Math

Modeling

&

Models

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What is Math - and Why Learn it?

"What is math - and why learn it?" Two questions you want me to answer, my dear nephew.

0. What does the word mathematics mean?

In Greek, 'mathematics' means 'knowledge'. The Pythagoreans used it as a common label for their four knowledge areas: Stars, music, forms and numbers. Later stars and music left, so today it only includes the study of forms, in Greek called geometry meaning earth-measuring; and the study of numbers, in Arabic called algebra, meaning to reunite. With a coordinate-system coordinating the two, algebra is now the important part giving us a number-language, which together with our word-language allows us to assign numbers and words to things and actions by using sentences with a subject, a verb and a predicate or object:

"The table is green" and "The total is 3 4s" or "T = 3*4". Our number-language thus describes Many by numbers and operations.

1. Numbers and operations are icons picturing how we transform Many into symbols

The first ten degrees of Many we unite: five sticks into one 5-icon, etc. The icons become units when counting Many by uniting unbundles singles, bundles, bundles of bundles. Operations are icons also:

Counting 8 in **2s** can be predicted by division, iconized by a broom pushing away **2s**: 8/2 = 4, so 8 = 4 **2s**.

Stacking the **2s** into a block can be predicted by multiplication, iconized by a lift pushing up the **2s**: 8 = 4x2.

Looking for unbundled can be predicted by subtraction, iconized by a rope pulling away the 4 **2s**: 8 – 4x2.

Uniting bundles and singles is predicted by addition, iconized by a cross, +, placing blocks next-to or on-top.

Recounting a total T in B-bundles is predicted by a 'recount-formula': **T = (T/B)*B**

saying 'From T, T/B times, B can be pushed away'.

9/2	4.some
9 – 4x2	1

Recounting 9 in **2s**, the calculator predicts the result

9 = 4B1= 4.1 **2s** = 4 **2s** + 1

Now, let us write out the total 345 as we say it when bundling in ones, tens, and ten-tens, or hundreds, we get T = 3*B^2 + 4*B + 5*1.

This shows that uniting takes place with four operations: number-addition unite unlike numbers, multiplication unite like numbers, power unite like factors, and block-addition (integration) unite unlike areas. So, one number is really many numberings united by calculations.

Thus, mathematics may also be called calculation on specified and unspecified numbers and formulas.

2. Placeholders

A letter like x is a placeholder for an unspecified number. A letter like f is a placeholder for an unspecified calculation formula. Writing 'y = f(x)' means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Writing y = f(2) is meaningless, since 2 is not an unspecified number. The first letters of the alphabet are used for unspecified numbers that do not vary.

3. Calculation formula predict

The addition calculation T = 5+3 predicts the result without having to count on. So, instead of adding 5 and 3 by 3 times counting on from 5, we can predict the result by the calculation 5+3 = 8.

Likewise, with the other calculations:

- The multiplication calculation 5*3 predicts the result of 3 times adding 5 to itself.
- The power calculation 5^3 predicts the result of 3 times multiplying 5 with itself.

4. Reverse calculations may also be predicted

'5 + 3 = ?' is an example of a forward calculation. '5 + ? = 8' is an example of a reversed calculation, often written as 5 + x = 8, called an equation that asks: which is the number that added to 5 gives 8? An equation may be solved by guessing, or by inventing a reverse operation called subtraction, x = 8 – 5; so, by definition, 8-5 is the number x that added to 5 gives 8. The calculator says that 8-5 is 3.

We now test to see if this is the solution by calculating separately the left and right side of the equation. The left side gives 5 + x = 5 + 3 = 8. The right side is already calculated as 8.

When the left side is equal to the right side, the test shows that x = 3 is indeed a solution to the equation.

Likewise, with the other examples of reverse calculations:

- $\frac{8}{5}$ is the number x, that multiplied with 5 gives 8. So, it solves the equation 5*x = 8.
- $\sqrt[5]{8}$ is the number x, that multiplied with itself 5 times gives 8. So, it solves the equation x^5 = 8.
- $\log_5(8)$ is the number x of times to multiply 5 with itself to give 8. So, it solves the equation 5^x = 8.

Thus, where the root is a factor-finder, the logarithm is a factor-counter.

Together we see that an equation is solved by 'moving to <u>opposite side</u> with <u>opposite sign</u>'

5 + x = 8	5*x = 8	x^5 = 8	5^x = 8
x = 8 – 5	x = $\frac{8}{5}$	x = $\sqrt[5]{8}$	x = $\log_5(8)$

5. Double-counting creates per-numbers and fractions

Double-counting in two units creates per-numbers as e.g. 3$ per 4kg or 3$/4kg or ¾ $/kg.

To bridge the units, we just recount the per-number: 15$ = (15/3)*3$ = (15/3)*4kg = 20kg.

With the same unit, a per-number becomes a fractions or percent: 3$/4$ = ¾, 3$/100$ = 3%.

Again, the per-number bridges: To find ¾ of 20, we recount 20 in 4s. 20 = (20/4)*4 gives (20/4)*3 = 15.

6. Change formulas

The unspecified number-formula T = a*x^2 + c*x + d contains basic change-formulas:

- T = c*x; proportionality, linearity
- T = c*x+d; linear formula, change by adding, constant change-number, degree1 polynomial
- T = a*x^2 + c*x + d; parabola-formula, change by acceleration, constant changing change-number, degree2 polynomial
- T = a*b^x; exponential formula, change by multiplying, constant change-percent
- T = a*x^b; power formula, percent-percent change, constant elasticity

7. Use

- Asking '3kg at 5$ per kg gives what?', the answer can be predicted by T = 3*5 = 15$.
- Asking '10 years at 5% per year gives what?', the answer can be predicted by the formula T = 105%^10 – 100% = 62.9% = 50% in plain interest plus 12.9% in compound interest.
- Asking 'If an x-change of 1% gives a y-change of 3%, what will an x-change of 7% give?', the answer can be predicted by the approximate formula T = 1.07^3 – 100% = 22.5% = 21% plus 1.5% extra elasticity.
- Asking 'Will 2kg at 3$/kg plus 4kg at 5$/kg total (2+4)kg at (3+5)$/kg?', the answer is 'yes and no'.

The unit-numbers 2 and 4 can be added directly, whereas the per-numbers 3 and 5 must first be multiplied to unit-numbers 2*3 and 4*5 before they can be added as areas.

Thus, geometrically per-numbers add by the area below the per-number curve, also called by integral calculus.

A piecewise (or local) constant p-curve means adding many area strips, each seen as the change of the area, p*Δx = ΔA, which allows the area to be found from the equation A = Δp/Δx, or A = dp/dx in case of local constancy, called a differential equation since changes are found as differences. We therefore invent d/dx-calculation also called differential calculus.

Geometrically, dy/dx is the local slope of a locally linear y-curve. It can be used to calculate a curve's geometric top or bottom points where the curve and its tangent are horizontal with a zero slope.

8. Conclusion.

So, my dear Nephew, Mathematics is a foreign word for calculation, called algebra in Arabic. It allows us to unite and split totals into constant and changing unit- and per-numbers. *Love, your uncle Allan.*

Algebra **unites**/*splits into*	**Changing**	**Constant**
Unit-numbers (meter, second, dollar)	**T = a + b** *T – b = a*	**T = a*b** $\frac{T}{b} = a$
Per-numbers (m/sec, m/100m = %)	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\log_a(T) = b$

A Short History of Mathematics

Mathematics has two main fields, Algebra and Geometry, as well as Statistics. Geometry means ’earth-measuring’ in Greek. Algebra means ’reuniting’ in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus, together algebra and geometry give an answer to the fundamental human question: how do we divide the earth and what it produces?

Originally human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus, trade was only possible with those highlanders that had silver in their mountains.

The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire. The dark Middle Ages came when the Greek silver mines were emptied and the Arabs conquered the Spanish mines. German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number-system and algebra.

The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: A+B+C = 180 and a side law: a^2+b^2=c^2. Pythagoras generalized these findings by claiming: All is numbers.

This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don’t know Geometry. However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation.

The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has an acceleration g; and the distance s and the time t are connected by the formula: s = ½*g*t^2. However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middle men. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine.

The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move?

The church said ’among the stars’. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth.

But why do things fall? The church said: everything follows the unpredictable will of our metaphysical lord only attainable through belief prayers and church attendance. Newton objected: It follows its own will, a force called gravity that can be predicted by a formula telling how a force changes the motion, which made Newton develop change-calculations, calculus. So instead of obeying the church, people should enlighten themselves by knowledge, calculations and school attendance.

Brahe used his life to write down the positions of the planets among the stars. Kepler used these data to suggest that the sun is the center of the solar system, but could not prove it without sending up new planets. Newton, however, could validate his theory by different examples of falling and swinging bodies.

Newton’s discoveries laid the foundation of the Enlightenment period realizing that when an apple follows its own will and not that of a metaphysical patronizor, humans could do the same. Thus, by enlightening themselves people could replace the double patronization of the church and the prince with democracy, which lead to two democracies, one in The US and one in France. Also, formulas could be used to predict and therefore gain control over nature, using this knowledge to build an industrial welfare society based upon a silver-free economy emerging when the English replaced the import silk and pepper from the Far East with production of cotton in the US creating the triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton.

Mathematics Before or Through Applications

Top-Down and Bottom-Up Understandings of Linear and Exponential Functions

Background
"Mathematics Before or Through Applications?" might seem a strange question. Of course, mathematics has to be learned before it can be applied! This seems to be a self-evident fact we have to take for granted. And that is precisely why postmodern research will start looking for hidden contingency. Are we caught by our own Top-Down phrasing? Would a rephrasing like "Grammar before or through language?" reveal the existence of a Bottom-Up alternative to be reflected upon, developed, tested, researched and considered an option? Being postmodern looking for alternative silenced discourses and understandings this study asks: Can mathematics be understood, taught and learned both ways, both Top-Down and Bottom-Up? Can a Bottom-Up strategy help to stop the exodus of students away from math-based educations as science and technology?[1]

Methodology
Out of the breakdown of premodern order, modernity saw the emergence of contingency. Scared by the idea of a contingent world modernity desperately began to reinstate order.[2] Where modernity means hiding contingency, postmodernity means accepting contingency, accepting that things might also be otherwise. Postmodern questions could be: Is contingency disguised necessity - or is necessity disguised contingency? Is the relationship between the world and its description one of correspondence or one of interpretation/construction? Do we catch - or are we caught by our catch? Do concepts make us conceive or be conceited? Are all phrasings contingent except this one? Is truth discovered or invented? Does the world only have one true perspective?

Postmodernity means realising that although the world may be able to speak in numbers through measuring instruments, it is unable to speak in words since no word-measuring instrument exists. So words and phrasings are in the end contingent human constructions.

Inspired by the above statements one possible postmodern research paradigm can be formulated: Where modernity sees humans in need of guidance on their way to fulfil the enlightenment dream of human emancipation, postmodernity sees humans clientified by hidden contingency. Where modern research tries to discover necessity (but ends up inventing it and phrasing its constructions as findings) to convince humans of the necessity of obedience, postmodern research tries to discover hidden contingency in practises, discourses and understandings to inspire to the possibility of de-clientification.

Thus, a postmodern study will try to identify a contingent tradition, use inspiration and imagination to identify alternatives, and report effects and inspirations resulting from practising this alternative.

Identify a Contingent Tradition
As to linear and exponential functions several "example-of" traditions exist all using a Platonic Top-Down understanding defining a concept as an *example of* a more abstract concept: Linear and exponential functions are *examples of* functions, that are *examples of* relations, that are *examples of* products, that are *examples of* sets. In a Top-Down tradition it seems self-evident that functions have to be taught (and learned, hopefully) before linear and exponential functions can be introduced.

These traditions often create learning and teaching problems.[3]

One tradition sees a linear function as an *example of* a polynomial function naturally to be followed up by a second degree polynomial. An exponential function is seen as an *example of* an inverse function, in this case of a logarithm, which is an *example of* an integral.

The modern abstract algebra tradition sees linear and exponential functions as *examples of* homomorphisms satisfying the two functional equations $f(x+y) = f(x)+f(y)$ and $f(x+y) = f(x)*f(y)$.

[1] See Jensen et al. 1998
[2] See Bauman 1992
[3] See Tarp 1998 a

A third tradition sees linear and exponential functions as *examples of* s functions with straight graphs on respectively linear and logarithmic paper.

Identify Hidden Alternatives to a Contingent Tradition

A hidden alternative to the Platonic Top-Down understanding is a nominalistic Bottom-Up understanding where a new concept is defined as a *name for* a common property of less abstract examples: "A function is a *name for* a calculation with variable quantities", or in Euler's own words:

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.[4]

In Arabic the word "Algebra" means reunite. If we buy five items in a store we don't have to pay all the single prices, we can ask for them to be united into a total. If the total is 17 $ we are allowed to pay e.g. 20 $. This new total is then split into the price and the change. So living in a money based culture means being engaged in a "social practice of totalling" consisting of reuniting and splitting totals.

The fundamental question: "How many totally?" can be answered in four different ways depending of the nature of the numbers involved:

The questions *lead to the calculation stories or equations*

"5 \$ and 3 \$ total ? \$"	$T = 5+3$	or $T = a+n$
"5 days @ 3 \$/day total ? \$"	$T = 5*3$	or $T = a*n$
"5 days @ 3 %/day total ? %"	$1+T = 1.03$^5	or $1+T = a$^n
"5 seconds @ 3 m/sec increasing to 4 m/sec total ? m"	$\Delta T = \int_0^5 (3+\frac{4-3}{5}x)\,dx$	or $\Delta T = \int_a^b f(x)dx$

The operations "+" and "*" unite variable and constant unit-numbers; "∫" and "^" unite variable and constant per-numbers. The inverse operations "–" and "/" split a total into variable and constant unit-numbers; "d/dx" and "√ and log" split a total into variable and constant per-numbers:

Totals unite/split into	variable	constant
unit-numbers \$, m, s, ...	$T = a+n$ $T–n = a$	$T = a*n$ $T/n = a$
per-numbers \$/m, m/100m=%, ...	$\Delta T = \int f dx$ $\frac{dT}{dx} = f$	T = a^n $\sqrt[n]{T} = a$ $\log_a T = n$

Inspired by this perspective an alternative can be designed by rephrasing and deconstructing linear and exponential functions into constant-change-stories emerging from questions as:

"100\$ plus n days @ 5\$/day total ? \$"; and "100\$ plus n days @ 5%/day total ? \$"
Likewise differential and integral calculus emerges as variable-change-stories from questions as:

"100\$ plus n times @ (10%/n)/time total ? \$"
"100m plus 5 seconds @ 3m/sec increasing to 4 m/sec total ?m".

Linear Change

Linear change answers questions like "100\$ plus n days @ 5\$/day total ? \$".

The Total T after n days can be calculated as $T = 100 + 5*n$, or more general: $T = B + a*n$ where B, T: initial, terminal capital; a: \$-addition/day; n: days

Exponential Change

Exponential change answers questions like "100\$ plus n days @ 5%/day total ? \$". Since we cannot add % to \$, we have to consider the initial capital B as 100%. After an addition of 5% the terminal value is 105% of B, i.e. B*105% = B*1.05. So we add 5% by multiplying with 105% or 1.05.

[4] See Euler 1748

The Total T after n days can be calculated as T = 100*1.05^n, or more general:

T = B*a^n , a=1+r ; B, T: initial, terminal capital; r: %-addition/day; n: days

An Alternative Micro-Curriculum

Inspired by the above Bottom-Up understanding of linear- and exponential change-stories and by several student comments over the years, a micro curriculum was designed and tested:

Linear Change and Exponential Change

• A change is linear if it is constant: +5$, +5$, +5$. Linear change is also called “+change”

• A change is exponential if it is constant in percent: +5%, +5%, +5%, i.e. if the change- multiplier is constant: *1.05, *1.05, *1.05. Exponential change is also called “*change” or interest change.

Linear change Exponential change

b+a*n = T = b*a^n a = 1+r

	+a	*a	
	etc.		
+a$ n times **+a*n**	+a	*a	***a^n** +r% n times
	+a	*a	
	b		

T:	final value
+a:	change per time
*a:	multiplier per time
+r:	%-change per time
n:	number of changes
b:	initial value

A calculation table can be used to report calculations.

A calculation table has two columns, a number column to the left containing number equations, and a calculation column to the right containing calculation equations. The two columns are divided into two parts splitting the calculation table into four sections.

Top-left shows the quantity to be calculated and top-right shows the equation to do the job. Bottom-left shows the values to be used in the calculation, and bottom-right shows the processing or solving of the equation. The two calculations below answer the two questions:

“80$ plus 4 days @ ? $/day total 100$”

a = ?	T = b+(a*n)
T = 100	T–b = a*n
b = 80	$\frac{(T-b)}{n} = a$
n = 4	$\frac{(100-80)}{4} = a$
	5 = a

“80$ plus 4 days @ ? %/day total 100$”

r = ?	T = b*(a^n)
T = 100	$\frac{T}{b} = a^n$
b = 80	$\sqrt[n]{(\frac{T}{b})} = a$
n = 4	$\sqrt[4]{(\frac{100}{80})} = a$
a = 1+r	1.057 = a = 1+r
	0.057 = r = 5.7%

Differential Calculus

Differential calculus answers questions like “100$ + n times @ (10%/n)/time total ? $”. The Total T after n additions can be calculated as T = 100*(1+0.1/n)^n. We can set up a T-table for the different n’s:

n	1	10	100	1000	10000	100000	1000000
T	110	110.4622	110.5116	110.5165	110.5170	110.5171	110.5171

The stabilised T-number 110.5171 can also be calculated by using the Euler number e:

100*e^0.1 = 110.5171.

Hence, we see that for n very large:

e^0.1	= (1+0.1/n)^n or	
e^t	= (1+t/n)^n	or
e^(t/n)	= 1+t/n	or by substituting dx=t/n making dx very small
e^dx	= 1+dx	true locally, i.e. for very small numbers dx

This local connection between something non-linear e^x and something linear 1+x, corresponds to the geometrical fact, that a bending curve is locally linear. This Bottom-Up alternative to the traditional Top-Down ε–δ approach can be used all over calculus:

Rule: (x^n)'=n*x^(n–1)		
Proof: Let y = x^n, and let dx be very small.		
If x->x+dx then y->	y+dy	= (x+dx)^n
		= (x*(1+dx/x))^n
		= (x*e^(dx/x))^n
		= x^n * e^(n*dx/x)
		= x^n * (1+n*dx/x))
		= x^n + n*dx*x^(n–1)
hence	dy	= n*dx*x^(n–1)
and	dy/dx	= n*x^(n–1)

Integral Calculus
Integral calculus answers questions like "100m plus 5 seconds @ 3m/sec increasing to 4 m/sec total ?m". We observe that the total change ΔF can be calculated in two different ways:

	ΔF = F2–F1	as a difference between the terminal and the initial values
	ΔF = ΣΔF	as the sum of the single changes, or
	ΔF = ∫dF	for very small single changes dF
If	dF/dx = f	
then	dF = fdx	
	∫dF = ∫fdx= ΔF = F2–F1	
Since	d/dx(3x+0.1x^2) = 3+0.2x	
	d(3x+0.1x^2) = (3+0.2x)dx	

$$\int_0^5 d(3x+0.1x^2) = \int_0^5 (3+0.2x)\,dx = \Delta(3x+0.1x^2) = (3*5+0.1*5^2) - 0 = 17.5$$

So, the answer is:

"100m plus 5 sec. @ 3m/sec increasing to 4 m/sec total $100+\int_0^5 (3+\frac{4-3}{5}x)\,dx = 117.5$ m"

Practice the Alternative
The Calculus change-curriculum has been tested in my own classes with a positive result, but it has not been researched. The linear and exponential change-curriculum has been tested and researched in my own classes and in two other classes with two other teachers over a period of three years.

Abroad I have given an introduction to the Bottom-Up understanding of linear and exponential functions at a new four-year Secondary Teacher Education College in South Africa. The college wanted to solve the local 1% success problem in mathematics: 90% of the students did not enter the final exam in mathematics, and 90% failed.

The mathematics curriculum at the college and at the high schools followed a Platonic Top-Down tradition. In the science education classes at the college the educational theory-tradition was that of curriculum 2005, OBE (Outcome Based Education) and Vygotskian constructivism.

Report the Consequences
After the introduction I asked the students to fill out a questionnaire. Three of the questions were:

1. What do you think about the idea of introducing Bottom-Up understanding in the classroom at the college?

2. What do you think about the idea of introducing Bottom-Up understanding in the classrooms of South African secondary schools?

3. I have asked you to use a special calculation table with four parts: One for the unknown, one for the equation, one for the numbers and one for the calculation. What is your opinion of this calculation table?

I received 127 answers from the five classes. Although some students were concerned about confusing the students with a different approach, the overall reaction was very positive. Graded on a scale from "very bad (–2)" to "very good (+2)" the average for the three question were 1.3, 1.4 and 1.4. The median was 2 in all three cases. The percentage of answers saying 1 or 2 were respectively 90%, 87% and 89%. The Danish classes also showed a positive attitude towards the calculation table.

The term "Bottom-Up Mathematics" spread on the campus, and I was also invited to other classes to give a Bottom-Up understanding of equations, trigonometry, differential calculus, integral calculus and the ε–δ definition. Also the teachers at the college became quite interested in the Bottom-Up approach. I ended up designing a Bottom-Up curriculum for all four years. However, it was not considered since the management had diagnosed the 1-% problem to be a result of teachers needing a formal background in traditional mathematics.

From the relative success of the Bottom-Up understanding of linear and exponential functions two questions arise: Why is a Bottom-Up understanding more user-friendly? Why is a Bottom-Up understanding not part of the tradition - why is the tradition hiding contingency?

Why is Bottom-Up Mathematics More User-Friendly?

According to the British sociologist Anthony Giddens routinisation is the cornerstone of a traditional society.[5] By echoing these routines actors build up tacit practical consciousness enabling them to become participants in the practice and providing them with a basic ontological security and an identity. Actors also possess a discursive consciousness by which they can verbalise and reflect upon the world.

Today's society, however, is a globalised post-traditional society[6]. It is especially the development of the information and communication technology that has made global communication possible e.g. through the multi-channel satellite and Internet connected television. This has made visible alternatives to existing traditions making these appear contingent and ambiguous. With the loss of an external identity to echo, identity becomes self-identity, a reflexive project, where the individual actor has to create his own biographical narrative or self-story looking for authenticity and shunning meaninglessness.[7]

Inspired by Giddens we can say: In the classroom the students become initiated into the traditions of a social or abstract system called mathematics by echoing the routinised social practices of the system's access point, the teacher.[8] The students can choose to accept the tradition and build up their practical consciousness and their basic trust and identity to become participants reproducing the social practice. Or the students can choose to accept the contingency of the tradition by questioning it with their discursive consciousness.

Modern traditional students might opt for gaining access to the social practise by becoming echo-learners and later echo-teachers. But with postmodern post-traditional students it is different. They are engaged in building their own biographical narratives looking for meaning. By referring upwards a Top-Down sentence ("a function is an example of (or child of) a relation") can give only one answer, thus creating echo teaching. And by referring upwards Top-Down sentences become "unknown-unknown" relations that cannot be anchored to the students' existing learning narrative. They become meaningless producing echo-resistance and become only accessible through echo-learning. Since a

[5] See Giddens 1984
[6] See Giddens in Beck et al. 1994
[7] Giddens 1991 p. 5
[8] Giddens 1990 p. 83

Bottom-Up sentence (“a function is a name for a calculation with variable quantities”) is an “unknown-known” relation it can be anchored to the students' existing narrative, thus extending this. The students have learned in the sense that they are “able to tell something new about something they already knew”. This definition of learning could be called postmodern Bottom-Up learning.

So it is through their ability/inability to extend the students existing narrative Bottom-Up/ Top-Down mathematics becomes user friendly/hostile in a post-traditional society.

Why Is Bottom-Up Mathematics Unrecognised?

It is a postmodern point, that a phrasing constructs the described and that humans are clientified by ruling phrasings and discourses.[9] Inspired by this we could ask: Are the actors (students and teachers) and the system clientified, caught and frozen in a “mathematics” discourse forcing them to subscribe to the "mathematics before mathematics application" conviction?

A rephrasing of this ruling discourse will reveal an alternative silenced "number-language" discourse in which mathematics becomes the meta-language or grammar of the number-language. This deconstruction turns the Top-Down "Mathematics before application" conviction upside-down to the opposite Bottom-Up "Language before grammar" conviction. Through its captivity modern mathematics implements a “grammar before language” practice, which would create global illiteracy if spread from the number language to the word language, thus preventing language to become a human right.

Deconstructing Top-Down to Bottom-Up Mathematics

When deconstructing Top-Down “metamatics” to Bottom-Up mathematics many traditional phrasings and convictions should be questioned. Kronecker’s famous phrase “God made the integers; all else is the work of man” should be met with the question: “Are the integers made by god or social constructions?”[10]

One deconstruction comes out of regarding mathematics as part of a language house. Humans communicate in languages: A word-language with sentences assigning words to things and actions. And a number-language with equations assigning numbers or calculations to things and actions. The world is described by a language, which is described by a meta-language. Grammar is describing words in sentences, and mathematics is describing numbers and operations in equations.

