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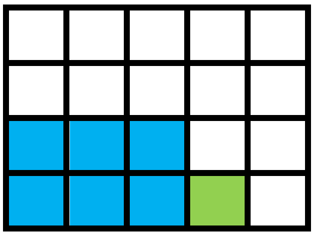
CORE MATH

THROUGH YOUR KID'S

TILE-MATH



T = 2**B**1 **3S**

RECOUNTING BUNDLE-NUMBERS

EARLY TRIGONOMETRY

CALCULUS

STEM

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LEARN CORE MATHEMATICS THROUGH YOUR KID'S TILE-MATH

Asked 'How old next time?', a 3-year-old says 'Four' showing four fingers; but objects when seeing them held together two by two: 'That is not four, that is two twos!' A child thus sees what exists in the world, bundles of 2s, and 2 of them. So, adapting to Many, children develop bundle-numbers with units as 2 2s having 1 1s as the unit, i.e. a tile, also occurring as bundle-of-bundles, e.g. 3 3s, 5 5s or ten tens.

Recounting 8 in 2s as 8 = (8/2)x2 gives a recount-formula $T = (T/B)\mathrm{x}B$ saying 'From the total T, T/B times, B can be pushed away' occurring all over mathematics and science. It solves equations: ux2 = 8 = (8/2)x2, so u = 8/2. And it changes units when adding on-top, or when adding next-to as areas as in calculus, also occurring when adding per-numbers or fractions coming from double-counting in two units. Finally, double-counting sides in a tile halved by its diagonal leads to trigonometry.

The following papers present close to 50 micro-curricula in **Mastering Many** inspired by the bundle-numbers children bring to school.

Learn Core Mathematics Through Your Kid's Tile-Math:
Recounting Bundle-Numbers and Early Trigonometry

This first paper is written for the conference 'The Research on Outdoor STEM Education in the digiTal Age (ROSETA) Conference' planned to take place between 16th and 19th June 2020 at Instituto Superior de Engenharia do Porto in Portugal.

The Power of Bundle- & Per-Numbers Unleashed in Primary School:
Calculus in Grade One – What Else?

This second paper is written for the International Congress for Mathematical Education, ICME 14, planned to be held in Shanghai from July 12th to 19th, 2020, but postponed one year.

Allan Tarp, Aarhus, Denmark, March 2020

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The Power of Bundle- & Per-Numbers Unleashed in Primary School:
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LEARN CORE MATHEMATICS THROUGH YOUR KID'S TILE-MATH: RECOUNTING BUNDLE-NUMBERS AND EARLY TRIGONOMETRY

Fifty years of mathematics education research has failed to create a mathematics for all. This raises the Cinderella question: are there hidden unnoticed alternatives that may make the prince dance? There are. Education may be different, and also math may be different from today's 'meta-matism'. Adapting to Many, children develop bundle-numbers with units as 2 *3s having 1 1s as the unit, i.e. a tile also occurring as bundle-of-bundles, e.g. 3 3s. Recounting a total in bundles gives a recount-formula to solve equations, to change units when adding on-top or when adding areas next-to as in calculus, also occurring when adding per-numbers or fractions coming from double-counting in two units. Double-counting sides in a tile halved by its diagonal leads to trigonometry. Asking 'What kind of mathematics may grow from tiles?' this paper uncovers* 25 *micro-curricula for outdoor STEM education.*

POOR PISA PERFORMANCE, A PERMANENT PANDEMIA?

When evaluating the effect of mathematics education, poor PISA performance occurs all over the world, despite 50 years of increasing research and funding. Thus, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life" (p. 3).

Thus, it seems that it is now demonstrated beyond any doubt that mathematics is indeed difficult to learn so research should continue to understand why this is the case. But then again, can we be sure that what is called mathematics education is not something else.

Education can be different as seen when comparing continental Europe using multi-year lines to prepare teenagers for offices in the public or private sector, with North America using self-chosen half-year blocks to support the identity work of their teenagers.

History shows that also mathematics can be different. The Pythagoreans used it as a label for their knowledge about Many by itself, in space, in time and in space and time, also called arithmetic, geometry, music and astronomy. Where North America still uses specific names, Europe teaches 'metamatism' combining set-derived 'meta-matics', defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples, with 'mathematism' true inside but seldom outside by adding without units (Tarp, 2018).

Finally, math education may suffer from a goal displacement (Bauman, 1990) by changing mathematics from being an inside means to master the outside goal, Many, to being a goal in itself, thus unable to develop the mastery of Many, children bring to school through adaption to Many. Seeing mastery of Many as the end goal, phenomenology has found that Many presents itself to children as bundle-numbers, e.g. 2 3s. (Tarp, 2018). This creates a basis for cycles of experiential learning (Kolb, 1984) and design research to design and test micro-curricula (MC) leading to new cycles. So, respecting children's own two-dimensional bundle-numbers we now ask: What kind of mathematics may grow from tiles?

MC01. Counting by Bundling and Iconizing

Chopsticks placed on tiles show that digits are icons with as many sticks as they represent if written less sloppy. Counting fingers in 3s, we may include the word bundle in the counting sequence by saying '0*B*1, 0*B*2, 0*B*3 no 1*B*0; 1*B*1, 1*B*2, 1*B*3 no 2*B*0; 2*B*1, 2*B*2, 2*B*3 no 3*B*0, 3*B*1'. Counting and stacking tiles and using a folding ruler or a rope to show a bundle, we see that 3 bundles is 1 bundle-of-bundles or a 3x3 square of tiles, so we should instead say '2*B*3 no 3*B*0 no 1*BB*0*B*0'.

To include time in bundle-counting in 3s, we place a cube on each of ten neighboring tiles e.g. on a chess board. A cube moves to the next tile, and both move on to the next tile where they unite to 1 bundle that moves to the tile above, from where it moves to the next tile. This is repeated until tile 9 where 3 bundles unite to 1 bundle-of-bundles that move to the tile above, from where it moves to the last tile, thus showing that ten recounts as 1*BB*0*B*1 3s. Now the same is repeated with bundle-counting in 4s, then in 2s.

MC02. Formulas Predict
Eight persons are bundle-counted in 2s by asking 2s to go to neighboring tiles 4 times. Observing the total 8 splits into 4 2s, we write T = 8 = 4*2. Using an uphill stroke to iconize a broom pushing away bundles, the action 'from 8, push away 2s' may be entered on a calculator as '8/2', thus predicting 4 before carrying out the action. This allows rewriting 8 = 4*2 to 8 = (8/2)*2, or $T = (T/B)*B$ using unspecified numbers, saying 'From *T*, *T*/*B* times, *B* can be pulled away'. This 'recount-formula' predicts changing units. And, rephrasing recounting to "how many 2s in 8?" allows formulating recounting as an equation $u*2 = 8$ solved by $u = 8/2$, i.e. by moving a number to opposite side with opposite calculation sign.

