Math Ed & Research 2017-2020

A Goal Displacement Makes a Means a fake Goal.

The Goal of Teaching Math is ~~to Learn Math~~ to Master Many.

- so, Count & Multiply before you Add.

OnTop & NextTo Addition Root Proportionality & Integral Calculus.

Reversed Addition Roots Equations and Differential Calculus.

As PerNumbers, Fractions are Operators, needing Numbers to become Numbers

- so, PerNumbers and Fractions Add by their Areas as Integral Calculus.

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Preface

01-02. To celebrate the 500year anniversary of the 95 Luther theses I decided to write two feature articles to a Danish newspaper. The first asks If Europe really need Compulsory School Classes, arguing that the North American self-chosen half-year blocks might be a better way to support adolescents in their complicated identity work after puberty. The second asks why mathematics, which was created as a straight forward natural science about the physical fact Many has to be presented as a metaphysical self-referring science that transforms many potential users to losers.

03. The first conference was the Congress of the European Society for Research in Mathematics Education, CERME, 10th congress in Ireland in February. I chose the thematic working group on Arithmetic and Number Systems, and wrote the paper 'CupCounting and Calculus in Early Childhood Education'. The paper points to the decreased PISA performance despite increased research, and the uses a method called 'institutional skepticism' to explain it. The paper then looks at mathematics as essence, and as existence containing chapters on creating number-icons, counting in icons, Re-counting in the same unit and in a different unit, Double-counting creates proportionality as per-numbers, added on-top or next-to, reversed adding on-top and next-to, and how schools use ten-counting only. The paper then describes a micro-curriculum designed for the outside goal 'to master Many'. Finally, literature on cup-counting is addressed before the conclusion and recommendation.

04. At the conference, a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is "Only one: to improve, mathematics education should ask, not what to do, but what to do differently." Thus, to be successful, research should not study problems but look for hidden differences that might make a difference. Research that is skeptical towards institutionalized traditions could be called difference research or contingency research or Cinderella research making the prince dance by looking for hidden alternatives outside the ruling tradition. The French thinker Lyotard calls it 'paralogy' inventing dissension to the reigning consensus. To give a more detailed answer I wrote the academic essay 'Fifty Years of Research without Improving Mathematics Education, Why?' Looking at the Outside Roots of Mathematics, the essay suggests a Rethinking of Mathematics from Below. Then it presents how School Teaches Mathematics in 12 points, and in 25 points how School Could Teach Mathematics. The conclusion is followed by an extract of my personal contributions to mathematics education research.

05. The second conference was the 9th International Mathematics Education and Society Conference or MES9 that took place in Greece in April. The conference focused on the social theorizing of mathematics education and the theme of MES9 was 'Mathematics Education and Life at Times of Crisis'. My paper proposal was called 'A 1year pre-engineer course for Young migrants, a job for critical or civilized math education.' It contained chapters on Background and question, Critical and civilized thinking, A historical background, Critical versus civilized mathematics education, Criticizing and civilizing rational numbers, 'Preschool calculus and multiplication before addition' as a 1year pre-engineer math course; and a final discussion and conclusion as well as an appendix showing: a critical and a civilized math curriculum.

06. My project proposal was called 'Online teacher training for curing math dislike: cup & re-counting & multiplication before addition. It contained chapters on the background, A Case: Peter, stuck in division and fractions, and a conclusion. And as appendices, 1day Skype Seminar on how to cure Math-dialike by BundleCounting, ReCounting & BundleWriting before adding, and a summary of the 4 primary and secondary 4 study units at the MATHeCADEMY.net

07. My symposium proposal was called a 'Debate on how to improve mathematics education.' It contained chapters on Mathematics Itself, Education in General, Mathematics Education, the Learner, the Teacher, the Political System, Research, and Conflicting Theories.

08. My poster proposal called ‘MigrantMath as CupCounting & PreSchool Calculus’ contained chapters on the background, Mathematics as an Essence, as well as Mathematics as ManyMath, a Natural Science about Many before the conclusion.

09. At the International Congress on Mathematics Education, ICME, in Germany in 2016 my contribution to the topic study group on Philosophy of mathematics education was called ‘From Essence to Existence In Mathematics Education’. My proposal for a book following the topic study group was called ‘A Heidegger View on How to Improve Mathematics Education.’ It contains chapters on introduction, What does ‘is’ Mean, the Heidegger Universe, Meeting Many celebrating the 500year anniversary for Luther’s 95 theses by describing meeting Many in 12 theses; and a conclusion.

10. Now the time had come to summarize the 2017 work in an article to be sent to a journal. The proposal is called ‘Count and Multiply Before You Add: Proportionality and Calculus for Early Childhood and Migrants.’ It contains chapters on Decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Institutional Skepticism, Mathematics as Self-Referring Gossip, Meeting Many, Comparing Many-matics with Mathe-matics, Testing a Many-Matics Micro-Curriculum, Ending the Dienes Era, as well as a Conclusion and a Recommendation.

11. Each second year the Swedish ‘MatematikBiennale’ takes place in relation to the MADIF conference. I send in 11 proposals for the 2018 Biennale. They were all rejected. The proposals were called as follows. Start-math for children and migrants: bundle-count and re-count before adding, Multiplication before addition strengthens the number sense in children and migrants, Dislike towards division cured with 5 sticks and 1 cup and bundle-writing, Fractions and percentages as per-numbers, Fractions and per-numbers add as integration, Proportionality as double-counting, with per-numbers, Equations solved by moving, reversing or re-counting, Calculus: Addition of and division into locally constant per-numbers, Calculus in primary, middle and high school, Stem-based core-math makes migrants pre-engineers, The teacher as a difference-researcher. At the end a comment was added called ‘Fifty years of research without improving mathematics education, why?’

12-13. To help Sweden cope with OECD report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change’ I wrote an article sent to the Swedish government and a university in southern Sweden. Swedish educational shortages challenge traditional mathematics education offered to migrants. Mathematics could be taught in its simplicity instead of as ‘meta-matsim’, a mixture of ‘meta-matics’ defining concepts as examples of inside abstractions instead of as abstractions from outside examples; and ‘mathe-matism’ true inside classrooms but seldom outside as when adding numbers without units. Rebuilt as ‘many-matics’ from its outside root, Many, mathematics unveils its simplicity to be taught in a STEM context at a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries. The article is called ‘The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants.’ It contains chapters on decreased PISA Performance Despite Increased Research, Social Theory Looking at Mathematics Education, Meeting Many, Meeting Many in a STEM Context with subchapters on a short World History, Nature Obeys Laws, but from Above or from Below?, Counting and DoubleCounting Time and Space and Matter and Force and Energy, Warming and Boiling water, Letting Steam Work, An Electrical circuit, How high up and how far out, How many turns on a steep hill, Dissolving material in water. Finally, a chapter on the Simplicity of Mathematics as well as a discussion: How does Traditional MatheMatics differ from ManyMatics was included before the conclusion. A shorter 10 page version for the MADIF11 Conference is included.

14. Introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015

OECD-report ‘Improving Schools in Sweden’. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

15. The KomMod report provides an alternative response to KOM-project terms of reference, in the expectation that the Science Board of education and the Ministry of education want to respect a common democratic IDC-tradition with Information and Debate between alternatives before a Choice is made. It was written in Danish in 2002 And translated into English in 2017.

16-18, 20. The third conference was the 9th conference of the Classroom Teaching Research for all Students, CTRAS, held in China in July. Having presented my contribution ‘Decreasing PISA Performance in spite of increasing research’ I was asked to give a keynote-presentation. I sent in twelve proposals formulated so they could also become proposals for 1day Skype Seminars: The Root of Mathematics, Many, dealt with by Block-Numbers, Bundle-Counting & Preschool Calculus; 12 Luther-like Theses about how ManyMath can Improve Math Education; Curing Math Dislike with one Cup and five Sticks; DoubleCounting rooting Proportionality - and Fractions and Percentages as PerNumbers; Algebraic Fractions made easy by Block-Numbers with Units; Algebra and Geometry, always Together, never Apart; Calculus in Middle School and High School; Mathematics, the Grammar of the Number-Language. But why teach Grammar before Language?; Quantitative Literature also has three Genres: Fact and Fiction and Fiddle; Distance Teacher Education in Mathematics by the CATS method: Count & Add in Time & Space; 50 years of Sterile Mathematics Education Research, Why?; and Difference-Research, a more Successful Research Paradigm?

The conference chose the latter, so I wrote the article ‘Difference-Research Powering PISA Performance: Count & Multiply before You Add’, containing chapters on Decreased PISA Performance Despite Increased Research, Difference-research Searching for Hidden Differences, Social Theory Looking at Mathematics Education, a philosophical Background for Difference Research, Meeting Many, Examples of Difference-research, Remedial Curricula, a Macro STEM-based Core Curriculum, Teaching Differences to Teachers, Being a Difference-Researcher, and a conclusion. Likewise, the keynote PowerPoint Presentation at the CTRAS 2017 Conference is included. At the end of the conference I summarized my reflections in a paper called: Reflections from the CTRAS 2017 Conference in China, Examples of Goal Displacements in Mathematics Education.

19. At the 8th ICMI-East Asia Regional Conference on Mathematics Education, the theme of the Conference was ‘Flexibility in Mathematics Education’. The website writes:

“Flexibility in Mathematics Education” has been chosen as the theme of the conference. Flexibility is highly related to creativity, multiplicity, and adaptation. In the current era, rapid changes in economy, environment and society have been facilitated by the rapid development of technology and engineering. Flexibility in mathematical thinking, problem solving, teaching methods, evaluation, teacher education and mathematics education research is a key to empowering learners, teachers, educators and researchers to tackle the complexity and uncertainty, and to giving them the capacity and motive to change in the innovative era.

The Topic Study Group themes were TSG 1: Flexibility in Mathematics Curriculum and Materials; TSG 2: Flexibility in Mathematics Classroom Practices; TSG 3: Flexibility in Mathematics Assessment; TSG 4: Flexibility in Mathematics Teacher Education and Development; TSG 5: Flexibility in the Use of ICT in Mathematics; TSG 6: Flexibility in the Use of Language and Discourse in Mathematics; and TSG 7: Flexibility in Mathematics Learning.

21-22. Next, an overview on how to cure Math Dislike is included together with concrete material called 'Migrant Math' containing 20 exercises inspired by the paper 'The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants.' The exercises are: From Sticks to Icons, Counting-sequences in Icons, BundleCount in Icons, BundleCount with Dices, ReCount in the Same Unit, ReCount in a New Unit, ReCount in BundleBundles, ReCount in Tens on Squared Paper or an Abacus, ReCount from Tens, ReCount Large Numbers in Tens, DoubleCount with PerNumbers, DoubleCount with Fractions and Percentages, ReCount PerNumbers, Fractions, Add OnTop, Reversed Adding OnTop, Add NextTo, Reversed Adding NextTo, Add Tens, Reversed Adding Tens, Recounting Solves Equations.

23. The compendium 'Mathematics Predicts, PreCalculus with a TI-82 or TI-84' contains the following chapters: Mathematics Predicts, Calculations Predict, Formulas Predict, Trigonometry, Statistics, Stochastic Variation, Polynomials and Calculus, Two Equations with Two Unknowns; and Three, Letter Calculation, Transposing Formulas, Homework, Project Forecasting, Project Distance to a Far-away Point, Project the Bridge, Project Golf, Project Driving, Project Vine Box, Revision Problems Using TI-84.

Allan Tarp, Aarhus Denmark, December 2017

01. Does Europe really need Compulsory School Classes?

Compulsory classes force children and young people to follow the year group and its schedule. Compulsory classes made sense when created in Prussia about 200 years ago in an agricultural society; and also in industrial society with its permanent life jobs. In an IT-society, compulsory classes make sense in primary school: with both mother and father in changing self-realizing jobs, the first 3-4 school-years children need a warm and loving nanny with only one class, quickly getting a gaze of each child's characteristics and needs.

On the other hand, compulsory classes mean disaster in secondary school with young people who have left childhood and started an extensive identity work to uncover and develop their personal potential and talent. Here a compulsory class is the last thing they need, which is evident when observing the seven sins of compulsory classes.

Noise. Having an activity imposed that you do not master or find interesting, you quickly switch to other activities, surfing the Web or chatting with others in the same situation. The result is noise, which can be so violent that the rest of the class must wear hearing protectors.

Absence. Once you have given up on learning you feel a desire for absence, perhaps even to drop out. But that will hurt the school’s economy, so you will not be allowed to leave the class regardless of your extent of absence.

Bullying. When you finally meet up again after an absence, it is tempting to bully those who meet every day.

Drinking. Especially if they do not want to participate in the extended weekend drinking starting in lower secondary school and coming to full expression in das Gymnasium, where many are sent to the hospital at the annual welcome parties or get hurt under excessive drinking on study tours.

Substitute teachers. Once you have conquered the territory, it is natural to bully also the various teachers who come to visit. Some can take it, others cannot and take a long-term sick leave. Skilled substitute teachers are expensive, so often a recent high school graduate is selected instead, or cleaning personnel.

Bottom marks. The extent of mental absence is shown by the written marks. Thus, in Denmark with 5 passing marks, the three lowest are given when answering correctly 16%, 33% or 50% at the final exam in mathematics at the end of lower secondary school. And here the international passing level at 70% gives the second-highest mark. The low level of learning can, however, be hidden by replacing written tests with oral, which is much more effective to increase the marks with floods of leading questions. Denmark is virtually the only country in the world maintaining an oral exam. Its credibility is illustrated by the joke, which is often exchanged over coffee table during an exam: With a friendly external examiner, a good teacher can examine a chair to a passing mark, provided the chair stays quiet.

War against boys. In a compulsory class, girls and boys are forced to go along, although the girls are two years ahead in development. It provides both with a skewed impression of the opposite sex, and school dislike makes boys leave school before upper secondary school, where there are two girls for every boy. In short, compulsory classes pump boys out of school to remain in the outskirts, while girls are pumped into the juggernaut universities in Copenhagen; and in Aarhus, where they then move to Copenhagen after graduation, since that is where the jobs are. With the absence of boys, girls find another girl and a sperm bank so that together they can get a single child.

Which creates the compulsory class’ most fatal consequence, a birth-rate in Europe at 1½ child per family. A quick calculation shows that with 0.75 child per woman, Europe's population will halve twice over the course of 100 years. A population decline unprecedented in history.

Unlike in the North American republics. Here young people do not have multi-year compulsory classes. Instead, they are welcomed to high school with recognition: “Inside, you carry a talent that it is our mutual job to uncover and develop through daily lessons in self-chosen half-year academical or practical blocks together with a teacher who only teaches one subject. If successful we say ‘good job, you have talent, try out more blocks’. If not we say ‘good try, you have courage to try out the unknown, now let’s find another block for you to try out. And at the last year you can try out college blocks.”

Thus, the absence of multi-year educational defeats allows you to enter a local block-organized college at 18 and get a two-year practical diploma degree or continue at a regional college and get a four-year job-directed bachelor's degree.

Without compulsory classes, Europe could do the same, so that every other boy could be an engineer at the age of 22; and at the age of 25 have a well-paid job, a family, and three children ensuring state survival: one for mother, one for father, and one for the state.

As demonstrated in North America, compulsory classes are not a biological necessity.

As mammals, we are equipped with two brains, one for routines and one for feelings. When we raised up on our hind legs, we developed a third brain to keep balance; and to hold concepts since we could now use the front legs to grab the food and eat it or share it with others. In this way, grapping could provide the holes in our head with our two basic needs, food for the body and information to the brain. For by assigning sounds to what we grasp, we develop language to transfer information between brains.

In fact, we have two languages, a word language and a number language. At home children learn to talk and to count. Then as an institution, the school takes over and teaches children to read and to write and to calculate, and to live together with others in a democracy.

The ancient Greek sophists saw enlightenment as a prerequisite for democracy: knowing the difference between nature and choice, we can avoid hidden patronization in the form of choice presented as nature. The philosophers had the opposite view: Choice does not exist, since all physical things are but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Consequently, people should give up democracy and accept the patronage of the philosophers.

The Christian Church eagerly took over the idea of metaphysical patronage and converted the academies into monasteries, until the Reformation recovered the academies. Likewise, nor emperors nor kings had anything against being inserted by the Lord's grace.

Metaphysical patronage ended with Newton’s three times no. “No, the moon does not move among the stars, it falls to the ground like an apple. No, moons and the apples do not follow a metaphysical unpredictable will; instead they follow their own will, which is predictable because it follows a formula. And no, a will does not maintain order, it changes it.”

Once Newton discovered the existence of a non-metaphysical changing will, this created the foundation for the 1700 Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. The result was two republics, one in the United States and one in France. The United States still has its first Republic, France its fifth, since Prussia tried to overthrow the French Republic again and again.

France first got upper hand by mobilizing the population with enlightenment and democracy. As a counter measure, Prussia created a strong central administration with an associated ‘Bildung’ education with three goals: The population must be kept unenlightened so it will not demand democracy. Instead, Bildung must install nationalism transforming the population into a 'people', Germans, obeying the almighty Spirit by fighting other 'people', especially the French with their democracy. Finally, from the population, its elite must be sorted out to form a new central administration; and receive classical Bildung to become a new knowledge-nobility to replace the old blood-nobility, which was unable to strangle French enlightenment and democracy.

The rest of Europe eagerly took over the Prussian Bildung education. One might expect that when Europe became republics, its school form would follow. Here is only to say that still it is not too late. But it requires a comprehensive school reform, for the two school forms are very different.

In continental Europe, compulsory classes are replaced by a mess of competing compulsory lines in upper secondary school and with a confusion of more or less coordinated lines at the tertiary level leading to a 3year bachelor degree, usable only if supplemented with a 2year university directed master degree.

In the North American republics, compulsory classes stop after primary school. With self-chosen half-year blocks, learners can try something new each half year and continue if the trial was successful; and, as important, get out if it turns out to be an area outside your personal talent.

At the same time, the mark reflects the personal effort. Thus, at a half-year math block you can collect 700 points. The daily assignments give 100 points based on neatness, completeness and correctness. Late delivery does not count. The final test counts 200 points; and 400 points come from five tests, of which the lowest is neglected.

The 700 points corresponds to 100%, and the characters A, B and C correspond to 90%, 80% and 70% of the points. A score below 70% means that the block must be retaken or be replaced by another block.

At 18 you can continue at a regional four-year college, or a local two-year community college, which is divided into quarters so it's easy to take blocks while you work or during summer holidays. Likewise, the block system makes it easier to change job in case of unemployment or a desire for new challenges.

But why don’t Europe do the same? Because Europe is so over-institutionalized, that it cannot imagine a society without institutions. And once you have chosen institutions, the school is used to create public servants through compulsory classes in primary school and in a myriad of compulsory lines at the secondary and university level.

And compulsory classes mean disappearance of the freedom to develop your personal potential. Instead school struggle with its well documented seven sins. Sins, Europe believes it can eliminate through its political system. If it has not died out before.

02. Mathematics, Banality or Evilness

Mathematics is steeped in evilness right from the first to the last class in the 12-year school, which we leave our children and young people to in the belief that the school will prepare them to master their environment and its two languages, the word language, and the number language called ‘math’ by the school. Strange, for we master our world through actions, by reading and writing and by counting and adding, so why is it necessary to learn to ' math'?

Thus the evilness of mathematics begins with its name; and by claiming that counting and adding are mere applications of mathematics, which, as such, of course, must first be learned before it can be applied; and which, unfortunately, is so difficult to learn, that it requires an extra effort leading still more to fail.

Also mathematics hides its origin. The ancient Greek Pythagoreans used the word as a common name for their four knowledge areas, music and stars and shapes and numbers, that constitutes ancient and medieval basic training, quadrivium, as recommended by the Greek philosopher Plato.

With music and astronomy out, today mathematics is just a common name for the two remaining areas, geometry, which in Greek means earth-measuring; and algebra, which in Arabic means to reunite numbers, and again hidden by the school, claiming instead that algebra means to search for patterns.

Algebra followed when the Renaissance replaced Roman numbers as CCXXXIV with the Arabic number 234 = 2 ten-tens and 3 tens and 4 ones = 2*10*10 + 3*10 + 4*1 showing algebra’s four ways to unite numbers. Addition unites unlike numbers such as 3 + 4. Multiplication unites like plus-numbers such as 3 + 3 + 3 + 3 = 3*4; power unites like multipliers such as 3*3*3*3 = 3^4; and the three number-blocks 200, 30 and 4 are united by next-to addition, also called times-plus calculation, or integration, the Latin word for uniting.

And blocks is exactly what children bring to school. Asking a three-year child "how old will you next time?" the answer is four with four fingers shown. But displaying four fingers held together two and two will prompt an immediate protest: "No, it's not four, that is two twos!"

So children come to school with two-dimensional block-numbers all carrying a unit, corresponding to Lego-blocks that stack as 1, 2, 3 or more 4ere. By combining geometry and algebra in their shapes and buds, blocks are highly suitable as a basis for connecting the starting point, children's block-numbers, with the final goal: algebra’s uniting block-numbers illustrated by geometrical shapes.

However, the school is ignoring this and instead it teaches one-dimensional line-numbers located on a number line with each their name; and where the system will only be visible in the late twenties, where many children count over by saying 'ten-and-twenty' instead of 'thirty'. This then allows the school to pass a dyscalculia-diagnose and to institutionalize a corresponding dyscalculia-treatment supported by a growing dyscalculia-research with an associated dyscalculia-industry.

Evilness occurs when the school itself installs dyscalculia in the child by teaching line-numbers instead of block-numbers, thus teaching today's two-dimensional Arabic numbers, used by communities and kids, as if they were one-dimensional ancient Roman numbers.

Both number systems count by bundling.

Roman numbers use linear bundling: in a row of sticks, 5 1s are bundled to a V, 2 V'er to an X, 5 X's to a L, 2 Ls to a C, and so on. So a Roman number remains a one-dimensional string of letters as I, V, X, L, C etc.

Arabic numbers use rectangular bundling: in a row of sticks, twelve 1s are bundled to 1 ten-bundle and 2 unbundled, written as 12. Bundles then stack to a block of e.g. 4 10s, until ten bundles of 10s create a new block with the unit ten-ten or hundred, which then again stack in a block until ten of them create the unit ten-ten-ten or one thousand, etc.

So, where Roman numbers never have units, Arabic numbers always have, just as in children's own number system.

Nevertheless, the school teaches only in numbers without units. Likewise, the school does not distinguish between 2 * 3 = 6 and 2+3 = 5. The former is always true since 2 3s can be recounted to 6 ones. The latter is true only if the omitted units are the same: 2 days + 3 days is 5 days, but 2 weeks + 3 days is 17 days, and 2 days + 3 weeks is 23 days. Mathema-tics without units should be called 'mathema-tism', something that is true inside, but seldom outside a classroom. This would allow seeing if its diagnoses are created by teaching mathematics as mathematism.

Its evilness begins when mathematics neglects children's own Arabic number system and impose on them a Roman number system. It continues by forcing children to add before counting; and by forcing upon children the four operations in the order addition, subttraction, multiplication and division, where the last is presented so difficult that it triggers new dyscalculia diagnoses.

It is in fact the opposite order that is the natural. We count by bundling, so 7 sticks are counted in 3s by removing 3s many times, which is division predicted by a calculator as '7/3 = 2. something'. Then the 2 3s are stacked, which is multiplication. Removing the stack to look for unbundled is subtraction, predicted by a calculator as '7 – 2*3 = 1'. So, the calculator prediction holds true: 7 = 2.1 3s. Which shows that a natural number is a decimal number with a unit where the decimal point separates bundles form the unbundled. In contrast to the school that writes 5.6 tens as 56, i.e. without a unit and with a misplaced decimal point, and even calls such a number a natural number. An effective way to create even more diagnoses.

So counting includes the three operations division, multiplication and subtraction, and in that order.

After counting, it is natural to learn re-counting, back-counting and double-counting to change unit, or to create or remove an overload occurring when removing or adding. Thus, 7 can be recounted in the same unit 3s with or without an overload as 1.4 3s or 2.1 3s.

Recounting in a new unit means asking e.g. ‘how many 4s is 2 3s?'. We get the answer by a manual recounting, or by asking the calculator for a prediction: 2*3/4 = 1. something and 2*3 – 1*4 = 2, so 2 3s = 1.2 4ere.

Recounting the tens is done by pure multiplication: 3 8ere = 3*8 = 24 = 2.4 tens.

Back-counting from tens leads to solving equations. The question '5 tens is how many 4ere?' becomes the equation 50 = 4*x. The solution is obtained by recounting 50 in 4s, x = 50/4. So an equation is just another word for a back-counting, which means using the opposite operation, i.e. moving a number to the opposite side with the opposite sign. A natural approach easy to understand.

But, again silenced by the school, instead postponing equations to later grade levels. Here equations are presented as examples of open statements expressing equivalence between two numbers-names, and which teachers learn to solve using an abstract neutralization method.

Double-counting in different colors leads directly to the most important numbers, 'per-numbers', used to change units: If 3 red corresponds to 4 blue then 5 red correspond to how many blue? Or later: If 3 kg cost 4 $ then what is the cost of 5 kg? To answer we use the per-number 4$/3kg to recount the kilo-number 5 in 3s, 5/3, so many times we must pay 4$.

Changing unit is one of the two core areas of mathematics. However, the school does not recognize words as re-counting, back-counting, double- counting, or per-numbers. Instead, it uses the word 'proportionality', and again postpones it to later grades and makes it so difficult that new diagnoses are issued.

Why must children not learn the different ways of counting already in pre-school, where they count by themselves, time after time? Why does the school hide the great advantages in counting before adding? After all, totals must be counted before they can be added?

In addition, addition is not well defined: Should two blocks be added on-top or next-to each other, also called integration, the Latin word for uniting?

On-top addition means recounting to a common unit. But the school insists on using a so-called carry-method, which creates new diagnoses.

At the same time, the school only works with totals counted in tens. It is therefore unnecessary to change unit and to do next-to addition, the second main area of mathematics, and therefore more important than on-top addition; and that can be learned as early as pre-school by posing Lego-blocks next to each other and ask '3 2s plus 5 4s total how many 6s?' Nevertheless, school postpones it to the last school year with the claim that only the very best can learn next-to addition.

Reversing next-to addition is called differentiation. It asks e.g. '3 2s plus how many 4s gives 7 6s?'. Here we first remove the 3 2s with a minus before we recount the rest in 4s by division. So in reversed next-to addition subtraction comes before division. Of course, for in next-to addition, multiplication comes before addition.

But, the school does not recognize the words next-to addition or reversed next-to addition, nor does it recognize the word times/plus calculation or minus/division calculation. Instead, it introduces the Latin words integral and differential calculus and postpone both to the upper-secondary level where they are presented in reversed order, i.e. reversed next-to addition before next-to addition. Which makes both hard to understand with a high failure rate as a consequence.

A sly way to sabotage any high school reform. The parliament would like everyone to learn forward and reversed next-to addition, but both teachers and their teachers, the university professors, protest loudly: It cannot be done!

Of course it can, you just need to teach what is in the world, blocks to be united or split, and in that order, i.e. integration before differentiation. It is that simple to make calculus accessible to all.

So if the school allowed children and young people to meet its root Many as it naturally occurs in the world, i.e. as block-numbers that are counted, re-counted, back-counted, and double-counted, to be added on-top or next-to and forward or reversed, then everyone would learn everything in mathematics.

However, then no longer can mathematics be used for exclusion, which is precisely the school's main task, according to the sociologist Bourdieu. We think we got rid of the nobility with its privileges, but instead of a blood-nobility we got a knowledge-nobility protecting its monopoly on today's most important capital form, knowledge capital, by using the school to exercise what he calls symbolic violence.

The word-language cannot be used for exclusion since it is learnt before school. In contrast to the number-language that school can make so hard that is will be accessible to the nobility's own children alone. In other words, the same technique as the mandarin class used when they made the Chinese alphabet so difficult that only their children could pass the state's official exams.

But why do teachers accept to teach evil mathematics? Because of the banality of evil as described by Arendt in her book about Eichmann in Jerusalem. Here Arendt points to the lurking evil stored in blindly following orders in institutions originally created to ensure that good thing happens.

To keep your job, you must obey orders, 'conform or die'. Institutions do not compete as does the private labor market where 'compete or die' ensures control by the users' needs.

Together with skeptical post-modern thinking, also Arendt finds inspiration in the last century’s great philosopher, Heidegger, who points out that to realize your existential potential you must have an authentic relationship with the surrounding things. To ensure this, we continually must ask if a thing’s true existence is shown or hidden by institutionalized essence claims.

So, as an institution, mathematics education should continually ask whether it mediates an authentic image of its subject, the physical fact Many. Or, whether the institution is caught in what the sociologist Baumann calls a ‘goal displacement’, where the initial goal is transformed into a subordinate instrument to a new target: the institution's self-preservation.

Mathematics education could be a framework for children's and young people's authentic meetings with its physical root, Many. Instead, it has become an attempt to cure self-created diagnoses.

To deal with Many is simple and banal, so why drown the banality of mathematics in evilness?

Sensory perception, experience and common sense are the worst enemies of evil mathematics. So practice existence before essence, also in mathematics education. Which instead should comply with the international PISA-intention: To equip a population with knowledge and skills for the realization of their individual potentials.

Consequently, please drop the evil mathematics. Allow the child to develop its existing number language through guided learning meetings with its root, Many. Remove the evil textbooks on line-numbers and addition before counting. Use blocks and playing cards to illustrate block-numbers and activities such as counting, re-counting, back-counting and double-counting followed by forward and reversed on-top and next-to addition; and swap differential and integral calculus in high school, so all young people learn next-to addition both forth and back.

Again, Luther is right: Contact can be established individually without an institutionalized intermediary.

03. CupCounting and Calculus in Early Childhood Education

To improve PISA results, institutional skepticism rethinks mathematics education to search for hidden alternatives to choices institutionalized as nature. Rethinking preschool and primary school mathematics uncovers cup-counting in bundles less than ten; as well as re-counting to change the unit, later called proportionality, and next-to addition, later called integration. As to ICT, information and communications technology, a calculator can predict re-counting results before being carried out manually. By allowing overloads and underloads when re-counting in the same unit, cup-writing takes the hardness out of addition, subtraction, multiplication and division. This offers preschool students a good start and special needs students a new start when entering or reentering ordinary classes only allowing ten-counting and on-top addition to take place.

Decreased PISA performance despite increased research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

> The highest performing education systems across OECD countries are those that combine excellence with equity. A thriving education system will allow every student to attain high level skills and knowledge that depend on their ability and drive, rather than on their social background. Sweden is committed to a school system that promotes the development and learning of all its students, and nurtures within them a desire for lifelong learning. PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD 2015, p. 3).

Created to help students cope with the outside world, schools institutionalize subjects as inside means to outside goals. To each goal there are many means, to be replaced if not leading to the goal; unless a means becomes a goal itself, thus preventing looking for alternative means that could lead to the real goal if difficult to access. So we can ask: Does mathematics education have a goal-means exchange seeing inside mathematics as the goal and the outside world as a means?

Once created as a means to solve an outside problem, not solving the problem easily becomes a means to necessitate the institution. So to avoid a goal/means exchange, an institution must be reminded constantly about its outside goal. Institutional skepticism is created to do precisely that.

Institutional skepticism

The ancient Greek sophists saw enlightenment as a means to avoid hidden patronization by Plato philosophy presenting choices as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, symbolic interactionism and Grounded theory (Glaser & Strauss, 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget, 1970). In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnoses, and education all presenting patronizing choices as nature (Lyotard, 1984; Tarp, 2004).

Building on Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers ‘existence precedes essence, or (..) that subjectivity must be the starting point’ (Marino, 2004, p. 344). Kierkegaard was skeptical to institutionalized Christianity seen also by Nietzsche as imprisoning people in moral serfdom until someone ‘may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.’ (Marino, 2004, pp. 186–187). Inspired by Heidegger, Arendt divided human activity into labor and work aiming at survival and reproduction, and action focusing on politics, creating institutions to be treated with utmost care to avoid the banality of evil by turning totalitarian (Arendt, 1963).

Since one existence gives rise to many essence-claims, the existentialist distinction between existence and essence offers a perspective to distinguish between one goal and many means.

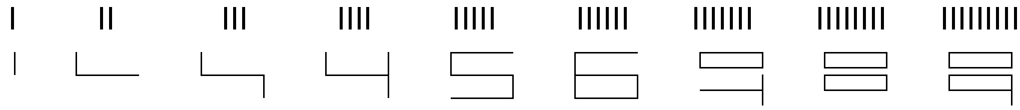
Mathematics as essence

In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra (Freudenthal, 1973) both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite Many’ in Arabic.

Then the invention of the concept SET allowed mathematics to be a self-referring collection of ‘well-proven’ statements about ‘well-defined’ concepts, i.e. as ‘MetaMatics’, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract essence. Thus SET transformed grounded mathematics into a self-referring ‘MetaMatism’, a mixture of MetaMatics and ‘MatheMatism’ true inside a classroom but not outside where claims as ‘1 + 2 IS 3’ meet counter-examples as e.g. 1 week + 2 days is 9 days. And, as expected, teaching numbers without units and meaningless self-reference creates learning problems.

Mathematics as existence

Chosen by the Pythagoreans as a common label, mathematics has no existence itself, only its content has, algebra and geometry. Algebra contains four ways to unite as shown when writing out fully the total T = 342 = 3*B^2 + 4*B + 2*1 = 3 bundles of bundles and 4 bundles and 2 unbundled singles = 3 blocks. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reversing splitting operation. So, with a human need to describe the physical fact Many, algebra was created as a natural science about Many. To deal with Many, we count by bundling and stacking. But first we rearrange sticks in icons. Thus five ones becomes one five-icon 5 with five sticks if written less sloppy. In this way we create icons for numbers until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle and one, one bundle and two etc.

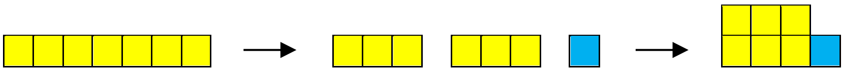


1 2 3 4 5 6 7 8 9

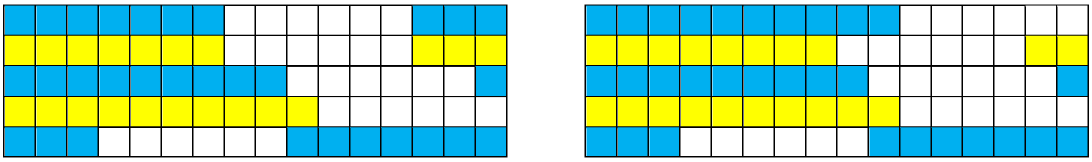
Holding 4 fingers together 2 by 2, a 3year old child will say ‘That is not 4, that is 2 2s. This inspires ‘cup-counting’ bundling a total in icon-bundles. Here a total T of 7 1s can be bundled in 3s as T = 2 3s and 1 where the bundles are placed in a bundle-cup with a stick for each bundle, leaving the unbundled outside. Then we describe by icons, first using ‘cup-writing’, T = 2]1, then using ‘decimal-writing’ with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T = 2.1 3s. Moving a stick outside or inside the cup changes the normal form to overload or underload form. Also, we can use plastic letters as B and C for the bundles.

T = 7 = IIIIIII → III III I → II] I → 2]1 3s = 1]4 3s = 3]-2 3s or BBI → 2BI

Using squares or LEGO blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to, thus showing the total as a double stack described by a cup-number or a decimal number, T = 7 = 2 3s & 1 = 2]1 3s = 2.1 3s.



We live in space and in time. To include both when counting, we introduce two different ways of counting: in space, geometry-counting, and in time, algebra-counting. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.



To predict the result we use a calculator. A stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s. As for the two icons for taking away, division shows the broom wiping away several times, and subtraction shows the trace left when taking away just once.

Thus by entering ‘7/3’ we ask the calculator ‘from 7 we can take away 3s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 3s by asking ‘7 – 2*3’. From the answer ‘1’ we conclude that 7 = 2]1 3s. Likewise, showing ‘7 – 2*3 = 1’, a display indirectly predicts that 7 can be recounted as 2 3s and 1, or as 2]1 3s.

7 / 3	2.some
7 – 2 * 3	1

A calculator thus uses a ‘recount-formula‘, T = (T/B)*B, saying that ‘from T, T/B times Bs can be taken away’; and a ‘restack-formula’, T = (T–B)+B, saying that ‘from T, T–B is left if B is taken away and placed next-to’. The two formulas may be shown by using LEGO blocks.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 4 2s as 3]2 2s with an outside overload; or as 5]-2 2s with an outside underload thus leading to negative numbers:

Letters	Sticks	Total T =	Calculator	
B B B B	II II II II	4]0 2s	4*2 – 4*2	0
B B B I I	II II II I I	3]2 2s	4*2 – 3*2	2
B B B B B B	II II II II II II	5]-2 2s	4*2 – 5*2	-2

To recount in a different unit means changing unit, also called proportionality or linearity. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2]2 5s.

IIII IIII IIII → IIIII IIIII I I → 2]2 5s. With letters, C = BI so that BBB → BB IIII → CC II

A calculator can predict the result. Entering ‘3*4/5’ we ask ‘from 3 4s we take away 5s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be recounted as 2 5s and 2, or as 2]2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Double-counting creates proportionality as per-numbers
Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘6$ = ?kg’ we use the per-number to recount 6 in 2s: 6$ = (6/2)*2$ = 3*3kg = 9kg. And vice versa: Asking ‘?$ = 12kg’, the answer is 12kg = (12/3)*3kg = 4*2$ = 8$.

Once counted, totals can be added on-top or next-to
Asking ‘3 5s and 2 3s total how many 5s?’ we see that to be added on-top, the units must be the same, so the 2 3s must be recounted in 5s as 1]1 5s that added to the 3 5s gives a total of 4]1 5s.

IIIII IIIII IIIII III III → IIIII IIIII IIIII IIIII I → 4]1 5s. With letters: 3B + 2C = 3B III III = 4BI.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. Thus we get 4]1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2]5 8s.

IIIII IIIII IIIII III III → IIIII III IIIII III IIIII → 2]5 8s → 2.5 8s

Using a calculator to predict the result we include the two totals in a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2]5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing adding on-top and next-to
Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2]6 8s?’, using sticks will give the answer 2]1 3s:

IIIII IIIII IIIII III III I ← IIIII III] IIIII III] IIIIII ← 2]6 8s

Using a calculator to predict the result the remaining is bracketed before counted in 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration means subtracting before dividing, as shown in the gradient formula $y' = \Delta y/t = (y_2 - y_1)/t$.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Primary schools use ten-counting only
In primary school numbers are counted in tens to be added, subtracted, multiplied and divided. This leads to questions as ‘3 4s = ? tens’. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount- and restack-formula above is impossible since the calculator has no ten button. Instead it is programmed to give the answer directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

3 * 4	12

Recounting icon-numbers in tens is called doing times tables to be learned by heart. So from grade 1, 3*4 is not 3 4s any more but has to be recounted in tens as 1.2 tens, or 12 in the short form.

Recounting tens in icons by asking ‘38 = ? 7s’ is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens, 0.3 7s must be written in fraction form as 3/7 and calculated as 0.428…, shown directly by the calculator, 38/7 = 5.428…

38 / 7	5.some
38 – 5 * 7	3

Without recounting, primary school labels the problem ‘38 = ? 7s’ as an example of a division, 38/7, which is hard to many, or as an equation ’38 = x*7’ to be postponed to secondary school.

Designing a micro-curriculum

With curriculum architecture as one of its core activities, the MATHeCADEMY.net was asked to design a micro-curriculum understandable and attractive to teachers stuck with division problems; and allowing special need students to return to their ordinary class. Two were designed.

In the ‘1 cup and 5 sticks’ micro-curriculum, 5 is cup-counted in 2s as 1]3 2s or 2]1 2s or 3]-1 2s to show that a total can be counted in 3 ways: overload, normal or underload with an inside and an outside for the bundles and singles. So to divide 336 by 7, 5 bundles are moved outside as 50 singles to recount 336 with an overload: 336 = 33]6 = 28]56, which divided by 7 gives 4]8 = 48.

Besides the ‘Cure Math Dislike by 1 cup and 5 sticks’, 8 extra micro-curricula were designed (mathecademy.net/preschool/) where cup-counting involves division, multiplication, subtraction and later next-to and on-top addition, in contrast to primary school that turns this order around and only allows on-top addition using carrying instead of overloads. Thus, if using cup-writing with overloads or underload instead of carrying, the order of operations can be turned around to respect that totals must be counted before being added.

	Carry	Bundle-writing	Words
Add	1 4 5 <u>1 7</u> 6 2	4]5 <u>1]7</u> 5]12 6]2 = 62	4 ten 5 <u>1 ten 7</u> 5 ten 12 5 ten 1 ten 2 6 ten 2 = 62
Subtract	1 4 5 <u>1 7</u> 2 8	4]5 <u>1]7</u> 3]-2 2]10-2 = 2]8 = 28	4 ten 5 <u>1 ten 7</u> 3 ten less2 2 ten 8 = 28
Multiply	4 <u>2 6 * 7</u> 1 8 2	7 * 2]6 14]42 18] 2 = 182	7 times 2 ten 6 14 ten 42 14 ten 4 ten 2 18 ten 2 = 182
Divide	<u>2 4 rest 1</u> 3 | 7 3 <u>6</u> 1 3 <u>1 2</u> 1	7]3 counted in 3s 6]13 6]12 + 1 2 3s]4 3s + 1 24 3s + 1 73 = 24*3 + 1	7ten3 6ten 13 6ten12 + 1 3 times 2ten4 + 1 3 times 24 + 1

In the first micro-curriculum the learner uses sticks and a folding rule to build the number-icons up to nine; and uses strokes to draw them thus realizing there are as many sticks and strokes in the icon as the number it represents, if written less sloppy. In the second the learner counts a given total in icons by bundling sticks and using a cup for the bundles; and reporting first with cup-writing and decimal numbers with a unit; then by using an abacus in algebra and geometry mode. In the third the learner recounts a total in the same unit thus experiencing creating or removing overloads and underloads. In the fourth the learner recounts a total in a different unit. In the fifth the learner adds two icon-numbers on-top of each other. In the sixth the learner adds two icon-numbers next-to each other. In the seventh the learner reverses on-top addition. And in the eights, the learner reverses next-to addition. Finally, the learner sees how double-counting creates per-numbers.

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2]1 3s → 2.1 3s	7/3 7 – 2*3	2.some 1
M3	'2.7 5s is also how many 5s?' IIIII IIIII IIIIIII = V V V II = V V V V <u>III</u> 2]7 = 2+1]7–5 = 3]2 = 3+1]2–5 = 4]-3 So 2.7 5s = 3.2 5s = 4.-3 5s	(2*5+7)/5 (2*5+7) – 3*5 (2*5+7) – 4*5	3.some 2 -3
M4	2 5s is how many 4s?' IIIII IIIII = IIII I IIII I = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2*4	2.some 2
M5	'2 5s and 4 3s total how many 5s?' IIIII IIIII III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2
M6	'2 5s and 4 3s total how many 8s?' IIIII IIIII III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 (2*5+4*3) – 2*8	2.some 6
M7	'2 5s and ? 3s total 4 5s?' IIIII IIIII IIIII IIIII = IIIII IIIII III III III I so 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*5	3.some 1
M8	'2 5s and ? 3s total how 2.1 8s?' IIIIIIII IIIIIIII I = IIIII III IIIII III I so 2 5s + 2.1 3s = 2.1 8s	(2*8+1 – 2*5)/3 (2*8+1 – 2*5) – 2*8	2.some 1

One curriculum used silent education where the teacher demonstrates and guides through actions only, not using words; and one curriculum was carried out by a substitute teacher speaking a foreign language not understood by the learner. In both cases the abacus and the calculator quickly took over the communication. For further details watch www.youtube.com/watch?v=IE5nk2YEQIA.

After these micro-curricula a learner went back to her grade 6 class where proportionality created learning problems. The learner suggested renaming it to double-counting but the teacher insisted on following the textbook. However, observing that the class took over the double-counting method, he finally gave in and allowed proportionality to be renamed and treated as double-counting. When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integration.

Thus cup-counting and a calculator for predicting recounting results allowed the learner to reach the outside goal, mastering Many, by following an alternative to the institutionalized means that because of a goal-means exchange had become a stumbling block to her; and performing and reversing next-to addition introduced her to and prepared her for later calculus classes.

Literature on cup-counting

No research literature on cup-counting was found. Likewise, it is not mentioned by Dienes (1964).

Conclusion and recommendation

As to theory, two genres exist; a master genre exemplifying existing theory, and a research genre developing new theory by including a question and a theoretical guidance to a valid answer based upon analyzing reliable data. To avoid indifference, this paper addresses the OECD report ‘Improving schools in Sweden’ by asking if mathematics education might have a goal-means exchange. As theoretical guidance, institutional skepticism allows using the existentialist existence-versus-essence distinction to distinguish outside goals from inside means, which leads to asking when mathematics is respectively existence and essence. Analyzing traditional math shows that by being set-based and by adding numbers without units, its concepts and statements are unrooted and little applicable to the outside world, thus being primarily essence. Then grounded theory helps showing how mathematics looks like if grounded in its physical root, Many. To tell the difference, two names are coined, ‘ManyMatics’ versus ‘MetaMatism’ mixing ‘MetaMatics’ defining concepts as examples of abstractions instead of as abstractions from examples, with ‘MatheMatism’ valid only inside classrooms. To validate its findings and again to avoid indifference, the paper includes a classroom test of a micro curriculum described in details to allow it to be tested in other classrooms. Its originality should welcome the paper for publishing since no literature on ManyMatics exists.

So, if a research conference fails to accept the paper for presentation or as a poster, an extra exchange can be added to help solving the paradox that the Swedish problems occur despite increased research and funding: Neglecting a genre analysis might exchange the master and research genres with the consequence that peer-review becomes unable to accept groundbreaking new paradigms. Such research conferences include master papers that, although career promoting, are unable to uncover alternative, hidden ways to guide solving problems in mathematics education.

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04. Fifty Years of Research without Improving Mathematics Education, Why?

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by the PISA studies organised by the Organisation for Economic Co-operation and Development, OECD, showing a low level and a continuing decline in many countries. Thus, to help the former model country Sweden, OECD wrote a critical 2015 report 'Improving Schools in Sweden, an OECD Perspective': "PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life."

Researchers in mathematics education meet in different fora. On a world basis, the International Congress on Mathematical Education, ICME, meets each four year. And on a European basis, the Congress of the European Society for Research in Mathematics Education, CERME, meets each second year.

At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is "Only one: to improve, mathematics education should ask, not what to do, but what to do differently." Thus, to be successful, research should not study problems but look for hidden differences that might make a difference. Research that is skeptical towards institutionalized traditions could be called difference research or contingency research or Cinderella research making the prince dance by looking for hidden alternatives outside the ruling tradition. The French thinker Lyotard calls it 'paralogy' inventing dissension to the reigning consensus. Difference research scarcely exists today since it is rejected at conferences for not applying or extending existing theory that is able to produce new researchers and to feed a growing research industry, but unable to reach its goal, to improve mathematics education.

To elaborate, mathematics education research is sterile because its three words are not well defined.

As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist: a continental European education serving the nation's need for public servants though multi-year compulsory classes and lines at the secondary and tertiary level; and a North American education aiming at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers.

As to research, academic articles can be written at a master level applying or exemplifying existing theories, or at a research level questioning them. Just following ruling theories is especially problematic in the case of conflicting theory as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible respectively.

Consequently, you cannot know what kind of mathematics and what kind of education has been studied, and you cannot know if research is following ruling traditions or searching for new discoveries. So, seeing education as an institutional help to children and youngsters master outside phenomena leads to the question: What outside phenomena roots mathematics?

The Outside Roots of Mathematics

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep balance and to store sounds assigned to what we grasped with our forelegs, thus providing the holes in the head with our two basic needs, food for the body and information for the brain.

The sounds developed into languages. In fact, we have two languages, a word-language and a number-language. Children learn to talk and to count at home. Then, as an institution, school takes over and teaches children to read and to write and to calculate.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’. Observing the existence of many chairs, we ask ‘how many totally?’ and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, ‘the total is 3 chairs’ or, if counting legs, ‘the total is 3 fours’, which we abbreviate to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’.

And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

And since we master outside phenomena through actions, learning the word-language means learning actions as how to listen, to read, to write and to speak. Likewise, learning the number-language means learning actions as how to count and to add. You cannot learn how to math, since math is not an action word, it is a label as is grammar. Thus, mathematics may be seen as the grammar of the number-language.

Using the phrasing ‘the number-language is an application of mathematics’ implies that then ‘of course mathematics must be taught and learned before it can be applied’. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

So, one way of improving mathematics education is to respect that language comes before meta-language. Which was also the case in continental Europe before the arrival of the ‘New Math’ that turned mathematics upside down to become a ‘meta-matics’ presenting its concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically and which would present mathematics as ‘many-matics’, a natural science about Many.

Thus, Euler defined a function as a common name for calculations with unspecified numbers, in contrast to calculations without that could be calculated right away without awaiting numbers to be specified. Defining all concepts as examples of the mother concept set, New Math turned a function into an example of a set-product where first-component identity implies second-component identity, which learners heard as ‘bublibub is an example of bablibab’.

Before New Math, Germanic countries taught counting and reckoning in primary school. Then the lower secondary school taught algebra and geometry, which are also action words meaning to reunite totals and to measure earth in Arabic and in Greek. 50 years ago, New Math made all these activities disappear. This means that what research has studied is problems coming from teaching how to math. So, one alternative presents itself immediately: Forget about New Math and, once again, teach mathematics as rooted in numbers and reckoning and reuniting totals and measuring earth.

Re-rooting mathematics resonates with its historic origin as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many.

As to New Math, its idea of deriving definitions from the mother concept set leads to meaningless self-reference as in the classical liar paradox 'This sentence is false', being true if false and false if true. This was shown by Russell looking at the set of sets not belonging to itself. Here a set belongs to the set if it doesn't, and does not belong if it does.

To avoid self-reference, Russell created a hierarchical type theory in which fractions could not be numbers if defined by numbers as done by New Math defining fractions as equivalence classes in a set of number-pairs. Insisting that fractions are numbers, New Math invented a new set-theory that by mixing sets and elements also mixes concrete examples and their abstract names, thus mixing concrete apples that can feed humans and the word 'apple' that cannot. By mixing things and their names, New Math and its meta-matics ceases to be a language about the real world. Still, it has entered universities worldwide as the only true version of mathematics.

So, to improve its education, mathematics should stop teaching top-down meta-matics from above and begin teaching bottom-up many-matics from below instead.

Rethinking Mathematics from Below

To improve it we must rethink mathematics. To rethink we seek guidance by one of the greatest thinkers of the 20th century, Heidegger, being very influential within existentialist thinking and French skeptical post-structural thinking.

Heidegger holds that to exist fully means to establish an authentic relationship to the things around us. To allow a thing to open its 'Wesen' and escape its gossip-prison created by reigning essence-claims we must use constant questioning. So, returning to the fundamental goal of education, preparing humans for what is outside, we must keep on asking to the Wesen of the root of the number language, the physical fact Many, and allow Many to escape from its New Math gossip, 'Gerede'.

With 2017 as the 500year anniversary for Luther's 95 theses, we can describe meeting Many in theses.

1. Using a folding ruler we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent.

2. Using a cup for the bundles we discover that a total can be 'cup-counted' in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and 'less 1' outside; or, if using 'cup-writing' to report cup-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s. We discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s.

On a folding ruler, distances are counted in tens. Here one centimeter is a bundle of ten millimeters, and ten centimeters gives a bundle of one decimeter. If the length of a hand is counted to 6 strokes after 1.7 tens, we write the length as T = 1.76 tens centimeters = 17.6 centimeters leaving the 6 unbundled millimeters outside.

3. Using recounting a total in the same unit by creating or removing overloads or underloads, we discover that bundle-writing offers an alternative way to perform and write down operations:

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92 ; and T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336 ; and T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

4. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved in counting by bundling and stacking. To count 7 in 3s we take away 3 many times iconized by an uphill stroke showing the broom wiping away the 3s. Showing

7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, dividing and subtracting a multiple, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate e.g. T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.

5. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. A ‘recount formula’ T = (T/B)*B saying that T/B times B can be taken away from T, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and a ‘restack formula’ T = (T–B)+B saying that T–B is left when B is taken away from T and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Thus we discover the nature of formulas: formulas predict.

6. Wanting to recount a total in a new unit, we discover that again a calculator can predict the result by bundling and stacking and dragging away the stack:

T = 4 5s = ? 6s. First (4*5)/6 = 3.some. Then (4*5) – (3*6) = 2. Finally T = 4 5s = 3.2 6s

Also, we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.

7. Wanting to recount a total in tens, we discover that a calculator can predict the result directly by multiplication. Only, the calculator leaves out the unit and misplaces the decimal point:

T = 3 7s = ? tens. Answer: T = 21 = 2.1 tens

Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.

And wanting to recount a total from tens to icons, we discover that this again is an example of recounting to change the unit:

T = 3 tens = ? 7s. First 30/7 = 4.some. Then 30 – (4*7) = 2. Finally T = 30 = 4.2 7s

Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged

8. Using the letter u for an unknown number, we can rewrite recounting from tens, e.g. 3 tens = ? 7s, as 30 = u*7 with the answer 30/7 = u. Officially this is called to solve an equation, so here we discover a natural way to do so: Move a number to the opposite side with the opposite sign. The equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, thus predicted by the restack-formula as 8 = (8–2)+2. Thus, the equation 8 = u + 2 has the solution is 8–2 = u, again moving a number to the opposite side with the opposite sign.

9. Once counted, totals can be added. But we discover that addition is not well defined. With two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to get the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. Thus next-to addition geometrically means to add areas, and algebraically it means to combine multiplication and addition. Officially this is called integration, a core part of traditional mathematics in high school and at the first year of university.

10. Also we discover that addition can be reversed. Thus, the equation above restacking 8 by moving 2, 8 = u+2, can also be read as reversed addition: u is the number that added to 2 gives 8,

which is precisely the formal definition of u = 8–2. So, we discover that subtraction is reversed addition. And, again we see that the equation u+2 = 8 is solved by u = 8–2, i.e. by moving to the opposite side with the opposite sign. Likewise, the equation recounting 8 in 2s, 8 = u*2, can be read as reversed multiplication: u is the number that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2? So, we discover that division is reversed multiplication. And, again we see that the equation u*2 = 8 is solved by u = 8/2, i.e. by moving to the opposite side with the opposite sign. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root and logarithm as u = 3√20 and u = log3(20). So, again we solve the equations by moving to the opposite side with the opposite sign. Reversing next-to addition, we can ask e.g. 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer, first we remove the initial total T1, then we count the rest in 5s: u = (T–T1)/5. Combining subtraction and division in this way is called differentiation. By observing that this is reversing multiplication and addition we discover that differentiation is reversed integration.

11. Observing that many physical quantities are ‘double-counted’ in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of ‘per-numbers’ serving as a bridge between the two units. Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

12. Observing that a quantity may be double-counted in the same unit, we discover that per-numbers may take the form of fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. Thus, to find 3 per 5 of 20, 3/5 of 20, we just recount 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3. To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. Thus, calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

Writing out a total T as we say it, T = 345 = 3*ten*ten + 4*ten + 5*1, shows a number as blocks united next-to each other. Also, we see algebra’s four ways to unite numbers: addition, multiplication, repeated multiplication or power, and block-addition also called integration. Which is precisely the core of mathematics: addition and multiplication together with their reversed operations subtraction and division in primary school; and power and integration together with their reversed operations root, logarithm and differentiation in secondary school. Including the units, we see there can only be four ways to unite numbers: addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers.

How School Teaches Mathematics

Before addressing how school guides children on their way to mastering Many let us look at the number-language children bring to school. Asking a three-year old child "how old will you next time?" the answer is four with four fingers shown. But displaying four fingers held together two and two will prompt an immediate protest: "No, that is not four, that is two twos!"

So, children come to school with two-dimensional ‘block-numbers’ all carrying a unit, corresponding to LEGO-bricks that stack as 1, 2, 3 or more 4s. Thus, by combining geometry and algebra in their shapes and knobs, they are an excellent basis for connecting the starting point, children's block-numbers, with the final goal, the Arabic numbers also being a collection of blocks of 1s, tens, ten-tens etc.

To emphasize that we count by bundling and stacking, the school could tell children that eleven and twelve is a special ‘Viking-way’ to say ten-1 and ten-2. Then they probably would count ‘2ten9, 3ten, 3ten1’ instead of saying ‘ten-and-twenty’ and risk being diagnosed with dyscalculia. In Danish, eleven and twelve mean ‘one left’ and ‘two left’, implying that the ten-bundle has been counted already. And, except from some French additions because of the Norman conquest, English is basically Anglish, a dialect from Harboøre on the Danish west coast where the ships left for Angland.

Now let us see how school prepare children and youngsters to meet Many by offering them what is called mathematics education. Again, we use the form of theses.

1. School could respect the origin of the word mathematics as a mere name for algebra and geometry both grounded in the physical fact Many and created to go hand in hand. Instead, school teaches mathematics as a self-referring ‘meta-matics’ defining concepts as examples of abstractions, and not as abstractions from examples. Likewise, school teaches algebra and geometry separately.

2. School could respect that a digit is an icon containing as many sticks as it represents. Instead, school presents numbers as symbols like letters. Seldom it tells why ten does not have an icon or why ten is written as 10; and seldom it tells why ten1 and ten2 is called eleven and twelve.

3. School could follow the word-language and use full sentences ‘The total is 3 4s or T = 3 4s or T = 3*4’. Instead, by only saying ‘3’, school removes both the subject and the unit from number-language sentence, thus indicating that what children should learn is not a number-language but a one-dimensional number system claimed to be useful later when meeting life’s two-dimensional numbers.

4. School could develop the two-dimensional block-numbers children bring to school and are supposed to leave school with. Instead, school teaches its one-dimensional line-numbers as names for the points along a number line, using a place-value system. Seldom numbers are written out as we say them with the unit ones, ten, ten-tens, etc. Seldom a three-digit number is taught as a short way to report three countings: of ones, of bundles, and of bundles of bundles. Seldom tens is called bundles; seldom hundreds is called ten-tens or bundles of bundles.

5. School could respect that a number is a horizontal union of vertical blocks of 1s, bundles, bundles of bundles etc., and that counting-on means going up one step in the 1-block until we reach the bundle level where a bundle of 1s is transformed into 1 extra bundle making the bundle block go up 1 while the 1-block falls back to zero; and school could respect that a natural number is a decimal number with a unit. Instead school represents numbers by a horizontal number-line, where counting-on means moving one step to the right and where a natural number is presented without unit and with a misplaced decimal point.

6. School could respect that totals must be counted and sometimes recounted in a different unit before being added. Instead, without first teaching counting, school teaches addition from the beginning regardless of units, thus transforming addition to mere counting-on. Seldom school teaches real on-top and next-to addition respecting the units.

7. School could respect that also operations are icons showing the three basic counting activities: division as bundling, multiplication as stacking the bundles, and subtraction as removing the stack to look for unbundled singles; and school could respect the natural order of operations: division before multiplication before subtraction before addition. Instead school reverses this order without respecting that addition has two meanings, on-top and next-to, or that division has two meanings, counted in and split between.

8. School could respect that 3*8 means 3 8s that may or may not be recounted in tens. Instead school insists the 3*8 IS 24 and asks children to learn the multiplication tables by heart. Seldom the geometrical understanding is included showing that recounting in tens means the stack increases its width and therefore must decrease its height to leave the total unchanged.

9. School could respect that basic calculations become understandable by recounting a total in the same unit to create or remove over- or underloads. Instead school does not allow over- and underloads and insists on using specific algorithms with a carry-technique.

10. School could respect that proportionality is just another word for per-numbers coming from double-counting, and that per-numbers are operators that need a number to become a number. Instead school renames per-numbers to fractions, percentages and decimal numbers and teach them as numbers that can be added without considering the unit, and teaches proportionality as an example of a linear function, which isn't linear since the b in y = a*x+b makes it an affine function instead.

11. School could respect that equations are just another name for reversed calculation rooted in recounting tens in icons and solved by moving to the opposite side with the opposite sign. Instead school teaches equations as statements expressing equivalence between two different number-names to be solved by performing the same operation to both sides aiming at using the laws of abstract algebra to neutralize the numbers next to the unknown.

12. School could respect that integrating means adding non-constant per-numbers to be taught in primary school as next-to addition of block-numbers, and in middle school as mixture tasks; and respect that reversed integration is called differentiation made relevant since adding many differences boils down to one single difference between the end- and start-number. Instead school neglects primary and middle school calculus; and it teaches differentiation before integration, that is reduced to finding an antiderivative to the formula to be integrated. Seldom continuity and differentiability are introduced as formal names for local constancy and local linearity. Seldom the units are included to make clear that per-numbers are integrated, and that differentiation cerates per-numbers.

How School Could Teach Mathematics

Seeing the goal of mathematics education as preparing students for meeting Many, doing so in a Heideggerian gossip-free space offers many differences to be tried out and studied. Again, we use a list form.

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to teach bundle-counting, recounting in the same and in a different unit, calculator prediction, on-top and next-to addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling and tripling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to Here a difference is to talk about 8/2 as ‘8 counted in 2s’ instead of as ‘8 divided between 2’; and to rewrite the number as ‘10 or 5 times less something’ and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 33-5B6+50 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of ‘mathe-matism’ true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes ‘middle school integration’ later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of reverse operations to establish the basic rule for solving equations ‘move to the opposite side with the opposite sign’: In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn’t have a reverse sign. This gives 3*u = 14 – 2. Since u doesn’t have a reverse sign, 3 moves to the other

side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. 'transform the formula T = a/(b-c) so that all letters become subjects.' A hymn can be created: "Equations are the best we know / they're solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don't want to stop."

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to 'pre-dict' a number, instead statistics can 'post-dict' its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5^{th} root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times

gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio Δy/Δx can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children's own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched, and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden

learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format.

The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial:

T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, with at least two alternative meanings for all three words, at least 2*2*2 i.e. 8 different forms of mathematics education research exist. The past 50 years has shown the little use of the present form applying theory to study meta-matics taught in compulsory multi-year classes or lines. So, one alternative presents itself directly as an alternative for future studies: to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many, and to teach it in self-chosen half-year block at the secondary and tertiary level; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining and trying to understand the misery coming from teaching meta-matism in compulsory classes. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research must search for differences and test if they make a difference, not in compulsory classes, but with daily lessons in self-chosen half-year blocks. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a sentence, the subject exists, but the sentence about it may be gossip; so, stop teaching essence and start experiencing existence.

