4 5s that can be recounted to another unit, e.g. 7s.

Math issues are prevented by bundle-numbers that can be trained as counting '6,..., 10' also as 'bundle less 4, B-3, B-2, B-1, Bundle'. And '10,..., 15' as 'Bundle, 1left, 2left, 3left, 4left, 5left' to show that 'eleven' and 'twelve' originate from Viking counting.

Tælling før addition styrker talsansen hos børn og migranter

Vi mestrer Mange med et tal-sprog med talsprogs-sætninger, formler, fx T = 4 5ere = 4*5. Som viser, at vi ital-sætter totaler T ved at bundte og stakke. Så 4*5 er altså 4 5ere, der kan om-tælles til en anden enhed, fx 7ere.

Matte-problemer forebygges med bundt-tal. Og kan indøves ved at ’6, …, 10’ også tælles som ’bundt pånær 4, B-3, B-2, B-1. B’. Og ’10, …, 15’ som ’bundt, 1levnet, 2levnet, 3levnet, 4levnet, 5levnet’ for at vise, at ’eleven’ og ’twelve’ stammer fra vikingetiden.

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Vi mestrer Mange med et tal-sprog med talsprogs-sætninger, formler, fx T = 4 5ere = 4*5. Som viser, at vi ital-sætter totaler T ved at bundte og stakke. Så 4*5 er altså 4 5ere, der kan om-tælles til en anden enhed, fx 7ere. Eller tiere, som er den internationale bundt-størrelse.

At se tal som bundt-formler gør matte let og forebygger matte-problemer og dyskalkuli. Og bør derfor indøves via forskellige tælleremser, så ’5, 6, 7, 8, 9, 10’ også tælles som ’5, bundt pånær 4, B-3, B-2, B-1. B', og som '½budt, ½bubndt&1, ½bB&2, ½B&3, ½B&4, Bundt. Og '10, 11, 12, 13, 14, 15' tælles som 'bundt, 1bundt&1, 1B&2, 1B&3,1B&4, 1B&5', og som 'bundt, 1levnet, 2levnet, 3levnet, 4levnet, 5levnet' for at vise, at 'elleven' og 'twelve' stammer fra vikingetiden.

Cifre samler mange streger til ét ikon: Fem streger i 5tallet, osv. indtil ti = 1bundt0, 10.

Med en kop til bundter kan en total T på 7'bundt-tælles' i ikon-bundter, fx T = 7 = 2B1 3ere. Herefter kan totalen om-tælles i samme enhed og skabe overlæs og underlæs: T = 7 = 2B1 3ere = 1B4 3ere = 3B-2 3ere. Tilsvarende med totaler optalt i tiere, T = 68 = 6B8 = 5B18 = 7B-2 tiere.

Før addition opøves talsansen med multiplikationstabellen, som reduceres til en 5x5-tabel ved at omskrive tal over 5, fx 6 = ½bundt&1 = bundt-4. Først fordobling, fx T = 2*7 = 2*(½bundt&2) = bundt&4 = 14, eller T = 2*7 = 2*(bundt-3) = 20-6 = 14. Herefter med bundt-tælling, fx T = 2*38 = 2*3B8 = 6B16 = 7B6 = 76. Så halvering, fx ½*38 = ½* 3B8 = ½*4B-2 = 2B-1 = 19.

At gange med 5 er at gange med halve bundter, 5*7 = ½bundt*7 = ½70 = ½ af 6B10 = 3B5 = 35.

For flere detaljer, se det web-baserede lærerakademi MATHeCADEMY.net og MrAlTarp YouTube videoer.

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Division dislike cured with 5 sticks and 1 cup and bundle-writing

A class dislikes division, e.g. 336/7. The solution is to see 336/7, not as 336 divided between 7, but as 336 counted in 7s; and to use bundle-writing 336 = 33B6 = 28B56, since a total can be recounted in three ways: normal and with overload or underload. Now, with T = 336 = 33B6 = 28B56, we have T/7 = 4B8 = 48.

Recounting may be trained with bundle-counting 5 sticks in 2s.

Normal: T = 5 = 2B1 2s. With overload: T = 5 = 1B3 2s. With underload: T = 5 = 3B-1 2s.

Likewise with: T = 74 = 7B4 = 6B14 = 8B-6.

Ulyst til division kureret med 5 pinde og 1 kop og bundt-skrivning

En klasse har problemer med division, fx 336/7. Løsningen er at opfatte 336/7, ikke som 336 delt mellem 7, men som 336 optalt i 7ere; samt benytte bundt-skrivning 336 = 33B6 = 28B56, idet totaler kan omtælles på tre måder: normal og med overlæs eller underlæs. Så med T = 336 = 33B6 = 28B56, er T/7 = 4B8 = 48.

Omtælling indøves med 5 pinde, som bundt-tælles i 2ere.

Normal: T = 5 = 2B1 2ere. Med overlæs: T = 5 = 1B3 2ere. Med underlæs: T = 5 = 3B-1 2ere.

Ligeledes med: T = 74 = 7B4 = 6B14 = 8B-6.

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

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Fokus: Ulyst til division kureret

En klasse har problemer med division, fx 336/7. Løsningen er at opfatte 336/7, ikke som 336 delt mellem 7, men som 336 optalt i 7ere; samt benytte bundt-skrivning 336 = 33B6, hvor koppen opdeler totalen i bundtede inden for koppen og u-bundtede udenfor.

Samt ved øvelser i at 'bundt-tælle' totaler på tre måder: normal og med overlæs eller underlæs.

Først med 5 pinde, som bundt-tælles i 2ere med en kop til bundterne.

Normal: T = IIIII = II II I = 2B1 2ere. Med overlæs: T = IIIII = II III = 1B3 2ere. Med underlæs: T = IIIII = II II I I̶ = 3B-1 2ere.

På samme måde hvis vi optæller i 10ere: T = 74 = 7B4 = 6B14 = 8B-6.

Så med en total på 336 (dvs. 33.6 tiere) er der 33 bundter indenfor koppen og 6 ubundtede udenfor. Men vi fortrækker 28 indenfor, så 5 bundter flytter udenfor som 50, dvs. nu med 56 udenfor, som divideret med 7 giver 4 indenfor og 8 udenfor:

T = 336 = 33B6 = 28B56, og T/7 = 4B8 = 48.

Bundt-skrivning kan bruges ved alle regne-operationer.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

Bundt-skrivning kan også bruges ved multiplikationstabellen:

T = 4*8 = 4*1B-2 = 4B-8 = 32 og 7*8 = 7*1B-2 = 7B-14 = 6B-4 = 5B6 = 56

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Fractions and percentages as per-number

Factions dislike disappear if viewing a fraction as a per-number obtained from double- counting in the same unit, 3/5 = 3$ per 5$; or as percent 2% = 2/100 = 2$ per 100$.

Recounting and double-counting uses the 'recount-formula' T = (T/B)*B, saying 'From the total T, T/B times, Bs can be pushed away.'

To find 2/3 of 12 means finding 2$ per 3$ of 12$. Here 12 recounts in 3s as 12$ = (12/3)*3$, giving (12/3)*2kr = 8$. So 2/3 of 12 is 8.

Brøker og procenter som per-tal

Problemer med brøker forsvinder ved at se en brøk som et per-tal, der fremkommer ved en dobbelt-tælling i samme enhed, 2/3 = 2kr per 3kr, eller som procent 2% = 2/100 = 2kr per 100kr.

Ved om-tælling og dobbelt-tælling bruges tælle-formlen T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere'.

Herved findes 2/3 af 12 som 2kr per 3kr af 12 kr. Altså ved at om-tælle i 12 i 3ere som 12kr = (12/3) * 3kr, der giver (12/3) * 2kr = 8kr. Så 2/3 af 12 er 8.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

Fantasi som kernen i sociologi er beskrevet af Mills (1959). Bauman (1990) er enig ved at sige, at sociologisk tænkning "genindfører fleksibilitet til en verden, der er fastfrosset i rutiner ved at vise en alternativ verden, som den kunne være forskellig fra hvad den er nu (s. 16). "

Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

Et sådant eksempel er at sige, at formålet med matematikuddannelse er at lære matematik, da en sådan målsætning er åbenbart meningsløs ved sin selvreferencemåde.

Forbindelsen mellem et mål og dets midler er også til stede i den eksistentialistiske filosofi, der er beskrevet af Sartre (2007) som at fastholde, at "Eksistens går forud for essens (s. 20)". På samme måde påpeger Arendt (1963) at det at praktisere et middel blindt uden at reflektere over sit mål kan føre til at praktisere "ondskabets banalitet".

Inspireret af de gamle græske sofister, der ønsker at undgå at blive patroniseret af valg præsenteret som natur, søger 'differensforskning' efter skjulte forskelle, der gør en forskel (Tarp, 2017). For at undgå en målforskydning i matematikuddannelsen spørger differensforskning: Hvordan ville matematikken se ud, hvis den grundfæstes i sin udvendige rod, den fysiske faktum Mange?

For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Brøker og procenter som per-tal.

En klasse har problemer med brøker. Dels med at finde en brøkdel af en total, dels med at forlænge og forkorte, hvor mange adderer og subtraherer i stedet for at multiplicere og dividere.

Løsningen er at se en brøk som et per-tal, der fremkommer ved en dobbelt-tælling i samme enhed, 2/3 = 2kr per 3kr, eller som procent 2% = 2/100 = 2kr per 100kr.

Ved investering forventes et afkast, der kan være højere eller lavere, fx 7kr pr 5kr eller 3kr per 5kr.

Ved om-tælling og dobbelt-tælling bruges tælle-formlen T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere'.

Herved findes 2/3 af 12 som 2kr per 3kr af 12 kr. Altså ved at om-tælle i 12 i 3ere som 12kr = (12/3) * 3kr, der giver (12/3) * 2kr = 8kr. Så 2/3 af 12 er 8.

Opgaven' Hvor mange procent er 3 per 5?' løses ved at om-tælle 100 i 5ere or erstatte 5kr med 3kr: T = 100kr = (100/5) * 5kr, der giver (100/5) * 3kr = 60kr. Så 3/5 er det samme som 60 pr. 100, eller 3/5 = 60%.

At forlænge eller forkorte brøker sker ved at indsætte eller fjerne den samme enhed ovenfor og nedenfor brøklinjen: T = 2/3 = 2 4ere / 3 4ere = (2*4)/(3*4) = 8/12; og T = 8/12 = 4 2ere / 6 2ere = 4/6.

Faktisk kan og bør brøker og decimaltal introduceres i første klasse i forbindelse med optælling i ikoner under ti. 7 optalt i 3ere giver en stak på 2 3ere samt 1. Anbringes denne ved siden af i sin egen stak, fås et decimaltal, T = 7 = 2.1 3ere. Anbringes den ovenpå optalt som 3ere, fås en brøk: T = 7 = 2 1/3 3ere.

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Fractions and per-numbers added as integration

Factions dislike disappear if viewing a fraction as a per-number obtained from double- counting in the same unit, 3/5 = 3$ per 5$; and by respecting that also fractions are added with units: 1/2 of 2 + 2/3 of 3 gives 3/5 of 5. And not 7/6, as the school says.

Per-numbers also ad with units: 2kg at 3$/kg + 4kg at 5$/kg gives 6kg at (2*3$ + 4*5$)/6kg. Thus, here 3$/kg+5$/kg = 4.33$/kg, where the per-number add as the area under the piecewise constant per-number graph, called integration) Similarly, with fractions.

Brøker og per-tal adderet som integration

Problemer med at addere brøker forsvinder ved at se en brøk som et per-tal fremkommet fra dobbelt-tælling i samme enhed, 3/5 = 3kr per 5kr. Samt ved at respektere, at brøker adderes med enheder: ½ af 2 + 2/3 af 3 giver 3/5 af 5. og ikke 7/6, som skolen siger.

Også per-tal adderes med enheder: 2kg á 3kr/kg + 4kg á 5kr/kg giver 6kg á (2*3kr + 4*5kr)/6kg. Da 3kr/kg + 5kr/kg = 4.33kr/kg, adderes per-tal som arealet under den stykkevis konstante per-tals kurve (integration). Tilsvarende med brøker.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

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For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Brøker og per-tal adderet som integration.

En klasse har problemer med at addere brøker. Mange adderer tæller og nævner hver for sig.

Løsningen er at se en brøk som et per-tal fremkommet fra dobbelt-tælling i samme enhed, 3/5 = 3kr per 5kr, eller som procent 3% = 3/100 = 3kr per 100kr.

Samt at begynde med at addere brøker med enheder, som fx ½ af 2 + 2/3 af 3, der netop giver 1+2 af 2+3, altså 3/5 af 5. Her er altså ½+2/3 = 3/5, som fås ved at addere tæller og nævner hver for sig.

Tilsvarende adderes per-tal med enheder: 2kg á 3kr/kg + 4kg á 5kr/kg. Her adderes styktallene 2kg og 4kg direkte til 6kg, medens pertallene skal opganges til styktal før de kan adderes: 3*2kr

+ 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg. Så her er 3kr/kg + 5kr/kg = 4.33kr/kg, kaldet det vægtede gennemsnit.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tals kurve. Tilsvarende med brøker.

At addere to brøker a/b og c/d uden enheder er i princippet meningsløst, men kan gives mening ved at brøkerne tages af den samme total, b*d. Man får da additionen:

a/b af b*d + c/d af b*d, hvilket giver en total på a*d + c*b af b*d. Altså er a/b + c/d = (a*d + c*b)/b*d.

At addere brøker og pertal med enheder giver en god introduktion til calculus. Som vist er multiplikation før addition det samme som integration. Og omvendt integration er det samme som differentiation: Opgaven 2kg á 3kr/kg + 4kg á ?kr/kg = 6 kg á 5kr/kg fører til 6 + 4*x = 30 eller T1 + 4*x = T2, som løses med subtraktion før division, altså differentiation: x = (T2-T1)/4 = ∆T/4.

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Proportionality as double-counting with per-numbers

Proportionality dislike disappears by renaming it to ‘unit-shift by double-counting’, which leads to ‘per-numbers’ such as e.g. 2$ per 3kg or 2$/3kg or 2/3 $/kg. Recounting uses the ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away.’

Thus, a total of 16$ is recounted kg as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg can be recounted in $ as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Proportionalitet som dobbelt-tælling med per-tal

Problemer med proportionalitet forsvinder ved at omdøbe proportionalitet til ’enheds-skift ved dobbelt-tælling’, som fører til ’per-tal’ som fx 2kr pr 3kg eller 2kr / 3 kg eller 2/3 kr/kg. Til om-tælling bruges ’omtællings-formlen’ T = (T/B)*B, der siger: ’Fra T kan vi T/B gange fjerne Bere’.

Herved kan 16kr om-tælles i 2ere som T = 16kr = (16/2) * 2kr = (16/2) * 3kg = 24 kg. Ligeledes kan de 12kg om-tælles i 3ere som T = 12kg = (12/3) * 3kg = (12/3) * 2kr = 8kr.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

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Fokus: Proportionalitet som dobbelt-tælling med per-tal

En klasse har problemer med proportionalitet. Prisen er 2kr / 3kg. Alle finder kr-tallet for 12kg, men kun få finder kg-tallet for 16kr. Løsningen er at omdøbe proportionalitet til 'enheds-skift' ved ' dobbelt-tælling', som fører til 'per-tal' som fx 2kr pr 3kg eller 2kr / 3 kg eller 2/3 kr/kg. Enhederne forbindes så ved at om-tælle det kendte antal i per-tallet.

Op-tælling og om-tælling bruger begge 'tælle-formlen' T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere'.

Herved kan 16kr om-tælles i 2ere som T = 16kr = (16/2) * 2kr = (16/2) * 3kg = 24 kg. Ligeledes kan de 12kg om-tælles i 3ere som T = 12kg = (12/3) * 3kg = (12/3) * 2kr = 8kr. Vil denne forskel gøre en forskel? I teorien, ja, da proportionalitet forbindes med optælling, en basal fysisk aktivitet.

Faktisk findes proportionalitet i første klasse ved at op-tælle totaler i ikon-bundter forskellig fra standardbundtet ti og ved bagefter at om-tælle i en ny enhed. Dette fører direkte til tælleformlen, der har samme form som y = k*x.

Således kan en total på 8 op-tælles i 2ere som T = (8/2)*2 = 4*2 = 4 2ere.

Og en total på 3 4ere kan om-tælles til 5ere som T = (3*4/5)*5 = 2*5 + 2.

Og per-tal fører direkte videre til brøktal, som fremkommer ved dobbelt-tælling i samme enhed, fx 2kr per 3kr = 2kr/3kr = 2/3 = 2 per 3.

2/3 af 15 svarer til at få 2kr per 3kr af 15kr fundet ved at om-tælle 15 i 3ere og deraf tage 2: T = 15kr = (15/3)*3kr giver (15/3)*2kr = 10 kr. Så 2/3 af 15 er 10.

Tilsvarende findes 20% af 15 ved at om-tælle 15 i 100ere: T = 15 = (15/100)*100 giver (15/100)*20 = 3.

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Equations solved by moving across, inverse reckoning or recounting

Equations such as 2+3u = 14 and 25–u = 14 and 40/u = 5 are easily solved by the rule for reverse operations: 'Move to opposite side with opposite calculation sign'.

The equation u+3 = 8 asks for a number u that added to 3 gives 8, which by definition is u = 8-3; thus +3 moves to the opposite side with the opposite calculation sign. Similarly with u*2 = 8 solved by u = 8/2; and with u^3 = 12 solved by u = 3√12, where the root is a 'factor-finder'; and with 3^u = 12 solved by u = log3(12), where the logarithm is a 'factor counter'.

Ekvationer løst ved overflytning, tilbageregning eller omtælling

Ekvationer som 2 + 3u = 14 og 25 – u = 14 og 40/u = 5' løses let ved reglen for omvendte operationer: 'Flyt til modsat side med modsat regnetegn'.

I u+3 = 8 søges det tal u, der adderet med 3 giver 8, hvilket pr. definition er u = 8-3; så +3 flytter til modsat side med modsat regnetegn. Tilsvarende med u*2 = 8, som løses af u = 8/2; og med u^ 3 = 12, som løses af u = 3√12, hvor roden er en faktor-finder; og med 3^u = 12, som løses af u = log3(12), hvor logaritmen er en faktor-tæller.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

Et sådant eksempel er at sige, at formålet med matematikuddannelse er at lære matematik, da en sådan målsætning er åbenbart meningsløs ved sin selvreferencemåde.

Forbindelsen mellem et mål og dets midler er også til stede i den eksistentialistiske filosofi, der er beskrevet af Sartre (2007) som at fastholde, at "Eksistens går forud for essens (s. 20)". På samme måde påpeger Arendt (1963) at det at praktisere et middel blindt uden at reflektere over sit mål kan føre til at praktisere "ondskabets banalitet".

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For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Ekvationer løst ved overflytning.

En klasse har problemer med ekvationerne 2 + 3u = 14 og 25 – u = 14 og 40/u = 5, hvor ekvationen er sammensat eller den ubekendte har omvendt regnetegn. Løsningen er at bruge definitionerne af de omvendte operationer til at fastlægge den grundlæggende løsningsregel: 'Flyt til modsat side med modsat regnetegn'.

I u+3 = 8 søges det tal u, der adderet med 3 giver 8, hvilket pr. definition er u = 8-3; så +3 flytter til modsat side med modsat regnetegn. Tilsvarende med u*2 = 8, som løses af u = 8/2; og med u^3 = 12, som løses af u = 3√12, hvor roden er en faktor-finder; og med 3^u = 12, som løses af u = log3(12), hvor logaritmen er en faktor-tæller.

Ekvationen 2 + 3*u = 14 kan ses som en dobbelt beregning, der reduceres til en enkelt af en parentes omkring den stærkere operation, 2 + (3*u). Flyttes 2 til modsat side med modsat regnetegn fås 3*u = 14-2. Så flyttes 3 til modsat side, hvor en parentes sættes om det, der først skal beregnes: u = (14-2)/3 = 12/3 = 4.

Ekvationen kan også løses ved frem-og-tilbage-gang: Frem ganges med 3 og adderes med 2. Tilbage subtraheres 2 og divideres med 3, så u = (14-2)/3 = 4.

I ekvationen 25 – u = 14 har u modsat regnetegn, og flytter derfor til modsat side for at få et normalt regnetegn. Herefter flyttes 14 til modsat side med modsat regnetegn: 25 = 14 + u; 25 – 14 = u; 11 = u.

Tilsvarende med 40/u = 5 som giver 40 = 5*u; 40/5 = u; 8 = u.

Har klassen lært bundt-tælling og om-tælling vil en dobbelt om-tælling gives 40 = (40/u)*u = 5*u, og 40 = (40/5)*5, så u = 40/5.

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Calculus: Adding and splitting into locally constant per-numbers

Calculus is made easy by beginning with integral calculus for adding piecewise or locally constant per-numbers by their areas. Adding ‘2kg á 3$/kg + 4kg á 5$/kg’, the unit numbers 2 and 4 add directly to 6, while the per-numbers 3 and 5 must be multiplied to unit-numbers before they can add: 3*2 + 5*4 = 26. Thus, the answer is 6 kg á 26/6 $/kg. However, multiplication creates areas, so per-numbers add by the area under the piecewise constant per-number graph.

Calculus: Addition af og opdeling i lokalt konstante per-tal

Calculus lettes ved at begynde med integralregning til addition af stykkevis eller lokalt konstant per-tal ved deres arealer.

I additionen ’2kg á 3kr/kg + 4kg á 5kr/kg’ adderes styktallene 2kg og 4kg direkte til 6kg, medens pertallene skal opganges til styktal før de kan adderes: 3*2kr + 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tal graf.