META2-language *META-grammar*	Chomsky	set product relation function	META2-language *META-matics*
META-language *grammar of* *the WORD-language*	subject verb object	calculation operation number	META-language *grammar of* *the NUMBER-language* *Mathematics*
WORD-language *applications of* *grammar*	WORD stories	NUMBER stories	NUMBER-language *applications of* *mathematics*
WORLD	THINGS &	ACTIONS	WORLD

Two language-houses – and two castles in the air

Being the grammar of the number-language mathematics can tell two types of stories. The abstract Top-Down story of modern mathematics portraying the set as the universal, Platonic creator of all of mathematics including numbers and operations. Or concrete Bottom-Up stories about social needs constructing numbers and operations and a number-language and a grammar, mathematics.

[9] See e.g. Foucault 1972

[10] See Tarp 1998, b

The close relationship between the word-language and the number-language could also be practised in the classroom: production and consumption of number stories about the world could be a central activity from the first day at school.[11] And the quantitative literature, i.e. calculations on points, particles, people, power, pecunia etc., could be as important for the number language as qualitative literature is for the word language. Also the central grammar concepts could be, not functions, but the genres "fact/fiction/fiddle": Fact-stories to be trusted, fiction-stories to be doubted and supplemented with parallel scenarios based upon alternative assumptions, and fiddle-stories to be rejected.[12]

Conclusion

A number language is a human right - but mathematics has problems delivering. Modern Platonic Top-Down mathematics cannot deliver, neither can its siblings, critical and constructivist mathematics.

Modern Platonic Top-Down mathematics relates concepts to more abstract concepts thus becoming not relatable to the students' existing learning narratives. Critical mathematics asks for a critical attitude towards applications of mathematics, but not towards the assumed Platonic nature of mathematics. Constructivist mathematics inspired by Piaget and Vygotsky is apparently adopting a Bottom-Up strategy advocating cognitive structures to be constructed through operation and communication. However, there seems to be a hidden Platonic agenda as to what concepts to be thus constructed. To Piaget the three mother structures of mathematics, the algebraic, ordering and topological structures "correspond fairly closely with the fundamental operational structures of thought."[13] Vygotsky had "a lifelong interest in Spinoza"[14], and to Spinoza the physical world is all manifestations of a metaphysical God or Nature.[15]

Being caught and frozen in a "mathematics" discourse forcing it to subscribe to a "mathematics before mathematics application" conviction the educational system is unable to recognise Bottom-Up mathematics as an alternative. Bottom-Up mathematics substitutes a Platonic dehumanised "metamatics" with a nominalistic rehumanised mathematics respecting the social roots of mathematics and offering sentences to the students that are relatable to their existing learning narratives. So if mathematics is to deliver a number language to all, a postmodern nominalistic Bottom-Up mathematics should be given a chance to be developed[16], tested and studied.

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[11] See Tarp 1998 a
[12] See Tarp: "Fact, Fiction, Fiddle -Three types of Models" in these proceedings
[13] Piaget 1969 p. 46
[14] Vygotsky 1934 p. xiv
[15] Russell 1945 p. 571
[16] See Tarp 1998 a

Fact, Fiction, Fiddle - Three Types of Models

Humans communicate in languages. A word language with sentences assigning words to things and actions. And a number language with equations assigning numbers or calculations to things and actions. "Word stories" are differentiated into different genres: Fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like "This sentence is false". "Number stories" are often called mathematical models. Also models can be differentiated into three genres: Fact, fiction and fiddle. Fact models quantify and calculate deterministic quantities. Fiction models quantify and calculate non-deterministic quantities. Fiddle models quantify qualities that cannot be quantified. As with word stories also different number stories should be treated different: Facts should be trusted, fiction should be doubted and fiddle should be rejected. This paper discusses typical examples of all three kinds of models from the classroom.

Algebra, the Four Mother Models

In Arabic the word "Algebra" means reunite. If we buy five items in a store, we don't have to pay all the single prices, we can ask for them to be united into a total. If the total is 17 $ we are allowed to pay e.g. 20 $. This new total is then split in the price and the change. So living in a money based culture means being engaged in a "social practice of totalling" consisting of reuniting and splitting totals.

The questions *lead to the calculation stories or equations*

"5 $ and 3 $ total ? $"	T = 5+3	or T = a+n
"5 days @ 3 $/day total ? $"	T = 5*3	or T = a*n
"5 days @ 3 %/day total ? %"	1+T = 1.03^5	or 1+T = a^n
"5 sec. @ 3 m/sec increasing to 4 m/sec total ? m"	$\Delta T = \int_0^5 (3+\frac{4-3}{5}x)\,dx$	or $\Delta T = \int_a^b f(x)dx$

The fundamental question: "How many total?" can be answered in four different ways: The operations "+" and "*" unite variable and constant unit-numbers; "∫" and "^" unite variable and constant per-numbers. The inverse operations "–" and "/" split a total into variable and constant unit-numbers; "d/dx" and "√ and log" split a total into variable and constant per-numbers:

Totals unite/split into	variable	constant
unit-numbers $, m, s, ...	T = a+n T–n = a	T = a*n T/n = a
per-numbers $/m, m/100m=%, ...	$\Delta T = \int f dx$ $\frac{dT}{dx} = f$	T = a^n $\sqrt[n]{T} = a \quad \log_a T = n$

A calculation table can be used to report calculations. In the following example we want to answer the question " 4 days @ ? $/day total 100$":

a = ?	T = a*n
T = 100	T/n = a
n = 4	100/4 = a
	25 = a

NI	CI
NII	CII

A calculation table has two columns. A number column N containing number equations of the form "<quantity> = <number>", e.g. "T=100". And a calculation column C containing calculation equations of the form "<quantity> = <calculation>", e.g. "T=a*n". The two columns are divided into two parts I and II splitting the calculation table into four sections: NI, NII, CI and CII.

NI shows the quantity chosen to be calculated and CI shows the equation chosen to do the job. NII shows the chosen values to be used in the calculation. CII shows the processing or solving of the equation: first by rephrasing the equation CI to isolate the unknown quantity, then by inserting into the calculation the values from NII to find the value of the unknown quantity in NI. Depending of the nature of the quantity in NI the equation in CI can be called a fact/fiction/fiddle model or a room/rate/risk model.

A model can be considered a co-operation between humans and technology with each their area of responsibility. The first three sections NI, NII and CI involve human choices. The last section CII does not involve choices and can be handled by technology, e.g. a computer equipped with suitable software, e.g. MathCad.

Fact Models
If the equation in CI is a fact, the model can be called a *fact* model, a “since-hence” model or a “room”-model. Fact models quantify quantities and calculate deterministic quantities: “What is the area of the walls in this room?”. In this case the calculated answer of the model is what is observed. Hence calculated values from a fact models can be trusted. The four algebraic mother models above and many models from basic science and economy are fact models.

Fiction Models
If the equation in CI is a fiction, i.e. if it is contingent and could look otherwise, the model can be called a *fiction* model, an “if-then” model or a “rate” model. Fiction models quantify quantities and calculate non-deterministic quantities: “My debt will soon be paid off at this rate!”. Fiction models are based upon contingent assumptions and produces contingent numbers that should be supplemented with calculations based upon alternative assumptions, i.e. supplemented with parallel scenarios.

Models from basic economy calculating averages assuming variables to be constant are fiction models, see e.g. the forecast models below. Models from economic theory showing nice demand and supply curves are fiction models.

Area	**Equations**			**Fact/Fiction/Fiddle**
Economy			**constant**	
Shopping	Cost = price*volume		*price*	Fact
Time-series	T = To+a*n	ΔT/Δn=a	*slope*	with constant numbers
	T = To*a^n, a=1+r	(ΔT/T)/Δn=r	*%change*	*else* Fiction
	T = To*n^a	(ΔT/T)/(Δn/n)=a	*elasticity*	(calculates averages)
	T = To+a*n	ΔT=a$	*$-input*	
Saving	T = To*a^n, a=1+r	ΔT=r%	*%-input*	
	T/R=a/r, 1+R=(1+r)^n	ΔT=a$+r%	*$&%-input*	
Theory	Demand = Supply			Fiction
Physics	acceleration = position”			Fact
Falling body	Force = Mass * acceleration			Fact
	Force = Mass * gravity			Fiction if air resistance
Electrical	Watt = Volt * Ampere			Fact
circuit	Volt = Ohm * Ampere			Fiction unless resistor
Statistics	Risk = Consequence * Probability			Fiction/Fiddle
Dice-game	Risk = 6*(1/6)			Fiction
Technology	Risk = Casualty*Probability + Death*Probability			Fiddle

Examples of fact, fiction and fiddle models

Fiddle Models
If the equation in CI is a fiddle, the model can be called a *fiddle* model or a “risk” model. Fiddle models quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence * probability.

In a card game and in insurance the consequence (money) can be quantified and so can the probability. Still the model is a fiction because it calculates not the factual but the average winning per play.

In evaluating the risk of a road statistics can provide the probabilities for the different casualties, but casualties cannot be quantified. Still in some cases they are quantified by the social cost of the different casualties to public institutions as hospitals etc. This is problematic since it is much cheaper to stay in a cemetery than in a hospital.

Evaluating the risk of a nuclear plant is even more problematic since in this case no statistics exists to provide the probabilities. Still such probabilities were calculated in the WASH-1400 report (The Reactor Safety Study) from 1975.

Fact and Fiction Models in the Classroom
In more pre-calculus classes, the linear and exponential forecasting models were introduced as follows: In statistical tables quantities often change over time. The total change over a period is a factual true number. The average change per year is a fictional number not necessarily true. It can however be used as an assumption in a forecast: "What happens if the change continues like this?"

				forecast	
Years after 1990	n	0	4	9	?
Year	x	1990	1994	1999	?
Capital in $	y	88 851	102 191	?	120 000

One assumption is that the average $-change per year stays constant. In this case the model is linear. Being fictional forecast numbers should be supplemented with scenarios based upon alternative assumptions. An alternative assumption is that the average %-change per year stays constant. In this case the model is exponential.

In all classes several students expressed content to see authentic use of linear and exponential functions.

The Grand Quantitative Narratives
Since the expulsion from the garden of Eden humans have felt the three fundamental needs of life on this planet: 1. You need your daily bread. 2. You need to get it yourself. 3. You need to co-operate with others. Phrased differently: all humans satisfy their needs by trying to control nature's flows of matter, energy and information through three fundamental cycles: An ecological cycle, an economic cycle and a social cycle.

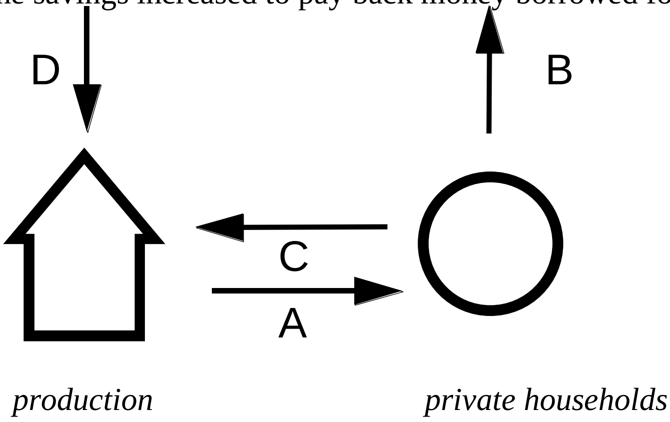
Quantification and calculation models are common within the first two cycles. The grand narratives of physics tell about how the flow of matter and energy is controlled by the forces of nature from the micro to the macro level. The grand narratives of economics tell about how the flow of goods and services are controlled by an opposite flow of money. An unbalanced flow creates big social problems and crises calling for a public control of the money flow.

The Basic Economic Cycle
The basic economic cycle consists of two sectors, the production and the private households.[17] Humans satisfy their needs by producing goods and services for others. As a return they receive money to buy what they need personally. This creates the two fundamental flows of money: production creates income A, which is used for consumption C implying new production. A balance between income and consumption will balance the cycle satisfying human needs through jobs, income and consumption. There is however a sink and a source to the basic cycle: savings B and investment D. Savings is money not spent on consumption. Investment is money spent on goods, that cannot be consumed, e.g. buildings, machines etc.

A balance between savings and investment will balance the cycle. But if the savings are bigger than the investment the cycle will dry out resulting in an increasing unemployment. This was the case after the first World War where Germany was forced to send money to France as war compensations without France being obliged to use this money to buy German goods. This made

[17] See e.g. Heilbroner et al. 1998

the British economist J. M. Keynes resign from the peace negotiations.[18] And it was the case in the great depression of the 30’s, where investment in stocks decreased dramatically after the Wall Street crash in 1929 and the savings increased to pay back money borrowed for speculation on the stock market.[19]



A model for a basic economic cycle could contain four equations per turn:

The first turn:

1. The initial income $Ao = 100$
2. The consumption is assumed to be a constant percentage of the income $Co = c*Ao$
3. What is not consumed is saved $Bo = Ao – Co$
4. The investment is assumed to be a constant percentage of the consumption $Do = d*Co$

The next turn:

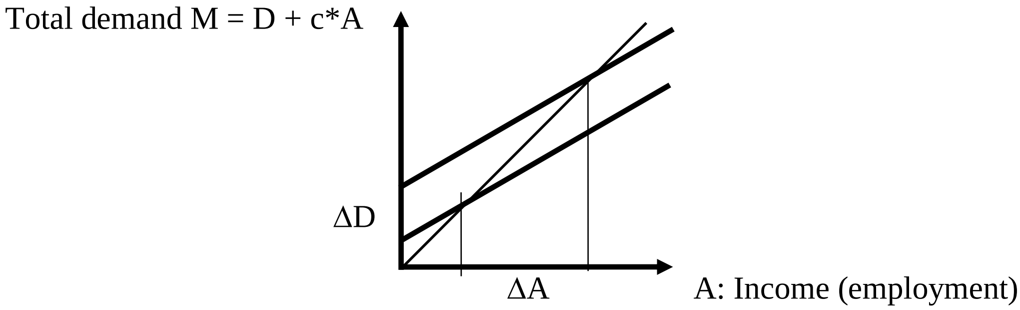
1. The next income is what was produced for consumption and investment $A1 = Co+Do$
2. etc.

The equations 1 and 3 are facts, and 2 and 4 are fiction. All in all the model is a fiction that should be paralleled with scenarios.

The Simple Keynes Model

The simple Keynes model assumes investment D to be a constant parameter implying a linear connection between the total demand M and income A: $M = D+C = D+c*A$.

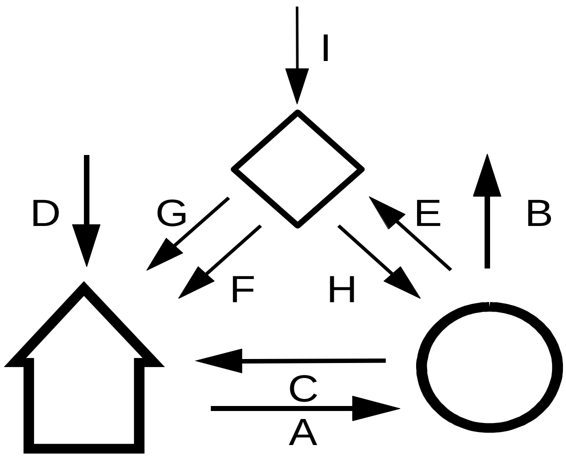
A change in D will change the balance of the cycle from an income (and employment) A to a new balance A’. The ratio of change is called the multiplier $m = \Delta A/\Delta D$. By increasing D through increasing public investment, the government will be able to increase the employment.[20] So the economic cycle needs a strong public sector to secure employment and welfare.



An Economic Cycle with a Public Sector

According to Keynes introducing a third public sector can solve the unbalance of the two-sector cycle. Financed by taxes E this public sector can balance the cycle through public investment G, public consumption F and transferals to unemployed H. A deficit I should not create concern since it can be paid back when the cycle is brought back to balance.

[18] See Keynes 1920
[19] See e.g. Galbraith 1987
[20] See Keynes 1973 and e.g. Screpanti et al. 1995



production *government* *private households*

A model for this 3-sector economic cycle could contain nine equations per turn:

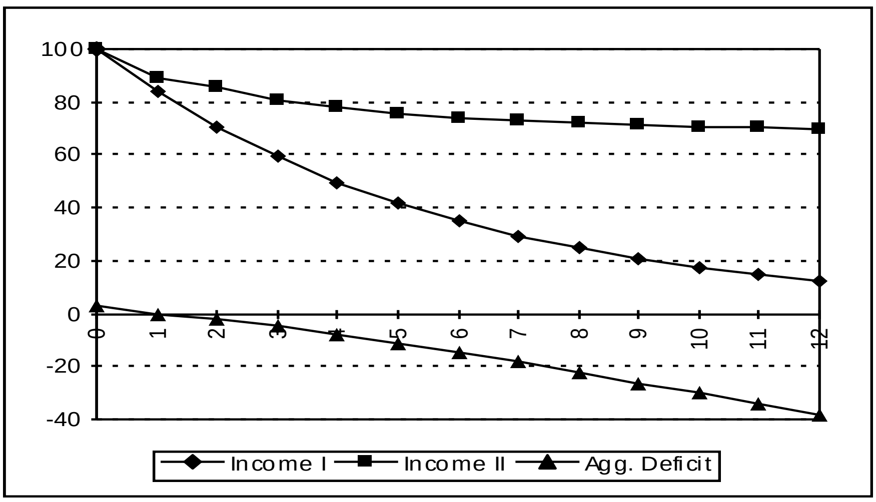
The first turn:

1. The initial income $Ao = 100$
2. The initial transferal $Ho = 4$
3. The taxes are assumed to be a constant percentage of the income and the transferals $Eo = e*(Ao+Ho)$
4. The consumption is assumed to be a constant percentage of what is left after tax $Co = c*(Ao+Ho–Eo)$
5. Savings is what is left after tax and consumption $Bo = Ao+Ho–Eo–Co$
6. Public consumption is assumed to be constant $Fo = \text{constant}$
7. The public investment is assumed to be a constant percentage of the investment gap $Go = g*(Bo–Do)$
8. The private investment is assumed to be a constant percentage of the public and private consumption $Do = d*(Co+Fo)$
9. The deficit is the difference between taxes and public expenditure $Io = Fo+Go+Ho-Eo$

The next turn:

1. The next income is what was produced for consumption and investment, private and public $A1 = Co+Do+Fo+Go$
2. The transferrals are assumed to be a constant percentage of the employment gap $H1 = h*(Ao-A1)$
3. etc.

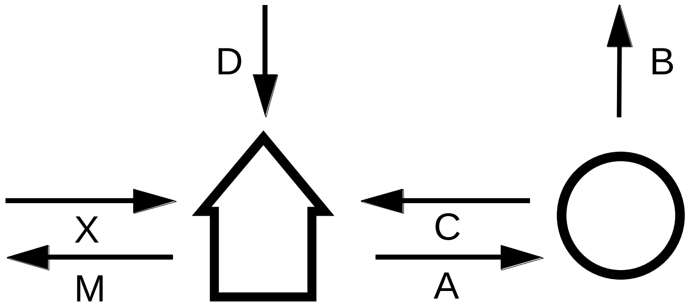
The equations 1, 2, 5 and 9 are facts, the rest are fiction. All in all the model is a fiction that should be paralleled with scenarios.



Spreadsheet Simulation
The two models above can be simulated on a spreadsheet e.g. Microsoft Excel. Above we see the income I and II without and with a public sector in the case of a 70% consumption and a 20% investment. Financed by a 30% tax the public sector covers 50% of the investment and employment gap, and consumes 20 units/year.

Through annual deficits an aggregated deficit builds up to be paid back when the investment and/or the export increases.

Further Extensions of the Economic Cycle Models
Introducing other types of connections between the variables can change the models above: linear, polynomial, exponential etc. Introducing more variables or more sectors can extend the model. A bank-sector will have an interest rate to stimulate savings and investment. A foreign sector will provide an additional sink and source through import M and export X. Import and export will if balanced be of mutual benefit in the case of exchange, but not in the case of exploitation as in economies based upon slavery, colonies or low salaries.



Many governments use economic cycle models to evaluate the effect of fiscal policy options. These models use many variables connected in systems of non-linear differential equations. These models will however still be calculating non-deterministic quantities since human choices will always be involved in economics. No matter how extended these models may become they will stay fiction models based upon contingent assumptions producing contingent numbers that should be paralleled with scenarios based upon alternative assumptions.

Mathematical Models in the Classroom
Should mathematics be learned before it can be applied? Should applications be left to the applying subjects such as physics, economics etc.? Are authentic applications and models to complicated to take up? There seems to be many problems introducing models into the classroom.

It might be impossible to introduce authentic complicated fact models from e.g. physics into the mathematics classroom. It is however possible to introduce authentic complicated fiction models from e.g. economics through one of the models basic scenarios build upon basic mathematics as in the basic economic cycles above. It is even possible to teach mathematics through applications by listening to the original meaning of the words algebra and geometry, i.e. reuniting and earth measuring, cf. the four algebraic mother models above.[21]

Conclusion
The close relationship between the word language and the number language suggests that number stories can be differentiated in the same way as word stories: Facts to be trusted. Fiction to be doubted and supplemented with parallel scenarios based upon alternative assumptions. And fiddle to be rejected.

It is possible to bring these concepts into the classroom in connection with very basic models to give students a realistic attitude towards mathematical models from the start.

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[21] See Tarp 1998

And it is possible to bring authentic fiction models into the classroom through one of its basic scenarios. It is even possible to teach mathematics not before but through applications.

Thus, there are many ways to try to solve today's exodus problem: students evading math-based educations as science and technology creating public concern for future productivity and welfare.[22]

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[22] See Jensen et al. 1998

Applying Mathe-Matics, Mathe-Matism or Meta-Matics

To solve the relevance paradox in mathematics education this paper takes a sceptical look at one of the taboos of mathematics education, the mathematical terminology. Two kinds of words are found, LAB-words abstracted from laboratory examples; and LIB-words exemplified from library abstractions, transforming mathe-matics to meta-matics. A third kind of mathematics is mathematism only valid in the library and not in the laboratory, and blending with meta-matics to meta-matism. This distinction suggests that the relevance paradox of mathematics education occurs when teaching and applying metamatism and disappears when teaching and applying mathematics.

The Relevance of Mathematics Applications

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998). And 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371). In order to make mathematics more relevant to the students it has been suggested that applications and modelling should play a more central role in mathematics education. However, when tested in the classroom the result is not always positive: At the Danish Preparation High School pre-calculus was changed from an application-free curriculum to an application-based curriculum by replacing e.g. quadratic functions with exponential functions. Still the student performance kept deteriorating to such a degree that at the present reform the teacher union and the headmaster union have suggested that pre-calculus should no more be a compulsory subject. Thus, the question can be raised: Why did this application-based curriculum not solve the relevance paradox? Is there an alternative unnoticed application-based curriculum that can make a difference? This paper argues that the relevance paradox is an effect of our terminology; it occurs when using LIB-words to teach LIB-mathematics and disappears when using LAB-words to teach LAB-mathematics.

Institutional Scepticism

Modern natural science has established research as a number-based 'LAB-LIB research' where the LIB-statements of the library are induced from and validated by reliable LAB-data from the laboratory as illustrated by e.g. Brahe, Kepler and Newton, where Brahe by studying the motion of the planets provided LAB-data, from which Kepler induced LIB-equations that later were deduced from Newton's LIB-theory about gravity.

Word-based research copies number-based research by operationalising its theories in order to validate them by reliable data. This however raises two questions: A quantitative theory can be operationalised through a calculation; how can a qualitative theory be operationalised? The reliability of quantitative data is checked through measuring, how is the reliability of qualitative data checked? These questions have led to scepticism towards modern word-based research.

This scepticism towards words is validated by a simple 'number&word-observation': placed between a ruler and a dictionary a thing can point to a number but not to a word, so a thing can falsify a number-statement in the laboratory but not a word-statement in the library; thus numbers are reliable, and words are unreliable; numbers represent natural correctness, and words represent political correctness; numbers carry research, words carry interpretations, which presented as research become seduction - to be counteracted by sceptical counter-seduction research using rewording to produce different words that are validated by being, not 'true', but 'Cinderella-differences' making a difference by solving problems and paradoxes (Tarp 2003).

Scepticism is part of democracy. The ancient Greek sophists used scepticism to distinguish between necessity and choice. Later it was revived as institutional scepticism in the Enlightenment and became part of its two democracies; the American in the form of pragmatism, symbolic interactionism and grounded theory; and the French in the form of post-structuralism and post-modernism.

In America Grounded Theory only allows the use of words grounded in data retrieved by never asking what you are looking for (Glaser 1992: 25). This creates two kinds of words, LAB-words abstracted from laboratory examples, and LIB-words exemplified from library abstractions. Thus, a function is a LAB-word if presented as a name for certain calculations as in the Euler-definition:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities (Euler 1748: 3).