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05. A 1year pre-engineer course for Young migrants, a job for critical or civilized math education

UN population forecasts predict a continuing migrant flow to Europe to benefit from its socialist welfare and educational systems. But a critical question could ask: Is European education ready to benefit from the engineer potential in young migrants allowing them to build up welfare and education in their own country? Is critical socialist thinking able to reform its European line-organized office directed education dating back to the Napoleon wars? A recent OECD report saying that Sweden should urgently reform its school system to improve quality and equity suggests that a solution might instead be provided by the civilized thinking of the North American Enlightenment republics, historically created to receive and integrate migrants through its half-year block-organized talent developing education.

Background and question

According to the numbers of hours spend there, education is by far the most extensive public intervention in private life; and with the basic human need for a word- and a number-language for communication, mathematics is one of its core subjects. Consequently, research in math education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Centre for Mathematics Education in Sweden that with its positive attitude to receiving male migrants now beats China with 123 boys/ 100 girls of the 16-17 years old. However, despite increased research and funding, Sweden has seen its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change' (OECD, 2015).

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life." (p. 3)

In the report OECD writes

> Sweden has the highest percentage of students arriving late for school among all OECD countries, especially among socio-economically disadvantaged and immigrant students, and the lack of punctuality has increased between 2003 and 2012. There is also a higher-than-average percentage of students in Sweden who skip classes, in particular among disadvantaged and immigrant students. Arriving late for school and skipping classes are associated highly negatively with mathematics performance in PISA and can have serious adverse effects on the lives of young people, as they can cut into school learning and also distract other students. (p. 69) The reforms of recent years are important, but evidence suggests they are also somewhat piecemeal, and simply too few, considering the serious situation of the Swedish school system. (p. 55) Sweden faces a serious deterioration in the quality and status of the teaching profession that requires immediate system-wide attention. This can only be accomplished by building capacity for teaching and learning through a long-term human resource strategy for the school sector. (p. 112)

Inspired by the OECD report we can ask: How to improve mathematics and its education to better serve the population and migrants? And more specifically: How to design a 1year pre-engineer course for young migrants beginning from scratch?

Critical and civilized thinking provide two kinds of answers.

Critical and civilized thinking

As to the content of critical thinking, the Oxford Dictionary of Philosophy writes:

> The title is specifically applied to the philosophical approach of the Frankfurt school. This owed its philosophical background to Hegel and to Marx, seeing social and cultural imperfections as defects of rationality, and comparing them with an ideal to which the progress of reason, embodied in pure and undistorting social arrangements, would ideally tend (pp. 88-89)

Civilized thinking mixes existentialism, seeing existence as preceding essence, with the thinking of the two Enlightenment republics, American pragmatism being skeptical towards any philosophical is-claim, and French post-structuralism warning against hidden patronization in choices presented as nature. But to more clearly see the difference between the two we need to go back in history.

A historical background

The distance from its energy source allows water in all three forms: solid, liquid and gas. Thus a continuous flow of incoming high order energy from the sun and outgoing low order waste energy to space during the night allows green cells to store energy to be exploited by grey cells coming in three forms: reptiles, mammals and humans. That by standing up allowed the brain to develop language by remembering sounds given to what the forelegs transformed to hands was grasping. Thus meeting the two fundamental needs shown by the holes in the head: to supply the stomach and the lungs and the brain with food and oxygen and information.

When humans left Africa some went east to the fertile river valleys, some went west to the mountains. Trade took place exchanging eastern silk and pepper with western silver. Its silver mines allowed ancient Greece to develop a culture where men could leave the daily routine work to women and slaves to discuss social matters as 'can adults live together on equal terms or is patronization needed as with children?'

Social theory thus has human interaction as its main focus. As to communication, the most basic interaction, Berne (1964) has developed a transactional analysis describing three different ego-states called Parent and Adult and Child to reflect the social fact that human interaction can be patronized and non-democratic, or it can be non-patronized and democratic. In a family the interaction between children and parents will typically be one of patronization. In a society adult interaction typically will be non-patronized, unless the society is a non-democratic autocracy where patronization is carried on into adulthood. In this way Berne describes the main problem in human interaction, the choice between patronization and self-determination or 'Mündigkeit'. The fact that the German word 'Mündigkeit' does not have an English equivalent indicates that social interaction is quite different outside continental Europe and inside where the presence of and resistance against patronization created the label 'Mündigkeit'.

The debate on patronization runs all the way though the history of social theory (Russell, 1945). In ancient Greece the sophists warned against hidden patronization coming from choices presented as nature. Hence to practice the three ingredients of a democracy, information and debate and decision, a population should be enlightened to tell choice from nature. Seeing the physical as examples of metaphysical forms only visible to philosophers from his academy, Plato labelled democratic debate as ignorance. Instead social power should be given to the philosophers who could make wise decisions based upon information coming from insight and knowledge, thus needing neither debate nor democracy. In this way Plato instituted the patronization that Foucault calls 'pastoral power' to be continued first by the Christian church and later by modern universities still using the scholastic research method of only allowing late opponents to already defended texts to be accepted as researchers.

The Greek silver mines lasted about hundred years. Then the Romans took over, financing their empire by silver mines in Spain, eventually captured by the Vandals and by the Arabs. The lack of silver made Europe descend into the dark Middle Age. Here the patronization question reappeared in the controversy on universals between the realists and the nominalists. The realist took the Plato standpoint by renaming his metaphysical forms to universals claimed to have independent existence and to be exemplified in the physical world, and consequently waiting to be discovered by philosophers. In contrast to this the nominalist saw universals as names invented to facilitate human interaction.

Then German silver transported to Italy reopened east-west trade financing the Renaissance, seeing a protestant uprising against the patronization of the Roman Catholic Church resulting in the bloody

30year war from 1618. To avoid the chaos of war, Hobbes in his book ‘Leviathan’ argued that to protect themselves against their natural egoistic state, humans would have a much better life if accepting the patronization of an autocratic monarch.

Seeing the laboratory as preceding the library, Brahe retrieved data for the motion of planets, which together with Kepler’s interpretation allowed Newton to discover that the moon doesn’t move among the stars, instead it falls towards the earth as does the apple, both following their own physical will and not the will of a metaphysical patronizor. This inspired Locke to argue against patronization. His chief work, the ‘An Essay Concerning Human Understanding’, was highly inspirational in the Enlightenment 1700-century, which resulted in two democracies being installed, one in the US and one in France. American sociology sees human interaction as based upon enlightenment and freed from patronization. Its ‘it is true if it works’ pragmatism expressed by Peirce and James leads on to symbolic interactionism and to the natural observation rooted research paradigm Grounded Theory resonating with the principles of natural learning expressed by Piaget. In harmony with this, the US enlightenment school, being organized in half-year blocks and aiming at developing the talent of the individual has set the international standard followed worldwide outside Europe.

Inside Europe counter-enlightenment came from Germany where Hegel reinstalled metaphysical patronization in the form of a Spirit expressing itself through the history of the people. Trying to end the French Republic by war resulted in French occupation of Berlin. To get Napoleon out, the king realized that as the French he could no more depend on the blood nobility. So he asked Humboldt to use Hegel to design a line-organized Bildung education with three goals: Bildung must not enlighten to keep the population from demanding democracy as in France; instead, by imposing upon it a feeling of nationalism, Bildung should transform the population into a people following the will of the Spirit by fighting other people, especially the French. Finally Bildung should use the Sprit expressing itself in romanticism to sort out a knowledge nobility among the people for a central administration (Berglar, 1970).

Opposing Hegel, Nietzsche argued that only by freeing itself from metaphysical philosophical hegemony, western individuals would be able to realize their full potentials. Following Hegel, Marx claimed that until a socialist utopia has been established, a socialist party serving the interest of the working people should patronize people through a dictatorship of the proletariat. Once in power, Hegel-based socialism saw no reason to replace the Hegel-based counter-enlightening line-organized education with the enlightening block- organized education of the American republics. Marxist thinking developed into the critical theory of the Frankfurter school infiltrating the 1968 student revolt to secure that Europe’s Bildung education could carry on its Hegel-based patronization.

Today, the sophist warning against unrooted is-claims is carried on by the existentialism of Kierkegaard and Nietzsche and Heidegger and Sartre, defining existentialism as holding that ‘existence precedes essence, or (..) that subjectivity must be the starting point’ (Marino, 2004: 344); and by French post- structuralism with Derrida and Lyotard and Foucault and Bourdieu showing skepticism towards hidden patronization in our most fundamental institutions: words, correctness, cures and education (Lyotard, 1984), (Tarp, 2004, 2). Foucault thus says:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006: 41)

In Germany, Arendt carried his Heidegger’s work further by dividing human activity into labor and work focusing on the private sphere and action focusing on the political sphere thus accepting as the first philosopher political action as a worthy human activity creating institutions that should be treated with care to avoid ‘the banality of evil’ if turning totalitarian by the sheer banality of just following orders (Arendt, 1963). Likewise, Bauman points out that by following authorized routines modernity can create both gas turbines and gas chambers (Baumann, 1989).

As to their meanings, the word ‘critical’ comes from Greek ‘kritike’ meaning to pass judgement; and civilized comes from latin ‘civis’ meaning a free citizen. So civilized thinking means republican thinking always being skeptical towards false is-claims; and critical thinking means passing judgements; but to pass a judgement you must be elected judge by a democratic process, or have insight in the difference between right and wrong as e.g. believing in the Hegel assumption that instead of being free to create their own history, humans are puppets on a string playing out the manuscript of the Spirit. So basically the contradiction between critical and civilized thinking is a replay of the ancient controversy between the Greek philosophers and sophists.

Critical versus civilized mathematics education

The difference between critical and civilized mathematics education is seen in a paper describing how to deal with teaching and learning problems in a Brazilian math class (Tarp, 2004, 1)

> In Brazil there is a research group, which has focused on issues related to new technologies and mathematics education. This research group has developed software and work with students at different levels and with teachers. A teacher from a nearby school approached the group (..) From the teacher perspective, she had some tough problems to face and she foresaw that the computers would be able to help her. The teacher was teaching a class of 5th graders, which in her view was really problematic. The kids were older (15 years average) than the expected age for this grade: 11. The kids felt humiliated somehow as they were put in a school with kids much younger than them and they had flunked many times, and at several instances they had to repeat all the subjects of a given school year because their ‘failure in mathematics’. The students transformed this humiliation into violence in class. The teacher was in fact considering the possibility of just quitting the job since she could not work with those kids in a way she found effective. (..) The teacher was enthusiastic about a software, which deals with rational numbers. (..) both researchers and teacher had the ‘intuition’ that the computer might have a positive effect in this class and maybe could avoid that the students had to repeat this grade again. (Sec. 2, par. 2-4)

The teacher is supposed to teach rational numbers to a class with a mixture of 11year old students and 15year old repeaters having given up rational numbers and turning to violence. The research group could have asked critical questions as ‘is rational numbers defined from below as an abstraction from concrete examples or from above as an example of an abstraction?’ and ‘ why teach addition when it is meaningless to add fractions without units?’ Instead the group uncritically assumed ‘that the computer might have a positive effect’.

The paper also describes how civilized thinking would work differently:

> The research group is working halftime in classrooms and halftime at the university. It focuses on the concerns of typical classrooms as expressed by students, teachers in their stories of complaints. The teacher complains about the violence in the class tempting her to quit the job since she cannot work in a way she finds effective. And the students complain about having to repeat the class because they don’t want to learn about fractions, since the teacher by just echoing the textbook is unable to explain to the students, why they shall learn fractions, and what they are useful for. Asked to comment this, the teacher says that mathematics education means education in mathematics, and since rational numbers is part of the mathematics textbook it must be taught and learned. Mathematics is difficult to learn, so the students have to work harder, or be supported by computers. Hence the problems will not disappear before schools can afford computers, or the students decide to become more engaged in mathematics.
>
> Based upon the motto “echo-phrasing is freezing, re-phrasing is freeing” postmodern thinking sees modern institutions frozen in echo-phrasings, that have to be discovered and rephrased. Since the teacher is echoing the textbook, the echoes can be found here. The textbook presents fractions as examples of rational numbers, being example of number sets, being examples of sets. This is the typical way of presentation within modern set-based mathematics explaining concepts as examples of more abstract concepts. This phrasing conflicts with the student demand for explanations relating fractions to their use.
>
> So instead of developing software to supplement, and thus support the existing top-down echo-phrasing of fractions, the group begins to look for alternative bottom-up approaches in journals, other textbooks, other countries, and in other time periods. Also they use their imagination by accessing the silent part of their ‘knowledge-iceberg’ developed through years of classrooms experience as mathematics educators. Using curriculum architecture they design examples of micro curricula, where fractions emerges from dividing

problems, that can be introduced into the ordinary classroom as e.g. games, where students work in pairs throwing dices and splitting the profit, or loss, proportional to their stakes shown by their dice-numbers.

This ‘proportional splitting’ approach leads to (and thus shows the authenticity and necessity of) fractions, and multiplication of fractions and integers. (Sec. 5, par. 2-5)

So where critical thinking shows no criticism towards the actual mathematics education tradition, civilized thinking asks if this tradition is nature or choice presented as nature and thus hiding alternatives.

Criticizing and civilizing rational numbers

In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, (Freudenthal, 1973) both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite Many’ in Arabic.

Meeting Many we ask ‘how many?’ Counting and adding gives the answer. We count by bundling and stacking as seen when writing a total T in its block form: T = 354 = 3·B^2 + 5·B + 4·1 where the bundle B is ten typically. This illustrates the four ways to unite: On-top addition unites variable numbers, multiplication unites constant numbers, power unites constant factors and per-numbers, and next-to addition, also called integration, unites variable blocks. As indicated by its name, uniting can be reversed to split a total into parts predicted by the reversed operations: subtraction, division, root & logarithm and differentiation.

Operations **unite**/*split* Totals in	Variable	Constant
Unit-numbers m, s, kg, $	**T = a + b** *T – b = a*	**T = a·b** *T/b = a*
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫a·db** *dT/db = a*	**T = a^b** $\sqrt[b]{T} = a \quad \log_a T = b$

Although presented as nature, ten-bundling is a choice. Bundling Many in a ‘icon-bundles’ less than ten means asking e.g. ‘T = 7 = ? 4s’. The answer is predicted on a calculator by two formulas, a recount-formula ‘T = (T/B)·B’ telling that from a total T, T/b times B s can be taken away, and a restack-formula ‘T = (T–B)+B’ telling that from a total T, T–B is left when B is taken away and placed next-to. First T = 7/4 gives 1.some. Then T = 7 – 1·4 leaves 3. So the prediction is T = 7 = 1 4s & 3 = 1.3 4s = 1 ¾ 4s. Thus with icon-counting, a natural number is a decimal number with a unit where the decimal point separates singles from bundles (Tarp, 2016)

Double-counting physical units creates per-numbers as 3$/4kg. With this, units can be changed by recounting $s in 3s or kgs in 4s: 15$ = (15/3)·3$ = (15/3)·4kg = 20kg. So as per-numbers, fractions are not numbers, but operators, needing a number to become a number. To add, per-numbers must be multiplied to unit-numbers, thus adding as areas, called integration: ½ of 4 + 2/3 of 3 = (1/2*4 + 2/3*3) of (4+3) = 4 of 7.

The root of geometry is the standard form, a rectangle, that halved by a diagonal becomes two right-angled triangles with sides and angles connected by three laws, A+B+C = 180, a^2+b^2 = c^2 and tanA = a/b. Being filled from the inside by triangles, a circle with radius r gets the circumference 2·π·r where π = n·tan(180/n) for n large.

Thus, as a label for algebra and geometry, mathematics is a natural science about the physical fact Many. However, the invention of the concept SET allowed mathematics to become a self-referring collection of ‘well-proven’ statements about ‘well-defined’ concepts, i.e. as ‘MetaMatics’, defined from above as examples from abstractions instead of from below as abstractions from examples. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel set-theory tries to avoid self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract essence. To avoid self-reference Russell introduced a type theory allowing reference only to lower degree types. Consequently, fractions cannot be numbers since they refer to numbers in their modern definition: In a set-product of integers, a fraction is an equivalence set created by the equivalence relation a/b ~c/d if a·d = b·c.

Thus SET transformed grounded mathematics into a self-referring 'MetaMatism', a mixture of MetaMatics and 'MatheMatism' true inside a classroom but seldom outside where claims as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So rational numbers is pure MetaMatism by also being MatheMatism: Inside a classroom, 1/2 + 2/3 = 7/6. Outside 1 coke out of 2 bottles and 2 cokes out of 3 bottles add up to 3 cokes out of 5 bottles, and not 7 cokes out of 6 bottles as taught inside.

Not criticizing rational numbers shows that critical thinking has taboos and that it lacks self-criticism by showing no criticism towards its own un-criticalness.

'Preschool calculus and multiplication before addition' as a 1year pre-engineer math course

As a label, mathematics has no content itself, only it ingredients have, algebra and geometry both rooted in the physical fact Many. To deal with Many we count & add. By counting a total T in bundles, bundle-counting creates numbers as blocks of bundles and unbundled occurring in three different ways, normal and overload and underload as in T = 2B1 3s = 1B4 3s = 3B-2 3s when recounted in the same unit. Recounted in a different unit roots proportionality through the recount formula T = (T/B) ·B allowing a calculator to predict the result. Recounting in and from tens means resizing blocks where the height and the base are inversely proportional as in 3 7s = 2)1 tens or 4 tens = 5 8s. Reversed addition is called equations solved by recounting: 2·x = 8 = (8/2)·2 so x = 8/2, showing the solving method 'move to opposite side with opposite sign'. With counting before adding, division and multiplication comes before addition.

Once counted, totals can be added on-top if the units are made the same by proportionality, and next-to as areas also called integration. A composite area always changes with the last block added: change in Area = height* change in base, or $\Delta A = h \cdot \Delta b$ or $h = \Delta A/\Delta b$. So areas can be found by developing $\Delta/\Delta x$-calculations, also called differentiation in the case of replacing interval changes with local changes: $y' = dy/dx = \Delta y/\Delta x$ for Δx arbitrarily small; as when the per-number is neither globally nor piecewise but locally constant (continuous) (Tarp, 2013).

Finally, double counting a physical quantity in two different units creates pre-numbers or fractions as 2$/3kg = 2/3 $/kg that must be multiplied to areas before being added. The difference between a full critical and civilized mathematics education curriculum is illustrated in the appendix.

Discussion and conclusion

We asked: wanting to design a 1year pre-engineer course for migrants beginning from scratch, should we use critical and civilized thinking?

Investigating its theoretical background shows that critical thinking is based on Marx, again based on Hegel counter-enlightenment going back to Greek Plato philosophy resonating with the Greek meaning of the word 'kritike', to pass judgement. For Plato, that was precisely what the philosophers were able to do since to them all physical was but examples of metaphysical forms only visible to them. Hegel replaced the forms with a Spirit expressing itself through the history of different people thinking they can decide their future themselves, but in reality just being puppets on a string playing out the masterplan of the Spirit. To Marx, the means to the Spirit's goal, a socialist society, was a proletarian dictatorship with a democracy in the form of a representative pyramid where the top central committee decided the correctness code that justified the judgement passed by critical thinking. Consequently, rational numbers cannot be criticized if part of this code. Likewise, criticizing

Hegel-based line-organized office directed education is out of the question. With its lack of self-criticism and dependence on the will of a metaphysical Spirit, critical thinking reminds of a totalitarian religion preaching political correctness instead of teaching enlightenment.

Being skeptical towards ungrounded is-claims, civilized thinking unmasks false nature by uncovering hidden alternatives to choices presented as nature. So categories and correctness are grounded in the outside world; and as means avoiding the banality of evil, its institutions accommodate to resistance from the outside goals they are created to meet. Consequently, mathematics is ManyMath, a natural science accommodating to the physical fact Many; and education must be organized in flexible half-year blocks aiming at uncovering and developing the talent of the individual learner.

So as a 1year pre-engineer course for migrants from scratch we will get to different answers. Uncritically accepting mathematics as meaningless MetaMatism, critical thinking will say it is impossible to learn a pre-engineer background in one year since mathematics is difficult to learn thus taking many hours of hard dedicated work.

Civilized thinking welcomes a course showing that while MetaMatism is difficult, ManyMath is quickly learned: To deal with many, we count and recount and double-count before performing next-top and on-top addition and reversed addition. First we count in ones to produce icons, then we bundle-count in normal, overload and underload form by recounting in the same unit thus realizing that numbers are 2dimensional blocks and not names to the points on a 1dimensional cardinality line as claimed by MetaMatism. Then we recount in a new unit to proportional numbers. Then we recount in and from tens to resize the number blocks. Then we double-count to create per-numbers and fractions. Then to add on-top we must change the unit by proportional recounting; and to add per-numbers we must add next-to as areas where a composite area changes with the last block added. And finally reversed addition leads to solving equations presenting 'opposite side with opposite sign' as a natural method.

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Appendix: a critical and a civilized math curriculum

Primary school

Critical mathematics curriculum	Civilized mathematics curriculum (Tarp, 2016)
1dim. Number-line with number-names No counting, only adding and next-to Addition & Subtraction before Multiplication & Division Multiplication tables to be memorized No calculator	2dim. Number-blocks with units Counting before adding, next-to before on-top Multiplication & Division before Subtraction & Addition Multiplication tables recount to & from tens Calculator from the start as predictors
One and two digit numbers Addition Subtraction Multiplication Division Simple fractions	BundleCount Many in BundleCups ReCount Many in same Unit & in new Unit (Proportionality) ReCount: In Tens & From Tens (Multiplication & Division) Calculator Prediction: RecountFormula Addition: NextTo (Integration) & OnTop Reversed addition: Equations

Middle school

Fractions are numbers that can be added without units. Letter-fractions must be factorized before added	Fractions are PerNumbers (operators needing a number to become a number) and added by areas (integration)
Negative numbers Fractions Percentages & Decimals Proportionality LetterNumbers Algebraic fractions Solve a linear equation Solve 2 equations w. 2 unknowns	DoubleCounting produces PerNumbers & PerFives (fractions) & PerHundreds (%) Geometry and algebra go hand in hand when working with letter-numbers and letter-formulas; and with lines and forms The coordinate system coordinates geometry and algebra so that length can be translated to D-change, and vice versa

High school

Functions are set-relations	Functions are formulas with two variables
Squares and square roots Solve quadratic equations Linear functions Quadratic functions Exponential functions Logarithm Differential Calculus Integral Calculus Statistics & probability	Integral Calculus as adding PerNumbers Change & Global/Piecewice/Local constancy Root/log as finding/counting change-factors Constant change: Proportional, linear, quadratic, exponential, power Simple and compound interest Predictable Change: Integral Calculus & Differential Calculus Unpredictable Change: Statistics & probability

06. Online Teacher Training for Curing Math Dislike: Cup&Re-Counting & Multiplication Before Addition

Set transformed Mathematics from a mere label for Algebra and Geometry into a self-referring subject changing the two from example-containers to examples of set, causing massive learning problems as shown by PISA. Re-rooting mathematics in the physical fact Many, the MATeCADEMY.net offers an alternative teacher training.

Background

In spite of increased mathematics education research, Swedish PISA results decrease as witnessed by the OECD 2015 report 'Improving Schools in Sweden'. Mathematics seems to be hard, but we could ask: Maybe it is not mathematics that is taught, and maybe there is a hidden mathematics that rooted in the outside world becomes meaningful? And if so, where can teachers learn about it? Existentialist thinking might provide an answer. Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism as holding that 'existence precedes essence' (Marino, 2004 p. 344). But how does essence-math differ from existence-math?

A Case: Peter, stuck in division and fractions

Being a mathematics teacher in a class of ordinary students and repeaters flunking division and fractions, Peter is about to give up teaching when he learns about the '1cup & 5sticks' method to cure mathematics dislike by watching 'CupCount and ReCount before you Add' (https://www.youtube.com/watch?v=IE5nk2YEQIAxx).

Here 5 sticks are CupCounted in 2s using a cup for bundles. He sees that a total can be recounted in the same unit in 3 different forms: overload, standard and underload:

T = 5 = I I I I I = <u>II</u> I I I = 1B3 2s = <u>II</u> <u>II</u> I = 2B1 2s = <u>II</u> <u>II</u> <u>II</u> ~~I~~ = 3B-1 2s

So counted in bundles, a total has an inside number of bundles and an outside number of singles; and moving a stick out or in creates an over-load or an under-load.

When multiplying, 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 inside and 56 outside as an overload that can be recounted: T = 7 x 4B8 = 28B56 = 33B6 = 336.

And when dividing, 336/7 is bundle-written as 33B6 /7 recounted to 28 inside and 56 outside according to the multiplication table. So 33B6 /7 = 28B56 /7 = 4B8 = 48.

To try it himself, Peter downloads the 'CupCount & ReCount Booklet'. He gives a copy to his colleagues and they decide to arrange a free 1day Skype seminar.

In the morning they watch the PowerPoint presentation 'Curing Math Dislike', and discuss six issues: first the problems of modern mathematics, MetaMatism; next the potentials of postmodern mathematics, ManyMath; then the difference between the two; then a proposal for a ManyMath curriculum in primary and middle and high school; then theoretical aspects; and finally where to learn about ManyMath.

Here MetaMatism is a mixture of MatheMatism, true inside a classroom but rarely outside where '2+3 = 5' is contradicted by 2weeks+3days = 17days; and MetaMatics, presenting a concept TopDown as an example of an abstraction instead of BottomUp as an abstraction from many examples: A function IS an example of a set-product.

In the afternoon the group works with an extended version of the CupCount & ReCount Booklet where Peter assists newcomers. At the seminar there are two Skype sessions with an external instructor, one at noon and one in the afternoon.

Bringing ManyMath to his classroom, Peter sees that many difficulties disappear, so he takes a 1year distance learning education at the MATHeCADEMY.net teaching teachers to teach MatheMatics as ManyMath, a natural science about Many. Peter and 7 others experience PYRAMIDeDUCATION where they are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. An external coach assists the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. In a pair each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays by coaching a new group of 8 teachers.

At the academy, the 2x4 sections are called CATS for primary and secondary school inspired by the fact that to deal with Many, we Count & Add in Time & Space.

At the academy, primary school mathematics is learned through educational sentence-free meetings with the sentence subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic 'grasp-to-grasp' learning.

Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic 'gossip-learning': Thank you for telling me something new about something I already knew.

Conclusion

An existentialist distinction between essence and existence shows that what is taught in schools in not mathematics, but a self-referring MetaMatism turning mathematics upside down and containing some statements that do not apply outside. As a common label for Algebra and Geometry meaning reuniting Many and measuring Earth in Arabic and Greek, mathematics should let existence precede essence and become ManyMatics, a natural science about how to divide the earth and its Many products.

Reference

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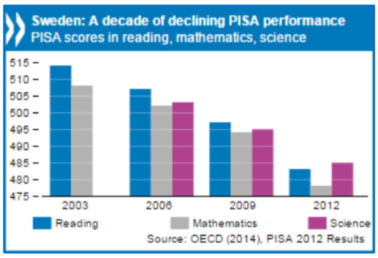
Cure MathDislike: CupCount 'fore you Add

1Day Skype Seminar on CupCounting, ReCounting & CupWriting

Action Learning on the child's own 2D NumberLanguage as observed when holding 4 fingers together 2 by 2 makes a 3-year-old child say 'No, that is not 4, that is 2 2s.'

<table>
<tr><td>09-11</td><td>Listening and Discussing: Curing Math Dislike, a PowerPointPresentation
To master Many, we Math?? No, first we Count, then we Add. Math is a label, not an action word.</td></tr>
<tr><td></td><td>1. The problems of Modern MatheMatics, or MetaMatism
2. The potentials of PostModern MatheMatics, or ManyMath
3. The Difference between MetaMatism and ManyMath
4. A ManyMath Curriculum for Primary and Middle and High school
5. Theoretical aspects, and 6. Where to learn about ManyMath?
Bad Math: MatheMatism, true inside but rarely outside classes: 2+3 IS 5, but 2weeks+3days = 17d? Adding 1D Line Numbers without units may create MathDislike.
Evil Math: MetaMatics, presenting a concept TopDown as an example of an abstraction instead of BottomUp as an abstraction from many examples: A function IS an example of a set-product.
Good Math: ManyMatics, a natural science about Many mastering Many by CupCounting, ReCounting & CupWriting: T = 5 = I I I I I = II I I I = 1]3 2s = II II I = 2]1 2s = II II II I = 3]-1 2s.</td></tr>
<tr><td>11-13</td><td>Skype Conference. Lunch</td></tr>
<tr><td>13-15</td><td>Doing: Trying out the CupCount 'fore you Add booklet to see proportionality and calculus and solving equations as golden LearningOpportunities in Cup- & Re-Counting and NextTo Addition.</td></tr>
<tr><td></td><td>RECOUNTING, in the same unit creates over- or under-load, in a new unit creates proportionality
Question: T = 2.1 3s = ? 3s. Answer: T = 2.1 = 2]1 = 1]4 = 3]-2 3s
Q: T = 2 3s = ? 4s A: T = 2 3s = III III = IIII II = 1]2 4s = 1]1 5s = 3] 2s = 1]1] 2s = 11.0 2s</td></tr>
<tr><td></td><td>CalculatorPrediction. Q: T = 2 4s = ? 5s. A: T = 1.3 5s since
RecountFormula T = (T/B)*B says 'From T, T/B times, Bs can be taken away'
2*4/5 1.some
2*4 – 1*5 3</td></tr>
<tr><td></td><td>RECOUNTING in and from Tens resizes blocks meaning teaching multiplication before addition:
Q: T = 3 7s = ? tens. A: T = 3*7 = 21 = 2.1 tens. Q: T = 47 = ? 6s. A: T = (47/6)*6 = 7 6s & 5</td></tr>
<tr><td></td><td>Multiply & Divide with CupWriting creating or removing overloads
Q: T = 7 * 463 = ? A: T = 7 * 4]6]3 = 28]42]21 = 28]44]1 = 32]4]1 = 3241
Q: T = 3241 / 7 = ? A: T = 32]4]1 / 7 = 28]44]1 / 7 = 28]42]21 / 7 = 4]6]3 = 463</td></tr>
<tr><td></td><td>ADD NextTo. Q: T = 2 4s + 3 5s = ? 9s. A: T = 2.5 9s (integration)
ADD OnTop. Q: T = 2 4s + 3 5s = ? 5s. A: T = 1.3 5s + 3 5s = 1]3 + 3] = 4]3 = 4.3 5s</td></tr>
<tr><td></td><td>DoubleCounting in two units creates PerNumbers
Q: T = 10$ = ?kg with 4$ per 5kg. A: T = 10$ = (10/4) * 4$ = (10/4) * 5 kg = 12.5 kg</td></tr>
<tr><td></td><td>Reversed Addition: Solving Equations by moving to Opposite Side with Opposite Sign
<table><tr><td>2 x ? = 8 = (8/2) x 2</td><td>2 + ? = 8 = (8-2) + 2</td><td>T = 2 3s + ? 5s = 3.2 8s</td></tr><tr><td>? = 8/2, ReCounting</td><td>? = 8-2, ReStacking</td><td>? = (3.2 8s – 2 3s)/5 = ΔT/5, Differentiation</td></tr></table></td></tr>
<tr><td>15-16</td><td>Coffee. Skype Conference.</td></tr>
</table>

Background



The effect of MathDislike is seen in the 2015 OECD report *Improving Schools in Sweden*: 'PISA 2012, however, showed a stark decline in the performance of 15-year-old students with more than one out of four students not even achieving the baseline Level 2in mathematics at which students begin to demonstrate competencies to actively participate in life'.
MATHeCADEMY.net offers UK or DK online Teacher Training based upon Action Learning and Research papers on CupCounting published at the ICME 2004-2012 *(mathecademy.net/papers/icme-trilogy)*. More details on MrAlTarp YouTube videos:

Summary of the 4 primary and secondary 4 study units at the MATHeCADEMY.net

	QUESTIONS	ANSWERS
C1 COUNT	How to count Many? How to recount 8 in 3s: T = 8 = ? 3s How to recount 6kg in \$: T = 6kg = ?\$ How to count in standard bundles?	By bundling and stacking the total T predicted by T = (T/b)*b T = 8 = ?*3 = ?3s, T = 8 = (8/3)*3 = 2)2 3s = 2.2 3s = 2*3 +2 = 2 2/3*3 If 4kg = 2\$ then 6kg = (6/4)*4kg = (6/4)*2\$ = 3\$ Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten 2ten 3 = 4*B^2 + 2*B + 3
C2 COUNT	How can we count possibilities? How can we predict unpredictable numbers?	By using the numbers in Pascal's triangle We 'post-dict' that the average number is 8.2 with the deviation 2.3. We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation)
A1 ADD	How to add stacks concretely? T = 27+16 = 2ten7+1ten6 = 3ten13 = ? How to add stacks abstractly?	By restacking overloads predicted by the restack-equation T = (T-b)+b T = 27+16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43 Vertical calculation uses carrying. Horizontal calculation uses FOIL
A2 ADD	What is a prime number? What is a per-number? How to add per-numbers?	Fold-numbers can be folded: 10 = 2fold5. Prime-numbers cannot: 5 = 1fold5 Per-numbers occur when counting, when pricing and when splitting. The \$/day-number a is multiplied with the day-number b before added to the total \$-number T: T2 = T1 + a*b
T1 TIME	How can counting & adding be reversed ? Counting ? 3s and adding 2 gave 14. Can all calculations be reversed?	By calculating backward, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign. x*3+2 = 14 is reversed to x = (14-2)/3 Yes. x+a = b is reversed to x = b-a, x*a = b is reversed to x = b/a, x^a = b is reversed to x = a√b, a^x = b is reversed to x=logb/loga
T2 TIME	How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable?	By using constant change-equations: If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44 If Ko = 30 and ΔK/K = r = 2%, then K7= Ko*(1+r)^n= 30*1.02^7= 34.46 By solving a variable change-equation: If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx
S1 SPACE	How to count plane and spatial properties of stacks and boxes and round objects?	By using a ruler, a protractor and a triangular shape. By the 3 Greek Pythagoras', mini, midi & maxi By the 3 Arabic recount-equations: sinA=a/c, cosA=b/c, tanA=a/b
S2 SPACE	How to predict the position of points and lines? How to use the new calculation technology?	By using a coordinate-system: If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14) Computers can calculate a set of numbers (vectors) and a set of vectors (matrices)

07. Debate on how to improve mathematics education

In this symposium, the author invites opponents to debate how to improve mathematics education inspired by the Chomsky-Foucault debate on human nature. The main question is: 'If research cannot improve Math education, then what can?

Bo: Today we discuss Mathematics education and its research. Humans communicate in languages, a word-language and a number-language. In the family, we learn to speak the word language, and we are taught to read and write in institutionalized education, also taking care of the number-language under the name Mathematics, thus emphasizing the three r's: Reading, Writing and Arithmetic. Today governments control education, guided by a growing research community. Still international tests show that the learning of the number language is deteriorating in many countries. This raises the question: If research cannot improve Mathematics education, then what can? I hope our two guests will provide some answers. I hope you will give both a statement and a comment to the other's statement before the floor will comment.

1. Mathematics Itself

Bo: We begin with Mathematics. The ancient Greeks Pythagoreans used this word as a common label for what we know, which at that time was Arithmetic, Geometry, Astronomy and Music. Later Astronomy and Music left, and Algebra and Statistics came in. So today, Mathematics is a common label for Arithmetic, Algebra, Geometry and Statistics, or is it? And what about the so-called 'New Math' appearing in the 1960s, is it still around, or has it been replaced by a post New-Math, that might be the same as pre New-Math? In other words, has pre-modern Math replaced modern Math as post-modern Math? So, I would like to ask: 'What is Mathematics, and how is it connected to our number-language?'

2. Education in General

Bo: Now let us talk about education in general. On our planet, life takes the form of single black cells, or green or grey cells combined as plants or animals. To survive, plants need minerals, pumped in water from the ground through their leaves by the sun. Animals instead use their heart to pump the blood around, and use the holes in the head to supply the stomach with food and the brain with information. Adapted through genes, reptiles reproduce in high numbers to survive. Feeding their offspring while it adapts to the environment through experiencing, mammals reproduce with a few children per year. Humans only need a few children in their lifetime, since transforming the forelegs to hands and fingers allows humans to grasp the food, and to share information through communication and education by developing a language when associating sounds to what they grasp. Where food must be split in portions, information can be shared. Education takes place in the family and in the workplace; and in institutions with primary, secondary and tertiary education for children, for teenagers and for the workplace. Continental Europe uses words for education that do not exist in the English language such as Bildung, unterricht, erziehung, didactics, etc. Likewise, Europe still holds on to the line-organized office preparing education that was created by the German autocracy shortly after 1800 to mobilize the population against the French democracy, whereas the North American republics have block-organized talent developing education from secondary school. As to testing, some countries use centralized test where others use local testing. And some use written tests and others oral tests. So, my next question is 'what is education?'

3. Mathematics Education

Bo: Now let us talk about education in Mathematics, seen as one of the core subjects in schools together with reading and writing. However, there seems to be a difference here. If we deal with the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. However, you cannot Math, you can reckon. At the European continent reckoning, called 'Rechnung' in German, was an independent subject until the arrival of the so-called new Mathematics around 1960. When opened up, Mathematics still contains subjects as fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc. Today, Europe only offers classes in Mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. Therefore, I ask, 'what is Mathematics education?'

4. The Learner

Bo: Now let us talk about at the humans involved in Mathematics education: Governments choose curricula, build schools, buy textbooks and hire teachers to help learners learn. We begin with the learners. The tradition sees learning taking place when learners follow external instructions from the teacher in class and from the textbook at home. Then constructivism came along suggesting that instead learning takes place through internal construction. Therefore, I ask 'what is a learner?'

5. The Teacher

Bo: Now let us talk about the teacher. It seems straightforward to say that the job of a teacher is to teach learners so that learning takes place, checked by written tests. However, continental Europe calls a teacher a 'Lehrer' thus using the same word as for learning. In addition, a Lehrer is supposed to facilitate 'unterrichtung and erziehung and to develop qualifications and competences. In teacher education, the subject didactics, meant to determine the content of Bildung, is unknown outside the continent. And until lately, educating lehrers took place outside the university in special lehrer-schools. Thus, being a teacher does not seem to be that well-defined. Therefore, my next question is 'what is a teacher?'

6. The Political System

Bo: Now let us talk about governments. Humans live together in societies with different degrees of patronization. In the debate on patronization, the ancient Greek sophists argued that humans must be enlightened about the difference between nature and choice to prevent patronization by choices presented as nature. In contrast, the philosophers saw choice as an illusion since physical phenomena are but examples of metaphysical forms only visible to philosophers educated at Plato's Academy who consequently should be accepted as patronizors. Still today, democracies come in two forms with a low and high degree of institutionalized patronization using block-organized education for individual talent developing or using line-organized education for office preparation. As to exams, some governments prefer them centralized and some prefer them decentralized. As to curricula, the arrival of new Mathematics in the 1960s integrated its subfields under the common label Mathematics. Likewise, constructivism meant a change from lists of concepts to lists of competences. However, these changes came from Mathematics and education itself. So my question is: 'Should governments interfere in Mathematics education?'

7. Research

Bo: Now let us talk about research. Tradition often sees research as a search for laws built upon reliable data and validated by unfalsified predictions. The ancient Greek Pythagoreans found three metaphysical laws obeyed by physical examples. In a triangle, two angles and two sides can vary freely, but the third ones must obey a law. In addition, shortening a string must obey a simple ratio-law to create musical harmony. Their findings inspired Plato to create an academy where knowledge meant explaining physical phenomena as examples of metaphysical forms only visible to philosophers educated at his academy by scholasticism as 'late opponents' defending their comments on an already defended comment against three opponents. However, this method discovered no new metaphysical laws before Newton by discovering the gravitational law brought the priority back to the physical level, thus reinventing natural science using a laboratory to create reliable data and test library predictions. This natural science inspired the 18th century Enlightenment period, which again created counter-enlightenment, so today research outside the natural sciences still uses Plato scholastics. Except for the two Enlightenment republics where American Pragmatism used natural science as an inspiration for its Grounded Theory, and where

French post-structuralism has revived the ancient Greek sophist skepticism towards hidden patronization in categories, correctness and institutions that are ungrounded. Using classrooms to gather data and test predictions, Mathematics education research could be a natural science, but it seems to prefer scholastics by researching, not Math education, but the research on Math education instead. To discuss this paradox I therefore ask, 'what is research in general, and within Mathematics education specifically?'

8. Conflicting Theories

Bo: Of course, Mathematics education research builds upon and finds inspiration in external theories. However, some theories are conflicting. Within Psychology, constructivism has a controversy between Vygotsky and Piaget. Vygotsky sees education as building ladders from the present theory regime to the learners' learning zones. Piaget replaces this top-down view with a bottom-up view inspired by American Grounded Theory allowing categories to grow out of concrete experiences and observations. Within Sociology, disagreement about the nature of knowledge began in ancient Greece where the sophists wanted it spread out as enlightenment to enable humans to practice democracy instead of allowing patronizing philosophers to monopolize it. Medieval times saw a controversy between the realists and the nominalists as to whether a name is naming something or a mere sound. In the late Renaissance, a controversy occurred between Hobbes arguing that their destructive nature forces humans to accept patronization, and Locke arguing, like the sophists, that enlightenment enables humans to practice democracy without any physical or metaphysical patronization. As counter-enlightenment, Hegel reinstalled a patronizing Spirit expressing itself through art and through the history of different people. This created the foundation of Europe's line-organized office preparing Bildung schools; and for Marxism and socialism, and for the critical thinking of the Frankfurter School, reviving the ancient sophist-philosopher debate by fiercely debating across the Rhine with the post-structuralism of the French Enlightenment republic. Likewise, the two extreme examples of forced institutionalization in 20th century Europe, both terminated by the low institutionalized American Enlightenment republics, made thinkers as Baumann and Arendt point out that what made termination camps work was the authorized routines of modernity and the banality of evil. Reluctant to follow an order, you can find another job in the private sector, but not in an institution. Here the necessity of keeping a job forces you to carry out both good and evil orders. As an example of a forced institution, this also becomes an issue in Mathematics Education. So I ask: What role do conflicting theories play in Mathematics education and its research?

References

Chomksky, N. & Foucault, M. (2006). *The Chomsky-Foucault Debate on Human Nature.* New York: The New Press.

The Chomsky-Foucault debate on human nature, https://www.youtube.com/watch?v=3wfNl2L0Gf8&feature=youtu.be

Paul & Allan debate on postmodern math education, https://www.youtube.com/watch?v=ArKY2y_ve_U

08. Poster: MigrantMath as CupCounting & PreSchool Calculus

Europe receives a continuing migrant flow to benefit from its welfare and educational systems. To benefit from the engineer potential in young migrants allowing them to build up welfare and education in their own country, Europe must rethink its line-organized office directed education dating back to the Napoleon wars; and must replace meaningless top-down MetaMatism with bottom-up ManyMath.

Background

Increased mathematics education research seems to create a decrease in Nordic PISA results as witnessed by the latest PISA study and the OECD 2015 report ‘Improving Schools in Sweden’. We ask: Can existentialism point to a possible solution?

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism as holding that ‘existence precedes essence’ (Marino, 2004 p. 344). Thus a hypothesis can be formulated: Mathematics performance will increase if replacing essence-math with existence-math.

Mathematics as an Essence

The Pythagoreans labeled their four knowledge areas by a Greek word for knowledge, mathematics. With astronomy and music now as independent areas, today mathematics is a common label for its two remaining activities both rooted in Many: Geometry meaning to measure earth in Greek, and Algebra meaning to reunite numbers in Arabic and replacing Greek Arithmetic (Freudenthal, 1973).

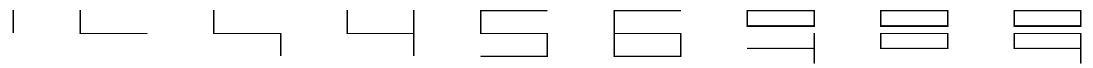
Then the set-concept transformed mathematics to ‘MetaMatics’ defining its concepts by self-reference as examples from internal abstractions instead of as abstractions from external examples. Looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence it false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

‘MetaMatism’ means mixing MetaMatics with ‘MatheMatism’ true inside but seldom outside the classroom as e.g. ‘the fraction paradox’ where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

Mathematics as ManyMath, a Natural Science about Many

A number as 345 = 3*B^2+4*B+5*1 shows that to deal with Many, first we iconize then we bundle and stack. Until ten we count in 1s by iconizing, i.e. by rearranging sticks in icons so five ones becomes one five-icon 5 with five sticks, etc.

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



1 2 3 4 5 6 7 8 9

With icons, a total can be ‘bundle-counted’ in icon-bundles so a total T of 7 is bundled in 3s as T = 2 3s & 1 shown with 2 sticks in a in a bundle-cup and 1 stick outside; reported with ‘bundle-writing’, T = 2B1 3s, then with ‘decimal-writing’ where a decimal point separates the bundles from the singles, and including the unit 3s, T = 2.1 3s.

A calculator can predict a counting result. A stack of 2 3s is iconized as 2x3 showing a lift used 2 times to lift the 3s. Taking away is iconized with ‘/3’ or ‘–3’ showing the broom or the trace when wiping away 3 several times or just once, called division and subtraction. Entering ‘7/3’, we ask the calculator ‘from 7 take away 3s’ and get the answer ‘2.some’. Entering ‘7 – 2x3’ we ask ‘from 7 take away 2 3s’ and get the answer 1 leftover. Thus the calculator predicts that 7 = 2B1 3s = 2.1 3s.

Once bundle-counted, totals are re-counted, double-counted or added next-to or on-top. To recount in the same unit, changing a bundle to singles creates over- or under-load as when recounting 4 2s as 3.2 2s, or as 5 less 2 2s leading to negative numbers:

T = 4 2s = 3.2 2s, or T = 4 2s = 5.-2 2s

To recount in a different unit means changing unit, called proportionality. Asking ‘3 4s is how many 5s?’ sticks give the result 2.2 5s as predicted by a calculator.

T = 3 4s = IIII IIII IIII → IIIII IIIII I I → 2B2 5s → 2.2 5s

Recounting in and from tens means resizing number-blocks where the height and the base are inversely proportional as in 3 7s = 2B1 tens or 4 tens = 5 8s.

Double-counting a physical quantity creates ‘per-numbers’ as 4$/5kg allowing 16$ to be recounted in 4s to bridge to the kg-numbers: 16$ = (16/4)*4$= (16/4)*5kg = 20kg.

Next-to addition of 2 3s and 4 5s as 3.2 8s means adding areas, called integration. To add on-top the units are made the same by recounting as 1.1 5s and 4 5s = 5.1 5s. Reversed addition is called equations solved by recounting: 2·x = 8 = (8/2)·2 so x = 8/2, showing the solving method ‘move to opposite side with opposite sign’.

The root of geometry is a rectangle that halved by a diagonal becomes two right-angled triangles where the sides and the angles are connected by three laws, A+B+C = 180, a^2+b^2 = c^2 and tanA = a/b. Being filled from the inside by such triangles, a circle with radius r gets the circumference 2·π·r where π = n·tan(180/n) for n large.

Conclusion

There is a fundamental difference between essence- and existence-math, MetaMatism and ManyMath. This means the latter has to be tested outside traditional school in preschool or in special courses for young migrants wanting to become engineers or teachers to help building welfare and education systems in their own country.

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09. A Heidegger View on How to Improve Mathematics Education

After 50 years of research, mathematics education still has learning problems as witnessed by the PISA studies. So, a suspicion arises: Can we be sure that what has been undertaken is mathematics and education and research? We seek an answer in philosophy by listening to Heidegger that, wanting to establish its meaning, finds two forms of Being: that what is, and how it is. In a Heidegger universe, the core ingredients are I and It and They, where I must neglect the gossip from They to establish an authentic relationship to It. Bracketing mathematics' gossip will allow its root, Many, to open itself and disclose a 'many-matics' as a grounded natural science different in many ways from the traditional self-referring set-based mathe-matics. So to improve its educational sentences, math should bring its subjects to the classroom, but leave its gossip outside.

Introduction

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by PISA studies showing a low level and a continuing decline in many countries. Thus, the former model country Sweden face that 'more than one out of four students not even achieving the baseline level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life." (OECD 2015, p. 3)

Researchers in mathematics education meet in different fora. In Europe, the Congress of the European Society for Research in Mathematics Education, CERME, meets each second year. At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? (http://cerme10.org/scientific-activities/plenary-sessions/)

By questioning its success, maybe the short answer is: How can mathematics education research be successful when its three words are not that well defined? As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist. In continental Europe, education serves the nation's need for public servants through multi-year compulsory classes and lines at the secondary and tertiary level. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers at the secondary level, and the tertiary level also has a flexible block organization allowing additional blocks to be taken in the case of unemployment or change of job.

As to research, academic articles can be written at a master level applying or exemplifying existing theories, or at a research level questioning them. Just following ruling theories is especially problematic in the case of conflicting theory as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible respectively.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and we cannot know if research is following ruling traditions or searching for new discoveries. So to answer the question 'How to improve mathematics education research', first we must try to make the three words well defined by asking: What is meant by Mathematics, what is meant by education and what is meant by research?

Common for all three questions is the word 'is', so let us begin by asking 'what is meant by 'is'?'.

What does 'is' Mean

'To be or not to be', 'Cogito, ergo sum', 'What is 'is'?'. Three statements about the nature of being that may or may not have been formulated by Hamlet, Descartes and Heidegger. Still they direct our attention to reflecting and discussing the most used word in sentences, to be.

In his book 'Being and Time', 'Sein und Zeit' in the original German version, Heidegger writes:

> Do we in our time have an answer to the question of what we really mean by the word 'being'? Not at all. So it is fitting that we should raise anew the question of the meaning of Being. (..) Our aim in the following treatise is to work out the question of the meaning of Being and to do so concretely. (Heidegger 1962, p. 1)

Going back in time, Heidegger says that the question 'provided a stimulus for the researches of Plato and Aristotle only to subside from then on as a theme for actual investigation. (p. 2).' Furthermore, Heidegger says, '(..) a dogma has been developed which not only declares the question about the meaning of Being to be superfluous, but sanctions its complete neglect. It is said that Being is the most universal and the emptiest of concepts. As such it resists every attempt at definition (p. 2).'

Heidegger sees this dogma based upon three presuppositions. As to seeing Being as the most universal concept, Heidegger writes 'In medieval ontology Being is designated as a 'transcendens'. Aristotle himself knew the unity of this transcendental 'universal' as a unity of analogy in contrast to the multiplicity of the highest generic concepts applicable to things (..) So if it is said that Being is the most universal concept, this cannot mean that it is the one which is clearest or that it needs no further discussion. It is rather the darkest of all (p. 3).'

As to seeing the concept of Being is indefinable Heidegger says that 'Being cannot be derived from higher concepts by definition, nor can it be presented through lower ones (..) We can infer only that Being cannot have the character of an entity (..) The indefinability of Being does not eliminate the question of its meaning (p. 4).'

As to seeing Being as a concept that of all concepts is the one that is self-evident, Heidegger says 'The very fact that we already live in understanding of Being and that the meaning of Being is still veiled in darkness proves that it is necessary in principle to raise this question again (p. 4).'

Heidegger concludes by saying that

> By Considering these prejudices, however, we have made plain not only that the question of Being lacks an answer, but that the question itself is obscure ad without direction. So if it is to be revived, this means that we must first work out an adequate way of formulating it (p. 4).

To do so, Heidegger says that 'We must therefore explain briefly what belongs to any question whatsoever, so that from this standpoint the question of Being can be made visible as a very special one with its own distinctive character (p. 5).'

Then Heidegger addresses the nature of a general question aiming at establishing a definition of M by answering the question 'What is M?'. Heidegger assigns to a question the word inquiry and says that 'Every enquiry is a seeking. Every seeking gets guided beforehand by what is sought. Inquiry is a cognizant seeking for an entity both with regard to the fact that it is and with regard to its Being as it is (p. 5).'

Here Heidegger describes the two different uses of being, one that establishes existence, 'M is', and one that establishes 'how M is' to others, since what exists is perceived by humans that begin to categorize it by naming or characterizing or analogizing it, in all three cases using the word 'is'.

Heidegger points to four different uses of the word 'is'. 'Is' can claim a mere existence of M, 'M is'; and 'is' can assign predicates to M, 'M is N', but this can be done in three different ways. 'Is' can point down as a 'naming-is' ('M is for example N or P or Q or …') defining M as a common name for its volume of more concrete examples. 'Is' can point up as a 'judging-is' ('M is an example of N') defining M as member of a more abstract category N. Finally, is can point over as an 'analogizing-is' ('M is like N') portraying M by a metaphor carrying over known aspects from another N.

Heidegger stresses the double meaning of being, 'that M is & how M is' by saying 'Everything we talk about, everything we have in view, everything towards which we comport ourselves in any way, is being; what we are is being and so is how we are. Being lies in the fact that something is and in its Being as it is (p. 6-7).'

To separate that which is from how it is, Heidegger coins the word 'Dasein' by saying 'This entity which each of us is in himself and which includes inquiring as one of the possibilities of its Being, we shall denote by the term "Dasein" (p.7).'

So here Heidegger transforms the 'cogito ergo sum' into 'Ich bin da, und Ich frage' (I exist here and I question). By connecting the word 'da' to existence, Heidegger places existence in time and space since 'da' can mean both there and then. Also, Heidegger sees questioning as the most important ability of Dasein.

Within existentialist thinking, existence and essence are core concepts (Marino 2004). Here Heidegger says

> [Dasein's] Being-what-it-is (essentia) must, so far as we can speak of it at all, be conceived in terms of its Being (existentia). (..) To avoid getting bewildered, we shall always use the Interpretative expression "presence-at-hand" for the term "existentia", while the term "existence", as a designation of Being, will be allotted solely to Dasein. The essence of Dasein lies in its existence. (p. 42)

Here Heidegger reformulates his basic statement 'that M is and how M is' to 'by existing, M has existentia described (by Others) by essentia'; or 'existing, M exists together with presence-at-hand.'

To tell if the essentia of existentia, that is, the characteristics of presence-at-hand, is determined by the Others or by Dasein itself, Heidegger later introduces the concept 'ready-at-hand'

> Equipment can genuinely show itself only in dealings cut to its own measure (hammering with a hammer, for example) (..) In dealings such as this, where something is put to use, our concern subordinates itself to the "in-order-to" which is constitutive for the equipment we are employing at the time; the less we just stare at the hammer-Thing, and the more we seize hold of it and use it, the more primordial does our relationship to it become, and the more unveiledly is it encountered as that which it is - as equipment. (..) The kind of Being which equipment possesses - in which it manifests itself in its own right - we call "readiness to-hand". (p. 69)

As to existence, Heidegger talks about authentic an unauthentic existence.

> In each case Dasein is its possibility, and it 'has' this possibility, but not just as a property, as something present-at-hand would. And because Dasein is in each case essentially its own possibility, it can, in its very Being, 'choose' itself and win itself; it can also lose itself and never win itself; or only 'seem' to do so. But only in so far as it is essentially something which can be authentic - that is, something of its own - can it have lost itself and not yet won itself. As modes of Being, authenticity and inauthenticity (these expressions have been chosen terminologically in a strict sense) are both grounded in the fact that any Dasein whatsoever is characterized by mineness. (p. 42-43)

As to the Other, Heidegger talks about a dictatorship.

> We have shown earlier how in the environment which lies closest to us, the public 'environment' already is ready-to-hand and is also a matter of concern. In utilizing public means of transport and in making use of information services such as the newspaper, every Other is like the next. This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (p. 126)

As to describing the present-at-hand, Heidegger warns against gossip in the form of idle talk, 'Gerede' in German.

> Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (..) Thus, by its very nature, idle talk is a closing-off, since to go back to the ground of what is talked about is something which it leaves undone. (..) Because of this, idle talk discourages any new inquiry and any disputation, and in a peculiar way suppresses them and holds them back. (p. 169)

The Heidegger Universe

Summing up, from a Heidegger viewpoint the question 'what is 'is'?' leads to two forms of being: that what is; and how it is. Which depends on how They see it: sentenced by a judging-is as an example of an above category, or accepted by a naming-is as a difference among other examples below, or facetted by an analogizing-is as artistically metaphorized by parallel examples.

By his two-fold statement 'that what is; and how it is', Heidegger suggests that an ordinary sentence as 'Peter destroys the apple' is in fact two sentences, on stating existence, 'Peter is', and one stating a judgement 'destroys the apple', that might be gossip since it can be questioned: Is Peter destroying the apple, or preparing it for food, or transforming it in an artistic process, or …?

As to existence statements, the language has seven basic is-statements: I am, you are, he/she is, it is, we are, you are, they are. Heidegger sees three of these as more basic, I am and it is and they are, describing the core of the meaning of being: I exist in a world together with Things and Others.

So, the core of a Heidegger universe is I and It and They. Or, using Heidegger's terms, Dasein is in a world together with Things and They; and to escape unauthenticity, Dasein must constantly question what is present-at-hand to set it free from its prison of ruling They-gossip, so it becomes ready-at-hand, allowing Dasein an authentic existence. Thus, Dasein should be sceptical towards the essence-claims produced by They using judging-is to trap existence in a predicate-prison. Instead, Dasein should ask the judged to open itself to allow alternative authentic terms to arise using naming-is and analogizing-is.

Traditionally, education means teaching learners about the outside world. Here Heidegger sees a learner as a Dasein having as possibility to transform the surrounding presence-at-hand to ready-at-hand; but being hindered by They, teaching presence-at-hand as examples of textbook gossip instead of arranging meetings allowing the transformation to take place.

As to mathematics education, Heidegger sees Dasein in a world with numbers as entities present-at-hand, but caught in essence-claims of idle talk called mathematics. So to establish an authentic ready-at-hand relationship to them, Dasein must meet numbers directly and replace the gossip's judgment statements pointing up with naming statements pointing down.

However, numbers come in different forms. Buildings often carry roman numbers, and number plates carry Arabic numbers in two versions, an Eastern and a Western. Apparently, numbers are local gossip about something behind, to be seen in the first three Roman numbers, I and II and III, that is, about different degrees of 'Many'.

So, in the sentence 'here are three apples', three is not in the world by itself, apples are, as well as other units as oranges, chairs, days, hours etc. all having the form of plural to signal the presence of Many. Consequently, what is in the world is Many, and it is Many that Dasein should ask to open itself to establish an authentic relationship free of the restrictions of the gossip called mathematics.

Meeting Many

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep balance and to store sounds assigned to what we grasped with our forelegs, freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into languages. In fact, we have two languages, a word-language and a number-language.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair'. Observing the existence of many chairs, we ask 'how many totally?' and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, 'the total is 3 chairs' or, if counting legs, 'the total is 3 fours', abbreviated to 'T = 3 4s' or 'T = 3*4'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is an operation'. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

With 2017 as the 500year anniversary for Luther's 95 theses, we can choose to describe meeting Many in 12 theses.

1. Using a folding ruler we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. (Thus, there are four sticks in the four icon, and five sticks in the five icon, etc. Transforming four ones to one fours allows counting with fours as a unit also.)

2. Using a cup for the bundles we discover that a total can be 'bundle-counted' in three ways: the normal way or with an overload or with an underload. (Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and 'less 1' outside; or, if using 'bundle-writing' to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Finally, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s.)

3. Using recounting a total in the same unit by creating or removing overloads or underloads, we discover that bundle-writing offers an alternative way to perform and write down operations. (Thus,

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48)

4. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved in counting by bundling and stacking. (Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate e.g. T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.)

5. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. (Thus, the 'recount formula' T = (T/B)*B says that T/B times B can be taken away from T, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the 'restack formula' T = (T–B)+B says that T–B is left when B is taken away from T and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict.)

6. Wanting to recount a total in a new unit, we discover that a calculator can predict the result when

bundling and stacking and dragging away the stack. (Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Also, we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.)

7. Wanting to recount a total in tens, we discover that a calculator predicts the result directly by multiplication; only leaving out the unit and misplacing the decimal point. (Thus, asking T = 3 7s = ? tens, the calculator predicts: T = 21 = 2.1 tens. Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.)

And wanting to recount a total from tens to icons, we discover that this again is an example of recounting to change the unit. (Thus, asking T = 3 tens = ? 7s. the calculator predicts: First 30/7 = 4.some; then 30 – (4*7) = 2; and finally T = 30 = 4.2 7s. Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged.)

8. Using the letter u for an unknown number, we can rewrite recounting from tens as 3 tens = ? 7s, as 30 = u*7 with the answer 30/7 = u, officially called to solve an equation; hereby discovering a natural way to do so: Move a number to the opposite side with the opposite sign. (Thus, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to; predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, again moving a number to the opposite side with the opposite sign.)

9. Once counted, totals can be added, but addition is ambiguous. (Thus, with two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to get the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. So next-to addition geometrically means to add areas, and algebraically it means to combine multiplication and addition. Officially this is called integration, a core part of traditional mathematics in high school and at the first year of university.)

10. Also we discover that addition and other operations can be reversed. (Thus, in reversed addition, 8 = u+2, we ask: what is the number u that added to 2 gives 8, which is precisely the formal definition of u = 8–2. And in reversed multiplication, 8 = u*2, we ask: what is the number u that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root, the factor-finder, and logarithm, the factor-counter, as u = 3√20 and u = log3(20). In all cases we solve the equations by moving to the opposite side with the opposite sign. Reversing next-to addition we ask 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer u, from the terminal total T we remove the initial total T1 before we count the rest in 5s: u = (T–T1)/5 = ΔT/5. Combining subtraction and division in this way is called differentiation, the reverse operation to integration combining multiplication and addition.)

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. (Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.)

12. Observing that a quantity may be double-counted in the same unit, we discover that per-numbers may take the form of fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. (Thus, to find 3 per 5 of 20, 3/5 of 20, we just recount 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3. To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and

fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.)