MC03. Unbundled Become Decimals, Fractions or Negative Numbers
Can we predict the result of rearranging in 4s persons placed on tiles as 2 7s? Entering 2*7/4, a calculator says '3.some'. Using a horizonal stroke to iconize a rope pulling away stacks, entering 2*7-3*4 gives the answer '2' thus predicting that 2 7s recount as 3 4s and 2. The unbundled 2 may be placed next-to the stack reported as a decimal number, T = 3*B*2 4s = 3.2 4s, or on-top counted as bundles, 2 = (2/4)*4 = 2/4 4s, reported as a fraction, T = 3 2/4 *B* 4s; or, if counting what is needed for an extra bundle, reported by a negative number, T = 4 *B*-2 4s = 4.-2 4s. The prediction is then tested with persons or cubes placed on tiles.

MC04. Double-counting Creates Per-numbers
Traveling through a row of tiles is rewarded with 3 cubes per 2 tiles thus creating the 'per-number' 3/2 cubes/tiles. Travelling 12 tiles thus gives 3 cubes 6 times, which can be precited by recounting in the per-number: *T* = 12 tiles = (12/2)*2 tiles = (12/2)*3 cubes = 18 cubes. Alternatively, we can equate the per-numbers in an equation $u/12 = 3/2$ solved by moving to opposite side with opposite sign, $u = 3/2*12 = 18$. Per-numbers are all over mathematics and science, e.g. meter = (meter/second)*second = speed*second.

Double-counting in the same unit creates fractions: Marking 2 tiles with a dot for each 3 tiles traveled thus creates a per-number 2 tiles/3 tiles = 2/3. Having travelled 12 tiles, we mark 2 dots 4 times. Again, this can be precited by recounting in the per-number: *T* = 12 tiles = (12/3)*3 tiles marking dots on (12/3)*2 tiles, i.e. 8 tiles with dots.

MC05: Bundle-Numbers Add Next-to or On-top, Directly or Reversed
Once counted as stacks, totals may unite next-to or on-top, iconized by a cross showing the two directions. Adding 2 3s and 4 5s next-to as 8s means adding the areas 2*3 and 4*5, called integral calculus where addition follows multiplication. Adding them on-top, first recounting must change the units to the same. This is called proportionality. Reversed addition asks e.g. '2 3s and how many 5s give 3 8s?'. Here, first the 2*3 stack is pulled away from the 3*8 stack, then recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called reversed integration or differentiation.

MC06: Next-to Addition & Subtraction of Per-numbers and Fractions is Calculus
Throwing a dice 8 times, the outcomes 1 and 6 place 4 cubes on a chess board, and the rest place 2 cubes. When ordered, we may have 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 = 8 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s. This average says that all per-numbers would be 2.6 if alike.

Per-numbers thus add by areas, i.e. by integration. Reversing the question to '2 5s + how many 3s total 3 8s' leads to the equation $2*5 + u*3 = 3*8$ solved by differentiation:

$2*5 + u*3 = 3*8$, so $u*3 = 3*8 – 2*5$, so $u = (3*8 – 2*5)/3 = 4\ 2/3$, or $u = (T2\text{-}T1)/3 = \Delta T/3$

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC07. Double-Counting Sides in a Rectangle Halved by its Diagonal

Two neighboring tiles form a rectangle, that halved by its diagonal creates a right triangle with base *b*, height *h* and diagonal *d*. Recounting pair of sides produces the trigonometry formulas: $h = (h/d)*d = \sin A*d$; $b = (b/d)*d = \cos A*d$, and $h = (h/b)*b = \tan A*b$ that allows a $(b,h) = (+3,+2)$ angle to be predicted by tan-1(2/3) to give 33.7 degrees. This again allows predicting the diagonal: $h = \sin A*d$, or $2 = \sin 33.7*d$, or $d = 2/\sin 33.7 = 3.60$.

MC08. Meeting Pythagoras

Four tiles with base *b* and diagonal *d* form a squared tile. Here 4 diagonals form a square containing 4 half-tiles, i.e. 2 tiles. Consequently $d*d = 2*b*b$, or d^2 = b^2 + b^2.

A tile-pair has base *b*, height *h*, and diagonal *d*. Turned 90 degrees a copy is placed on-top. Repeated three times, this creates a square with the side $b+h$. Inside we find a diagonal square and four half tile-pairs; as well as a $b*b$ square and a $h*h$ square and two tile-pairs. But 4 half tile-pairs is 2 full tile-pairs, so we see that $d*d = b*b + h*h$, or d^2 = b^2 + h^2, making it easy the add squares, you just square the diagonal.

The normal from the right angle divides the diagonal in *p* and *q*. Seeing *b* as $d*\cos A$, and *p* as $b*\cos A$, we get $b*b = (d*\cos A)*b = d*(\cos A*b) = d*p$. So, the extension of the normal divides the diagonal square in two parts equal to the squares of the neighboring rectangle side. Since only the angle *A* is involved this applies to all triangles with angles not above 90 degrees, thus leading to the extended Pythagoras: a^2 = b^2 + c^2 – 2*b*c*cos$A$, etc.

MC09. The Height of an Accessible Flagpole

The point *P* forms a right triangle with a flagpole of height *h*. From a point *Q* in the distance *s* from *P*, the vertical distance to the diagonal is *k*. The angle *P* may now be found in two ways, $\tan P = h/b = k/s$, allowing the unknown height *h* to be found by moving to opposite side with opposite sign, $h = k/s*b$. Solved in a tile-system (coordinate-system), the diagonal is a line passing through three points with the coordinates (0,0), (s,k) and (b,h) providing the slope $c = \tan P = k/s$ and the equation $y = k/s*x$ that with $x = b$ gives $y = k/s*b$.

MC10. The Height of an Inaccessible Flagpole

Three points *A*, *B* and *C* with distances $AB = r$ and $BC = s$ are placed on a line towards the foot of an inaccessible flagpole with height *h*. With the flagpole, *A* and *B* form two right triangles with diagonals *d1* and *d*2. The vertical distance is *p* from *d1* to *B*, and *q* from *d*2 to *C*. The distance from *C* to the foot of the flagpole pole is *c*. Using $\tan A = p/r$ and $\tan B = q/s$, the triangles give two formulas for *h*: $h = (r+s+c)*p/r = (s+c)*q/s$. Solved for *c*, this gives a formula that inserted in the *h*-formula gives $h = p*q*r/(p*r – p*s)$.

MC11. How High the Moon?

A vertical stick with height *h* helps finding the position of the moon or sun. If the shadow has the length *s*, the angle to the sun is predicted by $\tan A = s/h$. A compass helps finding a direction line segment north with the same length as the shadow. The segment between the two has the length *a*. The angle *A* then may be predicted by the formula $\sin(A/2) = ½a/s$.