Baggrund: Faldende PISA-resultater

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

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For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Calculus: Addition af og opdeling i lokalt konstante per-tal

En klasse har problemer med calculus. Løsningen er at udskyde differentialregning indtil efter integralregning er lært som et middel til at addere stykkevis eller lokalt konstant per-tal ved deres arealer.

I additionen '2kg á 3kr/kg + 4kg á 5kr/kg' adderes styktallene 2kg og 4kg direkte til 6kg, medens pertallene skal opganges til styktal før de kan adderes: 3*2kr + 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tal graf, altså som få arealstrimler, $S = \Sigma p*\Delta x$.

Et ikke-konstant per-tal kan anses som lokalt konstant (kontinuert), hvilket betyder addition af uoverskueligt mange arealstrimler, $S = \int p*dx$. Der dog kan lettes ved at omskrive strimlerne som tilvækster, $p*dx = dy$ eller $dy/dx = p$. For ved opsummering af tilvækster vil alle midterled forsvinde og blot efterlade differensen mellem y-slut og y-start.

Dette giver en ægte motivation for differentialregning: Kan strimlen 2x*dx skrives som tilvæksten d(x^2), bliver summen ∫2x*dx differensen mellem x^2-slut og x^2-start.

Tilvækst- formler kan observeres i et rektangel, hvor ændringerne Δb og Δh i basen b og højden h giver den samlede ændring af arealet ΔT som summen af en lodret strimmel, $\Delta b*h$, og en horisontal strimmel, $b*\Delta h$, og et hjørne, $\Delta b*\Delta h$, der kan negligeres ved små ændringer.

Dvs. d(b*h) = db * h + b * dh eller optalt i Tere: dT/T = db/b + dh/h, eller med T' = dT/dx, T'/ T = b'/ b + h'/ h.

Derfor er (x^2)'/ x^2 = x'/x + x'/x = 2*x'/x, hvilket giver (x ^ 2)' = 2*x, da x' = dx/dx = 1.

Så differentialregning er først og fremmest nyttig til hurtig opsummering af mange tal. Sidenhen også til at beskrive grafers vækstforhold med henblik på optimering.

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Calculus in primary school, in middle school and in high school

Counted and recounted, blocks may be added, but on-top or next-to? Next-to addition of 2 3s and 4 5s as 8s means integrating their areas, called integral calculus where multiplication precedes addition. Asked oppositely ‘2 3s +? 5s gives 3 8s’, the answer is obtained by letting subtraction precede division, called differential calculus. So, with block numbers, calculus occurs already in primary school.

In middle school calculus occurs when adding per-numbers in blending tasks as ‘2kg á 3$/kg + 4kg á 5$/kg = ?’

Calculus ved skolestart, i mellemskolen og i highskolen

Efter optælling følger addition af blokke, men skal de adderes ovenpå eller sidelæns?

Skal 2 3ere og 4 5ere adderes sidelæns som 8ere, sker det via deres areal, altså ved integration, hvor multiplikation kommer før addition.

Spørges modsat ’2 3ere + ? 5ere giver 3 8ere’, fås svaret ved at lade subtraktion komme før division, dvs. ved differentiation.

Så med bloktal, vil calculus forekomme allerede ved skolestarten.

I mellemskolen findes calculus i blandingsregning ’2kg á 3kr/kg + 4kg á 5kr/kg = ?’

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Calculus ved skolestart, i mellemskolen og i highskolen.

Matematik betyder viden på græsk, der valgte ordet som fællesbetegnelse for vidensområderne aritmetik og geometri og musik og astronomi, som de så som studiet af mange for sig selv, i rum, i tid og i tid og rum.

Med musik og astronomi ude, er matematik i dag blot en fællesbetegnelse for algebra og geometri, begge med rødder i mange, som det fremgår af deres betydning på arabisk og græsk: at genforene tal og at måle jord.

Når vi møder mange, spørger vi 'hvor mange totalt?' Svaret får vi ved at tælle, før vi regner. Tælling sker ved at bundte og stakke, forudsagt af tælleformlen T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere', fx T = 3 4ere = (3*4)/5*5 = 2 5ere & 2.

Efter optælling følger addition af stakke, men skal de adderes ovenpå eller sidelæns?

Skal stakkene 2 3ere og 4 5ere adderes sidelæns som 8ere, sker det via deres areal, altså ved integration, hvor multiplikation kommer før addition.

Spørges modsat '2 3ere + ? 5ere giver 3 8ere', fås svaret ved at lade subtraktion komme før division, dvs. ved differentiation.

Så ved at optælle totaler i stak-tal, vil calculus forekomme allerede ved skolestarten.

I mellemskolen optræder calculus ved blandingsregning:

I additionen '2kg á 3kr/kg + 4kg á 5kr/kg' adderes styktallene 2kg og 4kg direkte til 6kg, medens per-tallene skal opganges til styk-tal før de kan adderes: 3*2kr + 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tal graf, altså addition af arealstrimler, S = Σp*Δx, eller S = ∫p*dx i highskolen, hvor per-tallene er lokalt konstante (kontinuerte), og igen først skal adderes før de kan subtraheres ved differentiation.

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STEM-based core-math makes migrants pre-engineers

Our word- and number-language assign words and numbers to the world with sentences and formulas that contain a subject, a verb, and a predicate. With a number-language, young migrants can access core-math directly: Recounting in a new unit is predicted by a ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away’, e.g., T = 3 4s = (3*4)/5*5 = 2 5s & 2. After recounting and recounting, blocks may add next-to as areas (integration) or on-top if the units are made equal by recounting (proportionality).

STEM-baseret kerne-matte gør migranter til præ-ingeniører

Vore tale- og tal-sprog itale- og ital-sætter verden med sætninger og formler, som indeholder et subjekt, et verbum og et prædikat. Med tal-sprog får unge migranter adgang til med kerne-matte: Omtælling i ny enhed forudsiges af en ’tælle-formel’ T = (T/B)*B, der siger: ’Fra T kan vi T/B gange fjerne Bere’, fx T = 3 4ere = (3*4)/5*5 = 2 5ere & 2. Efter op- og omtælling kan blokke adderes sidelæns som arealer (integration) eller ovenpå hvis enhederne gøres ens ved omtælling (proportionalitet).

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

Fantasi som kernen i sociologi er beskrevet af Mills (1959). Bauman (1990) er enig ved at sige, at sociologisk tænkning "genindfører fleksibilitet til en verden, der er fastfrosset i rutiner ved at vise en alternativ verden, som den kunne være forskellig fra hvad den er nu (s. 16). "

Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

Et sådant eksempel er at sige, at formålet med matematikuddannelse er at lære matematik, da en sådan målsætning er åbenbart meningsløs ved sin selvreferencemåde.

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Inspireret af de gamle græske sofister, der ønsker at undgå at blive patroniseret af valg præsenteret som natur, søger 'differensforskning' efter skjulte forskelle, der gør en forskel (Tarp, 2017). For at undgå en målforskydning i matematikuddannelsen spørger differensforskning: Hvordan ville matematikken se ud, hvis den grundfæstes i sin udvendige rod, den fysiske faktum Mange?

For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: STEM-baseret kerne-matte.

Vi mestrer omverdenen med tale-sprog og tal-sprog, der itale-sætter og ital-sætter ting med sætninger og formler, som indeholder et subjekt, et verbum og et prædikat: 'Bordet er gult' og 'Totalen er 3 4ere'. Til et sprog hører et metasprog, en grammatik og en matematik, som bør læres efter sproget.

Unge migranter får direkte adgang til tal-sproget med kerne-matte: A) Cifre er ikoner med det antal streger, det repræsenterer. B) Regnearter er ikoner for bundt-tælling: division fjerner bundter, multiplikation stakker bundter, subtraktion fjerner stakken for at finde u-bundtede, addition forener stakke ovenpå eller sidelæns. C) Bundt-tælling og bundtskrivning viser bundter inden for koppen og u-bundtede udenfor, fx T = 4B5 = 4.5 tiere = 45. D) Totaler skal op-tælles og om-tælles og dobbelt-tælles før de adderes. E) Om-talt i samme enhed kan en total skrives på 3 måder: normal, med overlæs eller underlæs, fx T = 46 = 4B6 = 3B16 = 5B-4. F) Om-tælling i ny enhed (proportionalitet) forudsiges af en 'tælle-formel' T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere', fx T = 3 4ere = (3*4)/5*5 = 2 5ere & 2. G) Om-tælling fra tiere til ikoner giver ekvationer, fx x*5 = 40 = (40/5)*5 med løsning x = 40/5.

Dobbelttælling giver per-tal og proportionalitet med om-tælling i pertallet: med 2kr per 3 kg er 12 kg = (12/3)*3kg = (12/3)*2kr = 8kr. H) Efter optælling kommer addition, ovenpå og sidelæns, der fører til proportionalitet og integration. I) Omvendt addition fører til ligninger og differentiation. J) Per-tal fører til brøker, der som operatorer begge skal opganges for at blive tal, og derfor adderes som arealer, altså ved integration. K) Calculus er addition af og opdeling i lokalt konstante per-tal. L) Undervejs inddrages centrale STEM-områder under temaet 'vand i bevægelse'.

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Math blocks for the block-organized secondary school

In the EU, secondary school is a line-organized with compulsory classes forcing the teenagers to follow the year-group despite boys being two years behind in maturity. For economic reasons, low achievers are forced to stay in the class which cannot be repeated.

In the United States, secondary school supports the teenager’s identity work by welcoming them with esteem: ‘Inside, you carry a talent that together, we will now uncover and develop through daily homework in self-selected half-year blocks of a practical or theoretical nature together with teachers who have one subject only.’

Matte-blokke til den blok-opdelte sekundærskole

I EU er sekundærskolen linjeopdelt med ‘tvangsklasser’, hvor de unge følger årgangen til skade for drenge, der er to år bagud i modenhed. Af økonomiske grunde fastholdes langsomme unge, som får lave karakterer og lavt selvværd. I USA støtter den identitetsarbejdet ved at byde den unge velkommen med agtelse: ’Du bærer et talent, som vi i fællesskab nu afdækker og udvikler gennem daglige lektier i selvvalgte halvårsblokke af praktisk eller teoretisk art sammen med lærere, der kun har ét fag.’

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Fokus: Matte-blokke til den blok-opdelte sekundærskole.

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Så alle forlader halvårsblokken med ros. Og med lyst til som 18-årige at prøve kræfter med de tertiære jobrettede veje, hvor netop deres personlige talent kan udfoldes. Og som også er opdelt i halvårsblokke, så man hurtigt kan supplere eller udbygge sin grad med nye blokke ved jobskifte eller arbejdsløshed.

På en blokopdelt skole kan matte tilbydes i forskellige blokke med teoretisk eller praktisk udgangspunkt, så alle kan få kompetencer til at påbegynde en tertiær STEM-uddannelse (Science, Technology, Engineering, Math). Der kan også tilbydes blokke, som gentager primærskolens matte (tal, regnearter, addition, subtraktion, multiplikation, division, brøker, osv.). Og som kan kurere matte ulyst og give en introduktion til kerneområderne proportionalitet, ligninger og calculus ved at bruge en anderledes tilgang (mange-matik) med bloktal, bundt-tælling, om-tælling og dobbelt-tælling med per-tal før addition.

En blokopdelt skole er særlig effektiv for unge migranter som ønsker hurtigt at opnå STEM-kompetence for at bidrage til genopbygning af deres oprindelige hjemland.

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The teacher as a difference researcher

Difference research finds hidden differences that make a difference. It is used to solve problems in class. Or by teacher-researchers with shared time between academic work at a university and intervention research in a class. Or by full-time researchers working together with teachers: The teacher observes problems, the researcher identifies differences. A mutual micro-curriculum is created, tested by the teacher and reported by the researcher in a pretest-posttest study.

Læreren som differens-forsker

Differensforskning finder skjulte forskelle, der gør en forskel. Den bruges til at løse problemer i klassen, eller af lærer-forskere, der deler tid mellem akademisk arbejde på et universitet og interventionsforskning i en klasse. Eller af fuldtids forskere, der samarbejder med lærerne om fælles brug: læreren observerer problemer, forskeren identificerer forskelle. Sammen udarbejdes et mikro-curriculum, der testes af læreren og rapporteres af forskeren efter en pretest-posttest undersøgelse.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Læreren som differens-forsker.

Når traditioner giver problemer, kan differensforskning afdække skjulte forskelle der gør en forskel. Eksempelvis siger traditionen, at en funktion er et eksempel på en mængderelation, hvor førstekomponentidentitet medfører andenkomponent-identitet, hvilke du unge hører som ‘bublibub er et eksempel som bablibab’, som ingen finder meningsfyldt. En forskel er at bruge Eulers oprindelige definition, som alle unge godtager problemløst: ‘En funktion er et fællesnavn for regnestykker med både kendte og ukendte tal.’

Differensforskning kan bruges af lærere til at løse problemer i klassen, eller af lærer-forskere, der deler deres tid mellem akademisk arbejde på et universitet og interventionsforskning i en klasse. Eller af fuldtids forskere, der samarbejder med lærerne om fælles brug af differensforskning: læreren observerer problemer, forskeren identificerer forskelle. Sammen udarbejdes et mikro-curriculum, der testes af læreren og rapporteres af forskeren efter en pretest-posttest undersøgelse. En typisk differensforsker begynder som en almindelig lærer, der reflekterer over, om alternativer kan løse observerede læringsproblemer.

En differens-forsker kombinerer elementer fra aktionslæring og aktionsforskning og interventionsforskning og designforskning. Først identificeres en forskel, så designes et mikro-curriculum, der testes for at se, hvad der gør en forskel. Forløbet rapporteres internt og drøftes med kolleger. Efter at have gentaget denne cyklus af design, undervisning og intern rapportering, kommer den eksterne rapportering til magasiner eller konferencer eller tidsskrifter af forskellen og hvilken forskel den gør i en pretest-posttest undersøgelse.

Forskning bør skabe viden til at forklare naturen og forbedre sociale forhold. Men som institution risikerer den det, sociologen Bauman kalder en målforskydning, så forskningen bliver selvrefererende i stedet for at finde forskelle.

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MATHeCADEMY.net: Math as a Natural Science about Many, a Booth Exhibit

MATHeCADEMY.net provides online courses for teachers who want to teach mathematics as ‘Many-Math’, i.e. as a natural science about the physical fact many; as well as wanting to see mathematics as a number-language in family with the word-language, both using full sentences with a subject, a verb and a predicate, and where two competencies are sufficient: counting and adding in space and time. Many-Math respects the child's own number-language with flexible 2D bundle-numbers like T = 2*3 = 2 3s; and counting before addition.

MATHeCADEMY.net: Læreruddannelse i matte som naturvidenskaben om Mange, ideudstilling

MATHeCADEMY.net giver online-kurser til lærere, der ønsker at undervise i matematik som 'mange-matik', dvs. som en naturvidenskab om det fysiske faktum Mange; samt ønsker at se matematik som et tal-sprog i familie med tale-sproget, hvor begge bruger fulde sætninger med subjekt, verbum og prædikat, og hvor to kompetencer er tilstrækkeligt: at tælle og regne i rum og tid. Mange-matik respekterer barnets eget tal-sprog, med fleksible 2D bundt-tal som T = 2*3 = 2 3ere, og tælling før addition.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Læreruddannelse i matte som naturvidenskaben om Mange.

Peter er matematiklærer i en klasse af hvor mange har opgivet division og brøker, så Peter er ved at opgive at være lærer, da han hører om '1kop & 5pinde' metoden til at helbrede matematik-modvilje (Tarp, 2018), og da han ser 'CupCount and ReCount before you add' (www.youtube.com/watch?v=IE5nk2YEQIAxx).

Her bundt-tælles 5 pinde i 2ere med en kop til bundterne. Han ser, at en total kan optælles i samme enhed på tre måder: overlæs, standard og underlæs:

T = 5 = I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = IIII II I = 3B-1 2s

Så optalt i bundter giver indvendigt et antal bundter og et udvendigt et antal singler; og flyttes en pind ud eller ind skaber det overlæs eller underlæs.

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Her er MetaMatisme en blanding af MateMatism, sand inde i et klasseværelse, men sjældent udenfor hvor '2 + 3 = 5' modsiges af 2 uger + 3dage = 17dage; og MetaMatik, der præsenterer begreber TopDown som et eksempel på en abstraktion i stedet for BottomUp som en abstraktion fra mange eksempler: ”En funktion er et eksempel på et mængdeprodukt” i stedet for ”En funktion er en tal-sprogs sætning”

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For flere detaljer, se MrAlTarp YouTube videoer.

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Introducing Allan Tarp

As a curriculum architect at the MATHeCADEMY.net, Allan Tarp uses difference research to find differences that make a difference, such as icon-numbers, bundle-counting, recounting and double-counting and per-numbers, see MrAlTarp YouTube videos, and the article ‘Mastering Many’ in Journal of Mathematics Education, 11 (1), 103-117. As an instructor at

MATHeCADEMY.net, he teaches teachers to teach mathe-matics as ‘Many-Math’, a natural science about many, with two competencies only, count and add in time and space.

Som curriculum arkitekt ved MATHeCADEMY.net, bruger Allan Tarp ’Differens-forskning' til at finde forskelle, der gør en forskel, som fx ikontal, bundt-tælling, om-tælling og dobbelt-tælling og per-tal, se MrAlTarp YouTube videoer, samt artiklen ‘Mastering Many’ i Journal of Mathematics Education, 11(1), 103-117. Som instruktør på MATHeCADEMY.net lærer han lærere at undervise i mate-matik som ’mange-matik’, en naturvidenskab om Mange, med blot to kompetencer, tæl og regn i tid og rum

A Case: Peter, stuck in division and fractions

Being a mathematics teacher in a class of ordinary students and repeaters flunking division and fractions, Peter is about to give up teaching when he learns about the ‘1cup & 5sticks’ method to cure mathematics dislike by watching ‘CupCount and ReCount before you Add’ (https://www.youtube.com/watch?v=IE5nk2YEQIAxx).

Here 5 sticks are CupCounted in 2s using a cup for bundles. He sees that a total can be recounted in the same unit in 3 different forms: overload, standard and underload:

T = 5 = I I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = II II II ł = 3B-1 2s

So counted in bundles, a total has an inside number of bundles and an outside number of singles; and moving a stick out or in creates an over-load or an under-load.

When multiplying, 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 inside and 56 outside as an overload that can be recounted: T = 7 x 4B8 = 28B56 = 33B6 = 336.

And when dividing, 336/7 is bundle-written as 33B6 /7 recounted to 28 inside and 56 outside according to the multiplication table. So 33B6 /7 = 28B56 /7 = 4B8 = 48.

To try it himself, Peter downloads the ‘CupCount & ReCount Booklet’. He gives a copy to his colleagues and they decide to arrange a free 1day Skype seminar.

In the morning they watch the PowerPoint presentation ‘Curing Math Dislike’, and discuss six issues: first the problems of modern mathematics, MetaMatism; next the potentials of postmodern mathematics, ManyMath; then the difference between the two; then a proposal for a ManyMath curriculum in primary and middle and high school; then theoretical aspects; and finally where to learn about ManyMath.

Here MetaMatism is a mixture of MatheMatism, true inside a classroom but rarely outside where ‘2+3 = 5’ is contradicted by 2weeks+3days = 17days; and MetaMatics, presenting a concept TopDown as an example of an abstraction instead of BottomUp as an abstraction from many examples: A function IS an example of a set-product.

In the afternoon the group works with an extended version of the CupCount & ReCount Booklet where Peter assists newcomers. At the seminar there are two Skype sessions with an external instructor, one at noon and one in the afternoon.

Bringing ManyMath to his classroom, Peter sees that many difficulties disappear, so he takes a 1year distance learning education at the MATHeCADEMY.net teaching teachers to teach MatheMatics as ManyMath, a natural science about Many. Peter and 7 others experience PYRAMIDeDUCATION where they are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. An external coach assists the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. In a pair each teacher corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays by coaching a new group of 8 teachers.

At the academy, the 2x4 sections are called CATS for primary and secondary school inspired by the fact that to deal with Many, we Count & Add in Time & Space.

At the academy, primary school mathematics is learned through educational sentence-free meetings with the sentence subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic ‘grasp-to-grasp’ learning.

Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic ‘gossip-learning’: Thank you for telling me something new about something I already knew.

Et tilfælde: Peter, kørt fast i division og fraktioner

Peter er matematiklærer i en klasse af hvor mange har opgivet division og brøker, så Peter er ved at opgive at være lærer, da han hører om '1kop & 5pinde' metoden til at helbrede matematik-modvilje, og da han ser 'CupCount and ReCount before you add (https://www.youtube.com/watch?v=IE5nk2YEQIAxx).