Conclusion

To answer the questions 'what is mathematics, education and research' we looked for an answer in a Heidegger universe by allowing the root of mathematics, the physical fact Many, to open itself for us. This disclosed a 'many-matics' with digits as icons containing as many sticks as they represent; and where counting and recounting and double-counting totals come before adding them next-to and on-top, thus creating a natural order for the four basic operations, also being icons present in the counting process: first division draws away bundles then multiplication lift them to a stack that subtraction takes away to look for unbundled singles. This shows that natural numbers are two-dimensional blocks with a counting-number and a bundle-number as a unit, and with a decimal point to separate the bundles from the unbundled. Once counted, blocks can be added where next-to addition means adding areas, also called integration; and where on-top addition means recounting in the same unit to remove or create overloads. And where reversed addition next-to and on-top leads to differentiation and equations. Double-counting in different units leads to per-numbers being added or calculated in calculus, present in primary school as adding blocks, and in middle and high school, as adding piecewise and locally constant per-numbers. Finally, letters and functions are used for unspecified numbers and calculations.

Many-matics differs in many respects from traditional mathematics; that presents digits as symbols and numbers as names for points along a one-dimensional number-line; that neglects counting and recounting and double-counting and next-to addition and goes directly to on-top addition first, then subtraction, then multiplication and in the end division leading on to fractions that by being added without units becomes an example of 'mathe-matism' true inside but seldom outside classrooms: ½ + 2/3 is claimed to be 7/6 in spite of the fact that 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and certainly not 7 red of 6 apples. Being set-based, definitions use self-referring judging-is statements from above instead of naming-is statements from below, thus defining a concept as 'meta-matics', that is, as an example of an abstraction instead of as an abstraction from examples, as it was created historically. Thus a function is defined as an example of a set-relation where first-component identity implies second-component identity, instead of as a placeholder for an unspecified calculation with unspecified numbers. A closer look thus discloses traditional set-based mathematics as 'meta-matism', a mixture of meta-matics and mathe-matism.

Meta-matism as '2+3 = 5' adding numbers without units contradicts observations as 2weeks + 3 days = 17 days. And it makes a syntax error in the number-language sentence 'T = 2+3' by silencing the subject and the verb. By keeping the gossip part and leaving out the existence part, meta-matism ceases to be a number-language describing the real world. This contradicts the historic origin of mathematics as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many.

In Greek, geometry means earth measuring, which is done by dividing earth into triangles. In Arabic, algebra means to reunite numbers. Writing out a total T as we say it, T = 345 = 3*ten*ten + 4*ten + 5*1, shows a number as blocks united next-to each other. Also, we see algebra's four ways to unite numbers: addition, multiplication, repeated multiplication or power, and block-addition also called integration. Which is precisely the core of mathematics: addition and multiplication together with their reversed operations subtraction and division in primary school; and power and integration together with their reversed operations root, logarithm and differentiation in secondary school. Including the units, we see there can only be four ways to unite numbers: addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers.

As to traditional set-based mathematics, its idea of deriving definitions from the mother concept set leads to meaningless self-reference as in the classical liar paradox 'This sentence is false', being true if false and false if true. This was shown by Russell looking at the set of sets not belonging to itself. Here a set belongs to the set if it doesn't, and does not belong if it does.

To avoid self-reference, Russell created a hierarchical type theory in which fractions could not be numbers if defined by numbers, e.g. as equivalence classes in a set of number-pairs as done by set-based mathematics that consequently invented a new set-theory that by mixing sets and elements also mixes concrete examples and their abstract names, thus mixing concrete apples that can feed humans and the word 'apple' that cannot. By mixing things and their names, existence and gossip, set-based mathematics and its meta-matism fill the number-language with both semantic and syntax errors. Still, this language has entered universities worldwide as the only true version of mathematics to be transmitted through education that is improved using research to produce solid findings.

In a Heidegger universe, education means allowing I to meet It directly without They and its patronizing gossip; and to replace judging-is with naming-is when choosing how to label It. Likewise with research seen as a collective education replacing ungrounded categories with grounded ones.

So, maybe the answer to the question about solid findings in mathematics education research is 'Only one: to improve, mathematics education should ask, not what to do, but what to do differently.' Maybe research should not study problems but look for hidden differences that make a difference.

However, difference research scarcely exists today since it is rejected at conferences (Tarp 2015) for not applying or extending existing theory that might produce new researchers and feed a growing appliance industry, but being unable to reach its goal, to improve mathematics education.

In short, to be successful, mathematics education research must stop studying the misery coming from teaching meta-matism in compulsory classes. Instead, mathematics must respect its origin as a natural science grounded in Many. And research must search for differences and test if they make a difference, not in compulsory classes, but with daily lessons in self-chosen half-year blocks. Then learning the word-language and the number-language together may not be that difficult, so that all will leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a sentence, the subject exists, but the sentence about it may be gossip; so stop preaching essence and start teaching existence; or, bring the subject to the classroom and leave the sentence outside.

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10. Count and Multiply Before You Add: Proportionality and Calculus for Early Childhood and Migrants

Disappointing PISA results might be caused by a goal displacement seeing mathematics as the goal and its outside root, Many, as a means. Meeting Many free of a self-referring mathe-matics uncovers an alternative 'many-matics'; with digits and operations as icons containing as many sticks as they represent and showing the counting process; with multiplication before addition; with numbers as two-dimensional decimal numbers with units created by bundle-counting, and ready to be re-counted in the same unit to remove or create overloads to make operations easier, or in a new unit, later called proportionality, or to and from tens. Addition now occurs both on-top and next-to, later called integration. So, to improve, mathematics should again be a natural science about Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, where it was the lowest in the Nordic countries and significantly below the OECD average. This caused OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other countries also experience declining PISA results. Since mathematics education is a social institution, social theory might be able to explain the 50 years of unsuccessful research.

Social Theory Looking at Mathematics Education

As to the nature of sociology, Bauman talks about its role and about organizations:

> Sociological thinking (..) renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. (..) **Rational** action is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right: the new end against which the organization tends to measure the rationality of its performance. (Bauman, 1990, pp. 16, 79, 84)

As an institution, mathematics education is a public organisation with a 'rational action in which the end to be achieved is clearly spelled out', apparently aimed at educating students in mathematics: The goal of mathematics education is to teach mathematics. On the other hand, such a goal is self-referring; as is 'the goal of bublibub education is to teach bublibub'. So we ask: in mathematics education, **is** mathematics the end or **is** it a means; and if so then what **is** the end? Or, in other words: **is** there a goal displacement in mathematics education?

The answer to these is-questions may be found in what Bauman calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'? (Bauman, 1992, p. ix)

Institutional Skepticism

On the first page of his book 'Being and Time', Heidegger writes that his aim is 'to work out the question of the meaning of *Being* and to do so concretely'. As to looking for an answer, Heidegger says:

> Every enquiry is a seeking. Every seeking gets guided beforehand by what is sought. Inquiry is a cognizant seeking for an entity both with regard to the fact <u>that</u> it is and with regard to its Being <u>as</u> it is. (Heidegger, 1962, p. 5)

Here Heidegger describes two uses of 'is'. One claims existence, 'M is', one claims 'how M is' to others, since what exists is perceived by humans categorizing it by naming or characterizing or analogizing it to create 'M is N'-statements. This gives a total of four uses:

'Is' can claim a mere existence, 'M is'. 'Is' can point down as a 'naming-is' ('M is for example P or Q or …') defining M as an abstract name for its volume of concrete examples. 'Is' can point up as a 'judging-is' ('M is an example of N') defining M as member of a more abstract category N. Finally, 'is' can point over as an 'analogizing-is' ('M is like N') portraying M by a metaphor carrying over known aspects from a similar N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are', or, I exist in a world together with It and with They, with Things and with Others. The 'I' Heidegger calls 'Dasein', the 'It' he calls 'present-at-hand', and 'They' he calls 'the Others'. To have real existence, Dasein must create an authentic relationship to 'It' by transforming what is 'present-at-hand' to 'ready-at-hand'. However, this is made difficult by the dictatorship of 'They', shutting 'It' up in a predicate-prison of idle talk, gossip:

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

In France, Heidegger inspired the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu pointing out that society forces words upon you to diagnose you so it can offer you cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world. (Lyotard, 1984)

From a Heidegger view, education preparing learners for the outside world is seen as Dasein having the transformation from present-at-hand to ready-at-hand hindered by They hiding presence-at-hand in textbook sentences. To create an authentic relationship, Dasein therefore should insist on meeting the subject and neglect the rest of the sentence.

As to mathematics education, a Heidegger view sees Dasein in a world with numbers present-at-hand, but caught up in idle talk called mathematics. So, to establish an authentic ready-at-hand relationship to them, Dasein must meet numbers directly and replace the gossip's judgments pointing up with naming definitions pointing down.

However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. Apparently, numbers are local gossip about something behind, to be seen in the first three Roman numbers, I and II and III, that is, different degrees of 'Many'.

Consequently, what is in the world is Many, so Many should be the subject in the questions Dasein poses to obtain an authentic relationship free of the restrictions of the potential gossip called mathematics.

Mathematics as Self-Referring Gossip

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time; and together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts, self-referring defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

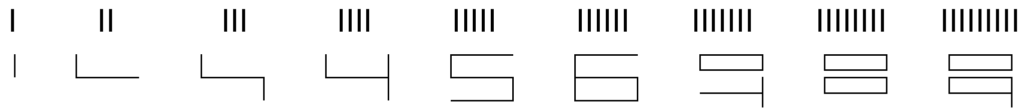
If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts. In this way SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

Meeting Many

Many exists all over time and space: Many days and years, many houses and cars etc. To deal with Many we ask, 'how many in total?' To answer, we count and add. We count by bundling and stacking as seen when holding 4 fingers together 2 by 2 makes a 3-year-old child say: 'That is not 4, that is 2 2s.'; and when writing out fully the total T = 456 = 4*B^2 + 5*B + 6*1 showing three stacks or blocks next-to each other: one with 4 bundles of bundles, one with 5 bundles, and one with 6 unbundled singles.

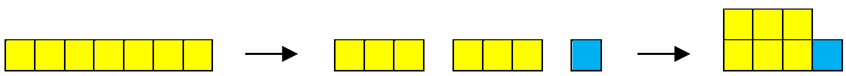
Digits occur by rearranging sticks into icons so that five ones become one five-icon 5 with five sticks, if written less sloppy. In this way, we create icons until ten since the bundle-number is counted as 10, one bundle and no unbundled, followed by eleven and twelve meaning 'one left' and 'two left' in 'Anglish', a western Danish dialect around Harboøre from where the Vikings sailed to 'Angland'.



1 2 3 4 5 6 7 8 9

A total may be counted in several ways. Some gather-hunter cultures count 'one, two, many'. Agriculture needed to differentiate between degrees of many: '1, 2, 3, 4, .., 10, 11, 12', etc. To include the bundle-size we can count '01, 02, .., B, 1B1, 1B2', etc.; or '0.1 tens, 0.2 tens', etc. To signal closeness to the bundle we can count '1, 2, …, 7, ten less 2, ten less 1, ten', etc.

A total is re-counted in another unit when asking '7 is how many 3s?' Using squares or LEGO-blocks or an abacus, we can stack the 3-bundles on-top of each other with an additional stack of unbundled 1s next-to so that a total T of 7 1s can be counted in 3s as T = 2 3s and 1:



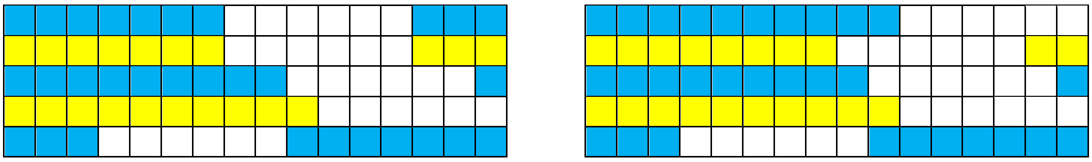
With 'bundle-counting' we place the bundles in a bundle-cup with a stick for each bundle, leaving the unbundled outside. Then, with icons, we report by 'bundle-writing', T = 2B1 3s, and by 'decimal-writing', T = 2.1 3s where a decimal point separates the bundles from the unbundled, always including the unit 3s. A stick moves outside the cup as a bundle of 1s, and inside as 1 bundle. This will change the ‘normal’ form to an ‘overload’ or an ‘underload’:

T = 7 = IIIIIII → III III I → II] I → 2B1 3s = 1B4 3s = 3B-2 3s

Using a plastic letter B for the bundles, we get

T = 7 = IIIIIII → III III I → BBI → 2BI 3s

We include space and time in the two ways of counting: ‘geometry-counting’ in space, and ‘algebra-counting’ in time. Counting in space, we count blocks and report the result on a ten-by-ten abacus in geometry-mode, or with squares. Counting in time, we count sticks and report the result on a ten-by-ten abacus in algebra-mode, or with strokes.



To predict the result, we use a calculator. A stack of 2 3s is iconized as 2x3 (or 2*3) showing a lift used 2 times to stack the 3s. As for taking away, subtraction shows the trace left when taking away just once, and division shows the broom wiping away several times.

So, by entering ‘7/3’ we ask the calculator ‘from 7, 3s can be taken away how many times?’ The answer is ‘2.some’. To find the leftovers we take away the stack of 2 3s by asking ‘7 – 2*3’. From the answer ‘1’ we conclude that 7 = 2B1 3s. Showing ‘7 – 2*3 = 1’, a display indirectly predicts that 7 can be re-counted as 2 3s and 1, or as 2.1 3s.

7 / 3	2.some
7 – 2 * 3	1

A calculator thus uses a ‘recount-formula‘, T = (T/B)*B, saying that ‘from T, T/B times, Bs can be taken away’; and a ‘restack-formula’, T = (T–B)+B, saying that ‘from T, T–B is left if B is taken away and placed next-to’. The two formulas may be shown by using LEGO blocks.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 4 2s as 3B2 2s with an outside overload; or as 5B-2 2s with an outside underload thus leading to negative numbers:

Letters	Sticks	Total T	Calculator	
B B B B	II II II II	4B0 2s	4*2 – 4*2	0
B B B I I	II II II I I	3B2 2s	4*2 – 3*2	2
B B B B B B	II II II II II II	5B-2 2s	4*2 – 5*2	-2

To re-count in a different unit means changing unit, also called proportionality. Asking ‘3 4s is how many 5s?’ we can use sticks or letters to see that 3 4s becomes 2B2 5s.

T = 3 4s = IIII IIII IIII → IIIII IIIII I I = 2B2 5s.

A calculator can predict the result. Entering ‘3*4/5’ we ask ‘from 3 4s, 5s can be taken away how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be re-counted as 2 5s and 2, or as 2B2 5s, or as 2.2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Re-counting in and from tens

Re-counting icon-numbers in the standard bundle-size tens leads to questions as ‘3 4s is how many tens’. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens.

T = 3 4s = IIII IIII IIII → IIII IIII II I I → 1B2 tens = 1.2 tens

Using the recount- and restack-formula above is impossible since the calculator has no ten button. Instead it gives the answer directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

3 * 4	12

Re-counting icon-numbers in tens is called multiplication tables to be learned by heart. However, seeing 3*4 as 3 4s means seeing multiplication as a geometrical block that re-counted in tens will increase its width and therefore decrease its height to keep the total unchanged. Furthermore, the ten-by-ten table can be reduced to a small 4-by-4 table since 5 is half of ten and 6 is ten less 4, 7 is ten less three etc.

Thus T = 4*7 = 4 7s that re-counts in tens as T = 4*7 = 4*(10 – 3) = 40 – 12 = 28 = 2.8 tens; and T = 6*9 = (10 – 4)*(10 – 1) = 100 – 40 – 10 + 4 = 54.

These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

Re-counting tens in icons by asking ‘38 = ? 7s’ is predicted by a calculator as 5.3 7s, i.e. as 5*7 + 3. Since the result must be given in tens, 0.3 7s must be written as what it is, 3 counted in 7s, also called a fraction 3/7, and calculated as 0.428…, shown directly by the calculator:

38 / 7	5.428

Once counted, totals can be added on-top or next-to

Asking ‘3 5s and 2 3s total how many 5s?’ we see that to add on-top, the units must be the same so the 2 3s must be re-counted in 5s as 1B1 5s that added to the 3 5s gives 4B1 5s.

T = 3 5s + 2 3s = IIIII IIIII IIIII + III III → IIIII IIIII IIIII + III II I = 4B1 5s = 4.1 5s.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. So we get 4B1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Since 3*5 is an area, adding next-to means adding areas, called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2B5 8s.

T = 3 5s + 2 3s = IIIII IIIII IIIII + III III → ~~IIIIIIII~~ ~~IIIIIIII~~ IIIII = 2B5 8s = 2.5 8s

Using a calculator to predict the result we use a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away 2 8s leaves 5. So we get 2B5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Reversing adding on-top and next-to

Reversed addition may be called backward calculation or solving equations. Reversing next-to addition may be called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2B6 8s?’, using sticks will give the answer 2B1 3s:

IIIII IIIII IIIII + <u>III</u> <u>III</u> I ← IIIII <u>III</u> IIIII <u>III</u> IIIII I ← II] IIIIII = 2B6 8s

Using a calculator to predict the result, the remaining is bracketed before counted in 3s.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Adding or integrating two areas next-to each other means multiplying before adding. Reversing integration, i.e. differentiation, then means subtracting before dividing, as shown in the gradient formula y' = Δy/t = (y2 – y1)/t.

Double-counting in different units creates per-numbers and proportionality
Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 5kg, or 2$/5kg. To answer the question '6$ = ?kg' we use the per-number to re-count 6 in 2s, that many times we have 5kg: 6$ = (6/2)*2$ = (6/2)*5kg = 3*5kg = 15kg. And vice versa: Asking '?$ = 20kg', the answer is 20kg = (20/5)*5kg = (20/5)*2$ =4*2$ = 8$.

Double-counting in the same unit creates fractions and percentages as per-numbers
Double-counting a quantity in the same unit, per-numbers take the form of fractions, 3$ per 5$ = 3/5, or percentages as 3$ per 100$ = 3/100 = 3%.

Thus, to find 3$ per 5$ of 20$, or 3/5 of 20, we just re-count 20 in 5s and take that 3 times: 20 = (20/5)*5 = 4 5s, which taken 3 times gives 3*4 = 12, written shortly as 20 counted in 5s taken 3 times, 20/5*3.

To find what 3$ per 5$ is per 100$, or 3/5 = ?%, we just re-count 100 in 5s, that many times we take 3: 100 = (100/5)*5 = 20 5s, and 3 taken 20 times is 60, written shortly as 3 taken 100-counted-in-5s times, 3*100/5. So 3$ per 5$ is the same as 60$ per 100$, or 3/5 = 60%.

Adding 3kg at 4$/kg to 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8$/kg. Likewise with adding fractions. Thus, per-numbers and fractions are not numbers, but operators needing a number to become a number, and adding by their areas; thus becoming integration as when adding block-numbers next-to each other. Thus, calculus appears at all school levels: at primary level, at lower and at upper secondary level, and at tertiary level.

Using letters and functions for unspecified numbers and calculations
We can set up a calculation with an unspecified number u, as T = 2 + ? = 2 + u. Also, we can set up an unspecified calculation with an unspecified number u, as T = 2 ? u = f(u), called a formula or a 'function'. Although we can write it, T = f(2) is meaningless since 2 is not an unspecified number.

Comparing Many-matics with Mathe-matics

Meeting Many free of gossip discloses a 'many-matics' quite different from traditional set-based mathe-matics:

As to digits, the former sees them as icons containing as many sticks as they represent; the latter sees them as symbols like letters.

As to natural numbers, the former sees them as two-dimensional blocks described as decimal numbers with a unit and a decimal point to separate inside bundles from outside singles; the latter sees them as examples of a place value system naming points along a one-dimensional number-line.

As to operations, the former sees them as icons describing the three parts of a counting process: bundling by division, stacking by multiplication, removing stacks by subtraction, and uniting stacks by on-top or next-to addition; the latter sees them as mappings from a set-product to a set, and in the opposite order: addition first, then subtraction, then multiplication, then division.

As to calculators, the former sees them as means to predict a re-counting result; the latter typically sees them as hindering understanding.

As to the different forms of counting (bundle-counting, re-counting in the same and a different unit, re-counting to and from tens) the former sees them as means to describe a total by a number to answer the basic question 'how many in total?'; the latter sees bundle-counting and re-counting as irrelevant because of the place value system with base ten.

As to addition, the former sees addition to be postponed till after the total has been bundle-counted and re-counted, and sees on-top and next-to addition to be treated at the same time; the latter sees addition as the first operation to introduce and in no need of re-counting or overloads or next-to addition since numbers are counted in tens only.

As to multiplication, the former sees a product as a number with a unit that may or may not be re-counted in tens, and sees overloads as a natural way to report the result; the latter sees it as a calculation following a specific algorithm.

As to division, the former uses bundle-writing to re-count a number with an overload to make the division easier; the latter sees it as a calculation following a specific algorithm.

As to fractions, the former sees them as per-numbers, both being operators needing a number to produce a number, thus being added by their areas, i.e. by integration; the latter sees fractions as rational numbers that can be added without considering units.

As to proportionality, the former sees it as double-counting in two units creating a per-number as a bridge between the units; the latter sees it as an example of a linear function.

As to equations, the former sees it as a name for reversed calculation to be solved by moving to opposite side with opposite sign; the latter sees it as an equivalence statement to be changed by performing the same operations to both sides aiming at neutralizing the numbers next to the unknown.

As to calculus, the former sees three types, preschool calculus adding blocks next-to each other, middle school calculus adding piecewise constant per-numbers and fractions by their areas, and high school calculus finding the are under a locally constant (continuous) graph; the latter sees differential calculus as preceding integral calculus.

Testing a Many-Matics Micro-Curriculum

A '1 cup and 5 sticks' micro-curriculum can be designed to help a class stuck in division. The intervention begins by bundle-counting 5 sticks in 2s, using the cup for the bundles. The results, 1B3 2s and 2B1 2s and 3B-1 2s, show that a total can be counted and written in 3 ways, overload and normal and underload. So, to divide 336 by 7, 5 bundles are moved outside as 50 singles to re-count 336 with an overload: 336 = 33B6 = 28B56, which divided by 7 gives 4B8 = 48. With multiplication singles move inside as bundles: 7* 4[8 = 28[56 = 33[6 = 336.

Ending the Dienes Era

No research literature on bundle-counting was found. However, similar ideas were found at Dienes, the inventor of Multi-base blocks. As to the place value system, Dienes says:

> I have been suggesting, for the past half century, that different bases be used at the start, and to facilitate understanding of what is going on, physical materials embodying the powers of various bases should be made available to children. Such a system is a set of multibase blocks (..) Educators today use the "multibase blocks", but most of them only use the base ten, yet they call the set "multibase". These educators miss the point of the material entirely. (Dienes, 2002, p. 1)

Dienes wants children to use physical blocks to understand counting with different bases and the role of power. So here mathematics is the goal and blocks are a means. Which Dienes makes clear by his 6 stages to real understanding of set-based mathematics: Free play, following rules, comparison, representation, symbolization, formalization. (Clouthier, 2010)

A Dienes-approach to mathematics thus is an example of a goal displacement having self-referring set-based mathematics as its goal, and dedicated blocks as a means.

Conclusion

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject and a verb and an object or predicate, ‘This is a chair’. As does the number-language assigning numbers to like things, ‘the 3 chairs have 4 legs each’, abbreviated to ‘the total is 3 fours’, or ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, and with the number-language in its original form where mathematics was a common name for algebra and geometry both rooted in the physical fact Many. But in its present self-referring set-based form, mathematics has turned into ‘meta-matism’, a grammar for the number-language seen as the goal in mathematics education, using both the outside world and the number-language as means. So, by its self-reference and by its difference from many-matics, mathe-matics has a goal displacement.

Recommendation

There is no second chance to make a first impression. So how learners meet mathematics matters, both in early childhood and as migrants. Consequently, education preparing for the outside world should bring into the classroom examples of Many, becoming Totals when bundle-counted as block-numbers with units, thus ready to be re-counted in the same unit or in another unit or in or from tens, always illustrated algebraically using bundle-writing and geometrically using LEGO-blocks. Keeping algebra and geometry together introduces negative numbers and proportionality before introducing addition, that might even be postponed to after double-counting in physical units have introduced per-numbers and fractions and percentages.

The core of mathematics education, proportionality and calculus and equations, lies in counting. So, to improve mathematics education, counting and multiplication come before adding; number-sentences are written out fully as ‘T = 6*7’ instead of just ‘6*7 = 42’ depriving the total of its true identity as 6 7s by forcing it to be re-counted in tens right away; bundle-counting and re-counting in tens precede the place value system introduced as a sloppy but quicker way to write natural numbers without units and with misplaced decimal point; addition includes both the on-top and next-to version, both followed by reversed addition leading to equations. Finally, double-counting leads to per-numbers and fractions and percentages, all added by their areas thus creating the right order in calculus by letting integration precede differentiation.

Thus, mathematics education should begin with number-language sentences as T = 6*7 = 6 7s, or its general form as the recount-formula T = (T/B)*B occurring all over mathematics: when re-counting in another unit; when solving equations by re-counting, leading to the ‘opposite side & sign’-method, u*3 = 24 = (24/3)*3 giving u = 24/3 = 8; when double-counting to change unit; when relating proportional quantities; in trigonometry as a = a/c*c = sinA*c; and in calculus as dy = dy/dx*dx = y’*dx. And addition should begin as calculus when adding blocks next-to before on-top.

In short, fixation of mathematics as a goal in itself, hides the real goal, its outside root Many. So stop teaching self-referring meta-matism and start teaching many-matics, a natural science about the physical fact Many.

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11 Proposals for the Mathematics Biennale 2018

01) Start-math for children and migrants: Bundle-count and re-count before adding

A 3year old sees 4 fingers held together 2 by 2: "It is not 4, it is 2 2s". So a child counts in the block-numbers as we do: 456 = 4 bundle-bundles + 5 bundles + 6 unbundled. The child's block-numbers lead directly to proportionality and equations. So we should count before we add.

Digits units many sticks in one icon: Five sticks in the 5-icon etc.; up to ten = 1bundle0 = 1B0 = 10.

With a cup for bundles, a total T of 7 sticks is bundle-counted in icon-bundles as T = 7 = 2 3s & 1 = 2B1 3s. Next, the total can be re-counted in the same unit to create overload or underload:

T = 7 = 2B1 3s = 1B4 3s = 3B-2 3s.

A total can also be re-counted in a new unit (proportionality), e.g. 2 4s = ? 5s, predicted by a calculator as 2*4/5 = 1 and 2*4 – 1*5 = 3, so T = 2 4s = 1B3 5s.

We count by bundling and stacking predicted by operations, also being icons: Counting a total 8 in the 2s, 8/2 shows the broom that from 8 sweeps 2s away. Multiplication 4x2 shows the lift that stacks the 4 2s, and subtraction 8 – 2 shows the trace created by from 8 dragging 2 away. The result may therefore be predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

Re-counting from icon-bundles to 10s leads to the multiplication table:

T = 3 4s = 3*4 = 12 = 1ten2 = 1B2 10s.

Reversing by re-counting from 10s to icons creates equations to be solved by re-counting: 'How many 5s give 40' leads to the equation ‘x*5 = 40’ solved by recounting 40 in 5s: 40 = (40/5)*5, giving x = 40/5. So an equation is solved by moving to the opposite side with the opposite sign.

02) Multiplication before addition strengthens the number sense in children and migrants

We master Many using a number-language with number-language sentences, formulas, e.g. T = 4 5s = 4*5, showing how we master Many by bundling and stacking. So 4*5 is 4 5s that may be re-counted in another unit, e.g. in 7s. Or in tens, the international bundle-size.

Viewing numbers as bundle-formulas makes math easy and prevents math-problems and dyscalculia; therefore, to be practiced with various counting rhymes where '5, 6, 7, 8, 9, 10' is counted also as ' 5, bundle less 4, B-3, B-2, B-1, bundle'; and as ' ½bundle, ½bundle & 1, ½, ½bB&2, ½B &3, ½B &4, bundle. Likewise, '10, 11, 12, 13, 14, 15 ' can be counted as 'bundle, 1bundle & 1, 1B & 2, 1B & 3, 1B & 4, 1B & 5 ', and as 'Bundle, 1left, 2left, 3left, 4left, 5left ' in order to show that 'eleven' and 'twelve' is derived from the Viking age.

Digits unite many sticks into one icon: Five sticks in the 5-icon, etc., up to ten = 1bundle0 = 1B0 = 10. With a cup for bundles, a total T of 7 sticks is bundle-counted in icon-bundles as T = 7 = 2 3s & 1 = 2B1 3s. Next, the total can be re-counted in the same unit to create overload or underload: T = 7 = 2B1 3s = 1B4 3s = 3B-2 3s. Likewise with totals counted in tens, T = 68 = 6B8 = 5B18 = 7B-2.

Before adding, the number sense is trained by the multiplication table, reduced to a 5 x 5 table by rewriting number above 5, e.g. 6 = ½bundle & 1 = bundle-4. First doubling, e.g. T = 2*7 = 2*(½ bundle & 2) = bundle & 4 = 14, or T = 2*7 = 2*(bundle-3) = 20-6 = 14. Then with bundle-counting, e.g. T = 2*38 = 2*3B8 = 6B16 = 7B6 = 76. Then halving, e.g. ½*38 = ½*3B8 = ½*4B-2 = 2B-1 = 19.

Multiplying with 5 means multiplying with half-bundles, 5*7 = ½bundle*7 = ½*70 = ½* 6B10 = 3B5 = 35.

03) Dislike towards division cured with 5 sticks and 1 cup and bundle-writing

A class has problems with division, e.g. 336/7. The solution is to see 336/7, not as 336 divided among 7, but as 336 counted in 7s; and to use bundle-writing 336 = 33B6, where the cup splits the total in bundled within the cup and unbundled outside.

And to bundle-count totals in three ways: normal and with overload or underload.

First with 5 sticks bundle-counted in 2s with a cup to the bundles.

Normal: T = IIIII = II II I = 2B1 2s.

With overload: T = IIIII = II III = 1B3 2s.

With underload: T = IIIII = II II I$\underline{\text{I}}$ = 3B-1 2s.

In the same way we count in 10s: T = 74 = 7B4 = 6B14 = 8B-6.

So, with a total of 336 (i.e. 33.6 tens) there are 33 bundles inside the cup and 6 unbundled outside. But we prefer 28 within, so 5 bundles move outside as 50 giving 56 outside that divided by 7 gives 4 inside and 8 outside:

T = 336 = 33B6 = 28B56, and T/7 = 4B8 = 48.

Bundle-writing can be used by all operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65-27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7 * 48 = 7 * 4B8 = 28B56 = 33B6 = 336

T = 7 * 48 = 7 * 5B-2 = 35B-14 = 33B6 = 336

T = 336/7 = 33B6/7 = 28B56/7 = 4B8 = 48

T = 338/7 = 33B8/7 = 28B58/7 = 4B8 + 2/7 = 48 2/7

Bundle-writing can also be used with the multiplication table:

T = 4*8 = 4*1B-2 = 4B-8 = 32 and 7*8 = 7*2 = 1B-7B-14 = 6B-4 = 5B6 = 56

04) Fractions and percentages as per-numbers

A class has trouble with fractions. Both to find a fraction of a total, and to expand and shorten fractions, where many add and subtract instead of multiplying and dividing.

The solution is to see a fraction as a per-number obtained by a double-counting in the same unit, 2/3 = 2$ per 3$ or as percentage 2% = 2/100 = 2$ per 100$.

Investment is expected to give a return, which may be higher or lower, e.g. 7$ per 5$ or 3$ per 5$.

With re-counting and double-counting we use a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

Now 2/3 of 12 is found as 2$ per 3$ of 12$. So we re-count 12 in 3s as 12$ = (12/3)*3$ giving (12/3)*2$ = 8$. So 2/3 of 12 is 8.

The task 'what is 3 in percent of 5? ' is solved by re-counting 100 in 5s and replace 5$ with 3$: T = 100$ = (100/5)*5$ giving (100/5)*3$ = 60$. So 3/5 is the same as 60 per 100, or 3/5 = 60%.

To expand or shorten a fractions is done by inserting or removing the same unit above and below the fraction bar: T = 2/3 = 2 4s/3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6.

Fractions and decimal numbers should be introduced in grade 1 relating to counting in icons under ten. 7 counted in 3s gives a stack on the 2 3s & 1. The unbundled 1 can be placed next-to as its own stack, a decimal number, T = 7 = 2.1 3s. Or it can be placed on-top counted as 3s, i.e. as a fraction: T = 7 = 2 1/3 3s.

05) Fractions and per-numbers add as integration

A class have problems with adding fractions. Many adds the numerators and the denominators separately.

The solution is to view a fraction as a per-number obtained from double-counting in the same unit, 3/5 = 3$ per 5$, or as the percentage of 3% = 3/100 = 3$ per 100$. As well as to begin with adding fractions with units, such as ½ of 2 + 2/3 of 3, that just gives 1 + 2 of 2 + 3, so 3/5 of 5. Here, then, ½ + 2/3 = 3/5, which is obtained by adding the numerators and denominators separately.

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg. So here is 3$/kg + 5$/kg = 4.33 $/kg, called the weighted average.

Geometrically, adding products means adding areas, called integration. So per-numbers add by their areas under the piecewise constant per-number graph. Corresponding with fractions.

Adding two fractions a/b and c/d without units is meaningless, but can be given meaning if taken of the same total, b*d: a/b of b*d + c/d of b*d gives a total on a*d + c*b of b*d.

So a/b + c/d = (a*d + c*b)/b*d.

Adding fractions and per-numbers with units provides a good introduction to calculus. As shown, multiplication before addition is the same as integration. And inverted integration is the same as differentiation:

The task ‘2kg at 3$/kg + 4kg at ? $/kg = 6kg á 5$/kg’ leads to the equation 6 + 4*x = 30 or T1 + 4*x = T2, solved with subtraction before division, called differentiation: x = (T2-T1)/4 = ΔT/4.

06) Proportionality as double-counting, with per-numbers

A class has a problem with proportionality. The price is 2$/3 kg. All will find the $-number for 12 kg, but only a few will find kg-number for 16$. The solution is to rename proportionality to ‘shifting units’ by 'double-counting', leading to 'per-numbers' such as 2$ per 3 kg or 2$/3kg or 2/3 $/kg. The units are connected by re-counting the known part of the per-number.

With re-counting and double-counting we use a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B'.

This allows re-counting 16$ in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24kg. Likewise, the 12kg re-counts in 3s as T = 12 kg = (12/3)*3 kg = (12/3)*2$ = 8$. Will this difference make a difference? In theory, yes, since proportionality is associated with counting, a basic physical activity.

In fact, proportionality takes place in grade 1 when counting totals in icon-bundles different from the standard bundle ten and by afterwards re-counting the total in a different unit. This leads directly to the re-count formula, which has the same shape as y = k*x.

Thus, a total of 8 re-counts in 2s as T = (8/2)*2 = 4*2 = 4 2s.

And a total of 3 4s re-counts in 5s as T = (3*4/5)*5 = 2*5 + 2.

And per-numbers lead directly on to the fractions, obtained by double-counting in the same unit, e.g. 2$ per 3$ = 2$/3$ = 2/3 = 2 per 3.

Getting 2/3 of 15 means getting 2$ per 3$ of 15$ found by re-counting 15 in 3s and take 215 thereof: T = 15$ = (15/3)*3$ giving (15/3)*2$ = 10$. So 2/3 of 15 is 10.

Likewise, 20% of 15 is found by re-counting 15 in 100s:

T = 15 = (15/100)*100 giving (15/100)*20 = 3.

07) Equations solved by moving, reversing or re-counting

A class has problems with the equations 2+3*u = 14 and 25 – u = 14 and 40/d = 5, where the equation is composite or where the unknown has a inverse sign. The solution is to use the definitions of the inverse operations to create the basic solution rule: 'move to the opposite side with the opposite sign'.

In $u+3 = 8$ we seek a numbers u that added to 3 gives 8, which is $u = 8-3$ by definition; so + 3 moves to the opposite side with the opposite sign. Corresponding with $u*2 = 8$, solved by $u = 8/2$; and with $u^3 = 12$, solved by $u = 3\sqrt{12}$, where the root is a factor-finder; and with $3^u = 12$, solved by $u = \log3 (12)$, where the logarithm is a factor-counter.

The equation $2+3*u = 14$ can be seen as a double calculation that is reduced to a single by a bracket around the stronger operation, $2+(3*u)$. Moving 2 to the opposite side with the opposite sign gives $3*u = 14–2$. Then 3 moves to the opposite side with opposite sign, but first a bracket is placed around what first must be calculated: $u = (14–2)/3 = 12/3 = 4$.

Equations can also be solved by walking forward and backward: Forward we first multiply with 3 and then add 2. Backwards, we first subtract 2 and then divide by 3, so $u = (14–2)/3 = 4$.

In the equation $25 – u = 14$, u has opposite sign and therefore moves to the opposite side to get a normal sign before 14 moves to the opposite side with opposite sign:

$25 = 14 + u$; $25–14 = u$; $11 = u$.

Corresponding with $40/u = 5$ giving $40 = 5*u$ and $40/5 = u$ or $8 = u$.

Having learned re-counting this can also be used:
$40 = (40/u)*u = 5*u$ and $40 = (40/5)*5$, giving $u = 40/5$.

08) Calculus: Addition of and division into locally constant per-numbers

A class has problems with calculus. The solution is to postpone differential calculus until after integral calculus is taught as a means of adding piecewise or locally constant per-numbers by their areas.

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg.

Geometrically, adding products means adding areas, called integration. So per-numbers add by their areas under the piecewise constant per-number graph, i.e. by adding a few area strips, $S = \Sigma p*\Delta x$.

A non-constant per-number is locally constant (continuous), meaning adding of countless many area strips, $S = \int p*dx$. Unless we can rewrite the strips as changes, $p*dx = dy$ or $dy/dx = p$. For when adding changes, all middle terms disappear leaving just the total change from the start to the end point.

This motivates differential calculus: If the strip $2*x*dx$ can be rewritten as a change, $d (x^2)$, then the sum $\int 2*x*dx$ is the change of x^2 from the start to the end point.

Change-formulas can be observed in a rectangle, where changes Δb and Δh in the base b and height h gives the total change of the area ΔT as the sum of a vertical strip, $\Delta b*h$ and a horizontal strip, $b*\Delta h$; and a corner, $\Delta b*\Delta h$ that can be neglected with small changes.

Therefore, $d(b*h) = db*h + b*dh$, or, if counted in Ts:

$dT/T = db/b + dh/h$, or with $T' = dT/dx$, $T'/T = b'/b + h'/h$.

So with $(x^2)'/x^2 = x'/x + x'/x = 2*x'/x$, $(x^2)' = 2*x$ since $x' = dx/dx = 1$.

So differentiation is a smart way to add many numbers; but also useful to describe growth and decay and optimization.

09) Calculus in primary, middle and high school

Mathematics means knowledge in Greek, who chose the word as a common name for their four areas of knowledge, arithmetic and geometry and music and astronomy, which they saw as the study of many by itself, in space, in time and in space and time.

With music and astronomy gone, today mathematics is just a common name for algebra and geometry, both rooted in Many as evidenced by their meaning in Arabic and Greek: to reunite numbers and to measure the earth. Meeting Many we ask 'How many in total?' The answer we get by counting, before we add. Counting is done by bundling and stacking,

predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B', e.g. T = 3 4s = (3*4)/5*5 = 2 5s & 2.

Once counted, stacks can be added, but on-top or next-to?

Next-to addition of the stacks 2 3s and 4 5s as 8s means adding their areas, i.e. by integration, where multiplication comes before addition.

Reversed, we ask '2 3s +? 5s gives 3 8s', now letting subtraction come before division, called differentiation.

So in primary school, calculus occurs with next-to addition of stacks.

In middle school calculus occurs with blending and average tasks:

When adding per-numbers with units, e.g. 2kg at 3$/kg + 4kg at 5$/kg, the unit-numbers 2kg and 4kg directly to 6kg, while the per-numbers must be multiplied before added: 3*2$ + 5*4$ = 26$. So the answer is 6kg á 26/6 $/kg.

Geometrically, adding products means adding areas, called integration.

Thus per-numbers add by the area under the piecewise constant per-number graph, i.e. by adding area strips, S = Σp*Δx, or S = ∫ p*dx in high school, where per-numbers are locally constant (continuous), and where per-numbers are added before they can be subtracted by differentiation.

10) Stem-based core-math makes migrants pre-engineers

We master the outside world by a word-language and a number-language, describing it by sentences and formulas containing a subject, a verb, and a predicate: 'the table is yellow' and 'the total is 3 4s'. The two languages both have a meta-language, a grammar and a mathematics, that should be learned after the language, otherwise causing dyslexia and dyscalculia.

Young migrants get direct access to the number-language with core-math curriculum:

A) Digits are the icons with the number of sticks, it represents.

B) Operations are icons for counting by bundling and stacking: division removes bundles, multiplication stack bundles, subtraction removes a stack to look for unbundled, addition unites stacks on-top or next-to.

C) Bundle-counting and bundle-writing shows the bundles inside the cup and the un-bundled outside, e.g. T = 4B5 = 4.5 tens = 45.

D) Totals must be bundle-counted and re-counted and double-counted before they can add.

E) Re-counted in the same unit, a total can be written in 3 ways: normal, with overload or with underload, e.g. T = 46 = 4B6 = 3B16 = 5B-4.

F) Re-counting in a new unit (proportionality) be predicted by a 're-count-formula' (T) = (T/B)*B, saying 'From T, T/B times we can remove B', e.g. T = 3 4s = (3*4)/5*5 = 2 5s & 2.

G) Re-counting from tens to icons creates equations, e.g. x*5 = 40 = (40/5)*5 with solution x = 40/5. Double-counting gives per-numbers and proportionality with re-counting in the per-number: with 2$ per 3kg, 12 kg = (12/3)*3 kg = (12/3)*2$ = 8$.

H) After counting comes addition, on-top and next-to, leading to proportionality and integration.

I) Reverse addition leads to equations and differentiation.

J) Per-numbers lead to fractions, both operators needing to be multiplied to become numbers, and therefore added by their areas, i.e. by integration.

K) Calculus means adding and splitting into locally constant per-numbers.

L) Core STEM-areas become applications under the theme 'water in movement'.

Details: ' A STEM-based Core Math Curriculum for Outsiders and Migrants ', http://mathecademy.net/papers/miscellaneous/

11) The teacher as a difference-researcher

When traditions give problems, difference research uncovers hidden differences that make a difference. For example, the tradition says that ‘a function is an example of a set relation where first component identity implies second component identity’, which the learner hears as ‘bublibub is an example as bablibab ', which nobody finds meaningful. A difference is to use Euler's original definition accepted by all: 'a function is a common name for calculations with both known and unknown numbers'.

Difference research can be used by teachers to solve problems in class, or by teacher-researchers sharing their time between academic work at a university and intervention research in a class. Or by full-time researchers, working with teachers to apply difference research: the teacher observes the problems, the researcher identifies the differences. Together they establish a micro-curriculum to be tested by the teacher and reported by the researcher after a pretest-posttest study. A typical difference researcher begins as an ordinary teacher who reflects on whether alternatives can solve learning problems observed.

A difference-researcher combines elements of action learning and action research and intervention research and design research. First a difference is identified, then a micro-curriculum is designed to be tested to see what kind of difference it makes. The effect will be reported internally and discussed with colleagues. After repeating this cycle of design, teaching and internal reporting, it is time for an external reporting of the difference and its effect in magazines or journals or at conferences.

Research should provide knowledge to explain nature and to improve social conditions. But as an institution in runs the danger of the what the sociologist Bauman calls a goal displacement, so research will be self-referencing instead of finding differences. Hargreaves, write for example: ‘What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads’ (Hargreaves, 1996, p. 7).

Hargreaves, D.H. (1996). *Teaching as a Research-based Profession: Possibilities and Prospects.* Cambridge: Teacher Training Agency Lecture.

Fifty years of research without improving mathematics education, why?

Within education, mathematics is in the front. Consequently, research has grown rapidly for fifty years to solve its many learning problems. The lack of success is shown by the PISA studies showing a low level and a continuing decline in many countries. Thus, to help the former model country Sweden, OECD wrote a critical 2015 report ‘Improving Schools in Sweden, an OECD Perspective’.

At the CERME 10 congress in February 2017 a plenary session asked: What are the solid findings in mathematics education research? To me, the short answer is “Only one: to improve, mathematics education should ask, not what to do, but what to do differently.”

Thus, to be successful, research should not study problems but look for differences that make a difference. Research that is skeptical towards institutionalized traditions could be called difference research. In France, Lyotard calls it ‘paralogy’ inventing dissension to the reigning consensus.

Difference research scarcely exists today since it is rejected at conferences for not applying or extending existing theory.

To elaborate, maybe mathematics education research is sterile because its three words are not that well defined?

As to mathematics, it has meant many different things in its almost 5000 years of history spanning from a natural science about the physical fact Many to a self-referring logic.

As to education, two different forms exist: a continental European education serving the nation's need for public servants though multi-year compulsory classes and lines at the secondary and tertiary level; and a North American education aiming at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks together with one-subject teachers.

As to research, academic articles can be written at a master level applying existing theories, or at a research level questioning them. Just following theories is problematic in the case of conflicting theories as within education where Piaget and Vygotsky contradict each other by saying teach as little and as much as possible.

Consequently, you cannot know what kind of mathematics and what kind of education has been studied, and you cannot know if research is following ruling traditions or searching for alternatives. So, if institutionalized education should help children and youngsters to master outside phenomena we must ask: What outside phenomena roots mathematics?

We master the outside world with two languages, a word-language and a number-language. Children learn to talk and to count at home. Then, as an institution, school takes over and teaches children to read and to write and to calculate.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, 'This is a chair', and 'T = 3*4'. Both languages have a meta-language, a grammar, describing the language, describing the world. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

So, one way of improving mathematics education is to respect that language comes before meta-language. Which was also the case in continental Europe before the arrival of the 'New Math' that turned mathematics upside down to become a 'meta-matics' presenting its concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically and which would present mathematics as 'many-matics', a natural science about Many.

Before New Math, Germanic countries taught counting and reckoning in primary school. Then the lower secondary school taught algebra and geometry, which are also action words meaning to reunite totals and to measure earth in Arabic and in Greek. 50 years ago, New Math made all these activities disappear.

Thus, one alternative immediately presents itself: Re-root mathematics in its historic origin as a common label chosen by the Pythagoreans for their fours knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about pure numbers, number in space, number in time, and number in space and time. The four combined in the quadrivium, a general curriculum recommended by Plato. So, with music and astronomy gone, today mathematics should be but a common label for algebra and geometry, both activities rooted in the physical fact Many by meaning 'reuniting numbers' and 'measuring earth' in Arabic and Greek respectively.

Consequently, to improve its education, mathematics should stop teaching top-down meta-matics from above and begin teaching bottom-up many-matics from below instead.

For details, see 'Difference-Research Powering PISA Performance', Fifty Years of Research without Improving Mathematics Education', and 'A STEM-based Core Math Curriculum for Outsiders and Migrants' at http://mathecademy.net/papers/miscellaneous/.

12. The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants

Swedish educational shortages challenge traditional mathematics education offered to migrants. Mathematics could be taught in its simplicity instead of as 'meta-matsim', a mixture of 'meta-matics' defining concepts as examples of inside abstractions instead of as abstractions from outside examples; and 'mathe-matism' true inside classrooms but seldom outside as when adding numbers without units. Rebuilt as 'many-matics' from its outside root, Many, mathematics unveils its simplicity to be taught in a STEM context at a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

Other countries also experience declining PISA results. Since mathematics education is a social institution, social theory might be able to explain 50 years of unsuccessful research in mathematics education.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16). A wish to uncover unnoticed alternatives motivates a 'difference-research' (Tarp, 2017) asking two questions: 'Can this be different – and will the difference make a difference?' If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

Natural sciences use difference-research to keep on searching until finding what cannot be different. Describing matter in space and time by weight, length and time intervals, they all seem to vary. However, including per-numbers will uncover physical constants as the speed of light, the gravitational constant, etc. The formulas of physics are supposed to predict nature's behavior. They cannot be proved as can mathematical formulas, instead they are tested as to falsifiability: Does nature behave different from predicted by the formula? If not, the formula stays valid until falsified.

Social sciences also use difference-research beginning with the ancient Greek controversy between two attitudes towards knowledge, called 'sophy' in Greek. To avoid hidden patronization, the sophists warned: Know the difference between nature and choice to uncover choice presented as nature. To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy. The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord deciding world behavior.

Today's democracies implement common social goals through institutions with means decided by parliaments. As to rationality as the base for social organizations, Bauman says:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called *goal displacement*. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

As an institution, mathematics education is a public organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts. In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about shapes and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers, operations and calculations. However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. And, being sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens, might speed up writing but might also slow down learning, together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, in Lincolns Gettysburg address, ‘Four scores and ten years ago’ shows that not all count in tens.

To get an answer to the questions ‘What is mathematics?’ and ‘How is mathematics education improved?’ we might include philosophy in the form of what Bauman calls ‘the second Copernican revolution’ of Heidegger asking the question: What is ‘is’? (Bauman, 1992, p. ix).

> Inquiry is a cognizant seeking for an entity both with regard to the fact <u>that</u> it is and with regard to its Being <u>as</u> it is. (Heidegger, 1962, p. 5)

Heidegger here describes two uses of ‘is’. One claims existence, ‘M is’, one claims ‘how M is’ to others, since what exists is perceived by humans wording it by naming it and by characterizing or analogizing it to create ‘M is N’-statements.

Thus, there are four different uses of the word ‘is’. ‘Is’ can claim a mere existence of M, ‘M is’; and ‘is’ can assign predicates to M, ‘M is N’, but this can be done in three different ways. ‘Is’ can point down as a ‘naming-is’ (‘M is for example N or P or Q or …’) defining M as a common name for its volume of more concrete examples. ‘Is’ can point up as a ‘judging-is’ (‘M is an example of N’) defining M as member of a more abstract category N. Finally, ‘is’ can point over as an ‘analogizing-is’ (‘M is like N’) portraying M by a metaphor carrying over known characteristics from another N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: ‘I am’ and ‘it is’ and ‘they are’; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the ‘I’ (Dasein) must create an authentic relationship to the ‘It’. However, this is made difficult by the ‘dictatorship’ of the ‘They’, shutting the ‘It’ up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one’s own Dasein completely into the kind of Being of ‘the Others’, in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the “they” is unfolded. (..) Discourse, which belongs to the essential state of Dasein’s Being and has a share in constituting Dasein’s disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

In France, Heidegger inspired the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu, pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Lyotard, 1984. Bourdieu, 1970. Chomsky et al, 2006).

From a Heidegger view a sentence contains two things: a subject that exists, and the rest that might be gossip. So, to discover its true nature hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its predicate-prison. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help rebuilding their original countries.

So, to restore its authenticity, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many

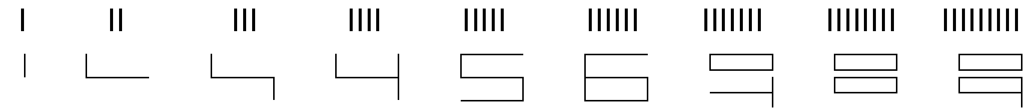
As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep the balance and to store sounds assigned to what we grasped with our forelegs, now freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into two languages, a word-language and a number-language.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’. Observing the existence of many chairs, we ask ‘how many in total?’ and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, ‘the total is 3 chairs’ or, if counting legs, ‘the total is 3 fours’, abbreviated to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

With 2017 as the 500year anniversary for Luther’s 95 theses, we can choose to describe meeting Many in theses.

01. Using a folding ruler, we discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the four icon, and five sticks in the five icon, etc. Counting in 5s, the counting sequence is 1, 2, 3, 4, Bundle, 1-bundle-1, etc. This shows, that the bundle-number does not need an icon. Likewise, when bundling in tens. Instead of ten-1 and ten-2 we use the Viking numbers eleven and twelve meaning 1 left and 2 left in Danish.



1 2 3 4 5 6 7 8 9

02. Transforming four ones to a bundle of 1 4s allows counting with bundles as a unit. Using a cup for the bundles, a total can be ‘bundle-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘bundle-writing’ to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, we discover that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s. Next, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Or, with tens: T = 234 = 23B4 = 2BB3B4.

03. Recounting in the same unit by creating or removing overloads or underloads, bundle-writing offers an alternative way to perform and write down operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

04. Asking a calculator to predict a counting result, we discover that also operations are icons showing the three tasks involved when counting by bundling and stacking. Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication, iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s. This prediction holds at a manual counting: I I I I I I I = III III I. Geometrically, placing the unbundled single next-to the stack of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the stack makes it 1/3 3s, so 1/3 3s = 0.1 3s. Likewise when counting in tens, 1/ten tens = 0.1 tens. Using LEGO bricks to illustrate T = 3 4s, we discover that a block-number contains two numbers, a bundle-number 4 and a counting-number 3. As positive integers, bundle-numbers can be added and multiplied freely, but they can only be subtracted or divided if the result is a positive integer. As arbitrary decimal-numbers, counting-numbers have no restrictions as to operations. Only, to add counting-numbers, their bundle-number must be the same since it is the unit, T = 3*4 = 3 4s.

05. Wanting to describe the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, we discover two formulas. The 'recount formula' T = (T/B)*B says that 'from T, T/B times B can be taken away' as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the 'restack formula' T = (T–B)+B says that from T, T–B is left when B is taken away and placed next-to, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. The recount or proportionality formula turns out to a very basic formula. It turns up in proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right-angled triangles, and in calculus as dy = (dy/dx)*dx = y'*dx when counting steepness on a curve.

06. Wanting to recount a total in a new unit, we discover that a calculator can predict the result when bundling and stacking and dragging away the stack. Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Also we discover that changing units is officially called proportionality or linearity, a core part of traditional mathematics in middle school and at the first year of university.

07. Wanting to recount a total in tens, we discover that a calculator predicts the result directly by multiplication, only leaving out the unit and misplacing the decimal point. Thus, asking T = 3 7s = ? tens, the calculator predicts: T = 21 = 2.1 tens. Geometrically it makes sense that increasing the width of the stack from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged. With 5 as half of ten, and 8 as ten less 2, a 10x10 multiplication table can be reduced to a 3x3 table including the numbers 2, 3 and 4. Thus, 4*8 = 4*(ten less 2) = 4ten less 8 = 32; 5*8 = half of 8ten = 4ten = 40; 7*8 = (ten less 3)*(ten less 2) = tenten, less 3ten, less 2ten, plus 6 = 56.

Wanting to recount a total from tens to icons, we discover this as another example of recounting to change the unit. Thus, asking T = 3 tens = ? 7s, the calculator predicts: First 30/7 = 4.some; then 30 – (4*7) = 2; and finally T = 30 = 4.2 7s. Geometrically it again makes sense that decreasing the width means increasing the height to keep the total unchanged.

08. Using the letter u for an unknown number, we can rewrite the recounting question '? 7s = 3tens' as 'u*7 = 30' with the answer 30/7 = u since 30 = (30/7)*7, officially called to solve an equation. Here we discover a natural way to do so: Move a number to the opposite side with the opposite calculation sign. Thus, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, obtained again by moving a number to the opposite side with the opposite calculation sign.

09. Once counted, totals can be added, but addition is ambiguous. Thus, with two totals T1 = 2 3s and T2 = 4 5s, should they be added on-top or next-to each other? To add on-top they must be recounted to have the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 =

3.2 8s. So next-to addition geometrically means adding areas, and algebraically it means combining multiplication and addition. Officially, this is called integration, a core part of traditional mathematics in high school and at the first year of university.

10. Also we discover that addition and other operations can be reversed. Thus, in reversed addition, 8 = u+2, we ask: what is the number u that added to 2 gives 8, which is precisely the formal definition of u = 8–2. And in reversed multiplication, 8 = u*2, we ask: what is the number u that multiplied with 2 gives 8, which is precisely the formal definition of u = 8/2. Also we see that the equations u^3 = 20 and 3^u = 20 are the basis for defining the reverse operations root, the factor-finder, and logarithm, the factor-counter, as u = 3√20 and u = log3(20). Again we solve the equation by moving to the opposite side with the opposite calculation sign. Reversing next-to addition we ask 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer u, from the terminal total T we remove the initial total T1 before we count the rest in 5s: u = (T–T1)/5 = ΔT/5, using Δ for the difference or change. Letting subtraction precede division is called differentiation, the reverse operation to integration letting multiplication precede addition.

11. Observing that many physical quantities are 'double-counted' in two different units, kg and dollar, dollar and hour, meter and second, etc., we discover the existence of 'per-numbers' serving as a bridge between the two units. Thus, with a bag of apples double-counted as 4$ and 5kg we get the per-number 4$/5kg or 4/5 $/kg. As to 20 kg, we just recount 20 in 5s and get T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. As to 60$, we just recount 60 in 4s and get T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

12. Economy is based upon investing money and expecting a return that might be higher or lower than the investment, e.g. 7$ per 5$ or 3$ per 5$. Here when double-counting in the same unit, per-numbers become fractions, 3 per 5 = 3/5, or percentages as 3 per hundred = 3/100 = 3%. Thus, to find 3 per 5 of 20, or 3/5 of 20, as before we just recount 20 in 5s and replace 5 with 3, T = 20 = (20/5)*5 giving (20/5)*3 = 12.

To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100 in 5s and replace 5 with 3: T = 100 = (100/5)*5 giving (100/5)*3 = 60. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%. Also we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly, but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise when adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

13. Halved by its diagonal, a rectangle splits into two right-angled triangles. Here the angles are labeled A and B and C at the right angle. The opposite sides are labeled a and b and c.

The height a and the base b can be counted in meters, or in diagonals c creating a sine-formula and a cosine-formula: a = (a/c)*c = sinA*c, and b = (b/c)*c = cosA*c. Likewise, the height can be recounted in bases, creating a tangent-formula: a = (a/b)*b = tanA*b

As to the angles, with a full turn as 360 degrees, the angle between the horizontal and vertical directions is 90 degrees. Consequently, the angles between the diagonal and the vertical and horizontal direction add up to 90 degrees; and the three angles add up to 180 degrees.

An angle A can be counted by a protractor, or found by a formula. Thus, in a right-angled triangle with base 4 and diagonal 5, the angle A is found from the formula cosA = a/c = 4/5 as cos-1(4/5) = 36.9 degrees.

The three sides have outside squares with areas a^2 and b^2 and c^2. Turning a right triangle so that the diagonal is horizontal, a vertical line from the angle C split the square c^2 into two rectangles. The rectangle under the angle A has the area (b*cosA)*c = b*(cosA*c) = b*b = b^2. Likewise, the rectangle under the angle B has the area $(a*\cos B)*c = a*(\cos B*c) = a*a = a^2$. Consequently $c^2 = a^2 + b^2$, called the Pythagoras formula.

This allows finding a square-root geometrically, e.g. $x = \sqrt{24}$, solving the quadratic equations $x^2 = 24 = 4*6$, if transformed into a rectangle. On a protractor, the diameter 9.5 cm becomes the base AB, so we have 6units per 9.5cm. Recounting 4 in 6s, we get 4units = (4/6)*6units = (4/6)*9.5 cm = 6.33 cm. A vertical line from this point D intersects the protractor's half-circle in the point C. Now, with a 4x6 rectangle under BD, BC will be the square-root $\sqrt{24}$, measured to 4.9, which checks: $4.9^2 = 24.0$.

A triangle that is not right-angled transforms into a rectangle by outside right-angled triangles, thus allowing its sides and angles and area to be found indirectly. So, as in right-angled triangles, any triangle has the property that the angles add up to 180 degrees and that the area is half of the height times the base.

Inside a circle with radius 1, the two diagonals of a 4sided square together with the horizontal and vertical diameters through the center form angles of 180/4 degrees. Thus the circumference of the square is $2*(4*\sin(180/4))$, or $2*(8*\sin(180/8))$ with 8 sides instead. Consequently, the circumference of a circle with radius 1 is $2*\pi$, where $\pi = n*\sin(180/n)$ for n large.

14. A coordinate system coordinates algebra with geometry where a point is reached by a number of horizontally and vertically steps called the point's x- and y-coordinates.

Two points A(xo,yo) and B(x,y) with different x- and y-numbers will form a right-angled change-triangle with a horizontal side $\Delta x = x-xo$ and a vertical side $\Delta y = y-yo$ and a diagonal distance r from A to B, where by Pythagoras $r^2 = \Delta x^2 + \Delta y^2$. The angle A is found by the formula $\tan A = \Delta y/\Delta x = s$, called the slope or gradient for the line from A to B. This gives a formula for a non-vertical line: $\Delta y/\Delta x = s$ or $\Delta y = s*\Delta x$, or $y-yo = s*(x-xo)$. Vertical lines have the formula x = xo since all points share the same x-number.

In a coordinate system three points A(x1,y1) and B(x2,y2)and C(x3,y3) not on a line will form a triangle that packs into a rectangle by outside right-angled triangles allowing indirectly to find the angles and the sides and the area of the original triangle.

Different lines exist inside a triangle: Three altitudes measure the height of the triangle depending on which side is chosen as the base; three medians connect an angle with the middle of the opposite side; three angle bisectors bisect the angles; three line bisectors bisect the sides and are turned 90 degrees from the side. Likewise, a triangle has two circles; an outside circle with its center at the intersection point of the line bisectors, and an inside circle with its center at the intersection point of the angle bisectors.

Since Δx and Δy changes place when turning a line 90 degrees, their slopes will be $\Delta y/\Delta x$ and $-\Delta x/\Delta y$ respectively, so that $s1*s2 = -1$, called reciprocal with opposite sign.

Geometrical intersection points are predicted algebraically by solving two equations with two unknowns, i.e. by inserting one into the other. Thus with the lines $y = 2*x$ and $y = 6-x$, inserting the first into the second gives $2*x = 6-x$, or $3*x = 6$, or $x = 2$, which inserted in the first gives $y = 2*2 = 4$, thus predicting the intersection point to be (x,y) = (2,4). The same answer is found on a solver-app; or using software as GeoGebra.

Finding possible intersection points between a circle and a line or between two circles leads to a quadratic equation $x^2 + b*x + c = 0$, solved by a solver. Or by a formula created by two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter round clockwise. With $k = m-x$, this creates 4 areas combining to $(x + k)^2 = x^2 + 2*k*x + k^2$. With $k = b/2$ this becomes $(x + b/2)^2 = x^2 + b*x + (b/2)^2 + c - c = (b/2)^2 - c$ since $x^2 + b*x + c = 0$. Consequently the solution formula is $x = -b/2 \pm\sqrt{((b/2)^2 - c)}$.