And a function is a LIB-word if presented as an example of a set-relation with the property that first-component identity implies second-component identity, as done in modern mathematics.

> In France Derrida uses the word 'logocentrism' to warn against believing that words represent the world, and Lyotard uses the word 'postmodern' to warn against believing that 'metanarratives' describes the world (Cahoone 1996: 343, 482). Foucault describes how words are used to cancel democracy by installing 'pastoral power' 'which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers' (Foucault in Dreyfus et al, 1982: 213, 215).

In this way Foucault opens our eyes to the salvation promise of the modern generalised church: 'You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction centre, the hospital, and do exactly what we tell you'.

Summing up, institutional scepticism suggests that we direct scepticism towards the taboo-part of mathematics education, the mathematical terminology, by asking 'what kind of words are used in mathematics education? Would other words make a difference and solve the relevance paradox?

Lab-Mathematics and Lib-Mathematics

Euclidean geometry is the model for modern set-based mathematics defining its concepts as examples of the concept set, and deducing its statements from axioms. Thus modern mathematics is presented from above as LIB-mathematics using LIB-words in spite of the fact that it developed from below as LAB-mathematics using LAB-words as e.g. in the golden Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

The success was so overwhelming that mathematicians feared that mathematics (called geometry at that time) had come to a standstill at the end of the 18th century:

> Physics and chemistry now offer the most brilliant riches and easier exploitation; also, our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities'. (Lagrange in Kline: 623)

However, in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. This thinking lead Cantor to introduce the word 'set' to distinguish between different degrees of infinity having the natural numbers as a unit, just as numbers were introduced to distinguish between different degrees of multiplicity having 1 as its unit. However changing infinity from a quality to a quantity was controversial:

> To Kronecker (..) Cantor's work on transfinite numbers and set theory was not mathematics but mysticism. Kronecker was willing to accept the whole numbers because these are clear to the intuition. These 'were the work of God.' All else was the work of man and suspect. (Kline 1972: 1197).

As to the paradoxes in set-theory even Cantor saw problems asking Dedekind in 1899 whether the set of all cardinal numbers is itself a set; because if it is, it would have a cardinal number larger than any other cardinal (Kline: 1003). Another paradox was Russell's paradox showing that talking about sets of sets leads to self-reference and contradiction as in the classical liar-paradox 'this sentence is false': If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. Russell solves this paradox by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Thus, if a fraction is defined as a set of numbers it cannot be a number itself making e.g. the addition '2+3/4' meaningless. And set-based mathematics defines a fraction as an equivalence set in a product set of two sets of numbers such that the pair (a,b) is equivalent to the pair (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same rational number ½.

Not wanting a fraction-problem, set-based mathematics has chosen to neglect Russell's type-theory by accepting the Zermelo-Fraenkel axiom system making self-reference legal by not distinguishing between an element of a set and the set itself. But removing the distinction between examples and abstractions and between different abstraction levels is hiding that mathematics historically developed through layers of abstractions. And by changing from being a LAB-word to being a LIB-word, set also changed mathematics from being LAB-mathematics to being LIB-mathematics.

Mathematism

Mathematism is mathematics that is valid only in the library, and not in the laboratory (Tarp 2004). The '2&3-paradox' illustrates the difference:

2*3 = 6 is a LAB-claim easily verified by recounting 2 3s as 6 1s: *** *** -> * * * * * *.

2+3 = 5 is a LIB-claim with countless counter-examples in the lab: T = 2 m + 3 cm = 203 cm, T = 2 weeks + 3 days = 17 days, T = 2 tens + 3 ones = 23 etc.

The US Mars-program crashing two probes by neglecting the units when adding cm to inches witnesses that seduction by mathematism is costly. The units must be alike and put outside a parenthesis; so, addition can only take place inside a parenthesis to be valid. Thus 20%+10% = 30% only if taken of the same total; else the 'fraction-paradox' applies containing 'killer-mathematics' only valid inside the classroom where it is 'applied' to kill the relevance of mathematics:

Inside the classroom	20/100 + 10/100 = = = 20% + 10% =	30/100 = 30%
Outside the classroom	20% + 10% = or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

Another example of mathematism and killer-mathematics is the modern function concept: If 'f(x) = x+2' means 'let f(x) be a place-holder for the calculation 'x+2' containing x as a variable number' then f(3) = 5 means that '5 is a calculation containing 3 as a variable number'. This is a multiple syntax error: 5 is a number, not a calculation; 5 does not contain 3; and 3 is a constant number, not a variable number (Tarp 2002).

Mathematics and Metamatism

Thus it seems we have three kinds of activities sharing the same name 'mathematics': LAB-mathematics abstracted from LAB-words below and validated in the laboratory, LIB-mathematics or 'meta-matics' exemplified from LIB-words above, and 'mathematism' containing statements that are valid in the library but not in the laboratory. By reducing the laboratory to an applier and not a creator of mathematics, LIB-mathematics cannot distinguish mathematics from mathematism as witnessed by the crashing Mars-probes, the 2&3-paradox, the fraction-paradox etc. Thus LIB-mathematics becomes a mixture of metamatics and mathematism that can be called 'metamatism'.

This makes it possible to formulate some hypothesis: Maybe the relevance paradox is an effect of teaching, not mathematics, but metamatism? Maybe the lacking success of introducing application of mathematics is an effect of introducing application of, not mathematics, but metamatism? Maybe ‘education in mathematics’ is rather ‘indoctrination in metamatism’ teaching ‘killer-mathematics’ only existing and valid inside classrooms, where it kills the relevance of mathematics? And maybe replacing application of metamatism with application of mathematics will turn out to be a ‘Cinderella-difference’ making a difference by solving the relevance paradox?

To test these hypotheses, we need a clear profile of LAB-mathematics: what words does it use, and what content does it have?

Constructing a Lab-Mathematics

A LAB-mathematics should respect two fundamental principles: A Kronecker-principle saying that only the natural numbers can be taken for granted. And a Russell-principle saying that we cannot use self-reference and talk about sets of sets. The appendix shows one example of a Kronecker-Russell mathematics based on the LAB-words ‘repetition in time’ and ‘multiplicity in space’ creating a set-free, fraction-free and function-free ‘Count&Add-laboratory’ where addition predicts counting-results making mathematics our language of prediction.

Set-based LIB-mathematics has different number sets as integer, rational numbers etc. Multiplicity-based LAB-mathematics only has stack-numbers and per-numbers. This has a consequence when talking about applications as illustrated by the trade problem ‘if 3kg = 2$, then 8$ = ?kg’. Writing ‘3kg = 2$’ seems problematic since kilos and dollars cannot be equal in a traditional sense. But in LAB-mathematics it expresses that a physical quantity can be counted in both kg and dollar. This use of the equation sign is more rational that the traditional use, making a syntax error when identifying a calculation with a number by writing ‘2*3 = 6’ instead of ‘2*3->6’ or ‘(2*3) = 6’.

Trade: Applying Per-Numbers, Fractions or Functions

In primary school the trade problem ‘if 3kg = 2$, then 8$ = ?kg’ is solved by applying the ‘recount-equation’ T = (T/b)*b: by recounting the 8$ in 2s we know how many times we have, not only 2$, but also 3kg since 2$ can be replaced with 3kg: T = 8$ = (8/2)*2$ = (8/2)*3kg = 12kg.

In secondary school this ‘recount&replace-method’ is supplemented with a method expressing corresponding quantities as a per-number: 3kg per 2$ = ‘3 per 2’ ‘kg per $’ = 3/2 kg/$. Then we recount the units: kg = (kg/$)*$ = (3/2)*8 = 12; or write a bill: 8$ at 3/2 kg/$ = 8*3/2kg = 12kg.

In all the cases per-numbers are applied; the only difference is the order of the calculations.

In traditional mathematics the trade question applies fractions and equations to express that the price can be calculated in two different ways by dividing the $ with the kg: 2/3 = 8/x.

In modern mathematics the trade question applies functions and equations to set up a linear model f(x) = b+a*x that translates the trade question to the equation f(x) = 8. But first a linear function must be understood as an example of a homomorphism f(x+y) = f(x)+f(y).

This definition is satisfied by the proportionality function f(x) = a*x; but not by the linear function f(x) = b+a*x, thus creating the bizarre mathematical theorem ‘the linear function is not linear’; indicating that the layer of rationality in modern mathematics is rather thin hiding a corpus of authorized routines so typical and dangerous for modern western society (Bauman 1989: 21).

So the trade question can be regarded as an application of per-numbers, of fractions&equations, or of functions&equations. By saying directly that ‘A is an application of B’ we indirectly say that B must be taught and learned before being applied to A. So in order to teach the trade problem we have to choose between three different curricula, one based on per-numbers, one based on fractions, and one based on functions. Since both fractions and functions are part of mathematism the choice is between a curriculum in mathematics and a curriculum in metamatism.

Testing a Per-Number Based Curriculum in the Classroom

The Danish Preparation High School has a function-based curriculum introducing functions before linear and exponential functions.

As an alternative the author designed a 20-lesson per-number based micro-curriculum including the first two of the three fundamental change-questions:

T is the total of 200$ + 5 days at 4 $/day
T is the total of 200$ + 5 days at 4 %/day
T is the total of 200$ + 5 days at 4 $/day constantly increasing to 6 $/day

Also the genre concepts of ‘fact models’ and ‘fiction models’ were included saying that a fact model is a ‘since-hence’ model calculating the reliable consequences of a constant per-number; while a fiction model is an ‘if-then’ model calculating the consequences of an assumed-constant per-number supplemented with scenarios based upon alternative assumptions (Tarp 2001 a & b).

When tested in different classes with different teachers the micro-curriculum turned out to be a Cinderella-difference by solving the relevance paradox (Tarp 2003).

The per-number approach was also tested in a science-class having the same problems as the preparation classes when confronted with typical science-questions as ‘if 3kg = 2liters, then 8liters = ? kg’. Again, it made a difference by excluding no one from the calculation laboratory.

The per-number approach has been developed into four full curricula in pre-calculus, calculus, science and economics (Tarp 2001 c). The first three curricula were sent to the Danish Education Ministry. The author had to ask for permission to test it in the classroom since Denmark, still deeply rooted in its continental feudal history, does not allow a High School to set up its own curriculum.

The Ministry however refused arguing that ‘it is a central part of these curriculum proposals that the students’ problems are caused by the terminology and can be changed by changing this. Modern educational research however points to a quite different direction, towards mathematical competences. The road forward does not go via a ‘softening’ of the terminology, but via a greater degree of teacher-insight into the students’ everyday thinking in order to lead them from this everyday thinking to scientific thinking’.

This Ministry reaction was to be expected from Foucault’s description of pastoral power. Only the pastoral terminology is accepted as ‘scientific’, and the relevance paradox can only be solved through introducing the new pastoral term ‘competence’ installing both students and teachers as ‘incompetent’. However, the students can be cured if they ‘develop competence’ in ‘scientific thinking’. And the teachers can be cured if they ‘develop competences’ in diagnosing the students’ way of thinking, and in guiding the students from the ‘wrong path’ to the ‘right path’.

By replacing ‘qualification’ with ‘competence’ a verb-based LAB-word is replaced by a LIB-word. This increases the pastoral power since students and teachers, knowing when they are qualifying themselves, cannot know when they are ‘competencing’ themselves, only the pastors can.

Again, we see how pastoral power is installed through words. At the same time a normality is worded, an abnormality is installed together with a normalizing institution meant to cure this abnormality. Failing its ‘cure’ it is ‘cured’ by the institution ‘research’ installing new ‘scientific’ words installing both a new normality and a new abnormality etc. In this way the research institution turns into an industry producing new pastoral LIB-words that are irrelevant to the LAB, but highly relevant to the survival and growth of the research industry (Tarp 2004).

Conclusion

Replacing authorized LIB-routines with the authentic LAB-routines can solve the relevance paradox in mathematics education. However, the educational institution might not be interested in replacing pastoral power with enlightenment, and the research institution might not want its funding to decrease by solving the relevance paradox.

So, to publish an alternative to the modern set-based LIB-mathematics a website has been set up called MATHeCADEMY.net offering free teacher PYRAMIDeDUCATION in multiplicity-based LAB-mathematics from below (Tarp 2004).

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Appendix. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with multiplicity in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Multiplicity can be counted in icons producing a stack of e.g. T = 3 4s = 3*4. The process ' from T take away 4' can by iconised as 'T-4'. The repeated process 'from T take away 4s' can by iconised as 'T/4, a 'per-number'. So the 'recount-equation' T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting and stacking: T = 8 = (8/4)*4 = 2*4, T = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation T=3*4= 12 is a prediction of the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal's length is predicted by the Pythagorean theorem a^2+b^2=c^2, and its angles are predicted by recounting the sides in diagonals: a = a/c*c = sinA*c, and b = b/c*c = cosA*c.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:

C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads (predicted by the 'restack-equation' T = (T-b)+b:

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = (5+1)ten7 = 6ten7 = 67.

10. Per-numbers can be added after being transformed to stacks. Thus the $/day-number b is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration		Reversed addition of per-numbers -> differentiation	
T2	= T1 + n*b	T2	= T1 + n*b
T2 - T1	= + n*b	(T2-T1)/n	= b
ΔT	= $\sum$ n*b	ΔT/Δn	= b
ΔT	= $\int$ b*dn	dT/dn	= b

Only in the case of adding constant per-numbers as a constant interest of e.g. 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers: 4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell multiplicity-based mathematics can be summarised as a 'count&add-laboratory' adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word 'algebra', reuniting.

ADDING	Constant	Variable
Stack-numbers or Totals m, s, kg, $	T = n*b T/n = b	T2 = T1 + n*b *T2-T1 = n*b*
Per-numbers m/s, $/kg, $/100$ = %	T = b^n $\sqrt[n]{T} = b$ $\log_b T = n$	T2 = T1 + $\int$b*dn *dT/dn = b*

The Count&Add-Laboratory

Applying Pastoral Metamatism or Re-Applying Grounded Mathematics

*When an application-based mathematics curriculum supposed to improve learning fails to do so, two questions may be raised: What prevents it from improving learning? And is 'mathematics applications' what it says, or something else? Skepticism towards wordings leads to postmodern thinking that, dating back to the ancient Greek sophists, warns against patronizing pastoral categories, theories and institutions. Anti-pastoral sophist research, identifying hidden alternatives to pastoral choices presented as nature, uncovers two kinds of mathematics: a grounded mathematics enlightening the physical world, and a pastoral self-referring mathematics wanting to 'save' humans through 'metamatism', a mixture of 'metamatics' presenting concepts as examples of abstractions instead of as abstractions from examples; and 'mathematism' true in the library, but seldom in the laboratory. Also 'applying' could be reworded to 're-applying' to emphasize the physical roots of mathematics. Three preventing factors are identified: 'ten=10'-centrism claiming that counting can only take place using ten-bundles; fraction-centrism claiming that proportionality can only be seen as applying fractions; and set-centrism claiming that modelling can only take place by applying set-based concepts as functions, limits etc. In contrast, an implying factor is grounded mathematics created through modelling the natural fact many by counting many in bundles &stacks; and by predicting many by a recount-formula T = (T/b)*b that can be re-applied at all school levels.*

Applying Mathematics Improves Learning – or Does it?

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998), and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994: 371). To improve learning, it has been suggested that applications and modelling should play a more central role in mathematics education. However, when tested in the classroom the result is not always positive: 30 years ago, the pre-calculus course at the Danish second-chance high school changed from being application-free to being application-based by replacing e.g. quadratic functions with exponential functions. Still student performance deteriorated to such a degree that at the 2005 reform the teacher union and the headmasters suggested that pre-calculus should no more be a compulsory subject. Thus, two questions can be raised: Why did this application-based curriculum not improve learning? And is 'mathematics applications' what it says, or something else that might make a difference? Postmodern thinking, dating back to the ancient Greek sophists, has identified the hidden patronization in fixed wordings.

Anti-Pastoral Sophist Research

Ancient Greece saw a struggle between the sophists and the philosophers as to the nature of knowledge. The sophists warned that to protect democracy people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers, seeing everything physical as examples of meta-physical forms only visible to them, patronization was a natural order if left to the philosophers (Russell 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spices. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea to avoid Portuguese forts was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people should do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. The US still has its first republic; France now has its fifth.

The German autocracy tried to stop the French democracy by sending in an army. However, the German mercenaries were no matches to the French conscripts only too aware of the feudal consequences of loosing. So, the French stopped the Germans, and later occupied Germany.

Unable to use the army, the German autocracy used the school to stop the enlightenment spreading from France. Humboldt was asked to create an elite school, and used Bildung as counter-enlightenment to create the self-referring Humboldt University (Denzin et al 2000: 85).

Inside the EU the sophist warning is kept alive in the French postmodern or post-structural thinking of Derrida, Lyotard and Foucault warning against patronizing categories, discourses and institutions presenting their choices as nature (Tarp 2004).

Derrida recommends that patronizing categories, called logocentrism, be 'deconstructed':

> Derrida encourages us to be especially wary of the notion of the centre. We cannot get by without a concept of the centre, perhaps, but if one were looking for a single 'central idea' for Derrida's work it might be that of decentring. It is in this very general context that we might situate the significance of 'poststructuralism' and 'deconstruction': in other words, in terms of a decentring, starting with a decentring of the human subject, a decentring of institutions, a decentring of the logos. (Logos is ancient Greek for 'word', with all its connotations of the authority of 'truth', 'meaning', etc.) (..) It is a question of the deconstruction of logocentrism, then, in other words of 'the centrism of language in general'. (Royle 2003: 15-16)

As to discourses Lyotard coins the term 'postmodern' when describing 'the crisis of narratives':

> I will use the term modern to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..) Simplifying to the extreme, I define postmodern as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

Foucault calls institutional patronization for 'pastoral power':

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1982: 213, 215)

In this way Foucault opens our eyes to the salvation promise of the generalized church: 'you are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will save you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and accept becoming a docile lackey'.

To Foucault, institutions building on discourses building on categories build upon choice, so they all have a history, a 'genealogy' that can be uncovered by 'knowledge archeology'.

The French skepticism towards words, our most fundamental institution, is validated by a 'number&word observation': Placed between a ruler and a dictionary a so-called '17 cm long stick' can point to '15', but not to 'stick'; thus it can itself falsify its number but not its word, which makes numbers nature and words choices becoming pastoral if hiding their alternatives.

On this basis a research paradigm can be created called 'anti-pastoral sophist research' deconstructing pastoral choices presented as nature by discovering hidden alternatives. Anti-pastoral sophist research doesn't refer to but deconstruct existing research by asking 'in this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?' To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the natural research method developed in the American enlightenment democracy and resonating with Piaget's principles of natural learning (Piaget 1970).

A Historical Background

The natural fact many provoked the creation of mathematics as a natural science addressing the two fundamental human questions ‘how to divide the earth and what it produces?’

Distinguishing the different degrees of many leads to counting that leads to numbers.

1.order counting counts in 1s and creates number-icons by rearranging the sticks so that there are five sticks in the five-icon 5 if written in a less sloppy way.

2.order counting counts by bundling&stacking using numbers with a name and an icon, resulting in a double stack of bundled and unbundled, e.g. T = 3 5s + 2 1s = 3)2) = 3.2 5s = 3.2*5 if using cup-writing and decimal-writing separating the left bundle-cup from the right single-cup. The result can be predicted by the ‘recount-formula’ T = (T/b)*b iconizing that counting in bs means taking away bs T/b times, e.g. T = (4*5)/7*7 = 2*7 + 6*1 = 2)6) = 2.6*7.

3.order counting counts in tens, having a name but not an icon since the bundle-icon is never used: counting in 5s, T = 5 1s = 1 5s = 1.0 bundle = 10 if leaving out the decimal and the unit.

In Greek, mathematics means knowledge, i.e. what can be used to predict with, making mathematics a language for number-prediction: The calculation ‘2+3 = 5’ predicts that repeating counting 3 times from 2 will give 5. ‘2*3 = 6’ predicts that repeating adding 2 3 times will give 6. ‘2^3 = 8’ predicts that repeating multiplying with 2 3 times will give 8. Also, any calculation can be turned around and become a reversed calculation predicted by the reversed operation: In the question ‘3+x = 7’ the answer is predicted by the calculation x = 7–3, etc.

Thus the natural way to solve an equation is to move a number across the equation sign from the left forward- to the right backward-calculation side, reversing its calculation sign.

3+x = 7 x = 7–3	3*x = 7 x = 7/3	x^3 = 7 $x = \sqrt[3]{7}$	3^x = 7 $x = \log_3(7)$

In Arabic, algebra means reuniting, i.e. splitting a total in parts and (re)uniting parts into a total. The operations + and * unite variable and constant unit-numbers; ∫ and ^ unite variable and constant per-numbers. The inverse operations – and / split a total into variable and constant unit-numbers; d/dx and √ & log split a total into variable and constant per-numbers:

Totals unite/split into	Variable	Constant
Unit-numbers $, m, s, ...	T = a + n *T – n = a*	T = a * n *T/b = a*
Per-numbers $/m, m/s, m/100m = %, ...	ΔT = ∫ f dx *dT/dx = f*	T = a ^ n $\sqrt[n]{T} = a$, $\log_a T = n$

In Greek, geometry means earth measuring. Earth is measured by being divided into triangles, again being divided into right-angled triangles, each seen as a rectangle halved by a diagonal.

Recounting the height h and base b in the diagonal d produces three per-numbers:

sinA = height/diagonal = h/d, tanA = height/base = h/b, cosA = base/diagonal = b/d.

Also a circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: C = 2 * r * (n*sin(180/n)) = 2 * r * π for n sufficiently big.

However, having to do without the Arabic numbers, Greek geometry turned into Euclidean geometry, freezing the development of mathematics until the Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. The seventeen-century men had broken both

of these bonds. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

The success was so overwhelming that mathematicians feared that mathematics (called geometry at that time) had come to a standstill at the end of the 18th century:

Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities. (Lagrange in Kline 1972: 623)

But in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. So in the 1870s the concept 'set' reintroduced rigor into mathematics.

Mathematics Versus Metamatics

Using sets, a function is defined 'from above' as a set of ordered pairs where first-component identity implies second-component identity; or phrased differently, as a rule assigning exactly one number in a range-set to each number in a domain-set. The Enlightenment defined function 'from below' as an abstraction from calculations containing a variable quantity:

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1748: 3)

So where the Enlightenment defined a concept as an abstraction from examples, the modern set-based definition does the opposite; it defines a concept as an example of an abstraction. To tell theses alternatives apart we can introduce the notions 'grounded mathematics' abstracting from examples versus 'set-based metamatics' exemplifying from abstractions, and that by proving its statements as deductions from meta-physical axioms becomes entirely self-referring needing no outside world. However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false:

Definition $M = \{ A \mid A \notin A \}$, statement $M \in M \Leftrightarrow M \notin M$.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Still, set-based mathematics soon found its way to the school even if it creates syntax errors:

A formula containing two variables becomes a function, e.g. y = 2*x+3 = f(x) where f(x) = 2*x+3 means that 2*x+3 is a formula containing x as the variable number. A function can be tabled and graphed, both describing if-then scenarios 'if x = 6 then y = 15'. But writing f(6) = 15 means that 15 is a calculation containing 6 as the variable number. This is a syntax error since 15 is a number, not a calculation, and since 6 is a number, not a variable. Functions can be linear, quadratic, etc., but not numbers. So claiming that at function increases is a syntax error.

Set-based metamatics defines a fraction as an equivalence set in a product set of two sets of numbers such that the pair (a,b) is equivalent to the pair (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same fraction ½. However, this definition conflicts with Russell's set paradox, solved by Russell by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Thus a fraction that is defined as a set of numbers is not a number itself, making additions as '2+3/4' meaningless.

Wanting fractions to be 'rational' numbers, set-based mathematics has chosen to neglect Russell's type-theory by accepting the Zermelo-Fraenkel axiom system making self-reference legal by not distinguishing between an element of a set and the set itself. But removing the distinction between examples and abstractions and between different abstraction levels means hiding that historically mathematics developed through layers of abstractions; and that mathematics can be defined through abstractions in a meaningful and uncontroversial way.

Mathematics Versus Mathematism

Traditionally, both 2+3 = 5 and 2*3 = 6 are considered universal true statements. The latter is grounded in the fact that 2 3s can be recounted as 6 1s. The first, however, is an example of 'mathematism' true in a library, but not in a laboratory where countless counter-examples exist: 2weeks + 3 days = 17 days, 2m + 3cm = 203cm etc. Thus addition only holds inside a bracket assuring that the units are the same: 2m + 3cm = 2*100cm + 3cm = (200 + 3)cm = 203cm.

Adding fractions without units is another example of mathematism:

Inside the classroom	20% (20/100) + 10% (10/100)	= 30% (30/100)
Outside the classroom e.g. in the laboratory	20% + 10%	= 32% in the case of compound interest = b% (10<b<20) in the case of a weighted average

Mathematics Modelling in Primary School

Having learned how to assign numbers to totals through counting by bundling&stacking, a real-world question as 'what is the total of 2 fours and 3 fives' can lead to two different models.