Her bundt-tælles 5 pinde i 2ere med en kop til bundterne. Han ser, at en total kan optælles i samme enhed på tre måder: overlæs, standard og underlæs:

T = 5 = I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = IIII II I = 3B-1 2s

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Title	Id	Submitted	Edit/View
Start-matte for børn og migranter: Kop-tæl og om-tæl før addition	3408	YES	View
Multiplikation før addition styrker talsansen hos børn og migranter	3412	YES	View
Ulyst til division kureret med 5 pinde og 1 kop og kop-skrivning	3413	YES	View
Brøker og procenter som per-tal	3414	YES	View
Brøker og per-tal adderet som integration	3417	YES	View
Proportionalitet som dobbelt-tælling med per-tal	3418	YES	View
Ekvationer løst ved overflytning, tilbageregning eller omtælling	3419	YES	View
Calculus ved skolestart, i mellemskolen og i highskolen	3420	YES	View
Calculus: Addition af og opdeling i lokalt konstante per-tal	3421	YES	View
STEM-baseret kerne-matte gør migranter til præ-ingeniører	3423	YES	View
Matte-blokke til den blok-opdelte skole	3426	YES	View
Læreren som differens-forsker	3427	YES	View
MATHeCADEMY.net & Mellemskolen.net: Matte-blokke til den blok-opdelte skole	3428	YES	View
Ulyst til division kureret med 5 pinde og 1 kop og kop-skrivning	3431	YES	View

+ Add new abstract

32. DE-MODELING NUMBERS, OPERATIONS AND EQUATIONS: FROM INSIDE-INSIDE TO OUTSIDE-INSIDE UNDERSTANDING

ABSTRACT

Adapting to the outside fact Many, children internalize social number-names, but how do they externalize them when communicating about outside numerosity? Mastering Many, children use bundle-numbers with units; and flexibly use fractions and decimals and negative numbers to account for the unbundled singles. This suggests designing a curriculum that by replacing abstract-based with concrete-based psychology mediates understanding through de-modeling core mathematics, thus allowing children to expand the number-language they bring to school.

Keywords: number; operation; equation; numeracy; proportionality; early childhood

1. Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish Centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA results caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life (p. 3)". In Germany the corresponding number is one of five students, according to a plenary address at the Educating Educators conference in Freiburg in October 2019.

This raises some questions: Is mathematics so hard that one out of four or five students cannot master even basic numeracy? Is it mathematics we teach? Do we use the proper psychological learning theories? Can we design a different mathematics curriculum where most students become successful learners? In short: could this be different?

2. Materials/ Subjects and Methods

To get an answer we use difference research (Tarp, 2018) to create a design research cycle (Bakker, 2018) consisting of reflection, design, and implementation.

2.1. Reflections on Different forms of Mathematics

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. So, as a label, mathematics has no existence itself, only its content has, algebra and geometry; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-math' rooted in Many was replaced by the 'New Mathematics'.

Here the invention of the concept Set created a 'setcentric' (Derrida, 1991), 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant self-reference defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. To avoid self-reference, Russell developed a type theory defining concepts from examples at the abstraction level below. This implies that fractions cannot be numbers, but operators needing numbers to become numbers. Wanting fractions to be rational numbers, the setcentric mathematics neglected Russell's paradox and insisted that to be well defined, a concept must be derived from the mother-concept set above.

In this way, the concept Set changed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism', true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2 x 3 = 6' stating that 2 3s can always be re-counted as 6 1s (Tarp, 2018).

Rejecting setcentrism as making mathematics to abstract, many Anglo-Saxon countries went back to basics and now teach what they call 'school mathematics' although this still is mathe-matism adding numbers and fractions without units.

So, today we have three different forms of mathematics: a pre setcentric mathe-matism saying that a function is a calculation with specified and unspecified numbers; a present setcentric meta-matism saying that a function is a subset of a set-product where first-component identity implies second-component identity; and a post setcentric many-math (Tarp, 2018) saying that a function is a number-language sentence with a subject, a verb and a predicate as in the word-language; and that a communicative turn learning language through communication instead of through grammar is needed in the number-language also (Widdowson, 1978).

In its pre and present setcentric versions, a mathematics curriculum typically begins with digits together with addition, later to be followed by subtraction as reversed addition, multiplication as repeated addition, and division as reversed multiplication - sometimes as repeated subtraction also. Then follows fractions, percentages and decimals as rational numbers. Then comes negative numbers, to be followed by expressions with unspecified letter numbers, and by solving equations.

Present setcentric meta-matics defines numbers by inside abstract self-reference as examples of sets. Zero is defined as the empty set. One is defined as the set containing the empty set as its only element. The next numbers then are generated by a follower principle.

With natural numbers defined, integers are defined as equivalence classes in the set- product of natural numbers created by the equivalence relation saying that (a,b) is equivalent to $(c.d)$ if cross addition holds, $a+d = b+c$. This makes (-2,0) equivalent to (0,2) thus geometrically forming straight lines with gradient 1 in a coordinate system.

With integers defined, rational numbers are defined as equivalence classes the set- product of integers created by the equivalence relation saying that (a,b) is equivalent to $(c.d)$ if cross multiplication holds, $a \text{ x } d = b \text{ x } c$, thus making (2,3) equivalent to (8,12) thus geometrically forming straight lines with various gradients in a coordinate system.

Equations are examples of open statements that may be transformed to a solution by using abstract algebra's group theory to neutralize numbers by their inverse numbers.

In geometry, halfplanes define lines that are parallel if a subset relation exists among their halfplanes. And an angle is the intersection set of two halfplanes.

Post setcentric many-math is grounded in the observation that when asked "How old next time?", a 3year-old will answer "4", but will object to 4 fingers held together 2 by 2: "That is not 4; that is 2 2s." So, when adapting to the outside fact Many children count in bundles, and use double-numbers to describe both the numbers of bundles and the bundle-unit. And it turns out that double-numbers contain the core of mathematics since recounting to change units implies proportionality and equations; and when adding double-numbers, on-top addition leads to proportionality making the units like, and next-to addition means adding areas, which leads to integral calculus.

2.2. Reflections on Different forms of Psychology

As institutionalized learning, education is meant to help human brains adapt to the outside world by accommodating schemas failing to assimilate it (Piaget, 1970); or to mediate institutionalized schemas that may colonize the brain (Habermas, 1981).

Adaption is theorized by psychology, often seen as the science of behavior and mind, thus being a sub-discipline of life science, where biology sees life as communities of green and grey cells, plants and animals. Plants stay and get the energy directly from the sun. Animals move to get the energy from plants or other animals, thus needing holes in the head for food and information, making the brain transform stimuli to behavior responses.

Besides the reptile and mammal brains for routines and feelings, humans also have a third human brain for balancing and for storing and sharing information, made possible by transforming forelegs to arms with hands that can grasp (and share) food and things that accompanied by sounds develop a language about the six core outside components: I, you, he-she-it, we, you, and they; or in German: ich, du, er-sie-es, wir, ihr, sie.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured e.g. by a social institution called education.

Educational psychology first focused on behavior by studying stimulus-response pairings, called classical conditioning where Pavlov showed how dogs would salivate when hearing a sound previously linked to food. Later Skinner (1953) developed this into an operant conditioning by adding the concepts of reinforcement and punishment as stimuli following a student response coming from building routines through repetition.

But, does correct responses imply understanding? So, the educational psychology called constructivism focuses on what happens in the mind when constructing inside meaning to outside stimuli. Here especially Piaget (1970), Vygotsky (1986), and Bruner (1977) have contributed in creating teaching methods and practices.

Piaget found four different development stages for children: the sensorimotor stage below 2 years old, the preoperational state from 2 to 7 years old, the concrete operational stage from 7 to 10 years old, and formal operational stage from 11 years old and up.

In philosophy, existentialism sees existence as preceding essence (Sartre, 2007). Where Piaget sees learning taking place through adaption to outside existence, Vygotsky focuses on adaption to inside institutionalized essence, i.e. through enculturation allowing learners to expand their ‘Zone of Proximal Development’ (ZPD) under guidance of a more knowledgeable other. “What a child can do today with assistance, she will be able to do by herself tomorrow” (azquotes.com).

Likewise pointing to the importance of good teaching, Bruner developed the concept of instructional scaffolding providing a ladder leading from the ZPD up to a school subject. This should be structured as its university version to help the teacher structure the subject in a way that would give the meaning that the students need for understanding.

Holding that no children master logical thinking before 11 years, and therefore needing to be taught using concrete objects and examples, Piaget instead warned against too much teaching by saying: “Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover for himself will remain with him visible for the rest of his life” (azquotes.com).

2.3. Merging Mathematics and Psychology

Behaviorism is the educational psychology of pre setcentric mathematics. Present setcentric mathematics instead uses Vygotskian constructivism offering scaffolding from the learners ZPD to the institutionalized setcentric university mathematics as defined by e.g. Freudenthal (1973). However, by its self-referring setcentrism, concepts are no longer defined from examples and counterexamples, but as examples themselves of the more abstract set concept. So now not both rules, procedures and concepts should be understood. Freudenthal therefore recommended a special conference be created called PME, Psychology of Mathematics Education, focusing on how to understand mathematics as described by Skemp (1971) saying “The first part of the book will be concerned with this most basic problem: what *is* understanding, and by what means can we help to bring it about? (p. 14)". Skemp then uses 123 pages to give an understanding of understanding, even if the inherent self-reference should make one skeptical towards such an endeavor.

Heidegger more directly points to four options when defining something by an is-statement: 'is for example' points down to examples and counter-examples, 'is an example of' points up to an abstraction, 'is like' points over to a metaphor, and 'is.' describes existence as something to experience without predicates.

Skemp understands numbers as equivalence cardinality classes in the set of sets being equivalent if connected by bijections. Consequently, children should begin drawing arrows between sets to see if they have the same cardinality that then can be named.

However, this approach met resistance in the classroom as illustrated by in this story:

Teacher: "Here is a set of hats and a set of heads. Is one bigger than the other?" Student: "There are more heads". Teacher: "Why?" Student: "There are six heads and only five hats." Teacher: "Can you please draw arrows from the hats to the heads!" Student: "No, then one person will not get a hat, and that is unfair."

In his book 'Why Johnny Can't Add', Morris Kline describes other examples on classroom resistance to the New Math, finally rejected by North America, choosing to go 'Back to Basics' even if this meant going back to mathe-matism.

Educational psychology thus has various schools. As an alternative, we might use the observation that children's initial language consists of words that are exemplified in the outside world, thus using personal names instead of a pronominal as I and you, and protesting when grandma is named 'Ann'.

Observing that brains easily takes in concepts naming outside examples allows formulating a research question: Can core mathematics as numbers, operations and equations be exemplified, de-modeled, or reified by concrete outside generating examples?

2.4. De-modelling Digits

Looking at a modern watch in front of an old building, we realize that Roman numbers and modern Hindu-Arabic numbers are different ways of describing Many.

The Romans used four icons to describe four, or they used one stroke to the left of the letter V iconizing a full hand. Modern numbers use one icon only when rearranging the four sticks or strokes into one 4-icon, which then serves as a unit when counting a total in fours as e.g. T = 3 4s.

We might even say that all digits from zero to nine are icons with as many sticks or strokes as they represent if written less sloppy, where the zero-digit iconizes a magnifying glass finding nothing.

The Romans bundled in 5, 10, 50, 100, 500 and 1000. Modern numbers bundle in tens only, which is written as 10 meaning 1 Bundle and none. However, in education we may want to symbolize ten with the letter B for 'Bundle'.

2.4.1. Designing and Implementing a micro-curriculum

Based upon the above reflections we now design and implement a micro-curriculum having as its goal to de-model and reify digits as icons. As means we ask the learners to rearrange four sticks in different connecting forms, then five sticks, then six sticks. This is followed by rearranging also other things in icons including themselves, and by walking the icons, etc. Then the learners build routine by exercising writing all digits as icons. As end product, the learner should be able to rearrange a collection of things in an icon and write down a report using a full number-language sentence with a subject, a verb and a predicate, e.g. "T = 5"; and writing T = B, B1, B2 etc. for ten, eleven, twelve etc.

2.5. Reflections on how to De-model Bundle-counting Sequences

From early childhood children memorize the inside sequence of number names 'one, two,…, ten, eleven, twelve, three-ten, four-ten' etc. Later they learn the symbols corresponding to the different number-names. In some languages they are lucky to word 'eleven, twelve, thirteen' as one-ten, two-ten, three-ten' etc. In English, number rationality begins with three-ten, making whole populations wonder what eleven and twelve means.

History shows that as most basic English words also these are 'Anglish' coming from the Danes settling in England long before the Romans arrived. Thus, with Danish you hear that eleven and twelve means 'one-left' and 'two-left' coming from Viking counting: 'eight, nine, ten, 1-left, 2-left, 3-ten'; and '1-twotens' where English shift to 'twenty-1'.

Likewise, many children and adults wonder why ten has no icon since it has its own name as the rest of the digits. Only few realize that when counting by bundling in tens, ten becomes 1 bundle, or 1B0, or 10 if leaving out the bundle when writing it; even if ten is included when saying it, as e.g. in 63 = sixty-three = 6ten3 = 6B3.

So, it may be an idea to practice different counting sequences that include the name 'bundle' so that 'ten, eleven and twelve' become '1 bundle none, 1 bundle 1, 1 bundle 2'. And it may also be an idea to also count in fives as did the Romans and several East Asian cultures as shown by Chinese and Japanese abacuses. So, we design a lesson about counting fingers first in 5s, then in tens, and later in 4s, 3s and 2s or pairs.

2.5.1. Designing and Implementing a micro-curriculum

One hand counted in 5s using B for Bundle: First 1, 2, 3, 4, 5 or B or 1B1; then 0B1, 0B2, 0B3, 0B4, 0B5 or B or 1B0; then 1Bundle less 4, 1B-3, 1B-2,1B-1,1B.

Two hands counted in 5s: First 1, 2, 3, 4, 5 or B or 1B0, 1B1, …, 1B4, 1B5 or 2B or 2B0; then 0B1, 0B2, 0B3, 0B4, 0B5 or B or 1B0, etc.; then 1Bundle less 4, 1B-3, 1B-2,1B-1,1B0, 2B-4, …, 2B-1, 2B or 2B0.

Two hands counted in tens: First 1, 2, 3, 4, 5 or half Bundle, 6, 7, 8, 9, ten or full Bundle or 1B0; then 0B1, 0B2,…, 0B9, 0B10 or B or 1B0; then 1B-9, 1B-8, …, 1B-1, 1B.

Two hands counted in 4s is similar to counting in 5s.

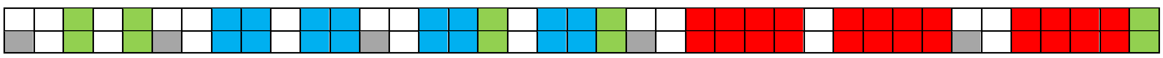
Two hands counted in 3s provides the end result T = ten = 3B1 3s. But 3 bundles, 3B, is also 1 bundle of bundles, making 9 = 1BB 3s. So we can also write:

T = ten = 3B1 3s = 1BB1 3s, or T = 1BB0B1 3s, or T = 101 3s.

Two hands counted in 2s provides the end result T = ten = 5B0 2s. But, 2 bundles, 2B, is also 1 bundle of bundles, making 4 = 1 BB 2s; and 2 bundles of bundles, 2BB, is also 1 bundle of bundles of bundles, making 8 = 1 BBB 2s. So we can also write:

T = ten = 5B0 2s = 1BBB 1B 2s = 1BBB 0BB 1B 0 2s = 1010 2s.

This can be illustrated with Lego bricks having different colors where a green 1x2 brick is B, a blue 2x2 brick is BB and a red 4x2 brick is BBB.



1	10	11	100	101	110	111	1000	1001	1010
1	2	3	4	5	6	7	8	9	ten

Figure 1. *Ten fingers counted by bundling in 2s and shown by Lego bricks.*

2.6. Reflections on how to De-model Operations

Counting a total of eight ones in 2s, we push away 2s using e.g. a playing card that may be iconized as a sloping stroke named division. So, the outside action 'from 8, push away 2s' may inside be iconized as '8/2' that gives an inside prediction of what will happen outside: 8/2 times we can perform the action 'from 8 push away 2', or T = 8 = 8/2 2s = 4 2s.

Once pushed away, the bundles of 2s may be lifted into a stack of 4 2s. An outside lifting process may be iconized inside by a wooden scissor lifting up things when compressed, and named multiplication. So, the outside action '4 times lifting 2s into a stack' may inside be iconized as 'T = 4 x 2 = 4 2s'. And, the reverse outside process 'de-stack 4 2s into ones' may inside be predicted by a multiplication T = 4 2s = 4 x 2 = 8.

The total outside process 'from 8 push away 2s to be stacked as 8/2 2s' then may be iconized inside as '8 = (8/2) x 2', or '*T* = *(T/B)* x *B*' if we use *T* for the total 8, and *B* for the bundle-unit. By changing units, this 'bundle-count' or 're-count to change unit' formula is perhaps the most fundamental formula in mathematics and science, also called the proportionality or linearity formula.

2.6.1. Designing and Implementing a micro-curriculum

Outside action	Inside prediction
From 8, 8/2 times 2s can be pushed away. So, 8 can be recounted in 2s as 8/2 2s And, *T* can be recounted in Bs as *T/B* Bs	8/2 = 4 8 = (8/2) x 2 *T* = *(T/B)* x *B*

Figure 2. *Counting 8 in 2s is an example of the recount formula T = (T/B) x B*

Asking "How many 2s will give 8" may be reformulated as an equation '*u* x 2 = 8' using a letter *u* for the unknown number; and solved by recounting 8 in 2s. So, an equation is solved by moving a number to the opposite side with the opposite calculation sign. Here, solving equations is just another name for recounting in icon-units.

Outside action	Inside prediction
How many 2s will give 8? To answer, we recount 8 in 2s	*u* x 2 = 8 = (8/2) x 2 so *u* = 8/2

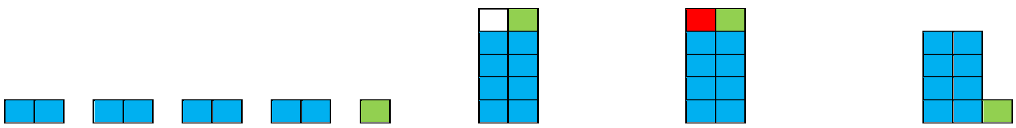
Figure 3. *An equation solved: to the opposite side with the opposite calculation sign*

Having pushed bundles away to stack, some unbundled may be left. So, from the total, we pull away the stack with a rope iconized as a horizontal stroke called subtraction.

Outside action	Inside prediction
From 9, 9/2 times 2s can be pushed away. From 9, pull away 4 2s leaves 1 Prediction: T = 9 = 4B1 2s	9/2 = 4.some 9 – 4 x 2 = 1 Prediction: 9 = 4 x 2 + 1

Figure 4. *To predict unbundled, we pull away the stack from the original total*

Counting a total of 9 in 2s thus is predicted by the division 9/2 = 4.some. To predict leftovers, we pull away the stack of 4 2s from the total 9, predicted by saying '9-4x2' giving the expected answer 1. So, a total of 9 may be counted in 2s as T = 9 = 4B1 2s, or as T = 4 x 2+1. Here a cross called addition iconizes the two ways to place the unbundled: next-to the stack iconized by a dot named a decimal point, T = 9 = 4.1 2s; or on-top of the stack counting in bundles as 1 = (1/2) x 2 giving T = 9 = 4B½ 2s, or counting what is missing in having a full bundle, T = 9 = 5B-1 2s, thus reifying fractions and negatives.



T = 9 = 4B 2s + 1 = 4B½ 2s = 5B-1 2s = 4B1 2s = 4.1 2s

Figure 5. *Unbundled singles may be placed on-top or next-to the stack of bundles*

Likewise, when counting in tens:

T = 48 = 4B tens + 8 = 4B8/10 tens = 5B-2 tens = 4B8 tens = 4.8 tens

Changing bundles to unbundled or vice versa gives ‘flexible bundle-numbers’ with or without an overload, or with an underload: T = 48 = 4B8 tens = 3B18 tens = 5B-2 tens.

2.7. Reflections on how to Recount into Tens

Once counted, a total may be recounted in the same unit, in a different unit, from tens into icons, or into tens. The first three cases are described in the chapter above.

Recounting from icons into tens, the recount formula cannot be used since there is no ten button or bundle button on the calculator. However, multiplication gives the result directly, only without units and with the decimal point moved one place.

Question: T = 3 8s = ? tens; answer: T = 3 8s = 3 x 8 = 24 = 2.4 tens

Recounting into tens includes multiplying two one-digit numbers, called setting up a multiplication table: a small table for the numbers 1-5, and a large for the numbers till 10.

2.7.1. Designing and Implementing a micro-curriculum

Turning over a stack will change e.g. 2 3s to 3 2s without changing the total. So, in multiplication, the order does not matter, the units may be commuted.

The small table follows directly from using fingers. It is obvious in the case of 2.

In the case of 3,

4x3 = 2x2x3 = 2 6s = 12 seeing a hand as a pawn with six extremities leaving it; and

5x3 = 3x5 = 3 5s = 3 hands = 1B5 = 15.

In the case of 4,

4x4 = 2x2x4 = 2 8s = 2 B-2s = 2B-4 = 1B6 = 16; and

5x4 = 4x5 = 4 hands = 2B0 = 20.

Finally, in the case of 5, 5x5 = 5 5s = 5 hands = 2B5 = 25.

In the large table we recount the numbers from 6 to 10 in bundles as B-4, B-3, etc.; and use a bead pegboard square with two rubber bands to show the actual stack as e.g.

6 7s = 6x7 = (B-4) x (B-3) = BB – 4B –3B + 4 3s removed twice = 3B12 = 4B2 = 42.

This roots the algebraic formula *(a – b)* x *(c – d) = a* x *c – a* x *d – b* x *c + b* x *d.*

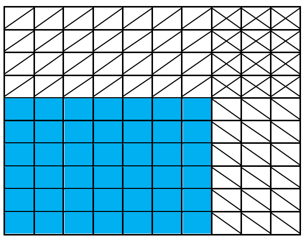


Figure 6. *A pegboard square shows that 6x7 = (B-4) x (B-3) = 10B – 4B – 3B + 4 x 3*

Recounting into tens also includes multiplying multi-digit numbers as e.g. 27 x 36 = 27 36s = ? tens. We may use a square or write the result in lines. It makes sense that changing the unit base from 36 to 10 will increase the height of the stack from 27 to 97.2.