MC12. The Slope of a Tile

A folding ruler allows creating a right triangle with the bottom line following a sloped tile. A lead line placed in the distance *d* along the ruler from the bottom line will mark on it a distance *b*. The slope of the tile is the same as the top angle *A*, thus predicted by $\tan A = b/d$.

MC13. How Many Turns on a Steep Tile?

On a 30-degree squared tile, a 10-degree road is constructed. How many turns will there be? We let *A* and *B* label the ground corners of the hillside. *C* labels the point where a road from *A* meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance $BC = u$. First, in the triangle BCD, the angle B is 30 degrees, and $BD = u*\cos(30)$. With Pythagoras we get $u^2 = CD^2 + BD^2 = CD^2 + u^2*\cos(30)^2$, or $CD^2 = u^2(1-\cos(30)^2) = u^2*\sin(30)^2$. Next, in the triangle ACD, the angle A is 10 degrees, and $AD = AC*\cos(10)$. With Pythagoras we get $AC^2 = CD^2 + AD^2 = CD^2 + AC^2*\cos(10)^2$, or $CD^2 = AC^2(1-\cos(10)^2) = AC^2*\sin(10)^2$. Finally, in the triangle ACB, $AB = 1$ and $BC = u$, so with Pythagoras we get $AC^2 = 1^2 + u^2$, or $AC = \sqrt{1+u^2}$.

Consequently, $u^2*\sin(30)^2 = AC^2*\sin(10)^2$, or $u = AC*\sin(10)/\sin(30) = AC*r$, or $u = \sqrt{1+u^2}*r$, or $u^2 = (1+u^2)*r^2$, or $u^2*(1-r^2) = r^2$, or $u^2 = r^2/(1-r^2) = 0.137$, giving the distance $BC = u = \sqrt{0.137} = 0.37$. So, two turns: 3.70 cm and 7.40 cm up the tile.

MC14. Rectangles as Extended Squares

A rectangle with base b and height $h = c*b$ may be called a 'c extended square'. It consists of a lower square, $b*b$, and an upper rectangle $(h-b)*b$, becoming a square also if c is 2. The two diagonals $d1$ and $d2$ are raised the angles $A1$ and $A2$ that may be predicted by $\tan A1 = h/b = c$, and $\tan A2 = (h-b)/b = h/b-1 = c-1$. The diagonal $d1$ is predicted by Pythagoras: $d1^2 = b^2 + h^2 = b^2 + c^2*b^2 = b^2*(1+c^2)$, or $d1 = b*\sqrt{1+c^2}$. The diagonal $d2$ is predicted by $d2 = b*\sqrt{1+(1-c)^2}$. Finally, the normal n to the diagonal $d1$ is predicted by $n*d1 = h*b = c*b^2$, or $n* b*\sqrt{1+c^2} = c*b^2$, or $n = b*c/\sqrt{1+c^2} = b* 1/\sqrt{1+1/c^2}$.

So, combining a tile with half of its neighbor will provide a rectangle as a 1.5 extended square where the diagonal angels and lengths may be predicted by proper formulas: $\tan A1 = 1.5$ giving $A1 = 56.3$, $\tan A1 = 0.5$ giving $A1 = 26.6$. Likewise with the two long diagonals: $d1 = b*\sqrt{1+c^2} = b*\sqrt{1+1.5^2} = b*1.80$; and $d2 = b*\sqrt{1+(1-c)^2} = b*\sqrt{1+1.5^2} = b* 1.12$. Finally, the normal: $n = b*1/\sqrt{1+1/c^2} = b*1/\sqrt{1+1/1.5^2} = b*0.832 = b*\sin A1$.

MC15. The Golden Factor Pervades Art

On a tile, a circle with center in the midpoint of an edge and passing through the opposite corners marks two points that extend the tile to one side or the other with the golden factor $\frac{1}{2}(1+\sqrt{5}) \approx 1.62$, i.e. 62%. This extends the original edge with the same factor as will extend the new edge to a length equivalent to adding an extra tile.

Likewise, with a radius half the edge the half-diagonal from the midpoint to the corner will mark a length that will divide the edge in two parts connected by the golden factor.

MC16. Meeting Pi on the Pavement

Two neighboring tiles are circumscribed by a semicircle, again circumscribed by two tiles. On the right tile, the diagonal creates two triangles enveloping a quarter of the semicircle, i.e. 180/4 degrees or pi/4. Consequently, $4*\sin(180/4) < \pi < 4*\tan(180/4)$. In other words, $\pi = n*\sin(180/n) = n*\tan(180/n) = 3.14\ldots$ for n sufficiently large.

MC17. Meeting Algebra

Half of a $b*b$ tile will extend a tile upwards to a $h*b$ playing card. Removing from a $h*h$ square two playing cards, and adding the bottom tile that has been removed twice will leave the square $(h - b)^2 = h^2 - 2*h*b + b^2$. And, removing from a $h*b$ playing card the bottom $b*b$ tile will leave the top $(h - b)*b = h*b - b*b$.

Four playing cards are arranged to form a $(h+b)*(h+b)$ square. Inside we find a $h*h$ square, a $b*b$ square and two playing cards, so, $(h+b)*(h+b) = (h+b)^2 = h*h + b*b + 2*h*b$.

Pulling away a $b*b$ tile from the $h*h$ square leaves a $(h-b)*h$ and a $(h-b)*b$ rectangles that add up to a $(h-b)*(h+b)$ rectangle. Consequently, $(h+b)*(h-b) = h^2 - b^2$.

To solve the quadratic equation $x^2+6x+8 = 0$ we use four tiles forming a square, labeling the first side x and the next 6/2. The $(x+6/2)$ square now contains two $6/2*x$ rectangles and two squares, x^2 and $(6/2)^2$ split in two parts, 8 below and $(6/2)^2-8$ above if possible, all disappearing except for last part. So $(x+6/2)^2 = (6/2)^2 – 8 = 1$ giving $x = -6/2 \pm 1 = -2$ and -4. Looking instead at $x^2+bx+c = 0$ gives the solution $x = -b/2 \pm \sqrt{((b/2)^2 - c)}$.

MC18. Predicting Change
Two $b*b$ tiles form a $h*b$ playing card that is extended with a tape on-top and to the left to show that a change in h and b, Δh and Δb, will give a change in the area, $\Delta(b*h) = \Delta b*h + b*\Delta h$, or with per-numbers, $\Delta(b*h)/(b*h) = \Delta b/b + \Delta h/h$. Thus, with products, the change-percentages almost just add: Changing a kg-number with 3% and a $/kg-number with 5% will make the $-number change with approximately 3% + 5% = 8%. This rule applies to changes less than 10% with decreasing precision. Here we neglect the upper right tape-corner, which is allowed for sufficiently small changes, giving $(b*h)'/(b*h) = b'/b + h'/h$. So, with $y = x^n$ we get that $dy/y = n*dx/x$, or $dy/dx = n*y/x = n*x^n/x = n*x^{(n-1)}$.

