ADDING PER-NUMBERS

A2

Questions	Answers
What is a per-number?	Double-counting in two units gives per-numbers; and fractions with like units
What serves a per-number?	Per-numbers bridge units; and adds as integral calculus
How to add per-numbers?	The $/day-number a is multiplied with the day-number b before added to the total $-number T, thus adding by areas as integral calculus: T2 = T1 + a*b

01 PER-NUMBERS COME FROM DOUBLE-COUNTING IN TWO UNITS

Question. What is a per-number?
Answer. A per-number is a double-number coming from counting in two units.
Example1. Apples may be double-counted as 2$ per 3kg, thus giving the per-number 2$/3kg or 2/3 $/kg, as well as the **reciprocal** per-number 3kg/2$ or 3/2 kg/$.
Example2. With 2 red among 3 apples, the red is double-counted as 2 of 3, thus giving the per-number 2/3. So, per-numbers become **fractions** with like units. The per-number 4$/6kg is the same as 2(2$)/3(2kg), so 4/6 = 2/3, called **reducing** or **expanding** a fraction by removing or adding a common unit to both numbers.

02 PER-NUMBERS BRIDGE UNITS

Question. What does per-numbers do?
Answer. Per-numbers bridge units, which is called proportionality, by simply recounting in the per-number: With 2meter per 5second, T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.
Six other ways to solve the two bridging problems
● Europe used ‘Regula-de-tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: ‘2m takes 5sec, 7m takes ?sec’ to get to the answer (7*5/2)sec = 17.5sec. Then we ask, Q2: ‘5sec gives 2m, 12sec give ?m’ to get to the answer (12*2)/5sec = 4.8m. Without calculators, multiplication is easier than division.
● Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter. Here division precedes multiplication, allowing the total to be counted in per-numbers.
● Equating the rates. The constant speed 2meter/5second allows setting up an equation equating the rates.
Q1: 2/5 = 7/x, where cross-multiplication gives 2*x = 7*5, which gives x = (7*5)/2 = 17.5.
Q2: 2/5 = x/12, where cross-multiplication gives 5*x = 12*2, which gives x = (12*2)/5 = 4.8.
● Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.
● Multiply with the per-number. Using the fact that T = 2m, and T = 5sec, division gives T/T = 2m/5sec = 1, and T/T = 5sec/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.
● Modeling with a linear function f(x) = c*x. Here f(2) = 5 gives f(x) = 5/2*x. Then f(7) = 5/2*7 = 17.5
And f(x) = 12 gives the equation 5/2*x = 12 solved by x = 12*2/5 = 4.8 with the opposite side&sign method.

03 FRACTIONS AND PERCENTAGES

Question. How to change fractions to percentages and vice versa?
Answer. By bridging the total and 100.
Example1. What is 3/4 in percent? We ask: 3 per 4 is what per 100. So we recount 100 in 4 as T = 100 = (100/4)*4 giving (100/4)*3 = 75, or 100/4 times giving 3. So, 3 per 4 is the same as 75 per 100 or 75%.
Example2. What is 3 per 4 of 20? We ask: 3 per 4 is what per 20. So we recount 20 in 4 as T = 20 = (20/4)*4 giving (20/4)*3 = 15, or 20/4 times giving 3. So, 3 per 4 is the same as 15 per 20.
Example3. What is 30% of 20? We ask: 30 per 100 is what per 20. So we recount 20 in 100 as T = 20 = (20/100)*100 giving (20/100)*30 = 6, or 20/100 times giving 30. So, 30% of 20 is the same as 6 per 20.
Example4. What is 2/3 of 4/5? We ask: What is 2 per 3 of 4 per 5 of a total 3*5. So first we recount 15 in 5s as T = 15 = (15/5)*5 giving (15/5)*4 or 12 that recount in 3s as T = 12 = (12/3)*3 giving (12/3)*2 or 8.
So, 2/3 of 4/5 of 15 is the same as 8 per 15 or 8/15 = (2*4)/(3*5), thus also found by direct multiplication.
Example5. Recounting 4 in 1/2s gives 8 predicted by T = (4/(1/2))*1/2 = 8*1/2 = 8 1/2s, so 4/(1/2) = 4*2 = 8.
Recounting 6 in 2/3s gives 9 predicted by T = (6/(2/3))*2/3 = 9*2/3 = 9 2/3s, so 6/(2/3) = 6*3/2 = 9.
Example6. Losing with dice-number 1, we win 5 of 6 times. So, we avoid losing 5/6*5/6*5/6 of 3 times, i.e. 5^3/6^3 = 125/216 = 58/100 = 58% of the times. How many times gives 50% is solved by (5/6)^x = ½.
Example7. In a 2x2 tile we take the bottom right quarter tile and mark its left border on the bottom line. Repeating this, we get marks creating a (quotient) series 1 + ½ + ¼ + 1/8 + etc. Adding up, we get 2.