Finding a tangent to a circle at a point, its slope is the reciprocal with opposite sign of the radius line.

15. A formula predicts a total before counting it. A formula typically contains both specified and unspecified numbers in the form of letters, e.g. T = 5+3*x. A formula containing one unspecified number is called an equation, e.g. 26 = 5+3*x, to be solved by moving to opposite side with opposite calculation sign, (26-5)/3 = x. A formula containing two unspecified numbers is called a function, e.g. T = 5+3*x. An unspecified function containing an unspecified number x is labelled f(x), T = f(x). Thus f(2) is meaningless since 2 is not an unspecified number. Functions are described by a table or a graph in a coordinate system with y = T = f(x), both showing the y-numbers for different x-numbers. Thus, a change in x, Δx, will imply a change in y, Δy, creating a per-number Δy/Δx called the gradient.

16. In a function y = f(x), a small change x often implies a small change in y, thus both remaining 'almost constant' or 'locally constant', a concept formalized with an 'epsilon-delta criterium', distinguishing between three forms of constancy. y is 'globally constant' c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is 'piecewise constant' c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is 'locally constant' c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio Δy/Δx can be globally, piecewise or locally constant, in the latter case written as dy/dx. Formally, local constancy and linearity is called continuity and differentiability.

17. As to change, a total can change in a predictable or unpredictable way; and predictable change can be constant or non-constant.

Constant change comes in several forms. In linear change, T = b + s*x, s is the constant change in y per change in x, called the slope or the gradient of its graph, a straight line. In exponential change, T = b*(1+r)^x, r is the constant change-percent in y per change in x, called the change rate. In power change, T = b*x^p, p is the constant change-percent in y per change-percent in x, called the elasticity. A saving increases from two sources, a constant $-amount per month, c, and a constant interest rate per month, r. After n months, the saving has reached the level C predicted by the formula C/c = R/r. Here the total interest rate after n months, R, comes from 1+R = (1+r)^n. Splitting the rate r = 100% in t parts, we get the Euler number e = (1+100%/t)^t = (1+1/t)^t if t is large.

Also the change can be constant changing. Thus in T = c + s*x, s might also change constantly as s = c + q*x so that T = b + (c + q*x)*x = b + c*x + q*x^2, called quadratic change, showing graphically as a line with a curvature, a parabola.

If not constant but still predictable, we have a change formula ΔT/Δx = f(x) or dT/dx = f(x) in the case of interval change or local change. Such an equation is called a differential equation which is solved by calculus, adding up all the local changes to a total change being the difference between the end and start number: T2-T1 = Σ ΔT = ∫ dT = ∫f(x)*dx. Thus, with dT/dx = 2*x, T2-T1 = Δ(x^2). Change formula come from observing that in a block, changes Δb and Δh in the base b and the height h impose on the total a change ΔT as the sum of a vertical strip Δb*h and a horizontal strip b*Δh and a corner Δb*Δh that can be neglected for small changes; thus d(b*h) = db*h + b*dh, or counted in T's: dT/T = db/b + dh/h, or with T' = dT/dx, T'/T = b'/b + h'/h. Therefore (x^2)'/x^2 = x'/x + x'/x = 2/x, giving (x^2)' = 2*x since x' = dx/dx = 1.

18. Unpredictable change can be exemplified by throwing a dice with two results: winning, +1, if showing 4 or above, and losing, 0, if showing 3 or below. Throwing a dice 5 times thus have 6 outcomes, winning from 0 to 5 times. The outcome is called an unpredictable or stochastic or random number or variable. Per definition, random numbers cannot be pre-dicted, instead they can be 'post-dicted' using statistics and probability.

Thus the outcome '0,0,0,1,1' can be described by three numbers. The mode is 0 since this number has the highest frequency, 3 per 5, or 3/5. The median is 0 since this is the middle number when aligned in increasing order. The mean u is the fictional number had all numbers been the same: u*5 = 0+0+0+1+1 with the solution u = 2/5 = 0.4. With the outcome '0,0,1,1,1', the mode and median and mean is 1 and 1 and 3/5 = 0.6.

To find the three numbers if the experiment is repeated many times we look at a ‘possibly tree’. The first toss has two results, win or lose, both occurring ½ of the times. Likewise with the following tosses: After two tosses we have three outcomes: 2 wins, 1 win and 0 wins. Here 2 wins and 0 wins occur half of half of the times, i.e. with a probability ¼. 1 win occurs twice, as win-lose or as lose-win, both with a probably of ¼, so the total probability for 1 win is 2*1/4 = ½. Continuing in this way we find that with 5 tosses there are 6 outcomes, winning from 0 to 5 times with the probabilities ½^5 a certain number of times: 1, 5, 10, 10, 5, 1. By calculations we find that the mode is 2 and 3, and that the median and the mean is 2.5, also found by multiplying the number of repetition with the probability for winning.

A spreadsheet random generator can show examples of other outcomes.

19. A sphere may be distorted into a cup. Even if distorted, a rectangle will still divide a sphere into an inside and an outside needing a bridge to be connected. And a sphere with a bridge may be distorted into a cup with a handle or into a donut. Distortion geometry is called topology, useful when setting up networks, thus able to prove that connecting three houses with water, gas and electricity is impossible without a bridge.

20. As qualitative literature, also quantitative literature has three genres, fact and fiction and ‘fiddle’, used when modeling real world situations. Fact is ‘since-then’ calculations using numbers and formulas to quantify and to predict predictable quantities as e.g. ‘since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20’. Fact models can be trusted once the numbers and the formulas and the calculation has been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter. Fiction is ‘if-then’ calculations using numbers and formulas to quantify and to predict unpredictable quantities as e.g. ‘if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20’. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made. Fiddle models is ‘what-then’ models using numbers and formulas to quantify and to predict unpredictable qualities as e.g. ‘since a graveyard is cheaper than a hospital, then a bridge across the highway is too costly.’ Fiddle models should be rejected and relegated to a qualitative description.

Meeting Many in a STEM Context

Having met Many by itself, now we meet Many in time and space in the present culture based upon STEM, described by OECD as follows:

> The New Industrial Revolution affects the workforce in several ways. Ongoing innovation in renewable energy, nanotech, biotechnology, and most of all in information and communication technology will change labour markets worldwide. Especially medium-skilled workers run the risk of being replaced by computers doing their job more efficiently. This trend creates two challenges: employees performing tasks that are easily automated need to find work with tasks bringing other added value. And secondly, it propels people into a global competitive job market. (..) In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature’s physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature’s three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone?

No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles, mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves. Nitrates and carbon-dioxide and water is waste for grey cells, but food for green cells producing proteins and carbon-hydrates and oxygen as food for the grey cells in return.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. To master plane or spatial shapes, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent.

So, a core STEM curriculum could be about cycling water. Heating transforms it from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we must build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

A short World History

When humans left Africa, some went west to the European mountains, some went east where the fertile valleys in India supplied everything except for silver from the mountains. Consequently, rich trade took place sending pepper and silk west and silver east. European culture flourished around the silver mines, first in Greece then in Spain during the Roman Empire. Then the Vandal conquest of the mines brought the dark middle age to Europe until silver was found in the Harz valley (Tal in German leading to thaler and dollar), transported through Germany to Italy. Here silver financed the Italian Renaissance, going bankrupt when Portugal discovered a sea route to India enabling them to skip the cost of Arab middlemen. Spain looked for a sea route going west and found the West Indies. Here there was neither pepper nor silk but silver in abundance e.g. in the land of silver, Argentine. On their way home, slow Spanish ships were robbed by sailing experts, the Vikings descendants living in England, now forced to take the open sea to India to avoid the Portuguese fortification of Africa's coast.

In India, the English found cotton that they brought to their colonies in North America, but needing labor they started a triangle-trade exchanging US cotton for English weapon for African slaves for US cotton. In the agricultural South, a worker was a cost to be minimized, but in the industrial North a worker was a consumer needed at an industrial market. During the civil war, no cotton came to England that then conquered Africa to bring the plantations to the workers instead. Dividing the world in closed economies kept new industrial states out of the world market that it took two world wars to open for free competition.

Nature Obeys Laws, but from Above or from Below?

In the Lord's Prayer, the Christian Church says: 'Thy will be done, on earth as it is in heaven'. Newton had a different opinion.

As experts in sailing, the Viking descendants in England had no problem stealing Spanish silver on its way across the Atlantic Ocean. But to get to India to exchange it for pepper and silk, the Portuguese fortification of Africa's cost forced them to take the open sea and navigate by the moon. But how does the moon move? The church had one opinion, Newton meant differently.

'We believe, as is obvious for all, that the moon moves among the stars,' said the Church, opposed by Newton saying: 'No, I can prove that the moon falls to the earth as does the apple.' 'We believe that when moving, things follow the unpredictable metaphysical will of the Lord above whose will is done, on earth as it is in heaven,' said the Church, opposed by Newton saying: 'No, I can prove they follow their own physical will, a physical force, that is predictable because it follows a mathematical formula.' 'We believe, as Aristotle told us, that a force upholds a state,' said the Church, opposed by Newton saying: 'No, I can prove that a force changes a state. Multiplied with the time applied, the force's impulse changes the motion's momentum; and multiplied with the distance applied, the force's work changes the motion's energy.' 'We believe, as the Arabs have shown us, that to deal with formulas you just need ordinary algebra,' said the Church, opposed by Newton saying: 'No. I need to develop a new algebra of change which I will call calculus.'

Proving that nature obeys its own will and not that of a patronizor, Newton inspired the Enlightenment century realizing that if enlightened we don't need the double patronization of the physical Lord at the Manor house and the metaphysical Lord above. Citizens only need to inform themselves, debate and vote. Consequently, to enlighten the population, two Enlightenment republics were created, in the US in 1776 and in France in 1789. The US still have their first republic allowing its youth to uncover and develop their personal talent through daily lessons in self-chosen half-year blocks, whereas the Napoleon wars forced France and the rest of continental Europe to copy the Prussians line-organized education forcing teenagers to follow their year-group and its schedule, creating a knowledge nobility (Bourdieu, 1970) for public offices, and unskilled workers, good for yesterday's industrial society, but bad for today's information society where a birth rate at 1.5 child per family will halve the population each 50 years since (1.5/2)^2 = 0.5 approximately.

Counting and DoubleCounting Time, Space, Matter, Force and Energy

Counting time, the unit is seconds. A bundle of 60 seconds is called a minute; a bundle of 60 minutes is called an hour, and a bundle of 24 hours is called a day, of which a bundle of 7 is called a week. A year contains 365 or 366 days, and a month from 28 to 31 days.

Counting space, the international unit is meter, of which a bundle of 1000 is called a kilometer; and if split becomes a bundle of 1000 millimeters, 100 centimeters and 10 decimeters. Counting squares, the unit is 1 square-meter. Counting cubes, the unit is 1 cubic-meter, that is a bundle of 1000 cubic-decimeters, also called liters, that split up as a bundle of 1000 milliliters.

Counting matter, the international unit is gram that splits up into a bundle of 1000 milligrams and that unites in a bundle of 1000 to 1 kilogram, of which a bundle of 1000 is called 1 tons.

Counting force and energy, a force of 9.8 Newton will lift 1 kilogram, that will release an energy of 9.8 Joule when falling 1 meter.

Cutting up a stick in unequal lengths allows the pieces to be double-counted in liters and in kilograms giving a per-number around 0.7 kg/liter, also called the density.

A walk can be double-counted in meters and seconds giving a per-number at e.g. 3 meter/second, called the speed. When running, the speed might be around 10 meter/second. Since an hour is a bundle of 60 bundles of 60 seconds this would be 60*60 meters per hour or 3.6 kilometers per hour, or 3.6 km/h.

A pressure from a force applied to a surface can be double-counted in Newton and in square meters giving a per-number Newton per square-meter, also called Pascal.

Motion can be double-counted in Joules and seconds producing the per-number Joule/second called Watt. To run properly, a bulb needs 60 Watt, a human needs 110 Watt, and a kettle needs 2000 Watt, or 2 kiloWatt. From the Sun the Earth receives 1370 Watt per square meter.

Warming and Boiling water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

Heating 1000 gram water 80 degrees in 167 seconds in a 2000 Watt kettle, the per-number will be 2000*167/80 Joule/degree, creating a double per-number 2000*167/80/ 1000 Joule/degree/gram or 4.18 Joule/degree/gram, called the specific heat of water.

Producing 100 gram steam in 113 seconds, the per-number is 2000*113/100 Joule/gram or 2260 J/g, called the heat of evaporation for water.

Letting Steam Work

A water molecule contains two Hydrogen and one Oxygen atom weighing 2*1+16 units. A collection of a million billion billion molecules is called a mole; a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. The volume should increase accordingly. But, if kept constant, instead the inside pressure will increase.

Inside a cylinder, the ideal gas law, $p*V = n*R*T$, combines the pressure, p, and the volume, V, with the number of moles, n, and the absolute temperature, T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

So, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus decreasing and increasing the pressure by letting steam in and out of the two holes. The leaving steam the is visible on steam locomotives. In the third generation used in power plants, two cylinders, a hot and a cold, connect with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back to the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical power to industries and homes.

An Electrical circuit

To work properly, a 2000 Watt water kettle needs 2000 Joule per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per charge-unit.

Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2 Volt/Ampere called Ohm.

Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula Watt = Volt*Ampere.

Thus, with a 60 Watt and a 120 Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So, the 120 Watt bulb only receives half of the energy of the 60 Watt bulb.

How high up and how far out

A ping-pong ball is send upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second.

To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the speed is split into a vertical and a horizontal part, both reducing it with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed by the formula: Horizontal distance to the top = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

The vertical distance is halved, but the vertical speed changes from 9.92 to 9.92*0.707 = 7.01 only. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the distance four times.

How many turns on a steep hill

On a 30-degree hillside, a 10 degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2.

In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2.

In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Dissolving material in water

In the sea, salt is dissolved in water. The tradition describes the solution as the number of moles per liter. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59gram = (100/59)*1mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

The Simplicity of Mathematics

Meeting Many, we ask ‘How many in total?’ To answer, we count and add. To count means to use division, multiplication and subtraction to predict unit-numbers as blocks of stacked bundles, but also to recount to change unit, and to double-count to get per-numbers bridging the units, both rooting proportionality.

Adding thus means uniting unit-numbers and per-numbers, but both can be constant or variable, so to predict, we need four uniting operations: addition and multiplication unite variable and constant unit-numbers; and integration and power unite variable and constant per-numbers. As well as four splitting operations: subtraction and division split into variable and constant unit-numbers; and differentiation and root/logarithm split into variable and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite. And it appears in Arabic numbers written out fully as T = 456 = 4 bundles-of-bundles & 5 bundles & 6 unbundled, showing all four uniting operations, addition and multiplication and power and next-to addition of stacks; and showing that the word-language and the number-language share the same sentence form with a subject and a verb and a predicate or object.

Shapes can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

So, in principle, mathematics is simple and easy and quick to learn if institutionalized education wants to do so; however, to preserve and expand itself, the institution might want instead to hide the simplicity of mathematics by leaving out the subject and the verb in the number-language sentences; and by avoid counting to hide the block-nature of numbers as stacked bundles in order to impose linear place-value numbers instead; and by reversing the natural order of operations by letting addition precede subtraction, preceding multiplication, preceding division; and by hiding the double nature of addition by silencing next-to addition; and by silencing per-numbers and present fractions as numbers instead of operators needing numbers to become numbers; and by adding fractions without units to hide the true nature of integration as adding per-numbers by their areas; and by postponing trigonometry to after ordinary geometry and coordinate geometry; and by forcing equations to be solved by obeying the commutative and associative laws of abstract algebra; and by hiding that a function is but another name for a number-language sentence; and by forcing differential calculus to precede integral calculus.

Discussion: How does Traditional MatheMatics differ from ManyMatics

But in the end, how different is traditional mathematics from ManyMatics? As their base they have Set and Many, but isn't that just two different words for the same? Not entirely. Many exists in the world, it is physical, whereas Set exists in a description, it is meta-physical. Thus, traditional mathematics defines its concepts top-down as examples, whereas ManyMatics defines its concepts bottom-up as abstractions. Still, the concepts might be the same, at least when taught? But a comparison uncovers several differences between the Set-derived tradition and its alternative grounded in Many.

The tradition sees the goal of mathematics education as teaching numbers and shapes and operations. In numbers, digits are symbols like letters, ordered according to a place value system, seldom renaming '234' to '2tentens 3tens 4'. There are four kinds of numbers: natural and integers and rational and real. The natural numbers are defined by a successor principle making them one dimensional placed along a number line given the name 'cardinality'. The integers are defined as equivalence classes in a set of ordered number-pairs where (a,b) is equivalent to (c,d) if a+d = b+c. Likewise, the rational numbers are defined by (a,b) being equivalent to (c,d) if a*d = b*c. Finally, the real numbers are defined as limits of number sequences.

The alternative sees the goal of mathematics education as teaching a number-language describing the physical fact Many by full sentences with the total as the subject, e.g. T = 2*3, thus having the same structure as the word-language, both having a language level describing the world, and a meta-language level describing the language. Digits are icons containing as many sticks as they represent if written less sloppy. Numbers occur when counting Many by bundling and stacking produces a block of bundles and unbundled, using bundle- or decimal-writing to separate the inside bundles from the outside unbundled. The bundle-number, typically ten, does not need an icon since it is counted as '1 bundle'. Thus, a natural number is a decimal number with a unit, illustrated geometrically as a row of blocks containing the unbundled, the bundles, the bundle of bundles etc. Counting includes recounting in the same unit to create overload or underload, as well as recounting in another unit, especially in and from tens. Double-counting in different units gives per-numbers and fractions; however, these are not numbers but operators needing a number to become a number. A diagonal divides a block into two like right-angled triangles where the base and the altitude can be recounted in diagonals or in each other. Real numbers as √2 are calculations with as many decimals as needed, since a single can always be seen as a bundle of parts.

The tradition sees operations in a number set as mappings from a set-product into the set. Addition is the basic operation allowing number sets to be structured with an associative and a commutative and a distributive law as well as a neutral element and inverse elements. Addition is defined as repeating the successor principle, and multiplication is defined as repeated addition. Subtraction and division is defined as adding or multiplying inverse numbers. Standard algorithms for operations are introduced using carrying. Electronical calculators are not allowed when learning the four basic operations. The full ten-by-ten multiplication tables must be learned by heart.

The alternative sees operations as icons describing the counting process. Here division is an uphill stroke showing a broom wiping away the bundles; multiplication is a cross showing a lift stacking the bundles into a block, to be dragged away to look for unbundled singles, shown by a horizontal track called subtraction. Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other. To add on-top, the blocks must be recounted in the same unit, thus grounding proportionality. Next-to addition means adding areas, thus grounding integration. Reversed adding on-top or next-to grounds equations and differentiation. A calculator is used to predict the result by two formulas, a recount-formula T = (T/B)*B, and a restack-formula T = (T-B)+B. A multiplication table shows recounting from icons to tens, and is used when recounting from tens to icons introduces equations as reversed calculations. When recounting a total to or from tens, increasing the base means decreasing the altitude, and vice versa. As to multiplication, the commutative law says that the total stays unchanged when turning over a 3 by 4 block to a 4 by 3 block. The associative law says that the total stays unchanged when including or excluding a factor from the unit, T = 2*(3*4) = (2*3)*4. The distributive law says that before adding, recounting must provide a common unit to bracket out, T = 2 3s + 4 5s = 1.1 5s + 4 5s = (1.1 + 4) 5s.

The tradition sees fractions as rational numbers to which the four basic operations can be applied. Thus, fractions can be added without units by finding a common denominator after splitting the numerator and the denominator into prime factors. Fractions are introduced after division, and is followed by ratios and percentages and decimal numbers seen as examples of fractions.

The alternative sees fractions as per-numbers coming from double-counting in the same unit. As per-numbers, fractions are operators needing a number to become a number, thus added by areas, also called integration. Double-counting is introduced before addition. With factors as units, splitting a number in prime factors just means finding all possible units.

After working with number sets, the tradition introduces working with letter sets and polynomial sets to which the four basic operations can be applied once more observing that only like terms can be added, but not mentioning that this is because it means the unit is the same.

The alternative sees letters as units to bracket out during addition or subtraction, and that when multiplied or divided gives a composite unit.

The tradition sees an equation as an open statement expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*x = 8	an open statement
(2*x)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(x*2)*(1/2) = 4	since multiplication is commutative
x*(2*(1/2)) = 4	since multiplication is associative
x*1 = 4	by definition of an inverse element
x = 4	by definition of a neutral element

As to the equation 2 + 3*x = 14, the same procedure as above is carried out twice, first with addition then with multiplication.

The alternative sees an equation as another name for a reversed calculation, to be reversed once more by recounting. Thus in the equation ‘2*x = 8’, recounting some 2s in 1s resulted in 8 1s, which recounted back into 2s gives 2*x = 8 = (8/2)*2, showing that x = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the opposite side & sign method.

The equation 2 + 3*x = 14, can be seen in two ways. As reversing a next-to addition of the two blocks, thus solved by differentiation, first removing the initial block and then recounting the rest in 3s: x = (14-2)/3 = 4. Or as a walk that multiplying by 3 and then adding by 2 gives 14,

x *(*3→)* 3*x *(+2→)* 3*x+2 = 14.

Reversing the walk by subtracting 2 and dividing by 3 gives the initial number:

x = 4 = (14-2)/3 *(←/3)* 14-2 *(←-2)* 14

The answer is tested by once more walking forward, 3*4 + 2 = 12 + 2 = 14.

The tradition sees a quadratic equation x^2 + b*x + c = 0 as a pure algebraic problem to be solved, first by factorizing, then by completing the square, and finally by using the solution formula.

The alternative sees solving a quadratic equation as a problem combining algebra and geometry, where a square with the sides x+b/2 creates fives areas, x^2 and b/2*x twice and c and (b^2/4-c) where the first four disappear and leaves (x+b/2)^2 to be the latter, b^2/4-c.

The tradition sees a function as an example of a relation between two sets where first-component identity implies second-component identity. And it gives the name 'linear function' to f(x) = a*x+b even if this is an affine function not satisfying the linear condition f(x+y) = f(x)*f(y), as does the proportionality formula f(x) = a*x.

The alternative sees a function as a name for a formula containing two unspecified numbers or variables, typically x and y. Thus, a function is a fiction showing how the y-numbers depends on the x numbers as shown in a table or by a graph.

The tradition sees proportionality as an example of a function satisfying the linear condition. The alternative sees proportionality as a name for double-counting in different units creating per-numbers.

The tradition sees geometry to be introduced in the order: plane geometry, coordinate geometry and trigonometry.

The alternative has the opposite order. Trigonometry comes first grounded in the fact that halving a block by its diagonal allows the base and the altitude to be recounted in diagonals or in each other. This also allows a calculator to find pi from a sine formula. Next comes coordinate geometry allowing geometry and algebra to always go hand in hand so that algebraic formula can predict intersection points coming from geometrical constructions.

The tradition has quadratic functions following linear functions, both examples of polynomials.

The alternative sees affine functions as one example of constant change coming in five forms: constant y-change per x-change, constant y-percent-change per x-change, constant y-percent-change per x-percent-change, constant y-change per x-change together with constant y-percent-change per x-change, and finally constantly changing y- change.

The tradition sees logarithm as defined as the integral of the function y = 1/x.

The alternative sees logarithm and root combined both solving power equations. Thus a^x = b gives x = loga(b); and x^a = b gives x = a√b. This shows the logarithm as a factor-counter and the root as a factor-finder.

The tradition sees differential calculus as preceding integral calculus, and the gradient y' = dy/dx is defined algebraically as the limit of Δy/Δx for Δx approaching 0, and geometrically as the slope of a tangent being the limit position of a secant with approaching intersection points. The limit is defined by an epsilon-delta criterium.

The alternative sees calculus as grounded in adding blocks next-to each other. In primary school calculus occurs when performing next-to addition of 2 3s and 4 5s as 8s. In middle school calculus occurs when adding piecewise constant per-numbers, as 2m at 3m/s plus 4m at 5m/s. In high school calculus occurs when adding locally constant per-numbers, as 5seconds at 3m/s changing constantly to 4m/s. Geometrically, adding blocks means adding areas under a per-number graph. In the case of local constancy this means adding many strips, made easy by writing them as differences since many differences add up to one single difference between the terminal and initial numbers, thus showing the relevance of differential calculus. The epsilon-delta criterium is a straight forward way to formalize the three ways of constancy, globally and piecewise and locally, by saying that constancy means an arbitrarily small difference.

Conclusion

With 50 years of research, mathematics education should have improved significantly. Its lack of success as illustrated by OECD report 'Improving Schools in Sweden' made this paper ask: Applying sociological imagination when meeting Many without having predicates forced upon it by traditional mathematics, can we design a STEM-based core math curriculum aimed at making migrants pre-teachers and pre-engineers in two years? This depends on what we mean by mathematics. And, looking back, mathematics has meant different things through its long history, from a common label for knowledge to today's 'meta-matism' combining 'meta-matics' defining concepts by meaningless self-reference, and 'mathe-matism' adding numbers without units thus lacking outside validity. So, inspired by Heidegger's 'always question sentences, except for its subject' we returned to the original Greek meaning of mathematics: Knowledge about Many by itself and in time and space.

Observing Many by itself allows rebuilding mathematics as a 'many-matics', i.e. as a natural science about the physical fact Many, where counting by bundling leads to block-numbers that recounted in other units leads to proportionality and solving equations; where recounting sides in triangles leads to trigonometry; where double-counting in different units leads to per-numbers and fractions, both adding by their areas, i.e. by integration; where counting precedes addition taking place both on-top and next-to involving proportionality and calculus; where using a calculator to predict the counting result leads to the opposite order of operations: division before multiplication before subtraction before next-to and on-top addition; and where calculus occurs in primary school as next-to addition, and in middle and high school as adding piecewise and locally constant per-numbers; and where integral calculus precedes differential calculus.

With water cycles fueled by the sun and run by gravity as exemplary situations, STEM offers various examples of Many in space and time since science and technology and engineering basically is about double-counting physical phenomena in different units.

The designed STEM-based core math curriculum has been tested in parts with success at the educational level in Danish pre-university classes. It might also be tested on a research level if it becomes known through publishing, i.e., if it will be accepted at the review process. It will offer a sociological imagination absent from traditional research seen by many teachers as useless because of its many references.

Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality used to collect reliable data and to test the validity of its findings by falsification attempts.

In a Swedish context, obsessive self-referencing has been called the 'irrelevance of the research industry' (Tarp, 2015, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics education and research are two examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead thus marginalizing or forgetting its original outside goal.

So, if a society as Sweden really wants to improve mathematics education, extra funding might just produce more researchers more eager to follow inside traditions than solving outside problems. Instead funding should force the universities to arrange curriculum architect compositions to allow alternatives to compete as to creativity and effectiveness, thus allowing the universities to rediscover their original outside rational goals and to change its routines accordingly. A situation described in several fairy tales; the Sleeping Beauty hidden behind the thorns of routines becoming rituals until awakened by the kiss of an alternative; and Cinderella making the prince dance, but only found when searching outside the established nobility.

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13. The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants

Educational shortages described in the OECD report 'Improving Schools in Sweden' challenge traditional math education offered to young male migrants wanting a more civilized education to return help develop and rebuild their own country. Research offers little help as witnessed by continuing low PISA scores despite 50 years of mathematics education research. Can this be different? Can mathematics and education and research be different allowing migrants to succeed instead of fail? A different research, difference-research finding differences making a difference, shows it can. STEM-based, mathematics becomes Many-based bottom-up Many-matics instead of Set-based top-down Meta-matics.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Math Education. But, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

To find an unorthodox solution let us pretend that a university in southern Sweden arranges a curriculum architect competition: 'Theorize the low success of 50 years of mathematics education research, and derive from this theory a STEM-based core mathematics curriculum for young male migrants.'

Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is a rational organization, 'in which the *end* is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Such a goal displacement occurs if saying 'The goal of mathematics education is to teach and learn mathematics'. Furthermore, by its self-reference such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts. In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2x3=6' stating that 2 3s can be re-counted as 6 1s.

So, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about shapes and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement, according to what Bauman (1992, p. ix) calls 'the second Copernican revolution' of Heidegger asking the question: What is 'is'?

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. (Heidegger, 1962, pp. 126, 169)

Heidegger has inspired existentialist thinking, described by Sartre (2007, p. 22) as holding that 'existence precedes essence'. In France, Heidegger inspired Derrida, Lyotard, Foucault and Bourdieu in poststructuralist thinking pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and to your world (Lyotard, 1984; Bourdieu, 1970; Foucault, 1995).

As to the political aspects of research, Foucault says:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006, p. 41; also on YouTube)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics and education and research are examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as an inside goal instead, thus marginalizing or forgetting its original outside goal.

To avoid this, difference-research is based upon the Greek sophists, saying ‘Know nature form choice to unmask choice masked as nature.’; and Heidegger saying ‘In sentences, trust the subject but question the rest.’; and Sartre saying ‘Existence precedes essence’; and Foucault, seeing a school as a ‘pris-pital’ mixing power techniques of a prison and a hospital by keeping children and adolescents locked up daily to be cured without being properly diagnosed. For it is differences that unmask false nature, and unmask prejudice in predicates, and uncover alternative essence, and cure an institution from a goal displacement.

Furthermore, difference-research knows the difference between what can be different and what cannot. From a Heidegger view an is-sentence contains two things: a subject that exists and cannot be different, and a predicate that can and that may be gossip masked as essence, provoking ‘the banality of Evil’ (Arendt, 2006) if institutionalized. So, to discover its true nature, we need to meet the subject, the total, outside its predicate-prison of traditional mathematics. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help develop or rebuild their original countries.

So, to restore its authenticity, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many, Children use Block-numbers to Count and Share

How to deal with Many can be learned from preschool children. Asked ‘How old next time?’, a 3year old will say ‘Four’ and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, ‘That is not 4, that is 2 2s. Children also use block-numbers when talking about Lego bricks as ‘2 3s’ or ‘3 4s’. When asked ‘How many 3s when united?’ they typically say ‘5 3s and 3 extra’; and when asked ‘How many 4s?’ they say ‘5 4s less 2’; and, placing them next-to each other, they say ‘2 7s and 3 extra’.

You don’t need research to observe how children love digital counting by bundling, replacing a bundle of 2 1s with 1 Lego Brick with 2 knobs to be placed in a cup for the bundles; and they don’t mind exchanging 2 2s with 1 Lego brick with 4 knobs to be placed in a cup for 4s. And they have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 1 4s &3, etc. And children don’t mind writing a total of 7 using ‘bundle-writing’ as T = 7 = 1B5 = 2B3 = 3B1 = 1BB0B3 = 1BB1B1. And with 1 plastic S for 1 borrowed, some children even writes T = 7 = 3B1 = 4BS = 5BSSS. Also, children love to count in 3s and 4s. Recounting in 5s is unfortunately not possible since Lego refuses to produce bricks with 5 knobs.

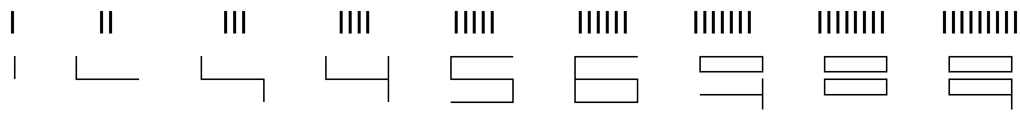
Sharing 9 cakes, 4 children takes one by turn as long as possible; with 4s taken out they say ‘I take 1 of each 4’, and with 1 left they say ‘let’s count it as 4’. And they smile when seeing that sharing 4 5s by 3 is predicted by asking a calculator 4*5/3. Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. So children master sharing, taking parts and splitting into parts before having learned about division and counting- and splitting-fractions, which they would like to learn before being forced to add.

Children thus show core mastery of Many before coming to school, allowing school to build upon this knowledge instead of rejecting it. So, school could ask research to design a curriculum, that counts totals in two-dimensional block-numbers instead of one-dimensional line-numbers; that counts and re-counts and double-counts totals before they are added, and then both on-top and next-to; that teaches 8/4 as 8 counted in 4s giving 2 4s instead of as 8 split between 4 giving 4 2s; and that root counting-fractions and splitting-fractions in per-numbers and re-counting. Difference-research gladly takes on such a curriculum design.

Meeting Many Creates a Count&Multiply&Add Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create a number-language sentence, T = 2 3s, containing a subject, a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents, icons can be used as units when counting: four strokes in the 4-con, five in the 5-icon, etc.



1 2 3 4 5 6 7 8 9

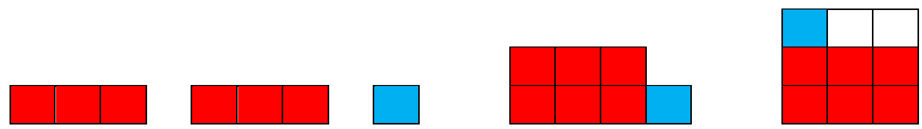
We count in bundles to be stacked as block-numbers to be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count we take away bundles (thus rooting division as a broom wiping away the bundles) to be stacked (thus rooting multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting subtraction as the trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)xB saying that 'from T, T/B times, B can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

Finally, bundle- or bundle-writing double-counts the bundles inside the bundle-cup and the singles outside, where an overload or underload is removed or created by re-counting in the same unit, T = 7 = 2B1 3s = 2B1 3s = 1B4 3s = 3B-2 3s.

Likewise, placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations to be solved by re-counting, resulting in moving numbers to the opposite side with the opposite sign: u x 7 = 42 = (42/7)x7 gives u = 42/7.

Double-counting in physical units creates per-numbers bridging the units, thus rooting proportionality. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. Then both on-top and next-to addition can be reversed, thus rooting equations and differential calculus.

In a rectangle split by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Meeting Many in a STEM Context

Having met Many by itself, now we meet Many in time and space in the present culture based upon STEM, described by OECD as follows:

> In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground since motion transfers through collisions, now present as increased motion in molecules; so the motion has lost its order and can no longer work.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making it combine to stars and planets and galaxies. Some planets have a size and a distance from its star that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles and mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves.

Technology is knowledge about ways to satisfy human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for motion and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language for predicting Many by calculation sentences, formulas, expressing counting and adding processes. First Many is double-counted in bundles and singles to create a total T that might be re-counted in the same or in a new unit or into or from tens; or double-counted in two physical units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. Finally, triangulation predicts spatial forms.

So, a core STEM curriculum could be about cycling water. Heating pumps in motion transforming water from solid to liquid to gas, i.e. from ice to water to steam; and cooling pumps motion out. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform moving water to moving electrons, electricity. To get to the dam, we build roads on hillsides.

The Electrical circuit, an Example

To work properly, a 2000Watt water kettle needs 2000Joule per second. The socket delivers 220Volts, a per-number double-counting Joules per charge-unit.

Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle has a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per charge-unit) *(charge-unit per second) we also have a second formula Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So surprisingly, the 120Watt bulb only receives half of the Joules of the 60Watt bulb.

Difference-research Differing from Critical and Postmodern Thinking

Together with difference-research, also critical thinking and postmodernism show skepticism towards knowledge claims, so how does difference-research differ?

As to critical thinking, Skovsmose & Borba (2000) describes a Brazilian research group that, focusing on issues related to new technologies and mathematics education, has developed software and work with students at different levels and with teachers. The group was approached by a teacher from a nearby school where she had some tough problems to face and hoped that the computers would be able to help her. She was teaching rational numbers to a class of 5th graders, with a mixture of 11year old students and 15year old repeaters having given up rational numbers and turning to violence.

> The teacher was enthusiastic about a software, which deals with rational numbers. (..) Both researchers and teacher had the ‘intuition’ that the computer might have a positive effect in this class and maybe could avoid that the students had to repeat this grade again. (p. 7)

By recommending computers, the researchers showed criticism, not towards fractions as such, but towards how they are taught. Critical thinking thus sees mathematics as an unquestionable goal, only how it is taught can be questioned.

Contrary to this, difference-research sees fractions as a means rooted in double-counting, and recommends fractions introduced as per-numbers via the ‘fraction-paradox’: 1 red of 2apples and

2red of 3apples total 3red of 5apples and not 7red of 6apples as says the textbook. Fractions thus add by their areas as integral calculus. Adding fractions of the same total can be treated later. Introducing fractions via per-numbers and separating core-mathematics from 'footnote-mathematics' will side the teacher with the learner against the textbook.

As to postmodern thinking, the book 'Mathematics Education within the Postmodern' (Walshaw, 2004) contains 12 chapters divided into three parts: thinking otherwise for mathematics education, postmodernism within classroom practices, and within the structures of mathematics education. The preface says:

> It is a groundbreaking volume in which each of the chapters develops for mathematics education the importance of insights from mainly French intellectuals of the post: Foucault, Lacan, Lyotard, Deluze. (p. vii)

Although the book wants to be skeptical towards both mathematics and its education, it is only the educational part that is scrutinized; and most authors describes how what is labeled postmodern thinking can be exemplified in educational contexts, they don't see mathematics itself as a social construction that could be questioned also. A central thinker as Derrida is mentioned only in the two survey chapters, and the core concept of deconstruction is not mentioned at all despite its fundamental importance to a postmodern perspective to mathematics education (Tarp, 2012).

By going behind French thinking to its root in Heidegger existentialism, difference-research is the only skeptical thinking raising the basic sociological question about a possible goal displacement in mathematics itself.

Conclusion and Recommendation

The task of the curriculum architect competition was 'Theorize the low success of 50 years of mathematics education research, and derive from this theory a STEM-based core mathematics curriculum for young male migrants.'

One explanation sees the situation caused by mathematics itself as very hard to teach and learn. This paper, however, sees it caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many. The two views lead to different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying 'To master Many, count to produce block-numbers and per-numbers that might be constant or variable, to be united by adding or multiplying or powering or integrating.

Thus, this simplicity of mathematics as expressed in a Count&Multiply&Add curriculum allows learners to keep their own block-numbers, and to acquire core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young male migrants learn core STEM subjects at the same time, thus allowing them to become pre-teachers or pre-engineers after two years to return help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

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14. Math Competenc(i)es - Catholic or Protestant?

Introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015 OECD-report 'Improving Schools in Sweden'. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other Scandinavian countries also have experienced declining PISA results. Which came as a surprise since they all adopted the idea of the eight mathematics competencies introduced by Niss (2003) as a means to solve poor mathematics performance. Of course, new ideas cannot work overnight, but after close to two decades it is time to ask: Why does math competencies not work?

Since education and textbooks are social constructions meant to solve important problems by common social institutions, maybe sociology can provide an answer to the lacking success of the eight mathematics competencies.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959), and by Bauman (1990) saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16). As to rationality as the base for social organizations, Bauman says (pp. 79, 84):

> **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right.

As an institution, mathematics education is a public organization with a rational action 'in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas, arithmetic and geometry and music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time, i.e. as a ‘Many-matics’. And together forming the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught ‘reckoning’ (Rechnung in German) in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here the invention of the concept SET created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming a meaningless language by mixing concrete examples and abstract concepts. In this way, SET transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘1 + 2 IS 3’ meet counter-examples as e.g. 1week + 2days is 9 days. So, mathematics has meant different things during its long history.

Defining Mathematics Competencies

In the paper ‘Mathematical Competencies and the Learning of Mathematics: The Danish Kom Project’ Niss writes (2003, p. 1):

> The fundamental idea of the project is to base the description of mathematics curricula primarily on the notion of a “mathematical competency”, rather than on syllabi in the traditional sense of lists of topics, concepts, and results. This allows for an overarching conceptual framework which captures the perspectives of mathematics teaching and learning at whichever educational level.

Niss writes (pp. 4-5) that the project was initiated in 2000 by the Danish Ministry of Education asking the following questions:

- To what extent is there a need for innovation of the prevalent forms of mathematics education?
- Which mathematical competencies need to be developed with students at different stages of the education system?
- How do we ensure progression and coherence in mathematics teaching and learning throughout the education system?
- How do we measure mathematical competence?
- What should be the content of up-to-date mathematics curricula?
- How do we ensure the ongoing development of mathematics as an education subject as well as of its teaching?
- What does society demand and expect of mathematics teaching and learning?
- What will mathematical teaching materials look like in the future?
- How can we, in Denmark, make use of international experiences with mathematics teaching?
- How should mathematics teaching be organised in the future?

Next, Niss defines what it means to master mathematics (pp. 5-6, 8):

> The Committee based its work on an attempt to answer the following question: *What does it mean to master mathematics?*’ (..) To master mathematics means to possess mathematical competence. (..)

> To possess a competence (to be competent) in some domain of personal, professional or social life is to master (to a fair degree, modulo the conditions and circumstances) essential aspects of life in that domain. *Mathematical competence* then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. (..) *A mathematical competency* is a clearly recognisable and distinct, major constituent of mathematical competence. (..) There are eight competencies which can be said to form two groups. The first group of competencies are to do with the ability to *ask and answer questions in and with mathematics*. (..) The other group of competencies are to do with the ability to deal with and *manage mathematical language and tools*:

Before writing that 'Possessing a mathematical competency (to some degree) consists in being prepared and able to act mathematically on the basis of knowledge and insight (p. 10)' Niss lists (pp. 7-9) and specify the two groups of four mathematical competencies

1. Thinking mathematically (mastering mathematical modes of thought)
2. Posing and solving mathematical problems
3. Modelling mathematically (i.e. analysing and building models)
4. Reasoning mathematically
5. Representing mathematical entities (objects and situations)
6. Handling mathematical symbols and formalisms
7. Communicating in, with, and about mathematics
8. Making use of aids and tools (IT included)

Discussing Mathematics Competencies

As to the definition of mathematics competencies, Niss is very clear: Mathematics competencies are the eight constituents of mathematics competence, defined as the ability to master mathematics. What is not so clear is what Niss means with these two words, mathematics and master.

What kind of mathematics

As to mathematics, at least two kinds of mathematics exits as shown above, a bottom-up and a top-down version, the original Greek grounded Many-matics and the modern self-referring meta-matism. Likewise, on the background of the science wars and mathematics wars in the previous decades, it would be relevant to clarify what kind of mathematics Niss is talking about: the original Greek version, the 'back to basics' pre-NewMath version, the set-based NewMath version, or a post-NewMath version in its constructivist or postmodern forms (Tarp, 1998, 2000).

Instead Niss refers to the fact that in Denmark, as one of the few countries if not the only, teacher education is not allowed to take place at universities where only research directed set-based mathematics is taught forcing students to include a master degree before being allowed to teach in upper secondary school.

Niss describes this difference in teacher background by saying that before upper secondary school, teachers 'are ambassadors of the student to the subject', whereas 'the university graduates who end up teaching mathematics see themselves as ambassadors of mathematics to the student' (pp. 2-3).

> A further aspect of the cultural and institutional differences that exist in Danish mathematics education is that mathematics is perceived and treated so differently at the different levels that one can hardly speak of the same subject, even if it carries the same name throughout the system. (..) The main problem is that the different educational levels tend to see themselves as competitors rather than as agents - acting at different sections of the education system - of the same overall endeavour and a common project, namely to increase and strengthen the mathematical competence of all students who receive some form of mathematics education.

On this background it seems clear that what Niss means with mathematics is the set-based university mathematics introduced with the NewMath. So what Niss points out is which competences are needed to master inside set-based university mathematics, not which are needed to master its outside root, Many. Thus, the question about what could be called quantitative competence is left unanswered.

What kind of Mastering

In the final report Niss left out two of the original Ministry questions, 'How can education take into account the new student type?' and 'What impact will a modified education have for teacher training?'. And in two questions, 'Which competences and qualifications can be acquired at the various stages of the education' and 'How can competences and qualifications be measured?', the word qualification is left out and the word mathematics is added. Likewise, the original term competence has replaced by his own term, competency (Tarp, 2002).

The difference between qualifications and competence might be illustrated by the fact that learning is a process shared by all three kinds of animals, reptiles and mammals and humans, all producing offspring to reproduce, but in different numbers since the chances of survival are different because of different learning abilities. Darwin's 'survival of the fittest' principle points to the fact that to survive you must fit to the surrounding outside world. Reptiles survive by their genes that might change over generations through mutations. Mammals feed their offspring until sexual maturity so they can adapt to the outside surroundings by guidance from their parents in an informal learning setting that could be called apprenticeship or learning from the master, providing the learner with tacit knowledge, also called abilities or know-how or competences. Likewise, humans learn basic living skills and the mother language as competences through apprenticeship guided by caring parents and adults. However, humans benefit from an additional learning possibility occurring when expanding the brain to keep the balance when standing up freed the forelegs to become graspers. Now the brain was also able to store sounds to mentally grasp what was grasped physically (in German: 'greifen & begreifen'), thus developing a word-language and a number-language for outside qualities and quantities allowing for life-long learning.

Language allowing information to be transferred between brains thus creates more competences quicker and more effective. And creates a formal learning setting called education or schooling using rational goal-means descriptions to qualify the learners to obtain the goal by following the means.

Thus, where animals develop competences from 'ex-ducational' informal learning outside school, humans learn additional qualification from 'in-ducational' formal learning inside schools. So human knowledge comes from two channels, from inside school as qualifications and from outside school as competences.

Inside teaching can take place through mediation to qualify or through guidance to develop competences. This discussion takes place between traditional teaching and constructivism; and within constructivism, between a social and a radical version where Vygotsky points to teaching, and Piaget to guidance.

Competence versus Capital

Niss uses no theoretical reference to mathematics or education, but points out that the report is supposed to be a response to question posed by the Ministry (p. 6).

Thus, there is no discussion of parallel and more developed or used concepts describing the same reality as does competences. As an example, Bourdieu (1977) has developed a theory on habitus and capital describing how in a social filed, your social or knowledge capital depends on your habitus within the field. Thus, it seems as if competence is a parallel concept to capital. If that is the case then, according to Bourdieu, capital is only obtainable by informal learning processes.

The Counter KomMod report

The KomMod report (Tarp, 2002) shows the original 12 Ministry questions and how they can be answered in a different way. In the end it compares the two reports by talking about a catholic and a protestant version of mathematics with eight and two competences respectively (p. 3):

> Defining competence as insight-based, the report assumes that mathematics is already learned, after which the rest of the time can be used to apply mathematics, not on the outside world, but on mathematics itself through eight internal competencies leading to exercising mathematical

professionalism. This makes it a report on ‘catholic mathematics’ with eight sacraments, through which the encounter with science can take place. In contrast to this, the counter-report portrays a ‘protestant mathematics’ that emphasizes the importance of a direct meeting between the individual and the knowledge root, Many, through two sacraments, count and add.

Quantitative Competence

In the outside world, Many often occurs in time and space. To master Many, you must have quantitative competence from informal learning or quantitative qualifications from formal learning.

Meeting Many, we ask ‘How many in total?’ To answer, we count and add to get a number for a number-language sentence telling that the total is e.g. T = 456, thus containing a subject and a verb and a predicate as in the word-language. By counting and adding you build different know-how as to how to master Many:

- A digit has as many strokes as it represents, e.g. four strokes in the 4-icon, etc.
- Counting the fingers on a hand, the total cannot be different, but how to count it can be different, e.g. T = 5 1s = 2 2s & 1 = 1 3s & 2 1s = 1 3s & 1 2s etc.
- The sentence T = 456 is a short way of writing T = 4*BB + 5*B + 6*1, describing what exists, three blocks with 6 1s and 5 bundles and 4 bundles-of- bundles, typically using ten as the bundle-size and therefore needing no icon since ten then is 1*B. This shows that a number is the result of several countings: of unbundled ones, of bundles, of bundles-of-bundles etc.; and shows that all numbers have units: ones, bundles, bundles-of-bundles, etc.
- Writing out fully, T = 456 also shows the four ways to unite totals: on-top addition creating a block described by multiplication as repeated addition, power describing repeated multiplication when forming bundles-of-bundles, and finally integration as next-to addition when juxtaposing blocks.
- Operations are icons also: division is iconized as a broom wiping away the bundles; multiplication as a lift stacking the bundles into a block; subtraction as a trace left when dragging away the blocks to look for unbundled singles; and addition as a cross since blocks may be added both on-top or next-to.
- To deal with leftover singles when bundling we introduce a decimal point to separate the bundles from the singles, e.g. T = 7 = 2B1 3s = 2.1 3s, or we count the singles in bundles also even if a part only, T = 7 = 2B1 3s = 2 1/3 3s.
- A total can be recounted to change unit. Recounting in the same unit creates overload or underload e.g. T = 42 = 4B2 = 3B12 = 5B-8. This is useful when performing standard operations as e.g. T = 5*43 = 5*4B3 = 20B15 = 21B5 = 215. Or, we just move the decimal point separating the bundle from the unbundled, e.g. T = 4.3 hundreds = 43 tens = 0.43 thousands.
- To recount in another bundle size we use a ‘recount formula’ T = (T/B)*B saying that ‘from T, T/B times B can be taken away’ as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the ‘restack formula’ T = (T–B)+B saying that ‘from T, T–B is left when B is taken away and placed next-to’, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. The recount formula turns out to be a very basic formula turning up repeatedly: In proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right-angled triangles, and in calculus as dy = (dy/dx)*dx = y’*dx when counting steepness on a curve.
- To recount icons in tens we use the multiplication table, e.g. T = 6 7s = 6*7 = 42. To recount tens in icons we solve equations, e.g. T = 42 = ? 7s = x*7 solved by x = 42/7, i.e. by moving numbers to opposite side with opposite sign.
- Double-counting a quantity in physical units creates per-numbers as e.g. 4$/5kg or 4/5 $/kg allowing the two units to be bridges by recounting in the per-number: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$, etc. With like units we get fractions, or percentages.

- Adding means uniting unit- and per-numbers, that can be constant or variable. So to predict, we need four uniting operations: addition and multiplication uniting variable and constant unit-numbers; and integration and power uniting variable and constant per-numbers. As well as four splitting operations: subtraction and division splitting into variable and constant unit-numbers; and differentiation and root/logarithm splitting into variable and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite.
- Blocks can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

Proportionality, an Example of Different Quantitative Competences

A question asks 'If 5kg costs 30$ what does 8kg cost; and what does 54$ buy?

A 1867 reguladetri 'long way-method' says: 'Make the outer units like, then multiply and divide, but from behind'. So, after reformulating the second question to '30$ buys 5kg, what does 54$ buy?' the first answer is 8*30/5$ = 48$; and the second answer is 54*5/30kg = 9kg.

A 1917 unit-method says: 1kg costs 30/5 = 6$, so 8 kg costs 6*8 = 48$.

A 1967 function-method says: With f(5) = 30, the linear function f(x) = c*x becomes f(x) = 6*x. So f(8) = 6*8 = 48. And 54 = 6*x is an equation. To neutralize 6, both sides are multiplied with its inverse element, 1/6, giving x = 54*1/6 = 9.

A 2017 back-to-basics method says 'cross-multiply' the price equation: 30/5 = x/8 gives 5*x = 8*30, so x = 48. And 30/5 = 54/x gives 30*x = 5*54, so x = 9.

A 2067 double-counting method recounts in the per-number 5kg/30$. So 8kg = (8/5)*5kg = (8/5)*30$ = 48$. And 54$ = (54/30)*30$ = (54/30)*5kg = 9kg.

Conclusion

Invented to improve mathematics education, the eight mathematics competencies inspired Scandinavian educational reforms that failed as witnessed by low PISA results decreasing until 2015. This paper asked why the competencies failed.

Formal education can use mediation to qualify or constructivism to create competences by guided meetings with the outside subjects for which education is supposed to prepare the learner. With Niss we can discuss which competences to create and how, but only in a constructivist setting that accepts the original Greek meaning of mathematics as knowledge about Many in time and space.

Niss may be right that his eight mathematical competences are needed to survive at a university that holds on to the original set-based version of mathematics introduced with the NewMath and recommended by Bruner to also be mediated in schools. But to master the outside goal Many, two competences will do, count & add, since they allow answering the standard question 'How many in total' by producing a number created by counting and adding as shown when writing out fully a number as a combination of blocks.

So the eight mathematics competences failed because university mathematics and school mathematics have different goals. At the university, education prepares you for the inside goal of staying at the university as a researcher; and in school, education prepares you for the task of mastering Many as it appears outside school in time and space.

Recommendation: Expand the Existing Quantitative Competence

By distinguishing between 4 and 2 2s at the 4^{th} birthday, a child shows that before formal learning begins in school, the informal learning of growing up makes the child develop the two core quantitative competences, counting and adding. By counting in 2dimensional block-numbers supplied with some leftovers, children show a basic competence in double-counting a total in bundles and unbundled. And, when adding blocks, they answer by using one of the units or by uniting the units, thus showing a basic competence in proportionality and calculus.

Seeing expanding the learner's quantitative competence as the goal of mathematics education, school may choose to use guiding 'footnote-teaching':

- Show that digits are icons with as many strokes as they represent by inviting the child to build up a 5-icon with five dolls or cars or animals, etc.
- Ask the child to use cups for the bundles when re-counting a total in icons thus emphasizing that counting means double-counting, first bundles to be placed in a bundle-cup, then unbundled singles to be left outside, allowing a total to be counted in three ways: normal, and with outside overload or underload.
- Show that the four operations are icons as well, created to allow a calculator to predict the result when recounting a total in another unit; especially from icons to tens predicted directly by the multiplication table; or from tens icons, becoming equations solved by recounting in the icon, and technically by moving numbers to opposite side with opposite sign.
- Accept overload or underload, quickly created or removed by recounting, with standard operations as adding, subtracting, multiplying or dividing.
- Show that totals can be added both on-top after recounting them in the same unit thus rooting proportionality, and next-to recounting them in the united unit thus rooting integral calculus.
- Show that reversed on-top addition roots equations, again solved by recounting, i.e. by moving to the opposite side with opposite sign; and that reversed next-to addition roots differential calculus by using subtraction to remove the initial block, and division to recount the rest.

Once school has allowed the child to use and develop its own quantitative competence, it will be possible to expand this by introducing double-counting in physical units to create per-numbers, becoming fractions if using the same physical unit. Adding per-numbers and fractions by their areas then becomes just another example of adding blocks next-to each other, also by their areas. (Tarp, 2017)

So, formal school mathematics education can choose to expand the child's existing two quantitative competences, to count and to add. Or it can choose to discard them and force upon the child eight mathematics competencies about one-dimensional number-names arranged in a place-value system, and about more or less obscure algorithms when adding, subtracting, multiplying and dividing, and about fractions as numbers that can be added without considering the units.

In short, the school can choose to strengthen or weaken the mastery of Many that the child brings to school. Wanting to improve mathematics education, maybe it would be a good idea to choose the former and stop practising the latter.

So, we can celebrate the 500year Luther anniversary by saying: The subject of mathematics education, Many, we can meet directly without being mediated by its 'latinized' version in the form of a self-referring meta-matism.

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15. The ‘KomMod Report’, a Counter Report to the Ministry’s Competence Report

Allan Tarp, 2002, translated into English in 2017.

The KomMod report provides an alternative response to KOM-project terms of reference, in the expectation that the Science Board of education and the Ministry of education want to respect a common democratic IDC-tradition with Information and Debate between alternatives before a Choice is made. The report replies to the following questions relating to mathematics education:

(a) What is the society's requirements for the education?

(b) To what extent is there a need to renew the existing education?

(c) How can education take into account the new student type?

(d) What content can contemporary mathematics education have?

(e) How can the future of education be organized?

(f) How to secure progression and consistency in education?

(g) What impact will a modified education have for teacher training?

(h) Which competences and qualifications can be acquired at the various stages of the education?

(i) How can competences and qualifications be measured?

(j) How can future teaching materials look like?

(k) How to secure a continuous development of the education?

(l) How can Denmark exchange educational experience with other countries?

Ad a. Our democratic society needs citizens and specialists to have a common number-language to communicate about quantities and calculations. Society needs mathematics as a human right, both as a discursive qualification and as silent competence.

Ad b. There is a need to renew the current mathematics education in order to solve its three main problems: 1. There is a widespread number-languages illiteracy, where many citizens are reluctant to use the number-language. 2. There are major transition issues between primary, secondary and tertiary education. 3. There is a decreasing enrolment to math-based education in science, technology and economy, as well as a large shortage of new secondary school teachers in mathematics.

Ad c. Future mathematics teaching should respect today's democratic, anti-authoritarian youth and its requirements on meaning and authenticity. This can be achieved if the subject respects its historical roots, and re-humanizes itself by presenting abstractions as abstractions and not as examples, i.e. as abstractions from examples (a function is a name for a formula with variable numbers), and not as examples of even more abstract abstractions (a function is an example of a set-relation). In short, the subject should portray itself as mathe-matics, recognizing its outside roots from which it has grown bottom up through abstractions. And the subject must say goodbye to the current ‘meta-matics’ and its belief that it has meta-physical roots and has grown top down as examples. Finally, the subject should respect the fact that people learn differently. Children learn by touching the world, i.e. by building competences. Young people learn by listening to the world, i.e. by building narratives and skills from the learning question "tell me something I don't know about something I know" (gossip-learning).

Ad d. Mathematics must respect its history as grown through abstractions, and thus also its construction as a number-language grammar, which can only be introduced after the number-language has been developed. The number-language has grown out of the meeting with quantity in time (repetition) and in space (many-ness). This meeting constructed numbers to describe the total, either through counting in pieces, bundles, bundles of bundles, bundles of bundles of bundles etc. Or faster by means of calculations to unite and divide unit-numbers (3$) and per-numbers (3$/day, 3%): Plus and minus unite and divide in variable unit-numbers (3 + 5 =?, 3 +? = 8). Multiplication and division unite and divide in constant unit-numbers (3 * 5 =?, 3 *? = 15). Potency and root & logarithm unite and divide in constant per numbers (3 times 5% =?%, 3 times ?% = 20%, ? times

5% = 20%). Integration and differentiation unite and divide variable per-numbers (5 seconds at 2 m/s growing evenly to 4 m/s =? m, 5 seconds of 2 m/s growing to 4 m/s = 18 m?). In short, the subject must respect the fact that geometry has grown out of what the word means in Greek, earth measurement; and respect that algebra has grown out of what the word means in Arabic, reunion, i.e. uniting and dividing constant and variable unit-numbers and per-numbers. Geometry and algebra must therefore respect their historical roots in an agricultural culture with two main questions: "How to share the Earth, and what it produces?" The number-language has a number of typical applications: Geometry deals with forms and shapes. Formulas deal with number levels. Growth deals with predictable change. Statistics/probability deals with change that is not pre-dictable but post-dictable. It is important to clean teaching of 'killer-Mathematics' (i.e. mathematics, that does not occur outside of the classroom, and that can only be used for one thing, killing students' interest). Addition should only occur within the parentheses, which ensures that the units are equal (T = 2 * 3 + 5 * 3 = (2 + 5) * 3 = 7 * 3 = 21). Fractions should only act together with their totals (1/2 of 2 plus 2/3 of 3 = 3/5 of 5). Equations should be solved by reversed calculation. Since the set concept cannot be well-defined it should be removed, and functions be postponed until it pops up historically after differential calculus.

Ad e. Future mathematics lessons can be organized in two main areas: Child math and youth math from respectively grade 1-7 and 8-12. Meeting the roots of mathematics roots, Many in time and space, will develop the learner's two core competences: to count and to add.

Ad f. Progression and consistency in teaching can be ensured by letting the child's math grow out of the local examples of Many, and of agricultural examples from rural and urban areas, and by letting the youth's math grow out of industrial culture and its global diversity. As well as by the child primarily working with unit-numbers, and young people primarily with per-numbers.

Ad g. By dividing education into the child's mathematics and the youth's mathematics, it will also be natural to divide teacher education in primary school teacher and secondary school teacher, as in the rest of the world approximately. This means that all future teacher-training takes place at a university. In the end, this will coincide with the division of the school into a primary school and secondary school that will take place within the next decade in connection with the high school collapse due to increased teacher retirement and decreasing enrollment of new teachers in mathematics and natural science.

Ad h. By meeting Many in time and space, the child develops competences in uniting and dividing constant and variable unit-numbers. In the countryside, bundling and re-bundling leads to multiplication and division. In the city, stacking and re-stacking leads to addition and subtraction. Calculating repetition and diversity develops the skills of young people to unite and divide constant and variable per-numbers. Totaling interest rates leads to power, root and logarithm. Totaling distances leads to integral and differential calculus.

Ad i. Competences are tacit knowledge and can therefore be neither described nor measured, but will evolve automatically through the meeting with meaningful and authentic situations, and grow from the many concrete experiences with Many in time and space, bundling and stacking, uniting and splitting, unit-numbers and per-numbers. Qualifications is measured as now through three types of tasks: Routine tasks, text tasks and projects.

Ad j. Future teaching materials should be short and concise so that time could be dedicated for student learning through self-activity. The material should respect that students have two brains, a reptile's brain for routines and a human brain for conceptual understanding. There should therefore be training tasks with responses, so learners can progress at their own pace and do as many exercises as wanted. As well as textbooks telling how mathematics has grown from practice through layers of abstractions, and accepting different names so concept may be named both bottom-up and top-down, as e.g. growth by adding and linear function etc.

Ad k. A continuous development of education can be ensured by continuously relating mathematics to its roots and not to the current political correctness.

Ad l. Exchange of experience with foreign countries can be done through establishing a Danish development research, in which practitioners can combine being researcher at a University with being attached to a teacher team at a school. This will avoid the current barren ‘ghost research’ performed by researchers without experience background in teaching practice. Development research should be difference-research (Cinderella-research) using practice based and sociological imagination to discover and try out hidden alternatives.

The Difference between the KOM- and KOMMOD Reports

In mathematics education, the two main question are: ‘How do concepts enter into the world and into the student's head - from the outside or from the inside?’ These questions give rise to different answers. Secondary school structuralism says ‘outside-outside’: Concepts exist in the meta-physical world, they are discovered by researchers and mediated by teachers. Primary school constructivism says ‘outside-inside’: Concepts exist in the meta-physical world, but are discovered through experimentation, in which each student construct their own knowledge and abilities (schemata and competences), both being silent and only to be observed through use. Post-structuralism says the ‘inside-outside’: Concepts are created through invention and social construction, and should be presented as such. Apprenticeship says ‘inside-inside’: Concepts are constructed by the apprentice during the participation in the master's practice.