MC19. Following Change Formulas e.g. when Playing Golf
Tiles form a coordinate system to move in. Person A starts a (+1,+1) trip in (0,3). Person B starts a (+1,-2) trip in (0,9). Predict where they meet. Person A starts a (+1,+*s*) trip in (0,0) where *s* decreases with 1 from +4 to -4. Person B starts a (+1,+*s*) trip in (10,0) where *s* increases with 1 from -4 to +4. Person A starts a (+1,+*s*) trip in (0,½) where *s* is doubled from ½ the first 4 steps, then halved the next 6 steps. Person B starts a (+1,+*s*) trip in 0,0 where *s* decreases with 1 from +3 to -3, then increases with 1 from -3 to +3. Person A and B start a (+1,+s) trip in (0,0) and (2,0) wanting to end closest to a golf hole in (10,0).

MC20. The Saving Formula
A saving combines a deposit amount *a* with an interest percent *r*, illustrated by two tiles, K1 and K2. K2 receives a one-time deposit a/r, and each period its interest amount $a/r*r = a$ is transferred to K1 after K1 has received its own interest amount. After *n* periods, K1 will contain a saving *A* growing from a deposit amount *a* and an interest percent *r*. But, at the same time, K1 will contain the total interest percent *R* of the initial amount a/r in K2, so $A = a/r*R$, or $A/R = a/r$, where $1+R = (1+r)^n$. Using the doubling-time as the period, a 1$ deposit will after 5 doubling periods save 31$ = 5$ deposit + 26$ compound interest.

MC21: Having Fun with a Tile System and with Bundle-bundle Squares
A dozen recounts as 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed in a tile system, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. With a per-number 2G/3R, a dozen R may be changed to 2G+9R, 4G+6R, 6G+3R, and 8G, thus traveling along a line sloping down with the per-number. With Bundle-Bundle squares we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas: $n*n + 2*n + 1 = (n+1)*(n+1)$; and $n*n – 2*n + 1 = (n-1)*(n-1)$; and $(n–1)*(n+1) = n*n – 1$.

MC22. Pascal's Triangle
A triangle of tiles consists of 1 tile in column 1, 2 in column 2, etc. until column 5. Traveling the triangle with a (+1,+1) win-step or a (+1,+0) loose-step we observe how many roads lead to each tile. Could it be predicted? Start over, but now let a coin decide the next step. Mark the tile with a short stroke. How many times did you win? Could it be predicted?

MC23. Game Theory
Two players A and B choose column and row at 2x2 tiles carrying the numbers 1, 2, 3, 4 in the top and bottom row, indicating what A pays back to B after having received a fix fee from B. Showing paper or stone means choosing the first or second strategy. Thus, if A chooses stone and B paper, A will pay back 2 to B. Which fee makes the game fair? Use cubes to show that the fee is 2.5 if 4 and 2 change places. In the first game, 3 is a stable 'saddle point' going up if A changes, and down if B changes, which they don't want. In the second game both players will be tempted to change, so both will mix strategies, but how?

MC24. Geometry with Handles
Graph theory and topology is geometry where neither distances nor angles matters, but only the relative positions between the points. A classic problem is the supply problem shown with two separated rows with 3 tiles each: How can three houses A, B and C be supplied with electricity, gas and water with no crossing wires? Hint: Connect A and B with gas and water. Conclusion: the task cannot be solved unless we add a bridge whereby the plan changes its topology to a torus which is a plane with a handle.

MC25. The Electric Circuit
To work properly, a device demands energy coming from a supplier, thus creating a circuit of carriers with the demander as a sink and the supplier as a source. If the demand is 16 energy-units per second, and the supplier provides 8 units per carrier, a flow of 2 carriers per second is needed, enabled by a device resistance at 4, calculated as the ratio between the supply and the flow. In technical terms: If a device needs 16 Joule per second (Watt) it needs 2 electrical carriers (Coulombs) per second (Ampere) as current from a battery delivering 8 Joule per Coulomb (Volt). To realize this, the device needs the resistance 4 Ohm. A circuit thus follows two formulas: Demand = Supply * Flow, or Watt = Volt * Ampere, or $P = U * I$; and Supply = Resistance * Flow, or Volt = Ohm * Ampere, or $U = R * I$.

Built into the device, the resistance cannot change, but the voltage can. Supplying a (4ohm, 16watt) device from a 4volt instead of an 8volt battery will give the current 4volt/4ohm or 1 ampere, supplying 4volt*1ampere = 4watt instead of 16watt, i.e. only a quarter of what is needed.

Supplying a (4ohm, 16watt) device from a 16volt instead of an 8volt battery will give the current 16volt/4ohm or 4ampere, supplying 16volt*4ampere = 64watt instead of 16watt, i.e. 4 times what is needed.

With a 12volt battery, a 3ohm device will produce the current 12volt/3ohm or 4ampere supplying 12volt*4ampere = 48watt. With a 1ohm it will be 12ampere and 144watt.

Using a 12volt battery to supply both a (3ohm, 48watt) and a (1ohm, 144watt) device, the total resistance 4ohm will produce the current 12volt/4ohm or 3ampere which on the first device will use 3ohm*3A = 9volt supplying 9volt*3A = 27W ≈ 60% of what is needed; and on the next device will use 1ohm*3A = 3volt supplying 3volt*3A = 9W ≈ 6% of what is needed. Thus, the bigger consumer receives the smaller part. Consequently, multiple devices are connected, not serial as here, but parallel increasing the current to 16A.

Three tiles serve at simulating how 2 cups supply a device labeled (4ohm, 16watt) with energy from an 8volt battery. LEGO-bricks serve as energy-units, and a slow metronome tells when a 'second' has passed. From the battery, 8ers are placed in 2 cups that move to the devise to deliver 2*8 units at the time signal, and then move back empty to refill.

In the case of a 4volt battery, 1 cup carries a 4er and delivers 1*4 watt, only 25% of what is needed. In the case of 1ohm, 8cups deliver 8*4 = 32watt.

With a 12volt battery supplying first a (3ohm, 48watt) and then a (1ohm, 144watt) device, 3 cups each supply 9 to the first device needing 48, and 3 to the second device needing 144 before returning to refill.

Discussion and Recommendation
This paper asked 'What kind of mathematics may grow from tiles?' The background was the phenomenological observation, that Many presents itself to children as bundle-numbers with units as e.g. 2 3s, thus having squared tiles as 1 1s or bundle-bundles as the unit, which allows geometry and algebra to go hand in hand from grade 1.