04 ADDING DIFFERENCE AND QUOTIENT SERIES

Question1. What is a difference (arithmetic) series? What is an quotient (geometric) series?
Answer1. In a difference/quotient series, the difference/quotient between two neighbors are constant.
Examples: 3,5,7,8 with +2 as the change number. 1,2,4,8 with 2 as the quotient, giving 100% as change%.
Question2. How to add difference and quotient series.
Answer2. Add a difference series to itself. A quotient series add bundles: S3 = 1+B+BB. With 9*(1+10+100) = 1000-1, we get S3 = (B^3 – 1)/(B – 1). Or we subtract the series from itself after adding an extra quotient.
Difference series. With S4 = 3+5+7+9, 2*S4 = (3+5+7+9)+(9+7+5+3) = 4*(3+9). So, S4 = 4*(3+9)/2.
Quotient series. S4 = 1+2+4+8, 2*S4 – S4 = (2+4+8+16) – (1+2+4+8) = 16 – 1. So, S4 = (2^4 – 1)/(2 – 1).
General sum formulas. Difference series: Sn = n*(first + last)/2. Quotient series: Sn = (q^n – 1)/(q – 1).

05 PER-NUMBERS IN SHARING

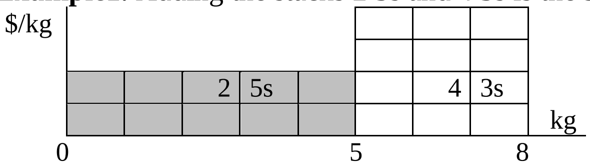
Question. How to share a winning?
Answer. By returning the stake several times. Or by receiving a proportional part of the winning.
Example1. The players A, B and C split a winning of 400$ from putting 2$, 3$ & 5$ into a pool creating 10$.
Method 1. The winning is counted in pools to get the odds: W = 400$ = (400/10)*10 = 40*10. Thus, the players get their stake back 40 times: A gets 2$ 40 times, i.e. 80$, etc.
Method 2. The winning is shared in the ratio 2:3:5. A gets 2 ten parts of the winning: A = 2/10*W = W/10*2.
Together A and B get 2/10 and 3/10 of W i.e. 2/10*W+3/10*W = (2/10+3/10)*W = ((2+3)/10)*W = 5/10*W.
Example2. An apartment is sold in 2 1/4-shares, 2 1/8-shares and 4 1/16-shares. No shares are left since 2/4*A + 2/8*A + 4/16*A = (2*4)/(4*4)*A + (2*2)/(8*2)*A + 4/16*A = 8/16*A + 4/16*A + 4/16*A = 16/16*A = 1*A.
B buys 1 1/8-share and 3 1/16-shares giving a total of (1*2)/(8*2)*A + 3/16*A = (2/16 + 3/16)*A = 5/16*A.
Example3. B receives 2/10 of a 200$-winning and 3/10 of a 100$-winning.
The total income is T = 2/10*200 + 3/10*100 = 40 + 30 = 70 = (70/300)*300 = 7/30 of 500. So, in this case 2/10+3/10 is not 5/10 = 15/30, but 7/30. Whereas 2/10 of a 200$ plus and 3/10 of 300$ total 11/50 of 500$.
Example4. ½ of 2 cokes + 2/3 of 3 cokes = 1 coke + 2 cokes = 3 cokes = (3/5)*5 cokes = 3/5 of 5 cokes.
So in this case 1/2+2/3 = 3/5. Thus, 1/2+2/3 = 7/6 only with the same unit, i.e. only if taken of the same total.
Example5. The fraction-paradox:

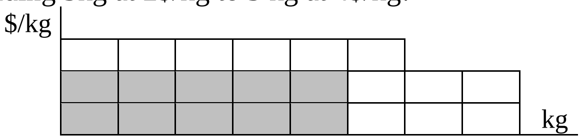
Inside the classroom	20/100 + 10/100 = = = 20% + 10% =	30/100 = 30%
Outside the classroom e.g. in the laboratory	20% + 10% = or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