Model 1 says: This question is an application of addition. The mathematical problem is to find the total of 2 and 3. Applying simple addition, the mathematical solution is 2 + 3 = 5, leading to the real-world solution 'the total is 5'. An answer that is useless and incorrect because it has left out the unit.

Model 2 says: This question is a re-application of recounting. The mathematical problem is to find the total of 2 4s and 3 5s. Re-applying recounting to find a mathematical solution, the units must be the same before adding so we recount the 2 4s in 5s, predicted by the recount-formula T = (2*4)/5*5 = 1 5s + 3 1s, giving the total of T = 4 5s + 3 1s = 4.3 5s, which leads to the real-world solution 'the total is 4 fives and 3 ones'. A prediction that holds when tested:

IIII IIII + IIIII IIIII IIIII -> IIIII III + IIIII IIIII IIIII -> IIIII IIIII IIIII IIIII III = 4)3) = 4.3 5s

This example shows that applying mathematism may lead to incorrect solutions when modelling addition problems. Whereas applying grounded mathematics creates the categories 'stack' and 'recounting', and allows practicing recounting by asking e.g. 2 fours = ? fives.

Recounting a stack in a different bundle-size is a brilliant example of a modelling process:

	Problem		Solution
Mathematics	T = 2*4 = ?*5	→	T = (2*4)/5*5 = 1.3 5s
	↑		↓
Real-world	IIII IIII -> ? 5s		IIII IIII -> IIIII III

Rephrasing the problem to 'what is the total of nines in 2 fours and 3 fives' introduces integration already in primary school. As a matter of fact, the core of mathematics can be introduced as application of recounting, using 1digit numbers alone (Zybartas et al 2005).

However, this is impossible in a 'ten=10'-curriculum that by presenting 10 as the follower of nine introduces at once the number ten as the standard bundle-size, a pastoral choice hiding that also other numbers can be used as bundle-size. 10 simply means bundle, i.e. 1.0 bundle if not excluding the unit. Thus, counting in 7s, 10 is the follower of 6, and the follower of nine is 13.

With ten as bundle-size, recounting-problems disappear, and all numbers loose their units, which creates the basis for teaching mathematism where 3 + 2 IS 5 without discussion.

Thus, in primary school an application-based curriculum using recounting to learn the modelling process is prevented by a pastoral choice, 'ten=10'-centrism, hiding that also other numbers can be used when counting by bundling&stacking. And prevented by mathematism claiming that 3+2 IS 5.

Mathematical Modelling in Middle School

Middle school introduces fractions as rational numbers and allows them to be added without units in spite of the fact that fractions are multipliers carrying units: 1/3 of 6 = 1/3*6.

The real-world question 'what is the total of 1 coke among 2 bottles and 2 cokes among 3 bottles?' can lead to two different models.

Model 1 says: This question is an application of adding fractions. The mathematical problem is to find the total of 1/2 and 2/3. Applying simple addition of fractions, the mathematical solution is 1/2 + 2/3 = 3/6 + 4/6 = 7/6, leading to the real-world solution '7 out of the 6 bottles are cokes'. An answer that is meaningless and useless because it has left out the unit: we cannot have 7 cokes if we only have 6 bottles; and we do not have 6 bottles, we only have 5.

Model 2 says: This question is a re-application of adding stacks by integrating their bundles. The mathematical problem is to find the total of 1/2 of 2 and 2/3 of 3. Re-applying integration, the mathematical solution is T = 1/2 * 2 + 2/3 * 3 = 3 = 3/5 * 5, giving the real-world solution 'the total is 3 cokes of 5 bottles'. A prediction that holds when tested on a lever carrying to the left 15 units in the distance 2 and 20 units in distance 3, and to the right 18 units in distance 5.

Sharing-problems asking 'the boys A, B and C paid \$1, \$2 and \$3 to a pool buying a lottery ticket. How should they share a 300\$ win?' can lead to two different models.

Model 1 says: This question is an application of fractions. The mathematical problem is to split a total of 300 in the proportions 1:2:3. Applying simple addition of fractions gives the answer: since boy A paid 1/(1+2+3) = 1/6 of the ticket he should receive 1/6 of the win, i.e. 1/6 of \$300 = 1/6*300 = 50; likewise with the other boys: boy B will get 2/6 and boy C 3/6 of 300\$. So, the real-world solution is: boy A \$50, boy B \$100, and boy C \$150. Of course, such questions can only be answered after fractions and its algebra has been taught and learned.

Model 2 says: This question is simply a re-application of recounting. The mathematical problem is to recount the win in pools, i.e. in 6s, which then can be paid back to the boys a certain number of times. Since 300 = (300/6)*6 = 50*6, the boys are paid back 50 times. So, the real-world solution is A: \$1*50 = \$50, B: \$2*50 = \$100, and C: \$3*50 = \$150.

Trade-problems as 'if the cost is 2\$ for 5kg, what then is the cost for 14kg, and how much can I buy for 6\$?' can lead to three different models.

Model 1 says: This question is an application of proportionality, fractions and equations. The mathematical problem is to set up an equation relating the unknown to the 3 known numbers. Applying proportionality, fractions and equations we can set up a fraction-equation expressing that the cost c and the volume v is proportional, c/v = k. Hence c1/v1 = c2/v2, or 2/5 = x/14 and 2/5 = 6/x. Now the x can be found by solving the equations, or by cross-multiplication. So, the real-world solution is 5.6\$ and 15 kg. Of course, such questions can only be answered after fractions and proportionality and equations has been taught and learned.

Model 2 says: This question is an application of linear functions. The mathematical problem is to set up a linear function expressing the price y as a function of the volume x, y = f(x) = m*x + c, given that the points (0,0) and (5,2) belongs to the graph of the function. The mathematical solution first finds c by inserting the point (0,0) in the formula: f(0) = m*0+c = 0, so c = 0; then we find m by inserting the point (5,2) in the formula: f(5) = m*5 = 2, so m = 2/5 = 0.4. Hence the linear formula is f(x) = 0.4*x. To answer the questions we insert the points (14,y) and (x,6) into the function: f(14) = 0.4*14 = y, and f(x) = 0.4*x = 6. Solving these equations give y = 5.6 and x = 15. So the real-world solution is 5.6\$ and 15 kg. Of course, such questions can only be answered when general functions, linear functions and equations has been taught and learned.

Model 3 says: This question is a re-application of recounting. The mathematical problem is to recount the 14kg in 5s and the 6$ in 2s since the cost is 2$ per 5kg. Thus, the mathematical solution is 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$, and 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg.

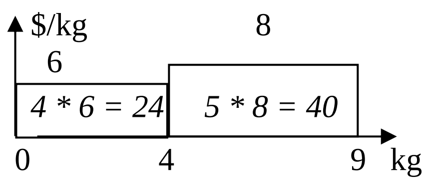
The examples show that many problems in middle school are re-applications of recounting from primary school; unless they are presented as being only solvable by applying fractions, proportionality and equations, in which case the modelling has to wait until these subjects has been taught and learned, which also excludes students unable to learn ungrounded mathematics.

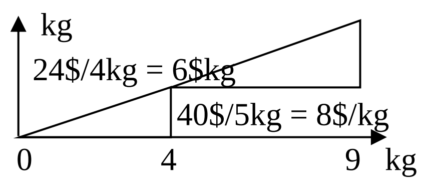
Again, unreflectively applying mathematism may lead to incorrect solutions. Whereas using grounded mathematics replaces fractions with the category per-numbers coming from double-counting in two different units, in 1s and 5s: 3 1s = (3/5)*5, or in $ and kg: 2$/5kg = 2/5 $/kg.

Adding numbers with units also occurs when modelling mixture situations, generalizing primary school's integrating stacks to middle school integral and differential calculus.

Thus asking 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg can be answered by using a table or a graph, realizing that integration means finding the area under the per-number graph; and vice versa, that the per-number is found as the gradient on the total-graph

4kg at 6$/kg	= 4 * 6 = 24
5kg at 8$/kg	= 5 * 8 = 40
9kg at ?$/kg	= 9 * ? = 64
	? = 64/9





Mathematical Modelling in High School

High school claims set-based functions to be its basis: a quantity growing by a constant number IS an example of a linear function; and a quantity growing by a constant percent IS an example of an exponential function; and both ARE examples of the set-based function concept.

Thus the real-world problem '200$ + ? days at 5$/day is 300$' leads to two different models: model 1 seeing the question as an application of linear functions; and model 2 seeing the question as a re-application of a formula stating that with constant change, the terminal number T is the initial number b added with the change m a certain number of times x: T = b + m*x. Inserting T = 300, b = 200 and m = 5 and using the Math Solver on a Graphical Display Calculator, the solution is found as x = 20. This prediction can be tested when graphing the function y = 200 + 5*x and observing that tracing x = 20 gives y = 300.

Cumulating a capital C by a yearly deposit p and interest rate r leads to two different models: model 1 seeing the question as an application of a geometric series; and model 2 setting up two accounts, one with the amount p/r from which the yearly interest p/r*r = p is transferred to the other, which after n years contains the cumulated interest p/r * R where by 1+R = (1+r)^n as well as the generated capital C. And C = p/r * R gives a beautiful a simple formula: C/p = R/r.

Two different models come out of the real-world problem 'Out driving, Peter observed the speed to be 6, 18, 11, 12 m/s after 5, 10, 15 and 20 seconds. What was the speed after 6 seconds? When was the speed 15m/s? When did he stop accelerating? When did he begin to accelerate again? What was the total distance travelled from 7 to 12 seconds?

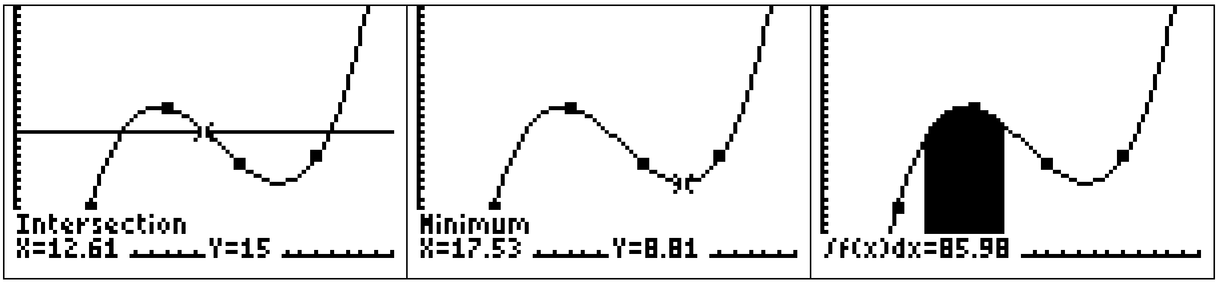
Model 1 says: This question is an application of matrices and differential and integral calculus. The mathematical problem is to set up a function expressing the distance y as a function of the time x, given that the function's graph contains the points (5,6), (10,18), (15,11), and (20,12). Applying matrices to solve 4 equations with 4 unknowns, the mathematical solution is y = 0.036 x^3 – 1.46 x^2 + 18 x – 52. Now the point (6,0) is inserted in the function to find y = 11.22. Inserting the point (x,15) in the function leads to a 3rd degree equation.

To solve this equation we guess a solution in order to factorize the 3rd degree polynomial to y = 0.036(x – 7.07)(x – 12.61)(x – 20.87). To find the turning points we must find the zeros of the derivative y' = 0.108 x^2 – 2.92 x + 18, i.e. x = 9.51 and x = 17.53, as well as the signs of the double-derivative y'' = 0.216 x – 2.92 changing sign from minus to plus in x = 13.52. Finally the distance travelled from 7 seconds to 12 seconds comes from the integral:

$$\int_7^{12} (0.036\,x^3 - 1.46\,x^2 + 18\,x - 52)\,dx = \left[\, 0.009\,x^4 - 0.49\,x^3 + 9\,x^2 - 52x \right]_7^{12} = 85.98.$$

Of course, this must wait till after matrices, polynomials and calculus are taught and learned.

Model 2 says: This question is a re-application of per-numbers. The mathematical problem is to find a per-number formula f(x) from a table of 4 data sets. On a Graphical Display Calculator Lists and CubicRegression do the job. Tracing x = 6 gives y = 11.22. Finding the intersection points with the line y = 15 using Calc Intersection gives x = 7.07, 12.61 and 20.87. Finding the turning points using Calc Minimum and Calc Maximum gives a local maximum at x = 9.51 and y = 18.10, and a local minimum at x = 17.53 and y = 8.81. The total meter-number from 7 to 12 seconds is found by summing up the m/s*s, i.e. by using Calc ∫f(x)*dx, which gives 85.98.



Change Equations

Solving any change-equation dy/dx = f(x,y) is easy when using technology. The change-equation calculates the change dy that added to the initial y-value gives the terminal y-value, becoming the initial y-value in the next period. Thus is dy = r*y, r = ro*(1 – y/M) is the change-equation if a population y grows with a rate r decreasing in a linear way with the population having M as its maximum. A spreadsheet can keep on calculating the formula y + dy -> y.

The Grand Narratives of the Quantitative Literature

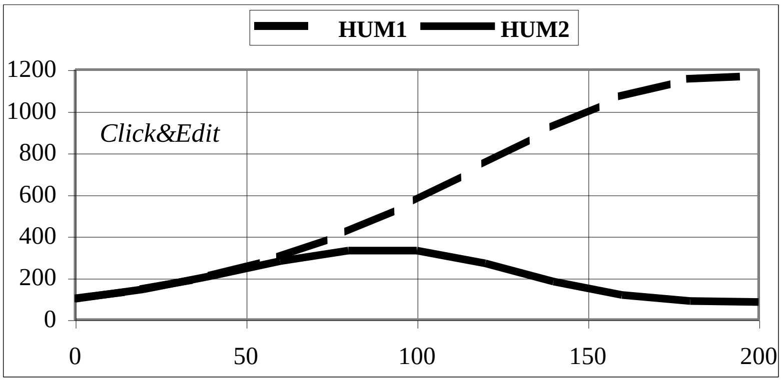
Literature is narratives about real-world persons, actions or phenomena. Quantitative literature also has its grand narratives. That an infinity of numbers can be added by one difference if the numbers can be written as change-numbers is a grand narrative: If y-changes dy are recounted in x-changes dx: dy = (dy/dx)*dx = y'*dx, then ∫y'dx = ∫dy = Δy = y2 – y1:

Since x^2 = (x^3/3)', then ∫x^2 dx = ∫(x^3/3)' dx = 7^3/3 – 2^3/3 if summing from 2 to 7.

In physics, grand narratives can be found among those telling about the effect of forces, e.g. gravity, producing parabola orbits on earth, and circular and ellipse orbits in space. Jumping from a swing is a simple example of a complicated model. Physics' grand narratives enabled the rise of the Enlightenment and the modern democracy replacing religion with science.

In economics, an example of a grand narrative is Malthus' 'principle of population' comparing the linear growth of food production with the exponential growth of the population; and Keynes' model relating demand and employment creating the modern welfare society. As are the macroeconomic models predicting effects of taxation and reallocation policies. Also limit-to-growth models constitute grand narratives predicting the future population depending on different assumptions as to e.g. food and pollution: leaving out pollution only food will restrict human growth, but including pollution the population level might be different:

Time	HUM1	HUM2
0	100	100
20	145	145
40	210	210
60	301	280
80	421	331
100	572	330
120	747	269
140	925	182
160	1071	118
180	1156	89
200	1171	84



Using Applied Ethnography to Test a Set-free Pre-calculus Curriculum

Working together with Mogens Niss 30 years ago, we discussed ways to include applications and modelling in the Danish mathematics curriculum. Niss stayed at the university, I chose to become an applied ethnographer in the classroom. By replacing set-based functions with formulas at the pre-calculus level, I designed and tested an application and modelling based curriculum that made all students learn everything. However, Denmark is the only country in the world still practicing oral exams in mathematics, and the ministry of education's examiner didn't like a set-free curriculum. Likewise, countless articles to the mathematics teachers' journal on the advantages of a set-free formula-based curriculum were neglected. It took 30 years before the Ministry of Education finally followed my advice: to improve learning, replace functions with formulas and make the curriculum application and modelling based. However, care is needed since the phrasing 'apply mathematics' installs as self-evident that 'of course mathematics must be learned before it can be applied'. This contradicts the historical fact that mathematics was created as layers of abstractions coming from modelling real-world problems.

Factors Preventing an Application & Modelling Based Curriculum

Four factors preventing an application and modelling based curriculum have been identified.

1. Applying mathematism instead of mathematics, answers proven correct in a library may not hold in a laboratory. This makes mathematics totally self-referential and impossible to use as a prediction of real-world situations. Adding numbers without units in primary school and adding fractions without units in middle school are examples of mathematism.

Different examples of 'centrism' claiming to have monopoly in certain modelling situations prevent or postpone many fruitful modelling situations, and exclude many potential learners.

2. In primary school, 'ten=10'-centrism conceals that 'ten IS 10' is a pastoral choice hiding its alternatives, e.g. five=10 in the case of counting&bundling in 5s. This prevents modelling recounting-situations as e.g. 4 5s = ? 7s predicted by the recount-formula T = (T/b)*b.

3. In middle school, fraction-centrism presenting fraction without units is a pastoral choice hiding its alternative, per-numbers. This forces proportionality to become an application of fractions, and prevents it from being a re-application of recounting. Being defined as sets, fractions become an example of 'metamatism' merging metamatics with mathematism.

4. In high school, set-centrism demands all concepts be defined as examples of the concept set. Solving equations by the set-based neutralizing method prevents equations from being solved by reversed calculation. Set-based functions and calculus prevent change situations from being modelled by re-applying per-numbers and using a Graphical Display Calculator.

Factors Implying an Application & Modelling Based Curriculum

Two factors implying an application and modelling based curriculum have been identified.

First, changing or deconstructing 'applying mathematics' to 're-applying mathematics' will signal that historically mathematics was created as an application modelling real-world problems, i.e. as a language describing and predicting the natural fact many. This distinction is useful when answering the question 'does 'applying mathematics' mean applying pastoral metamatics and mathematism, or re-applying grounded mathematics?'

Second, banning from school set-based metamatics and mathematism will bring back a new Enlightenment period with grounded mathematics. Thus, in primary school 'ten=10'-centrism is banned by practicing counting by bundling&stacking in 5-bundels, 7-bundles etc. before finally choosing ten as the standard bundle-size. In middle school fractions should be seen as per-numbers, and should always carry units; and equations should be introduced as reversed calculations. In high school functions and equations should be seen as formulas with two or one variables to be treated on a graphical display calculator using regression to produce formulas, describing per-numbers to be integrated to totals, or totals to be differentiated to per-numbers.

Conclusion

Now an answer can be given to the two initial questions. To improve learning, an application-based mathematics curriculum should re-apply grounded mathematics rooted in real-world problems; and not apply pastoral metamatism, a mixture of metamatics presenting concepts as examples of abstractions instead of as abstractions from examples, and mathematism true in a library but not in a laboratory and therefore unable to predict real-world situations.

Applying metamatism forces three cases of centrism upon mathematics as pastoral choices hiding their alternatives. The use of 'ten=10'-centrism hides that also other numbers than ten can be used as bundle-size when counting by bundling&stacking. Fraction-centrism hides that proportionality and many other applications of fractions can also be solved by instead re-applying recounting. And set-centrism hides that modelling change can take place without the use of set-based concepts as functions and limits. Finally, the wording 'apply mathematics' installs as self-evident that 'of-course mathematics must be taught and learned before it can be applied', thus hiding that historically mathematics is rooted in the real-world as a model.

A grounded approach will respect the historical nature of mathematics as a natural science rooted in the physical fact many. Here mathematics is created through its real-world roots and then re-applied to similar situations. To avoid 'ten=10'-centrism in primary school, before introducing 3.order counting installing ten as the only bundle-size, 2.order counting is used to emphasize that mathematics is a language for predicting real-world numbers, and to allow the learning of 1digit mathematics. To avoid fraction-centrism in middle school, proportionality is based upon recounting and per-numbers, and fractions always carry units when added. To avoid set-centrism in high school, the Graphical Display Calculator is used when modelling change, both the standard linear, exponential and polynomial models and the more complicated models.

Re-applying grounded mathematics invites the grand narratives of the quantitative literature into the mathematics curriculum, and allows the minor narratives to be introduced at an early stage. Applying pastoral metamatism means excluding the grand narratives and bringing the minor narratives to a halt until the metamatism applied is taught and learned. Does modelling want to generate grounded mathematics, or to be a docile lackey of pastoral metamatism?

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Saving Dropout Ryan with a TI-82

A principal asked for ideas to lower the high number of dropouts in pre-calculus classes. The author proposed using the cheap TI-82, but the teachers rejected the proposal saying students weren't even able to use a TI-30. Still the principal allowed buying one for a class. A compendium called 'Formula Predict' replaced the textbook. A formula's left- and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects.

The Task
The headmaster asked the mathematics teachers: "We have too many pre-calculus dropouts. What can we do?" I proposed buying the cheap TI-82 graphical calculator, but the other teachers rejected this proposal arguing that students weren't even able to use a simple TI-30. Still I was allowed to buy this calculator for my class allowing me to replace the textbook with a compendium emphasizing modeling with TI-82.

The Background
Enlightenment mathematics was as a natural science exploring the natural fact Many (Kline, 1972) by grounding its concepts as abstractions from examples, and by validating its statements by testing its deductions on examples. But after abstracting the set-concept, mathematics was turned upside down to modern mathematics or 'metamatism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathematism' true in the library, but not in the laboratory, as e.g. 2+3 = 5, which has countless counterexamples: 2m + 3cm = 203 cm, 2weeks + 3days = 17 days etc. This modern mathematics was completely self-referring not needing an outside world.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not belonging to itself, Russell proved that sets imply self-reference and self-contradiction as known from the classical liar-paradox 'this statement is false' being false when true and true when false:

If $M = \{ A \mid A \notin A \}$, then $M \in M \Leftrightarrow M \notin M$.

Likewise, Gödel proved that a well-proven theory is a dream since it will always contain statements that can be neither proved nor disproved.

In spite of being neither well-defined nor well-proved, mathematics still teaches metamatism. This creates big problems to mathematics education as shown e.g. by 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest that when counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

To design an alternative, mathematics can return to its roots, Many, guided by contingency research discovering alternatives to choices that presented as nature install hidden patronization.

Contingency Research
Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists warned that to practice democracy, the people must be enlightened to tell choice from nature in order to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should become the natural patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronization with democracy.

Two democracies were installed: one in the US still having its first republic; and one in France, now having its fifth republic. German autocracy tried to stop the French democracy by sending in an army. However, a German mercenary was no match to a French conscript aware of the feudal consequence of defeat. So, the French stopped the Germans and later occupied Germany.

Unable to use the army, the German autocracy instead used the school to stop enlightenment spreading from France. As counter-enlightenment, Humboldt used Hegel philosophy to create a patronizing line-organized Bildung school system based upon three principles: To avoid democracy, the people must not be enlightened; instead romanticism should install nationalism so the people sees itself as a 'nation' willing to fight other 'nations', especially the democratic ones; and the elite should be extracted and builded to become a knowledge-nobility for a new strong central administration replacing the blood-nobility that was unable to stop the French democracy.

Even as democracies, the EU still holds on to line-organized education instead of changing to block-organized education as in the North American republics allowing young students to uncover and develop their personal talent through individually chosen half-year knowledge blocks.

In France, the sophist warning against hidden patronization is kept alive in the post-structural thinking of Derrida, Lyotard, Foucault and Bourdieu. Derrida warns against ungrounded words installing what they label, such word should be 'deconstructed' into labels. Lyotard warns against ungrounded sentences installing political instead of natural correctness. Foucault warns against disciplines claiming to express knowledge about humans; instead they install order by disciplining both themselves and their subject. And Bourdieu warns against using education as symbolic violence to monopolize the knowledge capital for a knowledge nobility. (Tarp, 2004).

The difference between nature and choice is illustrated by the 'number & word dilemma': Placed between a ruler and a dictionary, a '17 cm long stick' can point to '15', but not to 'pencil', thus being able itself to falsify its number but not its word, which makes numbers nature, and words choices installing hidden patronization if hiding their alternatives.

Thus, contingency research does not refer to, but questions existing research by asking 'Is this nature or choice presented as nature?' To prevent patronization, categories should be grounded in nature using Grounded Theory (Glaser et al, 1967), the method of natural research developed in the first Enlightenment democracy, the American, and resonating with Piaget's principles of natural learning (Piaget, 1970).

The Case of Teaching Math Dropouts

Being our language about quantities, mathematics is a core part of education in both primary and secondary school. Parents accept the importance of learning mathematics, but many students fail to see the meaning of doing so. Consequently, special core courses for math dropouts are developed.

Traditions of Core Precalculus Courses for Dropouts

A typical core course for math dropouts is halving the content and doubling the text volume. So, in a slow pace the students work their way through a textbook once more presenting mathematics as a subject about numbers, operations, equations and functions applied to space, time, mass and money. To prevent spending time on basic arithmetic, a TI-30 calculator is handed out without instruction.