With units, a core question is how to change it, traditionally leading to proportionality in its classical regula-de-tri form multiplying before dividing, or to its modern form doing the opposite by first finding the per-unit-number. And thus postponed until after teaching all four operations from addition to division.

Recounting turns this order around. First division pushes away bundles to be stacked by multiplication to be pulled away by subtraction in order to find unbundled singles that become decimals, fractions or negative numbers depending on where they are placed. Recounting produces a formula $T = (T/B)*B$ present all over mathematics and science, and showing there things: How to change units, how to solve equations by moving to opposite side with opposite sign, and how per-numbers must be multiplied before being added.

Double-counting in two units produces per-numbers, becoming fractions with like units. As geometrical representations of bundle-numbers, squares and rectangles lead directly to double-counting the sides in rectangles halved by their diagonals, thus allowing trigonometry and tile or coordinate geometry to precede traditional plane geometry.

Addition comes last in two forms, on-top needing proportionality to change units, and next-to adding areas as integral calculus, also occurring when adding per-numbers and fractions with units to avoid mathematism. And reversed addition leads directly to differentiation.

Quadratic expressions, equations and functions also relate to tiles in a natural way, as does differential equations through change formulas directing trips through a tile system

So, it turns out that the core of mathematics springs from tiles once we accept the two-dimensional bundle-numbers children develop while adapting to Many before school. Education needs not teaching 'metamatism' as the only means leading to the end goal, Mastery of Many. Tiles will show the way directly in a concrete way including also outdoor and STEM education; and will perhaps be able to answer the Cinderella question with a yes, there is a hidden unnoticed alternative that makes the prince dance. Tile-mathematics may offer a new Kuhnian paradigm that will finally create a mathematics for all.

This paper has taken a first step in an experiential learning cycle by designing more than a score of micro-curricula to be tested inside and outside classrooms in the hope that after several cycles of redesigning they will become scores making math dislike evaporate.

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THE POWER OF BUNDLE- & PER-NUMBERS UNLEASHED IN PRIMARY SCHOOL: CALCULUS IN GRADE ONE – WHAT ELSE?

In middle school, fraction, percentage, ratio, rate, and proportion create problems to many students. So, why not teach it in primary school instead where they all may be examples of per-numbers coming from double-counting a total in two units. And bundle-numbers with units is what children develop when adapting to Many before school. Here children love counting, recounting, and double-counting before adding totals on-top or next-to as in calculus, also occurring when adding per-numbers. Why not accept, and learn from the mastery of Many that children possess until mathematics takes it away?

MATHEMATICS IS HARD, OR IS IT?

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature. That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969. Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries, and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Also among the countries with poor PISA performance, Denmark has lowered the passing limit at the final exam to around 15% and 20 % in lower and upper secondary school. And, at conferences as e.g. The Third International Conference on Mathematics Textbook Research and Development, ICMT3 2019, high-ranking countries admit they have a high percentage of low scoring students. Likewise at conferences, discussing in the breaks what is the goal of mathematics education, the answer is almost always 'to learn mathematics'. When asked to define mathematics, some point to schoolbooks, others to universities; but all agree that learning it is important to master its outside applications.

So, we may ask, is the goal of mathematics education to master outside Many, or to first master inside mathematics as a means to later master outside Many. Here, institutionalizing mathematics as THE only inside means leading to the final outside goal may risk creating a goal displacement transforming the means to the goal instead (Bauman, 1990) leading on to the banality of evil (Arendt, 1963) by just following the orders of the tradition with little concern about its effect as to reaching the outside goal. To avoid this, this paper will answer the question about the hardness by working backwards, not from mathematics to Many, but from Many to mathematics. So here the focus is not to study why students have difficulties mastering inside mathematics, but to observe and investigate the mastery of outside Many that children bring to school before being forced to learn about inside mathematics instead.

RESEARCH METHOD

Difference research searching for differences has uncovered hidden differences (Tarp, 2018c). To see if the differences make a difference, phenomenology (Tarp, 2018a), experiential learning (Kolb, 1984), and design research (Bakker, 2018) may create cycles of observations, reflections, and designs of micro curricula to be tested in order to create a new cycle for testing the next generation of curricula.

OBSERVATIONS AND REFLECTIONS 01

Asked "How old next time?" a three-year-old will say four showing four fingers, but will react to seeing the fingers held together two by two: "That is not four. That is two twos!" The child thus describes what exists, bundles of 2s, and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as $8 = (8/2)x2$ using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving a formula with unspecified numbers ‘$T = (T/B)xB$’ saying “from T, T/B times, B may be pushed away and stacked”. Consequently, search questions about ‘bundle-numbers’ and ‘recounting’ may be given to small groups of four preschool children to get ideas about how to design a generation-1 curriculum.

GUIDING QUESTIONS

The following guiding questions were used: “There seems to be five strokes in the symbol five. How about the other symbols?”, “How many bundles of 2s are there in ten?”, “How to count if including the bundle?”, “How to count if using a cup for the bundles?”, “Can bundles also be bundled, e.g. if counting ten in 3s?”, “What happens if we bundle to little or too much?”, “How to recount icon-numbers in tens?”, “How to manually recount 8 in 2s, and recount 7 in 2s?”, “What to do if a bundle is not full?”, “How to bundle-count seconds, minutes, hours, and days?”, “How to double-count lengths in centimeters and inches?”, “A dice decided my share in a lottery ticket, how to share a gain?”, “Which numbers can be folded in other numbers than 1s?”, “Asking how many 2s in 8 may be written as $u*2 = 8$, how can this equation be solved?”, “How to recount from tens to icons?”, “How to add 2 3s and 4 5s next-to?”, “How to add 2 3s and 4 5s on-top?”, “ 2 3s and some 5s gave 3 8s, how many?”, “How to add totals bundle-counted in tens?”, “How to subtract totals bundle-counted in tens?”, “How to add per-numbers?”, “How to enlarge or diminish bundle-bundle squares?”, “What happens when recounted stacks are placed on a squared paper?”, “What happens when turning or stacking stacks?”

Observations and reflections 02

Data and ideas allowed designing Micro Curricula (MC) with guiding questions and answers (Q, A).

MC31: Digits as Icons
With strokes, sticks, dolls, and cars we observe that four 1s can be bundled into 1 fours that can be rearranged into a 4-icon if written less sloppy. So, for each 4 1s there is 1 4s, or there is 1 4s per 4 1s. In this way, all digits may be iconized, and used as units for bundle-counting (Tarp, 2018b).

MC32: Bundle-counting Ten Fingers
A total of ten ones occurring as ten fingers, sticks or cubes may be counted in ones, in bundles, or with ‘underloads’ counting what must be borrowed to have a full bundle. Count ten in 5s, 4s, 3s, and 2s.