20% + 10% = 30% only when taken of the same total: 20% of 300 + 10% of 300 = 30% of 300. In all other cases the sum is different from 30%, so there is no general rule saying that 20% + 10% = 30%.
So per-numbers and fractions are not numbers, but operators, needing numbers to become numbers.
Exercise1. Split the winning 1600$ between the persons A, B and C in the ratio 3:1:4.
Exercise2. The king gets 2 7parts of the harvest, and the bishop get 1 9part. How much is left?
Exercise3. My two investments 200$ and 500$ gave 12% and 4% yield. What is the total yield percentage?

06 ADDING PER-NUMBERS I

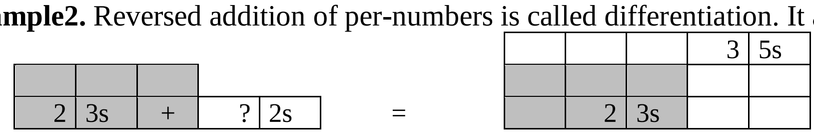
Question. How to add per-numbers and fractions?
Answer. As operators, per-numbers and fractions are multiplied to areas before adding as integral calculus
Example1. Adding the stacks 2 5s and 4 3s is the same as adding 5kg at 2$/kg to 3 kg at 4$/kg.



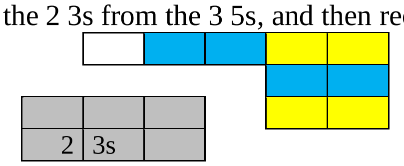


T = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8

Example2. Reversed addition of per-numbers is called differentiation. It asks e.g. 2 3s+? 2s = 3 5s:



The answer comes when first removing the 2 3s from the 3 5s, and then recounting the remaining 9 in 2s.



? = (3 5s – 2 3s)/2 = ΔT/2 = 9/2 = 4 1/2.

Or the answer can be obtained by a predication through a reversed calculation. In this way solving equations becomes another name for reversed calculations.

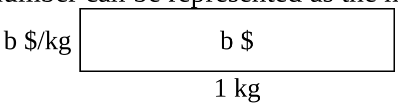
2 3s + ? 2s = 3 5s	The question	
2*3 + x*2 = 3*5	The equation	T1 + x*b = T2
x*2 = 3*5 – 2*3 = 9	To opposite side with opposite sign	x*b = T2 – T1 = ΔT
x = 9/2	To opposite side with opposite sign	$x = \frac{\Delta T}{b} = \frac{\Delta T}{\Delta n}$
x = 4 1/2	The answer	

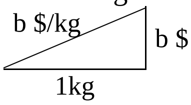
Exercise. Integrate 3 4s, 4 5s. Integrate 3 2s and 5 4s, etc. Differentiate 3 4s +? 5s = 6 7s, etc.

07 ADDING PER-NUMBERS II

Question. How to add per-numbers with units?

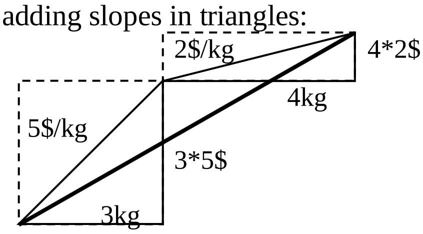
Answer. Per-numbers occur in Renaissance trade questions as price-numbers 4 $/kg or rent-numbers 4 $/day. The per-number can be represented as the height of a stack, or as the slope of the diagonal in a change-triangle.





Adding per-numbers from trade takes place in a table or by adding slopes in triangles:

a kg @	*b $/kg*	*= a*b $*
3 kg @	5 $/kg	= 3 * 5 = 15 $
4 kg @	2 $/kg	= 4 * 2 = 8 $
7 kg @	x $/kg	= 7 * x = ∑a*b = 23 $
	x	x = 23/7 $
		x = 3 2/7 $