Worldwide, two knowledge wars rage, a math-war between structuralism and constructivism, and a science-war between structuralism and post- structuralism. Instead of acknowledging this diversity, the report is trying to conceal it by taking over the core constructivist concept, competence, but giving it a structuralist content (insight-based action-readiness). The French philosopher Foucault has shown how new words create new clients: ‘Qualification’ creates the unqualified, and ‘competent’ creates the incompetent. But where the unqualified can cure themselves by qualifying themselves, the incompetent cannot cure themselves by ‘competencing’ themselves, and are thus left to be cured by others, the competence-competent. Adoption and modification of the word competence can therefore be interpreted as a structuralist attempt to win the math-war by a coup, instead of using it to a fruitful dialogue with equal partners.

First structuralism tried to solve the math-crisis through the wording ‘responsibility for your own learning’. Students took this seriously and turned their back to ‘meta-matics’ with its meaningless self-reference (a function is an example of a set-relation: bublibub is an example of bablibab).

Now instead the teachers are disciplined and incapacitated by constructing them as incompetent, with a consequent need for competence development through massive in-service training. Omitting the competence ‘experimenting’ shows that the report only respects science as an end-product, and neither the process nor its roots in the outside world. Neither does it respect the way in which young people and especially children acquire knowledge through self-activity and learning.

Defining competence as insight-based, the report assumes that mathematics is already learned, after which the rest of the time can be used to apply mathematics, not on the outside world, but on mathematics itself through eight internal competencies leading to exercising mathematical professionalism. This makes it a report on ‘catholic mathematics’ with eight sacraments, through which the encounter with science can take place. In contrast to this, the counter-report portrays a ‘protestant mathematics’ that emphasizes the importance of a direct meeting between the individual and the knowledge root, Many, through two sacraments, count and add; and emphasizes that linguistic competence precedes grammatical competence. Meaning that also with quantitative competence, the number-language comes before its grammar, mathematics; and as with the word-language, grammar remains a silent competence for most.

Will the math-war end with a KOM-coup? Or will it be settled through a democratic negotiation between opposing views? The choice is yours, and the KomMod report gives you an opportunity to validate the arguments, not from above from political correctness, but from below from the historic roots of mathematics. Best of luck.

SET-based ‘MetaMatics’, or Many-based ManyMatics: Learning by Meeting the Sentence or by meeting its Subject

Class 1-2

SETS are united: addition
2 + 3 = 5
47 + 85 = 135
82-65 = 17
PROBLEM:
Addition is a false abstraction: 2 m + 3 cm = 203 cm
2 weeks + 3 days = 17 days
2 C + 3 D = 23 D
3 stones = stone + stone + stone

Country: Bundle & ReBundle
Multiplication is true abstraction:
3 stones = 3 times stone = 3·stone
2·3·days = 6·days
2·m·3·cm = 6·m·cm = 600 cm ^ 2
Bundling and ReBundling:
Total = 6 1s =? 2s
Response: 6·1 = 6 = (6/2)·2 = 3·2
ReBundling-rule: T = (T/b)·b

6/2: Counted in 2s
6·2: Counting 2s

To find the total, count or calculate: ReBundling (division)

Multiplication rebundles in tens:
T = 8·3 = 24 = 2·ten & 4·1
= 2·D & 4·1
Multiplication is division!

Max-height 3:
T = 8 3s = overload
T = 8·3 = 2·3 ^ 2 & 2·3
Unbundled can also bundled in parts, for example in 5s:
T = 8·3 = (24/5)·5 = 4·5 & 4·1
= 4·5 & (4/5)·5 = (4 4/5)·5

Class 3-4

SETS are repeated: multiplication
2·3 = 6
7·85 = 595
372/7 = 53 1/7

City: Stacks & ReStack
T = 653 + 289 =?
653 = 6·C & 5·D & 3·1
279 = 2·C & 7·D & 9·1
T = 8·C & 13·D & 12·1
T = 8·C & (13 + 1)·D & (12-10)·1
T = (8 + 1)·C & (14-10)·D & 2·1
T = 9·C & 4·D & 2·1 = **942**
<u>ReStack rule: T = (T – b) + b</u>

T = 654-278 =?
653 = 6·C & 5·D & 4·1
278 = 2·C & 7·D & 8·1
T = 4·C & -2·D & -4·1
= (4-1)·(C) & (-2 + 10)·D & -4·1
= 3·(C) & (8-1)·(D) & (-4 + 10)·1
= 3·C & 7·D & 6·1 = 376

T = 7·653 =?
T = 7·(6·C & 5·D & 3·1)
= 42·C & 35·D & 21·1
= 42·(C) & (35+2)·(D) & (21-20) ·1
= (42 + 3)·(C) & (37-30)·D & 1·1
= 45·C & 7·D & 1·1 = 4571

T = 653/7 =?
T = 6/7·C & 5/5·D & 3/7
= 65/7·D & 3/7
= (65-2)/7·(D) & (20 + 3)/7
= 9·D & 23/7
= 9·D & 3 2/7 = 93 2/7
(double book-keeping)

Class 5-6

SETS are divided: fractions
$\frac{1}{2}+\frac{2}{3} = ?$
$\frac{1}{2}+\frac{2}{3} = \frac{3}{6}+\frac{4}{6} = \frac{7}{6}$
PROBLEM:
$\frac{1}{2}+\frac{2}{3} = \frac{1+2}{2+3} = \frac{3}{5}$ if
1 coke of 2 bottles plus 2 cokes of 3 bottles is (1 + 2) cokes of (2 + 3) bottles.

City: weighted average
$T = \frac{1}{2}\cdot 2+\frac{2}{3}\cdot 3 = 3 = \frac{3}{5}\cdot 5$, or
$T = \frac{1}{2}\cdot 4+\frac{2}{3}\cdot 3 = 4 = \frac{4}{7}\cdot 7$
So there are many different answers to the question
$\frac{1}{2}+\frac{2}{3} = ?$
But NEVER more than 1!

Trade calculations
5 kg cost 60 $, 3 kg cost ? $

ReBundle $	ReBundle kg
$ = ($/kg)·kg $ = (60/5)·3 $ = 36	3 kg = (3/5)·5kg 3 kg = (3/5)·60$ 3 kg = 36$

Percentages part 1
• 8 has 2, so 100 has ?
100 = (100/8)·8 has (100/8)·2 = 25
• 100 has 25, so 8 has ?
8 = (8/100)·100 has (8/100)·25 = 2
• 100 has 25,so ? has 2
2 = (2/25)·25 had by (2/25)·100 = 8

Class 6-7

Solution-**SETS**: open statements (equations)

2 + 3·x = 8 (2+3·x)-2 = 8-2 (3·x + 2)-2 = 6 3·x + (2-2) = 6 3·x + 0 = 6	3·x = 6 (3·x)/3 = 6/3 (x·3)/3 = 2 x·(3/3) = 2 x·1 = 2

L = {x ∈ R | 2+3·x = 8} = {2}
PROBLEM:
The weight-metaphor hides the count process, and creates many error possibilities as e.g.
If 2 + 3·x = 8, then 5·x = 8

Castle & Monastery: Coding 2 + (3·5) = 17 → 2 + (3·x) = T
DeCoding (solving an equation):
ReStacking 8 in two stacks:
2 + (3·x) = 8 = (8-2) + 2
3·x = 8-2 = 6
ReBundling from 1s to 3s:
3·x = 6 = (6/3)·3
x = 6/3 = 2
Forward- & back calculations:
To opposite side with opp. sign

<u>Forward</u>		<u>Back</u>
2 + 3·x	=	8
+ 2 ↑↓		*-2*
3·x	=	8-2 = 6
·3 ↑↓		*/3*
x	=	6/3 = 2

Percentage part 2
• 25% of 8 is ?
0.25·8 = x
• 25% of ? is 2
0.25·x = 2, så x = 2/0.25 = 8
• ? % of 8 is 2
x·8 = 2, so x = 2/8 = 0.25 = 25%

Class 8-9

SETS are connected: functions
Function: an example of a many-one set-relation
E.g. f (x) = 2 + 3·x
A function’s value and graph
PROBLEM:
The function came after calculus!
A syntax error to confuse the language and meta-language: the function’s value corresponds to the verb’s tie.

City: Trade and Tax
Per-numbers: Tax, custom, exchange and interest rates, profit, loss, bonds, assurance.
Adding per-numbers:
3 kg at 4$/kg + 5 kg at 6$/kg gives 8 kg at ? $/kg
Geometry: area and volume of plane and spatial forms. Right-angled triangles: Pythagoras, sine, cosine & tangent.
Linear funct.: growth by adding:
T = b + a + a + a + … = b + a·n
A function is a name for a calculation with variable numbers, such as. T = 2 + 3 x. (Euler 1748)
Calculations give fixed and functions give variable number.
The change of a function can be shown in tables or on curves.
The Inn: Redistribution by games
Winn on pools, lotto, roulette.
Statistics counts number of wins.
Risk = Consequence·propability.

Class 10	Class 11	Class 12
Set theory Function theory: Domains & values. Algebraic functions: Polynomials and polynomial fractions. First- & second-degree polynomials. Trigonometry. Analytical geometry.	Function theory: reverse and composite function. Non-algebraic functions: trigonometric functions. Logarithm- & exponential functions as homomorphisms: f (x*y) = f (x) # f (y) Stochastic functions. Core calculus.	Vector spaces. Main calculus. Simple differential equations.
The Renaissance: Constant per-numbers Numbers as many-bundles (polynomials): $T = 2345 = 2 \cdot B \wedge 3 + 3 \cdot B \wedge 2 + 4 \cdot B + 5 \cdot 1$ Reversed calculations with powers: $B \wedge 4 = 81$ $\quad$ $4 \wedge n = 1024$ $B = 4 \sqrt{81}$ $\quad$ $n = \log 1024/\log 4$ Interest rates: Single r, total R, compound RR $(1 + r)\wedge n - 1 = R = n \cdot r + RR$ Change with constant per-number and percentage: x: + 1 → T: + a$ $\quad$ linear change $\quad$ $T = b + a \cdot x$ x: + 1 → T: + r% $\quad$ exponential $\quad$ $T = b \cdot (1+r)\wedge x$ x: + 1% → T: + r% $\quad$ power change $\quad$ $T = b \cdot x \wedge r$ x: + 1 → T: + r% + a$ $\quad$ savings $\quad$ $T = a \cdot R/r$ Change with unpredictable (random) variation $\Delta T = ?$ $\quad$ $T = MID \pm 2 \cdot SPR$ Adding percentages by their areas (integration): 300$ at 4% and 500$ at 6% is 800$ at ? %. Change percentage: $T = a \cdot b$: $\quad$ $\Delta T/T \approx \Delta a/a + \Delta b/b$ $T = a/b$: $\quad$ $\Delta T/T \approx \Delta a/a - \Delta b/b$ Trigonometry: SIN & COS: short sides in percent of the long. TAN: the one short side in percent of the other.	**Industry: Variable per-numbers** Coordinate geometry: Geometry & algebra, always together, never apart. Curve fitting with polynomials: $T = A + B \cdot x + C \cdot x\wedge 2 + D \cdot x\wedge 3$ (or $y = A + B \cdot x + C \cdot x\wedge 2$) A: level, B: rise, C: curvature, D: counter-curvature Variable, predictable change: Differential calculus: $dT = (dT/dx) \cdot dx = T' \cdot dx$ The non-linear is locally linear: $(1+r)\wedge n \approx 1 + n \cdot r$ ($= 1 + n \cdot r + RR$: with a small interest, the compound-interest can be neglected) $T = x\wedge n$: $dT/T = n \cdot dx/x$, $dT/dx = n \cdot T/x = n \cdot x\wedge(n-1)$ Optimization tasks in engineering and economics. Integral calculus: $\Delta T = T2 - T1 = \int dT = \int f \cdot dx$, Total change = terminal – start = the sum of single changes, regardless of their number or size. Integration is done by rewriting to change form: Since $6 \cdot x\wedge 2 + 8 \cdot x = d/dx\,(2 \cdot x\wedge 3 + 4 \cdot x\wedge 2) = d/dx(T)$ then $\int(6 \cdot x\wedge 2 + 8 \cdot x)\,dx = \int d(2 \cdot x\wedge 3 + 4 \cdot x\wedge 2)$ $= \int dT = \Delta T = T2 - T1$ Accumulation tasks in engineering and economics.	**Major works in the Quantitative Literature:** Geometry, Trade, Economics, Physics, Biology. The three genres for quantitative literature: - *Fact or since-then calculations* quantifies the quantifiable, and calculates the calculable: since the price is 4$/kg, then the cost of 6 kg is 6·4$ = 24$. - *Fiction or if-then calculations* quantifies the quantifiable, and calculate the incalculable: if my income is 4m$/year, then 6 years of income will be 6·4m$ = 24 million $. - *Fiddle or so-what calculations* quantify the non-quantifiable: If the consequence 'broken leg' C is taken to be 2 million $, and if the probability p is taken to 30%, then the risk R will be R = C·p = 2m$·0.3 = 0.6 million $. The three courses of action: fact models are controlled especially for the units; fiction models are supplemented with alternative scenarios; fiddle models are referred to a qualitative treatment. Change equations solved by numerical integration. Functions of two variables. Differentiation and integration. Optimization and accumulation. Vectors used in trade and in the movement on a surface and in space.

16. Twelve Proposals for 1day Skype Seminars

01) The Root of Mathematics, Many, dealt with by Block-Numbers, Bundle-Counting & Preschool Calculus

"How old next time?" I asked the child. The answer was four with four fingers shown. But held together two by two created a protest: "That is not four, that is two twos!". That opened my eyes. Children come to school with two-dimensional block-numbers where all numbers have units. Instead, school teaches cardinality as a one-dimensional line with different number-names; thus disregarding the fact that numbers are two- dimensional blocks all having a unit as shown when writing out fully a total T = 345 = 3 BundleBundles + 4 Bundles + 5 Singles = 3*10^2 + 4*10 + 5*1. So, a number is blocks united (integrated) next-to each other, showing the four ways to unite numbers presented by Algebra, meaning reuniting in Arabic: Power and multiplication and 'on-top' and 'next-to' addition (integration).

Consequently, mathematics education should develop the two-dimensional block-numbers that children bring to school and allow them to practice counting before adding.

To master Many, we ask 'how many?' To answer, we bundle-count using a cup for the bundles. So, a number always has some bundles inside and some unbundled outside the cup.

Recounting in the same unit creates overloads or underloads by moving in or out of the cup, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. This makes calculation easy: T = 4 x 56 = 4 x 5B6 = 20B24 = 22B4 = 224.

Once counted and recounted, totals may de added. To have 2 3s and 4 5s added on-top as 5s, a unit must be changed, called proportionality. To add them next-to as 8s means adding their areas, called integration; which becomes differentiation when reversed by saying 2 3s + ? 5s = 3 8s', thus allowing calculus to take place in preschool.

02) 12 Luther-like Theses about how ManyMath can Improve Math Education

1. Digits are icons with as many sticks as they represent.

2. A total T can be 'bundle-counted' in the normal way or with an overload or underload: T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s.

3. 'Bundle-writing' makes operations easy: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

4. Counting T by bundling, T = (T/B)xB = (5/2)x2 = 2.1 2s, shows a natural number as a decimal number with a unit.

5. Operations are icons showing counting by bundling and stacking. -2 takes away 2. /2 takes away 2s. x2 stacks 2s. +2 adds 2 on-top or next-to.

6. A calculator predicts. Asking T = 4 5s = ? 6s, first (4x5)/6 = 3.some; then (4x5) – (3x6) = 2. So T = 4 5s = 3.2 6s

7. Recounting in tens, calculators leave out the unit and misplace the decimal point: T = 3 7s = 3*7 = 21 = 2.1 tens.

8. Recounting from tens, '? 7s = 3 tens', or 'u*7 = 30 = (30/7)x7', the answer u = 30/7 is found by 'move to opposite side with opposite sign'.

9. Adding totals is ambiguous: OnTop using proportionality, or NextTo using integration?

10. Operations are reversed with reverse operations: With u+3 = 8, u = 8–3; with ux3 = 8, u = 8/3; with u^3 = 8, u = 3√8; with 3^u = 8, u = log3(8); with T1 + u*3 = T2, u = ΔT/3.

11. Double-counting in different units gives 'per-numbers' as 4$/5kg, bridging the two units by recounting: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$

12. Double-counting in the same unit, per-numbers become fractions as operators, needing a

number to become a number, thus adding by their areas as integration.

03) Curing Math Dislike with one Cup and five Sticks

A class is stuck in division and gives up on 234/5. Having heard about '1cup & 5 sticks', the teacher says 'Time out. Next week, no division. Instead we do bundle-counting'. Teacher: 'How many sticks?' Class: '5.' Teacher: 'Correct, 5 1s, how many 2s?' Class: '2 2s and 1 left over'. Teacher: 'Correct, we count by bundling. The cup is for bundles, so we put 2 inside the cup and leave 1 outside. With 1 inside, how many outside? And with 3 inside, how many outside?' Class: '1 inside and 3 outside; and 3 inside and 1 lacking outside.' Teacher: 'Correct. A total of 5 sticks can be counted in 3 ways. The normal way with 2 inside and 1 outside. With overload as 1 inside and 3 outside. With underload as 3 inside and less 1 outside.' Class: 'OK'. Teacher. 'Now 37 means 3 inside and 7 unbundled 1s outside. Try recounting 37 with overload and underload. Class: '2 inside and 17 outside; and 4 inside and less 3 outside.'

Teacher: 'Now let us multiply 37 by 2, how much inside and outside?' Class: 6 inside and 14 outside. Or 7 inside and 4 outside. Or 8 inside and less 6 outside.'

Teacher: 'Now to divide 74 by 3 we recount 7 inside and 4 outside to 6 inside and 14 outside. Dividing by 3 we get 2 inside and 4 outside; plus 2 leftovers that still must be divided by 3. So 74/3 gives 24 and 2/3.'

Class: 'So to divide 234 by 5 we recount 234 as 20 inside and 34 outside. Dividing by 5 we get 4 inside and 6 outside; plus and 4 leftovers that still must be divided by 5. Thus 234/5 gives 46 and 4/5?'

Teacher: 'Precisely. Now try multiplication using bundle-counting'.

04) DoubleCounting rooting Proportionality - and Fractions and Percentages as PerNumbers

A class is stuck in fractions and percentages and gives up on ¾ = 75%. Having heard about 'per-numbers', the teacher says: Time out. Next week, no fractions, no percentage. Instead we do double-counting. First counting: 42 is how many 7s? The total T = 42 = (42/7)*7 = 6*7 = 6 7s. Then double-counting: Apples double-counted as 3 $ and 4 kg have the per-number 3$ per 4 kg, or 3$/4kg or ¾ $/kg. Asking how many $ for 10kg, we recount 10 in 4s, that many times we have 3$: The total T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Asking how many kg for 18$, we recount 18 in 5s, that many times we have 4kg: The total T = 18$ = (18/5)*5$ = (18/5)*4kg = 14.4kg. Double-counting in the same unit gives fractions and percentages as 3 per 4, ¾; and 75 per hundred, 75/100 = 75%.

¾ of 200$ means finding 3$ per 4$, so we recount 200 in 4s, that many times we have 3$: The total T = 200$= (200/4)*4$ gives (200/4)*3$ = 150$. 60% of 250$ means finding 60$ per 100$, so we recount 250 in 100s, that many times we have 60$: The total T = 250$= (250/100)*100$ gives (250/100)*60$ = 150$.

To find 120$ in percent of 250$, we introduce a currency # with the per-number 100# per 250$, and then recount 120 in 250s, that many times we have 100#: The total T = 120$ = (120/250)*250$ = (120/250)*100# = 48#. So 120$/250$ = 48#/100# = 48%. To find the end-result of 300$ increasing with 12%, the currency # has the per-number 100# per 300$. 12# increases 100# to 112# that transforms to $ by the per-number. The total T = 112# = (112/100)*100# = (112/100)*300$ = 336$.

05) Algebraic Fractions made easy by Block-Numbers with Units

A class is stuck in algebraic fractions insisting that (2b+4)/2b is 4. Having heard about 'Block-Numbers with units, the teacher says: 'Time out. Next week, no algebraic fractions. Instead we count totals with units.' Teacher, showing six sticks: 'How many sticks?' Class: '6.' Teacher: 'Correct, 6 1s, how many 2s?' Class: '3 2s'. Teacher: 'Correct, we count in 2s by taking away 2s, that is by dividing by 2, so T = 6 = (6/2) 2s = 3 2s = 3*2. So, factorizing 2b as 2*b, 2b is 2 bs or b

2s. Can 4 be written with a unit?' Class: '4 is 2 2s'. Teacher: 'Correct, so 2b and 4 can be written as b 2s and 2 2s totalling b+2 2s or (b+2)*2.' Class: 'OK'. Teacher: 'Now, 6 2s divided by 3 2s gives 6 divided by 3 or 2. And c 2s divided by 3 2s gives c divided by 3.' Class: 'OK'. Teacher: 'So, b+2 2s divided by b 2s gives b+2 divided by b.' Class: 'OK, and that gives 2?' Teacher: 'Well, division means removing a common unit. So, with b as b 1s and 2 as 2 1s we can remove the 1s. But b+2 1s divided by b 1s still gives b+2 divided by b, which is the result.' Class: 'OK'. Teacher: 'Now try (3c+9)/6c.' Class: 'We factorize to find a common unit 3: 3c is c 3s, 9 is 3 3s, and 6c is 2x3xc or 2c 3s. Removing the common unit we get (3c+9)/6c = (c+3)/2c.' Teacher: 'Correct. Now try (b^2c+ bd^3)/bc'. Class: 'We factorize to find a common unit b: b^2c is bxbxc or bc bs , bd^3 is d^3 bs, and bc is c bs. Removing the common unit, we get (b^2c+ bd^3)/bc = (bc + d^3)/c.'

06) Algebra and Geometry, always Together, never Apart

The ancient Greeks used mathematics as a common label for their four knowledge areas, arithmetic, geometry, music and astronomy, seen as many by itself, many in space, many in time and many in space and time. With music and astronomy gone, mathematics was a common label for algebra and geometry until the arrival of the 'New Math' that insisted that geometry must go and that algebra should be defined from above as examples of sets instead of from below as abstractions from examples. Looking at the set of sets not belonging to themselves, Russell showed that set-reference means self-reference as in the classical liar paradox 'this sentence is false' being true if false and vice versa. Still, the new set-based 'meta-matics' entered universities and schools as the only true mathematics; except for the US going 'back to basics', that by separating algebra and geometry crates learning problems that disappear if they are kept together as advocated by Descartes. Thus, in primary school, numbers should be two dimensional LEGO-blocks as 2 3s. And 3x6 should be a block of 3 6s that if recounted in tens must widen its width and shorten its height, so that 3 6s becomes 1.8 tens. And in secondary school bxc should mean b cs; and fractions should be operators needing a number to become a number thus by multiplication becoming areas that are added by integration. Likewise, Euclidean geometry should be introduced in a coordinate system allowing equations to predict the exact position of intersection points of lines in triangles before being constructed with ruler and compasses. And the quadratic equation x^2+bx+c=0 geometrically tells that since x^2+bx = -c, the four parts of a (x+b/2) square reduce to (b/2)^2-c = D, allowing x to be found easily as x = -b/2 ±√D.

07) Calculus in Middle School and High School

A class is stuck in differential calculus and gives up on d/dx(x^2) = 2x. Having heard about 'per-numbers', the teacher says 'Time out. Next week, no differentiation. Instead we go back to middle school and look at per-numbers.' Class: 'Per-numbers, what is that?' Teacher: 'Per-numbers are for example meter per second, dollar per kilo, or dollar per hour. Here is an example: What is the total of 2 kg at 3 $/kg + 4 kg at 5$/kg?'. Class: 'The kg-numbers add to 6, but how do we add per-numbers?' Teacher: 'Can we change $/kg-numbers to $-numbers?'. Class: 'We can multiply 2 and 3 to 6$, and 4 and 5 to 20$ that add up to 26$. But multiplication means adding areas?' Teacher: 'Precisely. Adding per-numbers by their areas is called integral calculus, also called finding the area under the per-number-graph.'

Class: 'But what if the per-number graph is not constant? Then there are too many strips to add!' Teacher: 'We use a trick. Adding 1000 numbers is difficult, but adding 1000 differences is easy since the middle numbers cancel out, so we are left with the difference between the end and the start number.' Class: 'But how can we write area-strips as differences?' Teacher: 'Well, if p is the per-number, then the area-strip with width dx is close to p*dx; but it is also the difference between the end area A2 and the start area A1, so p*dx = A2-A1 = dA, or p = d/dx(A).' Class: 'But that is differentiation?' Teacher: 'Precisely, so if we know that d/dx(x^2) = 2x, then we know that the area under the 2x graph is A2-A1 with A = x^2. So to find a quick way to area-formulas we need to learn to differentiate.' Class: 'OK.'

08) Mathematics, the Grammar of the Number-Language. But why teach Grammar before Language?

Humans have two languages, a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, 'This is a chair' and '3 chairs have a total of 3x4 legs', abbreviated to 'T = 3x4'. Both languages have a meta-language, a grammar, that describes the language that describes the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3x4' leads to a meta-sentence ''x' is an operation'. We master outside phenomena through actions, so learning a word-language means learning actions as how to listen, to read, to write and to speak. Likewise, learning the number-language means learning actions as how to count and to add. We cannot learn how to math, since math is not an action word, it is a label, as is grammar. Thus, mathematics can be seen as the grammar of the number-language. Since grammar speaks about language, language should be taught and learned before grammar. This is the case with the word-language, but not with the number-language. Saying 'the number-language is an application of mathematics' implies that then 'of course mathematics must be taught and learned before it can be applied'. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language. Instead school should follow the word-language and use full sentences 'The total is 3 4s' or 'T = 3x4'. By saying '3x4' only, school removes both the subject and the verb from number-language sentence, thus depriving it of its language nature.

09) Quantitative Literature also has three Genres: Fact and Fiction and Fiddle

Humans communicate in languages: A word language with sentences assigning words to things and actions. And a number language with equations assigning numbers or calculations to things and actions. 'Word stories' come in three genres: Fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like 'This sentence is false'. 'Number stories' are often called mathematical models. They come in the same three genres. Fact models can be called a 'since-then' models or 'room' models. Fact models quantify quantities and predict predictable quantities: "What is the area of the walls in this room?". Since the model's prediction is what is observed, fact models can be trusted. Algebra's four basic uniting models are fact models: T = a+b, T = axb, T = a^b and T = ∫y dx; as are many models from basic science and economy. Fiction models can be called 'if-then' models or 'rate' models. Fiction models quantify quantities but predict unpredictable quantities: "My debt is gone in 5 years at this rate!". Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based upon alternative assumptions. Models from statistics calculating averages assuming variables to be constant are fiction models; as are models from economic theory showing nice demand and supply curves. Fiddle models can be called 'then-what' models or 'risk' models. Fiddle models quantify qualities that cannot be quantified: "Is the risk of this road high enough to cost a bridge?" Fiddle models should be rejected asking for a word description instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence x Probability. It has meaning in insurance but not when quantifying casualties where it is cheaper to stay in a cemetery than at a hospital.

10) Distance Teacher Education in Mathematics by the CATS method: Count & Add in Time & Space

The MATHeCADEMY.net teaches teachers teach mathematics as 'many-math', a natural science about Many. It is a virus academy saying: To learn mathematics, don't ask the instructor, ask Many. To deal with Many, we Count and Add in Time and Space. The material is question-based.

Primary School. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives

14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

Secondary School. COUNT: How can we count possibilities? How can we predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? QUANTITATIVE LITERATURE, what is that? Does it also have the 3 different genres: fact, fiction and fiddle?

PYRAMIDeDUCATION organizes 8 teachers in 2 teams of 4 choosing 3 pairs and 2 instructors by turn.

The instructors instruct the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation.

The instructors correct the count&add assignments. In a pair, each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair.

11) 50 years of Sterile Mathematics Education Research, Why?

PISA scores are still low after 50 years of research. But how can mathematics education research be successful when its three words are not that well defined? Mathematics has meant different things in its 5000 years of history, spanning from a natural science about Many to a self-referring logic.

Within education, two different forms exist at the secondary and tertiary level. In Europe, education serves the nation's need for public servants through multi-year compulsory classes and lines. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks with one-subject teachers.

As to research, academic articles can be at a master level exemplifying existing theories, or at a research level questioning them. Also, conflicting theories create problems as within education where Piaget and Vygotsky contradict each other by saying 'teach as little and as much as possible'.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and if research is following traditions or searching for new discoveries. So to answer the question 'How to improve mathematics education research', first we must make the three words well defined by asking: What is meant by mathematics, and by education, and by research? Answers will be provided by the German philosopher Heidegger, asking 'what is 'is'?'

It turns out that, instead of mathematics, schools teaches 'meta-matism' combining 'meta-matics', defining concepts from above as examples of abstractions instead of from below as abstractions from examples; and 'mathe-matism' true inside but seldom outside class, such as adding fractions without units, where 1 red of 2 apples plus 2 red of 3 gives 3 red of 5 and not 7 red of 6 as in the textbook teaching 1/2 + 2/3 = 7/6.

So, instead of meta-matism, teach 'many-math' in self-chosen half-year blocks.

12) Difference-Research, a more Successful Research Paradigm?

Despite 50 years of research, many PISA studies show a continuing decline. Maybe, it is time for difference-research searching for hidden differences that make a difference:

1. The tradition teaches cardinality as one-dimensional line-numbers to be added without being counted first. A difference is to teach counting before adding to allow proportionality and integral calculus and solving equations in early childhood: bundle-counting in icon-bundles less than ten,

recounting in the same and in a different unit, recounting to and from tens, calculator prediction, and finally, forward and reversed on-top and next-to addition.

2. The tradition teaches the counting sequence as natural numbers. A difference is natural numbers with a unit and a decimal point or cup to separate inside bundles from outside singles; allowing a total to be written in three forms: normal, overload and underload: T = 5 = 2.1 2s = 2B1 2s = 1B3 2s = 3B-1 2s.

3. The tradition uses carrying. A difference is to use bundle-writing and recounting in the same unit to remove overloads: T = 7x 48 = 7x 4B8 = 28B56 = 33B6 = 336. Likewise with division: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

4. Traditionally, multiplication is learned by heart. A difference is to combine algebra and geometry by seeing 5x6 as a stack of 5 6s that recounted in tens increases its width and decreases its height to keep the total unchanged.

5. The tradition teaches proportionality abstractly. A difference is to introduce double-counting creating per-number 3$ per 4kg bridging the units by recounting the known number: T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Double-counting in the same unit transforms per-numbers to fractions and percentages as 3$ per 4$ = ¾; and 75kg per 100kg = 75/100 = 75%.

17. Difference-Research Powering PISA Performance: Count & Multiply before You Add

To explain 50 years of low performing mathematics education research, this paper asks: Can mathematics and education and research be different? Difference-research searching traditions for hidden differences provides an answer: Traditional mathematics, defining concepts from above as examples of abstractions, can be different by instead defining concepts from below as abstractions from examples. Also, traditional line-organized office-directed education can be different by uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks. And traditional research extending its volume of references can be different, either as grounded theory abstracting categories from observations, or as difference-research uncovering hidden differences to see if they make a difference. One such difference is: To improve PISA performance, Count and Multiply before you Add.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA performance decrease from 2003 to 2012, causing OECD to write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change’:

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015a, p. 3).

Other countries also experience low and declining PISA performance. And apparently research can do nothing about it. Which raises the question: Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? So, it is time to seek guidance by difference-research.

Difference-research Searching for Hidden Differences

Difference-research asks two questions: ‘Can this be different – and will the difference make a difference?’ If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

Natural sciences use difference-research to keep on searching until finding what cannot be different. Describing matter in space and time by weight, length and time intervals, they all seem to vary. However, including per-numbers will uncover physical constants as the speed of light, the gravitational constant, etc. The formulas of physics are supposed to predict nature’s behavior. They cannot be proved as can mathematical formulas, instead they are tested as to falsifiability: Does nature behave different from predicted by the formula? If not, the formula stays valid until falsified.

Social sciences can also use difference-research; and since mathematics education is a social institution, social theory might be able to explain 50 years of unsuccessful research in mathematics education.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman (1990, p. 16) agrees by saying that sociological thinking ‘renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.' Also, he talks about rationality as the base for social organizations:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called *goal displacement*. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

As an institution, mathematics education is a public organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'The goal of mathematics education is to teach mathematics'. However, by its self-reference such a goal is meaningless, indicating a goal displacement. So, if mathematics isn't the goal in mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{ A \mid A \notin A \}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

Thus, SET has transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '1 + 2 IS 3' meet counter-examples as e.g. 1 week + 2 days is 9 days.

So looking back, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about forms and numbers and operations? We all teach 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the sentence 'The total is 3 8s' we treat the predicate, 3 8s, as if it was the subject, which is a clear indication of a goal displacement.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers, operations and calculations. However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. And, being sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens, might speed up writing but might also slow down learning, together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, in Lincolns Gettysburg address, ‘Four scores and ten years ago’ shows that not all count in tens.

So, despite being presented as universal, many things can be different in mathematics, apparently having a tradition to present its choices as nature that cannot be different. And to uncover choice presented as nature is the aim of difference research.

A philosophical Background for Difference Research

Difference research began with the Greek controversy between two attitudes towards knowledge, called ‘sophy’ in Greek. To avoid hidden patronization, the sophists warned: Know the difference between nature and choice to uncover choice presented as nature. To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy. The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord using an unpredictable will to choose how the world behaves.

However, in the Renaissance difference research returns with Brahe, Kepler and Newton. Observations showed Brahe that planetary orbits are predictable in a way that did not falsify the church’s claim that the earth is the center of the universe. Kepler pointed to a different theory with the sun in the center. To falsify the Kepler theory a new planet had to be launched, which was impossible until Newton showed that planets and apples obey the same will, and a falling apple validates Kepler’s theory.

As experts in sailing, the Viking descendants in England had no problem stealing Spanish silver on its way home across the Atlantic Ocean. But to get to India to exchange it for pepper and silk, the Portuguese fortification of Africa’s cost forced them to take the open sea and navigate by the moon. But how does the moon move? The Church had one opinion, Newton had a different.

‘We believe, as is obvious for all, that the moon moves among the stars,’ said the Church; opposed by Newton saying: ‘No, I can prove that the moon falls to the earth as does the apple.’ ‘We believe that when moving, things follow the unpredictable metaphysical will of the Lord above whose will is done, on earth as it is in heaven,’ said the Church; opposed by Newton saying: ‘No, I can prove they follow their own physical will, a force that is predictable because it follows a mathematical formula.’ ‘We believe, as Aristotle told us, that a force upholds a state,’ said the Church; opposed by Newton saying: ‘No, I can prove that a force changes a state. Multiplied with the time applied, the force’s impulse changes the motion’s momentum; and multiplied with the distance applied, the force’s work changes the motion’s energy.’ ‘We believe, as the Arabs have shown us, that to deal with formulas we use algebra,’ said the Church; opposed by Newton saying: ‘No, we need a different algebra of change which I will call calculus.’

By discovering a physical predictable will Newton inspired a sophist revival in the Enlightenment Century: With moons and apples obeying their own physical will instead of that of a metaphysical patronizer, once enlightened about the difference between nature and choice, humans can do the same and do without a double patronization by the Lord at the manor house and the Lord above. Thus, two Enlightenment republics were installed, one in North America in1776 and one in France in 1789.

The US still has its first republic showing skepticism towards philosophical claims by developing American pragmatism, symbolic interactionism and grounded theory; and by allowing its citizens to uncover and develop talents through daily lesson in self-chosen half-year blocks in secondary and tertiary education.

France now has its fifth republic turned over repeatedly by their German neighbors seeing autocracy as superior to democracy and supporting Hegel’s anti-enlightenment thinking reinventing a metaphysical Spirit expressing itself through the history of different national people. To protect the republic, France established line-organized and office-directed elite schools, copied by the Prussia wanting to prevent democracy by Bildung schools meeting there criteria: The population must not be enlightened to prevent it asking for democracy as in France; instead a feeling of nationalism should be installed transforming the population into a people following the will of the Spirit by fighting other people especially the French; and finally the population elite should be extracted and receive Bildung to become a knowledge nobility for a new strong central administration to replace the inefficient blood nobility unable to stop democracy from spreading from France.

To warn against hidden patronization in institutions, France developed a post-structuralist thinking inspired by existentialist thinking (Tarp, 2016), especially as expressed in what Bauman (1992, p. ix). calls ‘the second Copernican revolution’ of Heidegger asking the question: What is ‘is’?

> Inquiry is a cognizant seeking for an entity both with regard to the fact that it is and with regard to its Being as it is. (Heidegger, 1962, p. 5)

Heidegger here describes two uses of ‘is’. One claims existence, ‘M is’, one claims ‘how M is’ to others, since what exists is perceived by humans wording it by naming it and by characterizing or analogizing it to create ‘M is N’-statements.

Thus, there are four different uses of the word ‘is’. ‘Is’ can claim a mere existence of M, ‘M is’; and ‘is’ can assign predicates to M, ‘M is N’, but this can be done in three different ways. ‘Is’ can point down as a ‘naming-is’ (‘M is for example N or P or Q or …’) defining M as a common name for its volume of more concrete examples. ‘Is’ can point up as a ‘judging-is’ (‘M is an example of N’) defining M as member of a more abstract category N. Finally, ‘is’ can point over as an ‘analogizing-is’ (‘M is like N’) portraying M by a metaphor carrying over known characteristics from another N.

Heidegger sees three of our seven basic is-statements as describing the core of Being: ‘I am’ and ‘it is’ and ‘they are’; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the ‘I’ (Dasein) must create an authentic relationship to the ‘It’. However, this is made difficult by the ‘dictatorship’ of the ‘They’, shutting the ‘It’ up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one’s own Dasein completely into the kind of Being of ‘the Others’, in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the “they” is unfolded. (..) Discourse, which belongs to the essential state of Dasein’s Being and has a share in constituting Dasein’s disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

Inspired by Heidegger, the French poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu points out that society forces words upon you to diagnose you so it can offer curing institutions including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Derrida, 1991. Lyotard, 1984. Bourdieu, 1970. Tarp, 2012).

From a Heidegger view a sentence contains two things: a subject that exists, and the rest that might be gossip. So, to discover its true nature hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its ‘predicate-prison’. We need to allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a different mathematics curriculum, e.g. one based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014), aiming at providing a background as pre-teachers or pre-engineers for young male migrants wanting to help rebuilding their original countries.

The philosophical and sociological background for difference research may be summed up by the Heidegger warning: In sentences, trust the subject but question the rest since it might be gossip. So, to restore its authenticity, we now return to the original subject in Greek mathematics, the physical fact Many, and use Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition

(Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many

As mammals, humans are equipped with two brains, one for routines and one for feelings. Standing up, we developed a third brain to keep the balance and to store sounds assigned to what we grasped with our forelegs, now freed to provide the holes in our head with our two basic needs, food for the body and information for the brain. The sounds developed into two languages, a word-language and a number-language. The ‘pencil-paradox’ observes that placed between a ruler and a dictionary, a pencil can itself point to its length but not to its name. This shows the difference between the two languages, the word-language is for opinions, the number-language is for prediction.

The word-language assigns words to things through sentences with a subject and a verb and an object or predicate, ‘This is a chair’. Observing the existence of many chairs, we ask ‘how many in total?’ and use the number-language to assign numbers to like things. Again, we use sentences with a subject and a verb and an object or predicate, ‘the total is 3 chairs’ or, if counting legs, ‘the total is 3 fours’, abbreviated to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

Thus, we can ask: What happens if looking at mathematics differently as a number-language? Again, difference-research might provide an answer.

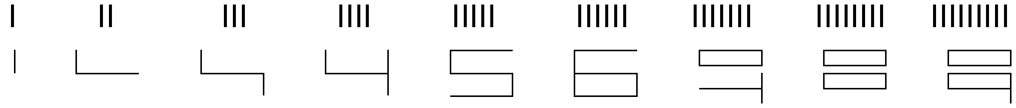
Examples of Difference-research

To prevent that mathematics becomes a meta-language that can be applied to describe and solve real-world problems, we must be careful with our language. Although it seems natural to talk about mathematics and its applications, this includes the logic that ‘of course mathematics must be learned before it can be applied’. Which is equivalent to saying ‘of course a grammar must be learned before it can be applied to describe a language’. This would lead to widespread illiteracy if applied to the word-language. And ‘grammar before language’ might be the cause of several problems in mathematics education. Of course, the subject must exist before the sentences can be made about it. So differences typically come from respecting that the number-language comes before its grammar and after meeting and experiencing the subject of its sentences, the total, describing the physical fact Many.

Digits as icons

A class of beginners, e.g. preschool or year 1 or migrants, is stuck in the traditional introduction of digits as symbols like letters. Some confuse the symbols, some have difficulties writing them, some can’t see why ten is written 10, some ask why eleven and twelve is not called ten-1 and ten-2.

Here a difference is to use a folding ruler to discover that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the four-icon, and five sticks in the five-icon, etc. Counting in 5s, the counting sequence is 1, 2, 3, 4, Bundle, 1-bundle-1, etc. This shows, that the bundle-number does not need an icon. Likewise, when bundling in tens. Instead of ten-1 and ten-2 we use the Viking numbers eleven and twelve meaning ‘1 left’ and ‘2 left’ in Danish, understood that the ten-bundle has already been counted.



1 2 3 4 5 6 7 8 9

Will this difference make a difference? In theory, yes, since rearranging physical entities into icons, e.g. five cars into in a five-icon, makes the icons physically before being formally written down. In his genetic epistemology, Piaget expresses a ‘greifen-vor begreifen’ principle, grasping physically before mentally. Thus, going from unordered cars to cars ordered into an icon to writing down the icon includes three of the four parts of his stage theory, the preoperational and the concrete operational and the formal operational stage. In practice, it works on a pilot study level thus being ready for a more formal study.

Counting sequences in different forms

A class of beginners have problems with the traditional introduction of the counting sequence and the place value system. Some count ‘twenty-nine, twenty-ten, twenty-eleven’. Some mix up 23 and 32.

Here a difference is to count a total of a dozen sticks in fives using different counting sequences: ‘1, 2, 3, 4, bundle, 1-bundle-1, …, 2 bundles, 2-bundles-1, 2-bundles-2’. Or ’01, 02, 03, 04, 10, 11, …, 22’. Or ‘.1, .2, .3, .4, 1., 1.1, …, 2.2’. Or ‘1, 2, bundle less 2, bundle less 1, bundle, bundle&1, bundle&2, 2bundle less 2, 2bundle less 1, 2bundles, 2bundles&1, 2bundles&2.’

Using a cup for the bundles, a total can be ‘bundle-counted’ in three ways: the normal way or with an overload or with an underload. Thus, a total of 5 can be counted in 2s as 2 bundles inside the bundle-cup and 1 unbundled single outside, or as 1 inside and 3 outside, or as 3 inside and ‘less 1’ outside; or, if using ‘bundle-writing’ to report bundle-counting, T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s. Likewise, when counting in tens, T = 37 = 3B7 tens = 2B17 tens = 4B-3 tens. Using a decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles, shows that a natural number is a decimal number with a unit: T = 3B1 2s = 3.1 2s; and t = 3B1 tens = 3.1 tens = 31 if leaving out the unit and misplacing the decimal point

Will this difference make a difference? In theory, yes, since counting by taking away bundles and placing one stick in a cup per bundle again combines the three operational parts of Piaget’s stage-theory allowing the learner to see, that a number has three parts: a unit, and some bundles inside the cup, and some unbundled outside. In practice, it works on a pilot study level thus being ready for a more formal study.

Multiplication tables made simpler

A class is stuck in multiplication tables. Some add instead of multiplying, some tries to find the answer by repeated addition, some just give random answers, and some have given up entirely to learn the tables by heart.

Here a difference is to see multiplication as a geometrical stack or block that recounted in tens increases its width and therefore decreases its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21.

Another difference is to begin by reducing the full ten-by-ten table to a small 2-by-2 table containing doubling and tripling, using that 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc.

Thus, beginning with doubling visualized by LEGO bricks, T = 2 6s = 2*6 = 2*(5&1) = 10&2 = 12, or T = 2*6 = 2*(10-4) = 20-8 = 12. And T = 2 7s = 2*7 = 2*(5&2) = 10&4 = 14, or T = 2*7 = 2*(10-3) = 20 – 6 = 14. Doubling then can be followed by halving that by counting in 2s will introduce a recount-formula T = (T/B)*B saying that T/B times B may be taken away from T: So when halving 8, 8 = (8/2)*2 = 4 2s, and 9 = (9/2)*2 = (<u>8&1</u>/2)*2 = (4&1/2)*2 = 4 & ½ 2s.

As to tripling, T = 3*7 = 3*(10-3) = 30 – 9 = 21.

Proceeding with factors after 2 and 3, 2-by-2 Medieval multiplication squares can be used to see that e.g. T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Division using bundle-writing and recounting
A class is stuck in short and long division. Some subtract instead of dividing, some invent their own algorithms typically time-consuming and often without giving the correct answers, some give up because they never learned the multiplication tables.

Here a difference is to talk about 8/2 as '8 counted in 2s' instead of as '8 divided between 2'; and to rewrite the number as '10 or 5 times less something' and use the results from a multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

As to long division, here a difference is to combine renaming numbers using bundle names, e.g. sixty-five as 6ten5, with bundle-writing allowing recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

Once bundle-writing is introduced, we discover that also bundles can be bundled, calling for an extra cup for the bundles of bundles: T = 7 = 3B1 2s = 1BB1B1 2s. Or, with tens: T = 234 = 23B4 = 2BB3B4.

Thus, by recounting in the same unit by creating or removing overloads or underloads, bundle-writing offers an alternative way to perform and write down all operations.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

Will this difference make a difference? In theory, yes, if the learner knows that a total can be recounted in the same unit to create an overload or an underload. In practice, it works on a pilot study level thus being ready for a more formal study.

Proportionality as double-counting creating per-numbers
A class stuck in proportionality. Nearly all find the \$-number for 12kg at a price of 2\$/3kg but some cannot find the kg-number for 16\$. Here a difference is to see the price as a per-number, 2\$ per 3kg, bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16\$ recounts in 2s as T = 16\$ = (16/2)*2\$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2\$ = 8\$.

Will this difference make a difference? In theory, yes, since proportionality is translated to a basic physical activity of counting and recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Fractions and percentages as per-numbers
A class is stuck in fractions. Rewriting fractions by shortening or enlarging, some subtract and add instead of dividing and multiplying; and some add fractions by adding numerators and denominators.

Here a difference is to see a fraction as a per-number coming from double-counting in the same unit, 3/5 = 3\$ per 5\$, or as percentage 3% = 3/100 = 3\$ per 100%. Thus 2/3 of 12 is seen as 2\$ per 3\$ of 12\$ that recounts in 3s as 12\$ = (12/3)*3\$ giving (12/3)*2\$ = 8\$ of the 12\$. So 2/3 of 12 is 8. Other examples are found in economy investing money and expecting a return that might be higher or lower than the investment, e.g. 7\$ per 5\$ or 3\$ per 5\$.

The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6.

To find what 3 per 5 is per hundred, 3/5 = ?%, we just recount 100$ in 5s and replace 5$ with 3$: T = 100$ = (100/5)*5$ giving (100/5)*3$ = 60$. So 3 per 5 is the same as 60 per 100, or 3/5 = 60%.

As per-numbers, also fractions are operators needing a number to give a number: a half is always a half of something as shown by the recount-formula T = (T/B)*B = T/B Bs. So also fractions must have units to be added.

If the units are different, adding fractions means finding the average fraction. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as the tradition teaches.

Taking fractions of the same quantity makes the unit the same, assumed to be already bracketed out, so that T = a/b + c/d really means T = (a/b + c/d) of (b*d). Thus adding 2/3 and 4/5 it is implied that the fractions are taken of the same total 3*5 = 15 that is bracketed out, so the real question is ‘T = 2/3 of 15 + 4/5 of 15 = ? of 15, giving T = 10 + 12 = 22 = (22/15)*15 when recounted in 15s.

Thus, adding fractions is ambiguous. If taken of the same total, 2/3 + 4/5 is 22/15; if not, the answer depends on the totals: 2/3 of 3 + 4/5 of 5 is (2+4)/(3+5) of 8 or 6/8 of 8, and 2/3 of 3 + 4/5 of 10 is 10/13 of 13, thus providing three different answers, 22/15 and 6/8 and 10/13, to the question ‘2/3+4/5 = ?’

Hiding the ambiguity of adding fractions makes mathematics ‘mathe-matism’ true inside but seldom outside classrooms.

As to algebraic fractions, a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

As when adding fractions, adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly, but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other. So adding fractions as the area under a piecewise constant per-number graph becomes ‘middle school integration’ later to be generalized to high school integration finding the area under a locally constant per-number graph. Thus calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level. n practice, it works on a pilot study level thus being ready for a more formal study.

Will this difference make a difference? In theory, yes, if first performing double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand.

Equations as walking or recounting

A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. when equations are composite or with a reverse sign in front of the unknown.

Here a difference is to use the definitions of reverse operations to establish the basic ‘OSS’-rule for solving equations, ‘move to the Opposite Side with the opposite Sign’. Thus, in the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise, with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12).

As to 2+3*u = 14, a difference is to see it as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn’t have a reverse sign. This gives 3*u = 14 – 2. Since u doesn’t have a reverse sign, 3 moves to the opposite side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4 since 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14.

Another difference is to see 2+3*u =14 as a walk, first multiplying u by 3 then adding 2 to give 14. To get back to u we reverse the walk by performing the reverse operations in reverse order. Thus, first subtracting 2 and then dividing by 3 gives u = (14-2)/3 = 4, checked by repeating the walk now with a known staring number: 4*3+2 = 14. Seeing an equation as a walk motivates using the terms 'forward and backward calculation sides' for 2+3*u and 14 respectively.

With 25 – u = 14, u moves to the opposite side to have its reverse sign reversed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5 giving 40 = 5*u; 40/5 = u; 8 = u. Alternatively, recounting twice gives 40 = (40/u)*u = 5*u, and 40 = (40/5)*5, consequently u = 40/5.

Pure letter-formulas build routine as e.g. 'transform the formula T = a/(b-c) so that all letters become subjects.' When building a routine, students often have fun singing:

"Equations are the best we know / they're solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don't want to stop."

Another difference is to introduce equations the foist year in primary school as another name for recounting from tens to icons, e.g. asking 'How many 9s are 45' or 'u*9 = 45' giving u = 45/9 since recounting 45 in 9s, the recount formula gives 45 = (45/9)*9, again showing the OppositeSide&Sign rule.

Likewise, the equation 8 = u + 2 describes restacking 8 by removing 2 to be placed next-to, predicted by the restack-formula as 8 = (8–2)+2. So, the equation 8 = u + 2 has the solution is 8–2 = u, again obtained by moving a number to the opposite side with the opposite calculation sign.

Will this difference make a difference? In theory, yes, since equations are related to something concrete, walking or recounting. In practice, it works on a pilot study level thus being ready for a more formal study.

Geometry and algebra, always together, never apart

A class is stuck in geometry. Some mix up definitions, some find the theorems to abstract to understand, some find proofs difficult and hard to remember, some find geometry boring.

Here a difference is to use a coordinate system to coordinate geometry and algebra so they go hand in hand always and never apart, thus using algebra to predict geometrical intersection points, and vice versa, to use intersection points to solve algebraic equations. Both in accordance with the Greek meaning of mathematics as a common label for algebra and geometry.

In a coordinate-system a point is reached by a number of horizontally and vertically steps called the point's x- and y-coordinates. Two points A(xo,yo) and B(x,y) with different x- and y-numbers will form a right-angled change-triangle with a horizontal side Δx = x-xo and a vertical side Δy = y-yo and a diagonal distance r from A to B, where by Pythagoras r^2 = Δx^2 + Δy^2. The angle A is found by the formula tanA = Δy/Δx = s, called the slope or gradient for the line from A to B. This gives a formula for a non-vertical line: Δy/Δx = s or Δy = s*Δx, or y-yo = s*(x-xo). Vertical lines have the formula x = xo since all points share the same x-number.

In a coordinate system three points A(x1,y1) and B(x2,y2)and C(x3,y3) not on a line will form a triangle that packs into a rectangle by outside right triangles allowing indirectly to find the angles and the sides and the area of the original triangle.

Different lines exist inside a triangle: Three altitudes measure the height of the triangle depending on which side is chosen as the base; three medians connect an angle with the middle of the opposite side; three angle bisectors bisect the angles; three line bisectors bisect the sides and are turned 90 degrees from the side. Likewise, a triangle has two circles; an outside circle with its center at the intersection point of the line bisectors, and an inside circle with its center at the intersection point of the angle bisectors.

Since Δx and Δy changes place when turning a line 90 degrees, their slopes will be Δy/Δx and -Δx/Δy respectively, so that s1*s2 = -1, called reciprocal with opposite sign.

As mentioned, geometrical intersection points are predicted algebraically by equating formulas. Thus with the lines y = 2*x and y = 6-x, equating formulas gives 2*x = 6-x, or 3*x = 6, or x = 2, which inserted in the first gives y = 2*2 = 4, thus predicting the intersection point to be (x,y) = (2,4). The same answer is found on a solver-app; or using software as GeoGebra.

Finding possible intersection points between a circle and a line or between two circles leads to a quadratic equation x^2 + b*x + c = 0, solved by a solver. Or by a formula created by two x-by-(x+k) playing cards placed on top of each other with the bottom left corner at the same place and the top card turned a quarter round clockwise. This creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution formula is x = -b/2 ±√((b/2)^2 – c).

Finding a tangent to a circle at a point, its slope is the reciprocal with opposite sign of the radius line.

Will this difference make a difference? In theory, yes, since coordinating geometry and algebra gives equations a geometrical form and allows geometrical situations to be predicted by equations. In practice, it works on a pilot study level thus being ready for a more formal study.

Trigonometry as right triangles with sides mutually recounted

A class is stuck in trigonometry. Some find the ratios to abstract to understand, some mix up the formulas, some find the algebra difficult to use.

A difference is to introduce trigonometry as blocks halved in two by its diagonal, making a rectangle split into two right triangles. Here the angles are labeled A and B and C at the right angle. The opposite sides are labeled a and b and c.

The height a and the base b can be counted in meters, or in diagonals c creating a sine-formula and a cosine-formula: a = (a/c)*c = sinA*c, and b = (b/c)*c = cosA*c. Likewise, the height can be recounted in the base, creating a tangent-formula: a = (a/b)*b = tanA*b

As to the angles, with a full turn as 360 degrees, the angle between the horizontal and vertical directions is 90 degrees. Consequently, the angles between the diagonal and the vertical and horizontal direction add up to 90 degrees; and the three angles add up to 180 degrees.

An angle A can be counted by a protractor, or found by a formula. Thus, in a right triangle with base 4 and diagonal 5, the angle A is found from the formula cosA = a/c = 4/5 as cos-1(4/5) = 36.9 degrees.

The three sides have outside squares with areas a^2 and b^2 and c^2. Turning a right triangle so that the diagonal is horizontal, a vertical line from the angle C splits the square c^2 into two rectangles. The rectangle under the angle A has the area (b*cosA)*c = b*(cosA*c) = b*b = b^2. Likewise, the rectangle under the angle B has the area (a*cosB)*c = a*(cosB*c) = a*a = a^2. Consequently c^2 = a^2 + b^2, called the Pythagoras formula.

This allows finding a square-root geometrically, e.g. x = √24, solving the quadratic equations x^2 = 24 = 4*6, if transformed into a rectangle. On a protractor, the diameter 9.5 cm becomes the base AB, so we have 6units per 9.5cm. Recounting 4 in 6s, we get 4units = (4/6)*6units = (4/6)*9.5 cm = 6.33 cm. A vertical line from this point D intersects the protractor’s half-circle in the point C. Now, with a 4x6 rectangle under BD, BC will be the square-root √24, measured to 4.9, which checks: 4.9^2 = 24.0.

A triangle that is not right-angled transforms into a rectangle by outside right triangles, thus allowing its sides and angles and area to be found indirectly. So, as in right triangles, any triangle has the property that the angles add up to 180 degrees and that the area is half of the height times the base.

Inside a circle with radius 1, the two diagonals of a 4sided square together with the horizontal and vertical diameters through the center form angles of 180/4 degrees. Thus the circumference of the square is 2*(4*sin(180/4)), or 2*(8*sin(180/8)) with 8 sides instead. Consequently, the circumference of a circle with radius 1 is 2*π, where π = n*sin(180/n) for n large.

Will this difference make a difference? In theory, yes, since in Greek, geometry means to measure earth, typically by dividing it into triangles, again divided into right triangles, which can be seen as rectangles halved by their diagonals; and recounting totals in new units leads directly to mutual recounting the sides in a right triangle, which leads on to a formula for calculating pi. Furthermore, the many applications of trigonometry might increase the motivation for learning more geometry where coordinate geometry uses right triangles to increase any triangle to a rectangle with horizontal and vertical sides. In practice, it works on a pilot study level thus being ready for a more formal study.

PreCalculus as constant change

A class is stuck in precalculus. Some find the function concept to abstract to understand, some sees f(2) as a variable f multiplied by 2, some cannot make sense of roots and logarithm. The tradition defines a function top-down from above as a set-relation where first-component identity implies second component identity.

A difference is to return to the original Euler-meaning of a function defining it bottom-up from below as a name for a formula containing specified and unspecified numbers. And to see a formula as the core concept of mathematics respecting that, whatever it means, in the end mathematics is but a means to an outside goal, a number-language.

As a number-language sentence, a formula contains both specified and unspecified numbers in the form of letters, e.g. T = 5+3*x. A formula containing one unspecified number is called an equation, e.g. 26 = 5+3*x, to be solved by moving to opposite side with opposite calculation sign, (26-5)/3 = x. A formula containing two unspecified numbers is called a function, e.g. T = 5+3*x. An unspecified function containing an unspecified number x is labelled f(x), T = f(x). Thus f(2) is meaningless since 2 is not an unspecified number. Functions are described by a table or a graph in a coordinate system with y = T = f(x), both showing the y-numbers for different x-numbers. Thus, a change in x, Δx, will imply a change in y, Δy, creating a per-number Δy/Δx called the gradient of the formula.

As to change, a total can change in a predictable or unpredictable way; and predictable change can be constant or non-constant.

Constant change comes in several forms. In linear change, T = b + s*x, s is the constant change in y per change in x, called the slope or the gradient of its graph, a straight line. In exponential change, T = b*(1+r)^x, r is the constant change-percent in y per change in x, called the change rate. In power change, T = b*x^p, p is the constant change-percent in y per change-percent in x, called the elasticity. A saving increases from two sources, a constant $-amount per month, c, and a constant interest rate per month, r. After n months, the saving has reached the level C predicted by the formula C/c = R/r. Here the total interest rate after n months, R, comes from the formula 1+R = (1+r)^n. Splitting the rate r = 100% in t parts, we get the Euler number e = (1+100%/t)^t = (1+1/t)^t if t is large.

Also the change can be constant changing. Thus in T = c + s*x, s might also change constantly as s = c + q*x so that T = b + (c + q*x)*x = b + c*x + q*x^2, called quadratic change, showing graphically as a bending line, a parabola.

The difference seeing functions as predicting number-language sentences also suggests that functions in the form of formulas should be introduced from the first class of mathematics to predict counting results by a calculator, allowing the basic operations to be introduced as icons showing the three tasks involved when counting by bundling and stacking. Thus, to count 7 in 3s we take away 3 many times iconized by an uphill stoke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times. To stack the 2 3s we use multiplication, iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1. To also bundle bundles, power is iconized as a cap, e.g. 5^2, indicating the number of times bundles themselves have been bundled. Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other. To add on-top, the blocks must be recounted in the same unit, thus grounding proportionality. Next-to addition means adding areas, thus grounding integration. Reversed adding on-top or next-to grounds equations and differentiation. Also, the four basic operations uncover the original meaning of the word algebra, meaning ‘to reunite’ in Arabic: Addition unites unlike numbers, multiplication unites like numbers into blocks, power unites like factors, and integration unite unlike blocks.

Thus, by bundling and dragging away the stack, the calculator predicts that 7 = 2B1 3s = 2.1 3s, using a cup or a decimal point to separate the ‘inside’ bundles from the ‘outside’ unbundled. This prediction holds at a manual counting:

T = 7 = I I I I I I I = III III I = 2 3s & 1.

Thus a calculator can predict a counting result by describing the three parts of a counting process, bundling and stacking and dragging away the stack, with unspecified numbers, i.e. with two formulas. The ‘recount formula’ T = (T/B)*B says that ‘from T, T/B times B can be taken away’ as e.g. 8 = (8/2)*2 = 4*2 = 4 2s; and the ‘restack formula’ T = (T–B)+B says that ‘from T, T–B is left when B is taken away and placed next-to’, as e.g. 8 = (8–2)+2 = 6+2. Here we discover the nature of formulas: formulas predict. Wanting to recount a total in a new unit, the two formulas can predict the result when bundling and stacking and dragging away the stack. Thus, asking T = 4 5s = ? 6s, the calculator predicts: First (4*5)/6 = 3.some; then (4*5) – (3*6) = 2; and finally T = 4 5s = 3.2 6s. Recounting a total in a new unit means changing unit, also called proportionality or linearity, a core concept in mathematics at school and at university level. Thus the recount formula turns up in proportionality as $ = ($/kg)*kg when shifting physical units, in trigonometry as a = (a/c)*c = sinA*c when counting sides in diagonals in right triangles, and in calculus as dy = (dy/dx)*dx = y’*dx when counting steepness on a curve by recounting a vertical change in a horizontal.

Will this difference make a difference? In theory, yes, since describing mathematics as the grammar of the number-language is a powerful metaphor uncovering the real outside goal of mathematics education, to develop a number-language having the same sentence structure as the word-language, which will demystify the nature of mathematics to many students. In practice, it works on a pilot study level thus being ready for a more formal study.

Calculus as adding locally constant per-numbers

A class is stuck in calculus. Some find the limit concept too abstract. Some find the applications too artificial. For some, their hate to differential calculus prevents them from learning integral calculus.