As to numbers, the tradition focuses on fractions and how to add fractions.

Then solving equations is introduced using the traditional balancing method isolating the unknown by performing identical operations to both sides of the equation. Typically, the unknown occurs on both sides of the equation as $2x + 3 = 4x – 5$; or in fractions as $5 = 40/x$.

Then relations between variables are introduced using tables, graphs and functions with special emphasis on the linear function $y = f(x) = a*x + b$.

In a traditional curriculum, a linear function is followed by the quadratic function. But a core course might instead go on to the exponential function $y = b*a^x$. To avoid solving its equations the solutions are given as formulas.

Problems in Traditional Core Courses

The intention of a traditional core course is to give a second chance to learners having dropped out op the traditional math course. However, from a skeptical viewpoint trying to avoid hidden patronization presenting choice as nature, several questions can be raised.

As to numbers, are fractions numbers or calculations that can be expressed with as many decimals as we want, typical asking for three significant figures? Is it meaningful to add fractions without units as shown by the fraction-paradox above?

As to equations, is the balancing method nature or choice presented as nature? The number x = 8-3 is defined as the number that added to 3 gives 8, x + 3 = 8. This can be restated as saying that the equation x + 3 = 8 has the solution x = 8-3; suggesting that the natural way to solve equations is the 'move and change calculation sign' method. This method that can be applied to all cases of reversed calculation:

x + 3 = 15 x = 15 – 3	x*3 = 15 $x = \frac{15}{3}$	x^3 = 125 $x = \sqrt[3]{125}$	3^x = 243 $x = \log_3(243)$

As to relation between variables, is the function nature or choice presented as nature? A basic calculation as 3 + 5 = 8 contains three numbers. If one of these is unknown, we have an equation to be solved, e.g. 3 + x = 5, if not already solved, 3 + 5 = x. With two unknown we have a formula as in 3 + x = y, or a relation as in x + y = 3 that can be changed into the formula y = 3 – x. So, the natural relation between two unknown variables seems to be a formula that when entered on a graphical calculator can be tabled and graphed.

As to avoiding solving exponential equations, is presenting solution formulas nature or choice presented as nature? Solving basic equations is just another way of defining inverse operations, so a natural thing is to define roots and logs as solution to the basic equations involving power as shown above.

Designing a Grounded Core Courses

So, a traditional core course seems to be filled with examples of choices presented as nature. This leads to the question: is it possible to design an alternative core course based upon nature instead of choices presented as nature? In other words, what would be the content of a core course in pre-calculus grounded in the roots of mathematics, the natural fact Many?

Mathematics as a Number-language

As to the nature of the subject itself, mathematics is a number-language that together with the word-language allows users to describe quantities and qualities in everyday life. Thus, a calculator is a typewriter using numbers instead of letters. A typewriter combines letters to words and sentences. A calculator combines figures to numbers that combined with operations becomes formulas. Thus, formulas are the sentences of the number-language typically telling how a total T can be calculated, e.g. T = a+b*c.

The Four Ways of Uniting

Rewriting T = 8765 as T = 8*10^3 + 7*10^2 + 6*10 + 5 shows a 4digit number as a short way of writing a 3^{rd} degree polynomial. Thus, polynomials are the fundamental formula uniting numbers by three operations: addition, multiplication and power. This gives meaning to the word 'algebra' meaning reuniting in Arabic. The fourth way is integration uniting per-numbers, e.g. 2 kg at 7 $/kg + 3 kg at 8 $/kg = 5 kg at ? $/kg.

Formulas Predict

One difference between the word and the number language is that sentences describe, whereas formulas use operations to predict the four different ways of uniting numbers:

Addition predicts the result of uniting unlike unit-numbers: uniting 2$ and 3$ gives a total that is predicted by the formula T = a+b = 2+3 = 5

Multiplication predicts the result of uniting like unit-numbers: uniting 2$ 5 times gives a total that is predicted by the formula T = a*b = 5*2 = 10.

Power predicts the result of uniting like per-numbers: uniting 2% 5 times gives a total that is predicted by the formula 1+T = a^b = 1.02^5, i.e. T = 0.104 = 10.4%.

Integration predicts the result of uniting unlike per-numbers: uniting 2kg at 7$/kg and 3kg at 8$/kg gives 5 kg at T$/5kg where T = 7*2 + 8*3 = the area under the per-number graph,

$T = \Sigma\ p*\Delta x = \int p*dx.$

Solving Equations with Solver
Inverse operations solve equations: x+2 = 6 is solved by x = 6-2. As do the TI-82 using a solver after entering the left hand and the right-hand side on the Y-list as Y1 = x+2 and Y2 = 6. Since any equation has the form Y1 = Y2, or Y1 – Y2 = 0, this only has to be entered to the solver once. After that, solving equations just means entering its two sides as Y1 and Y2. Using graphs, Y1 and Y2 becomes two curves having equal values in their intersection points.

If one of the numbers in a calculation is unknown, then so is the result. A formula with two unknowns can be described by a table answering the question 'if x is this, then what is y?' Graphing a table allows the inverse question to be addressed by reading from the y-axis.

Producing Formulas with Regression
Once a formula is known, it produces answers by being solved or graphed. Real-world data often come as tables, so to model real-world problems we need to be able to set up formulas from tables.

Simple formulas describe levels as e.g. cost = price*volume. Calculus formulas describe change and pre-calculus formula describes constant change.

If a variable y begins with the value b and changes by a number a x times, the y = b + a*x. This is called linear change and occurs in everyday trade and in interest-free saving.

If a variable y begins with the value b and changes by a percentage r x times, the y = b * (1+r)^x since adding 5% means multiplying with 105% = 1 + 5%. This is called exponential change and occurs when saving money in a bank and when populations grow or decay.

In the case of linear and exponential change a two-line table allows us to find the two constants b and a using regression on a TI-82.

Multi-line tables can be modeled with polynomials. Thus a 3-line table might be modeled by a degree 2 quadratic polynomial y = b + a*x + c*x^2 including also a bending-number c; and a 4-line table by a degree 3 cubic polynomial y = b + a*x + c*x^2 + d*x^3 including also a counter-bending-number d, etc.

Graphically, a degree 2 polynomial gives a bending ling, a parabola; and a third degree polynomial gives a double parabola. The top and the bottom of a bending curve as well as its zeros can be found directly by graphical methods of the TI-82

Fractions as Per-numbers
Fractions are rooted in per-numbers: 3$ per 5 kg = 3$/5kg = 3/5 $/kg. To be added, per-numbers first must be changed to unit numbers by being multiplied with their units:

3 kg at 4 $/kg + 5 kg at 6 $/kg = 8 kg at (3*4 + 5*6)/8 $/kg

Geometrically this means that the areas under their graphs add per-numbers.

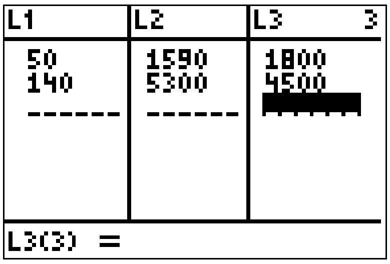
Her again TI82 comes in handy when calculating areas under graphs; also in the case where the graph is not horizontal but a bending line, representing the case when the per-number is changing continuously as e.g. in a falling body: 3 seconds at 4 m/s increasing to 6 m/s totals 15 m in the case of a constant acceleration.

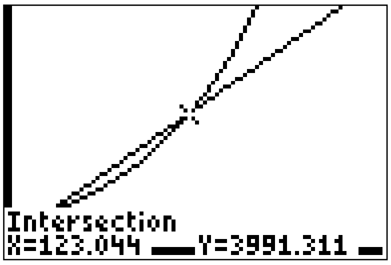
Models as Quantitative Literature

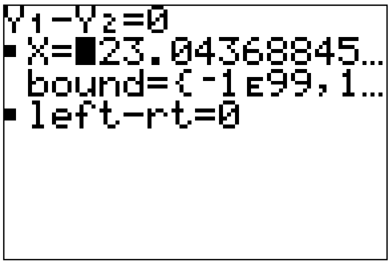
With the ability to use TI-82 as a quantitative typewriter able to set up formulas from tables and to answer both x- and y-questions, it becomes possible to include models as quantitative literature. All models share the same structure: A real-world problem is translated into a mathematical problem that is solved and translated back into a real-world solution.

	Problem		Solution
Mathematics		→	
	↑		↓
Real-world			

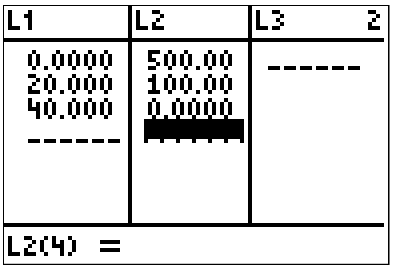
The project ‘Population versus food’ looks at the Malthusian warning: If population changes in an exponential and food in a linear way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2-line table, regression can produce two formulas: with x counting years after 1850, the population is modeled by Y1 = 815*1.013^x and the food by Y2 = 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

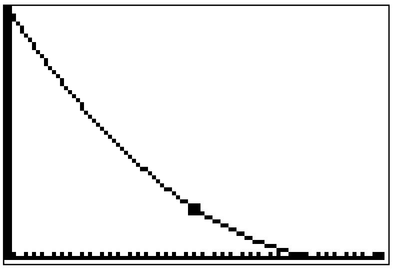


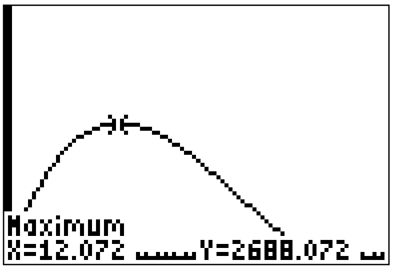




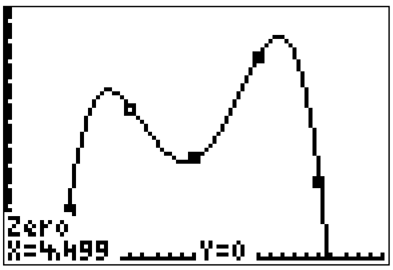
The project ‘Fundraising’ finds the revenue of a fundraising show assuming that all students will accept a free ticket, that 100 students will buy a 20$ ticket and that no one will buy a 40$ ticket. From this 3-line table the demand can be modeled by the quadratic formula Y1 = .375*x^2 – 27.5*x + 500. Thus, the revenue formula is the product of the price and the demand, Y2 = x*Y1. Graphical methods show that the maximum revenue will be 2688 $ at a ticket price of 12$.

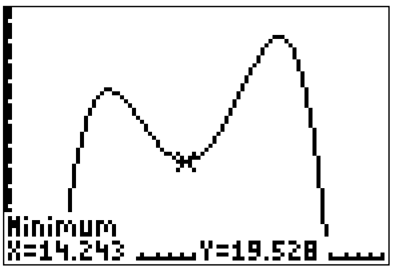


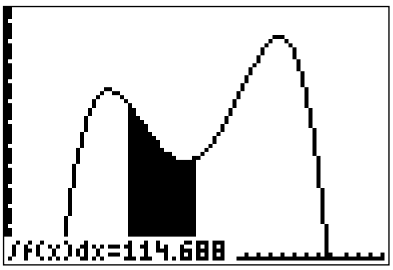




In the project ‘Driving with Peter’ his velocity is measured five times. A model answers two questions: When is Peter accelerating? And what distance did Peter travel in a given time interval? From a 5-line table the speed can be modeled by the 4th degree polynomial Y1 = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235. Visually, the triple parabola fits the data points. Graphical methods show that a minimum speed is attained after 14.2 seconds; and that Peter traveled 115 meters from the 10th to the 15th second.







Six other projects were included in the course. The project 'Forecasts' modeled a capital growing constantly in three different ways: linear, exponential and potential. The project 'Determining a Distance' uses trigonometry to predict the distance to an inaccessible point on the other side of a river. The project 'The Bridge' uses trigonometry to predict the dimensions of a simple expansion bridge over a canyon. The project 'Playing Golf' asks to predict the formula for the orbit of a ball that has to pass three given points: a starting point, an ending point and the top of a hedge. The project 'Saving and Pension' asks about the size of a ten years monthly pension created by a thirty years monthly payment of 1000$ at an interest rate of 0.4% per month. And the project 'The Takeover Try' asks how much company A has spent buying stocks in company B given an oscillating course described by a four-line table.

Testing the Core Course

The students expressed surprise and content with the course. Their hand-in was delivered on time. And they course finished before time allowing the inclusion of additional models from classical physics: vertical falling balls, projectile orbits, colliding balls, circular motion, pendulums, gravity points, drying wasted whisky with ice cubes and supplying bulbs with energy.

At the written and the oral exam, for the first time at the school, all the students passed. Some students wanted to move on to a calculus class, other were reluctant arguing that they had already learned the core of calculus.

Discussing the Result

The intention of a core course is to give a second chance to learners having dropped out from the traditional math course. Instead of using a traditional textbook, this approach used skepticism to uncover contingency in the pre-calculus curriculum by replacing choices presented as nature with nature itself. Also, up-to date technology was used to allow the users to cooperate with technology. The users provide the tables and the questions. Technology then provides the formulas and the answers.

Reporting Back to the Headmaster

The headmaster expressed satisfaction, but the teachers didn't like the textbook and its traditional math to be set aside. To encourage the teachers, the headmaster ordered TI-82 to be bought to all pre-calculus classes.

Conclusion: Making Losers Users

To give an extra chance to dropout students it gives good meaning to create a core course boiling the mathematics content down to its core. To be successful, the core should be deconstructed, i.e. the core should be grounded in its roots, the natural fact Many. Numbers should be presented as polynomials to show three of the four different operations uniting numbers according to Algebra's reuniting project. Also, direct and inverse operations should be presented as means to predict the united numbers or their parts. In this way the core of basic algebra becomes solving equations with the move and change sign method or with the solver of a graphical calculator. And the core of pre-calculus is to use regression to translate tables to formulas that can be processed when entered into the y-list of the TI82. Thus, grounding mathematics in its natural roots and a graphical calculator provides ordinary students with a typewriter that can be used to predict the behavior of real-world quantities.

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Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations

Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

A need for curricula for all students

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

Traditionally, a school system is divided into a primary school for children and a secondary school for adolescents, typically divided into a compulsory lower part, and an elective upper part having precalculus as its only compulsory math course. So, looking for a change we ask: how can precalculus be sustainably changed?

A Traditional Precalculus Curriculum

Typically, basic math is seen as dealing with numbers and shapes; and with operations transforming numbers into new numbers through calculations or functions. Later, calculus introduces two additional operations now transforming functions into new functions through differentiation and integration as described e.g. in the ICME-13 Topical Survey aiming to "give a view of some of the main evolutions of the research in the field of learning and teaching Calculus, with a particular focus on established research topics associated to limit, derivative and integral." (Bressoud et al, 2016)

Consequently, precalculus focuses on introducing the different functions: polynomials, exponential functions, power functions, logarithmic functions, trigonometric functions, as well as the algebra of functions with sum, difference, product, quotient, inverse and composite functions.

Woodward (2010) is an example of a traditional precalculus course. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

A Different Precalculus Curriculum

Inspired by difference research (Tarp, 2018) we can ask: Can this be different; is it so by nature or by choice?

In their ‘Principles and Standards for School Mathematics’ (2000), the US National Council of Teachers of Mathematics, NCTM, identifies five standards: number and operations, algebra, geometry, measurement and data analysis and probability, saying that “Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century (p. 2).” In the chapter ‘Number and operations’, the Council writes: “Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number (p. 7).”

Their biological capacity to adapt to their environment make children develop a number-language allowing them to describe quantity with two-dimensional block- and bundle-numbers. Education could profit from this to teach children primary school calculus that adds blocks (Tarp, 2018). Instead, it imposes upon children one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must be learned before it can be applied to operate on the functions introduced at the precalculus level.

However, listening to the Ausubel (1968) advice “The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly (p. vi).”, we might want to return to the two-dimensional block-numbers that children brought to school.

So, let us face a number as 456 as what it really is, not a one-dimensional linear sequence of three digits obeying a place-value principle, but three two-dimensional blocks numbering unbundled singles, bundles, bundles-of-bundles, etc., as expressed in the number-formula, formally called a polynomial:

T = 456 = 4*B^2 + 5*B + 6*1, with ten as the international bundle-size, B.

This number-formula contains the four different ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant ‘double-numbers’ or ‘per-numbers’. We might call this beautiful simplicity ‘the algebra square’ inspired by the Arabic meaning of the word algebra, to re-unite.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Figure 01. The ‘algebra-square’ has 4 ways to unite, and 5 to split totals

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers, creating the operations integration and its reverse operation, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power, and its two reverse operations, root and logarithm.

Precalculus, building on or rebuilding?

In their publication, the NCTM writes “High school mathematics builds on the skills and understandings developed in the lower grades (p. 19).”

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they may have from the lower grades?

Furthermore, what kind of mathematics has been taught? Was it ‘grounded mathematics’ abstracted ‘bottom-up’ from its outside roots as reflected by the original meaning of ‘geometry’ and ‘algebra’ meaning ‘earth-measuring’ in Greek and ‘re-uniting’ in Arabic? Or was it ‘ungrounded mathematics’ or ‘meta-matics’ exemplified ‘top-down’ from inside abstractions, and becoming ‘meta-matism’ if mixed with ‘mathe-matism’ (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept ‘function’, Euler saw it as a bottom-up name abstracted from ‘standby calculations’ containing specified and unspecified numbers. Later meta-matics defined a function as an inside-inside top-down example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may also be seen as an outside-inside bottom-up number-language sentence containing a subject, a verb and a predicate allowing a value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group operation applying the associative and commutative law as well as the neutral element and inverse elements, thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2:

2*u =6, so (2*u)*½ =6*½, so (u*2)*½ =3, so u*(2*½) =3, so u*1 =3, so u =3.

However, the equation 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula ‘T = (T/B)*B’ (Tarp, 2018), present all over mathematics as the proportionality formula, thus solved in one step: 2*u = 6 = (6/2)*2, giving u = 6/2.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology.

Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as a number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar.

So why not try designing a fresh start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction for an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

A Grounded Outside-Inside Fresh start Precalculus from Scratch

Let students see that both the word-language and the number-language provide ’inside’ descriptions of ‘outside’ things and actions by using full sentences with a subject, a verb, and an object or predicate, where a number-language sentence is called a formula connecting an outside total with an inside number or calculation, shortening ‘the total is 2 3s’ to ‘T = 2*3’;

Let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation. Writing ‘y = f(x)’ means that the y-number is found by specifying the x-number and the f-calculation. So with x = 3, and f(x) = 2+x, we get y = 2+3 = 5.

Let students see how calculations predict: how 2+3 predicts what happens when counting on 3 times from 2; how 2*5 predicts what happens when adding 2$ 5 times; how 1.02^5 predicts what happens when adding 2% 5 times; and how adding the areas 2*3 + 4*5 predicts adding the ‘per-numbers’ when asking ‘2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?’

Solving Equations by Moving to Opposite Side with Opposite Sign

Let students see the subtraction ‘u = 5-3’ as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers ‘to opposite side with opposite calculation sign’; a rule that applies also to the other reversed operations:

- the division u = 5/3 is the number u that multiplied with 3 gives 5, thus solving the equation u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, thus solving the equation u^3 = 5, and making root a ‘factor-finder’
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, thus solving the equation 3^u = 5, and making logarithm a ‘factor-counter’.

Let students see multiple calculations reduce to a single calculation by unhiding ‘hidden brackets’ where 2+3*4 = 2+(3*4) since, with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s.

This prevents solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5. Allowing to unhide the hidden bracket we get:

2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4

This solution is verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy a ‘Hymn to Equations’: “Equations are the best we know, they’re solved by isolation. But first the bracket must be placed, around multipli-cation. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don’t want to stop!”

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let students place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, three terms disappear if adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = b^2/4 – c.

Moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c, so u + b/2 = $\pm\sqrt{b^2/4 - c}$, so u = -b/2 $\pm\sqrt{b^2/4 - c}$

Recounting Grounds Proportionality

Let students see how recounting in another unit may be predicted by a recount-formula ‘T = (T/B)*B’ saying “From the total T, T/B times, B may be pushed away” (Tarp, 2018). In primary school this formula recounts 6 in 2s as 6 = (6/2)*2 = 3*. In secondary school the task is formulated as an equation u*2 = 6 solved by recounting 6 in 2s as u*2 = 6 = (6/2)*2 giving u = 6/2, thus again moving 2 ‘to opposite side with opposite calculation sign’.

Thus an inside equation u*b = c can be ‘de-modeled’ to the outside question ‘recount c from ten to bs’, and solved inside by the recount-formula: u*b = c = (c/b)*b giving u = c/b.

Let students see how recounting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. And see how filling a circle with right triangles from the inside allows phi to be found from a formula: circumference/diameter = π ≈ n*tan(180/n) for n large.

Double-counting Grounds Per-numbers and Fractions

Let students see how double-counting in two units create ‘double-numbers’ or ‘per-numbers’ as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number:

Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg.

Asking ‘9kg = ?$’ we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Let students see how double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 40%.

To find 40% of 20$ means finding 40$ per 100$, so we re-count 20 in 100s:

T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$:

T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

Let students see how double-counting physical units create per-numbers all over STEM (Science, Technology, Engineering and mathematics):

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter;

meter = (meter/second) * second = velocity * second;

joule = (joule/second) * second = watt * second

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, T = 456 = 4*B^2 + 5*B + 6*1, containing the formulas for constant change:

T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change. So where constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics to ‘post-dict’ what we can’t ‘pre-dict’; and using confidence for predicting intervals.

Combining linear and exponential change by n times depositing a$ to an interest percent rate r, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest percent rate R is predicted by the formula 1+R = (1+r)^n. This saving may be used to neutralize a debt Do, that in the same period changes to D = Do*(1+R).

This formula and its proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Uniting Constant Per-Numbers as Factors

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$, changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, the formula for change with a constant change-factor, also called exponential change.

Reversing the question, this formula entails two equations.

The first equation asks about an unknown change-percent. Thus, we might want to find which percent that added ten times will give a total change-percent 70%, or, formulated with change-factors, what is the change-factor, a, that applied ten times gives the change-factor 1.70. So here the job is 'factor-finding' which leads to defining the tenth root of 1.70, i.e. 10√1.70, as predicting the factor, a, that applied 10 times gives 1.70: If a^10 = 1.70 then a = 10√1.70 = 1.054 = 105.4%. This is verified by testing: 1.054^10 = 1.692. Thus, the answer is "5.4% is the percent that added ten times will give a total change-percent 70%."

We notice that 5.4% added ten times gives 54% only, so the 16% remaining to 70% is the effect of compound interest adding 5.4% also to the previous changes.

Here we solve the equation a^10 = 1.70 by moving the exponent to the opposite side with the opposite calculation sign, the tenth root, a = 10√1.70. This resonates with the 'to opposite side with opposite calculation sign' method that also solved the equations a+3 = 7 by a = 7-3, and a*3 = 20 by a = 20/3.

The second equation asks about a time-period. Thus, we might want to find how many times 7% must be added to give 70%, 1.07^n = 1.70. So here the job is factor-counting which leads to defining the logarithm log1.07(1.70) as the number of factors 1.07 that will give a total factor at 1.70: If 1.07^n = 1.70 then n = log1.07(1.70) = 7,84 verified by testing: 1.07^7.84 = 1.700.

We notice that simple addition of 7% ten times gives 70%, but with compound interest it gives a total change-factor 1.07^10 = 1.967, i.e. an additional change at 96.7%-70% = 26.7%, explaining why only 7.84 periods are needed instead of ten.

Here we solve the equation 1.07^n = 1.70 by moving the base to the opposite side with the opposite calculation sign, the base logarithm, n = log1.07(1.70). Again, this resonates with the 'to opposite side with opposite calculation sign' method.

A fresh start precalculus curriculum could 'de-model' the constant percent change exponential formula T = b*a^n to outside real-world problems as a capital in a bank, or as a population increasing or radioactive atoms decreasing by a constant change-percent per year.

De-modeling may also lead to situations where the change-elasticity is constant as in science multiplication formulas wanting to relate a percent change in T with a percent change in n.

An example is the area of a square T = s^2 where a 1% change in the side s will give a 2% change in the square, approximately:

With To = s^2, T1 = (s*1.01)^2 = s^2*1.01^2 = s^2* 1.0201 = To*1.0201.

Calculus Deals with Uniting Changing Per-Numbers as Areas

In mixture problems we ask e.g. '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?' Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before added, thus adding by areas. So here multiplication precedes addition.

Asking inversely '2kg at 3$/kg + 4kg at how many $/kg gives 6kg at 5 $/kg?', we first subtract the areas 6*5 – 2*3 before dividing by 4, a combination called differentiation, ΔT/4, thus meaningfully postponed to after integration.

Statistics Deals with Unpredictable Change

Once mastery of constant change is established, it is possible to look at time series in statistical tables asking e.g. "How has the unemployment changed over a ten-year period?" Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b + a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n.