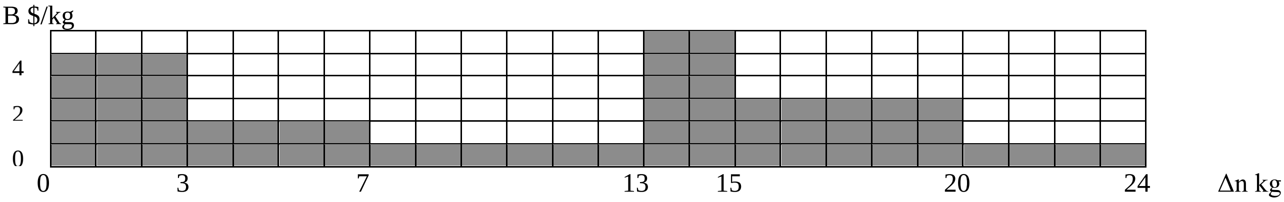
So per-numbers are added by their totals: 3 kg @ 5 $/ kg + 4 kg @ 2 $/ kg = (3+4) kg @ (∑a*b)/(3+4) $/ kg

The table can be supplemented with two columns showing the added values of both the kg-number Δn, and of the $-number ΔT, and of the per-number Σ b, as in this example where a teashop is adding different amounts with different prices to create a blending.

Δn kg	*b $/kg*	*Δn*b = ΔT*		*Σ Δn = Δn*	*Σ ΔT = ΔT*	*Σ b $/kg = ΔT/Δn*
3 kg @	5$/kg =	3 * 5 = 15		3	15$	15/3 = 5.0
4 kg @	2$/kg =	4 * 2 = 8		7	23$	23/7 = 3.3
6 kg @	1$/kg =	6 * 1 = 6		13	29$	29/13 = 2.2
2 kg @	6$/kg =	2 * 6 = 12		15	41$	41/15 = 2.7
5 kg @	3$/kg =	5 * 3 = 15		20	56$	56/20 = 2.8
4 kg @	1$/kg =	4 * 1 = 4		24	60$	60/24 = 2.5

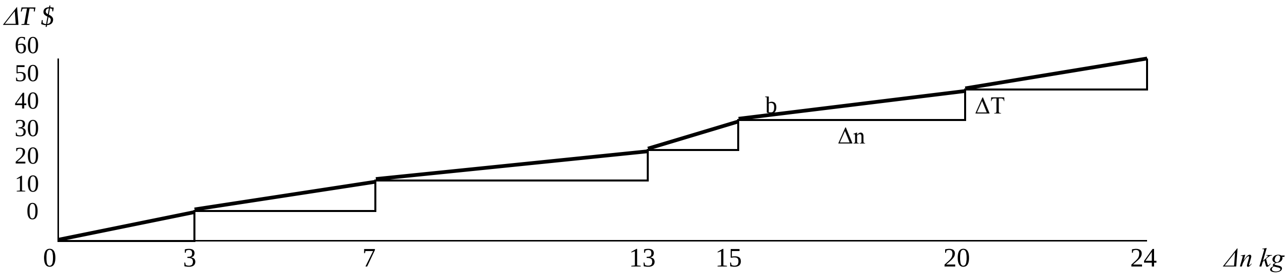
When plotting the per-number b $/kg against Δn kg in a coordinate system the total $-number is the area under the curve representing the sum of the stacks.

The Total as an area under the per-number curve



When plotting ΔT against Δn in a coordinate system the curve shows both the added kg-number Δn, the added total ΔT, and the single per-numbers b = ΔT/Δn as the slopes.

The per-number as the slope of the Total curve



Thus from blending tea in a shop we learn that:

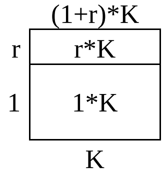
The Total is the area under the PerNumber curve predicted by an integration formula: T= ∑$/kg*kg = ∑b*Δn.

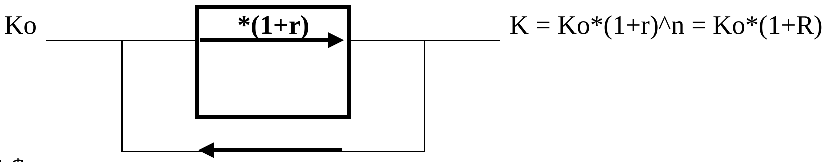
The PerNumber is the slope of the Total curve predicted by a differentiation formula: b = Δ$/Δkg = ΔT/Δn.

Exercise. Travel, first 5 seconds @ 4m/s, then 3 seconds @ 6m/s, then 4 seconds @ 2m/s etc.