Here a difference is to postpone differential calculus till after integral calculus is presented as a means to add piecewise or locally constant per-numbers by their areas. Thus, when adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 add by their areas as 3*2 + 5*4, meaning that per-numbers add by the area under the per-number-graph. With a piecewise constant per-number this mean a small number of area strips to add. But seeing a non-constant per-number as locally constant it means adding a huge amount of area strips, only possible if we can rewrite the strips as differences since the disappearance of the middle terms makes many differences add up to one single difference between the terminal and initial number. This of course makes rewriting a formula as a difference highly interesting, thus motivating a study of differential calculus. Thus, with the area strip 2*x*dx written as d(x^2), summing up the strips gives a single difference:

T2-T1 = Δ(x^2) = Σ ΔT = ∫ dT = ∫f(x)*dx = ∫2*x*dx .

Change formula come from observing that in a block, changes Δb and Δh in the base b and the height h impose on the total a change ΔT as the sum of a vertical strip Δb*h and a horizontal strip b*Δh and a corner Δb*Δh that can be neglected for small changes; thus d(b*h) = db*h + b*dh, or counted in T’s: dT/T = db/b + dh/h, or with T’ = dT/dx, T’/T = b’/b + h’/h. Therefore (x^2)’/x^2 = x’/x + x’/x = 2/x, giving (x^2)’ = 2*x since x’ = dx/dx = 1.

As to the limit concept, a difference is to rename it to ‘local constancy’: In a function y = f(x) a small change x often implies a small change in y, thus both remaining ‘almost constant’ or ‘locally constant’, a concept formalized with an ‘epsilon-delta criterium’, distinguishing between three forms of constancy. y is ‘globally constant’ c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is ‘piecewise constant’ c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is ‘locally constant’ c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in the latter case written as dy/dx. Formally, local constancy and linearity is called continuity and differentiability.

Finally, calculus allows presenting the core of the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Will this difference make a difference? In theory, yes, since presenting it as adding piecewise or locally constant per-numbers will ground integral calculus in meaningful real-world problems. Likewise, observing the enormous advantage in adding differences gives a genuine motivation for differential calculus that is lost if insisting that it comes before integral calculus. In practice, it works on a pilot study level thus being ready for a more formal study.

How Different is the Difference?

Difference research uses sociological imagination to revive the ancient sophist warning: Know nature from choice to discover choice presented as nature. Thus, true and false nature are separated by asking the tradition: Can this be different, and will the difference make a difference? Witnessed by 50 years of sterility, mathematics education research is a natural place to see if difference-research, DR, will make a difference.

The tradition says, ‘To obtain its goal, to learn mathematics, mathematics education must teach mathematics!’ DR objects, ‘No, to obtain its goal, mastery of Many, mathematics is a means to be replaced by another means if not leading to the goal, e.g. by ‘Many-matics’, defining its concepts from below as abstractions from examples instead of from above as examples of abstractions as does the traditional ‘meta-matics’.

The tradition says, ‘The core of mathematics is to operate on numbers!’ DR objects, ‘No, the core of mathematics is number-language sentences describing how totals are counted and recounted before being added; and having the same sentence structure as the word-language: a subject, a verb and a predicate.’

The tradition says, ‘Digits must be taught as symbols like letters!’ DR objects, ‘No, digits are icons containing as many strokes as they represent.’

The tradition says, ‘To describe cardinality, numbers must be taught as a one-dimensional number-line!’ DR objects, ‘No, numbers are two-dimensional blocks counting a total in stacks of bundles and unbundled singles.’

The tradition says, ‘Natural numbers must be taught as a place value system and ten-bundling is silently understood!’ DR objects, ‘No, numbers should be taught using bundle-writing to separate inside bundles from outside singles, making a natural number a decimal number with a unit. And ten-counting should be postponed until icon-counting and re-counting in the same and in a different unit has been experienced’.

The tradition says, ‘There are four kinds of numbers, natural and integer and rational and real numbers!’ DR objects, ‘No, a number is a positive or negative decimal number with a unit. Rational numbers are per-numbers, i.e. operators needing a number to become a number; and real numbers are calculations to deliver as many decimals as wanted.’

The tradition says, ‘Operations must be taught as functions from a set-product to the set supplying it with a structure obeying associative, commutative and distributive laws as well as neutral and inverse elements allowing equations to be solved by neutralization!’ DR objects, ‘Operations are icons showing the three processes of counting, bundling and stacking and removing stacks to look for unbundled singles; and adding stacks or blocks on-top or next-to. Solving equations is another word for reversing the processes by re-bundling or re-stacking’

The tradition says, ‘The natural order of teaching operations is addition before subtraction before multiplication before division allowing fractions to be introduced as rational numbers to which the same operations can be applied!’ DR objects, ‘No, since totals must be counted before they can be added, the natural order is the opposite: first division to take away bundles many times, then multiplication to stack the bundles, then subtraction to take away the stack once to look for unbundled singles, and finally addition in its two versions, on-top and next-to. And counting also implies recounting in the same or another unit, to and from tens, and double-counting producing per-numbers as operators needing numbers to become numbers, thus being added by their areas, i.e. by integration.’

The tradition says, ‘Calculators should not be allowed before all four operations are taught and learned!’ DR objects, ‘Calculators should be used from the start to predict counting and recounting results.’

The tradition says, ‘Operations must be taught using carrying!’ DR objects, ‘No, operations should be taught using bundle-writing allowing totals to be recounted with overloads or underloads.’

The tradition says, ‘Multiplication tables must be learned by heart!’ DR objects, ‘No, multiplication tables describe recounting from icon-bundles to ten-bundles; geometrically seen as changing a block by increasing the width and decreasing the height to keep the total unchanged; and algebraically sees as doubling or tripling totals written with an overload or an underload.’

The tradition says, ‘Division is difficult and must be taught using constructivism to allow learners invent their own algorithms!’ DR objects, ‘No, division should be taught as recounting from ten-bundles to icon-bundles using bundle-writing and recounting in the same unit to benefit from the multiplication tables.’

The tradition says, ‘Arithmetic comes before geometry, and they must be held apart until the introduction of the coordinate system!’ DR objects, ‘No, arithmetic should be seen as algebra kept together with geometry all the time and from the beginning, where numbers are a collection of blocks as well as a collection of numbers in cups; where recounting and multiplication means changing block-sizes as well as changing bundle-numbers; and where addition means adding blocks as well as bundle-numbers.’

The tradition says, ‘Proportionality must be postponed until functions have been introduced!’ DR objects, ‘No, as another name for changing units, proportionality occurs from the beginning as recounting in another unit; and is needed when adding on-top and next-to. And reoccurring when double-counting creates per-numbers as bridges between physical units.’

The tradition says, ‘Fractions must be introduced first as parts of something then as numbers by themselves!’ DR objects, ‘No, created by double-counting in the same unit, fractions are per-numbers and as such operators needing a number to become a number.’

The tradition says, ‘Prime-factorizing must precede adding fractions by finding a common denominator!’ DR objects, ‘No, prime-factorizing comes with recounting to another unit to find the units allowing a total to be recounted fully without any unbundled singles. And fractions should be added as operators, i.e. by integrating their areas.’

The tradition says, ‘Equations must be taught as statements about equivalent number-names, solved by the neutralizing method obeying associative, commutative and distributive laws!’ DR objects, ‘No, equations occur when recounting totals from tens to icons, and when reversing on-top and next-to addition.’

The tradition says, 'A function must be taught as an example of a set-relation where first-component identity implies second-component identity!' DR objects, 'No, a function should be taught as a formula with two unspecified numbers thus respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. Later polynomials can be introduced as the number-formula containing the different formulas for constant change: T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x.'

The tradition says, 'Linear functions must be taught before quadratic functions!' DR objects, 'No, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d.'

The tradition says, 'Quadratic equations must be solved by factorizing before introducing the solution formula!' DR objects, 'No, when solving the quadratic equation x^2+b*x+c = 0, algebra and geometry should go hand in hand to show that inside a square with the sides x+b/2, the equation makes three rectangles disappear leaving only (b/2)^2-c, allowing possible roots to be found and used in factorization if necessary.'

The tradition says, 'Differential calculus must be taught before integral calculus since the integral is defined as the anti-derivate.' DR objects, 'No, integral calculus comes before differential calculus. In primary school, next-to addition means multiplying before adding when asking e.g. T = 2 3s + 4 5s = ? 8s', while reversing the question by asking 2 3s + ? 5s = 6 8s, or T1 + ? 5s = T, leads to differential calculus subtracting before dividing to get the answer (T-T1)/5. In middle school, fractions and per-numbers add by their areas, i.e. by integration. And in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant predictable change.'

The tradition says, 'The epsilon-delta definition is essential in order to understand real numbers and calculus and must be learned by heart!' DR objects, 'No, it needs not be learned by heart. With units, it can be grounded in formalizing three ways of constancy; globally constant needing only the epsilon, piecewise constant with delta before epsilon, and locally constant with epsilon before delta.'

The tradition says, 'Statistics and probability must be taught separately!' DR objects, 'No, they should be taught together aiming at pre-dicting unpredictable numbers by intervals coming from 'post-dicting' their previous behavior.'

In continental Europe, the tradition says, 'Education means preparing for offices in the public or private sector. Hence the necessity of line-organized education with forced year-group classes in spite of the fact that teenage girls are two years ahead of the boys in personal development. Of course, boys and dropouts are to pity, but they all had the chance.' North American republics object: 'No, Education means uncovering and developing the learner's individual talents through daily lessons in self-chosen practical or theoretical half-year blocks together with a person teaching only one subject and praising the learner for having a talent or for having courage to test it.'

In mathematics education, the tradition says, 'Education means connecting learners to the canonical correctness through scaffolding from the learner's zone of proximal development as described in social constructivism by Bruner and Vygotsky.' DR objects, 'No, education means bringing outside phenomena inside a classroom to be assimilated or accommodated by the learners thus respecting that in a sentence, the subject is objective but the rest might be subjective as described in radical constructivism by Piaget and Grounded Theory and Heidegger existentialism.

In mathematics education, the tradition says, 'Research means applying or extending existing theory.' DR objects, 'No, where master level work means applying existing theory, research level means questioning existing theory, e.g. by asking if it could be different.'

How to Improve PISA Performance

PISA performance (Tarp, 2015a) can be improved in three ways: by a different macro-curriculum from class one, by remedial micro-curricula when a class is stuck, and by a STEM-based core-curriculum for outsiders.

Improving PISA performance means improving mathematics learning which can be done by observing three basic facts about our human and mammal and reptile brains.

The human brain needs meaning, so what is taught must be a meaningful means to a meaningful outside goal, mastery of Many; thus mathematics must be taught as 'Many-matics' in the original Greek sense as a common name for algebra and geometry both grounded in an motivated by describing Many in time and space; and not as 'meta-matism' mixing 'meta-matics', defining concepts from above as examples of internal abstractions instead of from below as abstractions from external examples, with 'mathe-matism', true inside but seldom outside classrooms as adding numbers without units.

The mammal brain houses feelings, positive and negative. Here learning is helped by experiencing a feeling of success from the beginning, or of suddenly mastering or understanding something difficult.

The reptile brain houses routines. Here learning is facilitated by repetition and by concreteness: With mathematics as a text, its sentences should be about subjects having concrete existence in the world, and having the ability to be handled manually according to Piagetian principle 'through the hand to the head'.

Also, we can observe that allowing alternative means than the tradition makes it not that difficult to reach the outside goal, mastery of many. Meeting Many, we ask 'How many in total?' To get an answer we count and add. We count by bundling and stacking and removing the stack to look for unbundles leftovers. This gives the total the geometrical form of a collection of blocks described by digits also having a geometrical nature by containing as many sticks as they represent. Counting also includes recounting in the same or in a new unit; or double-counting to produce per-numbers. Once counted, totals can be united or split, and with four kinds of numbers, constant and variable unit-numbers and per-numbers, there are four ways to unite: addition, multiplication, power and integration; and four ways to split: subtraction, division, root/logarithm and differentiation.

Thus, the best way to obtain good PISA performance is to replace the traditional SET-based curriculum with a different Many-based curriculum from day one in school, and to strictly observe the warning: Do not add before totals are counted and recounted – so multiplication must precede addition. However, this might be a long-term project. To obtain short-term improvements, difficult parts of a curriculum where learners often are stuck might be identified and replaced by an alternative remedial micro-curriculum designed by curriculum architecture using difference-research and sociological imagination. Examples can be found in the above chapter 'Examples of difference-research'.

Finally, in the case of teaching outsiders as migrants or adults or dropouts with no or unsuccessful educational background, it is possible to design a STEM-based core curriculum as described above allowing the outsiders become pre-teachers and pre-engineers in two years. Thus, applying sociological imagination when meeting Many without predicates forced upon it, allows avoiding repeating the mistakes of traditional mathematics.

The Tradition's 3x3 mistakes

Choosing learning mathematics as the goal of teaching mathematics has serious consequences. Together with being set-based this makes both mathematics education and mathematics itself meaningless by self-reference. Here a difference is to accept that the goal of teaching mathematics is mastering Many by developing a number-language parallel to the word-language; both having a meta-language, a grammar, that should be taught after the language to respect that the language roots the grammar instead of being an application of it; and both having the same sentence structure with a subject and a verb and a predicate, thus saying 'T = 2*3' instead of just '2*3'.

This goal displacement seeing mathematics as the goal of mathematics education leads to 3x3 specific mistakes in primary, middle and high school:

In primary school, numbers are presented as 1dimensional line numbers written according to a place value convention; instead of accepting that our Arabic numbers like the numbers children bring to school are 2dimensional block numbers. Together with bundle-counting and bundle-writing this gives an understanding that a number really is a collection of numbers counting what exists in the world, first inside bundles and outside unbundled singles, later a collection of unbundled and bundles and bundles of bundles etc.

Furthermore, school skips the counting process and goes directly to adding numbers without considering units; instead of exploiting the golden learning opportunities in counting and recounting in the same or in another unit, and to and from tens. This would allow multiplication to be taught and learned before addition by accepting that 4*7 is 4 7s that maybe recounted in tens as T = 4 7s = 2.8 tens = 28, to be checked by recounting 28 back to 7s, T = 28 = (28/7)*7 = 4*7 = 4 7s, using the recount-formula reappearing in proportionality, trigonometry and calculus. And giving division by 7 the physical meaning of counting in 7s.

Finally, addition only includes on-top addition of numbers counted in tens only and using carrying, a method that neglects the physical fact that adding or subtracting totals might crate overloads or underloads to be removed by recounting in the same unit. And neglecting the golden learning opportunities that on-top addition of numbers with different unit roots proportionality, and that next-to addition roots integration, that reversed roots differentiation thus allowing calculus to be introduced in primary school.

In middle school, fractions are introduced as numbers that can be added without units thus presenting mathematics as 'mathematism' true inside but seldom outside classrooms. Double-counting leading to per-numbers is silenced thus missing the golden learning opportunities that per-numbers give a physical understanding of proportionality and fractions, and that both per-numbers and fractions as operators need numbers to become numbers that as products add as areas, i.e. by integration.

Furthermore, equations are presented as open statements expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*u = 8	an open statement about two equivalent number-names
(2*u)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(u*2)*(1/2) = 4	since multiplication is commutative
u*(2*(1/2)) = 4	since multiplication is associative
u*1 = 4	by definition of an inverse element
u = 4	by definition of a neutral element

The alternative sees an equation as another name for reversing a calculation that stops because of an unknown number. Thus the equation '2*u = 8' means wanting to recount 8 in 2s: 2*u = 8 = (8/2)*2, showing that u = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the 'opposite side&sign' method. A method that allows the equation '20/u = 5' to be solved quickly by moving across twice; 20 = 5*u' and 20/5 = u', or more thoroughly by recounting 20 = (20/u)*u = 5*u = (20/5)*5 = 4*5, so u = 4.

Finally, middle school lets geometry precede coordinate geometry, again preceding trigonometry; instead of respecting that in Greek, geometry means to measure earth, which is done by dividing it into triangles again divided into right triangles. Consequently, trigonometry should come first as a mutual recounting of the sides in a right triangle. And geometry should be part of coordinate geometry allowing solving equations predict intersection points and vice versa, thus experiencing repeatedly that the strength of mathematics is the fact that formula predict.

In high school, a function is presented as an example of a set-relation where first-component identity implies second-component identity; and the important functions are polynomials with linear functions preceding quadratic functions; instead of respecting that a function is a name for a formula with two unspecified numbers, again respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. As to polynomials, they should be introduced as the number-formula containing the different forms of formulas for constant change, T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x. Consequently, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d and parallel to the other examples of constant change. Thus emphasizing the double nature of formulas that the can predict both level and change.

Furthermore, differential calculus is presented before integral calculus, presenting an integral as an antiderivative; instead of postponing differential calculus until after integral calculus is presented as adding locally constant per-numbers, i.e. as a natural continuation of adding fractions as piecewise constant per-numbers in middle school and next-to addition of blocks in primary school. Only in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant change.

Finally, high school presents algebra as a search for patterns, instead of celebrating the fact that calculus completes the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Uniting/ *splitting*	Variable	Constant
Unit-numbers	T = a + n *T – a = n*	T = a * n *T/n = a*
Per-numbers	T = ∫ a dn *dT/dn = a*	T = a^n, *loga(T) = n n√T = a*

Remedial Curricula

A remedial micro-curriculum might be relevant whenever learning problems are observed. Since you never get a second chance to create a first impression, especially remedial curricula in primary school are important to prevent mathematics dislike.

Thus, as described above in the chapter 'examples of difference-research', in primary school, problems might be eased by

- with digits, using a folding ruler to observe that a digit contains as many sticks or strokes as it represents if written in a less sloppy way.
- with counting sequence, using sequences that shows the role of bundling when counting to indicate that a given total as e.g. seven can be named in different ways: 7, .7, 0.7, bundle less 3, ½bundle&2, etc.

- with recounting, using a cup and 5 sticks to experience that at total of 5 can be recounted in 2s in three ways: with an overload, normal, or with an underload: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s, or T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s if using decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles.
- when learning multiplication tables, letting 3*7 mean 3 7s recounted in tens, i.e. a block that when increasing its width must decrease its height to keep the total unchanged.
- when learning multiplication tables, beginning by doubling and halving and tripling; and to recount numbers using half-ten and ten as e.g. 7 = half-ten&2 = 10less3 so that 2 times 7 is 2 times half-ten&2 = ten&4 = 14, or 2 times 10less3 = 20 less 6 = 14.
- when multiplying, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336, or T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336
- when dividing, using bundle-writing to create overloads or underloads according to the multiplication table, as e.g. T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48
- when subtracting, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38
- when adding, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

In middle school, problems might be eased by keeping algebra and geometry together and by re-describing

- proportionality as double-counting in different units leading to per-numbers
- fractions as per-numbers coming from double-counting in the same unit
- adding fractions as per-numbers by their areas, i.e. by integration
- solving equations as reversing calculations by moving to the opposite side with the opposite calculation sign

In high school, problems might be eased by re-describing

- functions as number-language sentences, i.e. formulas becoming equations or functions with 1 or 2 unspecified numbers
- calculus as integration preceding differentiation
- integration as adding locally constant per-numbers
- pre-calculus, calculus and statistics as pre- or post-dicting constant, non-constant and non-predictable change

A Macro STEM-based Core Curriculum

A macro-curriculum (Tarp, 2017) was designed as an answer to a fictitious curriculum architect contest set up by a Swedish university wanting to help the increasing number of young male migrants coming to Europe each year: 'The contenders will design a STEM-based core mathematics curriculum for a 2year course providing a background as pre-teacher or pre-engineer for young male migrants wanting to help rebuilding their original countries.'

The design was inspired by an article on STEM (Han et al, 2014). Thus te curriculum goal is mastery of Many in a STEM context for learners with no background. As to STEM, OECD writes:

> The New Industrial Revolution affects the workforce in several ways. Ongoing innovation in renewable energy, nanotech, biotechnology, and most of all in information and communication technology will change labour markets worldwide. Especially medium-skilled workers run the risk of being replaced by computers doing their job more efficiently. This trend creates two challenges: employees performing tasks that are easily automated need to find work with tasks bringing other added value. And secondly, it propels people into a global competitive job market. (..) In developed economies, investment in STEM disciplines (science,

> technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth. The importance of education in STEM disciplines is recognised in both the US and Europe. (OECD, 2015b)

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules, called heat; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles and mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves. Nitrates and carbon-dioxide and water is waste for grey cells, but food for green cells producing proteins and carbon-hydrates and oxygen as food for the grey cells in return.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or not. To master plane or spatial forms, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent. So, mastery of Many means counting and recounting and adding and reversing addition and describing change and spatial shapes.

A STEM-based core curriculum can be about cycling water. Heating transforms it from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

Teaching Differences to Teachers

A group of teachers wanting to bring difference-research findings to the classroom might want first to watch some YouTube videos at the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many.

Then to try out the 'Free 1day SKYPE Teacher Seminar: Cure Math Dislike by 1 cup and 5 sticks' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally, and at a Skype conference with a coach. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in the three genres of quantitative literature, fact and fiction and fiddle. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle + 2Bundle + 3 = 4tenten2ten3 = 4*B^2+2*B+3.

Being a Difference-Researcher

In mathematics education, difference-research can be used by teachers observing problems in the classroom, or by teacher-researchers splitting their time between academic work at a university and intervention research in a classroom. Or by full-time researchers cooperating with teachers both using difference-research, the teacher to observe problems, the researcher to identify differences, working out a different micro-curriculum together to be tested by the teacher and reported by the researcher conducting a pretest-posttest study.

Thus, a typical difference-researcher begins as an ordinary teacher observing learning problems in his classroom and wondering if he could teach differently. Personally, in a precalculus class I taught linear and exponential functions by following the textbook order presenting them as examples of functions, again presented as examples of relations between two sets assigning one and only one element in one set to each element in the other set. I realized that by defining concepts as examples of abstractions instead of as abstractions from examples, I basically taught that 'bublibub is an example of bablibab' which some learners just memorized while others refused to learn before I gave them some applications. Talking about the difference between saving at home and in a bank, some asked me: Instead of calling it linear and exponential functions, why don't you just call it change by adding and by multiplying since that is what it is?'

So here the students themselves invented a difference that makes sense since historically, functions came after calculus. And the difference made two differences. Nobody had problems with learning about change by adding and by multiplying. And the Ministry of Education followed my suggestion to replace functions with variables instead of making pre-calculus non-compulsory, which was the plan because of the high number of low marks.

So one way to become a difference-teacher is to combine elements from action learning and action research and intervention research and design research. First you identify a difference, then you design a micro-curriculum, then you teach it to learn what difference the difference makes, then you learn from reporting and discussing it internally with colleagues. After having repeated this cycle of teaching and reporting the difference, the difference and the difference it makes in a posttest or a pretest-posttest setting is reported externally to teacher magazines or to conferences or to research journals.

Research is an institution supposed to produce knowledge to explain nature and improve social conditions. But as an institution, research risks a goal displacement if becoming self-referring. This raises two questions: Can a teacher produce research, and can research produce teaching? (Hammersley, 1993, p. 215). Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality used to collect reliable data and to test the validity of its findings by falsification attempts.

Often sociological imagination (see e.g. Zybartas et al, 2005) seems to be absent from traditional research seen by many teachers as useless because of its many references. In a Swedish context, this has been called the 'irrelevance of the research industry' (Tarp, 2015b, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying in a YouTube debate with Chomsky on Human nature:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics education and research are two examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead thus marginalizing or forgetting its original outside goal.

Conclusion

With 50 years of research, mathematics education should have improved significantly. Its lack of success as illustrated by OECD report ‘Improving Schools in Sweden’ made this paper ask: Apparently half a century’s research in mathematics education has not prevented low and declining PISA performance. Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? Seeking guidance by difference-research searching traditions for hidden differences that make a difference, the answer is: Yes, mathematics can be different, education can be different, and research can be different.

Looking back, mathematics has meant different things through its long history, from a common label for knowledge in ancient Greece to today’s ‘meta-matism’ combining ‘meta-matics’ defining concepts by meaningless self-reference, and ‘mathe-matism’ adding numbers without units thus lacking outside validity. So, looking for a difference to traditional set-based meta-matism, one alternative is the original Greek meaning of mathematics: Knowledge about Many in time and space.

Observing Many, allows rebuilding mathematics as a ‘many-matics’, i.e. as a natural science about the physical fact Many, where counting by bundling and stacking leads to block-numbers that recounted in other units leads to proportionality and solving equations; where recounting sides in triangles leads to trigonometry; where double-counting in different units leads to per-numbers and fractions, both adding by their areas, i.e. by integration; where counting precedes addition taking place both on-top and next-to involving proportionality and calculus. And where using a calculator to predict the counting result leads to the opposite order of operations: division before multiplication before subtraction before next-to and on-top addition.

Observing classes in continental Europe and in North America shows that education can be line-organized with forced year-group classes aiming at fulfilling the nation’s need for officials for the public or private sector; or education can be block-organized with self-chosen half-year classes aiming at uncovering and developing the learner’s individual talent. In mathematics education, the tradition sees learning mathematics as the goal of teaching mathematics and defines its concepts from above as examples of abstractions, part of the ruling canonical correctness, to be reached by learners through scaffolding. Here a difference is to accept that concepts historically arose from below as abstractions from examples, thus allowing new concepts to connect to existing.

Observing conference proceedings, shows that research papers may instead be master level papers applying instead of questioning existing theory and aiming at explaining instead of solving educational problems. Here a difference is difference-research searching traditions for hidden differences that make a difference.

So yes, as to mathematics education research, all three components can be different. Bottom-up many-matics can replace top-down meta-matism. In teenage education, daily lessons in self-chosen half-year blocks can replace periodic lessons in forced year-group lines. And, searching for useable differences can replace attempts at understanding the lack of understanding non-understandable self-reference.

Consequently, PISA performance may increase instead of decrease, and Swedish schools might improve dramatically by respecting that education means preparing learners for the outside world, brought inside to change the classroom from a library with self-referring textbooks to be learned by hart into a laboratory allowing the learner to meet the educational subject directly instead of indirectly through textbook ‘gossip’. And by avoiding a goal displacement seeing mathematics as the goal for mathematics education, thus hiding the real goal, a number-language about Many in time and space.

To teach many-matics instead of meta-matism, big-scale in-service teacher training is needed, e.g. through the MATHeCADEMY.net, designed to teach teachers to teach mathematics as a natural science about Many by the CATS-approach, Count & Add in Time & Space, using PYRAMIDeDUCATION, where learners learn by being taught by the subject directly instead of indirectly by a sentence.

So, if a society as Sweden really wants to improve mathematics education, extra funding should force its universities to arrange curriculum architect contests to allow differences to compete as to imagination, creativity and effectiveness, thus allowing universities to rediscover their original external goal and to change their internal routines accordingly. A situation described in several fairy tales: The Beauty Sleeping behind the thorns of routines becoming rituals; and Cinderella making the prince dance, but only found when searching outside the canonical correctness.

With 2017 as the 500th anniversary of Luther’s 95 theses, the recommendation of difference-research to mathematics education research could be the following theses:

- To master Many, count and multiply before you add
- Counting and recounting give block-numbers and per-numbers, not line-numbers
- Adding on-top and next-to roots proportionality and integration, and equations when reversed
- Beware of the conflict between bottom-up enlightening and top-down forming theories.
- Institutionalizing a means to reach a goal, beware of a goal displacement making the institution the goal instead
- To cure, be sure, the diagnose is not self-referring
- In sentences, trust the subject but question the rest

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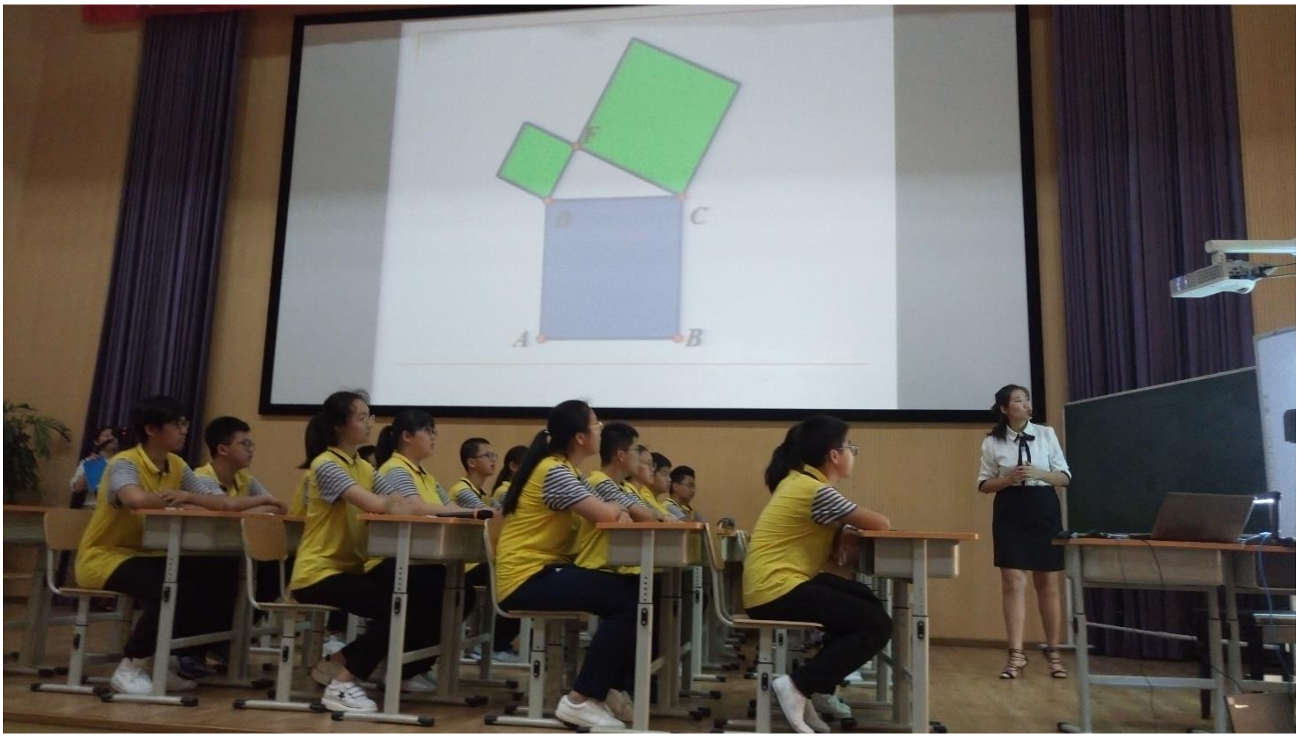
18. Reflections from the CTRAS 2017 Conference in China

Examples of Goal Displacements in Mathematics Education

At the annual conference of the Classroom Teaching for All Students Research Working Group (CTRAS), the 2017 conference theme was to promote classroom teaching research on exploring effective teaching strategies to support all students' mathematics learning. The two conference days contained half a day of plenary lectures. The first day also contained four examples of classroom teaching where a class of 5x3x2 students were taught in 30-40 minutes to illustrate examples of classroom lessons in China and the US. This paper reflects upon the lessons and some of the plenary lectures from a difference-research perspective looking for differences making a difference.



Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4^{th} year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA performance decrease from 2003 to 2012, causing OECD to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (OECD, 2015, p. 3).

Other countries also experience a low and declining PISA performance. And apparently research can do nothing about it. At a plenary discussion, it was mentioned that according to an American Educational Research Association report, many research studies on teacher education does not have value to classroom teachers and classroom teaching. So, to improve student performance, maybe a different kind of research is needed to rise questions as: Does it really have to be so, or can it be different? Can mathematics be different? Can education? Can research? So, maybe it is time to seek guidance by difference-research, searching for differences making a difference.

Searching for Hidden Differences, Difference-Research looks at Mathematics Education

Difference-research (Tarp, 2017) asks two questions: 'Can this be different – and will the difference make a difference?' Difference-research is inspired by the ancient Greek sophists looking for hidden differences to unmask choice masked as nature. If things work there is no need to ask for differences. But with problems, difference-research might provide a difference making a difference.

As to mathematics education, education is a social institution, and perhaps the most intervening one considering the numbers of hours spent there per week and during childhood and adolescence. As to institutions, Bauman talks about rationality and goal displacements in social organizations:

> **Max Weber,** one of the founders of sociology, saw the proliferation of organizations in contemporary society as a sign of the continuous rationalization of social life. **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (..) the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) It may happen in effect that the task originally seen as the reason to establish it is relegated to a secondary position by the all-powerful interest of the organization in self-perpetuation and self-aggrandizement. The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (Bauman, 1990, pp. 79, 84)

So, in a social institution, its goal cannot be different unless a means is masked as a fake goal, to be unmasked and replaced by the original goal by difference-research finding hidden differences.

As an institution, mathematics education is a social organization with a 'rational action in which the end to be achieved is clearly spelled out', apparently aiming at educating students in mathematics, 'we teach you mathematics so you can learn mathematics'. But this is a goal displacement created by meaningless self-reference (we teach you bublibub so you can learn bublibub). So, if mathematics isn't the goal in mathematics education, what is? And, how well-defined is mathematics after all?

How Well-Defined is Mathematics?

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, algebra, replacing arithmetic because of smarter numbers, and geometry, both rooted in the physical fact Many through their original meanings, 'to reunite' in Arabic and 'to measure earth' in Greek. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when all were replaced by the 'New Mathematics'.

Here the invention of the concept Set created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant definition by self-reference, i.e. defining a concept top-down as examples of abstractions instead of bottom-up as abstractions from examples. Thus the concept 'function', originally labeling a calculation containing both specified and unspecified numbers, was turned into a subset of a set-product where first-component identity implies second-component identity.

However, looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \left\{ A \mid A \notin A \right\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts: You can eat an example of an apple, but not the word 'apple'.

Thus, SET has transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics, defining concepts as examples of abstractions instead of as abstractions from examples; and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units, as '1 + 2 IS 3', meets counter-examples as e.g. 1 week + 2 days is 9 days.

So, looking back, mathematics has meant many different things during its more than 5000 years of history. But in the end, isn't mathematics just a name for knowledge about forms and numbers and operations? We all teach that 3*8 = 24, isn't that mathematics?

The problem is two-fold. We silence that 3*8 is 3 8s, or 2.6 9s, or 2.4 tens depending on what bundle-size we choose when counting. Also we silence that, which is 3*8, the total. By silencing the subject of the number-language sentence 'The total is 3 8s' or 'T = 3*8' we treat the predicate, 3 8s or 3*8, as if it was the subject, which is a clear indication of a goal displacement. Thus, the total of fingers on a hand cannot be different, but the way they are counted can: T = 5 1s = 2 2s &1 = 1 2s & 3 = 3 2s less 1 = 1 3s & 2 etc.

So, the goal of mathematics education is to learn, not mathematics, but to deal with totals, or, in other words, to master Many. The means are numbers and operations and calculations.

However, numbers come in different forms. Buildings often carry roman numbers; and on cars, number-plates carry Arabic numbers in two versions, an Eastern and a Western. Furthermore, we are sloppy by leaving out the unit and misplacing the decimal point when writing 24 instead of 2.4 tens. This might speed up writing, but might also slow down learning; together with insisting that addition precedes subtraction and multiplication and division if the opposite order is more natural. Finally, Lincolns Gettysburg address, 'Four scores and ten years ago' shows that not all count in tens. Thus in Denmark, seventy is called 'half four' with scores understood.

So, despite being presented as universal, many things can be different in mathematics, apparently having a tradition to present its choices as nature that cannot be different. And to unmask choice presented as nature is precisely the aim of difference-research.

How to find Hidden Differences?

Research is an institution supposed to produce knowledge to explain nature and improve social conditions. But as an institution, research risks a goal displacement if becoming self-referring. Questioning if traditional research is relevant to teachers, Hargreaves argues that

> What would come to an end is the frankly second-rate educational research which does not make a serious contribution to fundamental theory or knowledge; which is irrelevant to practice; which is uncoordinated with any preceding or follow-up research; and which clutters up academic journals that virtually nobody reads (Hargreaves, 1996, p. 7).

Here difference-research tries to be relevant by its very design: A difference must be a difference to something already existing in an educational reality, which then is used to collect reliable data and to test the validity of its findings by falsification attempts.

Hidden differences might be found by sociological imagination, seen as the core of sociology by Mills (1959); and by Negt (2016) using the term to recommend an alternative exemplary education for outsiders, originally for workers, but today also applicable for migrants.

As to the importance of sociological imagination, Bauman (1990, p. 16) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now.'

However, often sociological imagination (see e.g. Zybartas et al, 2005) seems to be absent from traditional research, seen by many teachers as useless because of its many references. In a Swedish context, this has been called the 'irrelevance of the research industry' (Tarp, 2015, p. 31), noted also by Bauman as hindering research from being relevant:

> One of the most formidable obstacles lies in institutional inertia. Well established inside the academic world, sociology has developed a self-reproducing capacity that makes it immune to the criterion of relevance (insured against the consequences of its social irrelevance). Once you have learned the research methods, you can always get your academic degree so long as you stick to them and don't dare to deviate from the paths selected by the examiners (as Abraham Maslow caustically observed, science is a contraption that allows non-creative people to join in creative work). Sociology departments around the world may go on indefinitely awarding learned degrees and teaching jobs, self-reproducing and self-replenishing, just by going through routine motions of self-replication. The harder option, the courage required to put loyalty to human values above other, less risky loyalties, can be, thereby, at least for a foreseeable future, side-stepped or avoided. Or at least marginalized. Two of sociology's great fathers, with particularly sharpened ears for the courage-demanding requirements of their mission, Karl Marx and Georg Simmel, lived their lives outside the walls of the academia. The third, Max Weber, spent most of his academic life on leaves of absence. Were these mere coincidences? (Bauman, 2014, p. 38)

By pointing to institutional inertia as a sociological reason for the lack of research success in mathematics education, Bauman aligns with Foucault saying in a YouTube debate with Chomsky on Human nature:

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky et al., 2006, p. 41)

Bauman and Foucault thus both recommend skepticism towards social institutions where mathematics and education and research are three examples. In theory, institutions are socially created as rational means to a common goal, but as Bauman points out, a goal displacement easily makes the institution have itself as the goal instead, thus marginalizing or forgetting its original outside goal.

Here Heidegger gives a tool to tell goals from means by pointing out, that in defining is-statements we should trust the subject but question the predicate since the subject, by its existence, cannot be different whereas the predicate is a judgement that might be a prejudice, i.e. one among several means that can be different, as illustrated above when reporting on the number of fingers on a hand.

Heidegger sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' (Dasein) must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

> This Being-with-one-another dissolves one's own Dasein completely into the kind of Being of 'the Others', in such a way, indeed, that the Others, as distinguishable and explicit, vanish more and more. In this inconspicuousness and unascertainability, the real dictatorship of the "they" is unfolded. (..) Discourse, which belongs to the essential state of Dasein's Being and has a share in constituting Dasein's disclosedness, has the possibility of becoming idle talk. And when it does so, it serves not so much to keep Being-in-the-world open for us in an articulated understanding, as rather to close it off, and cover up the entities within-the-world. (Heidegger, 1962, pp. 126, 169)

Inspired by Heidegger, the French poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu points out that society forces words upon you to diagnose you so it can offer you curing institutions including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to your world and yourself (Derrida, 1991. Lyotard, 1984. Bourdieu, 1970. Tarp, 2012).

Thus Foucault (1995) sees a school as a 'pris-pital', i.e. a mixture of a prison and a hospital. A school is prison-like by forcing students to stay together in classes for a long period of time, where continental Europe uses multi-year lines based upon age, in contrast to North America that from secondary school uses self-chosen half-year blocks.

And a school is hospital-like by wanting to cure the students by treating them for a diagnose that is not always that well-defined, and in many cases self-referring as when saying: we teach you mathematics so you can learn mathematics.

So, to make education a meaningful and civilized 'pris-pital', a diagnose must refer to a lack of or in knowledge about outside things or phenomena that students will meet when leaving school.

Thus, the original educational goal, to prepare children and adolescents for mastering the outside world, leads to two questions: What should the students meet in the classroom, the outside world brought inside, or descriptions of it in textbooks? And should all students meet the same in forced multi-year classes or be allowed to choose individually between half-year blocks?

And, to make its education a meaningful and civilized cure we must confront mathematics, seen as a collection of definitions and truth-claims, with two questions: Are the definitions self-referring or rooted in the outside goal, Many? Has the inside truth outside validity also?

Heidegger's warning 'In sentences, trust the subject but question the rest' implies that to discover the true nature of the subject hidden by the gossip of traditional mathematics, we need to meet the subject, the total, outside its 'predicate-prison'. By opening us, Many will appear with its nature undisguised, thus allowing us to construct different mathematics micro- and macro-curricula.

So we now return to the original subject in Greek mathematics, the physical fact Many, and use sociological imagination and Grounded Theory (Glaser et al, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties (Tarp, 2017). We do so to answer the question: How to find differences debunking mathematics from a goal to an inside means to the real outside goal, mastery of Many.

Meeting Many Creates a Count&Multiply&Add Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we count and add to create a number-language sentence, T = 2 3s, containing a subject, a verb and a predicate as in a word-language sentence. We count in bundles to be stacked as block-numbers to be re-counted and double-counted and processed by on-top and next-to direct or reversed addition. Thus, to count we take away bundles (thus rooting division) to be stacked (thus rooting multiplication) to be moved away to look for unbundled singles (thus rooting subtraction); finally we answer using bundle-writing for the bundles inside the bundle-cup and the singles outside, possibly with an overload or an underload to be removed or created by re-counting in the same unit, T = 7 = 2B1 3s = 1B4 3s = 3B-2 3s = 2 1/3 3s = 2.1 3s (thus rooting fractions and decimals to describe the singles). The result is predicted by a re-count formula T = (T/B)*B saying that 'from T, T/B times B can be taken away'. Re-counting in another unit roots proportionality. A total counted in icons can be re-counted in tens (thus rooting multiplication tables), or a total counted in tens can be re-counted in icons (thus rooting equations).

Double-counting in physical units creates per-numbers (again rooting proportionality) becoming fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas (thus rooting calculus).

Once counted or re-counted, totals can be added on-top after being re-counted in the same unit (again rooting proportionality); or next-to as areas (again rooting integral calculus). Then both on-top and next-to addition can be reversed (thus rooting equations and differential calculus).

In a rectangle split by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Thus, a Count&Multiply&Add curriculum differs from the tradition by presenting counting and multiplication before addition, and by using calculus to add fractions as per-numbers (Tarp, 2017).

Classroom Lessons

At the CTRAS 2017 conference, the first day contained four example of classroom lessons where a class of 5x3x2 students were taught in 30-40 minutes to illustrate examples of classroom teaching in China and the US.

B. China Teacher Lesson Display, Grade 5

The first lesson was a China teacher lesson display with a grade 5 class. The task was to fill a 3x3 square with the numbers 1-9 are so that they add up to 15 horizontally, vertically and on the diagonals, motivated by a video sequence from a fairy tale showing that this would lift a spell.

Personally, I found this an interesting task allowing the children to use their imagination and creativity. Likewise, a motivating video was a good idea. I observed that some students seemed to find the task difficult. This raises the question: ‘Will a different approach make a difference as to how many students succeed?’ So, from the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

Based upon the principle ‘algebra & geometry, always together, never apart’, symmetry is present on the geometry part, so it ought also to be present on the algebra part, e.g. by applying a counting sequence for the numbers 1-9 that counts the numbers as ‘Bundle less or plus’ using five as the bundle-number: Bundle less 4, B-3, B-2, B-1, B+0, B+1, B+2, B+3, B+4, inspired by the Roman numbers and a Chinese or Japanese abacus.

By its geometry, each sum will contain three numbers, so we can leave out the bundle B and redesign the task to ‘adding up to zero’. Because of the symmetry in geometry and algebra, 0 must be in the middle. Seeing zero as an even number, the three terms must be odd+odd+even, so the corners must be odd numbers.

Thus, the task could split up in several subtasks:

1. Starting by 5, find a symmetrical way to count from 1 to 9. Describe the symmetry.

2. Reformulate the task using these new numbers. Which number must be placed in the middle?

3. Adding two numbers to an odd number, how can the result be an even number?

4. Which numbers must be placed in the corners?

5. Show and test the answer using the numbers 1-9.

In a Count&Multiply&Add curriculum, re-counting the numbers from 1 to 9 in 5s is a routine task since the fingers on a hand is counted as ‘1 or bundle less 4; 2 or B-3 etc.’. Bundle-counting implies that you chose a bundle-size for the cup. In this case the sum 15 is obtained by three numbers, so 5 would be a natural choice as bundle-size allowing re-counting as T = 3 = 1B-2 5s, and T = 8 = 1B3 5s, etc. In such a class, the first subtask would be: ‘1. With the sum 15 obtained by three numbers, chose a bundle-size and reformulate the task.’

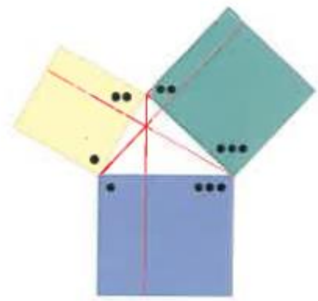
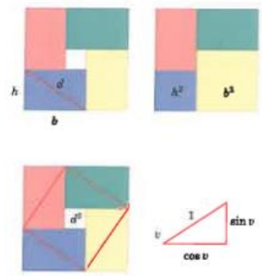
B. China Teacher Lesson Display, Grade 8

The second lesson was a China teacher lesson display with a grade 8 class. The task was to give a geometrical proof of the Pythagoras Theorem

Personally, I found this an interesting task allowing the adolescents to use their imagination and creativity. A proof is a core task in classical geometry; and choosing the Pythagoras Theorem as a core theorem is a good idea. I observed that some students seemed to find the task difficult. This raises the question: ‘Will a different approach make a difference as to how many students succeed?’ So, from the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if the Pythagoras Theorem is a goal or a means. Thus the Pythagoras Theorem may be seen as an inside means to the outside goal of adding travel-distances. If parallel, two distances add or subtract directly. But if perpendicular, they add by their squares: 3 steps over plus 4 steps up total 5 steps, since $3^2 + 4^2 = 5^2$.

Based upon the principle ‘algebra & geometry, always together, never apart’, the task could contain both a geometrical and an algebraical proof. If it is correct that the theorem can be proved in more than 100 ways, two easy proofs could be used first to include all students, and two more difficult proofs could be added later, as could a proof using trigonometry.

As to the background, three cases can be mentioned: an isosceles right-angled triangle, a right-angled triangle, and an arbitrary triangle leading to $a^2 + a^2 = c^2$ (or $c = a*\sqrt{2}$), $a^2 + b^2 = c^2$; and $a^2 + b^2 – 2*a*b*\cos C = c^2$, in its algebraic versions; or in its geometrical version: the three heights split the opposite squares in parts that are like to its outside neighbors. An easy algebraic proof is the one showing that the height splits the opposite square in parts that are like to its outside neighbors, which also holds for triangles that are not right-angled.	
An easy geometrical proof could be the one presented the next day in the plenary lecture ‘The wisdom of Traditional Mathematical Teaching in China’, shown with playing cards: Place four h-by-b playing cards after each other after turning them a quarter turn. The diagonals c also turn and now form a square with the area c^2. The full area can be expressed in two ways, as c^2 + 4 half cards, or as $h^2 + b^2$ plus two full cards. Consequently $h^2+b^2 = c^2$.	

Thus, the task splits up in several subtasks:

1. A 1.4-by-1.4 square is split into four triangles by the two diagonals. Prove that the triangles are isosceles and right-angled. Prove geometrically and algebraically that the Pythagoras Theorem $a^2+b^2 = c^2$ applies here. Measure the length of the diameter - are you surprised?

2. Draw a triangle with three angles less than 90 degrees. The three heights split the opposite squares in two parts. What can be said about the areas of two outside neighbors? Does this also apply to a right-angled triangle?

3. A geometrical proof of the Pythagoras Theorem uses four h-by-b playing cards placed after each other after turning them a quarter turn. The diagonals c also turn and now form a square with the area c^2. How can the total area be expressed?

4. Give an algebraic proof of the Pythagoras Theorem by using the result from question 2 and by splitting c in c1 and c2.

5. Tossing two dices gives the number of steps horizontally and vertically on a squared paper. Predict the length of the shortcut and test by measuring.

6. A 2meter bar is carried around a right-angled corner. How wide must the corridor be?

In a Count&Multiply&Add curriculum, counting includes a mutual re-counting of the sides in a right-angled triangle, seen as a rectangle halved by a diagonal. This allows trigonometry to be taught before geometry in accordance with the Greek meaning, earth-measuring. Thus, in a triangle ABC with C as the right angle and the side c split in c1 and c2 by the height, $\cos A = c1/b = b/c$, or $b^2 = c*c1$ and likewise with cosB. This shows that the height splits the opposite square in parts that are like to its outside neighbors, which also holds for triangles that are not right-angled. So in this case $c^2 = a^2 + b^2 – 2*a*b*\cos C$. The subtasks would be the same.

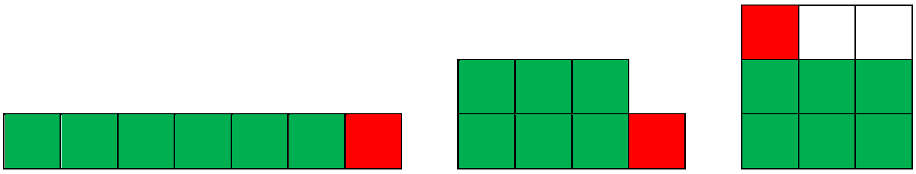
C. American Teacher Lesson Display, Grade 3

The third lesson was an American teacher lesson display with a grade 3 class. The task was to learn about and apply fractions, a core concept in algebra. I observed that some students seemed to find the task difficult. This raises the question: 'Will a different approach make a difference as to how many students succeed?' So, from the perspective of difference-research, asking 'Find a difference making a difference' I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if fractions is a goal or a means. Looking for the outside root of fractions we find double-counting in various contexts as e.g. icon-counting and switching units and parting.

Thus 'icon-counting fractions' occur when counting a total by bundling and stacking, which creates a double-counting of bundles and unbundled leftovers that can be placed in a separate stack for unbundled ones, separated by a decimal point, or on-top of the stack counted in 3s as a fraction creating a mixed number, since counting in 3s means taking away 3s, i.e. divide by 3.

Thus, a total of 7 can be counted as:



T = 7 = 2.1 **3s** = 2 1/3 **3s**

Re-counting 6 7s in tens gives T = 6 7s = 6*7 = 4 2/10 tens = 4.2 tens = 42 if leaving out the unit and misplacing the decimal point.

'Per-number fractions' or 'unit switching fractions' occur when double-counting something in the same or in different units. Counting in different units, per-numbers as 4$/5kg or 4/5 $/kg allows bridging the units by re-counting in the per-number:

10$ = (10/4)*4$ = (10/4)*5kg = 12.5kg; and 20kg = (20/5)*5kg = (20/5)*4$ = 16$.

'Parting fractions' are per-numbers coming from double-counting a part and the total in the same unit: If 5 of 7 apples are green, the fraction 5/7 of the 7 apples is the green part. Splitting a total in the ratio 2:3 means getting the fractions 2/5 and 3/5 of the total.

An outside sharing-situation can be a root or an application.

Sharing 8 apples between 4 persons not knowing division, they will repeat taking one each by turn as long as possible, e.g. by letting a mediator take away a bundle of 4s several times. In each bundle, a person then takes 1 of 4, or 1 per 4 or 1/4. In this case, the outside goal sharing roots the inside means icon-counting and per-numbers.

Or, sharing 8 apples between 4 persons may be presented as an application of getting the fraction ¼ of 8, found by dividing 8 by 4. Postponed to after division and fractions have been taught and learned, this is an example of a goal displacement, where the inside means, divisions and fractions, are treated as goals in need of outside applications as means for student motivation.

In education, a choice should be made as to which fraction should be taught first. In the actual lesson, the choice seemed to be teaching parting fractions as 2/7 by double-counting the part and the total, and to apply them to describe a self-designed packman, although the lesson also contained examples of sharing fractions when dividing a geometrical figure.

Observing the Piaget principle 'through the hand to the head (greifen vor begreifen)', one way to introduce parting fractions could be using the biological counters, the fingers: On my left hand, the fingers can be straight or bent. If 2 of the 5 fingers are bent I will say that the fraction 2 of 5 or 2/5 of my fingers are bent. If no fingers are bent, the fraction is 0/5. If all five fingers are bent, the fraction is 5/5. Thus, a fraction is used to describe a double-counting of a part in the total. In the fraction 2/5, 2 is called a numerator since it numbers the specials; and 5 is called a denominator since it names the total, and 'nomen' is 'name' in latin.

Later both hands can be used to illustrate fractions as 7/10, or 5/8 if excluding the thumbs.

Next step could be to discuss what is meant by saying that 3/5 of my ten fingers are bent. Here a choice must be made between parting and per-number fractions.

As to parting fractions, looking at the ten fingers I must apply my mathematical knowledge to say: I find the fraction 1/5 by splitting the total in 5 equal parts, which is done by dividing 10 by 5 giving 2. Now I can multiply with 3 to get 6. So I bend 6 fingers'.

As to per-number fractions, looking at the ten fingers I reformulate the task: the fraction 3/5 means taking 3 per 5, and with ten as two 5s, I just bend 3 fingers on both hands, i.e. 6 fingers'

As alternative means to the same goal, both should be presented in the class to observe differences as to effect.

Another option is to introduce parting fractions in a symmetry context using a dice and writing a cross if the dice shows an even number. Such a task splits up in several subtasks:

1. Put your left hand flat on the table with all finger straight.

2. On a dice, which numbers are even and why? The rest are called odd.

3. On this paper you find 10 rows with 5 squares in each row. Throw a dice five times. If even, bend a finger and write a cross; else leave the finger straight and write nothing. And report the number of crosses as a fraction 3 of 5 and as 3/5. Each time, mention which is the numerator and which is the denominator.

4. Please do the same with the next 9 rows.

5. At the bottom line, please fill in the report saying: The result 0/5 I got ?/10 times, etc.

6. Among the 10 rows, how many are identical? How many are symmetrical?

A third option is to introduce parting fractions in a probability context and continue with the following subtasks:

6. Use centi-cubes or double centi-cubes to show the answer to question 5.

7. In groups of fours, build your centi-cubes together vertically and write a report: The result 0/5 we got ?/40 times, etc.

8. In the class, arrange all the structures behind each other horizontally. Are they like?

In a Count&Multiply&Add curriculum, counting by bundling and stacking implies fractions and decimals to account for the unbundled singles placed on-top of or next-to the stack thus creating mixed numbers as e.g. T = 7 = 2 1/3 3s. Later fractions occur as per-numbers coming from double-counting in the same unit, as e.g. 2$/5$ = 2/5 = 2 per 5. Taking the fraction 2/5 of 20 means taking 2$ per 5$ of 20$, so we just re-count 20 in 5s as T = 20$ = (20/5)*5$ giving (20/5)*2$ = 8$.

As to addition, fractions add as per-numbers, both being operators needing a number to become a number. Multiplying before adding creates areas to be added, thus rooting integral calculus.

D. American Teacher Lesson Display, Grade 8

The fourth lesson was an American teacher lesson display with a grade 8 class. The task was to find a formula connecting the number of angles with the angle sum in a polygon.

Personally, I found this a core task in geometry, allowing the adolescents to use their imagination and creativity. I observed that some students seemed to find the task difficult. This raises the question: 'Will a different approach make a difference as to how many students succeed?' So, from the perspective of difference-research, asking 'Find a difference making a difference' I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if finding the angle sum in a polygon is a goal or a means. Looking for the outside root of angles we find changing direction under a closed journey with many turns. Thus the lesson could focus on a paper with three closed journeys with 3 and 4 and 5 turning points labeled A and B and C and D and E.

The triangle allows showing that the angle sum is 180 degrees from a new perspective: Inserting an extra point P between A and B transforms the line segment AB into a tri-angle APB where P adds 180 degrees to the angle sum zero. Pulling P out makes P decrease with what A and B increase, so the angle sum remains 180 degrees.

Likewise, on a triangle ABC, inserting an extra point P between A and B transforms the triangle into a four-angle APBC where B adds 180 degrees to the angle sum. Pulling P out makes P decrease with what A and B increase, so the angle sum remains added with 180.

And again the angle sum is increased by 180 degrees by inserting an extra point Q between A and P in the four-angle APBC. So each time an angle is added to the original 3, the angle sum gets 180 added to the original 180 degrees. Consequently, the total angel sum is 180 + 180*(angle number – 3).

Thus, as to teaching, the task could split up in several subtasks:

On this sheet, you see three different polygons. We would like to find a formula connecting the angle number with the angle sum in a polygon.

1. The word ‘polygon’ is Greek. What does it mean in English? In German a triangle is called a ‘Dreieck’. Are the words describing the same?

2. On a line segment AB, insert an extra point P between A and B to transform the line segment into a 3-angle APB. What is the angle sum in APB?

3. Pulling P away from the line segment makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APB?

4. On a triangle ABC, insert an extra point P between A and B to transform the 3-angle into a 4-angle APBC. What is the angle sum in APBC?

5. Pulling P away from the triangle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBC?

6. On a 4-angle polygon ABCD, insert an extra point P between A and B to transform the 4-angle into a 5-angle APBCD. What is the angle sum in APBCD?

7. Pulling P away from the four-angle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBCD?

8. On the 5-angle polygon ABCDE, insert an extra point P between A and B to transform the 5-angle into a 6-angle APBCDE. What is the angle sum in APBCDE?

9. Pulling P away from the four-angle makes P decrease and A and B increase. Are these numbers related? What is now the angle sum in APBCDE?

10. Try formulating a formula connecting the angle number n with the angle sum S.

11. Are any of these formulas correct?

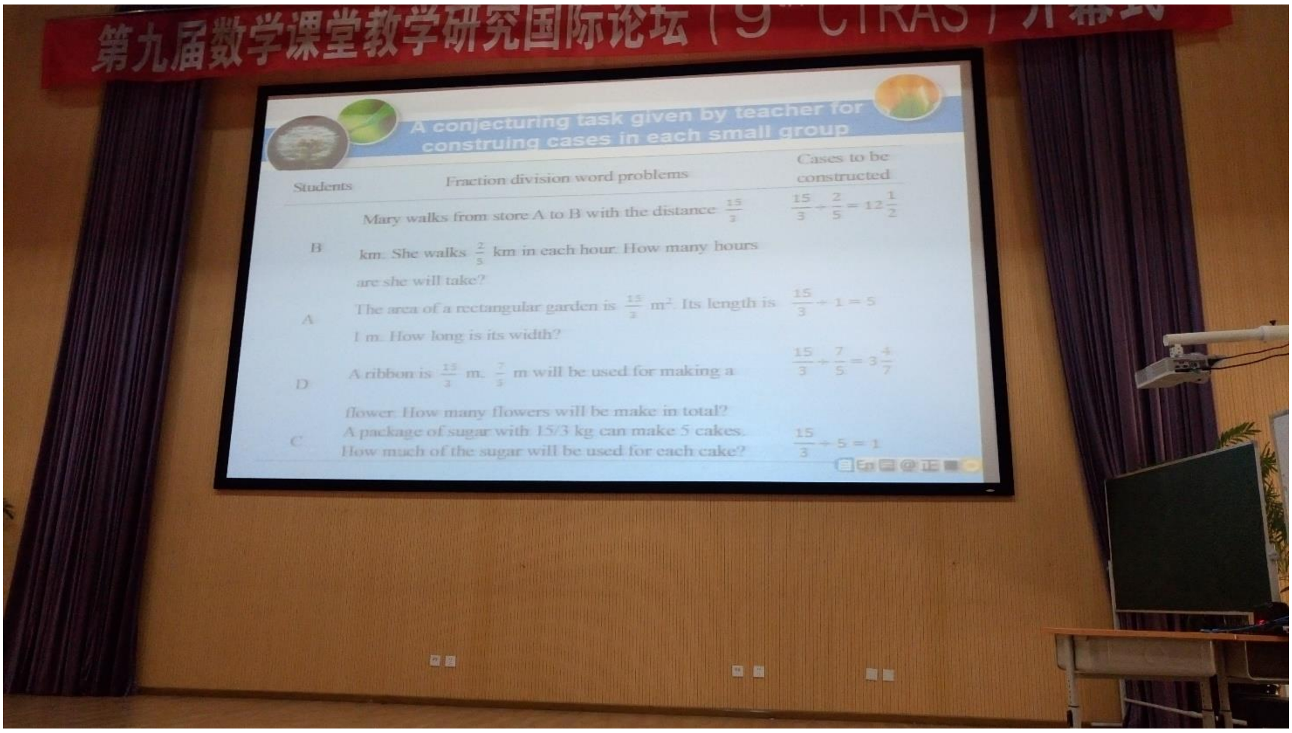
S = n*180 -3	S = n*180 - 360	S = 180 + 180*(n-2)

In a Count&Multiply&Add curriculum, the ‘Geometry & algebra, always together, never apart’ principle is observed. Thus a polygon will be lines connecting angles with given coordinates. So an angle is found by solving the equation tanA = slope. If all angles are to be found, in the end the rule for the angle sum can be used for checking.

Fractions and Mixed Numbers

Two plenary presentations contained mixed numbers. The ‘Using sharing brownies task for mixed number concept development’ presentation discussed the task: How to split 13 cookies between 4 children? The ‘The conjecturing contributing to the group argumentation in primary classrooms’ presentation contained a slide with three parts.

Students	Fraction division word problems	Cases to be constructed
B	Mary walks from store A to B with the distance 15/3 km. She walks 2/5 km in each hour. How many hours are she will take.	15/3 ÷ 2/5 = 12½
A	The area of a rectangular garden is 15/3 m^2. Its length is 1 m. How long is its width?	15/3 ÷ 1 = 15/3
C	A ribbon is 15/3 m. 7/5 m will be used for making a flower. How many flowers will be make in total?	15/3 ÷ 7/5 = 3 4/7
D	A package with 15/3 kg can make 5 cakes. How much of the sugar will be used for each cake?	15/3 ÷ 5 = 1

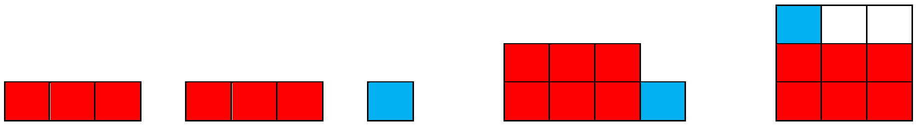


From the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if mixed numbers is a goal or a means. Looking for the outside root of mixed numbers we look for the roots of fractions, again rooted in division.