The average numbers allow calculating all totals in the period, assuming the numbers are predictable. However, they are not, so instead of predicting the numbers, we might 'post-dict' the numbers using statistics dealing with unpredictable numbers. This, in turn, offers a likely prediction interval by describing the unpredictable random change with nonfictional numbers, median and quartiles, or with fictional numbers, mean and standard deviation.

Calculus as adding per-numbers by their areas may also be introduced through cross-tables showing real-world phenomena as unemployment changing in time and space, e.g. from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add. First, they must multiply back to unit-numbers to find the total percent. However, multiplying creates areas, so per-numbers add by areas, which is what calculus is about. This procedure is later called Bayes' formula and conditional probability.

An example: in one region 10 persons have 50% unemployment, in another, 90 persons have 5% unemployment. To find the total, the unit-numbers can be added directly to 100 persons, but the percent-numbers must be multiplied back to numbers: 10 persons have 10*0,5 = 5 unemployed; and 90 persons have 90*0,05 = 4.5 unemployed, a total of 5 + 4.5 unemployed = 9.5 unemployed among 100 persons, i.e. a total of 9.5% unemployment, also called the weighted average. Later, this may be renamed to Bayes' formula for conditional probability.

Modeling in Precalculus Exemplifies Quantitative Literature
Furthermore, graphing calculators allows authentic modeling to be included in a precalculus curriculum thus giving a natural introduction to the following calculus curriculum, as well as introducing 'quantitative literature' having the same genres as qualitative literature: fact, fiction and fiddle (Tarp, 2001).

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator thus using a natural science proof, valid until finding counterexamples.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A' = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A' = x' = 1; and that if A = ½*x^2, then y = A' = (½*x^2)' = x; and that if A = 1/3*x^3, then y = A' = (1/3*x^3)' = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to 3, instead of adding small strips we just calculate the change in the area over this distance, later called the fundamental theorem of calculus.

This makes sense since adding many small strips means adding many small changes, which gives just one final change since all the in-between end- and start-values cancel out:

$\int_1^3 y * dx = \int_1^3 dA = \Delta_1^3\, \mathrm{A} = \Delta_1^3\, (\frac{1}{3} * x^3)$ = end – start = $\frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$

On the calculus course we just leave out the area by renaming it to a ‘primitive’ or an ‘antiderivative’ when writing

$\int_1^3 y * dx = \int_1^3 x^2 * dx = \Delta_1^3\, (\frac{1}{3} * x^3)$ = end – start = $\frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$

A graphing calculator show that this suggestion holds. So, finding areas under per-number graphs not only allows adding per-numbers, it also gives a grounded and natural introduction to integral and differential calculus where integration precedes differentiation just as additions precedes subtraction.

A Literature Based Compendium

From the outside, regression allows giving a practical introduction to calculus by analyzing a road trip where the per-number speed is measured in five second intervals to respectively 10 m/s, 30 m/s, 20 m/s 40 m/s and 15 m/s. With a five data-set table we can choose to model with a degree4 polynomial found by regression. Within this model we can predict when the driving began and ended, what the speed and the acceleration was after 12 seconds, when the speed was 25m/s, when acceleration and braking took place, what the maximum speed was, and what distance is covered in total and in the different intervals.

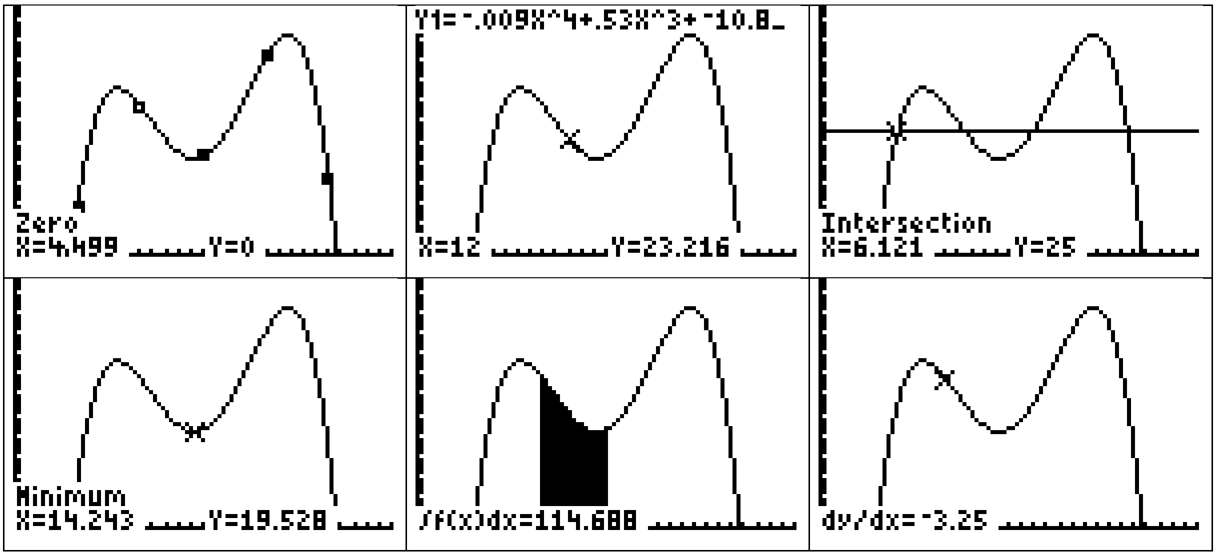

Figure 02. Analyzing Peter’s drive using regression and a graphing calculator

Another example of regression is the project ‘Population versus food’ looking at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2- line table regression can produce two formulas: with x counting years after 1850, the population is modeled by Y1= 815*1.013^x and the food by Y2= 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

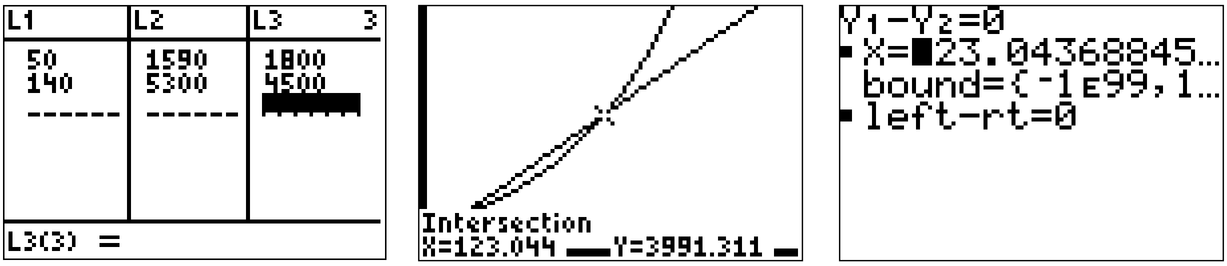

Figure 03. A Malthusian model of population and food levels

An example of an fresh start precalculus curriculum is described in 'Saving Dropout Ryan With a Ti-82' (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' (Tarp, 2019) replaced the textbook. A formula's left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

Precalculus in STEM

With the increased educational interest in STEM, modeling also allows including science-problems as e.g. the transfer of heat taking place when placing an ice cube in water or in a mixture of water and alcohol, or the transfer of energy taking place when connecting an energy source with energy consuming bulbs in series or parallel; as well as technology problems as how to send of a golf ball to hit a desired hole, or when to jump from a swing to maximize the jumping length; as well as engineering problems as how to build a road inclining 5% on a hillside inclining 10%.

Furthermore, precalculus allows students to play with change-equations, later called differential equations since change is calculated as a difference, $\Delta T = T2 - T1$. Using a spreadsheet, it is fun to see the behavior of a total that changes with a constant number or a constant percent, as well as with a decreasing number or a decreasing percent, as well as with half the distance to a maximum value or with a percent decreasing until disappearing at a maximum value. And to see the behavior of a total accelerating with a constant number as in the case of gravity, or with a number proportional to its distance to an equilibrium point as in the case of a spring.

Conclusion

So, by focusing on uniting and splitting into constant per-numbers, the fresh start precalculus curriculum has constant change-percent as its core. This will cohere with a previous curriculum on constant change-number or linearity; as well as with the following curriculum on calculus focusing on uniting and splitting into locally constant per-numbers, thus dealing with local linearity. Likewise, such a precalculus curriculum is relevant to the workplace where forecasts based upon assumptions of a constant change-number or change-percent are frequent. This curriculum is also relevant to the students' daily life as participants in civil society where tables presented in the media are frequent.

An Example of a Fresh start Precalculus Curriculum
This example was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

An Example of an Exam Question
Authentic material was used both at the written and oral exam. The first had specific, the second had open questions as the following asking 'What does the table tell?

Agriculture: Number of agricultural farms allocated over agricultural area

	1968	1969	1970	1971	197?	1973	1974	1975	1976	1977
Farms in total	**161142**	**154 694**	**148 512**	**144 070**	**143093**	**141 137**	**137712**	**13424S**	**130 7S3**	**127117**
0,0- 4,9 ha	25 285	23 493	21 533	21623	22123	21872	21093	19915	18 852	17 833
5.0- 9.9-	34 644	32129	30 235	28 404	27693	26 926	26109	25072	24066	23152
10,0-19.9-	48 997	46482	43 971	41910	40850	39501	38261	36 702	35 301	34 343
20.0-29.9-	25670	25 402	25181	24 472	24 195	23 759	23 506	23134	22737	22376
30,0-49.9-	18 505	18 779	18 923	18 705	18 968	18 330	19 095	19 304	10 305	19 408
50,0-99.9-	6 552	6 852	7 076	7 275	7 549	7956	7 847	8247	8 556	8723
100.0 ha and over	1489	1 557	1611	1681	1 715	1791	1801	1871	1934	1882

Figure 04. A table found in a statistical survey used at an oral exam.

Discussion and Conclusion
Asking "how can precalculus be sustainably changed?" an inside answer would be: "By its nature, precalculus must prepare the ground for calculus by making all function types available to operate on. How can this be different?"

An outside answer could be to see precalculus, not as a goal but as a means, an extension to the number-language allowing us to talk about how to unite and split into changing and constant per-numbers. This could motivate renaming precalculus to per-numbers calculations.

In this way, precalculus becomes sustainable by dealing with adding, finding and counting change-factors using power, roots and logarithm. Furthermore, by including adding piecewise constant per-numbers by their areas, precalculus gives a natural introduction to calculus by letting integral calculus precede and motivate differential calculus since an area changes with the last strip, thus connecting the unit number, the area, with the per-number, the height.

Finally, graphing calculators allows authentic modeling to take place so that precalculus may be learned through its use, and through its outside literature.

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01. Project Forecasting

Problem: How to set up a forecast assuming constant growth?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

A capital is assumed to grow constantly. From two data sets we would like to establish a forecast predicting the capital at a certain time and when a certain level will be reached.

2. The Mathematical Problem

We set up a table showing the capital to two different times. x are years, y is 1000 $

x	y = ?		
2	10	1. Linear ++ growth y = a*x + b	x: +1, y: +a (gradient, slope)
5	30	2. Exponential +* growth y = a*b^x = a*(1+r)^x	x: +1, y: + r% (interest rate, b=1+r)
8	?	3. Power ** growth y = a*x^b	x: +1%, y: + r% (elasticity)
?	60		

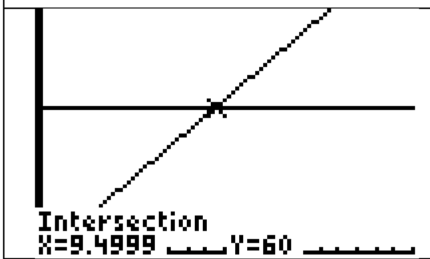
3. Solution to the Mathematical Problem

First we find the y-formulas using regression. We enter the table as lists L1 and L2 und STAT.
'LinReg Y1' produces a linear model transferred to the y-list as Y1
'ExpReg Y1' produces an exponential model transferred to the y-list as Y1
'PowerReg Y1' produces a power model transferred to the y-list as Y1

Linear growth

y = ?	y = 6.667*x – 3.333
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 6.667*8 – 3.333 = 50
Test	Trace x = 8 gives y = 50

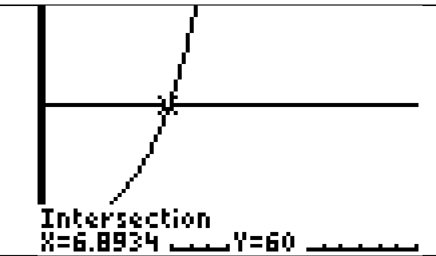
x = ?	y = 6.667*x – 3.333
y = 60	60 = (6.667*x)– 3.333 60 + 3.333 = 6.667*x 63.333/6.667 = x 9.5 = x
Test1	60 = 6.667*9.5– 3.333 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 9.5
Test3	CALC Intersection with y2=60 gives x = 9.5



Exponential growth

y = ?	y = 4.807 * 1.442^x
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.807 * 1.442^8 = 89.9
Test	Trace x = 8 gives y = 89.9

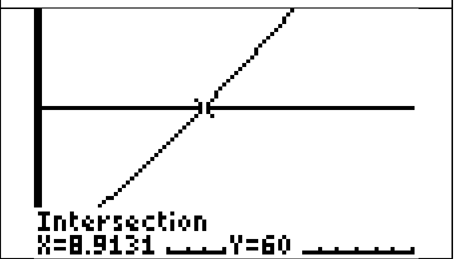
x = ?	y = 4.807 * 1.442^x
y = 60	60 = 4.807 * (1.442^x) 60/4.807 = 1.442^x log(60/4.807)/log1.142 = x 6.89 = x
Test1	60 = 4.807 * 1.442^6.89 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 6.89
Test3	CALC Intersection with y2=60 gives x = 6.89



Power growth

y = ?	y = 4.356*x^1.199
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.356*8^1.199 = 52.7
Test	Trace x = 8 gives y = 52.7

x = ?	y = 4.356*x^1.199
y = 60	60 = 4.356*(x^1.199) 60/4.356 = x^1.199 1.199√(60/4.356) = x 8.91 = x
Test1	60 = 4.356*8.91^1.199 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 8.91
Test3	CALC Intersection with y2=60 gives x = 8.91



4. Solution to the Real-world Problem

We see that forecast can be made by using technology's regression lines.
The forecasts give different answers to the same questions since different forms of growth is assumed.
Linear growth assumes that the gradient is constant.
Exponential growth assumes that the interest rate is constant.
Power growth assumes that the elasticity is constant.

02. Project Population and Food Growth

Problem: When will the population exceed food supply

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

Around the year 1800, the English economist Malthus (1766-1834) predicted a future food crisis, "since the world's population grows exponentially and food supply linearly, the population will one day overtake the food supply with famine to follow" (Malthus ' principle of population). Is Malthus right?

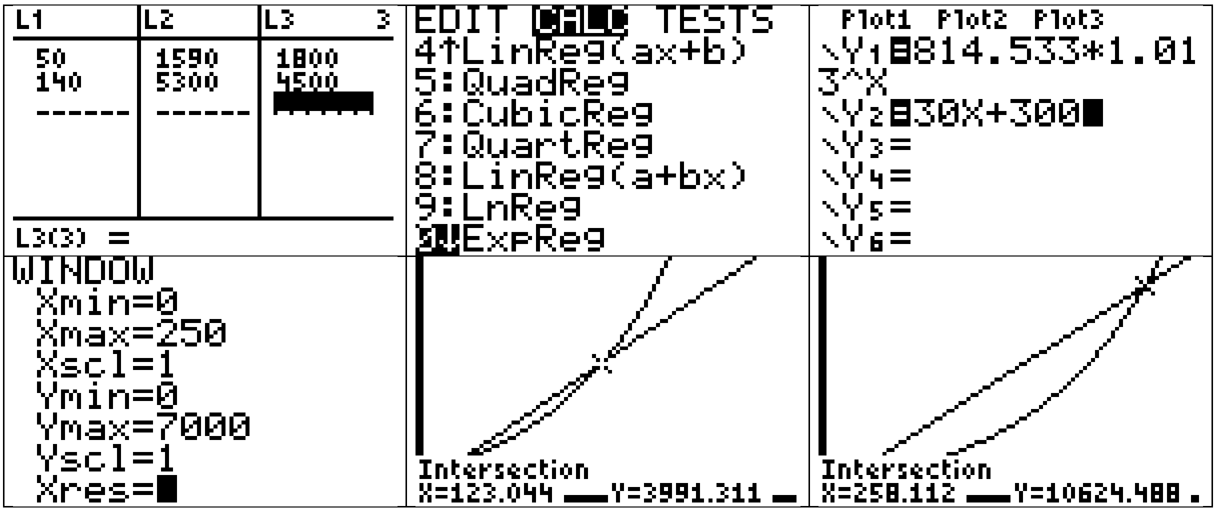
2. The Mathematical Problem

We set up a table of time x as the number of years after 1850; and the world's population, which is assumed to be 1.59 billion in 1900 and 5.3 billion in 1990; and world food production, which is assumed to be 1,800 billion daily rations in 1900 and 4.5 billion daily rations in 1990. The population is assumed to grow exponentially, and food quantity is assumed to grow linearly. The table scope is assumed to be 0 < x < 250.

x Years after 1850	Y1 World population in mio.	Y2 World food supply in mio. in daily rations
(1900) 50	1590	1800
(1990) 140	5300	4500

3. Solving the mathematical problem

On a TI-82, the x-numbers are on list L1, and the y-numbers are on lists L2 and L3. The formula for the population are determined by L1, L2, ExpReg Y1. The result is y1 = 815 * 1,013 ^ x. So, when x is 0 in 1850, the population is y = 815; and when x increases by 1, the population increases by 1.3% 1. The formula for the volume of food is determined by LinReg L1, L3, Y2. The result is y1 = 300 + 30 x. So, when x is 0 in the 1850, the food amount is y = 300; and when x increases by 1, the food increases by 30.
Famine occurs where Y1 is greater than Y2. Y1 and Y2 are the intersection found with 'Calc Intersection ' to around x = 30 and 123. So, as to the model there was famine from 1850 to 1880, and again after 1973.
Assume instead that the world population is growing by 1% a year, and the amount of food with 40 per year. Then the formulas become Y3 = 815 * 1.01^ x, and Y4 = 300 + 40 x. They will intersect at approximately x = 16 and x = 258. I.e. in this case, there was famine from 1850 to 1866, and again after 2108.
Intersections points can be controlled using MathSolver with a guess.



4. Solving the real-world problem

Malthus is right in saying that there will be famine if the world's population continues to grow exponentially, and the quantity of food continues to grow linearly, for a curved path will always outpace a straight. If food grows by 30 million daily rations per year, then famine will occur years 1973 if the world's population is growing by 1.3% per year, and in the year 2108 if the world's population is growing by 1% per year and food production grows by 40/year. However, it will be earlier if a part of the food is used for fueling cars instead.

03. Project Saving and Pension

Problem: How much pension will a saving provide?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

A saving comes from sending in a fixed amount each month to a bank. In the end, a saving can be used to drawing out a fixed amount each month. What is the relationship between the monthly saving input and the pension output?

2. The Mathematical Problem

If paying $1000 monthly for 30 years, what will the monthly pension be for 10 years? The interest rate is 0.4% monthly.

By saving two formulas apply, the first applies to a single deposit, the second for many monthly deposits:

1) K = Ko*(1+R), 1+R = (1+r)^n, K/Ko: terminal/initial capital, r: monthly rate, R: total interest, n numb. months.

2) K/a = R/r, K: terminal capital, a. monthly deposit, r: monthly rate, R: total interest,

3. Solving the mathematical problem

First we find the total interest rate per year R: 1+R = (1+r)^n = (1+0.004)^12, so R = 1.049 –1 = 0.049 = 4.9% per year.

Then we find the total interest rate for 30 years R: 1+R = (1+r)^n = (1+0.004)^(30*12) = 4.209. So. R = 4.209-1 = 3.209 = 321%.

The simple interest rate is 30*12*0.4% = 144%. So, the effect of compound interest is 321% - 144% = 177%.

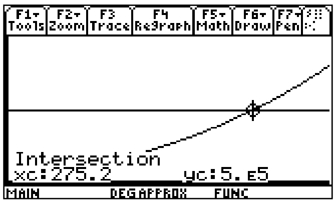
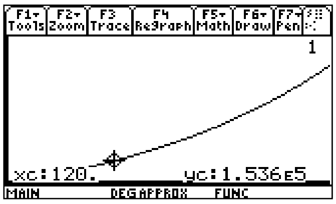
With a = 1000, r = 0.4% the saving after x deposits will be K = a*R/r = 1000*(1.004^x –1) / 0.004.

We find the saving after 10, 20 and 30 years:

Months	120	240	360
Saving	153632	401675	802147

The total deposit after 30 years is 1000*360 = 360000. The effect of compound interest is 802147 –360000 = 442147.

We observe that the saving is 500000 after 275 deposits.

x = ?	$\frac{1000*(1.004\text{^}x-1)}{0.004} = 500000$	Test 2	x = 120, K = ?
	$1.004\text{^}x - 1 = \frac{500000*0.004}{1000}$ 1.004^x = 2+1 $x = \frac{\ln(3)}{\ln(1.004)} = 275.2$		
Test1	$\frac{1000*(1.004\text{^}275.2-1)}{0.004} = 500000$ 499994 = 500000		

To use the saving for a 10years pension we use two accounts.

On the first, the saving grows from 10 years of interest to K = Ko (1 + R) = 802147 * (1 + 0.004) ^ 120 = 1295089.

The second is used for a 'negative saving' with a monthly redraw, a, that makes the two accounts balance after 10 years: K = a*R/r = Ko*(1+R), giving the equation a*(1,004^120 – 1)/0.004 = 1295089 that is solved by a = 8430.

Thus the relationship between output and input is (10*12*8430)/(30*12*1000) = 2.8.

Repeating the calculations with a monthly interest rate of 0.3% and 0.5% gives a different relationship:

Monthly %	Yearly %	Saving	Monthly pension	Relationship between output and input
0.3%	3.7%	646640	6425	(10*12*6425)/(30*12*1000) = 2.1
0.4%	4.9%	802147	8430	(10*12*8430)/(30*12*1000) = 2.8
0.5%	6.2%	1004515	11152	(10*12*11152)/(30*12*1000) = 3.7

4. Solving the real-world problem

When saving, an account grows from three monthly sources: a deposit, and an interest rate of the total deposit as well as of the total interest amount. When terminated, a saving continues to grow, but, used as a pension fund, the monthly input is replaced by a monthly output, the pension. Depositing $1000 monthly for 30 years allows taking out monthly $8430 in 10 years. With a monthly interest rate of respectively 0.3%, 0.4% and 0.5% the output-input ratio is respectively. 2.1, 2.8 and 3.7. However, it should be remembered that 40 years of inflation will reduce this factor.

Proof of the saving-formula: Account 1 contains the amount a/r. Each month we transfer the interest amount, r*a/r = a, to account 2, also receiving the monthly interest of its own amount. Account 2 thus will contain a saving K, but at the same time it contains total interest amount R of account 1, i.e. R*a/r. Therefore K = R*a/r, or K/a = R/r.

04. Project Supply, Demand and Market Price

Problem: How does supply and demand determine the market price?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

We assume we know supply curve and the demand for a given commodity, e.g. apples. That is, we know the market price determines supply and demand. If supply is larger than demand, a decrease in price should increase demand and decrease supply. If supply is less than demand, an increase in price should decrease demand and increase supply. The equilibrium price therefore should occur where supply equals demand.

2. The Mathematical Problem

We set up a table showing the relationship between price and demand and supply. The table is supposed to be valid for prices between 0 and 10, $0 < x < 10$. Regression allows finding the two formulas, becoming an equation when set to be equal, thus finding the point of intersection that determines the equilibrium price.

Linear graphs

Price x	Supply S	Demand D
2	40	80
4	60	50

Bending graphs

Price x	Supply S	Demand D
2	40	80
4	60	50
6	75	30

3. Solving the mathematical problem

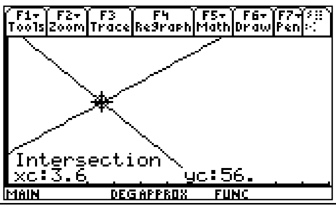
Entering a table to the data/matrix-editor of a graphic display calculator allows finding regression formulas.
With 2 data-sets we choose LinREg, giving a degree 1 polynomial without bending.
With 3 data-sets we choose QuadREg, giving a degree 2 polynomial with bending.
The intersection point is found graphically, or algebraically by solving two equations with two unknowns.