Question: T = 27 x 36 = 27 36s = ? tens. Answer: T = 2B7 x 3B6 = (2B+7) x (3B+6) = 6BB+12B+21B+42 = 6BB + 33B + 4B2 = 6BB + 37B + 2 = 9BB7B2 = 972 = 97.2 tens.

Vice versa, recounting from tens includes division as the opposite of multiplication. Asking 16.8 tens is how many 7s thus gives the division 168/7. We may use the square bottom-up, or write the result in lines using flexible bundle-numbers. Here, it makes sense that changing the base from 10 to 7 will increase the height of the stack from 16.8 to 24.

Question: ? 7s = 16.8 tens.

Answer: u x 7 = 168; u = 168 /7 = 16B8 /7 = 14 B28 /7 = 2B 4 = 24. So 24 7s = 16.8 tens.

T = 27 x 36 = ? tens			
	2B	7	
	6BB	21B	3B
	12B	42	6
6BB	33B	4B2	
6BB	37B	2	
9BB	7B	2	
9	7	2	

? x 7 = 168, or 168/7 = ?; answer: 2B4 = 24			
		7	
		14B	? = 2B
		28	? = 4
	14B	28	
	16B	8	
1BB	6B	8	
1	6	8	

Figure 7. *A 2D schema used top-down for multiplication and bottom-up for division*

2.8. Reflections on how to Model Double-counting with Per-numbers and Fractions

On a Lego brick we can double-count the dots and the rows, e.g. giving 2 rows per 8 dots on a 2x4 brick, thus producing the 'per-number' 2r/8d, or 2/8 r/d.

Double-counting a basket filled with 3red per 5 apples, the like units makes the per-numbers a fraction, both being operators needing numbers to become numbers.

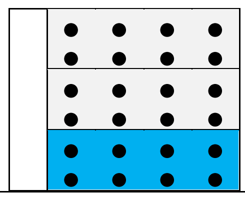
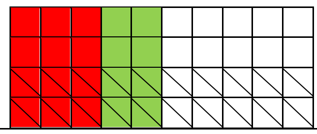
Asking "6 rows gives how many dots?", we recount 6 rows in the per-number as: T = 6r = (6/2) x 2r = (6/2) x 8d = 24d Likewise, when asking "how many rows gives 56 dots?": T = 56d = (56/8) x 8d = (56/8) x 2r = 14r	
To find the number of red apples among 20 apples, we set up a per-number equation u/20 = 3/5, or recount 20 in 5s: u/20 = 3/5; so u = 3/5 x 20 = 12. 20 a = (20/5) x 5a giving (20/5) x 3r = 12 red apples	

Figure 8. *Double-counting in two units creates per-numbers bridging the units*

Likewise with percent. 5% of 40 asks '5 per 100 is what per 40', giving the equation u/40 = 5/100; solved by moving 40 to opposite side with opposite sign: u = 5/100 x 40 = 2.

The unit-changing recount formula may also be used on units as e.g.: \$ = (\$/kg) x kg. Thus with the per-number 2\$/3kg, we may ask '?\$ = 12kg' and '10\$ = ?kg'. Recounting then gives: \$ = (\$/kg) x kg = 2/3 x 12 = 8; and kg = (kg/\$)x\$ = 3/2 x 10 = 15.

2.9. Reflections on how to De-model Trigonometry

In an a x b rectangle halved by its diagonal c, double-counting the sides creates the per-numbers sin $A = a/c$, cos $A = b/c$, and tan $A = a/b$. Filling a circle from the inside, we find a formula for the per-number perimeter per radius: $\pi = n$ x tan(180/n) for n large.

2.10. Reflections on how to Add Next-to and On-top, and how to Add Per-numbers

Once counted or recounted, blocks may be added next-to or on-top. Here adding 2 3s and 4 5s next-to as 8s means adding by areas, called integral calculus. Whereas adding 2 3s and 4 5s on-top means making the units like, using the recount formula to change units.

Adding 2kg at 3\$/kg with 4kg at 5\$/kg, the unit-numbers add directly as 2+4 = 6, whereas the per-numbers add by their areas (2x3 + 4x5)\$ / 6kg, which is integral calculus where multiplication precedes addition. Vice versa, asking what to add to 4kg at 5\$/kg to have 6 kg at 4\$/kg, we subtract the initial block 4x5 from 6x4 before counting in 2s, which is differential calculus where subtraction precedes division.

3. Results and Discussion

This study asked: Can core mathematics as numbers, operations and equations be exemplified, de-modeled, or reified by concrete outside generating examples?

The answer is yes, if we de-model digits as icons with as many sticks as they represent if written less sloppy; if we use the flexible bundle-numbers children develop when adapting to Many; if we de-model operations as means for bundle-counting 8 as 8/2 2s, leading directly to the recount formula $T = (T/B) \times B$, used to change units, and to solve the question ‘How many 2s in 8?’ by recounting 8 in 2s: $u \times 2 = 8 = 8/2 \times 2$, so $u = 8/2$.

The operations are de-modeled as bundle-counting where division pushes away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to find unbundled singles to be placed next-to or on-top of the stack as decimal numbers, negative numbers or fractions; and later added with other stacks next-to as integral calculus, or on-top after making the units the same by using the proportionality of the recount formula.

Exemplifying assigns concrete meaning to abstract concepts, so de-modeling and reifying needs no psychological learning theories about how meaning is constructed.

As expected, concrete meaning makes mathematics easy to learn, as confirmed when tested in pilot projects in preschool, special education, and in adult and migrant education.

Of course, the effect of using flexible bundle-numbers and recounting operations should be studied in detail in other cases also to open up a completely new research paradigm (Kuhn, 1962), that may make obsolete all single-number material and research on mathematics in its grammar-based form before undergoing a communicative turn.

4. Conclusion

As to the questions asked in the introduction, the answers are: Yes, mathematics is hard if taught as pre setcentric mathe-matism true inside but seldom outside classrooms; and if taught as present setcentric meta-matism that by defining its concepts by abstract self-reference forces learners to construct a meaning themselves. And no, mathematics is not hard if taught as post setcentric many-math, allowing learners to further develop what they bring to school, a quantitative competence created by adaption to outside quantity.

When writing a mathematics curriculum, we must ask: Is its goal to master inside mathematics first as the only means to master outside Many later? Because then Vygotskian constructivism is needed to assign meaning to what is made meaningless by abstract self-reference or by tradition. Or is its goal to master outside Many directly, or via other means if the mathematics tradition is hindering 1 out of 4 or 5 students in acquiring basic numeracy? Because then Piagetian constructivism is better suited to allow students to accommodate existing schemas created by natural adaption to outside numerosity.

As to research, maybe the time has come to reread the Marx inscription in the entrance hall at the Humboldt mother-university in Berlin: “The philosophers have hitherto only interpreted the world in various ways. The point, however, is to change it.”

Changing from abstract single- to concrete double-numbers, mathematics education will meet The Universal Declaration of Human Rights Article 26, saying “Everyone has the right to education. (..) Education shall be directed to the full development of the human personality”; as well as Article 4, saying “No one shall be held in slavery or servitude; slavery (..) shall be prohibited in all their forms.” This also applies to the slavery of an abstract-referring mathematics wanting to colonize children’s own number-language.

Likewise, mathematics education should respect the UN Global Goals for Sustainable Development, where goal 4 about quality education writes in target 4.6 on universal literacy and numeracy: “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy.”

So why teach children different, if they already know?

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34. Visit to Ho Chi Minh City University of Education December 7-13 2019

In December 2019 I paid a visit to the Ho Chi Minh City University of Education. Via Amsterdam and Kuala Lumpur, I arrived at Ho Chi Minh City Friday December 6 in the evening and was welcomed by Le Thai Bao Thien Trung, PhD Associate Professor, Didactic of Mathematics Team, Department of Mathematics, Ho Chi Minh City University of Education who took me to my hotel, a ten minutes' walk from the University.

Saturday, I gave a keynote address at the conference 'Psychology and Mathematics Education' at the University of Education. Sunday, I gave a talk on modeling to a group of master students. Monday, I gave a talk to a class of senior students on a poster presentation from the 'Educating the Educators' conference in Freiburg, Germany, in October, based upon my paper (http://mathecademy.net/educate-educators-2019/) and handed out the notes 'What is Math - and Why Learn it?' and 'Bundle Counting Table'. Tuesday, I gave a talk to the staff on research in mathematics education and networks to join and design research as a methodology to use when researching the implementation of the new activity-based curriculum inspired by Kolb's experimental learning theory.

Saturday December 7: keynote address to the conference

At the conference on Psychology and Mathematics education I gave a keynote address called 'De-modelling Numbers, Operations, and Equations' (http://mathecademy.net/demodel-mathematics/) where I introduced the concept of 'Adaptive Mathematics: Kids' own BundleNumbers with Units, Bundle- & Per-Numbers in Primary & Secondary School', and where I raised the M&M-question: The Goal of Math Education is that to Master outside Many, or Master inside Mathematics (as a means to later, possibly, Master outside Many).

I began by answering the question 'Is mathematics well defined?' by pointing to its three different versions: MetaMatics defining concepts as examples of abstractions instead of as abstractions form examples, MatheMatism true inside but seldom outside classrooms as when adding without units, and ManyMath, a natural science about the outside fact Many.

Then I described the observation that when adapting to Many, Children create Flexible BundleNumbers with units as e.g. 2 2s or 3 4s, which I used to point out that Bundle-Numbers can Shift unit and create a Recount-Formula 8 = (8/2)*2, or T = (T/B)*B recounting a total T in B-bundles by T/B times pushing away Bs; a formula that allows solving equations as u*2 = 8 by recounting 8 in 2s, thus moving 2 to opposite side with opposite sign; and that leads direct to core STEM formulas all expressing proportionality.

Next, I talked about green cells uniting to plants, and grey cells uniting to animals as reptiles, mammals or humans having one, two or three brains for routines, feeling and language. I ended by pointing to Darwin saying: "To SURVIVE, you must ADAPT to the outside world", adding the question "and so must math?"

Thus, as to math education, humans use the reptile brain for routines and rote learning; they use the mammal brain for developing positive or negative feelings towards mathematics; and they use the human brain for creating formulas that allows predicting the behavior of quantities.

As to how a brain adapts to the outside world, I mentioned stimulus and response allowing the brain to develop a word- and a number-language both using full sentences with a subject, a verb and a predicate.

Then I described three kinds of how brains adapt through learning:

Inside-outside Skinner-learning using the reptile brain for rote learning, and using the mammal brain for wishing reward and for fearing punishment. Although widespread in time and space, it is constantly met with the objection: Skinner-learning might lead to skills, but understanding is typically lacking.

Then I described two kinds of learning focusing on understanding, one adapting to inside culture, the other adapting to outside nature.

Inside-Inside Vygotsky-learning focuses on enculturation to the institutionalized university mathematics that by choosing abstract top-down understanding uses the reptile brain for learning definitions. Here Bruner says that school subjects must mirror university subjects to structure a good teaching, providing a scaffolding as the ladder down to the learner's ZPD, Zone of Proximal Development, as described by Vygotsky. Likewise, Skemp says that to understand mathematics, first you must understand understanding; numbers thus must be seen as cardinality, defined as equivalence classes in the set of sets. So, children first must draw arrows between sets before learning number-names. Finally, Vygotsky says that "What a child can do today with assistance, she will be able to do by herself tomorrow". So, learning comes from teaching by a more knowledgeable other. Therefore, good teacher education and good professional development matters. Peer brains must be taught by a major brain even if this might infer colonization as to the nature of numbers when 'preaching' that 0 = Ø, 1 = {Ø}, 2 = {Ø,{Ø}}, etc; and that functions are defined as subset of a set-product {(x,y)} where first-component identity implies second-component identity: if x1 = x2, then y1 = y2.

Outside-inside Piaget-learning focuses on bottom-up understanding through meeting concrete outside examples creating inside schemata meant to assimilate the outside examples, or to accommodate if not validated by meeting resistance. Adaption thus creates flexible schemata about what is outside: I, you, he, she, it, we, you, and they.

Piaget makes the point that "Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover for himself will remain with him visible for the rest of his life". In research, Grounded Theory means creating collective flexible schemata. So, when adapting to Many, learners should use grounded theory to answer the guiding learning questions listed in the curriculum. And teaching should be minimized to supplying concrete material and extra guiding questions, and to be opponents on the learners' findings negotiated through peer-brain communication and learning.

These three ways of adapting brains lead to asking: Can children discover/invent mathematics themselves to obtain a concrete exemplified understanding? To look for an answer we now look for outside roots for inside mathematics by re-rooting or de-modeling digits, operations, equations, functions, and fractions.

First, we de-model digits as icons with as many sticks as they represent: 4 sticks in the 4-icon, etc.

Then, we de-model division, multiplication, subtraction, and addition as icons also: From 9 push away 4s we write 9/4 iconized by a broom, called division. 2 times lifting the 4s to a stack we write 2x4 iconized by a lift called multiplication. From 9 pull away 2 4s' to find un-bundled we write 9 – 2x4 iconized by a rope, called subtraction. Uniting two stacks or blocks A and B next-to or on-top we write A+B iconized by the two directions, called addition.

Now, we bundle-count a Total of 9 in 2s: From 9, 9/2 times we push away 2; then from 9, we pull away 4 2s, leaving 1. Thus, the prediction by the recount-formula is T = 9 = 4B1 2s.

The unbundled can be placed in three ways: next-to the stack iconized by a dot named a decimal point, 4.1 2s; or on-top of the stack counted in bundles as 1 = (1/2) x 2 giving 4½B 2s; or counting what is missing in a full bundle, 5B-1 2s. This de-models decimals, fractions & negatives.

Now, bundle-counting ten fingers in 3s counting singles, bundles, and bundle-bundles gives the counting-sequence: 0B1, 0B2, 0B3 no 1B0, 1B1, 1B2, 1B3 no 2B0, 2B1, 2B2, 2B3 no 3B0, 3B1 or ten.

But 3 Bundles, is 1 Bundle-of-Bundles. So, T = 9 = 1BB 3s or T = ten = 3B1 3s = 1BB1 3s = 1BB0B1 3s = 1BB1B-2 3s = 101 3s

With stacks, bundle-counting ten fingers in 3s create three ways: over-load, normal, under-load.

Likewise, bundle-counting fingers in 2s makes 8 a bundle-bundle-bundle, and ten 1BBB0B1B0, or 1010.

Double-counting in two units creates per-numbers & proportionality. Double-counting in kg & $, we get a per-number 4kg per 5$ = 4kg/5$ = 4/5 kg/$.

With 4kg bridged to 5$, we recount in the per-number. Or we recount the units directly. Or we equate the per-numbers. Or we use the before 1900 'Rule of 3' (regula-de-tri) alternating the units, and, from behind, first multiply, then divide.

Typical question one: 12kg = ?$;

Answers: 12kg = (12/4) x 4kg = (12/4) x 5$ = 15$; or $ = ($/kg) x kg = 5/4 x 12 = 15; or *u*/12 = 5/4, so *u* = 5/4 x 12 = 15; or 'If 4kg is 5$, then 12kg is ?$'; regula-de-tri answer: 12x5/4 = 15

Typical question two: 20$ = ?kg

Answers: 20$ = (20/5) x 5$ = (20/5) x 4kg = 16kg; or kg = (kg/$) x $ = 4/5 x 20 = 16; or u/20 = 4/5, so u = 4/5 x 20 = 16; or 'If 5$ is 4kg, then 20$ is ?kg'; regula-de-tri answer: 20x4/5 = 16

With like units, per-numbers become fractions, both operators needing numbers to become numbers.

In a box filled with 3 red per 5 apples, double-counting reds and apples gives the fraction 3/5 reds/apples.

Question: ? red in 20 apples. Answer: Recount 20 in 5s (the per-number): T = 20 a = (20/5) x 5a gives (20/5) x 3r = 12 red apples.

Or, we equal the per-numbers: *u*/20 = 3/5; so *u* = 3/5 x 20 = 12 found by moving 20 to opposite side with opposite sign

Double-counting the sides in a block. Geometry means to measure earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula. And connected with the angles by per-number formulas double-counting the sides pairwise.

a = (a/c) x c = sinA x c; tanA = a/b = Δy/Δx = gradient

Circle: circumference/diameter = π = n*tan(180/n) for n large = n*sin(180/n) for n large

Counted and re-counted, Totals may be added, but how: next-to or on-top

Thus next-to addition of 4 5s to 2 3s to 3B2 8s means adding areas, and adding areas is integration. And on-top addition of 4 5s to 2 3s to 5B1 5s means making the units like, and changing units is proportionality.

Likewise, per-numbers add as areas, i.e. as integration: Asking "2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg?", the unit-numbers add on-top, but the per-numbers must be multiplied to unit-numbers, thus adding as areas under the per-number graph. Here, multiplication before addition

Reversely, subtracting per-numbers is called differential calculus: Asking "2kg at 3$/kg + 4kg at what = 6kg at 5$/kg?", first we remove the initial 2x3 block and recount the rest in 4s. So here subtraction giving a change, Δ, comes before division, as expected when being the opposite of integration.

Flexible bundle-numbers ease operations since over-load and under-load come in handy:

T = 65 + 27 = 6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92
T = 65 - 27 = 6**B**5 - 2**B**7 = 4**B**-2 = 3**B**8 = 38
T = 7 x 48 = 7 x 4**B**8 = 28**B** 56 = 33**B**6 = 336
T = 336 /7 = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48
T = 336 /7 = 33**B**6 /7 = 35**B**-14 /7 = 5**B**-2 = 48

Adding or subtracting unspecified numbers, we look for a common unit. So, finding T = 4ab^2 + 6abc we remember that factors are the units, and we use a factor-filter to find the common unit 2*a*b, thus giving

T = 4ab^2 + 6abc = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b) + 3*c*(2*a*b) = (2b+3c)*2ab = 2b+3c 2abs

Conclusion: Asking, can children discover/invent mathematics themselves to obtain a concrete exemplified understanding, the answer is YES, if we

- de-model digits as icons with as many sticks as they represent
- use the flexible bundle-numbers children develop when adapting to Many
- de-model operations as means for bundle-counting 8 as 8/2 2s, leading directly to the recount-formula T = (T/B) x B, used to change units, and to
- solve equations as ‘How many 2s in 8?’ by recounting 8 in 2s
- use double-counting to construct per-numbers, fractions and trigonometry
- add both next-to and on-top, making calculus be addition of per-numbers

Discussion: What is the Difference?

		Traditional math	**Adaptive math**
Digits	4	Symbol	Icon with four strokes
Numbers	456	One number	Three numberings, 4**BB**5**B**6
Division	8/2	8 split in 2	8 counted in **2s**
Multiplication	6 x 7	42	6 **7s** or 4**B**2 **tens**
Addition	2+3	2+3 = 5	2 **4s** + 3 **5s** = 2**B**3 **9s** 2 **4s** + 3 **5s** = 4**B**1 **5s**
Equations	3*u = 12	Neutralize (3*u)*1/3 = 12*1/3 (u*3)*1/3 = 4 u*(3*1/3) = 4 u*1 = 4 u = 4	Opposite side & sign u*3 = 12 = (12/3)*3 u = 12/3 = 4
Fractions	2/3	Numbers 1/2 + 2/3 **IS** 7/6	Per-numbers, i.e. operators, needing numbers to become numbers: 1/2 of 2 + 2/3 of 3 **IS** 3/5 of 5

Four Ways to Unite and Split a Total

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2 + 4***B** + 5 (a polynomial) shows the 4 ways to add: +, *, ^, next-to block-addition (integration). Addition and multiplication add changing and constant unit-numbers. Integration and power add changing and constant per-numbers. We might call this beautiful simplicity the ‘Algebra Square’ since in Arabic, algebra means to reunite.

The 4 uniting operations each has a reverse splitting operation:

Addition has subtraction (–), and multiplication has division (/). Power has factor-finding (root, √) and factor-counting (logarithm, log). Integration has per-number finding (differentiation dT/dn = T‘).

Reversing operations is solving equations, done by moving to **opposite side** with **opposite sign**.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $T/n = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫a*dn** $dT/dn = a$	**T = a^n** $n\sqrt{T} = a \quad log_a T = n$

The 'algebra-square' shows the four ways to unite or split numbers.

Recommendation: Learners should be researchers, Extending their already existing adaption to many

- To survive, also math must adapt to the outside world. So, it should adopt the double-numbers children develop before school; and accept fractions as per-numbers, both operators needing numbers to become numbers.
- Hence to survive math must learn from children, not the other way around.
- Designing a micro- or macro-curriculum we should always ask: What is it out there that the learners need to adapt to?
- When adapting, learners should use grounded theory to answer the guiding learning questions listed in the curriculum.
- Teaching should be minimized to supplying concrete material and extra guiding questions, and to be opponents on the learners' findings.

Question Guided Teacher Education may be found at the MATHeCADEMY.net offering free teacher training in many-math, a natural science about Many, using the **CATS** approach, **C**ount & **A**dd in **T**ime & **S**pace; and using PYRAMID**e**DUCATION where a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

- Each pair works together to solve **C**ount&**A**dd problems.
- The coach assists the instructors when instructing their team and when correcting the **C**ount&**A**dd assignments.
- Each teacher pays by coaching a new group of 8 teachers.

Theoretical Background

Tarp, A. (2018). Mastering Many by counting, recounting and double-counting before adding on-top and next-to. *Journal of Math Education, March 2018, 11*(1), 103-117.