Seeing ‘Mastering Many’ as the outside root of mathematics, meeting Many leads to the question ‘How many in total?’. To answer we total by counting and adding. We count by bundling and stacking, predicting the resulting number-block by a re-count formula, T = (T/B)*B, saying ‘from T, T/B times B can be taken away’, thus rooting division and multiplication. Thus, a total of 8 can be re-counted in 4s as T = 8 = (8/4)*4 = 2*4 = 2 4s. So the root of division is counting by bundling.

Counting 7 in 3s gives T = 7 = 2B1 = 2.1 3s if the singles are placed next-to the stack of 3s as a stack of 1s; or T = 2 1/3 3s if the singles are placed on-top of the stack of 3s, counted in 3s as part of a 3-bundle.



T = 7 = 2 3s & 1 = 2.1 3s = 2 1/3 3s

So the root of mixed numbers is double-counting a total in bundles and parts, expressing the part as a fraction or by a decimal point. Counting in icon-bundles different from ten, the fraction remains unchanged, T = 7 = 2 1/3 3s. But counting in tens, the fraction is changed into decimals: T = 6 7s = 4 2/10 tens = 4.2 tens = 42 if leaving out the unit and the decimal point.

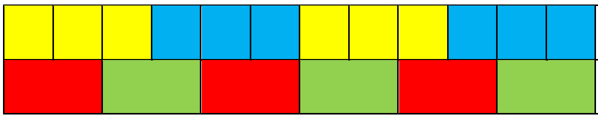
Sharing 8 apples between 4 persons not knowing division, they will repeat taking one each by turn as long as possible, e.g. by letting a mediator take away a bundle of 4s several times. In each bundle, a person then takes 1 of 4, or 1 per 4 or 1/4. Thus the root of fractions is per-numbers.

So, the sharing question can be reformulated to ‘How manty times can 4 be served by 8 items?’ or ‘8 = ?*4’ or ‘8 = u*4’ which is an equation solved by re-counting 8 in 4s: u*4 = 8 = (8/4)*4 giving u = 8/4 = 2, showing that an equation is solved by moving a number to the opposite side with the opposite sign.

Seeing 8/4 as ‘8 counted in 4s’ thus reflects what takes place psychically when sharing. However, the tradition says that ‘8/4 = 2’ means ‘8 shared between 4’ giving 4 2s and not 2 4s.

Thus, seeing division as THE sharing tool will exclude students unable to learn division, normally considered the difficult of the four operations; and introduced as the last operation, despite introducing it as the first is the natural approach if respecting that the natural way to share is to count the total in shares.

Furthermore, the sharing-understanding of division does not allow problems as ‘4 shared between 1/3’, to which the counting-understanding has the natural answer T = 4/(1/3) = 4*3 = 12. This resonates with the re-count formula saying T = (4/(1/3))*1/3, so 4/(1/3) must mean 4*3. Likewise, counting 4 in 2/3s halves the result, so T = 4/(2/3) = 4*3/2 = 6, or k/(2/3) = k*(3/2).



4 counted in 1/3s gives 12, so 4/(1/3) = 4*3
4 counted in 2/3s gives 6 , so 4/(2/3) = 4*3/2

Here the difference-research question is ‘Will presenting division as a counting means instead of as a sharing means make a difference?’

Returning to the discussion about outside goals and inside means we can ask: With ten as the international standard for bundles, does mixed numbers occur outside or only inside classrooms?

Sharing 13 brownies between 4, each get 3 ¼ brownie, which makes sense since a brownie can split in 4 equal parts. However, the answer could also be ‘3 each and 1 leftover’ as would be the case if sharing 13 cats instead. So whether 13 shared by 4 is 3 ¼ or ‘3 & 1 left’ depends on the unit.

To study the difference in concept development, difference-research would arrange two additional introductions ’13 brownies are arranged in boxes of fours; how many boxes are needed?’; and ’13 brownies are served in quarters, how many can be served? How many boxes are needed?’

Measuring lengths in inches, it makes sense to talk about 3 ¼ inch since 1inch splits into parts by repeated halving. Whereas 3 1/5 inch makes no sense.

However, internationally, length is measured in meters that splits into ten-parts, that split into ten-parts etc., making fractions of tens transform into decimals.

In the first slide task, a distance of 15/3 km only makes sense if sharing 15 km between 3 persons or parts. And presenting a velocity as 2/5 km per hour only makes sense if presented as a per-number 2 km per 5 hours. But both are rare cases that should be presented as footnotes to the typical outside problems using decimal numbers.

The next slide tasks also contain mixed numbers: 'The area of a rectangular garden is 15/3 m^2'; and 'A ribbon is 15/3 m'; and '7/5 m will be used for making a flower'; and 'A package with 15/3 kg can make 5 cakes'. By geometrical constructions it is possible to construct 15/5 in the case of a garden and a ribbon. It is however not possible to find precisely 1/3 of 15 kg without first calculating 15/3. So, again we can ask: Are these typical situations in need of mixed numbers, or will decimal numbers be more frequently used in such situations?

Likewise, we can ask if problems describing outside phenomena with mixed numbers are examples of a goal displacement where the outside goal has become a means to motivate learning an inside means presented as a goal?

Basically, a mixed number as 23 ¼ is a mixture of two different bundle-sizes, 23 is counted in tens and ¼ is counted in 4s. This only has meaning when measuring length in inches but since meters has become the international unit, maybe mixed numbers should play only a minor role as footnotes to decimal numbers especially if mathematics education should include all.

Adding mixed numbers directly have meaning when adding inches. Elsewhere, by containing fractions, which are not numbers but operators needing a number to become a number, they should be added by areas, i.e. by integration.

In a Count&Multiply&Add curriculum, mixed numbers thus occur from day one when counting a total by bundling leaves some unbundles singles described by a fraction or a decimal point. Later when ten bundling takes over, mixed numbers become decimal numbers. And from the beginning, all numbers are seen as mixed decimal numbers in disguise, T = 43 = 4.3 tens just as it is said, 4ten3, and as it is written in China.

Fractions: Numbers or Operators

At the end of the first day plenary presentation a discussion took place about the nature of fractions. Arguing that fractions are not numbers but operators needing a number to become a number, I used 5 water bottles for illustration: 'To my right I have 2 bottles, 1 is horizontal since it is empty; to my left I have 3 bottles, 2 are empty. So to the right ½ of my 2 bottles are empty, and to the left 2/3 of my 3 bottles are empty. In total 1+2 = 3 of my 2+3 = 5 bottles are empty, so in this case, adding ½ and 2/3 gives 3/5 of my bottles, the same answer as many students give when adding fractions by adding the numerators and adding the denominators. But the school teaches that ½ + 2/3 = 7/6, meaning that when added, I have 6 bottles and 7 of them are empty. This is perhaps why students dislike fractions. We teach fractions as if they are numbers. But fractions are not numbers, fractions are operators needing a number to become a number. So maybe we should teach fractions that way.'

A gentleman gave as a counter argument that I was mixing fractions with ratios: The example should be described, not by fractions but by ratios, to the right the ratio of empty bottles is 1:2, and to the left it is 2:3, and since ratios do not add it was meaningless to ask for the total. I replied that ratios describe sharing situations which was not the case here. But I thanked him for disagreeing and asked the conference organizers to include in the next conference a debate between persons with different views on mathematics and its education, e.g. on the nature of fractions, or on other issues. In the break, we continued the discussion and agreed on considering writing a common paper on fractions and ratios.

Decimal Multiplication in Grade 5

The second day, a plenary presentation presented a study om fifth graders' learning of decimal multiplication, a core task in algebra, but causing problems to some students when asked to do the multiplication ‘110*2.54’. From the perspective of difference-research, asking ‘Find a difference making a difference’ I wrote down the following reflection:

As to the question about a possible goal displacement, we can ask if multiplying and decimal numbers is a goal or a means. Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding. To count, we take away bundles to be stacked, thus rooting division and multiplication, allowing the result to be predicted by a re-count formula T = (T/B)*B saying ‘from T, T/B times, B can be taken away’. A total of e.g. 8 can be re-counted in 4s as a block-number T = (8/4)*4 = 2*4 = 2 4s.

Multiplication thus is a means to stack six 7s as T = 6 7s = 6*7; and a means to re-count 6 7s in tens: T = 6 7s = 6*7 = 42 = 4.2 tens if including the unit and the decimal point. So, looking for the outside root of multiplication we find stacking and shifting units. Thus, the present task is to re-count 110 2.54s in tens, or to re-count 2,54 110s in tens.

Based upon the principle ‘algebra & geometry, always together, never apart’, this task can be reformulated to changing the size of a number block: Re-counted in tens, a block of 110 2.54s will increase the base 2.54 with a factor close to 4 and decrease the height with the same factor, so the result will be close to 110/4 tens or 27.5 tens or 275. Or Re-counted in tens, a block of 2.54 110s will decrease the base 110 with a factor 11 and increase the height with the same factor, so the result will be close to 2.54*11 tens close to 27.5 tens or 275.

Using pure algebra, the ten-units can be shown as factors: 110*2.54 = 11 tens*2.54 = 11*25.4 = 1.1 tens 25.4 = 1.1*254 = 279.4’

Thus, the task could split up in several subtasks:

1. Geometrically, show the product 110*2.54 as two number-block with a base and a height.

2. In each case, what is the factor needed to change the base to tens.

3. How will this factor change the height?

4. Factorize the product to show the ten-units.

6. Include the ten-factors in the other factor.

7. Write the product with and without the unit tens.

Recommending counting and multiplying before adding, multiplication and decimal numbers are part of counting in a Count&Multiply&Add curriculum seeing mastering Many as the outside goal of mathematics. Meeting Many, we ask ‘How many in Total?’ Counting 7 in 3s gives T = 7 = 2B1 = 2.1 3s if the singles are placed next-to the stack of 3s as a stack of 1s, or 2 1/3 3s if the singles are placed on-top of the stack of 3s, counted in 3s as part of a 3-bundle.

To answer the question ‘How many in Total?’ we use a number-language sentence with a subject and a verb and a predicate as has word-language sentences. Thus T = 6*7 means that the total is counted by bundling and stacking as a block of 6 7s, that may or may not be re-counted in tens as T = 6 7s = 6*7 = 4ten2 = 4Bundle2 = 4B2 = 4.2 tens = 4 2/ten tens, or 42 if we ask a calculator, leaving out the unit and the decimal point.

We see that a decimal point is an inside means to the outside goal of separating parts from bundles. Thus, counting in 3s, 1 single is described by a decimal number or a fraction as 0B1 or 0.1 or 1/3. And, when counted in tens, 1 single becomes 0B1 or 0.1 or 1/10.

Counting in tens, a bundle-of-bundles, a BB, is called a ten-tens or a hundred; and a bundle-of-bundles-of-bundles, a BBB, is called a ten-ten-tens or a thousand. A bundle-of-bundles-of-bundles-of-bundles, a BBBB, is called a wan in Chinese probably describing a standard army unit of hundred hundreds.

Thus, a total of 2 thousands and 3 hundreds and 4 tens and 5 ones, written shortly as T = 2345 with 1s as the unit, can also be written as T = 2BBB3BB4B5 = 234.5 tens = 234.5*10. We see that multiplying with the bundle-number 10 moves the decimal point one place to the right. And reversely, dividing with (or counting in) the bundle-number 10 moves the decimal point one place to the left.

Using cups for the bundles and the bundles-of-bundles etc. allows a total to be reported by bundle-writing, where T = 2345 = 2BBB3BB4B5 tens = 234.5 tens.

Changing the unit to hundreds where H = BB, we get T = 2345 = 2BH3H4B5 = 2BB3B45 hundreds = 23.45 hundreds = 23.45*100. Changing the unit to thousands where M = BBB, we get T = 2345 = 2M3BB4B5 = 2B345 thousands = 2.345 thousands = 2.345*1000. Again, we see that the decimal point moves one place to the right each time we multiply with the bundle-number 10.

With a ten-bundle as a ten-part of a hundred-bundle we can write T = 10 = 0H1P = 0.1 hundreds, again using the decimal point to separate the parts. And with 1 as a ten-part of a ten-part, we can write T = 1 = 0H0P1PP = 0.01 hundreds. So counting in hundred-bundles, T = 345 = 3B4P5PP = 3.45 hundreds = 3.45*100.

Some physical units can be divided in parts. The length 1meter divides into ten ten-parts called a decimeter, dm, that divides into ten ten-parts called a centimeter, cm, that divides into ten ten-parts called a millimeter, mm. Thus T = 2345 mm = 234.5 cm = 23.45 dm = 2.345 m. Or counted in decimeters, T = 23.45 dm = 2B3.4P5PP, again using a decimal point to separate the parts.

So a number can change to a number between 1 and 10 by factoring ten-units in or out:

T = 2.3*75.6 = 2.3*7.56*10 = 17.388*10 = 173.88

T = 0.023*7560 = 2.3/10/10*7.65*10*10*10 = 17.388*10 = 173.88

The multiplication table is an inside means to the outside goal to change the unit from icons to tens by asking e.g. T = 6 7s = ? tens, or T = 6*7 = ?*10.

One way is to memorize the full ten-by-ten table, another way is to reduce it to a small 2-by-8 table containing doubling (and halving) and tripling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus

T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14, or 2*(½B2) = 1B4 = 14.

T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21, or 3*(B-3) = 3B-9 = 21.

T = 6*7 = 6*(½B2) = 3B12 = 4B2 = 42, or 6*7 = 6*(B-3) = 6B-18 = 4B2 = 42.

T = 6*7 = (5+1) * (10-3) = 50 – 15 +10 – 3 = 42, or

T = 6*7 = (10-4)*(10-3) = 100 – 30 – 40 + 12 = 42.

These results generalize to a*(b – c) = a*b – a*c and vice versa; and to (a – d)*(b – c) = a*b – a*c – b*d + d*c.

5	1	
50	10	B
-15	-3	-3

Multiplying often creates an overload to be removed by stepwise bundling

T = 3 57s = 3*57 = 3* 5ten7 = 15ten21 =15ten2ten1 = 17ten1 = 171, or

T = 3 57s = 3*57 = 3* 5B7 = 15B21 = 15B2B1 = 17B1 = 171

T = 13 57s = 13*57 = 13* 5B7 = 65B91 = 74B1 = 741, or

T = 13 57s = 13*57 = 1B3 * 5B7 = 5BB+7B+15B+21 = 5BB22B21 = 5BB24B1 = 7BB4B1 = 741.

The same result comes from using Renaissance-multiplication, also useful with multi-digit multiplication and when multiplying polynomials.

	1	3	
	/5	1/5	5
	/7	2/1	7
7	4	1	

	2a	-3b	
	8a^2	-12ab	4a
	10ab	-15b^2	+5b
8a^2	-2ab	-15b^2	

*Renaissance-mult. showing that 13*57 = 741 and that (2a-3b)*(4a+5b) = 8a^2-2ab-15b^2*

Creating and removing overloads also applies for decimal numbers as 523.47 = 5BB2B3.4P7PP.

T = 6* 523.47 = 6* 5BB2B3.4P7PP = 30BB12B18.24P42PP = 30BB12B18.28P2PP = 30BB12B20.8P2PP = 30BB14B0.8P2PP = 3140.82

Or, with bundle-writing: T = 6* 523.47 = 6* 5]2]3.4]7] = 30]12]18.24]42] = 30]12]18.28]2] = 30]12]20.8]2] = 30]14]0.8]2] = 3140.82

The same when multiplying multi-digit numbers:

T = 2.3*75.6 = 2.3P * 7B5.6P = 14B10.12P + 21BP+15P+18PP = 14B + (10+21) + (12+15)P + 18PP = 14B31.27P18PP = 7B3.8P8PP= 173.88 since BP = 1

The same result comes from using Renaissance multiplication also useful with many-digit multiplication and multiplying polynomials.

			2	.3	
		/	1/4	2/1	7
		/	1/0	1/.5	5
		/	1/.2	.1/8	.6
1	7	3	.8	8	

The same using B for bundles and P for parts, and where 1BP = 1:

T = 2.3*75.6 = 75.6*2.3 = 7B5.6P * 2.3P

		7B	5	6P	
		14B	10	12P	2
		21	15P	18PP	3P
	14B	31	27P	18PP	
	14B	31	28P	8PP	
	14B	33	8P	8PP	
	17B	3	8P	8PP	
1BB	7B	3	8P	8PP	
1	7	3.	8	8	

Difference-Research Presentation

My own plenary presentation (Tarp, 2017) was called ‘Difference-Research Powering PISA Performance: Count and Multiply before you Add’. Seeing poor PISA performance as the result of 50 years of low-performing Mathematics Education Research, I asked if this could be different.

First I talked about different education, comparing two types of classrooms: Half-year self-chosen blocks in North America versus multi-year forced lines in Continental Europe.

Then I talked about different kinds of mathematics, comparing bottom-up Many-based ‘Many-matics’ from below with top-down Set-based ‘meta-matics’ from above.

Next, I pointed to ancient Sophism, Renaissance natural science, and (post)modern existentialism as the inspiration for difference-research searching for differences making a difference.

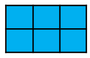
Finally, I talked about a different mathematics education, showing the beauty of the simplicity of mathematics: To master Many, count and re-count and multiply before you add; and when you add forwards & reverse, add block-numbers next-to & on-top, and add per-numbers and fractions by their areas, i.e. by calculus present in both primary and middle and high school.

Inspired by The Greek Sophist saying ‘Beware of choice masked as nature’, I warned against a Goal Displacement in mathematics education, occurring when a means becomes the goal; and unmasking means masked as goals is what difference-research is aiming at.

As to the main finding of difference-research, I showed the following slide unveiling the simplicity of mathematics when presented as tales of Many:

Difference-Research, Main Finding:
The Simplicity of Math – Math as Tales of Many

Meeting Many we ask: 'How Many in Total'



- To answer, we math. *Oops, sorry, math is not an action word but a predicate.*
- Take II. To answer, we **Count & Add**. And report with Tales of Many (Number-Language sentences): T = 2 **3s** = 2*3

Three ways to Count: CupCount & ReCount & DoubleCount

- CupCount gives units. ReCount changes units. Double-count bridges units by per-numbers as 2$/3kg

Recount to & from tens gives Multiplication & Equations, coming before Addition

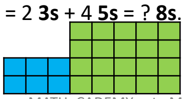
- To tens: T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens** . From tens: T = ? **7s** = u*7 = 42 = (42/7)*7 = 6 **7s** (ReCount-Formula)

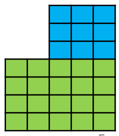
Counting gives variable or constant unit- or per-numbers, to be Added in 4 ways

- Addition & multiplication unites variable & constant unit-numbers.
- Integration & power unites variable & constant per-numbers.

Adding NextTo & OnTop roots Early Childhood Calculus & Proportionality

- EarlyChildhood-Calculus: T = 2 **3s** + 4 **5s** = ? **8s**. EarlyChildhood-Proportionality: T = 2 **3s** + 4 **5s** = ? **5s**





MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 5

As to the main warning of difference-research, the following slide shows the 3x3 goal displacements in mathematics education in primary, middle and high school:

Difference-Research, Main Warning:
The 3x3 Goal Displacements in Math Education

Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
	Equations	Could: be introduced in primary as recounting from ten-bundles to icon-bundles; and as reversed on-top and next-to addition Instead: Defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra
	Proportionality	Could: be introduced in primary as recounting in another unit when adding on-top; be double-counting producing per-numbers Instead: defined as linear functions, or as multiplicative thinking supporting the claim that fractions and ratios are rational numbers
High	Trigonometry	Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.
	Functions	Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate Instead: are introduced as set-relations where first-component identity implies second-component identity
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

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As to a different mathematics, the following slide shows the beauty of the simplicity of mathematics in 8 areas:

20. Different Mathematics

The Beauty of the Simplicity of Mathematics

21. The Goal & Means of Mathematics Education
22. Totals as Blocks. Digits as Icons. Operations as CupCounting Icons
23. ReCounting gives Proportionality & Multiplication & Equations
24. Multiplication tables simplified by ReCounting
25. DoubleCounting in different & same units creates PerNumbers & Fractions
26. Geometry: Counting Earth in HalfBlocks
27. Once Counted, Totals can be Added. But counting and double-counting gives 4 number-types (constant & variable unit-numbers & per-numbers) to add in 4 ways
28. How Different is the Difference? Set-based versus Many-based Mathematics

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As to the goals and means of mathematics education, the following slide shows the difference between the Set-based top-down tradition and the Many-based bottom-up difference:

21. Different Mathematics

The Goal and Means of Mathematics Education

The Set-based Top-Down Tradition:

- Mathematics exists as a collection of well-proven statements about well-defined concepts, all derived from the mother concept SET
- Mathematics is surprisingly useful to modern society
- Consequently, mathematics must be taught and learned

The Many-based Bottom-Up Difference:

- Many exists; to master Many we develop a number-language with Tales of Many, a 'ManyMatics'.
- **Many-matics**, defining concepts from below as **abstractions from examples**, is a more successful means to the goal of mastering Many than
- '**Meta-matics**' defining concepts from above as **examples from abstractions**

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The following slide compares the Set-based top-down tradition and the Many-based bottom-up difference:

28a. Different Mathematics

How Different is the Difference?

Set-based Math versus Many-based Math

	SET-based Tradition	**Many-based Difference**
Goal/Means	Learn Mathematics / Teach Mathematics	Learn to master Many / Math as Tales of Many
Digits	Symbols as letters	Icons with as many sticks as they represent
Numbers	Place-value number line names. Never with units	A union of blocks of stacked singles, bundles, bundle-bundles etc. Always with units
Number-types	Four types: Natural, Integers, Rational, Real	Positive and negative decimal numbers with units
Operations	Mapping from a set-product to the set	Counting-icons: /,*,-,+ (bundle, stack, remove, unite)
Order	Addition, subtraction, multiplication, division	The opposite
Fractions	Rational numbers, add without units	Per-numbers, not numbers but operators needing a number to become a number, so added by integration
Equations	Statement about equvalent number-names	Recounting from tens to icons, reversing operations
Functions	Mappings between sets	Number-language sentences with a subject, a verb and a predicate
Proportionality	A linear function	A name for double-counting to different units
Calculus	Differential before integral (anti-differentiation)	Integration adds locally constant per-numbers.

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Finally, a slide showed the main parts of a curriculum in 'ManyMath' seeing mathematics as a natural science about the physical fact Many

28b. Different Mathematics

Main Parts of a ManyMath Curriculum

Primary School – respecting and developing the Child's own 2D NumberLanguage

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- CupCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: / x - +

Middle school – integrating algebra and geometry, the content of the label math

- DoubleCounting produces PerNumbers as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always so length becomes change and vv.

High School – integrating algebra and geometry to master CHANGE

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

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The PowerPoint presentation was supplemented with a paper (Tarp, 2017) carrying the same title, describing in detail how PISA performance can improve in three ways: by a different macro-curriculum from class one, by remedial micro-curricula when a class is stuck, and by a STEM-based (Han et al, 2014) core-curriculum for outsiders. The next three chapters are extracts from the paper.

How to Improve PISA Performance

Improving PISA performance means improving mathematics learning which can be done by observing three basic facts about our human and mammal and reptile brains.

The human brain needs meaning, so what is taught must be a meaningful means to a meaningful outside goal, mastery of Many; thus mathematics must be taught as ‘Many-matics’ in the original Greek sense as a common name for algebra and geometry both grounded in an motivated by describing Many in time and space; and not as ‘meta-matism’ mixing ‘meta-matics’, defining concepts from above as examples of internal abstractions instead of from below as abstractions from external examples, with ‘mathe-matism’, true inside but seldom outside classrooms as adding numbers without units.

The mammal brain houses feelings, positive and negative. Here learning is helped by experiencing a feeling of success from the beginning, or of suddenly mastering or understanding something difficult.

The reptile brain houses routines. Here learning is facilitated by repetition and by concreteness: With mathematics as a text, its sentences should be about subjects having concrete existence in the world, and having the ability to be handled manually according to Piagetian principle ‘through the hand to the head’.

Also, we can observe that allowing alternative means than the tradition makes it not that difficult to reach the outside goal, mastery of many. Meeting Many, we ask ‘How many in total?’ To get an answer we count and add. We count by bundling and stacking and removing the stack to look for unbundles leftovers. This gives the total the geometrical form of a collection of blocks described by digits also having a geometrical nature by containing as many sticks as they represent. Counting also includes recounting in the same or in a new unit; or double-counting to produce per-numbers. Once counted, totals can be united or split, and with four kinds of numbers, constant and variable unit-numbers and per-numbers, there are four ways to unite: addition, multiplication, power and integration; and four ways to split: subtraction, division, root/logarithm and differentiation.

Thus, the best way to obtain good PISA performance is to replace the traditional SET-based curriculum with a different Many-based curriculum from day one in school, and to strictly observe the warning: Do not add before totals are counted and recounted – so multiplication must precede addition. However, this might be a long-term project. To obtain short-term improvements, difficult parts of a curriculum where learners often are stuck might be identified and replaced by an alternative remedial micro-curriculum designed by curriculum architecture using difference-research and sociological imagination. Examples can be found in the above chapter ‘Examples of difference-research’.

Finally, in the case of teaching outsiders as migrants or adults or dropouts with no or unsuccessful educational background, it is possible to design a STEM-based core curriculum as described above allowing the outsiders become pre-teachers and pre-engineers in two years. Thus, applying sociological imagination when meeting Many without predicates forced upon it, allows avoiding repeating the mistakes of traditional mathematics.

The Tradition’s 3x3 mistakes

Choosing learning mathematics as the goal of teaching mathematics has serious consequences. Together with being set-based this makes both mathematics education and mathematics itself meaningless by self-reference. Here a difference is to accept that the goal of teaching mathematics is mastering Many by developing a number-language parallel to the word-language; both having a meta-language, a grammar, that should be taught after the language to respect that the language roots the grammar instead of being an application of it; and both having the same sentence structure with a subject and a verb and a predicate, thus saying ‘T = 2*3’ instead of just ‘2*3’.

This goal displacement seeing mathematics as the goal of mathematics education leads to 3x3 specific mistakes in primary, middle and high school:

In primary school, numbers are presented as 1dimensional line numbers written according to a place value convention; instead of accepting that our Arabic numbers like the numbers children bring to school are 2dimensional block numbers. Together with bundle-counting and bundle-writing this gives an understanding that a number really is a collection of numbers counting what exists in the world, first inside bundles and outside unbundled singles, later a collection of unbundled and bundles and bundles of bundles etc.

Furthermore, school skips the counting process and goes directly to adding numbers without considering units; instead of exploiting the golden learning opportunities in counting and recounting in the same or in another unit, and to and from tens. This would allow multiplication to be taught and learned before addition by accepting that 4*7 is 4 7s that maybe recounted in tens as T = 4 7s = 2.8 tens = 28, to be checked by recounting 28 back to 7s, T = 28 = (28/7)*7 = 4*7 = 4 7s, using the recount-formula reappearing in proportionality, trigonometry and calculus. And giving division by 7 the physical meaning of counting in 7s.

Finally, addition only includes on-top addition of numbers counted in tens only and using carrying, a method that neglects the physical fact that adding or subtracting totals might crate overloads or underloads to be removed by recounting in the same unit. And neglecting the golden learning opportunities that on-top addition of numbers with different unit roots proportionality, and that next-to addition roots integration, that reversed roots differentiation thus allowing calculus to be introduced in primary school.

In middle school, fractions are introduced as numbers that can be added without units thus presenting mathematics as ‘mathematism’ true inside but seldom outside classrooms. Double-counting leading to per-numbers is silenced thus missing the golden learning opportunities that per-numbers give a physical understanding of proportionality and fractions, and that both per-numbers and fractions as operators need numbers to become numbers that as products add as areas, i.e. by integration.

Furthermore, equations are presented as open statements expressing equivalence between two number-names containing an unknown variable. The statements are transformed by identical operations aiming at neutralizing the numbers next to the variable by applying the commutative and associative laws.

2*u = 8	an open statement about two equivalent number-names
(2*u)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(u*2)*(1/2) = 4	since multiplication is commutative
u*(2*(1/2)) = 4	since multiplication is associative
u*1 = 4	by definition of an inverse element
u = 4	by definition of a neutral element

The alternative sees an equation as another name for reversing a calculation that stops because of an unknown number. Thus the equation ‘2*u = 8’ means wanting to recount 8 in 2s: 2*u = 8 = (8/2)*2, showing that u = 8/2 = 4. And also showing that an equation is solved by moving to the opposite side with opposite calculation sign, the ‘opposite side&sign’ method. A method that allows the equation ‘20/u = 5’ to be solved quickly by moving across twice; 20 = 5*u’ and 20/5 = u’, or more thoroughly by recounting 20 = (20/u)*u = 5*u = (20/5)*5 = 4*5, so u = 4.

Finally, middle school lets geometry precede coordinate geometry, again preceding trigonometry; instead of respecting that in Greek, geometry means to measure earth, which is done by dividing it into triangles again divided into right triangles. Consequently, trigonometry should come first as a mutual recounting of the sides in a right triangle. And geometry should be part of coordinate geometry allowing solving equations predict intersection points and vice versa, thus experiencing repeatedly that the strength of mathematics is the fact that formula predict.

In high school, a function is presented as an example of a set-relation where first-component identity implies second-component identity; and the important functions are polynomials with linear functions preceding quadratic functions; instead of respecting that a function is a name for a formula with two unspecified numbers, again respecting that a formula is the sentence of the number-language having the same form as in the word language, a subject and a verb and a predicate. Formulas should be used from the first day at school to report and predict counting results as e.g. T = 2 3s = 2*3 and T = (T/B)*B. As to polynomials, they should be introduced as the number-formula containing the different forms of formulas for constant change, T = a*x, T = a*x+b, T = a*x^2, T = a*x^c and T = a*c^x. Consequently, linear and quadratic functions should be taught together as constant change T = a*x+b and constant changing change T = a*x+b where a = c*x+d and parallel to the other examples of constant change. Thus emphasizing the double nature of formulas that the can predict both level and change.

Furthermore, differential calculus is presented before integral calculus, presenting an integral as an antiderivative; instead of postponing differential calculus until after integral calculus is presented as adding locally constant per-numbers, i.e. as a natural continuation of adding fractions as piecewise constant per-numbers in middle school and next-to addition of blocks in primary school. Only in high school, adding locally constant per-numbers means finding the area under the per-number graph as a sum of a big number of thin area-strips, that written as differences reduces to finding one difference since the middle terms cancel out. This motivates the introduction of differential calculus, also useful to describe non-constant change.

Finally, high school presents algebra as a search for patterns, instead of celebrating the fact that calculus completes the algebra project, meaning to reunite in Arabic: Counting produces two kinds of numbers, unit-numbers and per-numbers, that might be constant or variable. Algebra offers the four ways to unite numbers: addition and multiplication add variable and constant unit-numbers; and integration and power unites variable and constant per-numbers. And since any operation can be reversed: subtraction and division splits a total in variable and constant unit-numbers; and differentiation and root & logarithm splits a total in variable and constant per-numbers.

Uniting/ *splitting*	Variable	Constant
Unit-numbers	T = a + n *T – a = n*	T = a*n *T/n = a*
Per-numbers	T = ∫ a dn *dT/dn = a*	T = a^n, *loga(T) = n* *n√T = a*

Remedial Curricula

A remedial micro-curriculum might be relevant whenever learning problems are observed. Since you never get a second chance to create a first impression, especially remedial curricula in primary school are important to prevent mathematics dislike.

Primary school. Here problems might be eased by

- with digits, using a folding ruler to observe that a digit contains as many sticks or strokes as it represents if written in a less sloppy way.
- with counting sequence, using sequences that shows the role of bundling when counting to indicate that a given total as e.g. seven can be named in different ways: 7, .7, 0.7, bundle less 3, ½bundle&2, etc.

- with recounting, using a cup and 5 sticks to experience that at total of 5 can be recounted in 2s in three ways: with an overload, normal, or with an underload: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s, or T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s if using decimal point instead of a bracket to separate the inside bundles from the outside unbundled singles.
- when learning multiplication tables, letting 3*7 mean 3 7s recounted in tens, i.e. a block that when increasing its width must decrease its height to keep the total unchanged.
- when learning multiplication tables, beginning by doubling and halving and tripling; and to recount numbers using half-ten and ten as e.g. 7 = half-ten&2 = 10less3 so that 2 times 7 is 2 times half-ten&2 = ten&4 = 14, or 2 times 10less3 = 20 less 6 = 14.
- when multiplying, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336, or T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336
- when dividing, using bundle-writing to create overloads or underloads according to the multiplication table, as e.g. T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48
- when subtracting, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38
- when adding, using bundle-writing to create overloads to be removed by recounting in the same unit, as e.g. T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

Middle school. Here problems might be eased by keeping algebra and geometry together and by re-describing

- proportionality as double-counting in different units leading to per-numbers
- fractions as per-numbers coming from double-counting in the same unit
- adding fractions as per-numbers by their areas, i.e. by integration
- solving equations as reversing calculations by moving to the opposite side with the opposite calculation sign

High school. Here problems might be eased by re-describing

- functions as number-language sentences, i.e. formulas becoming equations or functions with 1 or 2 unspecified numbers
- calculus as integration preceding differentiation
- integration as adding locally constant per-numbers
- pre-calculus and calculus as describing constant and variable predictable change; and statistics as post-dicting non-predictable change allowing it to be predicted by confidence intervals.

Conclusion

Mathematics education is a social institution with one goal and many means; and as such running the risk of a goal displacement where the original goal becomes a means to a means becoming the goal instead, seduced by a persuasive logic: Mathematics is highly applicable to the outside world, but of course, mathematics must be learned before it can be applied. So of course, mathematics, as defined by the mathematicians, is the goal, and outside applications may be included as a means to motivate the students for learning mathematics even if it is a hard subject demanding a serious commitment, as witnessed by poor PISA results even after 50 years of mathematics education research.

To this compelling argument, difference-research, searching for differences making a difference, will ask: maybe it is the other way around. Maybe there are several forms of mathematics and has been so during its long history, all leading to the same outside goal described in ancient Greece as four knowledge areas about Many in time and space, together labeled ‘mathematics’.

So maybe mathematics becomes simple and easy to learn for all, if once again it accepts itself as a means to the outside goal, mastery of Many, accessible through a Many-matics answering the basic question ‘How many in total?’ by number-language sentences with a subject and a verb and a predicate in the form of a calculation uniting constant or variable unit- or per-numbers.

Therefore, if mathematics for all is a social goal, society must remind mathematics about its role as a means serving the outside goal, mastery of Many, by constantly asking the basic question from the fairy tale Cinderella: Are there other alternatives outside the saloons of present correctness? This precisely is the aim of difference-research searching for differences making a difference. This entails two tasks, to find differences and to test them in a classroom. In this paper only the first task was conducted. In doing so, hidden differences were located within:

- Number sequences. The tradition counts the fingers on a hand as 1, 2, 3, 4, 5. A difference is to count 1, 2, 3, 4, B (bundle); or 1, 2, 3, B less 1, B; or B less 4, B-3, B-2, B-1, B. Emphasizing the word ‘bundle’ allows showing the nature of counting as bundling, might make a difference in micro-studies.
- Multiplication. The tradition says that 6*7 is 42. A difference is to say that 6*7 is 6 7s that may stay as it is or be recounted in another unit. If recounted in tens, 6 7s is 4.2 tens, shown geometrically as a block where an increase of the base from 7 to ten means a decrease of the height from 6 to 4.2 to keep the total unchanged. Multiplication thus becomes an inside means for two outside goals, to stack bundles and to change the unit to tens. Presenting multiplication before addition as a means to stack and change unit might make a difference in micro-studies.
- Multiplication tables. The tradition says that 6*7 is 42, which is a part of a ten-by ten multiplication table. A difference is to include the total behind and to recount 6 and 7 by saying T = 6*7 = 6 7s to be recounted in tens = (ten less 4)*(ten less 3) = tenten, less 4ten, less 3ten, and 4 3s = 100 – 40 – 30 + 12 = 42; or T = 6*7 = (½ten & 1)*(ten less 3) = ½tenten, ten, less ½ 3ten, less 3 = 50 + 10 – 15 – 3 = 42; or with bundle-writing, T = 6*(1B-3) = 6B-18 = 4B2 = 42; or counting in 5s, T = 6*(½B2) = 3B12 = 4B2 = 42. Allowing numbers to be recounted before multiplied might make a difference in micro-studies.
- Multiplying decimal numbers. The tradition says that multiplying decimal numbers is like multiplying numbers, only keeping track of the place of the decimal point. A difference is to see both factors as numbers between 1 and 10 with ten-units factored in or out. Another difference is to use bundle-writing and allow overloads in the different cups by gradual re-counting. Presented in this way it might make a difference in micro-studies.
- Division. The tradition says that 9/4 is 9 shared by 4 giving each the mixed number 2 ¼ . A difference is to say that 9/4 is 9 counted in 4s giving a total of T = (9/4)*4 = 2*4 +1 = 2 ¼ 4s = 2.1 4s. And to realize that sharing 9 between 4 involves two take-steps. First 4-bundles are taken away from 9 to re-count 9 in 4s; then, in a 4-bundle, each takes 1 part of 4, i.e. ¼. Sharing thus does not root the traditional division-understanding, instead sharing roots both counting in icons and taking fractions. Presented in this way it might make a difference in micro-studies.
- Fractions. The tradition says that the fraction 3/5 is a rational number describing 3 as a part of 5. A difference is to say that the fraction 3/5 is a per-number coming from double-counting in the same unit; and as per-numbers, fractions are not numbers but operators needing a number to become a number thus adding by their areas as in calculus. Using the fingers on both hand, you quickly learn about 2/5 of 5 and 2/5 of ten. Presenting fractions as per-numbers occurring in sharing situations might make a difference in micro-studies
- Pythagoras. The tradition says that the Pythagoras Theorem allows calculating the hypotenuse from the two other sides in a right-angled triangle. A difference says that parallel distances add

directly but perpendicular distances add by their areas. Presented in this way it might make a difference in micro-studies.

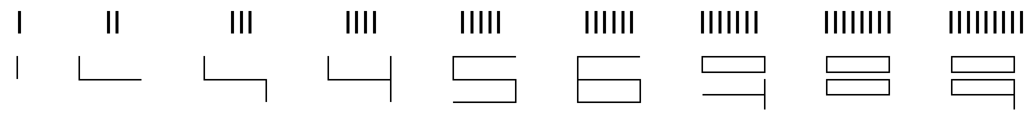
When teaching children to obtain mastery of Many, two options are available.

One option is to see mathematics as an unavoidable means that therefore might be a goal as well, leading to traditional teaching of line-numbers to be added, subtracted, multiplied and divided; and to fractions as rational numbers to be added directly without units, etc.

Another option is to build on what the children already know about mastering Many from being exposed to Many for several years before beginning school. Asking ‘How old will you be next time?’ a 3year old child answers ‘four’ with four fingers shown; but reacts to four fingers held together 2 by 2 with a ‘That is not four, that is two twos.’

So children come to school with 2dimensional number-blocks where all numbers have a unit as with the Arabic numbers they are supposed to learn, T = 345 = 3*10^2 + 4*10 + 5*1.

This allows school to practice guided discovery so the child can see that the digits are, not symbols as letters, but icons with as many strokes as they represent if written less sloppy, thus allowing the child to discover the transition from 4 1s to 1 4s, that can serve as a bundle when counting and re-counting.



1 2 3 4 5 6 7 8 9

Then school can practice double representation of totals using Lego blocks and bundle-writing with full number-language sentences as T = 2*3; and practice re-counting in the same unit to create or remove overloads, allowing the child to see that a total can be counted in different ways, as e.g. T = 7 = 5&2 = ten less 3 = ½bubndle & 2 = bundle less 3; or T = 12 = bundle & 2 = 2 ½bundles & 2 = 1 ½bundle & 7 = 3 ½bundles less 3 = 2 left (twelve = two left, ‘twe levnet’ in Viking language)

Then school can practice re-counting in a different unit so the child can experience the operations as means for a calculator-prediction using the re-count formula T = (T/B)xB, saying that ‘from T, T/B times B can be taken away’, presenting division as a broom wiping away the bundles, and multiplication as a lift stacking the bundles in a block to be removed by subtraction to count the unbundled singles: Asking ‘7 is how many 3s’, first we take 3s a number of times, predicted by 7/3 as 2. Then we take away the stack of 2 3s to count the leftovers, predicted by 7 – 2x3 as 1:

T = 7 = ? 3s. First 7/3 gives 2.some; next 7 – 2x3 gives 1; so T = 7 = 2.1 3s = 2 1/3 3s

Then school can practice recounting between icon-bundles and ten-bundles. Recounting in ten-bundles allows the multiplication table to be built slowly by beginning with doubling and halving and tripling. And recounting from ten-bundles to icon bundles allows the child to solve equation by recounting. So to answer the question ‘how many 8s is 24’ we juts re-count 24 in 8s to get the answer 3, thus moving a number to the opposite side with opposite sign:

?*8 = 24 = (24/8)*8 = 3*8 = 3 8s; so ? = 24/8 = 3

Then school can practice double-counting to create per-numbers bridging countings in different units, and becoming fractions if the units are the same.

Finally, once counted and re-counted, totals can add; either on-top after being re-counted to the same unit, later called proportionality, or next-to as areas also used when adding per-numbers and fractions, later called integral calculus. And then addition can be reversed, later called equations and differential calculus.

Thus, if the school allows children to develop their own number-language they will learn core subjects as proportionality and calculus and solving equations in the first year or two.

So why not celebrate the beauty of the simplicity of the child's own mathematics? Why replace the child's own 'Many-matics' with the school's traditional 'meta-matism,' mixing 'meta-matics', defining concepts as example of abstractions instead of as abstraction form examples, with 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units meet countless counterexamples: T = 2weeks + 3days is not 5 but 17 days; in contrast to this, T = 2*3 = 6 says that 2 3s can be re-counted as 6 1s which is universally true by including the unit 3.

Of course, an ethical issue occurs when depriving the child of its natural number-language, and forcing upon the child an alien language consisting of self-referring definitions and statements with uncertain validity.

In the second enlightenment republic France, Bourdieu calls this 'symbolic violence'; and Foucault, seeing the school as a 'pris-pital' mixing power techniques from a prison and a hospital, would warn against curing children not properly diagnosed, and against accepting self-reference when diagnosing (Bourdieu, 1970. Foucault, 1995).

So, wanting mathematics education to be for all, Many-based Many-matics from below should be preferred to set-based meta-matism from above.

This is how the Count&Multiply&Add curriculum was designed to allow children to develop their own number-language by the natural tasks of counting and re-counting and double-counting and multiplying before performing on-top and next-to addition and reversed addition.

So, a conjecture to be tested and researched is: PISA-like testing will improve if letting a Many-based Bottom-up Count&Multiply&Add curriculum replace the traditional Set-based top-down curriculum presenting 1dimensional numbers to be treated by addition firsts, then subtraction, then multiplication, then division leading to fractions added without units.

Of course, it will take many years to see the effect of a full curriculum, so in the meantime micro-curricula can be designed and tested via intervention research. Or the full curriculum can be tested as a 1year 'migrant-mathematics' course allowing young male migrants coming to Europe in high numbers to acquire competence as a pre-teacher or a pre-engineer to return help develop or rebuild their homeland after two years (Tarp, 2017).

As to online in-service teacher education, the MATHeCADEMY.net has been designed to teach teachers to teach mathematics as Many-matics, a natural science about Many, using the CATS-approach, Count&Add in Time&Space, partly described in DrAlTarp YouKu and YouTube videos; and organizing learners in groups of 8 using PYRAMIDeDUCATION.

Recommendation

With only a small percentage of mathematics education research having value to the classroom we must ask if research can be conducted differently. Here difference-research is a difference that might make a difference. Difference-research goes to the classroom to observe problems, allowing it to ask its basic question: Find a difference that makes a difference. Seeing education as preparing students for the outside world leads to accepting mathematics as it arose historically, an inside means to an outside goal, mastery of Many. This allows using intervention research to construct a different micro-curriculum to be tested and adapted in the classroom to see if it makes a difference.

Becoming a difference-researcher is straight forward. You begin as a teacher wanting to teach mathematics for all. At the master level, you read conflicting theory within sociology, philosophy and psychology. In sociology, you focus on the difference between patronizing and enlightening societies as described e.g. by Bauman and Giddens. In philosophy, you focus on the difference between a Platonic top-down view and a sophist bottom-up view as described e.g. by existentialism and post-structuralism. In Psychology, you focus on the difference between mediation and discovery as described e.g. by Vygotsky and Piaget. At the research level, you focus on the difference between testing existing theory and generating new theory as described e.g. by top-down deductive operationalization and a bottom-up inductive grounded theory. And you conduct intervention research by deigning different micro-curricula inspired by thinking differently within sociology, philosophy and psychology.

So, to improve mathematics education worldwide, China could educate ten-ten difference-researchers to spread along the coming new silk road where they each educate ten difference-researchers to help the local population implement a mathematics education for all, rooted in everyday experiences, thus allowing all to enjoy the beauty of the simplicity of mastering many.

With 2017 as the 500year anniversary for Luther's 95 theses, the recommendation can be given as 12 or 20 theses (Tarp, 2017), here reduced to 7 theses:

- To master Many, count and multiply before you add
- Counting and recounting give block-numbers and per-numbers, not line-numbers
- Adding on-top and next-to roots proportionality and integration, and equations when reversed
- Beware of the conflict between bottom-up enlightening and top-down forming theories.
- Institutionalizing a means to reach a goal, beware of a goal displacement making the institution the goal instead
- To cure, be sure, the diagnose is not self-referring
- In sentences, trust the subject but question the rest

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19. Sixteen Proposals for the 8th ICMI-East Asia Regional Conference on Mathematics Education

Theme of the Conference

"Flexibility in Mathematics Education" has been chosen as the theme of the conference. Flexibility is highly related to creativity, multiplicity, and adaptation. In the current era, rapid changes in economy, environment and society have been facilitated by the rapid development of technology and engineering. Flexibility in mathematical thinking, problem solving, teaching methods, evaluation, teacher education and mathematics education research is a key to empowering learners, teachers, educators and researchers to tackle the complexity and uncertainty, and to giving them the capacity and motive to change in the innovative era.

The Topic Study Group themes are:

TSG 1: Flexibility in Mathematics Curriculum and Materials

TSG 2: Flexibility in Mathematics Classroom Practices

TSG 3: Flexibility in Mathematics Assessment

TSG 4: Flexibility in Mathematics Teacher Education and Development

TSG 5: Flexibility in the Use of ICT in Mathematics

TSG 6: Flexibility in the Use of Language and Discourse in Mathematics

TSG 7: Flexibility in Mathematics Learning

The Simplicity of Mathematics Revealing a Core Curriculum (TSG 01)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add. First we take away bundles, thus rooting division; then we stack the bundles, thus rooting multiplication; then we move the stack away to look for singles, thus rooting subtraction; finally we answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences.

A calculator predicts the result by the recount-formula T = (T/B)*B saying 'from T, T/B times, B can be taken away', thus rooting fractions and decimals to describe the singles, e.g. T = 7 = 2 1/3 3s = 2.1 3s. Recounting in another unit roots proportionality. Changing units between icons and tens roots multiplication tables and equations.

Once counted, totals add on-top after being recounted in the same unit, again rooting proportionality; or totals add next-to, thus rooting integration. Reversing on-top and next-to addition roots equations and differentiation.

Double-counting in different physical units creates per-numbers, again rooting proportionality, where per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, again rooting integration.

Now in a rectangle split by a diagonal, recounting the side mutually creates the per-numbers sine, cosine and tangent. And traveling in a coordinate system, parallel distances add directly whereas perpendicular distances add by their squares. Recounting the y-change in the x-change creates linear formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Looking at constant and variable predictable change roots pre-calculus and calculus; and looking at unpredictable change roots statistics to post-dict the behavior of numbers by a mean and a deviation, again allowing probability to predict, not numbers but intervals. (Tarp, 2017)

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A STEM-based Math Core-Curriculum for migrants (TSG 01)

Seeing 'Mastery of Many' as the outside goal, we can construct a core math curriculum based upon exemplary situations of Many in a STEM context, having a positive effect on learners with a non-standard background (Han et al, 2014), thus allowing young male migrants to help their original countries as pre-teachers or pre-engineers.

Science is about nature itself. How three different Big Bangs, transforming motion into matter and anti-matter and vice versa, fill the universe with motion and matter interacting with forces making matter combine in galaxies, star systems and planets. Some planets have a size and a distance from its sun that allows water to exist in its three forms, solid and gas and liquid, bringing nutrition to green and grey cells, forming communities as plants and animals: reptiles, mammals and humans. Animals have a closed interior water cycle carrying nutrition to the cells and waste from the cells, and kept circulating by the heart. Plants have an open exterior water cycle carrying nutrition to the cells and kept circulating by the sun forcing water to evaporate through leaves.

Technology is about satisfying human needs. First by gathering and hunting, then by using knowledge about matter to create tools as artificial hands making agriculture possible. Later by using knowledge about motion to create motors as artificial muscles, combining with tools to machines making industry possible. And finally using knowledge about information to create computers as artificial brains combining with machines to artificial humans, robots, taking over routine jobs making high-level welfare societies possible.

Engineering is about constructing technology and power plants allowing electrons to supply machines and robots with their basic need for energy and information; and about how to build houses, roads, transportation means, etc.

Mathematics is our number-language allowing us to master Many by calculation sentences, formulas, expressing counting and adding processes. First Many is bundle-counted in singles, bundles, bundles of bundles etc. to create a total T that might be recounted in the same or in a new unit or into or from tens; or double-counted in two units to create per-numbers and fractions. Once counted, totals can be added on-top if recounted in the same unit, or next-to by their areas, called integration, which is also how per-numbers and fractions add. Reversed addition is called solving equations. When totals vary, the change can be unpredictable or predictable with a change that might be constant or variable. To master plane or spatial shapes, they are divided into right triangles seen as a rectangle halved by its diagonal, and where the base and the height and the diagonal can be recounted pairwise to create the per-numbers sine, cosine and tangent. So, a core STEM-based curriculum could be about formulas controlling cycling water cycles (Tarp, 2017).

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50 years of Sterile Mathematics Education Research, Why? (TSG 01)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

PISA scores still are low after 50 years of research. But how can mathematics education research be successful when its three words are not that well defined? Mathematics has meant different things in its 5000 years of history, spanning from a natural science about Many to a self-referring logic.

Within education, two different forms exist at the secondary and tertiary level. In Europe, education serves the nation’s need for public servants through multi-year compulsory classes and lines. In North America, education aims at uncovering and developing the individual talent through daily lessons in self-chosen half-year blocks with one-subject teachers.

Academic articles can be written at a master-level exemplifying existing theories, or at a research-level questioning them. Also, conflicting theories create problems as within education where Piaget and Vygotsky contradict each other by saying ‘teach as little and as much as possible’.

Consequently, we cannot know what kind of mathematics and what kind of education has been studied, and if research is following traditions or searching for new discoveries. So to answer the question ‘How to improve mathematics education research’, first we must make the three words well defined by asking: What is meant by mathematics, and by education, and by research? Answers will be provided by the German philosopher Heidegger (1962), asking ‘what is ‘is’?’

It turns out that, instead of mathematics, schools teach ‘meta-matism’ combining ‘meta-matics’, defining concepts from above as examples of abstractions instead of from below as abstractions from examples; and ‘mathe-matism’ true inside but seldom outside class, such as adding fractions without units, where 1 red of 2 apples plus 2 red of 3 gives 3 red of 5 and not 7 red of 6 as in the textbook teaching 1/2 + 2/3 = 7/6.

So, instead of meta-matism, teach mathematics as ‘many-math’, a natural science about Many, in self-chosen half-year blocks.

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The Center of Math Education: Its Sentences or its Subjects? (TSG 02)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal?

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask ‘How many in Total?’. To answer, we count and add and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences. However, a controversy exists as to what is the center of mathematics education, the predicate 2*3 or the subject T.

Seeing reproducing textbook knowledge as the goal, Vygotsky points to good teaching as the best means and recommends teaching as much as possible. Seeing individual sentences about the outside fact Many as the goal, Piaget points to good guidance as the best means and recommends teaching as little as possible.

Thus, where a Vygotsky class follows a textbook strictly, a Piaget class brings the subject of its sentences to the class to allow the learner to create individual sentences to be adapted through sharing, thus respecting Many as the outside goal of mathematics. Which resonates with Heidegger (1962) saying: In a sentence, the subject exists, but the rest might be gossip.

Flexibility in a primary classroom thus means using full sentences where the total exists as sticks and where the predicate can be flexible by using bundle-counting to count inside bundles and outside singles, e.g. T = IIIIIII = III III I = 2B1 3s or T = 1B4 3s = 3B-2 3s if allowing overloads and underloads outside the cup; which becomes useful when multiplying, T = 5* 67 = 5* 6B7 = 30B35 = 33B5 = 335; and when dividing: T = 335 /5 = 33B5 /5 = 30B35 /5 = 6B7 = 67.

The ‘geometry and algebra, always together, never apart’ principle allows learners to develop a flexible double number-concept, seeing the total T = 2*3 geometrically as number-block with 2 3s, that may or may not be recounted as 6 1s. Recounting 4*5 = 2 tens says that doubling its width, a block of 4 5s must halve its height to keep the total unchanged. Likewise, equations become tangible when recounting from tens to icons.

That totals must be counted and recounted before they add allows multiplication to precede addition.

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DoubleCounting roots Proportionality - and Fractions and Percentages as Per-Numbers (TSG 02)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

A school sees fractions as goals instead of means making a class stuck. Having heard about difference-research and per-numbers (Tarp, 2017), the teacher says: Time out. Next week, no fractions. Instead we do double-counting. First counting: 42 is how many 7s? The total T = 42 = (42/7)*7 = 6*7 = 6 7s. Then double-counting: Apples double-counted as 3 $ and 4 kg have the per-number 3$ per 4 kg, or 3$/4kg or ¾ $/kg. Asking how many $ for 10kg, we recount 10 in 4s, that many times we have 3$: The total T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Asking how many kg for 18$, we recount 18 in 5s, that many times we have 4kg: The total T = 18$ = (18/5)*5$ = (18/5)*4kg = 14.4kg. Double-counting in the same unit gives fractions and percentages as 3 per 4, ¾; and 75 per hundred, 75/100 = 75%.

¾ of 200$ means finding 3$ per 4$, so we recount 200 in 4s, that many times we have 3$: The total T = 200$= (200/4)*4$ gives (200/4)*3$ = 150$. 60% of 250$ means finding 60$ per 100$, so we recount 250 in 100s, that many times we have 60$: The total T = 250$= (250/100)*100$ gives (250/100)*60$ = 150$.

To find 120$ in percent of 250$, we introduce a currency # with the per-number 100# per 250$, and then recount 120 in 250s, that many times we have 100#: The total T = 120$ = (120/250)*250$ = (120/250)*100# = 48#. So 120$/250$ = 48#/100# = 48%. To find the end-result of 300$ increasing with 12%, the currency # has the per-number 100# per 300$. 12# increases 100# to 112# that transforms to $ by the per-number. The total T = 112# = (112/100)*100# = (112/100)*300$ = 336$.

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Assessing Goals Instead of Means (TSG 03)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and compare totals; and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences.

Counting includes bundle-counting to separate a total in bundles inside the cup and singles outside; and recounting in the same unit to create an outside overload or underload needed to ease operations, e.g. T = 4*56 = 4* 5B6 = 20B24 = 22B4 = 224.

Recounting in another unit, called proportionality, is predicted by a recount-formula T = (T/B)*B saying 'from T, T/B times, B can be taken away', thus rooting fractions and decimals to describe the singles, e.g. T = 7 = 2 1/3 3s = 2.1 3s. Changing units between icons and tens roots multiplication tables and equations.

Once counted, totals add on-top after being recounted in the same unit, again rooting proportionality; or next-to thus rooting integration. Reversing on-top and next-to addition roots equations and differentiation.

Double-counting in different physical units creates per-numbers, becoming fractions if the units are the same. Since per-numbers and fractions are operators needing a number to become a number, they add by their areas, again rooting integration.

In a rectangle split by a diagonal, recounting the side mutually creates the per-numbers sine, cosine and tangent. And traveling in a coordinate system, parallel distances add directly whereas perpendicular distances add by their squares. Recounting the y-change in the x-change creates linear formulas, algebraically predicting geometrical intersection points.

To avoid a goal displacement, assessment should test goals instead of means; and always use totals with units. With proportionality formulas in science as a core root for mathematics, several tasks should include per-numbers, e.g. taken from classical word problems. Numbers without units should be excluded, since adding numbers and fractions without units are examples of 'mathematism' true inside but seldom outside classrooms, where claims as 2+3 = 5 meet counterexamples as 2 weeks + 3 days = 1 7days. And where 1 red of 2 apples + 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as taught in school.

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The 2 Core Math Competences, Count & Add, in an e-learning Teacher Development (TSG 04)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal?

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3. Counting and double-counting in two units creates 4 number-types: variable and constant unit- and per-numbers that unite by addition, multiplication, integration and power.

That this simplicity typically is unknown to teachers created the MATHeCADEMY.net, teaching teachers to teach mathematics as 'ManyMath', a natural science about Many suing the CATS-approach: Count & Add in Time & Space. It is a virus academy saying: To learn mathematics, don't ask the instructor, ask Many. The material is question-based.

Primary School. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives 14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

Secondary School. COUNT: How to count possibilities? How to predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? Quantitative Literature, what is that? Does it also have the 3 genres: fact, fiction and fiddle?

PYRAMIDeDUCATION organizes 8 teachers in 2 teams of 4 choosing 3 pairs and 2 instructors by turn. The instructors instruct the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The instructors correct the count&add assignments. In a pair, each teacher corrects the other teacher's routine-assignment. Each pair is the opponent on the essay of another pair.

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12 Theses not Taught in Teacher Education (TSG 04)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying ‘You are taught mathematics to learn mathematics’, this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, mastering Many is the outside goal. As means, we iconize and bundle by digits, operations and formulas, becoming goals if forgetting the real goal.

1. Digits are icons with as many sticks as they represent.

2. A total T can be ‘bundle-counted’ in the normal way or with an overload or underload: T = 5 = 2B1 2s = 1B3 2s = 3B-1 2s.

3. ‘Bundle-writing’ makes operations easy: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

4. Counting T by bundling, T = (T/B)xB = (5/2)x2 = 2.1 2s, shows a natural number as a decimal number with a unit.

5. Operations are icons showing counting by bundling and stacking. -2 takes away 2. /2 takes away 2s. x2 stacks 2s. +2 adds 2 on-top or next-to.

6. A calculator predicts. Asking T = 4 5s = ? 6s, first (4x5)/6 = 3.some; then (4x5) – (3x6) = 2. So T = 4 5s = 3.2 6s

7. Recounting in tens, calculators leave out the unit and misplace the decimal point: T = 3 7s = 3*7 = 21 = 2.1 tens.

8. Recounting from tens, ‘? 7s = 3 tens’, or ‘u*7 = 30 = (30/7)x7’, the answer u = 30/7 is found by ‘move to opposite side with opposite sign’.

9. Adding totals is ambiguous: On-top using proportionality, or next-to using integration?

10. Operations are reversed with reverse operations: With u+3 = 8, u = 8–3; with ux3 = 8, u = 8/3; with u^3 = 8, u = 3√8; with 3^u = 8, u = log3(8); with T1 + u*3 = T2, u = □T/3.

11. Double-counting in different units gives ‘per-numbers’ as 4$/5kg, bridging the two units by recounting: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$

12. Double-counting in the same unit, per-numbers become fractions as operators, needing a number to become a number, thus adding by their areas as integration.

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Difference-Research at Work in a Classroom (TSG 04)

The CTRAS (Classroom Teaching Research for All Students) wants all students to benefit. The 2017 conference contained example of classroom lessons. Difference-research (Tarp, 2017) looks for a different approach based upon outside goals to see if more students benefit. Inspired by Greek sophists looking for hidden differences to unmask choice masked as nature, e.g. means presented as goals, difference-research asks two questions: 'Can this be different – and will the difference make a difference?'

The first task in a grade 5 class was to fill a 3x3 square with the numbers 1-9 so that they add to 15 horizontally, vertically and on both diagonals. Based upon the principle 'algebra & geometry, always together, never apart', the outside goal could be to give symmetry to both, e.g. by applying a counting sequence for the numbers 1-9 that counts the numbers as 'Bundle less or plus' using 5 as the bundle-number: Bundle less 4, B-3, B-2, B-1, B+0, B+1, B+2, B+3, B+4. By its geometry, each sum will contain three numbers, so we can leave out the bundle B and redesign the task to 'add to zero'. Thus, each sum must contain 2 odd numbers, placed in the corners.

The second task in a grade 8 class was to give a geometrical proof of the Pythagoras Theorem. Here an outside goal could be to add travel-distances. If parallel, two distances add or subtract directly. If perpendicular, they add by their squares: 3 steps over plus 4 steps up total 5 steps, since 3^2 + 4^2 = 5^2.

The third task in a grade 3 class was to learn about and apply fractions. Looking for the outside root of fractions we find double-counting in various contexts as e.g. icon-counting, statistics, splitting, per-numbers, changing. Double-counting bent and unbent fingers roots fractions as 2/5 of 5 and 2/5 of 10.

The fourth task in a grade 8 class was to find a formula connecting the number of angles to the angle sum in a polygon. Looking for the outside root of angles we find changing direction under a closed journey with many turns. Thus, the lesson could focus on a paper with three closed journeys with 3 and 4 and 5 turning points labeled from A to E. On the triangle, inserting an extra point P between A and B transforms the triangle ABC into a four-angle APBC where B adds 180 degrees to the angle sum. Pulling P out makes P decrease with what A and B increase, so the angle sum remains added with 180.

A plenary address discussed decimal multiplication in a grade 5 class exemplified by 110*2.54. Here a difference is to see multiplication as shifting units. Here a total of 110 2.54s is to be recounted in tens. Factorizing will show how the ten-units can change place: T = 110*2.54 = 1.1*10*10*2.54 = 1.1*254 = 279.4

References

Tarp, A. (2017). *Math Ed & Research 2017. Retrieved from http*://mathecademy.net/2017-math-articles/.

Pre-schoolers and Migrants Predict Recounting by a Calculator (TSG 05)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer we count and add. Asking T = 7 = ? 3s, first we take away bundles, thus rooting division iconized as a broom wiping away the bundles; then we stack the bundles, thus rooting multiplication iconized as a lift stacking the bundles; then we move the stack away to look for unbundled singles, thus rooting subtraction iconized as a trace left by the stack; and finally, we answer with a number-language sentence, containing a subject and a verb and a predicate as does word-language sentences.