In 5s with bundles: 0*B*1, …, 0*B*4, 0*B*5 no 1*B*0, 1*B*1, …, 1*B*4, 1*B*5 no 2*B*0.

In 5s with bundles and underloads: 1*B*-4, 1*B*-3, …, 1*B*0, 2*B*-4, … , 2*B*0.

MC33: Counting Sequences Using Tens and Hundreds
In oral counting-sequences the bundle is present as tens, hundreds, thousands, ten thousand (wan in Chinese) etc. By instead using bundles, bundles of bundles etc. it is possible to let power appear as the number of times, bundles have been bundled thus preparing the ground for later writing out a multi-digit number fully as a polynomial, $T = 345 = 3BB4B5 = 3*B^2 + 4*B + 5*1$.

Count 10, 20, 30, …, 90, 100 etc. Then 1*B*, 2*B*, …, 9*B*, ten*B* no 1*BB*.

Count 100, 200, 300, …, 900, ten-hundred no thousand. Then 1*BB*, 2*BB*, …, 9*BB*, ten*BB* no 1*BBB*.

Count 100, 110, 120, 130, …, 190, 200 etc. Then 1*BB*0*B*, 1*BB*1*B*, …, 1*BB*9*B*, 1*BB*ten*B* no 2*BB*0*B*.

A dice shows 3 then 4. Name it in five ways: thirty-four, three-ten-four, three-bundle-four, four-bundle-less6, and forty less 6. Travel on a chess board while saying 1*B*1, 2*B*1, 3*B*1, 3*B*2, …; 3*B*4.

MC34: Cup-counting and Bundle-bundles
When counting a total, a bundle may be changed to a single thing representing the bundle to go to a cup for bundles, later adding an extra cup for bundles of bundles. Writing down the result, bundles and unbundled may be separated by a bundle-letter, a bracket indicating the cups, or a decimal point.

Q. T = two hands, how many 3s?

A. With 1 3s per 3 1s we count 3 bundles and 1 unbundled, and write *T* = 3*B*1 3s = 3]1 3s = 3.1 3s showing 3 bundles inside the cup, and 1 unbundled outside. However, 3 bundles are 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 1*BB*0*B*1 3s = 1]0]1 3s = 10.1 3s with an additional cup for the bundle-bundles.

Q. *T* = two hands, how many 2s?

A. With 1 2s per 2 1s we count 5 bundles, *T* = 5*B*0 2s = 5]0 2s = 5.0 2s. But, 2 bundles is 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 2*BB*1*B*0 2s = 2]1]0 2s = 21.0 2s. However, 2 bundles-of-bundles is 1 bundle-of-bundles-of-bundles, 1*BBB*, so with bundle-bundle-bundles we write *T* = 1*BBB*0*BB*1*B*0 2s= 1]0]1]0 2s = 101.0 2s with an extra cup for the bundle-bundle-bundles.

MC35: Recounting in the Same Unit Creates Underloads and Overloads

Recounting 8 1s in 2s gives *T* = 4*B*0 2s. We may create an underload by borrowing 2 to get 5 2s. Then *T* = 5*B*-2 2s = 5]-2 2s = 5.-2 2s. Or, we may create an overload by leaving some bundles unbundled. Then *T* = 3*B*2 2s = 2*B*4 2s = 1*B*6 2s. Later, such 'flexible bundle-numbers' will ease calculations.

MC36: Recounting in Tens

With ten fingers, we typically use ten as the counting unit thus becoming 1*B*0 needing no icon.

Q. *T* = 3 4s, how many tens? Use sticks first, then cubes.

A. With 1 tens per ten 1s we count 1 bundle and 2, and write *T* = 3 4s = 1*B*2 tens = 1]2 tens = 1.2 tens, or *T* = 2*B*-8 tens = 2.-8 tens using flexible bundle-numbers. Using cubes or a pegboard we see that increasing the base from 4s to tens means decreasing the height of the stack. On a calculator we see that 3x4 = 12 = 1.2 tens, using a cross called multiplication as an icon for a lift stacking bundles. Only the calculator leaves out the unit and the decimal point. Often a star * replaces the cross x.

Q. *T* = 6 7s, how many tens?

A. With 1 tens per ten 1s we count 4 bundles and 2, and write *T* = 6 7s = 4*B*2 tens = 4]2 tens = 4.2 tens. Using flexible bundle-numbers we write *T* = 6 7s = 5*B*-8 tens = 5]-8 tens = 5.-8 tens = 3*B*12 tens. Using cubes or a pegboard we see that increasing the base from 7s to tens means decreasing the height of the stack. On a calculator we see that 6*7 = 42 = 4.2 tens.

Q. *T* = 6 7s, how many tens if using flexible bundle-numbers on a pegboard?

A. *T* = 6 7s = 6*7 = (*B*-4) * (*B*-3) = *BB*-3*B*-4*B*+4*3 = 10*B*-3*B*-4*B*+1*B*2 = 4*B*2 since the 4 3s must be added after being subtracted twice.

MC37: Recounting Iconizes Operations and Creates a Recount-formula for Prediction

A cross called multiplication is an icon for a lift stacking bundles. Likewise, an uphill stroke called division is an icon for a broom pushing away bundles. Recounting 8 1s in 2s by pushing away 2-bundles may then be written as a 'recount-formula' 8 = (8/2)*2 = 8/2 2s, or *T* = (*T*/*B*)**B* = *T*/*B* *B*s, saying "From *T*, *T*/*B* times, we push away *B* to be stacked". Division followed by multiplication is called changing units or proportionality. Likewise, we may use a horizontal line called subtraction as an icon for a rope pulling away the stack to look for unbundled singles.

These operations allow a calculator predict recounting 7 1s in 2s. First entering '7/2' gives the answer '3.some' predicting that pushing away 2s from 7 can be done 3 times leaving some unbundled singles that are found by pulling away the stack of 3 2s from 7. Here, entering '7-3*2' gives the result '1', thus predicting that 7 recounts in 2s as 7 = 3*B*1 2s = 3]1 2s = 3.1 2s.

Recounting 8 1s in 3s gives a stack of 2 3s and 2 unbundled. The singles may be placed next-to the stack as a stack of unbundled 1s, written as *T* = 8 = 2.2 3s. Or they may be placed on-top of the stack counted in bundles as 2 = (2/3)*3, written as *T* = 8 = 2 2/3 3s thus introducing fractions. Or, as *T* = 8 = 3.-1 3s if counting what must be borrowed to have another bundle.