Monday, December 8: meeting a group of master students

My talk focused on modelling, which plays a core role in the Vietnamese 2018-curriculum. In the morning I discussed my booklet Mathematics, Modelling and Models (http://mathecademy.net/math-modeling-models/) with professor Trung Le.

I began by repeating core points from my lecture the day before, that humans adapt to the outside world with three brains where the third brain develops two languages, a word-language and a number-language assigning words and numbers to outside things and actions, both using sentences with a subject, a verb and a predicate.

Then I repeated that psychology works with three kinds of learning: inside-outside Skinner-learning and inside-inside Vygostky-learning and outside-inside Piaget-learning. This corresponds to three kinds of mathematics that exists, MetaMatics defining concepts as examples of abstractions instead of as abstractions form examples, MatheMatism true inside but seldom outside classrooms as when adding without units, and ManyMath, a natural science about the outside fact Many.

I recalled that in ancient Greece, the Pythagoreans used the word ‘mathematics’ meaning ‘what we know’ in Greek as a common label for their four areas of knowledge: Music, astronomy, geometry and arithmetic studying many in time, in time and space, in space and by itself.

Then I suggested that since the yesterday lecture focused on primary school, today we could talk about upper secondary school and focus on the pre-calculus level and discuss how modelling could be used at this level. Precalculus is the level where the international ICMI 24 curriculum conference the year before in Japan discussed if all students should have the same curriculum with different degrees of details or whether a totally different curriculum focusing on applications should be offered to students not wanting to continue in a STEM-direction.

Instead I proposed a different approach: Start from scratch and give a new outside-inside understanding or definition of the operations addition, multiplication and power occurring when writing out fully Arabic numbers as 345 = 3*B^2 + 4*B + 5*1.

This allows the inverse operations subtraction and division and root and logarithm to be understood as solutions to equations solved by moving numbers to opposite side with opposite sign.

With 15-3 as the number x that added to 3 gives 15, the equation x+3 = 15 thus is solved by x = 15-3.

With 15/3 as the number x that multiplied with 3 gives 15, the equation x*3 = 15 is solved by x = 15/3.

With 3√15 as the base x that powered to 3 gives 15, the equation x^3 = 15 is solved by x = 3√15.

With log3(243) as the exponent x that with base 3 gives 15, the equation 3^x = 15 is solved by x = log3(243).

This provides a grounded understanding of root as a ‘factor-finder’, and logarithm as a ‘factor-counter’.

This allows modeling right away the two core forecasting questions: Given start and end values in a time-series, what will a future value be; and when will a certain level be reached?

Here the two standard models assume a constant yearly change-number or change-percent.

Adding 5$/year to an initial number 200$ will after 7 years give a total T = 200 + 5*7; or with unspecified numbers, T = b + a*n where the constants b and a may be found by linear regression using an IT-tool.

Adding 5%/year to an initial number 200$ will after 7 years give a total T = 200 *105%^7 since 5% will change 100% to 105% as enlarging-factor; or with unspecified numbers, T = b*a^n with a = 1+ r where the constants b and a may be found by exponential regression using an IT-tool.

Now the two question about a future value or time may be found by solving equations as T = 200 + 5*7 and 300 = 200 + 5*x in the case of a constant yearly change-number; and T = 200*1.05^7 and 300 = 200*1.05^x in the case of a constant yearly change-percent.

The equations may now be solved manually by moving to opposite side with opposite sign, and by using technology to apply the solver-function algebraically, or to identify intersection points geometrically.

Furthermore, the basic number formula T = 345 = 3*B^2 + 4*B + 5*1 contains the core formulas for constant change: proportionality and linear change, T = a*x and T = b+a*x; exponential and power change T = b*a^x and T = b*x^a; as well as accelerated change T = a*x^2 + b*x + c, graphically shown as a bending line, that might also have a cubic term, k*x^3, for counter-bending.

As the next example of modelling I chose the model called 'Project Collection, LafferCurve'. Here the Real-world problem is that we want to collect a charity fund among the school's 500 students by selling tickets at a fixed price. We therefore ask which of the following three collection models will provide the highest contribution: no marketing, marketing without or with a lottery modelled by three different demand-formulas all assuming that The demand would be 500 and 0 at the prices 0$ and 40$, but differing in assuming that the demand would fall quickly or slowly without marketing or with a lottery. This allows three different demand tables to be created with regression giving three different demand formulas showing how the demand Y1 would change with changing unit price x, thus giving a collected fund Y2 = Y1*x that should be maximized.

As the last example I talked about modeling a table containing per-numbers taken from my paper 'Saving Dropout Ryan by a TI-82' included in the booklet 'Math, modeling and models'. Here the Real-world problem is that while driving, a camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s and 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

Here the table contains 5 data sets that allows quartic regression, i.e. a 4. degree 4 polynomial with a 3-fold parabola changing curvature 3 times, to provide the formula y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235. Now the question asked can be answered using formula tables, or using technology, i.e. graphical readings or solver calculations.

Here a need for adding per/numbers arise. Consequently, an example is introduced to draw inspiration from saying: 2 kg at 3$/kg plus 4kg at 5$/kg. Here the unit-numbers 2 and 4 add directly to 6 since the units are like. However, the per-numbers 3 and 5 must be multiplied to $-numbers before being added, but the moment you multiply you create areas, so per-numbers add by their area under the pe-number graph showing how the per-number change with the kg-number. Consequently, per-numbers add by the formula T = Σp*Δx in the case of piecewise constancy, or t = ∫p*dx in the case of local constancy. Likewise, the change of y may always be recounted in the change of x as Δy = (Δy/Δx)*Δx = p*Δx in the case of linearity, or dy = (dy/dx)*dx = y'*dx in the case of local linearity.

Then as an example of different understandings of mathematical concepts I described the content of my phd-work: At the Danish second chance high school, many students didn't pass the precalculus exam making the ministry consider canceling mathematics as a mandatory subject at the 2005 reform. Then I showed that what was taught was, not mathematics, but meta-matics; and that if they changed this to mathematics, everybody would pass. I showed that the curriculum was built on top-down Vygotsky learning saying that before being applied, student must be taught about linear and exponential functions as examples of functions, defined top-down by saying that a function is an example of a set-relation assigning to each element in one set one and only one element in anther set. Which the students heard as "bublibub is an example of bablibab", something the was meaningless and therefore uninteresting unless you wanted to learn it by heart to pass the exam.

By looking at the history of the function concept I saw that Euler defined a function as a name for a calculation contain both specified and unspecified numbers. This allows defining a function by exemplifying bottom up as e.g. y = 2+x but not y = 2+3; which again allowed linear and exponential functions to be defined by outside examples, e.g. saving at home where an initial amount at b$ grows by adding a$ per month, or in a bank where an initial amount at b$ grows by adding r% per month, which let the students ask if this could be called change by adding and change by multiplying as well. Allowing the students own phrasings as parallel labels to the official labels turned out to be so successful that I recommended that the function concept be replaced by variables at the pre-calculus level despite fierce teacher resistance. The Ministry followed my recommendation and kept per-calculus as a mandatory subject.

Later, using regression and a graphical display calculator to transform tables to formulas, the learning process became so quick that a great part of physics could be included in the curriculum as reported in the paper 'Saving Dropout Ryan with a TI-82' (http://mathecademy.net/math-modeling-models/).

I ended by congratulating the students for becoming teachers in a country with perhaps the best curriculum in the world since it emphasizes modeling. A student teacher asked what to do in big classrooms with 40 students. I recommended dividing the class in groups of 5s working as private math consultants modeling the problems in the modeling compendium above thus practicing peer-brain learning.

Tuesday, December 9: A Senior Class Visited to a Poster Exhibition in the Staff Room

Wrong Numbers

~~LineNumbers~~

~~with place values~~ ☹

IconNumbers

BundleNumbers

PerNumbers ☺

Respect & Develop
Kids' own Flexible BundleNumbers

~~T is 48~~ No:
T is **4***B***8** = **3***B***18** = **5***B***-2**

Looking at a calculator we observe the core of mathematics as digits, operations, and equations. However, we teach numbers wrong by presenting them as combinations of digits using a place value system since writing out fully T = 345 = 3*B^2 + 4*B + 5*1 shows that 345 is three numberings of singles, bundles and bundles of bundles, i.e. a bundle-number. Likewise, we silence that digits are icons with as many sticks as they represent; and that fractions are per-numbers, both operators needing numbers to become numbers, as do digits.

A 3year old child describing four fingers held together two by two as two twos shows that when adapting to Many, children develop bundle-numbers that are flexible by accepting that five fingers may be counted both as 1B3, 2B1 and 3B-1 2s, i.e. with an overload, normal or with an under-load. Thus, school should teach flexible bundle-numbers instead of line-numbers.

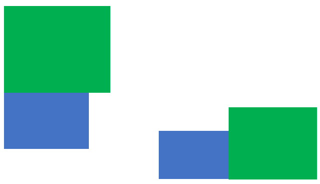
Wrong Operations

~~8/2 is 8 split by 2~~ NO: 8/2 is 8 counted in **2s**

~~5x8 is 40~~ NO: 5x8 is 5 **8s**

~~9-8 = 1~~ NO: 9 – 4x2 = 1, so 9 = 4**B**1 **2s**

2 **3s** + 4 **5s** = ???



OnTop or NextTo

Wrong Math = **Dislike**

As to operators, the tradition teaches addition as repeated adding 1, multiplication as repeated addition; and subtraction and division as reversed addition and multiplication using a carry-principle. This means that 5x8 is taught as 40 and that 8/2 is taught as 8 split in 2.

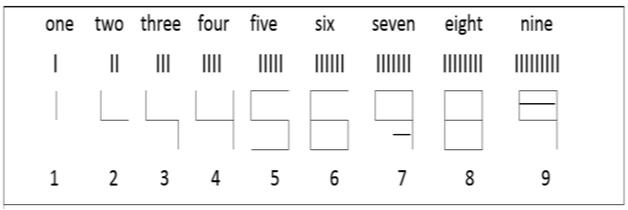
However, counting 8 in 2s means pushing away 2s 8/2 times, making division an icon for a broom pushing away bundles to be stacked making multiplication an icon for lifting bundles, so that 5x8 is 5 8s and only 40 if recounted in tens later.

Counting 9 in 2s, we pull away the stack 4 2s to look for unbundled singles, which makes subtraction an icon for a rope pulling stacks away.

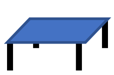
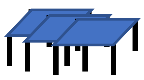
As to addition, it is not well defined since the two totals 2 3s and 4 5s may be added both on-top and next-to.

Numbers are Icons

5 sticks in the 5-icon etc.



	WORD language	NUMBER language
language, grammar	*'is' is a verb*	*'x' is an operation*
Language	This is a table	T = 3x4

To describe the world, we use a word-language combining letters to words, and words to sentences with a subject, a verb and a predicate; and a number-language, combining sticks to icons with as many sticks as they represent if written less sloppy, four sticks in the 4-icon etc., and combining digits to numbers, and combining numbers and operators to formulas or sentences, also containing a subject, a verb and a predicate.

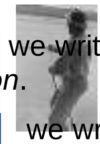
Both languages have a meta-language, a grammar describing the language that describes the world. The word-language teaches the language before the meta-language, the number-language does the opposite thus needing a communicative turn as the one that took place in foreign language education in the 1970s.

Operations are Icons

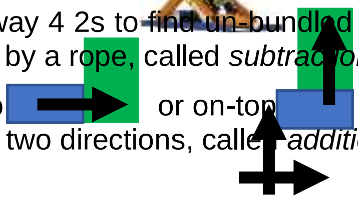
From 9 PUSH away 2s we write 9/2 iconized by a broom, called *division*.

4 times LIFTING 2s to a stack we write 4x2 iconized by a lift called *multiplication*.

From 9 PULL away 4 2s to find un-bundled we write 9 – 4x2 iconized by a rope, called *subtraction*.



UNITING next-to or on-top we write A+C iconized by two directions, called *addition*.

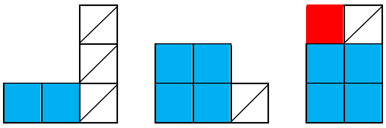


Counting 9 in 2s means pushing away 2s making division an icon for a broom pushing away bundles to be stacked making multiplication an icon for lifting bundles, so that 5x8 is 5 8s and only 40 if recounted in tens later.

Counting 9 in 2s, we pull away the stack 4 2s to look for unbundled singles, which makes subtraction an icon for a rope pulling stacks away.

As to addition, it is not well defined since the two totals 2 3s and 4 5s may be added both on-top and next-to.

Flexible Bundle-Numbers



Overload *Normal* *Underload*

I I I I I = ǁ I I I = ǁ ǁ I = ǁ ǁ ǁ

5 = 1***B***3 = 2***B***1 = 3***B***-1 **2s**

5 = 1.3 = 2.1 = 3.-1 **2s**

= 2 ½ **2s**

48 = 4***B***8 = 3***B***18 = 5***B***-2

T = 65 + 27 = ? = 6***B***5 + 2***B***7 = 8***B***12 = 9***B***2 = 92

T = 65 – 27 = ? = 6***B***5 – 2***B***7 = 4***B***-2 = 3***B***8 = 38

T = 7* 48 = ? = 7* 4***B***8 = 28***B***56 = 33***B***6 = 336

T = 336 /7 = ? = 33***B***6 /7 = 28***B***56 /7 = 4***B***8 = 48

We count 5 in 2s by pushing away bundles of 2s.

Pushing away 1 bundle leaves 3 unbundled as an overload that might be placed next to the stack of bundles as a stack of unbundled singles described as 1**B**3 2s or 1.3 2s thus rooting decimal numbers.

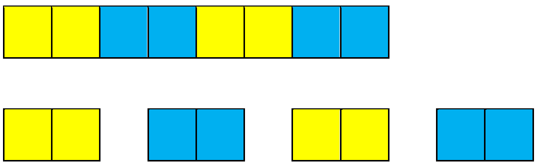
Pushing away 2 bundles leaves 1 unbundled that might be placed next to the stack as 2**B**1 2s or 2.1 2s again using a decimal number.

Or, the unbundled might be placed on-top of the stack counted as bundles as 2 ½ **B** 2s thus rooting fractions; or rooting negative numbers if counting what is missing for an additional bundle, 3**B**-1 2s or 3.-1 2s.

Allowing recounting using overloads or underloads eases calculations. Thus re-counting 336 to 28**B**56 makes division by 7 easier.

The Recount-formula

recounts a total T in B-bundles, e.g. 8 in 2s



8 = (8/2)*2 = 4*2

T = (T/B)*B

From T, T/B times, push B away

The Recount-formula solves equations:

u*2 = 8 = (8/2)*2

u = 8/2 (opposite side & sign)

u + 2 = 8	u*2 = 8	u^8 = 2	2^u = 8
u = 8 – 2	u = 8/2	u = $\sqrt[8]{2}$	u = $\log_2(8)$

The Recount-formula is used in STEM-formulas

m = (m/sec)*sec = speed*sec

$ = ($/hour)*hour = rate*hour

Recounting 8 in 2s means pushing away 2s 8/2 times giving the equation 8 = (8/2)*2, becoming a **recount-formula** using T for the total and B for the bundle: **T = (T/B)*B** saying "from T, T/B times, push B away".

The recount formula solves multiplication equations as u*2 = 8, since recounting 8 in 2s gives u*2 = 8 = (8/2)*2 thus giving the solution u = 8/2 obtained by moving to the opposite side with opposite sign. Thus 8/2 is the number that multiplied with 2 gives 8.

Likewise, the equations u+2 = 8, u^8 = 2 and 2^u = 8 define 8-2, 8√2 and log2(8), making root a 'factor-finder', and logarithm a 'factor-counter'; and again, solving the equations by the opposite side & sign.

STEM-formulas typically involve recounting in another unit thus also using the recount-formula.

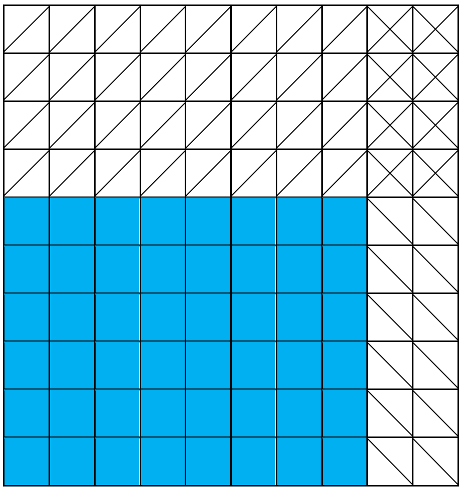
Recounting from icons to ten-numbers

Tables: recount from icons to tens

6 8s = ? tens

longer base - shorter height:

Using underloads:



T = 6 **8s** = 6*8

= (***B***-4)*(***B***-2)

= ***BB*** - 4***B*** - 2***B*** - - 8

= 10***B*** - 6***B*** + 8

= 4***B***8 = 4.8 **tens** = 48

Turning over a stack will change e.g. 2 3s to 3 2s without changing the total. So, in multiplication, the order does not matter, the units may be commuted.

The small table follows directly from using fingers. It is obvious in the case of 2.

In the case of 3, 4x3 = 2x2x3 = 2 6s = 12 seeing a hand as a pawn with six extremities leaving it; and 5x3 = 3x5 = 3 5s = 3 hands = 1B5 = 15.

In the case of 4, 4x4 = 2x2x4 = 2 8s = 2 B-2s = 2B-4 = 1B6 = 16; and 5x4 = 4x5 = 4 hands = 2B0 = 20.

Finally, in the case of 5, 5x5 = 5 5s = 5 hands = 2B5 = 25.

In the large table we recount the numbers from 6 to 10 in bundles as B-4, B-3, etc.; and use a bead pegboard square with two rubber bands to show the actual stack as e.g.

6 8s = 6*8 = (B-4)*(B-2) = BB – 4B –2B + 4 2s removed twice = 4B8 = 42.

So, increasing the base from 8 to 10 means decreasing the height from 6 to 4.2.

This roots the algebraic formula

(a – b)*(c – d) = a*c – a*d – b*c + b*d.

Per-numbers

DoubleCounting in kg & $ gives

a **Per-number 2$/3kg**

<u>**8$ = ?kg**</u>

8$ = (8/2) x 2$

= (8/2) x 3kg = 12kg

<u>**9kg = ?$**</u>

9kg = (9/3) x 3kg

= (9/3) x 2$ = 6$

Like units make per-numbers fractions: **2$/3$ = 2/3**

STEM-formulas typically contain per-numbers coming from double-counting:

m = (m/sec) * sec = speed * sec

kg = (kg/m^3) * m^3 = density * m^3

Joule = (Joule/sec) * sec = Watt * sec

Double-counting in two physical units, we observe that this creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg.

To bridge units, we recount in the per-number:

Asking '6$ = ?kg' we recount 6 in 2s:

T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and

T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Double-counting in the same unit creates fractions and percentages as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%.

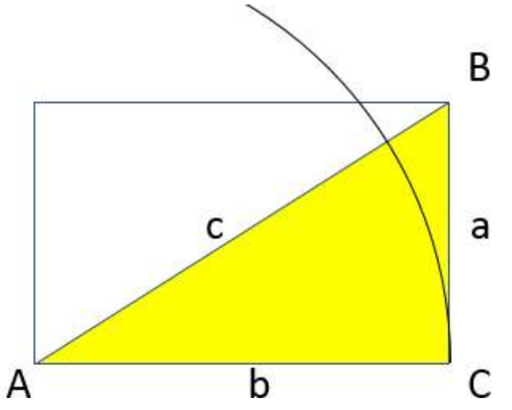
Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%.

We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number.

Recounting Sides in a Box: Trigonometry

Recount sides in a box halved by its diagonal



$T = (T/B)*B$

$a = (a/c)*c$ = **sinA***c

$a = (a/b)*b$ = **tanA***b, or

$\Delta y = (\Delta y/\Delta x)*\Delta x$ = **tanA** $* \Delta x$ = gradient $* \Delta x$

$\pi = n*\sin(180/n)$ for n large

$c*c = a*a + b*b$

Geometry means to measure earth in Greek. The earth can be divided into triangles that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula.

The sides connect with the angles by formulas recounting one side in the other side or in the diagonal:

$a = (a/c)*c = \sin A * c$; or $\sin A = a/c$

$b = (b/c)*c = \cos A * c$; or $\cos A = b/c$

$a = (a/b)*b = \tan A * b$; or $\tan A = a/b$

In a circle, the circumference recounted in diameters is called phi.

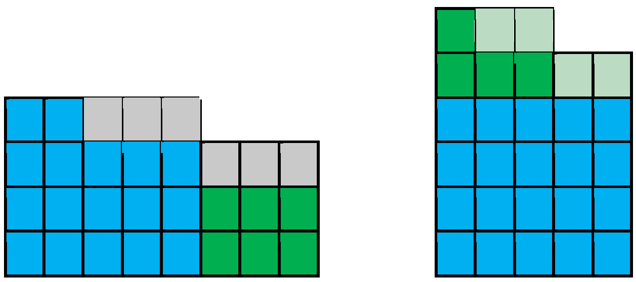
Filling the circle from the inside by right triangles allows phi to be found from a formula:

$\pi \approx n*\sin(180/n)$ for n large.

Addition is not Well Defined

Counted & Recounted, Totals may be Added

BUT: NextTo ➡	or OnTop ⬆
4 **5s** + 2 **3s** = 3**B**2 **8s**	4 **5s** + 2 **3s** = 5**B**1 **5s**
The areas are integrated *Adding areas = Integration*	The units changed to the same *Change unit = Proportionality*



Once counted and recounted, totals may be added, but should they be added next-to or on-top? So, addition is not well defined.