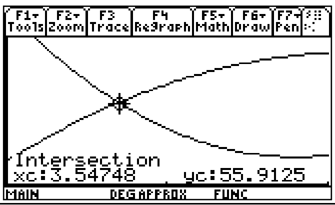
Degree one polynomial

x = ?	Supply = Demand
S = 10x+20 D = -15x+110	10x+20 = -15x+110 10x + 15x = 110-20 25x = 90 x = 90/25 = 3.6
Test1	y1(x) x=3.6 gives y=56 y2(x) x=3.6 gives y=56
Test2	Solve(y1(x)=y2(x),x) gives x=3.6
Test3	Graphical reading gives (x,y)=(3.6,56) (intersection)

Degree two polynomial

x = ?	Supply = Demand
S = -0.625x^2+13.75x+15 D = 1.25x^2-22.5x+120	-0.625x^2+13.75x+15 = 1.25x^2-22.5x+120 -1.875x^2+36.25x-105 = 0
Faktorising Zero rule	-1.875*(x-15.79)*(x-3.55) = 0 x = 15.79 og x = 3.55 15.79 is outside the validity area
Test1	y1(x) x=3.55 gives y=55.91 y2(x) x=3.55 gives y=55.91
Test2	Solve(y1(x)=y2(x),x) gives x=3.55
Test3	Graphical reading gives (x,y)=(3.55,55.91) (intersection)





4. Solving the real-world problem

We see that with linear supply and demand curves, the equilibrium price is \$3.6 resulting in an equilibrium level at 56 units. And we see that with bending supply and demand curves, the equilibrium price is \$3.55 resulting in an equilibrium level at 55.9 units. The solution assumes that the tables are unchanged. If changed, the regression formulas will change accordingly, and so will the solutions.

05.Project Collection, LafferCurve

Problem: How Which ticket price will optimize a collection income?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

We want to collect a charity fund among the school's 500 students by selling tickets at a fixed price. Which of the following three collection models will provide the highest contribution?

A) No marketing.	B) Marketing.	C) Marketing and lottery.

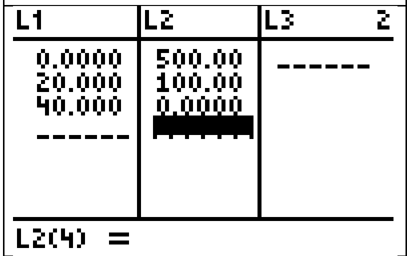
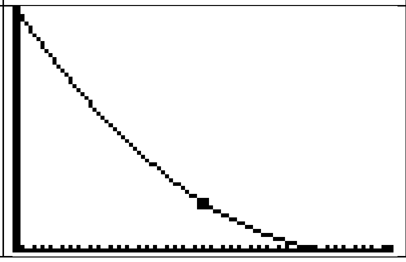
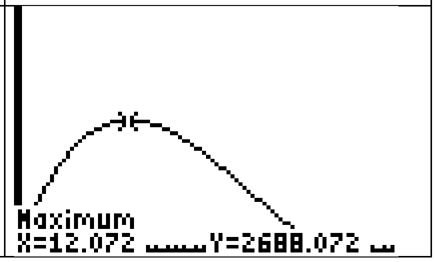
2. The Mathematical Problem

The demand Y1 will depend on the price x. The collected fund then will be Y2 = Y1*x.

3. Solution to the Mathematical Problem

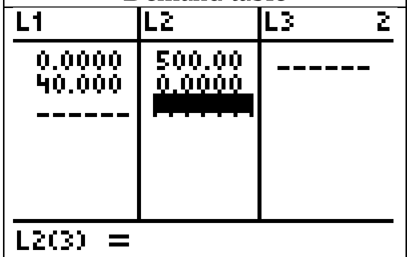
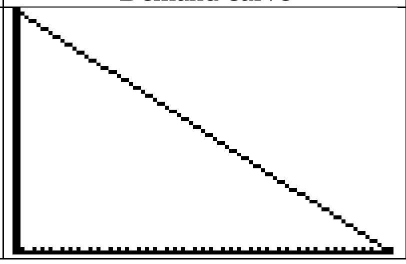
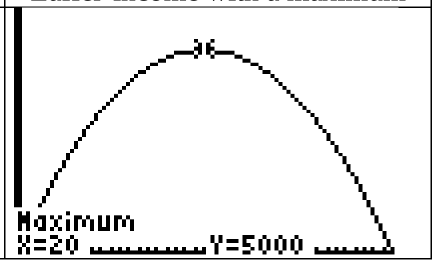
Model A. We assume that all 500 customers will buy a ticket at the price \$0, that no one will provide over \$40; and that demand is falling rapidly as only 100 customers will give \$20.

Demand: 3 data-sets gives a degree2 polynomial. 'QuadReg' gives Y1 = .375*x^2 – 27.5*x + 500

Demand table	Demand curve	Laffer-income with a maximum
		

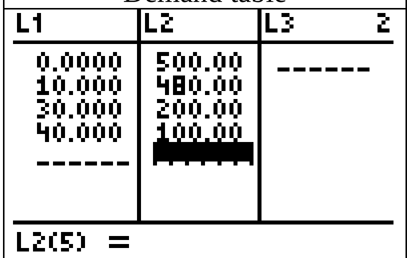
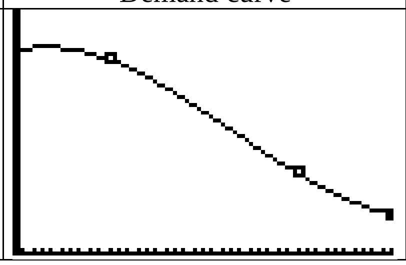
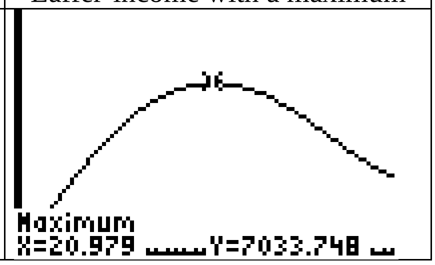
Test: Calc dy/dx ≈ 0 in x = 12.07. dy/dx = gradient, slope

Model B. We assume that with marketing, 500 will buy a ticket at \$0, no one will give more than \$40, demand is supposed to fall evenly. Demand: 2 data-sets gives a degree1 polynomial. 'LinReg' gives Y1 = -12.5*x + 500

Demand table	Demand curve	Laffer-income with a maximum
		

Test: Calc dy/dx ≈ 0 in x = 20. dy/dx = gradient, slope

Model C. Here marketing includes a lottery with a Grand Prize \$500 and 3 extra prizes of \$200. We assume this will result in all 500 customers will buy a ticket at the price \$0, 480 customers will give \$10, 400 customers will give \$20, 200 customers will give \$30 and 100 customers will give \$40. Demand: 4 data-sets gives a degree3 polynomial. 'CubicReg' gives Y1 = 0.013*x^3 – 0.933*x^2 + 6*x +500

Demand table	Demand curve	Laffer-income with a maximum
		

Test: Calc dy/dx ≈ 0 in x = 21.28. dy/dx = gradient, slope

4. Solution to the Real-world Problem

Collection without marketing will provide an income of \$2688 at a ticket price of \$12.
Marketing without lottery will give an income of \$5000 at a price at of \$20.
Marketing with lottery will provide an income of 7034 – (500 + 3 * 200) = 5934 at a price at \$21.
The Laffer-curve is an argument for increasing income tax together with decreased tax percentage.
The demand curve tells then that with growing tax rate, undeclared work will also grow.

06. Project Linear Programming

Problem: How to optimize a product mix?

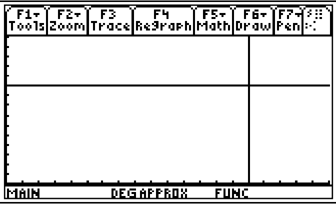
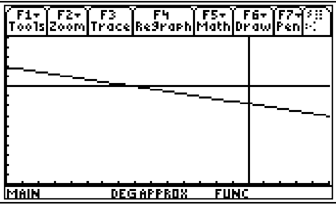
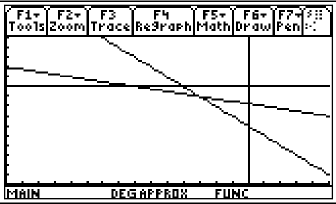
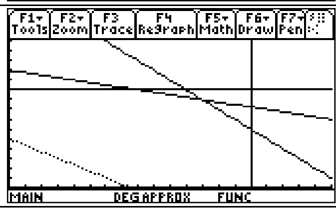
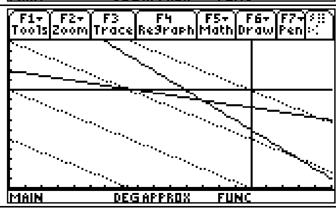
A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

A market booth sells water (max 15 boxes) and beer (max 10 boxes). The cost for water and beer is per box \$25 and \$100 respectively. A maximum of \$1200 DKK can be invested. At most 21 boxes can be sold during opening hours. Income is \$80/120 per box beer/water. How to optimize the income?

2. The Mathematical Problem

Outside world	Inside equations	Graphics
Boxes of water Boxes of beer	x y	y beer x water
Restriction on goods: The booth has place for a maximum of 15 boxes of water 10 boxes of beer	$0 \le x \le 15$ $0 \le y \le 10$	
Restriction on capital \$1200, given the cost: \$25 per box of water \$100 per box of beer	$25*x + 100*y \le 1200$ $(100*y \le -25*x + 1200$ $y \le -\frac{1}{4}*x + 12)$	
Restriction on labor: At most 21 boxes can be sold during opening hours.	$x + y \le 21$ $(y \le -x + 21)$	
Total income T is \$80/120 per box water/beer.	$T = 80*x + 120*y$ $y = -\frac{2}{3}*x + \frac{T}{120}$ N0: T = 0: $y = -2/3*x$ N600: T = 600: $y = -2/3*x + 5$	
Solution Buying 12 boxes of water and 9 boxes of beer will maximize the total income to \$2040.	'Solve($-\frac{1}{4}*x+12 = -x+21,x$)' gives x = 12 'y =-x + 21|x=12' gives y = 9 T = 80*x + 120*y | x=12 and y = 9' gives T = 2040	

3. Solution to the Mathematical Problem

Graphically, the restrictions give a polygon. The level-lines are since the income T only influences the intersection with the y-axis. A parallel translation of the level-lines across the polygon thus will increase or decrease T. So, the optimal value comes where a level-line leaves or is a tangent to the polygon, which will always happen in a corner. We can therefore predict the optimal situation by calculating all corner points (n equations with n unknowns), as well as T's values in these points (simplex method). This method is used when the number of variables is greater than 2.

4. Solution to the Real-world Problem

We see that the maximum income will be \$2040 when12 boxes of water and 9 boxes of beer are sold. Furthermore, we see that the effective restrictions are the opening hours and the invested capital. Linear programming is used to optimize a particular quantity (to maximize profit, to minimize cost, etc.) within a series of restrictions on other quantities.

07. Project Game Theory

Problem: Which strategy will maximize my outcome?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

Two players A and B choose between different strategies. The outcome table shows what B must pay to A. The 2person game is called a ZeroSum game, since what A wins, B loses, and vice versa.

2. The Mathematical Problems

We look at two different games:

I

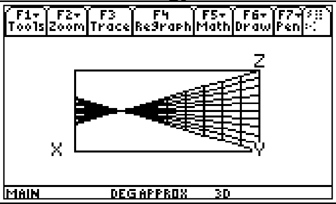
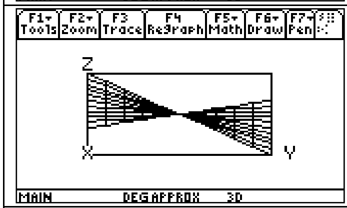
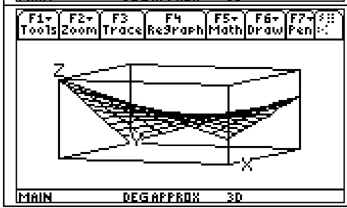
		B: b1	b2
A	a1	5	0
	a2	15	10

II

			B: *y* b1	*1-y* b2
A	*x*	a1	5	0
	1-x	a2	-5	10

3. Solution to the Mathematical Problems

The two players analyze the game:
A: With a1 is I risk outcome 0, with a2 I risk 10. In order to maximize my minimum-number I should choose a2. And, if I was B, I would choose b2, so I choose a2.
B: With b1 is I risk to pay 15, with b2 I risk 10. In order to minimize my maximum-pay I should choose a2. And, if I was A, I would choose a2, so I choose b2
The pair (a2, b2) is called the game **equilibrium** with the game value 10. No player will gain from not choosing the equilibrium strategy: A risks to get 0 instead of 10, so A sticks with the **maximin**-strategy. B risks paying 15 instead of 10, so B sticks with the **minimax**-strategy.

	'Solve (10*x-5 = -10*x+10, x)' gives x= 0.75
	'Solve (-15*y + 10 = 5*y, y)' gives y= 0.5
	Hold down the arrow for 1 second to watch the surface rotate

The two players analyze the game:
A: With a1 is I risk outcome 0, with a2 I risk -5. In order to maximize my minimum-number I should choose a1.
B: With b1 is I risk to pay 5, with b2 I risk 10. In order to minimize my maximum-pay I should choose b1.
But the maximin and minimax strategies a1 and b1 are not in equilibrium, since B gain by choosing b2, making A choose a2, making B choose b1, making A choose a1, etc.
Thus, with no equilibrium point, A should mix a1 and a2 in the ratio x to 1-x, randomly, and likewise B.
If B chooses b1 or b2, the outcomes will be respectively
P1 = 5*x – 5*(1-x) = 10*x – 5.
P2 = 0*x + 10*(1-x) = -10*x +10.
The two outcomes are like with 10*x – 5 = -10*x +10, or 20*x = 15, or x = 15/20 = 3/4 giving P = 10*3/4-5 = 2.5.
So by mixing the strategies a1 and a2 in the ration 3 to 1, player A will secure an average outcome 2.5 no matter what B chooses. So this game has the value 2.5.
Likewise we find that by mixing the strategies b1 and b2 in the ration 1 to 1, player B will secure an average loss 2.5 no matter what A chooses.
With A and B randomly mixing their strategies in the ratios x to 1-x, and y to 1-y respectively, the outcome will be
P = 5*x*y + 0*x*(1-y) – 5*(1-x)*y + 10*(1-x)*(1-y)
P = -10*x – 15*y + 20*x*y + 10
On a graphical display calculator P becomes a surface called a **saddle point**, going down one way, and up the other way.

4. Solution to the Real-world Problem

In game I, A and B should choose strategy a2 and b2 respectively resulting B losing 10 per game. In game II, the solution x =. 75, y = .5, P = 2.5 means that with a deck of cards, A should choose a2 when dragging clubs, otherwise a1. And that B should choose b2 when dragging black, otherwise b1. B will have an average loss at 2.5 per game.
We see that a 2person ZeroSum-game always has an equilibrium point in a saddle point going up to one side and down to the other. With the saddle point in a corner, this corner is the solution to the problem. With an inside saddle point, the strategies must be mixed randomly in a ration found by looking at the saddle point from the side.

08. Project Distance to a Far-away Point

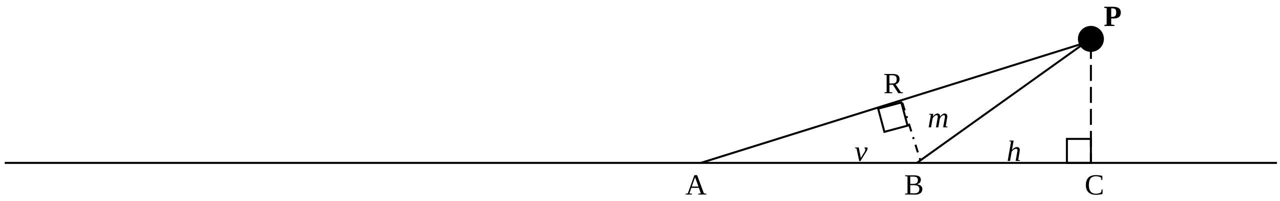
Problem: How to determine the distance to an inaccessible distant point?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

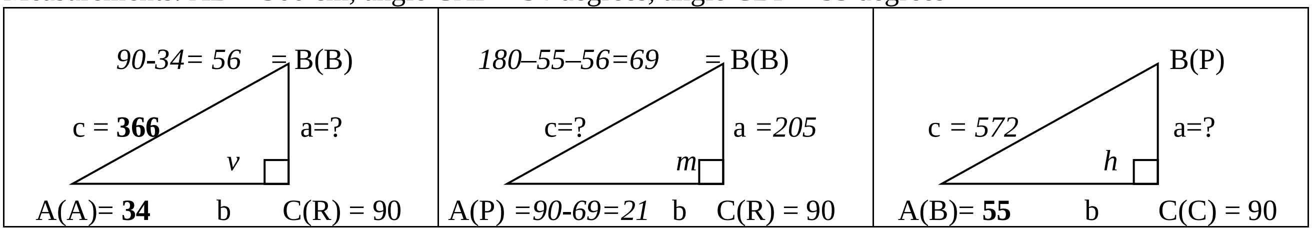
From a given baseline we want to determine the distance to a far-away inaccessible point P.



2. The Mathematical Problem

From a known baseline AB we measure the angles A and B to the inaccessible point P.
From the three right angled triangles ABR, BRP and BCP we calculate RB, BP as well as the distance PC.

Measurements: AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees



3. Solution to the Mathematical Problem

We set up three formula tables

Triangle ABR

a = ?	$\sin A = \frac{a}{c}$
A = 34 c = 366	$\sin 34 = \frac{a}{366}$ sin34 * 366 = a 205 = a
Test1 ☺	$\sin 34 = \frac{205}{366}$ 0.559 = 0.560
Test2 ☺	Math Solver $0 = \frac{x}{366} - \sin 34$ gives x = 205

Triangle PBR

c = ?	$\sin A = \frac{a}{c}$
A = 21 a = 205	$\sin 21 = \frac{205}{c}$ c * sin21 = 205 $c = \frac{205}{\sin 21}$ c = 572
Test1 ☺	$\sin 21 = \frac{205}{572}$ 0.358 = 0.358
Test2 ☺	Math Solver $0 = \frac{205}{x} - \sin 21$ gives x = 572

Triangle PBC

a = ?	$\sin A = \frac{a}{c}$
A = 55 c = 572	$\sin 55 = \frac{a}{572}$ sin55 * 572 = a 469 = a
Test1 ☺	$\sin 55 = \frac{469}{572}$ 0.819 = 0.820
Test2 ☺	Math Solver $0 = \frac{x}{572} - \sin 55$ gives x = 469

4. Solution to the Real-world Problem

By using trigonometry, we are able to determine the distance to the inaccessible point P to 469 cm.

09. Project Bridge

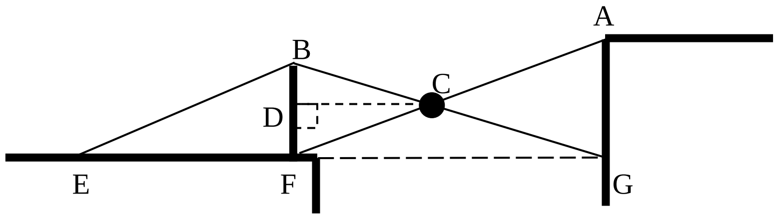
Problem: How to determine the dimensions of a bridge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

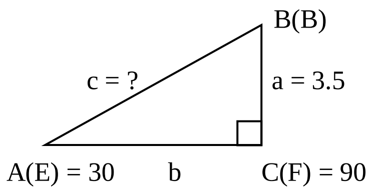
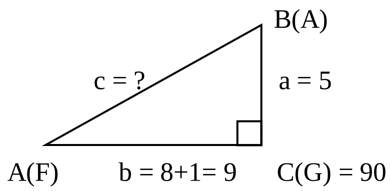
Over a canyon a suspension bridge made of steel i fastened to the cliff and to a vertical upright. We want to determine the length of the 3 beams as well as the welding point. The left fixing angle must be 30 degrees.



2. The Mathematical Problem

From the right-angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between the lines BG and FA.

Measurements: angle FEB = 30 degrees, FB = 3.5m, FG = 8m + 1m = 9m and AG = 5m.

		I a coordinate system with F as zero the following coordinates emerge: F: (0,0) and A: (9,5), as well as B: (0,3.5) and G: (9,0). Using linear regression, we determine the equations for the lines FA and BG.

3. Solution to the Mathematical Problem

We set up formula tables

Triangle EFB

c = ?	$\sin A = \frac{a}{c}$
A = 30 a = 3.5	$\sin 30 = \frac{3.5}{c}$ sin30*c = 3.5 c = 3.5/sin30 c = 7.0
Test1 ☺	$\sin 30 = \frac{3.5}{7}$ 0.5 = 0.5
Test2 ☺	Math Solver $0 = \frac{3.5}{x} - \sin 30$ gives x = 7

Triangle FGA og GFB

c = ?	$a^2 + b^2 = c^2$
a = 5 b = 9	$5^2 + 9^2 = c^2$ √(106)= c 10.30 = c
Test1 & Test2	
c = ?	$a^2 + b^2 = c^2$
a = 3.5 b = 9	$3.5^2 + 9^2 = c^2$ √(93.25) = c 9.66 = c

Lines BG and FA

BG: ?	y = ax + b
Test	y = – 0.389x + 3.5 Found by LinReg L1, L2, Y1 Trace x=0 gives 3.5 Trace x=9 gives 0 StatPlot fits

Likewise, we find
FA: ? y = 0.556x

Calc Intersection gives
x = 3.71 and y = 2.06
In the triangle FDC, DC = 3.71 and FD = 2.06
Pythagoras gives:
FC = √(3.71^2 + 2.06^2) = 4.24
In the triangle BDC, DC = 3.71 and
FD = 3.6-2.06 = 1.54
Pythagoras gives:
BC = √(3.71^2 + 1.54^2) = 4.02

4. Solution to the Real-world Problem

Using trigonometry, we found the lengths of the three steel beams as EB = 7.00 m, FA = 10.30 m and BG = 9.66
The welding point is determined by FC = 4.24 m and BC = 4.02 m.
As an extra control the bridge can be drawn and build by pipe cleaners in the ration 1:100.

10. Project Driving

Problem: How far and how did Peter drive?

A mathematical model

Mathematics	Problem	→	Solution
Real-world	Problem		Solution

1. The Real-world Problem

When driving, the velocity 100 km/t is 100*1000/(60*60) = 27.8 m/s. A camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s and 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

2. The Mathematical Problem

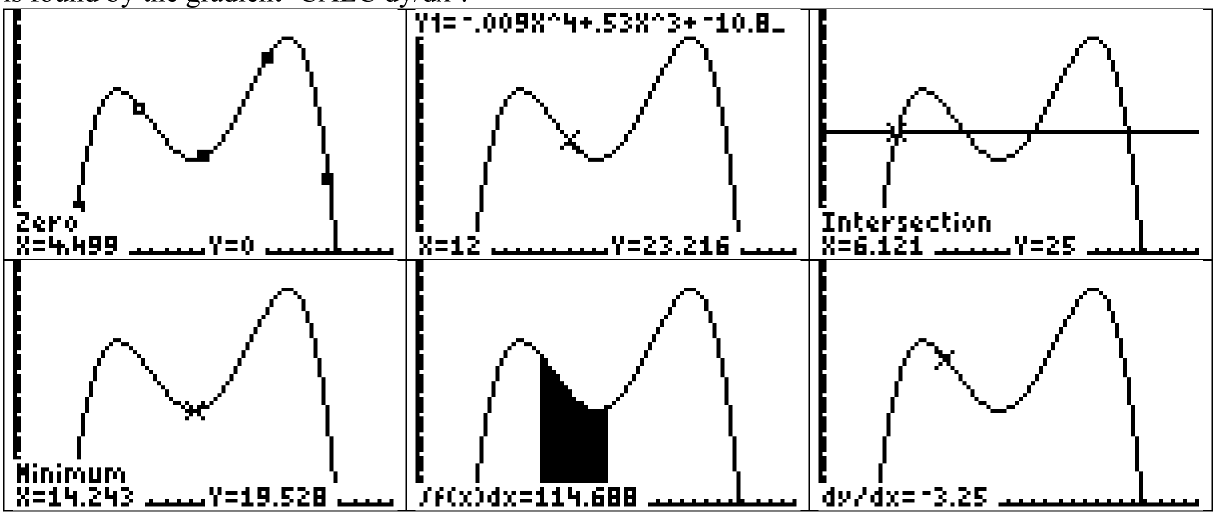
We set up a table showing time x and velocity y. The domain of the table is taken to be 0<x<30.

Time x sec	Velocity y m/s	Accel. dy/dx
5	10	?
10	30	?
15	20	?
20	40	?
25	15	?

3. Solution to the Mathematical Problem

On TI-84 the table is entered as the lists L1 and L2. 5 data sets allow quartic regression (a 4. degree polynomial with a 3-fold parabola) providing the formula y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235 placed as Y1. Now the question asked can be answered using formula tables, or using technology, i.e. graphical readings or calculations.