Q. *T* = 9, 8, 7; use the recount-formula to predict how many 2s, 3s, 4s, 5s before testing with cubes.

MC38: Recounting in Time

Counting in time, a bundle of 7days is called a week, so 60days may be recounted as T = 60days = (60/7)*7days = 8B4 7days = 8weeks 4days. A bundle of 60 seconds is called a minute, and a bundle of 60 minutes is called an hour, so 1 hour is 1 bundle-of-bundles of seconds. A bundle of 12hours is called a half-day, and a bundle of 12months is called a year.

MC39: Double-counting in Space Creates Per-Numbers or Rates

Counting in space has seen many units. Today centimeter and inches are common. ‘Double-counting’ a length in inches and centimeters approximately gives a ‘per-number’ or rate 2in/5cm shown with cubes forming an L. Out walking we may go 3 meters each 5 seconds, giving the per-number 3m/5sec. The two units may be bridged by recounting in the per-number, or by physically combining Ls.

Q. T = 12in = ?cm; and T = 20cm = ?in

A1. T = 12in = (12/2)*2in = (12/2)*5cm = 30cm; and A2. T = 20cm = (20/5)*5cm = (20/5)*2in = 8in

MC40: Per-numbers Become Fractions

Double-counting in the same unit makes a per-number a fraction. Recounting 8 in 3s leaves 2 that on-top of the stack become part of a whole, and a fraction when counted in 3s: T = 2 = (2/3)*3 = 2/3 3s.

Q. Having 2 per 3 means having what per 12?

A. We recount 12 in 3s to find the number of 2s: T = 12 = (12/3)*3 giving (12/3) 2s = (12/3)*2 = 8. So, having 2/3 means having 8/12. Here we enlarge both numbers in the fraction by 12/3 = 4.

Q. Having 2 per 3 means having 12 per what?

A. We recount 12 in 2s to find the number of 3s: T = 12 = (12/2)*2 giving (12/2) 3s = (12/2)*3 = 18. So, having 2/3 means having 12/18. Here we enlarge both numbers in the fraction by 12/2 = 6.

MC41: Per-numbers Become Ratios

Recounting a dozen in 5s gives 2 full bundles, and one bundle with 2 present, and 3 absent: T = 12 = 2B2 5s = 3B-3 5s. We say that the ratio between the present and the absent is 2:3 meaning that with 5 places there will be 2 present and 3 absent, so the present and the absent constitute 2/5 and 3/5 of a bundle. Likewise, if recounting 11 in 5s, the ratio between the present and the absent will be 1:4, since the present constitutes 1/5 of a bundle, and the absents constitute 4/5 of a bundle. So, splitting a total between two persons A and B in the ration 2:4 means that A gets 2, and B gets 4 per 6 parts, so that A gets the fraction 2/6, and B gets the fraction 4/6 of the total.

MC42: Prime Units and Foldable Units

Bundle-counting in 2s has 4 as a bundle-bundle. 1s cannot be a unit since 1 bundle-bundle stays as 1. 2 and 3 are prime units that can be folded in 1s only. 4 is a foldable unit hiding a prime unit since 1 4s = 2 2s. Equal number can be folded in 2s, odd numbers cannot. Nine is an odd number that is foldable in 3s, 9 1s = 3 3s. Find prime units and foldable units up to two dozen.

MC43: Recounting Changes Units and Solves Equations

Rephrasing the question “Recount 8 1s in 2s” to “How many 2s are there in 8?” creates the equation ‘u*2 = 8’ that evidently is solved by recounting 8 in 2s since the job is the same:

If u*2 = 8, then u*2 = 8 = (8/2)*2, so u = 8/2 = 4.

The solution u = 8/2 to u*2 = 8 thus comes from moving a number to the opposite side with the opposite calculation sign. The solution is verified by inserting it in the equation: u*2 = 4*2 = 8, OK.

Recounting from tens to icons gives equations: “42 is how many 7s” becomes u*7 = 42 = (42/7)*7.

MC44: Next-to Addition of Bundle-Numbers Involves Integration

Once recounted into stacks, totals may be united next-to or on-top, iconized by a cross called addition.

To add bundle-numbers as 2 3s and 4 5s next-to means adding the areas 2*3 and 4*5, called integral calculus where multiplication is followed by addition.

Q. Next-to addition of 2 3s and 4 5s gives how many 8s?

A1. *T* = 2 3s + 4 5s = (2*3+4*5)/8 8s = 3.2 8s; or A2. *T* = 2 3s + 4 5s = 26 = (26/8) 8s = 3.2 8s

MC45: On-top Addition of Bundle-Numbers Involves Proportionality
To add bundle-numbers as 2 3s and 4 5s on-top, the units must be made the same by recounting.

Q. On-top addition of 2 3s and 4 5s gives how many 3s and how many 5s?

A1. *T* = 2 3s = (2*3/5)*5 = 1.1 5s, so 2 3s and 4 5s gives 5.1 5s

A2. *T* = 2 3s + 4 5s = (2*3+4*5)/5 5s = 5.1 5s; or *T* = 2 3s + 4 5s = 26 = (26/5) 5s = 5.1 5s

MC46: Reversed Addition of Bundle-Numbers Involves Differentiation
Reversed addition may be performed by a reverse operation, or by solving an equation.

Q. Next-to addition of 2 3s and how many 5s gives 3 8s?

A1: Removing the 2*3 stack from the 3*8 stack, and recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called differentiation.

A2: The equation 2 3s + *u**5 = 3 8s is solved by moving to opposite side with opposite calculation sign

*u**5 = 3 8s – 2 3s = 3*8 – 2*3, so *u* = (3*8 – 2*3)/5 = 18/5 = 3 3/5, giving 3.3 5s.

MC47: Adding and Subtracting Tens
Bundle-counting typically counts in tens, but leaves out the unit and the decimal point separating bundles and unbundled: *T* = 4*B*6 tens = 4.6 tens = 46. Except for e-notation with a decimal point after the first digit followed by an e with the number of times, bundles have been bundled: *T* = 468 = 4.68e2.

Calculations often leads to overloads or underloads that disappear when re-bundling:

Addition: 456 + 269 = 4*BB*5*B*6 + 2*BB*6*B*9 = 6*BB*11*B*15 = 7*BB*12*B*5 = 7*BB*2*B*5 = 725.