Next-to addition of 4 5s and 2 3s as 8s means adding areas, which is also called integral calculus.

And the reverse question as e.g. 4 5s and ? 3s total 5 8s leads to differential calculus since first we pull away the 4 5s before we recount the rest in 3s by division:

$? = (5*8 – 4*5)/3 = \Delta T/3$.

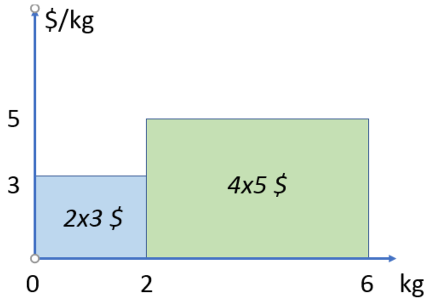
On-top addition of 4 5s and 2 3s means recounting one or both so that the units are changed to the same unit.

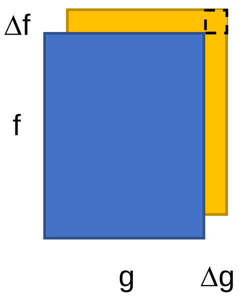
And changing units is proportionality.

Adding fractions and per-numbers: Calculus

2 kg	at	3 $/kg
+ 4 kg	at	5 $/kg
(2+4) kg	at	? $/kg

Unit-numbers add on-top. Per-numbers add next-to as areas under the per-number graph:





Before adding totals, the units must be made the same. With unit-numbers, recounting will do the job. Per-numbers must be multiplied to unit numbers first. But multiplication creates areas. So, per-numbers add by the area under the per-number graph, which is called integral calculus.

With piecewise constant per-numbers this is a quick job. But with locally constant (or continuous) per-numbers this means adding extremely many area-strips.

However, the area always changes with the last strip, so the area can be found by the formula DA = p*Dx, becoming dA = p*dx or d/dx(A) = p or A' = p in the case of locally constancy.

This motivates the development of differential calculus studying what is the derivative d/dx of known formulas.

Pushing slightly the top of two playing cards, we see that

$\Delta(f*g) \approx \Delta f*g + f*\Delta g$ leading to

$(f*g)' = f'*g + f*g'$, leading to

$(x^2)' = 2*x$ and $(x^n)' = n*(x^{n-1})$.

MATHeCADEMY.net

- Teaches Teachers to Teach MatheMatics as **ManyMath**, a natural science about **Many**
- Cures **Math Dislike** when counting fingers in flexible bundle-numbers
- YouTube videos
- Free 1day Skype Seminars

IconNumbers • ReCounting 7 in **5s** & **3s** & **2s**

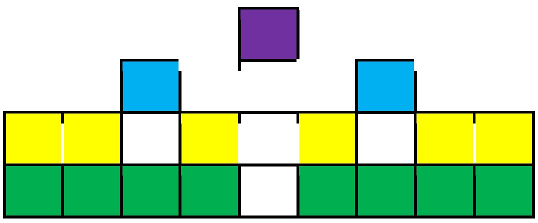
MATHeCADEMY.net offers free Question Guided Teacher Education by Teaching Teachers to Teach MatheMatics as ManyMath, a Natural Science about MANY.

To learn Math, Count & Add MANY, using the CATS method: Count & Add in Time & Space. Primary: C1 & A1 & T1 & S1. Secondary: C2 & A2 & T2 & S2

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

Each pair works together to solve Count & Add problems. The coach assists the instructors when instructing their team and when correcting the Count&Add assignments.

Each teacher pays by coaching a new group of 8 teachers.



4 Ways to Unite & Split

Operations unite/ *split into*	changing	constant
Unit-numbers *m, s, \$, kg*	**T = a + n** *T – a = n*	**T = a * n** *T/n = a*
Per-numbers *m/s, \$/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $log_a T = n$, $\sqrt[n]{T} = a$

We call this beautiful simplicity the '**Algebra Square**' since in Arabic, algebra means to reunite.

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2+4***B**+5 (a polynomial) shows the four ways to add: +, *, ^, next-to block-addition (integration.)

<u>Add</u> & <u>multiply</u> add changing and constant unit-numbers. <u>Integrate</u> & <u>power</u> add changing and constant per-numbers.

The 4 uniting operations have a reverse splitting operation:

Add has <u>subtract</u> (–), and multiply has <u>divide</u> (/).

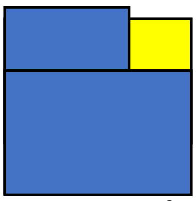
Power has factor-find (<u>root</u>, √) and factor-count (<u>logarithm</u>, log).

Integrate has per-number find (<u>differentiate</u> dT/dn = T').

Reversing operations solve equations by moving to **opposite side** with **opposite sign**.

Quadratic Equations
with 2 playing Cards

Solve the quadratic equation

3  8

u

u 3

u^2 + 6u + 8 = 0

(u+3)^2 = u^2 + 6u + 8 + 1

(u+3)^2 = 0 + 1

u+3 = ± 1

u = -3 ± 1

Solution: **u = -4, u = -2**

In the quadratic equation x ^ 2 + 6*x + 8 = 0 there are two unknown x's so it needs to be rewritten, so there is only one x.

Two playing cards has the width k and the height x + k. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that

(x+k)^2 = x^2 + 2*k*x + k^2, or, have the unknown x only once on the right side:

(x+k)^2 – k^2 = x^2 + 2*k*x, or 'x plus k squared, minus k squared gives x squared + double-k x.

We now rewrite the equation x^2 + 6x + 8 = 0 first to (x^2 + 2*3*x) + 8 = 0, then to (x+3)^2 – 3^2 + 8 = 0, and then to (x+3)^2 – 1 = 0, solved by three times moving to the opposite side.

Solving Equations

ManyMath: Recount

2 x u = 6 = (6/2) x 2	Solved by recounting 6 in 2s
u = 6/2 **= 3**	Test: 2 x 3 = 6 OK

MatheMatics: Neutralize with Abstract Algebra

2 x u = 6	Multiply has 1 as neutral element, and 2 has ½ as inverse element
(2 x u) x ½ = 6x½	Multiply 2's inverse element to both number-names
(u x 2) x ½ = 3	Apply the commutative law to ux2, 3 is the short number-name for 6x½
u x (2 x ½) = 3	Apply the associative law
u x 1 = 3	Apply the definition of an inverse element
u = 3	Apply definition of a neutral element. *With arrows, a test is not needed*

WHAT IS MATH - AND WHY LEARN IT?

"What is math - and why learn it?" Two questions you want me to answer, my dear nephew.

0. What does the word mathematics mean?

In Greek, 'mathematics' means 'knowledge'. The Pythagoreans used it as a common label for their four knowledge areas: Stars, music, forms and numbers. Later stars and music left, so today it only includes the study of forms, in Greek called geometry meaning earth-measuring; and the study of numbers, in Arabic called algebra, meaning to reunite. With a coordinate-system coordinating the two, algebra is now the important part giving us a number-language, which together with our word-language allows us to assign numbers and words to things and actions by using sentences with a subject, a verb and a predicate or object:

"The table is green" and "The total is 3 4s" or "T = 3*4". Our number-language thus describes Many by numbers and operations.

1. Numbers and operations are icons picturing how we transform Many into symbols

The first ten degrees of Many we unite: five sticks into one 5-icon, etc. The icons become units when counting Many by uniting unbundles singles, bundles, bundles of bundles. Operations are icons also:

Counting 8 in **2s** can be predicted by <u>division</u>, iconized by a broom pushing away **2s**: 8/2 = 4, so 8 = 4 **2s**.

Stacking the **2s** into a block can be predicted by <u>multiplication</u>, iconized by a lift pushing up the **2s**: 8 = 4x2.

Looking for unbundled can be predicted by <u>subtraction</u>, iconized by a rope pulling away the 4 **2s**: 8 – 4x2.

Uniting bundles and singles is predicted by <u>addition</u>, iconized by a cross, +, placing blocks next-to or on-top.

Recounting a total T in B-bundles is predicted by a 'recount-formula':	**T = (T/B)*B**
saying 'From T, T/B times, B can be pushed away'.	9/2 4.some
Recounting 9 in **2s**, the calculator predicts the result	9 – 4x2 1
9 = 4B1= 4.1 **2s** = 4 **2s** + 1	

Now, let us write out the total 345 as we say it when bundling in ones, tens, and ten-tens, or hundreds, we get T = 3*B^2 + 4*B + 5*1.

This shows that uniting takes place with four operations: number-addition unite unlike numbers, multiplication unite like numbers, power unite like factors, and block-addition (integration) unite unlike areas. So, one number is really many numberings united by calculations.

Thus, mathematics may also be called calculation on specified and unspecified numbers and formulas.

2. Placeholders

A letter like x is a placeholder for an unspecified number. A letter like f is a placeholder for an unspecified calculation formula. Writing 'y = f(x)' means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Writing y = f(2) is meaningless, since 2 is not an unspecified number. The first letters of the alphabet are used for unspecified numbers that do not vary.

3. Calculation formula predict

The addition calculation T = 5+3 predicts the result without having to count on. So, instead of adding 5 and 3 by 3 times counting on from 5, we can predict the result by the calculation 5+3 = 8.

Likewise, with the other calculations:

- The multiplication calculation 5*3 predicts the result of 3 times adding 5 to itself.
- The power calculation 5^3 predicts the result of 3 times multiplying 5 with itself.

4. Reverse calculations may also be predicted

'5 + 3 = ?' is an example of a forward calculation. '5 + ? = 8' is an example of a reversed calculation, often written as 5 + x = 8, called an equation that asks: which is the number that added to 5 gives 8?

An equation may be solved by guessing, or by inventing a reverse operation called subtraction, x = 8 – 5; so, by definition, 8-5 is the number x that added to 5 gives 8. The calculator says that 8-5 is 3.

We now test to see if this is the solution by calculating separately the left and right side of the equation. The left side gives 5 + x = 5 + 3 = 8. The right side is already calculated as 8.

When the left side is equal to the right side, the test shows that x = 3 is indeed a solution to the equation.

Likewise, with the other examples of reverse calculations:

- $\frac{8}{5}$ is the number x, that multiplied with 5 gives 8. So, it solves the equation 5*x = 8.
- $\sqrt[5]{8}$ is the number x, that multiplied with itself 5 times gives 8. So, it solves the equation x^5 = 8.
- $\log_5(8)$ is the number x of times to multiply 5 with itself to give 8. So, it solves the equation 5^x = 8.

Thus, where the root is a factor-finder, the logarithm is a factor-counter.

Together we see that an equation is solved by 'moving to <u>opposite side</u> with <u>opposite sign</u>'

5 + x = 8	5*x = 8	x^5 = 8	5^x = 8
x = 8 – 5	x = $\frac{8}{5}$	x = $\sqrt[5]{8}$	x = $\log_5(8)$

5. Double-counting creates per-numbers and fractions

Double-counting in two units creates per-numbers as e.g. 3$ per 4kg or 3$/4kg or ¾ $/kg.

To bridge the units, we just recount the per-number: 15$ = (15/3)*3$ = (15/3)*4kg = 20kg.

With the same unit, a per-number becomes a fractions or percent: 3$/4$ = ¾, 3$/100$ = 3%.

Again, the per-number bridges: To find ¾ of 20, we recount 20 in 4s. 20 = (20/4)*4 gives (20/4)*3 = 15.

6. Change formulas

The unspecified number-formula T = a*x^2 + c*x + d contains basic change-formulas:

- T = c*x; proportionality, linearity
- T = c*x + d; linear formula, change by adding, constant change-number, degree1 polynomial
- T = a*x^2 + c*x + d; parabola-formula, change by acceleration, constant changing change-number, degree2 polynomial
- T = a*b^x; exponential formula, change by multiplying, constant change-percent
- T = a*x^b; power formula, percent-percent change, constant elasticity

7. Use

- Asking '3kg at 5$ per kg gives what?', the answer can be predicted by T = 3*5 = 15$.
- Asking '10 years at 5% per year gives what?', the answer can be predicted by the formula T = 105%^10 – 100% = 62.9% = 50% in plain interest plus 12.9% in compound interest.
- Asking 'If an x-change of 1% gives a y-change of 3%, what will an x-change of 7% give?', the answer can be predicted by the approximate formula T = 1.07^3 – 100% = 22.5% = 21% plus 1.5% extra elasticity.
- Asking 'Will 2kg at 3$/kg plus 4kg at 5$/kg total (2+4)kg at (3+5)$/kg?', the answer is 'yes and no'.

The unit-numbers 2 and 4 can be added directly, whereas the per-numbers 3 and 5 must first be multiplied to unit-numbers 2*3 and 4*5 before they can be added as areas.

Thus, geometrically per-numbers add by the area below the per-number curve, also called by integral calculus.

A piecewise (or local) constant p-curve means adding many area strips, each seen as the change of the area, p*Δx = ΔA, which allows the area to be found from the equation A = Δp/Δx, or A = dp/dx in case of local constancy, called a differential equation since changes are found as differences. We therefore invent d/dx-calculation also called differential calculus.

Geometrically, dy/dx is the local slope of a locally linear y-curve. It can be used to calculate a curve's geometric top or bottom points where the curve and its tangent are horizontal with a zero slope.

8. Conclusion.

So, my dear Nephew, Mathematics is a foreign word for calculation, called algebra in Arabic. It allows us to unite and split totals into constant and changing unit- and per-numbers. *Love, your uncle Allan.*

Algebra **unites**/*splits into*	**Changing**	**Constant**
Unit-numbers (meter, second, dollar)	**T = a + b** $T - b = a$	**T = a*b** $\frac{T}{b} = a$
Per-numbers (m/sec, m/100m = %)	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T = a^b** $\sqrt[b]{T} = a$ $\log_a(T) = b$

Bundle Counting Table

Bundle-counting clarifies that we count by bundling, typically in tens

Example 01. Counting Mikado Sticks
The Mikado sticks are positioned next to each other to the right. Counting is done by taking one stick at a time to the left and assembling them in a bundle with an elastic band when we reach ten.

When counting, we say: "0 Bundle 1, 0 bundle 2,. .. "

"Why 0 bundle?" "Because we don't have a bundle yet, before we'll reach ten."

"..., 0 bundle 8, 0 bundle 9, 0 bundle ten, well no, 1 bundle 0".

Example 02. Counting matches
The box says 39, which we read as '3 bundles 9'. We bundle-count as with Mikado sticks.

Extra-option
Some children may find it fun later to count ' 1 bundle less 2, 1 bundle less 1, 1 bundle and 0, 1 bundle and 1 ' as a new way to count ' 0 bundle 8, 0 bundle 9, 1 bundle 0, 1 bundle 1 '. Later again, some children may find It fun to say ' 1 bundle-bundle 0 ' instead of ' ten bundles 0 ' or ' hundred '.

Example 03. Counting ten fingers or ten matches
The ten fingers (or ten matches) bundle are counted in 4s and in 3s while saying "The total is..." and possibly writing "T =..."

Ten counted in 4s	Ten counted in 3s
T = I I I I I I I I I I = ten 1s	T = III I I I I I I I = 1B7 3s
T = IIIIIIIIII = 1 tens = 1B0 tens	T = III III I I I I = 2B4 3s
T = IIII I I I I I I = 1B6 4s	T = III III III I = 3B1 4s
T = IIII IIII I I = 2B2 4s	T = III III III III = 4B-2 3s
T = IIII IIII IIII = 3B-2 4s	T = III III III I = 1BB 0B 1 3s

Table for counting ten tens, or 1 bundle bundles, or 1 hundred:

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~

Wednesday, December 10: talk to the staff

What is research and how to do research? Broadly speaking, research is a search for knowledge, or a search for knowing about something and for knowing about how to do something. So research about mathematics education research is search for knowing about mathematics, about teaching and about learning.

Once you have conducted a research project you would like to publish it. You might send it to a journal, but typically it is very difficult to have your paper accepted here and it often takes a long time. Furthermore, research accepted in journals does not necessarily benefit practice as witnessed by the peer-review crises and replication-crisis described later.

So it might be a bettr idea to send your contribution to specific network conferences to try to have it accepted for presentation and included in the proceedings.

An overview over existing networks can be find in the invitation to and proceedings of the 4year global conferences called ICME, International Congress on Mathematical Education, held for the first time in 1969 and next time in 2020 in Shanghai arranged by the ICMI, International Commission on Mathematics Instruction that also arranges local 2 or 3year conferences as e.g. EARCOME, the ICMI-East Asia Regional Conferences in Mathematics Education, held last time in 2018 in Taiwan, and next time in 2021 in Seoul.

There are also yearly conferences as e.g. CTRAS, Classroom Teaching Research for All Students Conference, held yearly in China, or before the ICME conference each 4 year. This is an including conference that allows networks to be created, and possibly the 2021 conference could take place in Vietnam since that may attract more participants attracted by not having to apply for and pay for a visa to China. Also, it is connected to a welcoming Journal called 'Journal of Mathematics Education', see e.g. my contribution 'Tarp, A, 2018, "Mastering Many", Journal of Mathematics Education, vol 11(1), pp. 103-117'. Its focus on mathematics for all students may be of extra interest to the Vietnamese curriculum described below.

As to deadlines, the ICME has expired, but the CTRAS 2020 has a deadline in February, and the EARCOME 9 2021 in Seoul the deadline is probably the summer 2020. (http://www.ksicmi.org/about/about_01.php).

The coming years Vietnam will implement a new curriculum (issued with Circular No. 32/2018/TT-BGDDT Dec. 26, 2018 of Minister of Education and Training). The curriculum put emphasis on experiments, modelling, STEM and mathematics for everyone:

> Mathematics has more and more applications in life, knowledge and basic mathematical skills helped people solve problems in real life in a systematic and accurate way, contributing to the promotion of social development. Mathematics at schools contributes to the formation and development of key qualities, general competence and mathematical competence for students; develop knowledge, key skills and create opportunities for students to experience and apply mathematics to their practices; establish a connection between mathematical ideas, between mathematics and practice, between mathematics with subjects and other educational activities, in particular with science, natural sciences, physics, chemistry, biology, technology, information to implement STEM education. Mathematics content is often logical, abstract and essential. Therefore, in order to understand and learn mathematics, the mathematics program in the school should ensure the balance between "learning" knowledge and "applying" knowledge into specific problem solving. In the course of learning and applying mathematics, students always have the opportunity to use modern technology, teaching equipment, especially electronic computers and handheld computers to support the performances, explore knowledge, solve mathematical problems. (..) Mathematics helps students to have a relatively general view of mathematics, understand the role and applications of mathematics in practice, the professions related to mathematics so that students have a career orientation, as well as having the ability to explore issues that are related to mathematics throughout life. In addition to core education content, during each academic year, students (especially those with a natural science and technology orientation) are choosing to learn some learning topics. These topics are aimed at enhancing mathematics knowledge, and carefully applying mathematical knowledge to practical practices that meet students ' interests,

needs, and career orientation.(..) The program is the spirit of "Mathematics for everyone", but everyone can learn math in a way that suits personal interests and abilities. The mathematics program attaches importance to the application, cohesion with practices or subjects, other educational activities, especially with subjects aimed at the implementation of STEM education, tied to the modern trend of economic development, science, social life, and global-level issues (such as climate change, sustainable development, financial education,,...). This is also reflected in the practical activities and experiences in mathematics education in various forms such as: implementing mathematical topics and learning projects, especially in practical applications and projects of mathematics.

The new Vietnamese curriculum is based upon Kolb's experimental learning cycle that applies both to the learning of students and teachers. The website 'https://www.simplypsychology.org/learning-kolb.html' writes that the cycle has four phases: 1. Concrete Experience - a new experience or situation is encountered, or a reinterpretation of existing experience; 2. Reflective Observation of the New Experience - of particular importance are any inconsistencies between experience and understanding; 3. Abstract Conceptualization reflection gives rise to a new idea, or a modification of an existing abstract concept (the person has learned from their experience); 4. Active Experimentation - the learner applies their idea(s) to the world around them to see what happens.

Being heavily inspired by Piaget, the four phases may be condensed to two core phases of adaption, assimilation and accommodation, where outside experiences create inside schemata assimilating the outside world, or being accommodated in case of outside resistance.

A Kolb-based curriculum allows teachers to become learners also by doing research in their own classroom: Work out plan A. Observe strong and weak points when implementing plan A. Use this to modify plan A to plan B. Observe strong and weak points when implementing plan B. Use this to modify plan C. Observe strong and weak points when implementing plan C. Etc.

Here Design Research (see e.g. A. Bakker's Design Research in Education Design Research in Education, Routledge 2018) presents itself as a natural method to use in master level and phd level research. To meet the genre-claim of research, the data gathered must be reliable, and the conclusion must be tested for validity. And in design research reliability comes when making systematic observations through notes, interviews, questionnaires etc. And test for validity here means holding on to the strong parts of the actual micro curriculum and changing the weak parts.

At a master level it suffices to work out and test plan A and B. A phd level should also include plan C as well as a detailed collection of data using mixed methods with written leaner reactions leading to questionnaires allowing everybody to react to exemplary statements on an agreement scale 1-5 with 3 as neutral, followed by focus group interviews, and individual learning trajectories.

A plan reflects and writes down goals and means: What are the goals as to skills, understandings, and modeling? What are the means chosen, and not chosen?

Typically, plan A is determined by the choice of the textbook and the teaching tradition. Here difference research (see Tarp Math ed & research 2017. Retrieved from //mathecademy.net/2017-math-articles/) is a method to identify the choices made in a traditional plan A as well as alternatives that may be included and tested in plan B.

