DEVELOPING THE CHILD'S OWN MASTERY OF MANY

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I. MEETING MANY, CHILDREN BUNDLE AND COUNT WITH UNITS

How children master Many I observed from my three Korean foster girls. Asked "How old next time?", a 3year old would say "Four" and show 4 fingers; but would react strongly if held together 2 by 2: 'That is not four, that is two twos', thus insisting that the outside existing bundles should inside be predicated by a 'bundle-number' including the unit. When asked "How many 3s when uniting 2 3s and 3 4s they would say '5 3s and 3'; and when asked "How many 4s?" they would say '5 4s less 2'; and, integrating them next-to each other, they typically said '2 7s and 4'.

Children have fun 'bundle-counting' their fingers in 3s in various ways: as 1Bundle7 3s, 'bundle-written' as T=1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Children thus master numbering before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the mastery of Many that children bring to school, counting before adding?

II. MATERIALS FOR QUESTION GUIDED COUNTCURRICULUM

Typically, a 'mediating curriculum' sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a 'developing curriculum' sees mathematics as an exoteric means to develop the children's existing ability to master Many by numbering outside totals and stacks with inside two-dimensional bundle-numbers. This calls for different materials from grade 1 that don't mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Tarp (2018, 2020) holding that only two competences are needed to master Many, counting and adding. The corresponding pre-service and in-service teacher education may be found at the MATHeCADEMY.net.

QC01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the 'learning opportunity (L.O.) is to change many ones to one icon with as many sticks or strokes as it represents if written less sloppy. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

QC02, counting sequences: "How to count fingers?" Here the L.O. is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H".

QC03, icon-counting: "How to count fingers by bundling?" Here the L.O. is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate: T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s, called an 'inside bundle-number' describing the 'outside stack-number'. Turning over a two- or three-dimensional stack or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+ 3)*4 = 2*4 + 3*4.

QC04, calculator-prediction: "How can a calculator predict a counting result?" Here the L.O. is to see the division sign as an icon for a broom pushing away bundles: 7/2 means 'from 7, push away bundles of 2s'. The calculator says '3.some', thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles. Finally, the subtraction sign iconizes a rope pulling away the stack to look for unbundled singles. By showing '7-3*2 = 1' the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional L.O. is to write 8 = (8/2)*2 as a 'recount-formula' T = (T/B)*B, saying "From T, T/B times B can be pushed away", to predict counting and recounting.

QC05, recounting in another unit: "How to change a unit?" Here the L.O. is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives fractions: T = 3 = (3/5)*5. Another L.O. is to observe how double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

QC06, unbundled becomes decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the L.O. is to see that the unbundled occur in three ways: Next-to the stack as a stack of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside stack 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit..

QC07, prime or foldable units: "Which stacks can be folded?" Here the L.O. is to examine the symmetry of a stack. The stack T = 2 4s = 2*4 has 4 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a 'prime unit' and 4 a 'foldable unit', 4 = 2 2s. A number is called even or symmetrical if it can be folded in 2s, else the number is called odd..

QC08, finding units: "What are possible units in T = 12?". Here the L.O. is that units come from factoring in prime units, 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

QC09, recounting from tens to icons: "How to change unit from tens to icons?" Here the L.O. is that asking 'T = 2.4 tens = 24 = ? 8s' can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

QC10, recounting from icons to tens: "How to change unit from icons to tens?" Here the L.O. is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional L.O. uses 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3*(ten less 3) = 3*ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. So, 'less less 1' means adding 1.

QC11, recounting stack-sides. "How to recount sides in a stack halved by its diagonal?" Here, in a stack with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

QC12. On squared paper a point has an out-number x and and up-number y, A(x,y). The per-number Δy/Δx allows moving on a line.

With A(2,5) and B(4,6), the line per-number is Δy/Δx = (6-5)/(4-2) = ½. Changing position to C(8,y) gives Δy = (Δy/Δx)* Δx = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

III. MATERIALS FOR QUESTION GUIDED ADD CURRICULUM

Counting ten fingers in 3s gives T = 1BundleBundle1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

QA01, next-to addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?" Here the L.O. is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

QA02, reversed next-to addition: "If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?" Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

QA03, on-top addition: "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?" Here the L.O. is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

QA04, reversed on-top addition: "If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?" Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

QA05, adding tens: "With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?" Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

QA06, subtracting tens: "If T1 = 23 and T2 add to T = 71, what is T2?" Here, recounting removes an underload: T2 = 71 – 23 = 7B1 –

2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

QA07, multiplying tens: "What is 7 43s recounted in tens?" Here the L.O. is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

QA08, dividing tens: "What is 348 recounted in 6s?" Here the L.O. is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349/6 = 34B9/6 = 30B49/6 = (30B48 + 1) /6 = 58 + 1/6.

QA09, adding per-numbers: "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" Here the L.O. is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

QA10, subtracting per-numbers: "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" Here the L.O. is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

QA11, solving the quadratic equation x^2 + 6x + 8 = 0. Two playing cards have the width 3 and the height x + 3. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that (x+3)^2 = x^2 + 2*3*x + 3^2, or, (x+3)^2 = x^2 + 6*x + 8+1, or (x+3)^2 = 1 since x^2 + 6x + 8 = 0. So x = -3 ± 1 = -4 and -2.

QA12, finding common units: "Only add with like units, so how add T = 4ab^2 + 6abc?". Here units come from factoring:

T = 2*2*a*b*b + 2*3*a*b*c = (2b+3c) *2ab.

IV. DISCUSSION

Meeting Many makes children bring flexible bundle-numbers to school with core math as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many.

References

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