Starting and ending points are found using 'CALC Zero'. Y-numbers are found using 'TRACE'. X-numbers are found using 'CALC Intersection'. Maximum and minimum are found with 'CALC Maximum/Minimum'. The total meter-number is obtained by summing up the m/s*s = ∫Y1dx. Acceleration is found by the gradient 'CALC dy/dx'.



y = ?	y = y1
x=12	y = y1(12) = 3.667
Test	TRACE x = 12 gives y = 23.216

x = ?	y = y1
y = 25	MATH Solver 0 = y1 - 25 gives x =6.12 and …
Test1	y1(3) = 6, y1(8) = 6
Test2	CALC intersection gives x = 6.12, 11.44, 16.86 and 24.47

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx = 0 at x = 5.5

4. Solution to the Real-world Problem

The driving began after 4.50 sec. and ended after 25.62 sec. After 12 sec the velocity was 23.2 m/s. And it was 25m/s after 6.12 sec, 11.44 sec, 16.86 sec and 24.47 sec. Acceleration took place in the time-intervals (4.50; 8.19) and (14.24; 21.74). Deceleration in the intervals (8.19;14.24) and (21.74;25.62). Max-velocity was 44.28 m/s = 159 km/t. after 21.7 sec. In the time-intervals (5;10), (10;15), (15;20) and (20;25) the distance travelled was 142.8 m, 114.7 m, 142.8 m and 189.7 m. The acceleration in the beginning of these time-intervals were 17.75, -3.25, 1.25, 4.25, -21.25 m/s^2. The total distance travelled was 597.4 m.

11. Project Vine Box

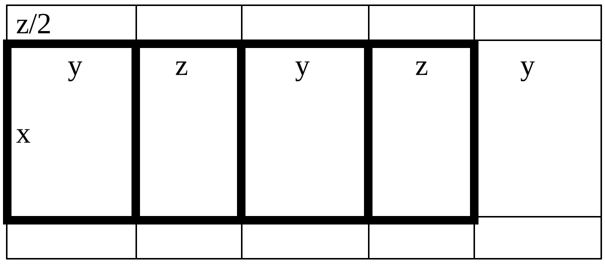
Problem: What are the dimensions of a 3 liters vine bag with the least surface area?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

Vine is sold in bottles or in boxes. A 3liter bag will be constructed by cutting out a piece of cardboard.



2. The Mathematical Problem

The cardboard dimensions are called x, y & z all in dm. We express the volume V and the Surface S as formulas:

V = x*y*z = 3, S = x*(3y+2z) + 2*z/2*(3y+2z)

3. Solution to the Mathematical Problem

We expand the S-formula: S = x*(3y+2z) + 2*z/2*(3y+2z) = 3xy + 2xz + 3yz + 2z^2

We now insert z = 3/(x*y) so that S only depends on two variables x and y:

S = 3xy + 2xz + 3yz + 2z^2 and z = 3/(x*y) gives $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x\text{^}2*y\text{^}2}$

Scenario A. We assume that y should be half the length of x: y = 0.5*x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x\text{^}2*y\text{^}2} = 1.5x\text{^}2 + \frac{21}{x} + \frac{72}{x\text{^}4}$, which gives $\frac{dS}{dx} = 3x - \frac{21}{x\text{^}2} - \frac{288}{x\text{^}5} = 0$ for x = 2.4

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point x = 2.4 and S = 19.56, so y = 0.5*x = 0.5*2.4 = 1.2, and z = 3/(2.4*1.2) = 1.0

Scenario B. We assume that y should be the same length of x: y = x. This restriction is inserted:

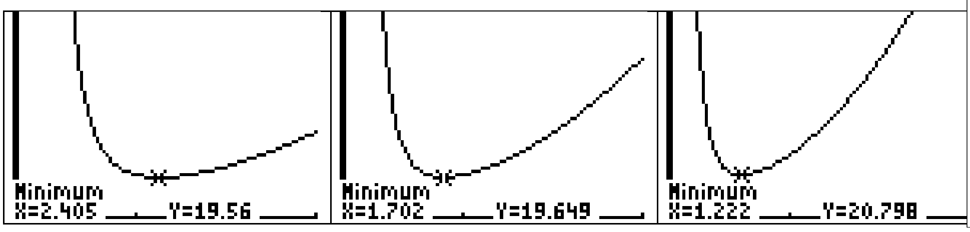
$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x\text{^}2*y\text{^}2} = 3x\text{^}2 + \frac{15}{x} + \frac{18}{x\text{^}4}$, which gives $\frac{dS}{dx} = 6x - \frac{15}{x\text{^}2} - \frac{72}{x\text{^}5} = 0$ for x = 1.7

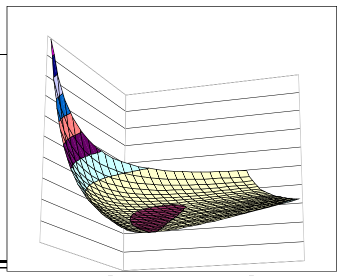
Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point x = 1.7 and S = 19.65, so y = x = 1.7, and z = 3/(1.7*1.7) = 1.0

Scenario C. We assume that y should be double the length of x: y = 2*x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x\text{^}2*y\text{^}2} = 6x\text{^}2 + \frac{12}{x} + \frac{4.5}{x\text{^}4}$, which gives $\frac{dS}{dx} = 12x - \frac{12}{x\text{^}2} - \frac{18}{x\text{^}5} = 0$ for x = 1.2

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point x = 1.2 and S = 20.80, so y = 2x = 2*1.2 = 2.4, and z = 3/(1.2*2.4) = 1.0





Scenario A   Scenario B   Scenario C   3 dim. in Excel

4. Solution to the Real-world Problem

We see that the minimum surface area is a little above 19 dm^2. Using an Excel-spreadsheet we can find the optimal solution to be x = 2.1 and y = 1.4 and z = 1.0, giving a minimum surface area at 19.47 dm^3.

(Graphing $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x\text{^}2*y\text{^}2}$ does not give a curve but a surface as shown on the Excel-picture.)

12. Project Golf

Problem: How to hit a golf hole behind a hedge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The Real-world Problem

From a position on a 2meter high flat hill we want to send a golf ball over a 3meter hedge 2meter away on the hill to hit a hole situated 12 meters away at level zero.
What is the orbit of the ball? How high is the ball at the distance 10 meters? When does the ball have a height of 6 meters? How high does the ball go? What is the direction of the ball in the beginning, at 10 meters distance and at the impact?

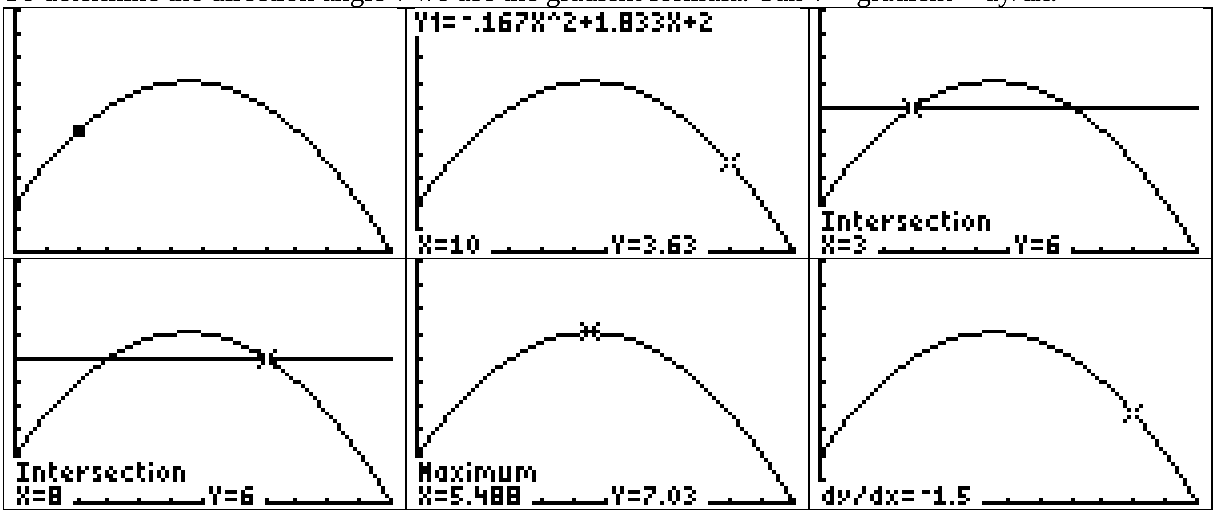
2. The Mathematical Problem

We set up a table with the length x and the height y having the domain 0<x<12.

Length x	Height y	Direction v
0	2	?
2	5	
12	0	?
10	?	?
?	6	

3. Solution to the Mathematical Problem

We insert the table as lists L1 and L2. Three data-sets allows a 2nd degree polynomial, quadratic regression, which produces the formula y= -0.167x^2 + 1.833x + 2, which is transferred to Y1.
Now the questions asked can be answered using formula tables and a calculator for graphing or calculating. The Y-number can be found by CALC Value, the x-number by CALC Intersection, the maximum by CALC Maximum, and the gradient by CALC dy/dx.
To determine the direction angle v we use the gradient formula: Tan v = gradient = dy/dx.



y = ?	y = y1
x=10	y = y1(10) = 3.667
Test	Trace x=10 gives y=3.67

x = ?	y = y1
y = 6	Math solver 0 = y1 - 6 gives x =3 & x = 8
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC Intersection gives x = 3 and 8

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx ≈ 0 at x = 5.5

v = ?	tan v = dy/dx
x = 0	tan v = 1.833 v = tan-1(1.833) v = 61.4

v = ?	tan v = dy/dx
x = 12	tan v = -2.167 v=tan-1(-2.167) v =- 65.2

v = ?	tan v = dy/dx
x =10	tan v = -1.5 v = tan-1(-1.5) v = -56.3

4. Solution to the Real-world Problem

The orbit of the ball is a parabola. The height of the ball at the distance 10 meters is 3.67 meters? At the distances 3 meters and 8 meters the ball has a height of 6 meters. The ball goes to the maximum height 7.04 meters? The direction of the ball in the beginning, at 10 meters distance and at the impact are 61.4 grader, -65.2 grader and -56.3 grader.

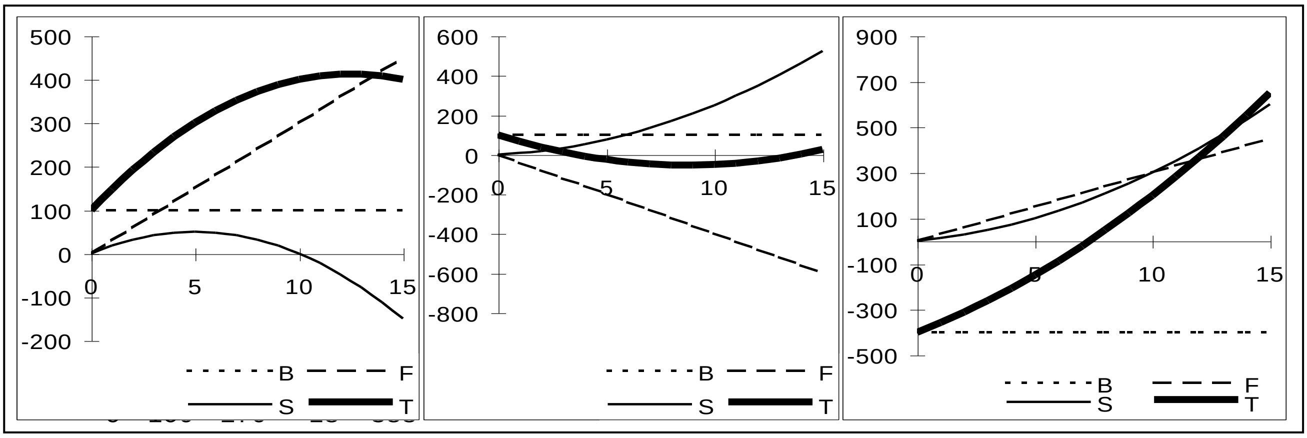
Project Family Firm

A family firm builds its total capital T through income from three generations. The grandfather has retired and has left the capital To \$. The father has established a routine which earns b \$/day. The son has just come back from the university, where he has learned a new technology enabling him to slowly raise the daily income d\$/day to s = so + d*n. The total capital after n days, since the family is calculated as a sum of polynomials:

Grandfather B	To	degree 0 polynomial, a constant
Father F	b*n	degree 1 polynomial, a line
Son S	s*n = (so + ½*d*n)* n = so*n + ½*d*n^2	degree 2 polynomial, a bended line
Total T	T = To + (b+so)*n + d*n^2	degree 2 polynomial, a parabola

Gatherers and Spreaders

Some families have both a spreader and a gatherer. The spreader may be the son, the father or the grandfather.



Spreader: Son Father Grandfather

Pricing the tea

In the family firm they discuss if increasing the unit price by necessity will make the sale decrease.

Grandfather: The sale will decrease with an increasing unit price. I believe in a linear relation y = a + b*x found from the table	Father: The sale will decrease slower at high unit prices. I believe in a degree 2 polynomial y = a + b*x + c*x^2 found from the table	Son: The sale will decrease more at low and high unit prices. I believe in a degree 3 polynomial y = a + b*x + c*x^2+ d*x^3 found from the table

price x	sale y
0	100
10	0

price x	sale y
0	100
5	80
10	0

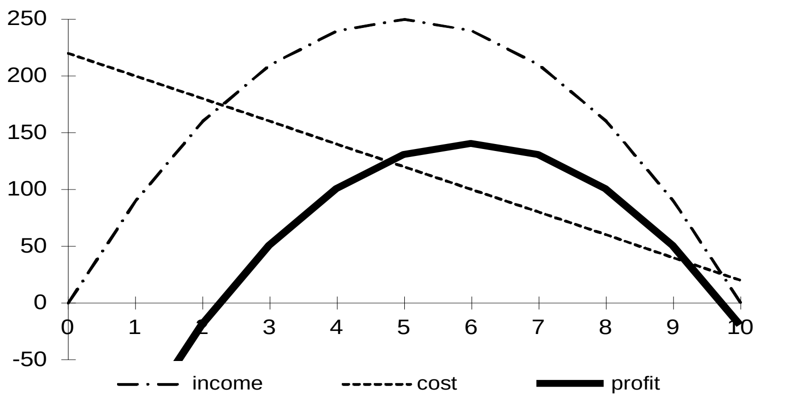
price x	sale y
0	100
2	60
8	40
10	0

The grandfather scenario

The sale will be y = 100-10*x found by regression. The total income T is T = unit price*sale = x*y = x*(100-10 x) = 100*x-10*x ^ 2, a degree 2 polynomial. The cost C to produce y units consists of a fixed cost co = 20 and a variable unit-cost m = 2. So, C = co + m*y = = 20 + 2*y = 20 + 2*(100–10*x) = 20 + 200–20*x = 220–20*x.

The profit P will be when P = T – C = (100*x–10*x^2) – (220–20*x) = -220 + 80x – 10*x^2, i.e. again a degree 2 polynomial.

price	sale	income	cost	profit
0	100	0	220	-220
1	90	90	200	-110
2	80	160	180	-20
3	70	210	160	50
4	60	240	140	100
5	50	250	120	130
6	40	240	100	140
7	30	210	80	130
8	20	160	60	100
9	10	90	40	50
10	0	0	20	-20



Exercises

1. Set up the father's scenario
2. Set up the son's scenario

Classic Word Problems

Examples of Babylonian math problems

Give me a number as combined with its reciprocal gives the number B.

I have multiplied the length with the width and got the area 10. I have multiplied the length by itself and got an area which is the same as if I multiply the difference between the length and the width with itself and with 9.

One man can dig 3 ditches in 1 day. How many men do I need to dig 100 ditches in 6 days?

How many lunar months in 29 days are needed to give a whole number of solar years of 365 days?

The ratio between the area and the circumference square is 12 for a circle.

Examples of Egyptian mathematics problems

The Egyptians wrote on papyrus. There are two preserved papyrus writings from approx. 1700 before Christ, the Rhind-Papyrus in London and Moscow-Papyrus in Moscow. Both contain problems and their solution, 85 at Rhind, and 25 in Moscow.

A quantity is sought so 2/3 of it, 1/2 of it, 1/7 of it and the total is 33.

Of 2 barrels of good grain can be made 5 bottles of ordinary beer. Of 3 barrels of good grain can be made 8 bottles of ordinary beer. 3 bottles of good beer correspond to 2 bottles of strong beer. How much grain to use to make 20 bottles of regular beer? And to make 30 bottles of strong beer?

Build a pyramid of cubic stones to be lifted by a lever. 1 man can lift a stone if the rod is 30 meters long. How many men should lift if the rod is only 12 meters long?

Proportionality

Proportionality occur wherever you want to change between two different units, that is, tasks where there is a constant per-number between two units. I.e. situations where we can count in two different units.

	Direct (trade)	Inverse (ditches)
Question	3 kg. = 4 \$ 5 kg. = ? \$? kg. = 10 \$	5 men dig a ditch in 7 days 3 men dig a ditch in ? days ? men dig a ditch in 4 days
Answer	T = 5 kg. = (5/3)*3 kg. = (5/3)*4 \$ = 6.67 \$ T = 10 \$ = (10/4)*4 \$ = (10/4)*3 kg = 7.5 kg	ManDays = 5*7 = 35 = (35/3)*3 = 11.67*3 ManDays = 5*7 = (35/4)*4 = 8.75*4

Standard Word Problems

Use the following procedure when solving standard word problems:

- Make a quick read through to see what type of task it is.
- Find the question mark, which shows what the unknown x is.
- If there are more unknowns, always leave x to be the smallest unknowns. The second can then either be expressed by x, or called y.
- Rephrase the text so that it begins with 'Let x = <e.g. the kg-number>' and only use the word 'is ', which can be translated directly to the equal sign ' = '.
- Translate the task from text to equations, solve the equations, and translate back.

Unit Problems

Numbers

Two numbers add up to 72, and one is twice as big as the other. What are the numbers?

Text	*Number*	*ANSWER*	*Equation*
Number1 Number2	x = ? y = 2*x	*24* *48*	x + y = 72 x + 2*x = 72 3*x = 72 x = 72/3 = 24

Coins

Mr. C pays a bill of 210 $ with three types of coins: 1s, 2s and 5s. There are 4 times as many 1s as 2s, and 20 fewer 2s than 5s. How many coins of each type were used?

Text	*Number*	*ANSWER*	*Equation*
5s 2s 1s	x = ? x-20 4*(x-20)	*30* *10* *40*	x*5 + (x-20)*2 + 4*(x-20)*1 = 210 5*x + 2*x – 40 + 4*x – 80 = 210 11*x = 210 + 120 x = 330/11 x = 30

Age

Mr. A is 4 times as old as B. 5 years ago was A 7 times as old as B. How old is A and B now?

Text	*Number*	*ANSWER*	*Equation*
B’s age now A’s age now B’s age then A’s age then	x = ? 4*x x - 5 4*x - 5	*10* *40*	7*(x-5) = 4*x – 5 7*x – 35 = 4*x – 5 7*x – 4*x = – 5 + 35 3*x = 30 x = 30/3 x = 10

Geometry

A rectangle has a circumference of 224 meters. The length is 4 meters shorter than 3 times the width. What is length and width?

Text	*Number*	*ANSWER*	*Equation*
Width Length	x = ? meter 3*x-4 meter	*29* *83*	2*x + 2*(3*x-4) = 224 2*x + 6*x – 8 = 224 8*x = 224 + 8 x = 232/8 x = 29

Levers

Mr. A, B and C put themselves on a tilt, B and C on the same side. They weigh, respectively, 100kg, 80 kg and 40 kg. A and B sit both 3 meters from the focal point. Where should C sit to obtain equilibrium?

Text	*Number*	*ANSWER*	*Equation*
C’s meter-number A’s contribution B’s contribution C’s contribution	x = ? 100*3 80*3 40*x	*1.5*	100*3 = 80*3 + 40*x 300 = 240 + 40*x 300 – 240 = 40*x 60/40 = x 1.5 = x

Tasks

1. Two numbers add up to 48, and one is twice as large as the other. What are the numbers?
2. Two numbers add up to 48 and one is three times the other. What are the numbers?
3. Mr. A pays a bill of 290 kr. With three types of coins: 1s, 2s and 5s. There are 5 times as many 1ere as 2ere, and 10 fewer 2s than 5s. How many coins of each type were used?
4. Mr. A pays a bill of 200 kr. With three types of coins: 1s, 2s and 5s. There are 3 times as many 1ere as 2ere, and 20 more 2s than 5s. How many coins of each type were used?

5. Mr. A is 5 times as old as B. 4 years ago, Mr. A was 6 times as old as B. How old is A and B now?
6. A is 8 times as old as B. 5 years ago, Mr. A was 9 times as old as B. How old is A and B now?
7. The circumference of a rectangle is 128 meters. The length is 4 meters longer than 5 times the width. What is length and width?
8. The circumference of a rectangle is 110 meters. The length is 5 meters shorter than 4 times the width. What is length and width?
9. Mr. A, B and C put themselves on a tilt, B and C on the same side. They weigh, respectively, 120kg, 60 kg and 50 kg. A and B both sit 4 meters from the focal point. Where should C sit to obtain equilibrium?
10. Mr. A, B and C put themselves on a tilt, B and C on the same side. They weigh, respectively, 90kg, 70 kg and 20 kg. A and B sit both 2 meters from the focal point. Where should C sit to obtain equilibrium?

PerNumber Problems

In per-numbers tasks, a per-number is multiplied to a unit-number before formulating the equation.

Traveling

Train1 runs from A to B with A speed of 40 km/h. Two hours later, train2 runs from A to B with speed 60 km/h. When do they meet?

Text	*Per-number*	*Unit-number*	*ANSWER*	*Equation*	
Hours		x = ?	*4*	40*(x+2)	= 60*x
Speed1	40 km/h			40*x + 80	= 60*x
Speed2	60 km/h			80	= 60*x – 40*x = 20*x
Km-number1		40*(x+2) km	*240*	80/20	= x
Km-number2		60*x km	*240*	4	= x

Train1 runs from A to B with speed 40 km/h. At the same time train2 runs from B to A with speed 60 km/h. When do the two trains meet when the distance from A to B is 300 km?

Text	*Per-number*	*Unit-number*	*ANSWER*	*Equation*	
Hours		x = ?	*4*	40*x + 60*x	= 300
Speed1	40 km/h			100*x	= 300*x
Speed2	60 km/h			x	= 300/100
Km-number1		40*x km	*120*	x	= 3
Km-number2		60*x km	*180*		

The same distance takes 3 hours upstream, and 2 hours downstream. What is the speed of the motorboat when the speed of the current is 5 km/h?

Text	*Per-number*	*Unit-number*	*ANSWER*	*Equation*
Speed	x = ? km/h		*25*	km = km/h*h = (x-5)*3 = (x+5)*2
Speed upstream	x – 5 km/h		*20*	3*x-15 = 2*x+10
Speed downstream	x + 5 km/h		*30*	3*x-2*x = 10+15
Hours		3 hours		x = 25

Mixture

How many liter 40% alcohol + 3 liter 20% alcohol gives ? liter 32% alcohol

Text	*Per-number*	*Unit-number*	*ANSWER*	*Equation*		
Liter		x = ? liter	*4.5*	0.4*x + 0.2*3	= 0.32*(x+3)	
Liter 3		x+3 liter	*7.5*	0.4*x + 0.6	= 0.32*x + 0.96	
Alcohol1	40%	0.4*x liter		0.4*x - 0.32*x		= 0.96- 0.6
Alcohol2	20%	0.2*3 liter		0.08*x	= 0.36	
Alcohol3	32%	0.32*(x+3)	liter	x	= 0.36/0.08	
				x	= 4.5	

Finance

Mr. A invests a win of $400.000 as follows: Some is set to return to 3% p.a., the remainder goes to 8% bonds. How much did he invest in each when the annual yield is $20.000?

Text	*Per-number*	*Unit-number*	*ANSWER*	*Equation*	
Bank in 1000		x = ? $	*240*	3%*x + 8%*(400-x)	= 20
Bonds in 1000		x+3 $	*160*	0.03*x + 32 – 0.08*x	= 20
Bank interest	3%			32 - 20	= 0.08*x – 0.03*x
Bond interest	8%			12	= 0.05*x
Bank contribution		3%*x $		12/0.05	= x
Bond contribution		8%*(400-x) $		240	= x

Work

Mr. A can dig a trench in 4 hours. B can dig the same ditch in 3 hours. How long does it take to dig it together?

Text	*Per-number*	*Unit-number*	*ANSWER*	*Equation*	
Hours		x = ? hours	*12/7*	¼*x + 1/3*x	= 1
A's speed	1/4 ditch/h			(¼ + 1/3)*x	= 1
B's speed	1/3 ditch/h			7/12*x	= 1
A's part		¼*x		x	= 12/7
B's part		1/3*x			

Tasks

1. Train1 runs from A to B with the speed 50 km/h. Three hours later, the train2 runs from A to B with the speed 60 km/h. When do they meet?
2. Train1 runs from A to B with the speed 50 km/h. At the same time train2 runs from B to A with the speed 60 km/h. When do the two trains meet when the distance from A to B is 400 km?
3. The same distance takes 4 hours upstream, and 3 hours downstream. What is the speed of the motorboat when the speed of the current is 6 km/h?
4. How many liter 35% alcohol + 4 liter 22% alcohol gives ? liter 30% alcohol
5. Mr. A invests a win of $500.000 as follows: Some is set to return to 4% p.a., the remainder goes to 7% bonds. How much did he invest in each when the annual yield is $25.000?
6. A can dig a trench in 5 hours. B can dig the same ditch in 4 hours. How long does it take to dig it together?