As an example of difference research including also STEM in the pre-calculus curriculum I pointed to my paper "Saving Dropout Ryan with a TI-82" in my booklet 'Math, modelling and models'.

There are many examples on alternatives micro-curricula. In primary school: flexible bundle-numbers, next-to and on-top addition. In middle school: per-numbers integrating rates and ratios and fractions, trigonometry before plane geometry, integral calculus used to add per-numbers and fractions. In high school: Regression used to model tables with polynomials. Integral calculus before differential calculus. Other examples on alternatives micro-curricula may be found in my trilogy: Math Ed & Research 2017-2019 found on MATHeCADEMY.net.

The talk also included the present crisis in research replication and peer review. Thus in the article "How Science goes Wrong", The Economist writes:

A rule of thumb among biotechnology venture-capitalists is that half of published research cannot be replicated. Even that may be optimistic. Last year researchers at one biotech firm, Amgen, found they could reproduce just six of 53 "landmark" studies in cancer research. (..) The most enlightened journals are already becoming less averse to humdrum papers. (..) But these trends need to go much further. Journals should allocate space for "uninteresting" work, and grant-givers should set aside money to pay for it. Peer review should be tightened - or perhaps dispensed with altogether, in favour of post-publication evaluation in the form of appended comments. That system has worked well in recent years in physics and mathematics (The Economist, 19 Oct. 2013).

The replication crisis thus comes from the ‘metascience’ observation that many research studies are difficult or impossible to replicate or reproduce. It applies to different fields, e.g. psychology where Pashler and Wagenmakers (2012) writes:

Is there currently a crisis of confidence in psychological science reflecting an unprecedented level of doubt among practitioners about the reliability of research findings in the field? It would certainly appear that there is (p. 528).

The authors refer among others to Ioannidis (2005) who writes:

Scientists in a given field may be prejudiced purely because of their belief in a scientific theory or commitment to their own findings. Many otherwise seemingly independent, university-based studies may be conducted for no other reason than to give physicians and researchers qualifications for promotion or tenure. (..) Prestigious investigators may suppress via the peer review process the appearance and dissemination of findings that refute their findings, thus condemning their field to perpetuate false dogma (p. 0698).

As to the peer review process, LeBel (2015) writes:

In recent years, there has been a growing concern regarding the replicability of findings in psychology (..) I propose a new replication norm that aims to further boost the dependability of findings in psychology (p. 1).

Addressing case series studies, Horton (1996) writes:

The importance of the case series in surgical research is beyond doubt. Therefore, it seems reasonable to ask whether we can trust this study method to yield a valid result. According to conventional epidemiological wisdom, the answer is no (p. 984).

The quality of research was also questioned by Lyotard (1984) distinguishing between consensus and dissension:

Consensus is a component of the system, which manipulates it (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. (..) Returning to the description of scientific pragmatics, it is now dissension that must be emphasized (p. 60-61).

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HO CHI MINH CITY
UNIVERSITY OF EDUCATION
MINISTRY OF EDUCATION AND TRAINING
ENHANCING TEACHER EDUCATION PROGRAM
THE WORLD BANK
INTERNATIONAL CONFERENCE
PSYCHOLOGY AND MATHEMATICS EDUCATION

35. Review 01 ICMT3

The paper contains 6 pages in the specified format. It is reviewed as an empirical study. For each category, the proposal is evaluated as -2: below the standard -1: slightly below the standard 0: meets the standard 1: good 2: excellent

1) Rationale, aim/goals, research questions (0)

There seems to be three research questions for the six pages paper, thus raising the question: Is there room for addressing three research questions in a six pages paper?

The first two is on page 1, chapter 2 section 2: "In our study, we ask how artefacts can be designed for inclusive education, and which inclusive practice can be reconstructed in lessons, when teachers use of the designed artefacts."

The third is on page 3, chapter 2 sections 2: "RQ1: How are plenaries in inclusive classes characterised?"

2) Theoretical framework and related literature (-1)

It would be nice with definitions, discussions and literature on the core concepts: artefacts, on inclusive education and on practices.

On page 2, the paper emphasizes the concept of "differential sensitivity", which does not seem to be a theoretical concept but one found in a textbook and used to allow dividing a lesson into two parts, a common part, and a part differentiating slow and quick learners with adaptive and in-depth activities respectively.

Part of the literature is not related to the research question, if any, but has more the character of name dropping.

3) Methodology / statement of authors position and argumentation (-1)

It would be nice to see how the artefact the 'twenty ten frame' was used by the three categories of students.

It would be nice to discuss or design alternative artefacts or alternative applications for subtraction tasks.

It would be nice to discuss the truth regimes adempted when choosing subtraction tasks as more relevant than giving meaning to the artefact 'minus' occurring both when subtracting as a process and when describing a lack in a total as e.g. T = 8 = Bundle less 2 = B-2 where B is the actual bundle-size ten used when counting.

It would be nice to discuss what kind of mathematics is served by serving the subtraction regime: a North American pre-setcentric version, a present set-centric version or a post-setcentric version.

The paper contains very few data if any on plenaries, mentioned in research question three.

4) Results (-2)

In relation to the first research question you would expect a detailed description and a discussion of different artefacts designed for inclusive education, together with additional adaptive and in-depth questions and tasks. As to the second research question you would expect a description of a wide variety of practices as well as arguments for them being inclusive.

None of this is given here. In the table 1, classroom situation starter plenary, the horizontal and vertical headings are not explained in details, consequently it is difficult to estimate the relevance of these data in connection with the research questions.

The discussion is not related to the research question(s). Furthermore, it introduces new concepts as 'complexity reduction' and 'holistic nature' that should have been introduced and discussed before choosing a methodology. Finally, the discussion ends with postulating several expectations without any grounding and with little relevance to the research question.

5) Clarity (0)

The chapter headings are: INTRODUCTION, DESIGN PRINCIPLES FOR INCLUSIVE MATHEMATICS EDUCATION, DESIGN AND METHODS, ANALYSIS OF AN EPISODE: INCLUSIVE PRACTICES, DISCUSSION

Thus, there is no chapter called 'conclusion' or 'findings'. The word 'design' seems to be used in two different ways, as a design of material, and as a design for the study where the term 'study-plan' might prevent confusion. The method described in the method chapter seems to be one chosen from a more general research project, and not for this study specifically. Thus, the chapters seem to lack coherence and relevance to the research question.

Proof-reading by a native speaker would improve the semantical and grammatical quality.

Question: Can textbooks be artefacts? The 'twenty ten frame' is an outside artefact with independent existence in the world, but is an inside description of it an artefact? According to Heidegger an Existentialism, in a verdict sentence 'A is B', the outside fact A exists, but the inside predicate B is an essence that is chosen and could be different. So including inside descriptions and outside existence as being both artefacts seems to water out the root of the word artefact, to distinguish between 'nature-fact' and human-fact', between a stone and a stone-ax. One of the main points about textbook is that they work best if accompanied by concrete artefacts that exist outside the text in the real world.

A clear recommendation, ACCEPT for presentation without further modification; TO BE CORRECTED as detailed below; REJECT

My recommendation will be "To be corrected"

Suggestions for improvement in a free text field for the authors

The first part of the focus and the curiosity of the paper is very important: Materials for inclusive mathematics education, likewise is asking "how artefacts can be designed for inclusive education".

Since there seems to be little time to collect data for a quality empirical paper, why not change it to a theoretical essay using sociological imagination and 'difference research' to include also narratives (tales about change in time and space), so the research question would be something like: "how can artefacts and narratives be integrated within inclusive education"

This would allow testing on a small scale a new innovative approach possibly with some in-depth descriptions of examples on "making losers users"

An example. The textbook example shows the subtraction 10-3 illustrated by an artefact 'a twenty ten frame'. Integrating a narrative means re-embedding and bridging the inside calculation in an outside problem by using a full number-language sentence with a subject and a verb and a predicate as in the word-language: the total is 7, abbreviated to T = 7.

This allows formulating several outside narratives:

01. "Yesterday there was ten, what happened?"; expecting the answer, "3 has been taken away" leading to the bridging or modeling narrative "T = 7 = 10 – 3" enacted by adding additional 3 to be immediately taken away. "See, we get 7 by taking 3 away from ten"

02. "Tomorrow there will be ten. Can we tell about this by writing differently what we have today?"; expecting the answer "today we have the ten less 3" leading to the bridging or modeling narrative "T = / = 10 less 3" or "T = 10 – 3" enacted by adding additional 3 in the ten-line below to be added first thing tomorrow. "See, we only lack three. So tomorrow we get the ten by then adding the three to the seven we have today"

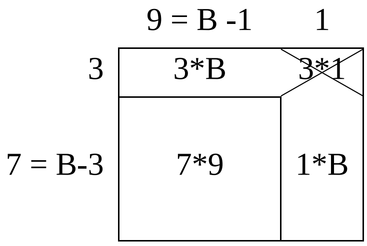
03. “How can a calculator predict what is needed tomorrow?”; expecting the answer “We enter 10-7”

04. “How can we tell a robot what the problem is?”; expecting the answer “Tell me what I need in order to enlarge 7 to ten?” abbreviated to “7 + what = 10” or “7 + ? = 10” or “7 + u = 10” called an unsolved question or equation to be answered or solved by the subtraction u = 10-7, thus isolating the unknow by “moving the other number, to the other side, with the other sign”, a method that works for all future equations with other operations.

Renaissance used a vertical and a horizontal stroke to separate negative numbers and divisors. The horizontal stroke still exists, but the vertical has disappeared except within book-keeping. Using full number-language sentences contains more narrative information about where the result came from so what has importance is not only the end result but also from where it came and why. By bridging the outside existence with the inside essence full number-language sentences also serve as modeling.

In this way we don’t serve a special inside ‘truth regime’ called ‘teaching subtraction’. Instead we help the learners to bridge meaning to the two outside ways the inside horizontal stroke occurs: as ‘taking-away’ in time, or as ‘lack’ in space, thus also introducing negative numbers together with subtraction, i.e. in a natural way. Thus being prepared to allow learning multiplication tables using ‘flexible bundle-numbers” where B stands for the bundle B used when counting totals, in this case tens.

Example 7*9 = 7 9s = (B-3)*(B-1) = BB, less 3B, less 1B, lessless3 = (ten-3-1)B +3 = 6B3 = 63, thus experiencing that ‘lessless’ or negative times negative gives positive as shown by illustrating the product by a ten by ten square where 7*9 is 7 9s. Here the 7 9s come from taking away the horizontal block 3B and the vertical block 1B, only we must add the corner block 3 since it has been taken away twice.



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Tarp, A. (2018b). A Heidegger view on how to improve mathematics education. *Philosophy of Mathematics Education Journal, 33*.

Comment to the second version of the paper

The authors have now revised their papers or poster proposals according to reviewer queries and suggestions. Every author, whose paper or poster proposal required corrections was asked to revise the paper and upload a) a revised version of the paper, b) a revised version with tracked changes and c) a letter to the reviewers with a comment on required changes to the latest revision.

Now, it is time to decide if the revisions have improved the papers or poster proposals in a way that they can be accepted for publication in the conference proceedings. Therefore, we kindly ask you to go to the reviewing area and check if the authors have improved their papers or poster proposals sufficiently according to your suggestions. In the reviewing area you can access the revisions of the same papers that were assigned to you for reviewing.

As to the paper, the second version includes some minor changes as sentences left out or reformulated, and the language is now more fluent; but the original structure is unchanged.

I have received no “letter to the reviewers with a comment on required changes to the latest revision” in order to discuss the relevance of my original comments.

When describing the results, I gave the grade -2 mainly because “The discussion is not related to the research question(s). “ as you would expect of a research paper.

Likewise, we have not have an opportunity to discuss my suggestion as to how the paper could be improved: “Since there seems to be little time to collect data for a quality empirical paper, why not change it to a theoretical essay using sociological imagination and ‘difference research’ to include also narratives (tales about change in time and space), so the research question would be something like: “how can artefacts and narratives be integrated within inclusive education”. This would allow testing on a small scale a new innovative approach possibly with some in-depth descriptions of examples on “making losers users”.”

I am certain that, for the next conference, it will be possible to write an extraordinary interesting paper on materials for inclusive mathematics education; and, if wanted I would like to contribute with offering a cooperation on some of the ideas mentioned in the review.

However, in its present state and missing the dialogue part in its second version, I fail to see that the paper is to be accepted for publication in the conference proceedings.

36. Review 02 ICMT3

The paper contains 6 pages, almost in the specified format. For each category, the proposal is evaluated as -2: below the standard -1: slightly below the standard 0: meets the standard 1: good 2: excellent.

1) Rationale, aim/goals, research questions (-1)

There seems to be no explicit research question. Implicitly, however, the research question might be: ”But you can start to give them feeling and knowledge about the dimensions of this activity [how to write good books] and to let them practise their first steps to become an author.” (page 1)

2) Theoretical framework and related literature (-2)

There are no references in the end even if there is space for it within the 6 pages, especially if the five-six examples of a double page break is removed which will give an additional half page. Some references to authors are given in the text without specifying the work referred to (Example of a lecture, page 3, Reader, page 5).

Several parts have no reference: The importance section, page 1-2; The art of writing section, page 2; Theoretical Component, page 2-3.

3) Methodology / statement of authors position and argumentation (-1)

It would be nice to have a discussion about how to set up a research design and how to choose a research methodology to answer the research question.

Likewise, it would be nice to have some reflections on the fact that the project seems to be an example of export from a highly developed country to less developed countries thus calling for discussions based upon post-colonial theory: Can we be sure that the course will not be another example of a new-colonizing project?

Especially, Germans with direct access to the rich German social theories, are somewhat obliged to include Habermas' theory about the system-world colonizing the life-world into this colonization discussion.

It would be nice to discuss or design alternative courses and to argue why they have not been chosen.

It would be nice to discuss the truth regimes adopted, e.g. when choosing subtraction tasks as more relevant than giving meaning to the artefact 'minus' occurring both when subtracting as a process and when describing a lack in a total as e.g. T = 8 = Bundle less 2 = B-2 where B is the actual bundle-size ten used when counting.

It would be nice to discuss what kind of mathematics is served by serving the subtraction regime: a North American pre-setcentric version, a present set-centric version or a post-setcentric version.

The paper contains very few data if any. Thus it would be nice to have the following statement substantiated: “Experiences (for example in Kosovo) show me that teaching without rich textbooks does not work properly.” (page 2)

Likewise, it would be nice to know what supports the following claim: “Textbooks generate and grant • Steadiness, • Consistency, • Perpetualness, • Continuity, • Sustainability for students and for teachers.“ How are these concepts defined? From what theory are they derived or taken? And what is meant by claiming “There is no Teaching 4.0 or Learning 4.0 or higher”; and how can this claim be supported?

In short, it would be nice to know the reasons for what choices have been made, and between which alternatives.

4) Results (-1)

As results is given a description of the contents of the course with an example of a lecture, again without discussing which alternatives have not been chosen and why, thus in Denmark a discussion has taken place between Mogens Niss' 8 competecies and Allan Tarp's 2 competences, counting and adding.

A short description of three sections called Practical component, Examples of assignments, and the Reader is also enclosed. Here the Reader section perhaps should inform about the alternatives a textbook writer must choose between by informing about the controversies in the grand theories surrounding mathematics and its education: in philosophy between structuralists and existentialists, where German can benefit from their direct access to Heidegger and Nietzsche; in psychology between Vygotsky and Piaget disagreeing about the need for extensive teaching; and within sociology where the agent-structure debate discusses if systems should adapt to actors or the other way around.

Finally, a chapter is included called "THE ORGANISATION AND THE PRAXIS OF THE WORKSHOP".

As to a discussion related to the research question, very little data and discussion is included on the aim "to let them practise their first steps to become an author" . It seems as if the course is more eager to supply the participants with a certain (subject-ideological?) background instead of allowing them to write themselves exemplary texts on core primary and secondary arithmetic, algebra and geometry and allowing their voices to be heard as to how they experienced this writing experience. Instead only a short evaluation to the course is included, which coheres only little to the research question.

5) Clarity (0)

The chapter headings are: INTRODUCTION, THE DESIGN AND FRAMEWORK OF THE WORKSHOP, and The Reader (not following the format given). Inside these chapters there are several sections. Thus in the Reader chapter there are two additional chapters: THE ORGANISATION AND THE PRAXIS OF THE WORKSHOP, SOME BUT IMPORTANT RESULTS AND EXPERIENCES, FURTHER STUDIES AND PROJECTS.

Thus, there is no chapter named after the four review criteria, nor is there a chapter with references even if there is half a page for it if complying with the format criteria. Thus, the chapters seem to lack coherence and relevance to the research question.

Proof-reading by a native speaker would improve the semantical and grammatical quality.

A clear recommendation, ACCEPT for presentation without further modification; TO BE CORRECTED as detailed below; REJECT

My recommendation will be "To be corrected"

Suggestions for improvement in a free text field for the authors

The aim of the report is very essential, to design and realize a workshop in writing math textbooks.

However, the moment we want to export a national workshop to a foreign culture a lot of consideration have to be made, first by informing us about culture encounters, especially in the case of a possible postcolonial situation as here, then to reflect if the participants should be informed about the choices they have to made before writing, or if is better to choose for them a specific knowledge regime.

So, the paper would gain immensely in relevance and quality if the research question could focus on this bey asking e.g. "which considerations should be made when exporting workshops in writing math textbooks to less developed countries?"

And again, here we that are living outside Germany could benefit enormously from scholars with direct access to the great variety of leading German speaking theorists. Also including the Glocksee project with its sociological imagination and exemplary education.

So maybe the paper should be rewritten as a theoretical essay, perhaps based upon Marx Feuerbach these 11 about changing instead of interpreting. After all, with close to 50 years of math education research still leaving many problems unsolved, maybe traditional setcentric mathematics is not the best thing to export if the less developed countries if they should have a chance to meet the OECD 2030 learning framework and supply all citizens with basic numeracy.

Thus, a core question in future workshops could be: how can we write textbooks “making losers users”

An example on the use of ‘difference research’ in textbook writing. In the textbook ‘Das Zahlenbuch’ an example shows the subtraction 10-3 illustrated by an artefact ‘a twenty ten frame’. Integrating a narrative means re-embedding and bridging the inside calculation in an outside problem by using a full number-language sentence with a subject and a verb and a predicate as in the word-language: the total is 7, abbreviated to T = 7.

This allows formulating several outside narratives:

01. “Yesterday there was ten, what happened?”; expecting the answer, “3 has been taken away” leading to the bridging or modeling narrative “T = 7 = 10 – 3” enacted by adding additional 3 to be immediately taken away. “See, we get 7 by taking 3 away from ten”

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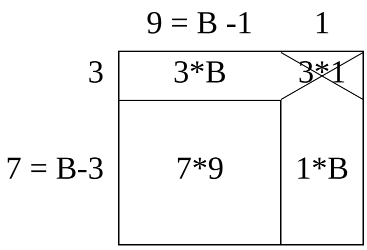
Renaissance used a vertical and a horizontal stroke to separate negative numbers and divisors. The horizontal stroke still exists, but the vertical has disappeared except within book-keeping. Using full number-language sentences contains more narrative information about where the result came from so what has importance is not only the end result but also from where it came and why. By bridging the outside existence with the inside essence full number-language sentences also serve as modeling.

In this way we don’t serve a special inside ‘truth regime’ called ‘teaching subtraction’. Instead we help the learners to bridge meaning to the two outside ways the inside horizontal stroke occurs: as ‘taking-away’ in time, or as ‘lack’ in space, thus also introducing negative numbers together with subtraction, i.e. in a natural way. Thus being prepared to allow learning multiplication tables using ‘flexible bundle-numbers” where B stands for the bundle B used when counting totals, in this case tens.

Example 7*9 = 7 9s = (B-3)*(B-1) = BB, less 3B, less 1B, lessless3 = (ten-3-1)B +3 = 6B3 = 63, thus experiencing that ‘lessless’ or negative times negative gives positive as shown by illustrating the product by a ten by ten square where 7*9 is 7 9s. Here the 7 9s come from taking away the horizontal block 3B and the vertical block 1B, only we must add the corner block 3 since it has been taken away twice.



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Tarp, A. (2018b). A Heidegger view on how to improve mathematics education. *Philosophy of Mathematics Education Journal*, *33*.

Tarp, A. (2019). *A new curriculum - but for which of the 3x2 kinds of mathematics education*. http://mathecademy.net/the-3x2-kinds-of-math-education/

<table>
<tr><td>I received two reviews of my paper "Report on a Workshop 'Writing Maths Textbooks'". I appreciate these reviews and I am grateful to get these comments. I will think about them and change my paper trying to follow their proposals and ideas as far as I am able to do that and to understand them.

But I like to comment those reviews. When I did two reviews myself I discovered that the "Review Guidelines" are not extremely useful in their standardized and normalised perspective on research to judge on every kind of paper:

1) Rationale, aim/goals, research questions

2) Theoretical framework and related literature

3) Methodology / statement of authors position and argumentation

4) Results

5) Clarity

Even so I was quite sceptical about these review categories I thought it is not up to me to criticize them. But being reviewed myself I think it is worth to discuss them</td><td>Thank you for wanting to discuss the criteria of the research genre.

I hope we agree on the importance of constructing labels to differentiate between texts reporting on physical things and actions, and texts portraying physical things and actions as examples of meta-physical constructs, and texts proposing meta-physical constructs for physical things and actions. In short that differentiates between journalism, bachelor or master level essays, and research.

Also within research genre a distinction should be made between top-down and bottom-up research, the former working inside-outside or top-down by testing hypotheses about operationalized theoretical constructs, and the latter working outside-inside or bottom-up by creating new categories from observations.