Subtraction: 456 – 269 = 4*BB*5*B*6 – 2*BB*6*B*9 = 2*BB*-1*B*-3 = 2*BB*-2*B*7 = 1*BB*8*B*7 = 187

Multiplication: 2 * 456 = 2 * 4*BB*5*B*6 = 8*BB*10*B*12 = 8*BB*11*B*2 = 9*B*1*B*2 = 912

Division: 154 /2 = 15*B*4 /2 = 14*B*12 /2 = 7*B*6 = 76

MC48: Next-to Addition & Subtraction of Per-Numbers and Fractions is Calculus
Throwing a dice 8 times, the outcome 1 and 6 places 4 cubes on a chess board, and the rest 2 cubes. When ordered it may be 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s called the average: If alike, the per-numbers would be 2.6 cubes per square. Thus per-numbers add by areas, i.e. by integration. Reversing the question to 2 5s + ? 3s total 3 8s then leads to differentiation: 2 5s + ? 3s = 3 8s gives the equation

2*5 + *u**3 = 3*8, so *u**3 = 3*8 – 2*5, so *u* = (3*8 – 2*5)/3 = 4 2/3, or *u* = (*T*2-*T*1)/3 = Δ*T*/3

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC49: Having Fun with Bundle-Bundle Squares
On a pegboard we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas:

$n*n + 2*n + 1 = (n+1)*(n+1)$; and $n*n – 2*n + 1 = (n-1)*(n-1)$; and $(n–1)*(n+1) = n*n – 1$.

Two $s*s$ bundle-bundles form two squares that halved by their diagonal d gives four half-squares called right triangles. Rearranged, they form a diagonal-square $d*d$. Consequently, $d*d = 2*s*s$

Four $c*b$ playing cards with diagonal d are placed after each other to form a $(b+c)*(b+c)$ bundle-bundle square. Below to the left is a $c*c$ square, and to the right a $b*b$ square. On-top are 2 playing cards. Inside there is a $d*d$ square and 4 half-cards. Since 4 half-cards is the same as 2 cards, we have the formula $c*c + b*b = d*d$ making it easy the add squares, you just square the diagonal.

MC50: Having Fun with Halving Stacks by its Diagonal to Create Trigonometry
Halving a stack by its diagonal creates two right triangles. Traveling around the triangle we turn three times before ending up in the same direction. Turning 360 degrees implies that the inside angles total 180 degree, and that a right angle is 90 degrees. Measuring a 5up_per_10out angle to 27 degrees we see that tan(27) is 0.5 approximately. So, the tan-number comes from recounting the height in the base.

MC51: Having Fun with a Squared Paper
A dozen may be 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed on a squared paper with the lower left corners coinciding, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. Traveling by saying "3up_per_1out, 2up_per_1out, …, 3down_per_1out" allows the end points to follow a parabola. With a per-number 2G/3R, a dozen R can be changed to 2G+9R, 4G+6R, 6G+3R, and 8G. Plotted on a square paper with R horizontally and G vertically will give a line sloping down with the per-number.

MC52: Having Fun with Turning and Combining Stacks
Turned over, a 3*5 stack becomes a 5*3 stack with the same total, so multiplication-numbers may commute (the commutative law). Adding 2 7s on-top of 4 7s totals (2+4) 7s, 2*7+4*7 = (2+4)*7 (the distributive law). Stacking stacks gives boxes. Thus 2 3s may be stacked 4 times to the box T = 4*(2*3) that turned over becomes a 3*(2*4) box. So, 2 may freely associate with 3 or 4 (the associative law).

DISCUSSION AND RECOMMENDATION

This paper asks: what mastery of Many does the child develop before school? The question comes from observing that mathematics education still seems to be hard after 50 years of research; and from wondering if it is hard by nature or by choice, and if it is needed to achieve its goal, mastery of Many.

To find an answer, phenomenology, experiential, and design research is used to create a cycle of observations, reflections, and testing of micro curricula designed from observing the reflections of preschoolers to guiding questions on mastering Many. The first observation is that children use two-dimensional bundle-numbers with units instead of the one-dimensional single numbers without units that is taught in school together with a place value system. Reflecting on this we see that units make counting, recounting, and double-counting core activities leading to proportionality by combining division and multiplication, thus reversing the order of operations: first division pulls away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to identify unbundled singles that becomes decimal, fractional or negative numbers. And that recounting between icons and tens leads to equations when asking e.g. 'how many 5s are 3 tens?' And that units make addition ambiguous: shall totals add on-top after proportionality has made the units like, or shall they add next-to as an example of integral calculus adding areas, and leading to differentiation when reversed? Finally, we see that flexible bundle-numbers ease traditional calculations on ten-based numbers.

Testing the micro curricula will now show if mathematics is hard by nature or by choice. Of course, investments in traditional textbooks and teacher education, all teaching single numbers without units, will deport testing to the outskirts of education, to pre-school or post-school; or to special, adult, migrant, or refugee education; or to classes stuck in e.g. division, fractions, precalculus, etc. All that is needed is asking students to count fingers in bundles. Recounting 8 in 2s thus directly gives the proportionality recount-formula 8 = (8/2)*2 or $T = (T/B)*B$ used in STEM, and to solve equations.

Likewise, direct and reversed next-to addition leads directly to calculus. Furthermore, testing micro curricula will allow teachers to practice action learning and action research in their own classroom.

Phenomenologically, it is important to respect and develop the way Many presents itself to children thus providing them with the quantitative competence of a number-language. Teaching numbering instead of numbers thus creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education (Widdowson, 1978). The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not lean from children instead?

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A BUNDLE COUNTING TABLE

A guide to bundle-counting in pre-school.

Bundle-counting clarifies that we count by bundling, typically in tens.

Example 01. Counting Mikado Sticks

The Mikado sticks are positioned next to each other to the right. Counting is done by taking one stick at a time to the left and assembling them in a bundle with an elastic band when we reach ten.

When counting, we say: "0 Bundle 1, 0 bundle 2,. .. "

"Why 0 bundle?" "Because we don't have a bundle yet, before we'll reach ten."

"..., 0 bundle 8, 0 bundle 9, 0 bundle ten, well no, 1 bundle 0".

Example 02. Counting matches

The box says 39, which we read as '3 bundles 9'. We bundle-count as with Mikado sticks.

Extra-option

Some children may find it fun later to count ' 1 bundle less 2, 1 bundle less 1, 1 bundle and 0, 1 bundle and 1 ' as a new way to count ' 0 bundle 8, 0 bundle 9, 1 bundle 0, 1 bundle 1 '. Later again, some children may find It fun to say ' 1 bundle-bundle 0 ' instead of ' ten bundles 0 ' or ' hundred '.

Example 03. Counting ten fingers or ten matches

The ten fingers (or ten matches) bundle are counted in 4s and in 3s while saying "The total is..." and possibly writing "T =..."

Ten counted in 4s	Ten counted in 3s
T = I I I I I I I I I I = ten 1s	T = III I I I I I I I = 1B7 3s
T = IIIIIIIIII = 1 tens = 1B0 tens	T = III III I I I I = 2B4 3s
T = IIII I I I I I I = 1B6 4s	T = III III III I = 3B1 4s
T = IIII IIII I I = 2B2 4s	T = III III III III = 4B-2 3s
T = IIII IIII IIII = 3B-2 4s	T = III III III I = 1BB 0B 1 3s

Table for counting ten tens, or 1 bundle bundles, or 1 hundred:

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~