08 ADDING CONSTANT PERCENTAGES

Question. How can we add constant per-numbers?
Answer. By repeated multiplications with a multiplier.
Example. Adding the interest amount r*K to a capital K gives T = K + r*K = 1*K + r*K = (1+r)*K.
Repeated n times, the terminal capital is K = Ko*(1+r)^n = Ko*(1+R), where the total interest R = (1+r)^n.





5 days @ 9 $/day = 5*9 = 45 $
5 days @ 9 %/day = 5*9% (i.e. 45 %) + CR = simple interest + compound interest, or R = n*r + CR, so
5 days @ 9 %/day = 5*9% (i.e. 45 %) + 8.9% = 53.9% since (1+9%)^5 = 1.539.
Exercise. Predict 6 days @ 7%/day, 4 days @ -8%/day etc.

09 SAVING AND INSATLLMENT PLANS

Question. How does a savings grow? **Answer.** By adding a fixed amount and percent each period.
Depositing n times a$ with an interest rate r% gives a saving A$ predicted by the formula, A/a = R/r. Such a saving may be used to pay off a debt Do, that in the same period has grown to D = Do*(1+R).
The formula and its proof both are elegant: in a bank, an account contains the amount a/r. Each period, a second account receives the interest amount from the first account, r * a/r = a, and its own interest, thus containing a saving A that is the total interest amount R * a/r. Since A = R * a/r, we get A/a = R/r.

10 PER-NUMBERS IN WORD-PROBLEMS

Question. How do we treat per-numbers in word problems?
Answer. Also, in word problems the per-number must be multipliesd to a unit-number before being added.
Example1: Train1 travels from A to B at 40 km/h. Two hours later train2 travels from A to B at 60 km/h. When do they meet?

Per-numbers	*Text*	*Unit-numbers*	*Prediction*		*ANSWERS*
	Hours	x = ? hours	40*(x+2)	= 60*x	*4 hours*
40 km/h	Speed1		x + 2	= 60*x/40 = 1.5*x	
60 km/h	Speed2		2	= 1.5*x – x = 0.5*x	
	Distance1	40*(x+2) km	2/0.5	= x	*240 km*
	Distance2	60*x km	4	= x	*240 km*

Example2: Train1 travels from A to B at 40 km/h. At the same time train2 travels from B to A at 60 km/h. When do they meet if the distance from A to B is 300km?

Per-numbers	*Text*	*Unit-numbers*	*Prediction*		*ANSWERS*
	Hours	x = ? hours	40*x + 60*x	= 300	*4 hours*
40 km/h	Speed1		100*x	= 300	
60 km/h	Speed2		x	= 300/100	
	Distance1	40*x km	x	= 3	*120 km*
	Distance2	60*x km			*180 km*

Exercise1. Repeat the train problems with other numbers.
Exercise2. It takes a motor boat 2 hours downstream and 3 hours upstream to travel the same distance. The current runs with 5 km/h. What is the speed of the boat?
Exercise3. ? liter 40% alcohol + 3 liter 20% alcohol gives ? liter 32% alcohol
Exercise4. ? $ @ 3%/$ + ? $ @ 8%/$ = 200$/4000$
Exercise5. Mr. A can dig a ditch in 4 hours, Mr. C in 3 hours. How long time does it take if they work together?

11 FINDING SQUARE ROOTS

Question. How to transform a rectangle into a square?
Answer. Squeeze the rectangle until it is between two neighbor squares with sides p and p+1.
Example1: Finding √70 means squeezing 7 tens until becoming a square (8+t)^2 situated between 8^2 and 9^2. This square has four parts as shown by two playing cards placed like an L: 8^2, and 8*t twice, and t^2.
Neglecting t^2 gives the equation 2*8*t = 70–8^2, or t = (70–8^2)/(2*8), giving 8+t = (70+8^2)/(2*8) = 8.375.
Finally, 8.375^2 = 70,14 = 70 approximately. Since (3B)^2 = 9B^2 we may find √2 by finding first √200 ≈ 14.
To solve the quadratic equation x^2+6x+8 = 0 we use four tiles forming a square, labeling the first side x and the next 6/2.
The (x+6/2) square contains two 6/2*x rectangles and two squares, x^2 and (6/2)^2 split in two parts, 8 below and (6/2)^2-8 above if possible, all disappearing except for last part. So (x+6/2)^2 = (6/2)^2– 8 = 1 giving x = -6/2 ± 1 = -2 and -4.
Looking instead at x^2+bx+c = 0 gives the solution x = -b/2 ± √((b/2)^2 –c).