Thus in the former, the theoretical framework and related literature comes before the search begins, whereas in the latter it comes when discussing if the finding represents new knowledge.

Furthermore, since research is a search for new knowledge, such a search should be guided by a question and a method to guide the search for an answer of some form

The former then uses the order: Focus, existing theory, expanding methodologically, results.

The latter using the order Focus, observations generating categories to be refined by additional observations, comparing with existing categories, result.</td></tr>
<tr><td>In maths education you find research where something is analysed maybe in a theoretical or an empirical way. But there is also mathematics education as a 'design science' (Erich Christian Wittman, Educational Studies in Mathematics December 1995, Volume 29, Issue 4, pp 355–374).</td><td>Thank you very much for referring to the Wittman paper MATHEMATICS EDUCATION AS A 'DESIGN SCIENCE' where he points to the danger that "However, there is the risk that by adopting standards, methods and research contexts from other well-established disciplines, the applied nature of mathematics education may be undermined. In order to preserve the specific status and the relative autonomy of mathematics education, the suggestion to conceive of mathematics education as a 'design science' is made."</td></tr>
</table>

	In the chapter 'THE 'CORE' AND THE 'RELATED AREAS' OF MATHEMATICS EDUCATION' Wittman begins by phrasing Goethe: "The sciences should influence the outside world only by an enlightened practice; basically they all are esoteric and can become exoteric only by improving some practice. Any other participation leads to nowhere." This warning can be used to warn against being too rigorous when applying grand theory to mathematics education; but it can also be a warning against being too rigorous when applying contemporary university theory to mathematics education having as the goal to improve the practise of mastering Many, the outside root of mathematics disciplines as geometry and algebra, both rooted in the outside world as shown by their Greek and Arabic meaning: to measure earth and to reunite quantities. Wittman apparently choses the former understanding by saying "Generally speaking, it is the task of mathematics education to investigate and to develop the teaching of mathematics at all levels (..) Scientific knowledge about the teaching of mathematics (..). This is a shame because what Goethe meant was that such a mathematics becomes the esoteric science he warns against, hoping that instead mathematics would see itself as becoming "exoteric only by improving some practice", in this case the practice of mastering the outside physical fact Many.
I tried to report about a workshop I developed on 'Writing Maths Textbooks'. Now to those review categories:	To make a report part of a research genre it could ask questions as e.g. "Which options exist when designing a workshop on writing Maths textbooks, and which reflections made me chose some options to others that I rejected."
The aim was to develop and deliver such a workshop. I think this is research even so I did not come up with a bundle of research questions. I think the simple triple jump 'research questions -> methods -> results' is more a persiflage on research than research.	As mentioned above developing a workshop becomes research if options and reflexions are included. Likewise, to report on delivering a workshop may become research if also reporting on how the participants reacted to the number of options to choose between letting their own voice be heard and possibly creating categories from these reactions.
While I used in my workshop a lot of literature, I did not find any example or model in the literature how design such a workshop. There is no related literature in narrower sense and no theoretical framework beside my own one	Also, within literature it would be interesting to report on options and reflexions behind the choices made. Especially since there are conflicting views within both philosophy and psychology and sociology. Which raises the question if these controversies should be mediated to the participants or not. And especially if the basic question should be raised, if mathematics education has mathematics as its goal or as a means to the outside goal of mastering many thus informing about the sociological warning against a goal displacement in mathematics education (Bauman, Weber).

I know it is nowadays quite common to use 'methods'. But I am quite sceptical if this is the only warrant for real research or a warrant for research at all. Often these methods are just a thinking corset. Developing my workshop I could try to make the (my) underlining philosophy more explicit. But I did not work with any kind of method – beside thinking.	
The result of my efforts is my design of the workshop. It is not a bundle of answers to a bundle of questions.	
If I could not explain my design it is of course my fault. But I am totally unable to explicate my approach to such a workshop and all my decision on six pages.	
Finally I like to comment the critic: "Proof-reading by a native speaker would improve the semantical and grammatical quality" (Allan Tarp). You are right! But wait a minute: By globalisation of educational research we all should speak now English. If you are not a native speaker you are suddenly disabled. I am born and brought up in Germany. As far as thinking is language based I am thinking German. And a lot of concepts like for example "BIldung" and "Erziehung" and "Stoff" and "Stoffdidaktik" as we use them in the German tradition and history of philosophy and humanities are based and established in our German language. So it is nearly impossibly to translate them into English. My imperfect English should always remind you that I think German even so I try to formulate my thoughts in English.	Thank you for pointing to the question about a common language. A hundred years ago we would be writing and discussing in German because of the great work and fundamental influence of German work in mathematics. As a Dane I had to learn both English and German as foreign languages. English was not so difficult because it is basically 'Anglish' i.e. a western Danish dialect around the Harbor region on the Danish west coast from where the settlers sailed, and some French on top because of the Norman invasion by Normans, also Danes but bilingual and thus importing French parallel words to show superiority. As I see it, when using a foreign language I should respect its meaning and its spelling and its grammar, which a native speaker can help me correcting without knowing anything about the content. But besides I should inform about the existence of special concepts from my own language and gladly accept learning new concepts from another language. Especially, I eagerly listen to and try to read Germans talking about the core concept of mathematics education, called Bildung at the continent and having a history from Herder and Hegel and Humboldt as a sister to marxism and nationalism. Likewise I envy Germans for having direct access to the great thinkers as Nietzsche, Husserl, Heidegger, Weber and Habermas just to mention a few. It would be a great gift to mathematic education if Germans could bring light from these thinkers into it.

37. Comments to ICMT3 Reviewers

Hereby, we forward the reviewers' comments to you. The reviewers indicate weaknesses and make suggestions for improvement of your paper. You are kindly requested to react to their objections and consider their suggestions in the revision of your submission. Please upload

a) a revised version of your submission with tracked changes,

b) a comment to the reviewers,

c) a final version of the paper

before 15 May 2019.

Please note that the organizers of the conference take no liability for the comments of the reviewers. If there are any questions regarding the reviews you might contact the reviewers directly through the reviewing area.

Comments to reviewers by Allan Tarp, April 29, 2019

Below quotes from reviewer comments are inside "", and my comments are inside <>.

Part 01 is a self-review written to clarify the comments. Part 02 is comments to reviewer R2. Part 03 is comments to reviewer R3

<Thank you very much for the two reviews they have been very inspirational in different ways when rewriting the paper.>

Part 01. Self-review of the paper 'Developing the Child's own Mastery of Many'.

1) Rationale, aim/goals, research questions

On the background of the fact demonstrated by OECD studies that 'Research thus still leaves many issues unsolved after half a century (p. 1)', and of the similarity of the word-language and the number-language, the paper asks the question 'IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE? (p. 1)'

2) Theoretical framework and related literature

A theoretical framework is chosen within the grand theories within which mathematics education resides. In philosophy the framework is existentialism giving precedence to existence over essence; and within sociology it is the warning against a goal-displacement where an institutionalized means establishes itself as a goal instead.

As related literature is chosen the 23rd ICMI Study 'Building the Foundation: Whole Numbers in the Primary Grades' to describe and compare with the present tradition, as well as my own article from the Journal of Mathematics Education 11(1).

3) Methodology / statement of authors position and argumentation

As methodology is chosen observations examining and discussing the mastery of Many that children bring to school, thus uncovering a different number type: two-dimensional bundle numbers with units that implies a need for recounting in another unit both icons and tens; as well as a different order of operations that postpones addition until after counting and recounting have taken place.

4) Results

Examining how bundle-numbers can be developed and systematized, a twofold curriculum is designed allowing the children and the teacher to co-examine how research-like guiding questions leads to developing the children's already existing mastery of Many. This leads to the conclusion that 'the number-language children bring to school contain core mathematics as proportionality and calculus, which allows designing a curriculum for all students without splitting it up into tracks (p.6)'.

5) Clarity

The paper respects the genre-criteria of the research genre wanting to generate new knowledge by asking a question, by choosing a method to find an answer, by comparing the answer with already existing knowledge, and by discussing the answer as to its character and as to consequences as to further research.

Part 02. Comments to reviewer R2:

1) Rationale, aim/goals, research questions

"The aim of the paper is to characterise typical questions for a new kind of textbook curriculum, which aims to enable children to develop mathematical knowledge by counting, bundling and adding activities. The paper contains 24 listed questions and accompanying learning opportunities (13 for counting and 11 for adding)."

<I would say that the aim of the paper is to answer the research question 'IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE?'>

2) Theoretical framework and related literature

"The theoretical framework seems to consist in the differentiation between number-language and world-language, but it is not pointed out comprehensibly how these languages can be characterised. Related literature is missing, too. "

< I would say that the theoretical framework is chosen within the grand theories within which mathematics education resides. And that as the related literature is chosen the 23rd ICMI Study 'Building the Foundation: Whole Numbers in the Primary Grades' as well as my own article from the Journal of Mathematics Education 11(1). >

"Additionally, some links are made to philosophical theories (Satre & Marx). However, the theories are neither linked to mathematics education nor are their meanings explained in the context of the paper. Mainly names and phrases are dropped without comprehensible connection to the text."

< Thank you for this remark. Consequently, I have added to the conclusion the point that mathematics education always resides in a greater philosophical and sociological context and therefore should be able to answer to core philosophical and sociological questions as to whether existence precedes essence or whether a goal displacement has taken place in mathematics education. That such questions are rarely if ever addressed or answered traditionally does not mean that they are not important. On the contrary, an activity calling itself scientific research should always have a dialogue with its overarching grand theories.>

"There is a of lack of theories and empirical findings dealing with counting, grouping or the place value system."

<I think that some can be found in the reference in the Journal of Mathematics Education 11(1), 103-117. Furthermore, the page format of six pages make you restrict yourself to answer the research question stated.>

3) Methodology/statement of authors position and argumentation

"The author advocates for a change in curriculum, but the suggested questions/tasks cannot be identified as a curriculum. Some questions seem to be new and contain interesting ideas, but the connection between them is insufficient. They are labelled as "textbook for a question guided curriculum", but information about the grade, competencies or learning principles is missing. Questions/tasks for a "recounting" and "adding curriculum" are listed over almost three pages. However, it is neither clear how the questions are related to each other nor how they are connected to the "language theories". The educational goal does not become clear to me."

<Thank you for commenting on the lack of clarity. This made me rewrite the final chapter. As to the aim of the textbook I write on page 3: 'This calls for a different kind of textbooks that instead of

mediating institutionalized knowledge allow the students to develop their existing knowledge trough guiding outside research-like questions.’ This means that the textbook is given the form of a series of ‘research-like questions’ to ‘allow the students to develop their existing knowledge’, which again means that we are talking about first year students. As to their internal connection, in many cases the following question occurs when working with the pervious question, thus counting sequences leads to asking what bundle to use which leads to what to do with the unbundled which leads to asking how to recount in another unit etc.>

“The bundling activities use non-decimal grouping as well as decimal grouping. It could be a good idea to deal with non-decimal grouping in school but an explanation or a discussion about the aim of non-decimal grouping is missing.”

<To me the important thing is to observe that children use grouping in their number-language so numbers become two-dimensional bundle-numbers, and then to respect this as a fact so that the development is guided by their own numbers instead of by an external abstract theory. Of course, in a longer paper with a different question it could be interesting to compare the two.>

“The “addition curriculum” is introduced by a short explanation of the relation between the operations and followed by a list of questions towards addition, subtraction, multiplication and division. It is not clear if the lists of questions contain the total of school activities. Furthermore, we do not get any information about how the ideas can be transformed into tasks and in which grades they should be realised. “

<Again, detail may be found in the reference Journal of Mathematics Education 11(1), 103-117. Here the important thing is to observe, that with two-dimensional bundle-numbers instead of one-dimensional line-numbers, adding has two forms, next-to addition and on-top addition leading directly to integral calculus and proportionality in grade one. This is a new and original result as reported in details in the journal reference, which leads to a need of designing alternatives textbooks and learning materials that accepts this and also accepts that addition is postponed to after counting and recounting has been practiced, and that organizes mathematics education as a coworking situation between the teacher and the students when investigating mastery of Many guided by research-like questions.>

4) Results/implications for research in the respective area

“The author does not make suggestions how the curriculum ideas could be evaluated or how the questions/tasks could be tested or developed to concrete textbooks task following the idea of design research.”

< Thank you for this remark. Consequently, I have changed and rewritten the conclusion by adding ‘Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees.’>

5) Clarity

“The paper lacks clarity in it´s argumentation, as mentioned above. The limit of pages is exceeded.”

<Thank you for being so detailed and structured in your comments about a lacking clarity. As to the limit of pages, the version I send in and the new version as well contains 6 pages as asked for, so since there is no specification as to the nature of exceedance, I fail to see how this should be the case.>

Part 03. Comments to reviewer R3:

1) Rationale, aim/goals, research questions

“The article is interesting and thought provoking and should be published as a discussion paper presenting different theses and assumptions. But I would not call it a research paper. It is maybe a paper to push or induce research.”

<As shown in the self-review, I would say that the paper is written in the research genre by stating and trying to answer the research question 'IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE?'>

2) Theoretical framework and related literature

"The quotes come from a quite vast extent: PISA, Sartre and Marx "

< As shown in the self-review, I would say that the theoretical framework is chosen within the grand theories within which mathematics education resides. And that as the related literature is chosen the 23rd ICMI Study 'Building the Foundation: Whole Numbers in the Primary Grades' as well as my own article from the Journal of Mathematics Education 11(1). >

"The concepts of mathematics ('goal displacement') and number-language (?) are use in a wrong way. Probably the authors means 'numeracy' as other papers call it."

<Thank you for this remark. Consequently, I have added on page 2 'Numeracy as 'the ability to understand and work with numbers' (Oxford Dictionary) thus has an outside interpretation by the child's own mastery of Many that contrasts the inside interpretations seeing numeracy as applying institutionalized mathematics. '

"But mathematics is neither numeracy nor a number language."

<The nature of mathematics has been left undecided after centuries of competing and conflicting views. Of course, a present dogmatic exist as it has always done and will always do in the future probably. But to think that the present dogmatic is the final is to lack a sense of historic humility. As to the goal of mathematics education, it cannot be to learn mathematics because of the meaningless self-reference in such a statement. Mathematics is an inside discourse of an outside source, the physical fact Many, as expressed by the ancient Greeks choosing the label mathematics for their four areas of knowledge about Many by itself, in space, in time and in time and space, also called arithmetic, geometry, music and astronomy. So, of course, the goal of mathematics education is to develop mastery of Many, or more precisely to develop the mastery of Many the children already possess when coming to school, anything else would be an attempt to brainwash and may be the reason for fifty years of less successful research in mathematics education.>

3) Methodology/statement of authors position and argumentation

<No comments is found as to this criterium>

4) Results/implications for research in the respective area

"I do not understand Q1 to Q24 on the pages 3 to 6. Are they examples for the text of a primary textbook? I would not give them to children in this way."

< Thank you for this remark. Consequently, I have changed and rewritten the first part of page 3 to 'Typically, a mediating curriculum sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a developing curriculum sees mathematics as an exoteric means to develop the children's existing ability to master Many by numbering outside totals and blocks with inside two-dimensional bundle-numbers. This calls for different textbooks from grade 1 that don't mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).'

"The questions in your CONCLUSION "Is the educational goal ...? What to choose ...? Must textbooks ...?" Could be answered in various ways. Your approach is certainly not the only one or only correct one."

< Thank you for this remark. Consequently, I have rewritten the conclusion.>

5) Clarity

“And you should clarify conceptually the area of:

School-mathematics, primary maths, mathematics, application of mathematics, daily life of children and their use of numbers.”

< Thank you for this remark. I can only say that Instead of a top-down conceptualization of these concepts I have chosen a bottom-up approach and methodology that is grounded in the outside educational need to master Many as well as in the mastery of Many that children already possess.>

38. Educating Educators Reviews

Submission ID: 34 Abstract title: Addition-free Math Make Migrants and Refugees STEM Educators Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Declined Reviews: Submission 34 | Review 1 Numerical criteria Content: 1 Significance: 2 Originality: 3 Relevance: 2 Style: 1 Comment I suggest that this is reviewed by a Mathematics Education person because there were some things that I could not follow and this could be because I'm from science education. However I do not feel that the proposal was clear enough. My first reaction is that it is not a good proposal. Having said that, if this is going to be presented as a poster, the presenter can be asked to clarify by participants on a one to one basis according to their needs and backgrounds. The proposal states that it is research-based but no research is reported in the description. Recommend for acceptance or rejection Not suitable Recommend as oral or poster Poster	Submission 34 | Review 2 Numerical criteria Content: 2 Significance: 1 Originality: 4 Relevance: 1 Style: 2 Comment Recommend for acceptance or rejection Not suitable Recommend as oral or poster Poster Reviewers' comments: The proposal states that it is research-based but no research is described in the proposal. It is also not clear how refugees will benefit as a result of the addition-free recounting math. You also need to clarify terms such as Many, for participants who are not familiar. No details are included about what will be presented in the actual poster. Although the approach of ManyMath as described in this proposal is original and interesting it does not fit the conference theme of scaling up PD. Neither the role of the teacher nor PD is addressed in the proposal. Chair's comment: Sorry, your contribution does not fit the theme of the conference (scaling up) professional development of science and mathematics teachers. We suggest you to try to submit your contribution at another conference. Success!
Submission ID: 36 Abstract title: Recounting Before Adding makes Teachers Course Leaders and Facilitators Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Accepted Reviews: Submission 36 | Review 1 Numerical criteria Content: 2 Significance: 2 Originality: 3 Relevance: 2 Style: 2 Comment The idea of the inquiry-based material repository Mathecademy is interesting. It's a pity you only have one reference (only by yourself), we know (worldwide) more teachers/researchers try to broaden the educational possibilities of blended learning (combinations of real meetings and online support). Recommend for acceptance or rejection	Submission 36 | Review 2 Numerical criteria Content: 3 Significance: 3 Originality: 4 Relevance: 3 Style: 3 Comment interesting idea of the self sustaining learning community Recommend for acceptance or rejection Suitable Recommend as oral or poster Oral Reviewers' comments: We think it is better to make a poster of this proposal, we are not convinced this proposal fits the format of a workshop, but are positive to have this contribution at the Ete. Please further specify the target group and what they gain from this presentation for their daily practice. Format of the presentation is not specified is this a

Suitable Recommend as oral or poster Poster	PowerPoint? or web-based? Chair's comment: Please update your proposal for a poster presentation and take account of the comments from the reviewers.
Submission ID: 37 Abstract title: Self-explanatory Learning Material to Improve your Mastery of Many Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Accepted Reviews: Submission 37 | Review 1 Numerical criteria Content: 1 Significance: 2 Originality: 3 Relevance: 3 Style: 2 Comment Topic 2, material dimension, research-based workshop. Reevant materials, but workshop not clearly described in proposal. Recommend for acceptance or rejection Suitable Recommend as oral or poster Oral	Submission 37 | Review 2 Numerical criteria Content: 1 Significance: 2 Originality: 3 Relevance: 1 Style: 3 Comment Not really a workshop described in the proposal, that is why I have to reject it. Recommend for acceptance or rejection Not suitable Recommend as oral or poster Poster Reviewers' comments: In the current form it does not describe the role of teachers or PD in the workshop. Creative and interesting approach does not fit the conference theme. The way this proposal is set up I think it will be better suited as a poster rather than a workshop. Relevant materials to present at a workshop, well linked to the conference theme and the materials dimension. But, the proposal do not give any theoretical grounding, or any structure for the workshop. On the other hand it gives lots of examples, but these are more or less presented in a too complicated way. Do you mean that you have 8 topics to present at the workshop? What about paragraph 10? What do you try to say with this? I assume that the examples are more understandable in practice (hands-on), then written here. I suggest that you add the meta level as an introduction to the workshop (incl reasons behind the examples, and add references if this is meant to be research-based), and some experiences with use of the materials at the end. Chair's comment: Please take account of the comments from the reviewers to update your proposal.
Submission ID: 42 Abstract title: Can Grounded Math and Education and Research Become Relevant to Learners Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Accepted Reviews: Submission 42 | Review 1 Numerical criteria Content: 3 Significance: 3 Originality: 4	Submission 42 | Review 2 Numerical criteria Content: 2 Significance: 2 Originality: 3 Relevance: 2 Style: 3 Comment Seems to be a discussion group proposal. Not clear what Grounded math would be. Recommend for acceptance or rejection Not suitable

Relevance: 3 Style: 3 Comment The paper of the discussion group addresses a very interesting (philosophical) topic. The paper contains also many good ideas, but also a few questionable statements. I suggest the authors to 1) link the DG more clearly to the overall conference theme (scaling up) 2) reflect whether (well) defining education, teacher and learner is the main challenge (I doubt that we will never reach a world-wide accepted solution) 3) avoid stating that 'Lehrer' means using the same word as for learning (Lehrer=Teacher, lehren=teaching; Schüler=learner/student, lernen=learning); by the way: use "Unterricht" not "Unterrichtung" 4) avoid stating that constructism means only working with peers or with manupulatives (this is a misuse; Glasersfeld et al. mean that also lecturing by a teacher leads to students' constructions; teachers, peers etc. can only send sonics, there is no direct transmission of knowledge (however, some teachers and teacher educators think so). Constructivism is understood by Glasersfeld etc. as an epistemological stance, not as a didactical principle) Recommend for acceptance or rejection Suitable Recommend as oral or poster Oral	Recommend as oral or poster Poster Reviewers' comments: Chair's comment: Please update your proposal for an oral presentation and take account of the comments from the reviewers.