To have the calculator predict the result we enter '7/3'. The answer 2.some tells us that 2 times 3s can be taken away. To look for unbundled singles we stack the 2 3s as 2*3 to be removed, so we enter '7 – 2*3'. The answer 1 tells us that 7 can be counted in 3s as 2 3s and 1, written as T = 7 = 2 1/3 3s if the single is placed on-top of the stack counted in 3s, or as T = 7 = 2.1 3s if the single is placed next-to the stack as a stack of unbundled.

This shows that a natural number is decimal number with a unit where the decimal point separates the bundles from the unbundled.

A calculator thus predicts the result by the recount-formula T = (T/B)*B saying 'from T, T/B times, B can be taken away'.

Recounting in tens means just multiplying. Recounting from tens to icons means asking 30 = ? 6s. Here we use the recount-formula to recount 30 in 6s, T = 30 = (30/6)*6 = 5*6. This shows, that an equation is solved by moving to the opposite side with opposite sign.

The totals 2 3s and 4 5s can add on-top as 3s or 5s, or next-to as 8s. Again, a calculator can predict the result: Entering (2*3+4*5)/8 gives 3.some and then (2*3+4*5) – 3*8 gives 2 so the prediction is T = 3.2 8s.

Also, the recount-formula can bridge units when double-counting has created a per-number as s4$/5kg. Here T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. Likewise with 18$.

References

Bauman, Z. (1990). *Thinking Sociologically.* Oxford, UK: Blackwell.

Mathematics as a Number-Language Grammar (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015).

The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra, to measure earth and to reunite numbers. Thus the outside goal of mathematics is to master Many.

Humans describe qualities and quantities with a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, 'This is a chair' and 'The total is 3x4 legs', abbreviated to 'T = 3x4'. Both are affected by the Heidegger (1962) warning: 'In is-sentences, trust the subject but question the predicate'.

Both languages also have a meta-language, a grammar, that describes the language that describes the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is a verb'. Likewise, the sentence 'T = 3x4' leads to a meta-sentence ''x' is an operation'.

We master outside phenomena through actions, so learning a word-language means learning actions as how to listen and read and write and speak. Likewise, learning the number-language means learning actions as how to count and add. We cannot learn how to math, since math is not an action word, it is a label, as is grammar. Thus, mathematics can be seen as the grammar of the number-language. Since grammar speaks about language, language should be taught and learned before grammar. This is the case with the word-language, but not with the number-language.

Saying 'the number-language is an application of mathematics' implies that 'of course mathematics must be taught and learned before it can be applied'. However, this corresponds to saying that the word-language is an application of its grammar that therefore must be taught and learned before it can be applied. Which, if implemented, would create widespread illiteracy, as with the present widespread innumeracy resulting from teaching grammar before language in the number-language.

Instead school should follow the word-language and use full sentences 'The total is 3 4s' or 'T = 3x4'. By saying '3x4' only, school removes both the subject and the verb from number-language sentence, thus committing a goal displacement.

References

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Deconstructing the Vocabulary of Mathematics (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra, to measure earth and to reunite numbers. Thus the outside goal of mathematics is to master Many.

Humans describe qualities and quantities with a word-language and a number-language, assigning words and numbers to things through sentences with a subject and a verb and an object or predicate, 'This is a chair' and 'The total is 3x4 legs', abbreviated to 'T = 3x4'.

Inspired by the Heidegger (1962) warning 'In is-sentences, trust the subject but question the predicate', Derrida (1991) to recommends deconstructing labels by destructing and reconstructing them inspired by the subject itself.

Thus Mathematics could be renamed to Many-matics, Many-math, Many-ology, or number-language. Geometry could be renamed to 'earth-measuring'; and algebra to 'reuniting numbers' according to its Arabic meaning.

Counting could split into its different forms: bundle-counting, using a cup for the bundles; re-counting to change the unit; and double-counting to bridge two units by a per-number.

In division, 'divided between 5' could be renamed to 'counted in 5s'; and 'to multiplied by 3' could be renamed 'to change the unit from 3s to tens' by reshaping the number block, widening the base and shorting the height.

Addition could split into on-top addition using proportionality to change the units, and next-to addition adding by areas as in integration.

Solving equations could be renamed to reversing calculations.

Fractions could be renamed to per-numbers coming from double-counting in the same unit.

Proportionality could be renamed changing units; and proportional could be renamed to 'the same except for units'.

Linear and exponential functions could be renamed change by adding and multiplying.

A function y = f(x) could be renamed to a formula or a 'number-language sentence'.

A root and a logarithm could be renamed to a factor-finder and a factor-counter.

Continuous could be renamed locally constant, and differentiable could be renamed locally linear

Integration could be renamed added by area.

In a right-angled triangle, the hypotenuse could be renamed the diagonal.

Finally, mathematical models could be named quantitative literature, having the same genres as qualitative literature, fact and fiction and fiddle (Tarp, 2017).

References

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Derrida, J. (1991). *A Derrida Reader: Between the Blinds.* Ed. P. Kamuf. New York: Columbia University Press

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Will Difference-Research Make a Difference? (TSG 06)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? Saying 'You are taught mathematics to learn mathematics', this meaningless self-reference creates a goal displacement preventing 50 years of research from solving the problems of mathematics education as witness by PISA studies (OECD, 2015). So maybe it is time for a different research approach, e.g. Difference-Research (Tarp, 2017).

Inspired by Greek sophists looking for hidden differences to unmask choice masked as nature, e.g. means presented as goals, difference-research asks two questions: 'Can this be different – and will the difference make a difference?' The philosophical background is the Heidegger warning 'In is-sentences, trust the subject but question the rest since it might be gossip.'

Looking for outside goals to inside means presented as goals, we see:

1. The tradition teaches cardinality as one-dimensional line-numbers to be added without being counted first. A difference is to teach counting before adding to allow proportionality and integral calculus and solving equations in early childhood: bundle-counting in icon-bundles less than ten, recounting in the same and in a different unit, recounting to and from tens, calculator prediction, and finally, forward and reversed on-top and next-to addition.

2. The tradition teaches the counting sequence as natural numbers. A difference is natural numbers with a unit and a decimal point or cup to separate inside bundles from outside singles; allowing a total to be written in three forms: normal, overload and underload: T = 5 = 2.1 2s = 2B1 2s = 1B3 2s = 3B-1 2s.

3. The tradition uses carrying. A difference is to use bundle-writing and recounting in the same unit to remove overloads: T = 7x 48 = 7x 4B8 = 28B56 = 33B6 = 336. Likewise with division: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

4. Traditionally, multiplication is learned by heart. A difference is to combine algebra and geometry by seeing 5x6 as a stack of 5 6s that recounted in tens increases its width and decreases its height to keep the total unchanged.

5. The tradition teaches proportionality abstractly. A difference is to introduce double-counting creating per-number 3$ per 4kg bridging the units by recounting the known number: T = 10kg = (10/4)*4kg = (10/4)*5$ = 12.5$. Double-counting in the same unit transforms per-numbers to fractions and percentages as 3$ per 4$ = ¾; and 75kg per 100kg = 75/100 = 75%.

References

Bauman, Z. (1990). *Thinking Sociologically.* Oxford, UK: Blackwell.

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Calculus in Primary and Middle and High School (TSG 07)

As a social institution, education has outside goals and inside means as pointed out by Baumann (1990), also warning against a goal displacement where a means becomes the goal instead. So means must be kept flexible by constantly asking: What is the outside goal to this inside means; and might a different means lead more to the goal? The ancient Greeks chose mathematics as a common label for arithmetic, geometry, music and astronomy, seen as knowledge about Many by itself, in space, in time and in time and space, resonating with the Greek and Arabic meaning of geometry and algebra: to measure earth and to reunite numbers. Thus, the outside goal of mathematics is to master Many.

Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3 = 2 3s, seeing that natural numbers are block-numbers with units.

Once counted, totals can be added, but addition is not well-defined: Two totals T1 = 2 3s and T2 = 4 5s may add on-top or next-to as 8s: T1 + T2 = 2 3s + 4 5s = 3.2 8s. Thus next-to addition means adding areas by combining multiplication and addition, called integration.

Reversing next-to addition, we ask e.g. 2 3s + ? 5s = 3 8s or T1 + ? 5s = T. To get the answer, first we remove the initial total T1, then we count the rest in 5s: u = (T–T1)/5. Combining subtraction and division in this way is called differentiation or reversed integration.

'Double-counting' a total in two physical units creates 'per-numbers' as 4$/5kg, or fractions as 4$/5$ = 4/5 if the units are the same. Per-numbers and fractions are not numbers, but operators needing a number to become a number: Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6 giving the total 8 kg at (3*4+5*6)/8 $/kg. Likewise with adding fractions. Adding by areas means that adding per-numbers and adding fractions become integration as when adding block-numbers next-to each other.

In high school calculus occurs when adding locally constant per-numbers, as 5seconds at 3m/s changing constantly to 4m/s. This means adding many strips under a per-number graph, made easy by writing the strips as differences since many differences add up to one single difference between the terminal and initial numbers, thus showing the relevance of differential calculus, and that integration should precede differentiation.

The epsilon-delta criterium is a straight forward way to formalize the three ways of constancy, globally and piecewise and locally, by saying that constancy means that the difference can be made arbitrarily small. (Tarp, 2013)

References

Bauman, Z. (1990). *Thinking Sociologically.* Oxford, UK: Blackwell.

Tarp, A. (2013). *Deconstructing Calculus.* Retrieved from: https://www.youtube.com/watch?v=yNrLk2nYfaY

Curing Math Dislike With 1 Cup and 5 Sticks (TSG 07)

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Meeting Many, we ask 'How many in Total?'. To answer, we count and add and answer with a number-language sentence, T = 2*3, containing a subject and a verb and a predicate as does word-language sentences, both affected by the Heidegger (1962) warning: 'In is-sentences, trust the subject but question the predicate'. However, by neglecting the subject and presenting the predicate as the goal, the tradition creates widespread dislike in math classes especially with division. To get the class back on track, the total must be reintroduced physically and in the sentence.

A class is stuck in division and gives up on 237/5. Having heard about '1cup & 5 sticks', the teacher says 'Time out. Next week, no division. Instead we do bundle-counting'. Teacher: 'How many sticks?' Class: '5.' Teacher: 'Correct, and how many 2s?' Class: '2 2s and 1 left over'. Teacher: 'Correct, we count by bundling. The cup is for bundles, so we put 2 inside the cup and leave 1 outside. With 1 inside, how many outside? And with 3 inside, how many outside?' Class: '1inside-3outside; and 3inside-less1outside.' Teacher: 'Correct. A total can be counted in 3 ways. The normal way with 2inside-1outside. With overload as 1inside-3outside. With underload as 3inside-less1outside.' Class: 'OK'. Teacher. 'Now 37 means 3inside-7outside if we count in tens. Try recounting 37 with overload and underload. Class: '2inside-17outside; and 4inside-less3outside.'

Teacher: 'Now let us multiply 37 by 2, how much inside and outside?' Class: 6inside-14outside. Or 7inside-4outside. Or 8inside-less6outside.'

Teacher: 'Now to divide 78 by 3 we recount 7inside-8outside to 6inside-18outside. Dividing by 3 we get 2inside-6outside or 26. With 79 we get 1 leftover that still must be divided by 3. So 79/3 gives 28 and 1/3.'

Class: 'So to divide 235 by 5 we recount 235 as 20 inside and 35 outside. Dividing by 5 we get 4 inside and 7 outside, or 47; With 237 we get 2 leftovers that still must be divided by 5. Thus 237/5 gives 47 and 2/5?'

Teacher: 'Precisely. Now let us go back to multiplication and division and use bundle-counting'.

References

Bauman, Z. (1990). *Thinking Sociologically.* Oxford, UK: Blackwell.

Heidegger, M. (1962). *Being and Time.* Oxford, UK: Blackwell.

Quantitative Literature Also has 3 Genres: Fact and Fiction and Fiddle (TSG 07)

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Humans communicate in languages: A word-language with sentences assigning words to things and actions; and a number-language with formulas assigning numbers or calculations to things and actions. ‘Word stories’ come in three genres: Fact, fiction and fiddle. Fact/fiction are stories about factual/fictional things and actions. Fiddle is nonsense like ‘This sentence is false’. ‘Number stories’ are often called mathematical models. They come in the same three genres.

Fact models can be called a ‘since-then’ models or ‘room’ models. Fact models quantify quantities and predict predictable quantities: “What is the area of the walls in this room?”. The model’s prediction is what is observed, so fact models can be trusted when units are checked. Algebra’s four basic uniting models are fact models: T = a+b, T = axb, T = a^b and T = ∫y dx; as are many models from basic science and economy.

Fiction models can be called ‘if-then’ models or ‘rate’ models. Fiction models quantify quantities but predict unpredictable quantities: “My debt is gone in 5 years at this rate!”. Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based upon alternative assumptions. Models from statistics calculating averages assuming variables to be constant are fiction models; as are models from economic theory showing nice demand and supply curves.

Fiddle models can be called ‘then-what’ models or ‘risk’ models. Fiddle models quantify qualities that cannot be quantified: “Is the risk of this road high enough to cost a bridge?” Fiddle models should be rejected asking for a word description instead of a number description. Many risk-models are fiddle models: The basic risk model says: Risk = Consequence x Probability. It has meaning in insurance but not when quantifying casualties where it is cheaper to stay in a cemetery than at a hospital.

References

Bauman, Z. (1990). *Thinking Sociologically.* Oxford, UK: Blackwell.

OECD. (2015). *Improving Schools in Sweden: An OECD Perspective.* Retrieved from: www.oecd.org/edu/school/improving-schools-in-sweden-an-oecd-perspective.htm.

20. Plenary PowerPointPresentation at the CTRAS 2017 July Conference in China

Difference-Research
Powering PISA Performance:
Count and Multiply before you Add

5 ~~Luther~~ Tarp Theses
II II ~~= 4~~ = 2 **2s**
3*5 ~~= 15~~ = 3 **5s**
8/4 ~~= 2 each~~ = 2 **4s**
2_w+3_d ~~= 5~~ = 17_d
1/2+2/3 ~~= 7/6~~ = 3/5

Curriculum Architect, Allan.Tarp@MATHeCADEMY.net
Teaches Teachers to Teach MatheMatics as ManyMatics, Tales of Many
a Heidegger-inspired VIRUS-Academy:
To learn, ask the subject, not the instructor
Full 31 page article: http://mathecademy.net/difference-research/

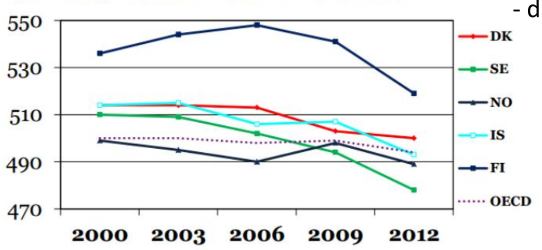
(Fractions ~~= numbers~~ = per-numbers = operators needing numbers to become numbers)

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 1

Poor Pisa Performance in Scandinavia

www.uvm.dk/~/media/UVM/Filer/Udd/Folke/PDF13/Dec/131203%20PISA%20Resultatnotat.pdf

Figur 2. Udvikling i matematikresultaterne i nordiske lande (2000-2012).

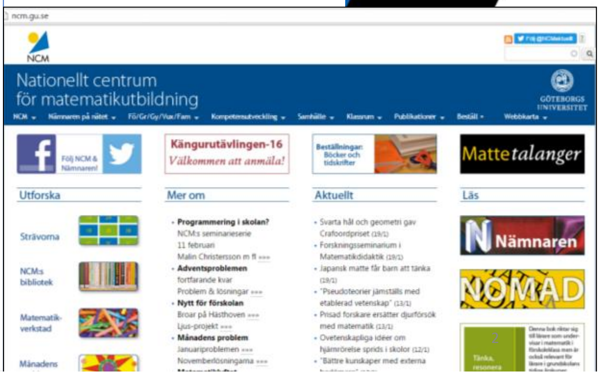


Ser man bort fra Finland (519 point), er Danmark det eneste af de nordiske lande, som er placeret i gruppen, der ligger signifikant over det internationale gennemsnit. Eleverne i Island (493 point) og i Norge (489 point) præsterer omkring gennemsnittet, mens den svenske score (478 point) er signifikant lavere end gennemsnittet. I tabel 1 nedenfor vises tallene bag figur 2.

Tabel 1. Gennemsnit for nordiske lande 2003-2012

	2003	2006	2009	2012	2012-2009	2012-2003
Finland	544	548	541	519	-22	-25
Danmark	514	513	503	500	-3	-14
Island	515	506	507	493	-14	-22
Norge	495	490	498	489	-9	-6
Sverige	509	502	494	478	-16	-31
OECD	500	498	499	494	-5	-6

All go down, Sweden especially
- despite increased research funding
Can Difference-Research
make a Difference
by finding a Difference?



Different Differences

Background

- Poor PISA Performance, witnessing 50 years of low-performing Math Education Research

10. Different Education

- Classroom: Half-Year Self-Chosen Blocks versus Multi-Year Forced Lines

20. Different Mathematics

- BottomUp Many-based Math from Below, versus TopDown Set-based Math from Above

30. Different Research

- Ancient Sophism, Renaissance Natural Science, (Post)Modern Existentialism

40. Different Math Education, showing the Beauty of the Simplicity of Math

- To master Many, Count & Multiply before you Add, Add next-to & on-top, and forwards & backwards

Powering PISA Performance - in a Nutshell

The Greek Sophists: Beware of choice masked as nature.

A Number-Language Sentence (a Tale of Many): **the Total is five, T = 5**

I I I I I T = 5 = 1Bundle3 **2s** = 2B1 **2s** = 3B-1 **2s,** or 1B2 **3s** = …

The **predicate** can be different (choice with alternatives)

The **subject** cannot be different (nature without alternatives)

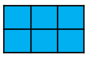
One Goal - many Means; Goal Displacement: When a Means becomes the Goal

Difference-Research unmasks Means masked as Goals, and says:

Use Full Sentences, if not, predicates becomes subjects and a means the goal

Difference-Research, Main Finding: The Simplicity of Math – Math as Tales of Many

Meeting Many we ask: 'How Many in Total'



- To answer, we math. *Oops, sorry, math is not an action word but a predicate.*
- Take II. To answer, we **Count & Add**. And report with Tales of Many (Number-Language sentences): T = 2 **3s** = 2*3

Three ways to Count: CupCount & ReCount & DoubleCount

- CupCount gives units. ReCount changes units. Double-count bridges units by per-numbers as 2$/3kg

Recount to & from tens gives Multiplication & Equations, coming before Addition

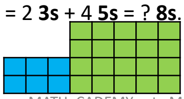
- To tens: T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens** . From tens: T = ? **7s** = u*7 = 42 = (42/7)*7 = 6 **7s** (ReCount-Formula)

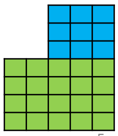
Counting gives variable or constant unit- or per-numbers, to be Added in 4 ways

- Addition & multiplication unites variable & constant unit-numbers.
- Integration & power unites variable & constant per-numbers.

Adding NextTo & OnTop roots Early Childhood Calculus & Proportionality

- EarlyChildhood-Calculus: T = 2 **3s** + 4 **5s** = ? **8s**. EarlyChildhood-Proportionality: T = 2 **3s** + 4 **5s** = ? **5s**





Education & Mathematics & Research

Education: a Social Institution

- In sociology, Bauman warns against ‘the danger of so-called *goal displacement*. The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right.’

Mathematics & Research: Truth claims

- In philosophy, Sartre says: ‘In existentialism, existence precedes essence.’
- In philosophy, Heidegger warns against true sentences with a subject & verb & predicate: ‘Trust the subject; but doubt the predicate, it could be different.’
- In counter-philosophy, the Greek sophists said: ‘Beware of choice masked as nature.’

Difference-Research asks 1 Question only: find a Difference that makes a Difference
- to **unmask** claimed goals, existence, subjects, nature as masked means, essence, predicates, choice.

Difference-Research, Main Recommendation: Visible and Tangible BUNDLES in Tales of Many

China: Educate Wans of DifferenceResearch Professors for the New SilkRoad & Africa

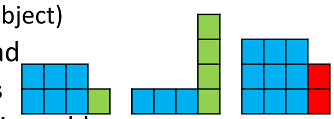
To improve PISA Performance, the Outsider (Child, Migrant) must touch & see & write the BUNDLE and use full number-language sentences in Tales of Many. (Bundles = units)

And must Count & Multiply before Adding.

- Several counting sequences:

T = I I I I I I I = 7 = B-3 (BUNDLE less 3) = ½B&2 (The Total is the goal, the subject)

- Recount in the same unit, **3s**, to create/remove over- or underload



T = 7 = I I I I I I I = III III I = 2B1 or T = III I I I I = 1B4 or T = III III III II = 3B-2 **3s**

Seeing T = 47 = 4B7 = 3B17 = 5B-3 makes a difference in multiplication tables:

T = 2*7 = 2*(½B&2) = B&4 = 14, or T = 2*7 = 2*(B-3) = 20-6 = 14

- A calculator predicts by the RecountFormula, where the operations (/, *, –) are icons for bundling & stacking & removing stacks to find unbundled: T = 7 = (7/3)*3 = 2B1 **3s**

7/3	2.some
7 – 2*3	1

Difference-Research, Main Warning: The 3x3 Goal Displacements in Math Education

Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
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High	Trigonometry	Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.
	Functions	Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate Instead: are introduced as set-relations where first-component identity implies second-component identity
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

11. Different Education

EU: Line-organized & Office-directed Schools

From secondary school, continental Europe uses **line-organized** education with forced classes and forced schedules making teenagers stay together in age groups even if girls are two years ahead in mental development.

The classroom belongs to the class. This forces teachers to change room and (in lower secondary school) to teach several subjects outside their training.

Tertiary education is also **line-organized** preparing for offices in the public or private sector. This makes it difficult to change line in the case of unemploy-ment, and it forces the youth to stay in education until close to 30 making reproduction fall to 1.5 child/family, causing the European population to die out very quickly by decreasing it to 25% in 100 years.

12. Different Education

US: Block-organized & talent-directed

Alternatively, North America uses **block-organized** education saying to teenagers:

“Welcome, inside you carry a **talent**! Together we will uncover and develop your personal talent through daily lessons in self-chosen half-year blocks, academical or practical, together with 1subject teachers. If successful the school will say ‘**good job**, you have a **talent**, you need some more’. If not, the school will say ‘**good try**, you have **courage** to try out the unknown, now try something new’”.

The classroom belongs to the teacher teaching one subject only.

Likewise, college is **block-organized** easy to supplement with additional blocks in the case of unemployment.

At the age of 25, most students have an education, a job and a family with three children, 1 for mother, 1 for father, and 1 for the state to secure reproduction.

20. Different Mathematics

The Beauty of the Simplicity of Mathematics

21. The Goal & Means of Mathematics Education

22. Totals as Blocks. Digits as Icons. Operations as CupCounting Icons

23. ReCounting gives Proportionality & Multiplication & Equations

24. Multiplication tables simplified by ReCounting

25. DoubleCounting in different & same units creates PerNumbers & Fractions

26. Geometry: Counting Earth in HalfBlocks

27. Once Counted, Totals can be Added. But counting and double-counting gives 4 number-types (constant & variable unit-numbers & per-numbers) to add in 4 ways

28. How Different is the Difference? Set-based versus Many-based Mathematics

21. Different Mathematics

The Goal and Means of Mathematics Education

The Set-based Top-Down Tradition:

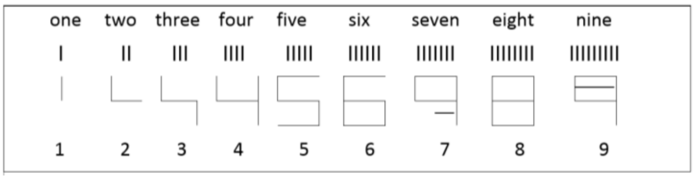
- Mathematics exists as a collection of well-proven statements about well-defined concepts, all derived from the mother concept SET
- Mathematics is surprisingly useful to modern society
- Consequently, mathematics must be taught and learned

The Many-based Bottom-Up Difference:

- Many exists; to master Many we develop a number-language with Tales of Many, a 'ManyMatics'.
- **Many-matics**, defining concepts from below as **abstractions from examples**, is a more successful means to the goal of mastering Many than
- '**Meta-matics**' defining concepts from above as **examples from abstractions**

22a. Different Mathematics

Digits as Icons. Totals as Blocks to be Cup-Counted



Icon-numbers. A folding ruler shows: digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent. Thus, there are four sticks in the 4-icon, etc.

Counting-sequences. A total of a dozen sticks counted in **5s** gives different counting sequences:

'1, 2, 3, 4, Bundle, 1B1, …, 2 Bundles, 2B1, 2B2', or

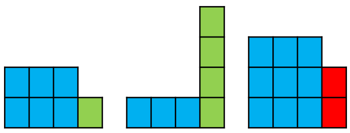
'01, 02, 03, 04, 10, 11, …, 22' , or '.1, .2, .3, .4, 1., 1.1, …, 2.2' , or

'1, 2, Bundle less 2, B-1, Bundle, B&1, B&2, 2B-2, 2B-1, 2Bundles, 2B&1, 2B&2.'

Cup-Counting. With a cup for the bundles, a total can be 'cup-counted' with inside bundles & outside singles in 3 ways: normal, with Overload or with Underload: T = 7 = 2]1 **3s** = 1]4 **3s** = 3]-2 **3s**

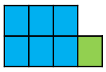
Or, when counting in tens :

T = 37 = 3]7 **tens** = 2]17 **tens** = 4]-3 **tens**



22b. Different Mathematics

Operations as CupCounting Icons



We count by bundling and stacking: T = I I I I I I I = III III I = [III / III] I = 2B1 **3s** = 2]1 **3s** = 2.1 **3s**

Thus, to count 7 in **3s** we take away 3 many times, iconized by an uphill stroke showing the broom wiping away the 3s. With 7/3 = 2.some, the calculator predicts that 3 can be taken away 2 times.

To stack the 2 **3s** we use multiplication, iconizing a lift, 2x3 or 2*3, transforming the bundles into a stack.

To look for unbundled singles, we drag away the stack of 2 **3s** iconized by a horizontal trace: 7 – 2*3 = 1.

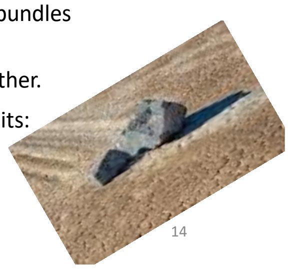
The prediction 'T = 7 = 2 **3s** & 1 = 2B1 **3s** = 2]1 **3s**' provides the

7/3	2.some
7 – 2*3	1

ReCount-formula: T = (T/B)*B saying 'from T, T/B times, B can be taken away'.

To also bundle bundles, power is iconized as a cap, e.g. 5^2, indicating the number of times bundles themselves have been bundled.

Finally, addition is a cross showing that blocks can be juxtaposed next-to or on-top of each other.



Counting thus provides the number-formula called a polynomial, where all numbers have units:

T = 456 = 4*BundleBundle + 5*Bundle + 6*1 = 4*B^2 + 5*B + 6*1

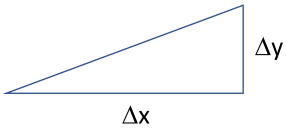
So counting creates 3 operations: to divide & to multiply & to subtract.

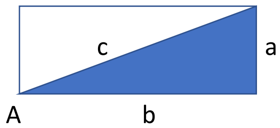
22c. Different Mathematics

The ReCount Formula is all over Mathematics

ReCount-formula: T = (T/B)*B saying 'from T, T/B times, B can be taken away'

ReCounting	T = (T/B)*B	8 = (8/2)*2 = 4*2 = 4 **2s**	
Proportionality	\$ = (\$/kg)*kg	\$ = p*kg	= price*kg, Economy
Coordinate Geometry	$\Delta y = (\Delta y/\Delta x)*\Delta x$	$\Delta y = m*\Delta x$	= slope*Δx
Differential Calculus	dy = (dy/dx)*dx	dy = y'*dx	= gradient*dx
Trigonometry	a = (a/c)*c; a = (a/b)*b	a = sinA*c; a = tanA*b	
Linearity	y = k*x	F = m*a, dist = vel*time, ..	Physics
Eigenvalues	Hψ = E*ψ	Schroedinger Equation in	Quantum mechanics





23. Different Mathematics

ReCounting gives
Proportionality & Multiplication & Equations

ReCounting in the same unit creates overloads & underloads

- T = IIIIIII = III III I = 2]1 **3s** = 1]4 **3s** (Overload III I I I I) = 3]-2 **3s** (Underload III III III II)

ReCounting in different units means changing units (**Proportionality**)

- T = 4 **5s** = ? **6s**. Calculator predicts with ReCount-formula T = (T/B)*B, T = 3]2 **6s**

4*5/6	3.some
4*5 – 3*6	2

ReCounting from icons to tens gives **Multiplication** :

- T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens**, predicted by multiplication

ReCounting from tens to icons gives **Equations** :

- T = ? **7s** = u*7 = 42 = (42/7)*7 = 6 **7s** with solution

u*7	= 42 = (42/7)*7
u	= 42/7 = 6

An equation is solved by moving to opposite side with opposite sign

24. Different Mathematics

Multiplication Tables Simplified by ReCounting

Geometry: Multiplication means that, recounted in tens, a block increases its width and therefore decreases its height to keep the total unchanged.

Thus T = 3*7 means 3 **7s** that may be recounted in tens as T = 2.1 **tens** = 21.

Algebra: The full ten-by-ten table can be reduced to a small 2-by-2 table containing doubling and tripling, using that 4 is doubling twice, 5 is ½Bundle, 6 is 5&1 or Bundle less 4, 7 is 5&2 or Bundle less 3, etc.

Beginning with doubling and halving visualized by CentiCubes

- T = 2 **6s** = 2*6 = 2*(½B&1) = B&2 = 12, or
- T = 2 **6s** = 2*6 = 2*(B-4) = 20-8 = 12.
  - T = 5 **7s** = 5*7 = 5*(B-3) = 5B – 15 = 50 – 15 = 35
  - T = 8 **7s** = 8*7 = (B-2)*(B-3) = BB – 2B – 3B + 6 = 100 – 20 – 30 + 6 = 56

25. Different Mathematics

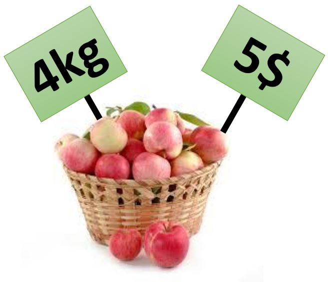
DoubleCounting in 2 units creates PerNumbers (Proportionality)
DoubleCounting in the same unit creates Fractions

Apples are double-counted in kg and in $.

With **4kg = 5$** we have 4kg/5$ = 4/5 kg/$ = a per-number

Questions: *4kg/100kg = 4/100 = 4%*

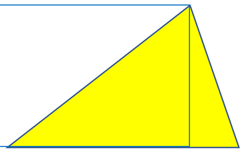
7kg = ?$	**8$ = ?kg**
7kg = (7/4)*4kg = (7/4)*5$ = 8.75$	8$ = (8/5)*5$ = (8/5)*4kg = 6.4kg



Answer: *Recount in the per-number*

26. Different Mathematics

Geometry: Counting Earth in HalfBlocks



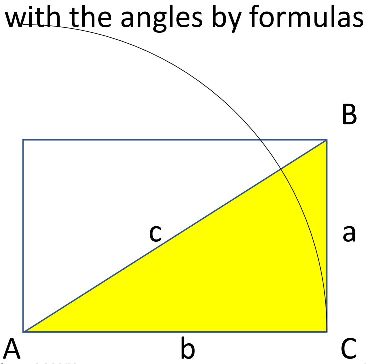
Geometry means to count earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras theorem. And connected with the angles by formulas recounting the sides in sides or in the diagonal:

A+B+C = 180 and a*a + b*b = c*c

a = (a/c)*c = sinA*c

a = (a/b)*b = tanA*b

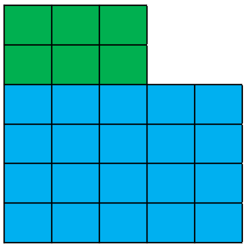
Circle: circum./diam. = **π = n*tan(180/n)** for n large

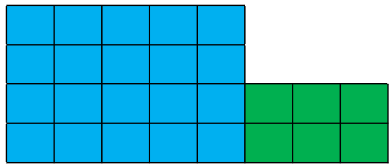


27a. Different Mathematics

Once Counted & ReCounted, Totals can be Added

OnTop	**NextTo**
2 **3s** + 4 **5s** = 1]1 **5s** + 4 **5s** = 5]1 **5s**	2 **3s** + 4 **5s** = 3]2 **8s**
The units are changed to be the same. *Change unit = Proportionality*	The areas are added. *Adding areas = Integration*





27b. Different Mathematics

Adding PerNumbers as Areas (Integration)

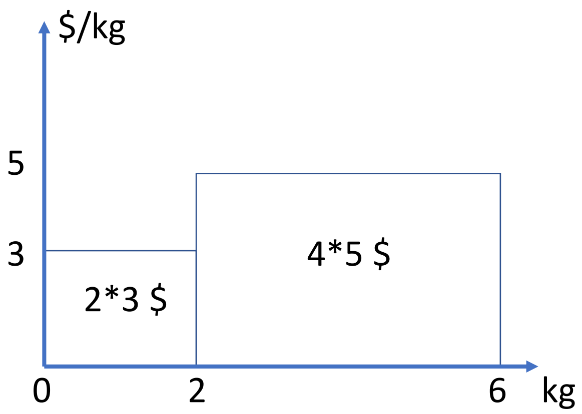
2 kg at 3 $/kg

+ 4 kg at 5 $/kg

(2+4)kg at (2*3 + 4*5)/(2+4) $/kg

Unit-numbers add on-top.

Per-numbers add next-to as **areas** under the per-number graph.



27c. Different Mathematics

With 2x2 different number-types we Add in 4 ways

Counting produces variable or constant **unit-numbers** or **per-numbers**

- Addition & Multiplication unites variable & constant unit-numbers
  - Subtraction & division splits into variable & constant unit-numbers
- Integration & Power unites variable & constant per-numbers
  - Differentiation & root/logarithm splits into variable & constant unit-numbers

Operations **unite** / *split into*	Variable	Constant
Unit-numbers *m, s, \$, kg*	**T = a + n** *T – a = n*	**T = a*n** *T/n = a*
Per-numbers *m/s, \$/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $log_a T = n$, $\sqrt[n]{T} = a$

28a. Different Mathematics

How Different is the Difference?

Set-based Math versus Many-based Math

	SET-based Tradition	Many-based Difference
Goal/Means	Learn Mathematics / Teach Mathematics	Learn to master Many / Math as Tales of Many
Digits	Symbols as letters	Icons with as many sticks as they represent
Numbers	Place-value number line names. Never with units	A union of blocks of stacked singles, bundles, bundle-bundles etc. Always with units
Number-types	Four types: Natural, Integers, Rational, Real	Positive and negative decimal numbers with units
Operations	Mapping from a set-product to the set	Counting-icons: /,*,-,+ (bundle, stack, remove, unite)
Order	Addition, subtraction, multiplication, division	The opposite
Fractions	Rational numbers, add without units	Per-numbers, not numbers but operators needing a number to become a number, so added by integration
Equations	Statement about equvalent number-names	Recounting from tens to icons, reversing operations
Functions	Mappings between sets	Number-language sentences with a subject, a verb and a predicate
Proportionality	A linear function	A name for double-counting to different units
Calculus	Differential before integral (anti-differentiation)	Integration adds locally constant per-numbers.

28b. Different Mathematics

Main Parts of a ManyMath Curriculum

Primary School – respecting and developing the Child's own 2D NumberLanguage

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- CupCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: / x - +

Middle school – integrating algebra and geometry, the content of the label math

- DoubleCounting produces PerNumbers as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always so length becomes change and vv.

High School – integrating algebra and geometry to master CHANGE

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

31. Different Research

Ancient Greece: Sophist vs. Philo-Sophists

Difference research began with the Greek controversy between two attitudes towards knowledge, called 'sophy' in Greek. To avoid hidden patronization, the sophists warned: 'Know the difference between nature and choice to uncover choice presented as nature.'

To their counterpart, the philosophers, choice was an illusion since the physical was but examples of metaphysical forms only visible to them, educated at the Plato academy.

The Christian church transformed the academies into monasteries but kept the idea of a metaphysical patronization by replacing the forms with a Lord using an unpredictable will to choose world behavior.

32. Different Research

Renaissance Natural Science

Background: Viking descendants in UK know how to sail, how to steal Spanish silver, how to follow the moon to go to India on open sea to buy silk and pepper:

How does the moon move?

Tradition: Between the stars. Newton: No, falling

Why does moons and apples fall?

Tradition : Following an metaphysical unpredictable will. Newton: No, a physical will predictable, following formulas.

What is the effect of a will or force

Tradition : Aristotle: a force maintains order . Newton: No, a force changes order.

How to use formulas?

Tradition : Arabic algebra. Newton: No, different algebra about change, Calculus

33. Different Research

Enlightenment Century 1700-1800

Newton’s physical will inspired the Enlightenment century (Locke) with its two republics

The US: Skepticism towards philosophy, US pragmatism, Symbolic Interactionism, Grounded Theory, Action Learning & Research

The French 5th : post-structuralism inspired by German thinking:

- Counter-enlightenment: Hegel’s metaphysical Spirit, the basis for Marxism and EU line-organized office-directed Bildung-education
- Existentialism: (Kierkegaard), Nietzsche, Heidegger, (Sartre: In Existentialism, existence precedes essence)

34. Different Research

French Post-Structuralism

Inspired by Heidegger's: 'In sentences, trust the subject & doubt the predicate'

- Derrida: Words can be different (DeConstruction)
- Lyotard: Truth can be different (PostModern skepticism towards meta-narratives)
- Foucault: Diagnoses can be different, Curing institutions also (a school is really a 'pris-pital' mixing power techniques from a prison and a hospital by fixing and diagnosing students at the same time)
- Bourdieu: Education can be different, and stop using symbolic violence and mathematics especially to create outsiders accepting power be given to a new knowledge-nobility

35. Different Research

Difference-Research finds Differences making a Difference

Difference-Research, inspired by its historical roots,

- Questions traditional words & truths & institutions
- Designs different micro-curricula & macro-curricula
- Reports if a difference makes a difference

<u>Examples</u>

Micro-curricula: MATHeCADEMY.net with YouTube/YouKu videos (MrAlTarp/DrAlTarp)

Macro-curriculum: 'The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Outsiders and Migrants', http://mathecademy.net/stem-based-core-math-for-migrants/

36. Different Research

Difference-Research: For whom?

- For teachers observing problems in the classroom
- For teacher-researchers splitting their time between academic work at a university and intervention research in a classroom.
- For full-time researchers cooperating with teachers both using difference-research, the teacher to observe problems, the researcher to identify differences, together working out a different micro-curriculum, to be tested by the teacher, and reported by the researcher conducting a pretest-posttest study.
- Difference-research begins by observing learning problems and wondering if we could teach differently, e.g. a child saying ‘II II, that is not 4, but 2 **2s**’, showing that children bring 2dimensional block-numbers to school where 1dimensional cardinal line-numbers then are forced upon them.

Conclusion

The 3x3 Goal Displacements in Math Education

Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
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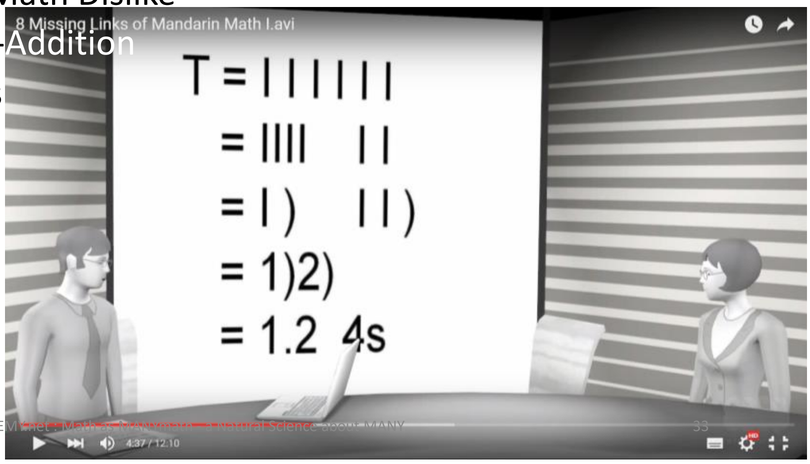
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

ManyMath is Different
But does it make a Difference? Try it out.

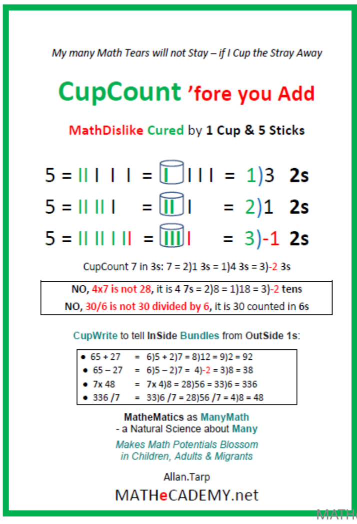
- Watch some YouTube or YouKu videos (MrAlTarp/DrAlTarp)
- Try the **CupCount before you Add** Booklet
- Try a 1day free Skype seminar **How to Cure Math Dislike**
- Try Action Learning and Action Research, e.g. **1Cup, 5Sticks**
- Collect data and Report on its 8 **MicroCurricula**, M1-M8
- Try a 1year online InService TeacherTraining at the MATHeCADEMY.net using PYRAMIDeDUCATION to teach teachers to teach MatheMatics as **ManyMatics**, a Natural Science about the root of mathematics, **Many**

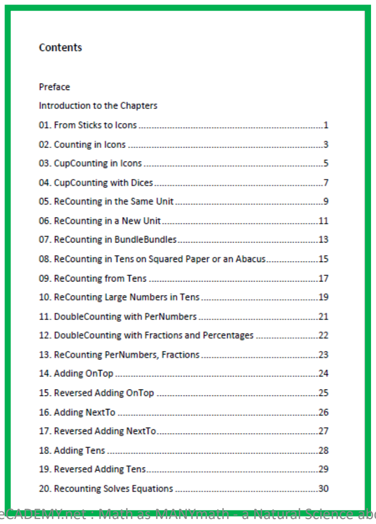
Some MrAlTarp YouTube Videos
Screens & Scripts on MATHeCADEMY.net

- Postmodern Mathematics Debate
- CupCounting removes Math Dislike
- IconCounting & NextTo-Addition
- PreSchool Mathematics
- Fractions
- PreCalculus
- Calculus
- Mandarin Mathematics
- World History



CupCount 'fore you Add Booklet, free to Download

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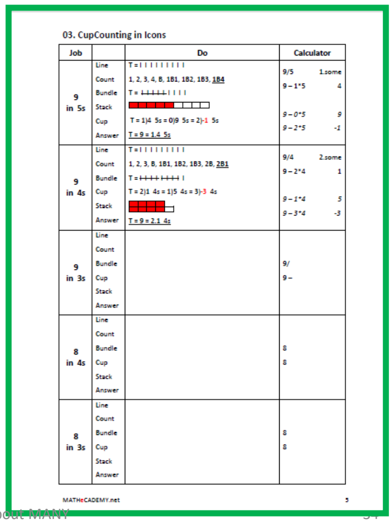

1day free Skype Seminar:
To Cure Math Dislike, CupCount before you Add

Action Learning based on the Child's own 2D NumberLanguage

09-11. Listen and Discuss the PowerPointPresentation
<u>To Cure MathDislike, replace MetaMatism with ManyMath</u>

- **MetaMatism** = MetaMatics + MatheMatism
- **MetaMatics** presents a concept TopDown as an example instead of BottomUp as an abstraction
- **MatheMatism** is true inside but rarely outside classrooms
- **ManyMath**, a natural science about Many mastering Many by CupCounting & Adding NextTo and OnTop.

11-13. Skype Conference. Lunch.

13-15. Do: Try out the <u>CupCount before you Add</u> booklet to experience proportionality & calculus & solving equations as golden LearningOpportunities in CupCounting & NextTo Addition.

15-16. Coffee. Skype Conference.

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A Primary School Test Curriculum, before
Math Dislike CURED by 1 Cup & 5 Sticks

Having problems in a division class, the teacher says: “Timeout, class. Next week no division, instead we take a field trip back to day 1 to learn CupCounting”

Let’s recount 5 in 2s by bundling, using a cup for the bundles:

5 = II I I I = [I] I I I = 1)3 **2s** = 1 Bundle & 3 **2s** *overload*

5 = II II I = [II] I = 2)1 **2s** = 2 Bundles & 1 **2s** *normal*

5 = II II II = [III] I = 3)-1 **2s** = 3 Bundles less 1 **2s** *underload*

Now we know that numbers can be ReCounted in 3 ways:
Normal, overload or underload if we move a stick **OUTSIDE** or **INSIDE**.
Now CupCount 7 in **3s**:
7 = I I I I I I I = 2)1 **3s** = 1)4 **3s** = 3)-2 **3s**

336**/7**
= 33)6 **/7**
= 28)56 **/7** = 4)8
☺ ✌ ⚐

A Primary School Test Curriculum, after
Math Dislike CURED by 1 Cup & 5 Sticks

When counting in TENS, before calculating, we cup-write the number to separate the **INSIDE** bundles from the **OUTSIDE** singles. Later we recount.

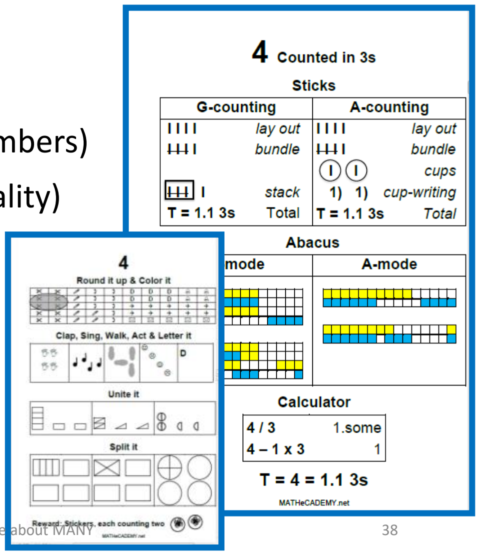
- **65 + 27** = 6)5 + 2)7 = 8)12 = 9)2 = **92**
- **65 – 27** = 6)5 – 2)7 = 4)-2 = 3)8 = **38**
- **7x 48** = 7x 4)8 = 28)56 = 33)6 = **336**
- **336 /7** = 33)6 /7 = 28)56 /7 = 4)8 = **48**

*With 336 we have 33 **INSIDE**, so to get 28, so we move 5 **OUTSIDE** as 50.*
Now try 456 / 7.

- **456 /7** = 45)6 /7 = 42)36 /7 = 6)5 + 1 = **65 1/7**

8 MicroCurricula for Action Learning & Research

C1. Create Icons
C2. Count in Icons (Rational Numbers)
C3. ReCount in the Same Icon (Negative Numbers)
C4. ReCount in a Different Icon (Proportionality)
A1. Add OnTop (Proportionality)
A2. Add NextTo (Integrate)
A3. Reverse Adding OnTop (Solve Equations)
A4. Reverse Adding NextTo (Differentiate)



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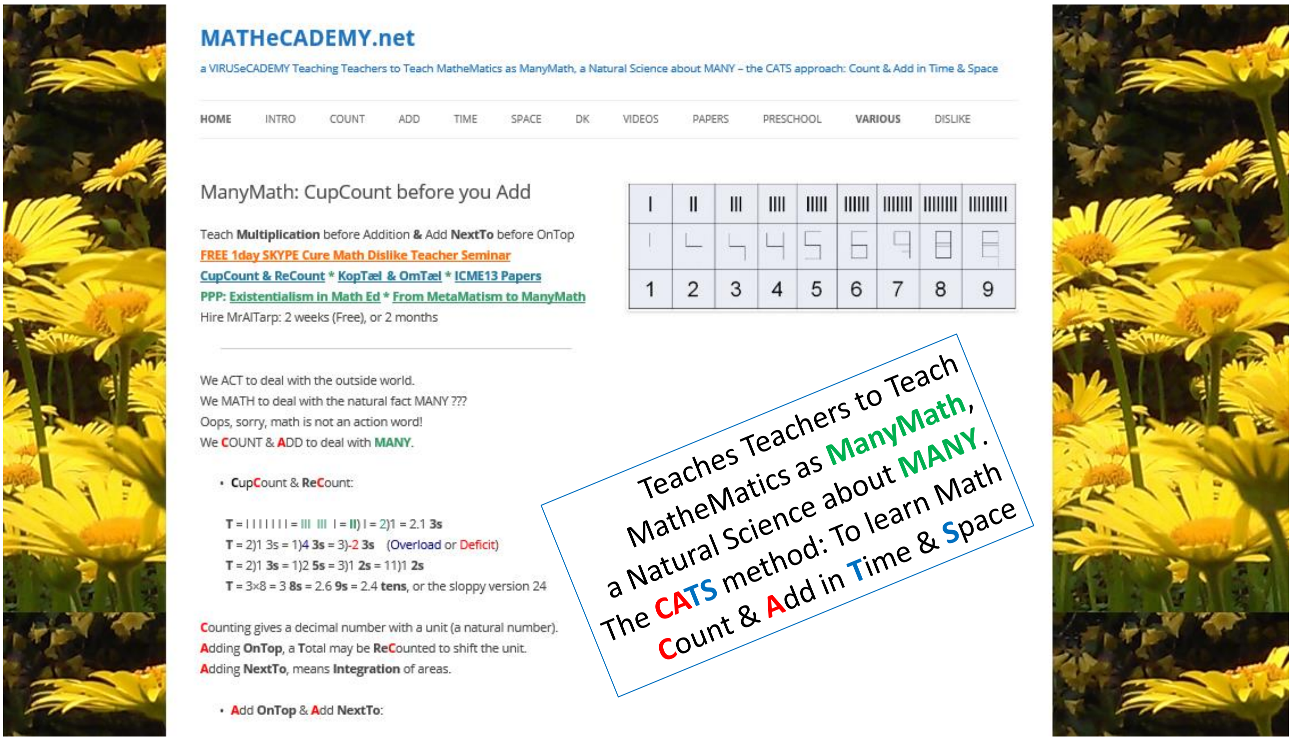
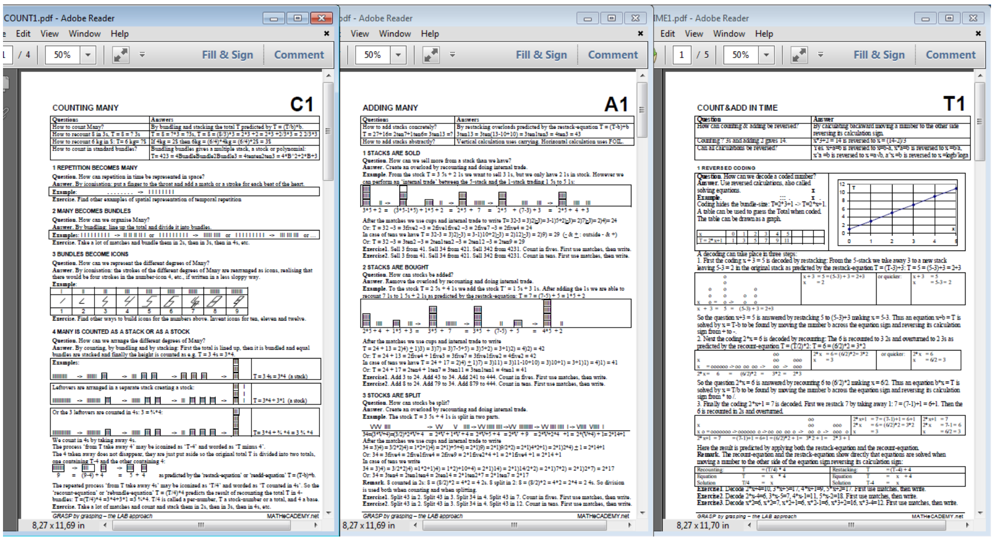


Teacher Training in CATS ManyMath
Count & Add in Time & Space



40

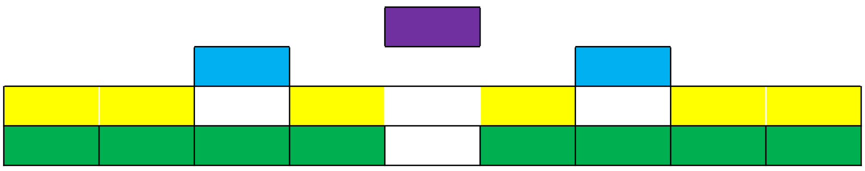
PYRAMIDeDUCATION

To learn MATH: Count&Add MANY
Always ask Many, not the Instructor
MATHeCADEMY.net - a VIRUSeCADEMY

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

- Each pair works together to solve Count&Add problems.
- The coach assists the instructors when instructing their team and when correcting the Count&Add assignments.
- Each teacher pays by coaching a new group of 8 teachers.

1 Coach
2 Instructors
3 Pairs
2 Teams



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Main Main Point: Respect the Child's own 2D BlockNumbers
allowing ReCounting & Multiplying before Adding

Difference-Research
Powering PISA Performance:
Count and Multiply before you Add
Think Things

IIII III III III = T = 4 **3s** = 1 **dozen** = 1.2 **tens** = 12 = twelve = 'two left' in Wiking Danish

This talk has been in Anglish, a dialect from the Wiking area on the Danish WestCost

Full 31 page article: http://mathecademy.net/difference-research/

Thank you for listening (Tak do for lytningen)

Allan.Tarp@MATHeCADEMY.net, Denmark

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21. Math Dislike

CURED

by 1 Cup & 5 Sticks

My Many Math Tears will not Stay – if I Cup the Stray Away

CupCOUNT before you ADD

5 = II I I I = I I I I = 1]3 2s, 5 = II II I

= II I = 2]1 2s

5 = II II I II = III I = 3]-1 2s

3 ways to CupCount: Overload, Normal, Underload

ReCount 7 in **3s**: 7 = 2]1 **3s** = 1]4 **3s** = 3]-2 **3s**

NO, 4x7 is not 28, it is 4 **7s** = 2]8 = 1]18 = 3]-2 **tens**
NO, 30/6 is not 30 divided by 6, it is **3tens** counted in **6s**

CupWriting tells **InSide Bundles** from **OutSide 1s**

• **65 + 27**	=	6]5 + 2]7 = 8]12 = 9]2 =	**92**
• **65 – 27**	=	6]5 – 2]7 = 4]-2 = 3]8 =	**38**
• **7x 48**	=	7x 4]8 = 28]56 = 33]6 =	**336**
• **336 /7**	=	33]6 /7 = 28]56 /7 = 4]8 =	**48**

MatheMatics as ManyMath - a Natural Science about Many

Makes Math Potentials Blossom in Children, Adults & Migrants

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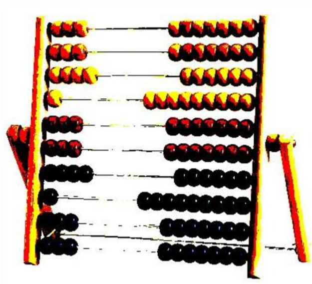
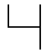
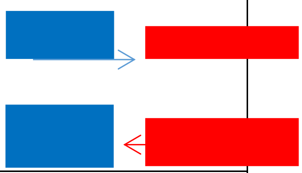
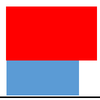


CupCount 'fore you Add

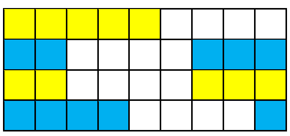
MatheMatics as **Many**Math

a Natural Science about **MANY**

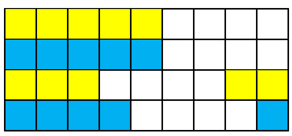
MATH**e**CADEMY.net Cure **Math Dislike**: Use Children's own 2D Numbers with Units

Count In *Icons* In *BundleCups*	T = I I I I =  = 4 T = 7 = I I I I I I I = III III I = 2]1 **3s** = 2 **Bundles** & 1 **3s**
ReCount In same Unit In new Unit	**ReBundle** to create Overload & Underload T = 7 = I I I I I I I = 2]1 **3s** = 1]4 **3s** = 3]-2 **3s** T = 2]1 **3s** = 1]3 **4s** = 1]2 **5s** = 3]1 **2s** = 1]1]1 **2s** = 11]1 **2s**
ReCount In Tens From Tens	3 **7s** = ? **tens** Answer: 3x7 = 21 = 2]1 **tens** ? **7s** = 3 **tens** Answer: (30/7)x7 = 4]2 **7s** 
DoubleCount in *PerNumbers* in *PerFive, 3/5* in *PerHundred, %*	With 4$ per 5kg, T = 20kg = (20/5) x 5kg = (20/5) x 4$ = 16$ 3 per 5 of 200$ = ?$. 200$ = (200/5) x 5$ gives (200/5) x 3$ = 120$ 70% of 300$ = ?$. 300$ = (300/100) x 100$ gives (300/100) x 70$ = 210$
Calculator Prediction RecountFormula	T = 2 **4s** = ? **5s** = 1]3 **5s** since 2x4/5 1.some; 2x4 – 1x5 3 T = (T/B) x B = T/B **Bs**
Add NextTo OnTop	T = 2 **3s** + 4 **5s** = 3]2 **8s**  *Integration* T = 2 **3s** + 4 **5s** = 1]1 **5s** + 4 **5s** = 5]1 **5s**  *Proportionality*
Multiply, Divide Use *CupWriting*	**7 x** 463 = **7 x** 4]6]3 = 28]42]21 = 28]44]1 = 32]4]1 = 3241 3241 **/7** = 32]4]1 **/7** = 28]44]1 **/7** = 28]42]21 **/7** = 4]6]3 = 463

T = 7 = 2]1 **3s** on an **Abacus**:



Geometry-mode



Algebra-mode

MrAlTarp
YouTube Videos

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MATH**e**CADEMY.net **FREE** 1day Skype Seminar
*Teaching Teachers to Teach MatheMatics as **Many**Math*

Piaget: Grasping with Fingers leads to Grasping Mentally

Four as an icon built by four cars, four rhinos, four sticks, a ruler folded in four parts, four beads on an abacus, LEGO blocks, pearls on a pearl board, etc.

Seven sticks bundle-counted as 1]2 5s, or as 2]1 3s or as 3]1 2s

The MATHeCADEMY.net stand at the MatematikBiennale in Sweden, 2014

Migrant Math: Core Math for Late Beginners

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Mathematics as ManyMath

a Natural Science about Many

Preface

”How old will you be next time?” I asked the child. “Four”, he answered and showed me four fingers. “Four, you said?” I asked and showed him four fingers held together two by two. “No, that is not four, that is two twos!” the child replied. That opened my eyes. Children come to school with two-dimensional block numbers where all numbers have units. However, the school does not allow the children to count the numbers before being added. Instead the school teaches cardinality as a one-dimensional number line where numbers have different names; thus disregarding the fact that numbers are two dimensional blocks where all numbers have a unit as shown when writing out fully

T = 345 = 3 BundleBundles + 4 Bundles + 5 Singles = 3*10^2 + 4*10 + 5*1.

This booklet allows schools and parents to choose an education that develops the 2D number blocks that the children bring to school instead of forcing a 1D number line upon them. Also, the booklet allows the children to practice ‘counting before adding’ and to include bundle-counting and re-counting to different units. The booklet thus is an answer to the question ‘How to Save and Develop a Child’s Math Potential?’

To master Many we ask ‘how many?’ To answer, we count by bundling and stacking to get a total T. Once counted, first a total can be recounted in the same unit to create overload or underload, or to create a different unit; next totals can be added NextTo, or OnTop if the units are the same.

Counting a total T of 7 ones in 3s we get the result T = 7 = 2 **3s** & 1 = 2B1 **3s** where B means Bundles.

We separate the *inside* bundles from the *outside* unbundled singles by a *cup* becoming a bracket when reporting the result with *bundle-writing*: T = III III I = II B I = 2B1 **3s**

Once counted, a total can be *recounted* to create *overload* or *underload*, deficit. To create an overload, we move a stick from the inside to the outside: T = II B I = I B III I = 1B4 **3s**.

To create an underload, we borrow foreign sticks to move a bundle from the outside to the inside

T = II B I = II B I II II = III B II = 3B-2 **3s**.

Thus a given total can be *recounted* in three ways: normal, with overload and with underload.

T = 7 = 2B1 **3s** = 1B4 **3s** = 3B-2 **3s**.

A total of 68 can be recounted in four different ways as T = 68 = 6B8 **tens** = 5B18 **tens** = 7B-2 **tens**.

Recounting and bundle-writing come in handy when we add, subtract, multiply or divide numbers:

Using bundle-writing to add 65 and 27 we get an overload outside the bundle cup allowing us to move 10 **1s** from the outside to the inside as 1 **tens**

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

Using bundle-writing to subtract 27 from 65 we get an underload outside the bundle cup allowing us to move a bundle of 1 **tens** from the inside to the outside as 10 **1s** to remove the underload.

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

Alternatively, before subtracting we can create an overload outside by moving 1 **tens** from the inside to the outside as 10 **1s**

T = 65 – 27 = 6B5 – 2B7 = 5B15 – 2B7 = 3B8 = 38

Using bundle-writing to multiply 48 with 7 we get an overload outside the bundle cup allowing us to move 50 **1s** from the outside to the inside as 5 **tens**

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

Alternatively, before multiplying we can create an underload outside by borrowing 2 **1s**. Later the underload can be removed by moving 2 **tens** outside as 20 **1s**

T = 7* 48 = 7* 4B8 = 7* 5B-2 = 35B-14 = 33B6 = 336

Using bundle-writing to divide 336 with 7 we prefer to have 28 instead of 33 inside the bundle cup, so we create an overload outside by moving 5 bundles outside as 50 **1s**

T = 336 = 33B6 = 28B56; so T /7 = 4B8 = 48

Alternatively, we can create an underload outside before dividing

T = 336 = 33B6 = 35B-14; so T /7 = 5B-2 = 4B8 = 48

To divide 338 by 7 we get 2 single leftovers that counted in 7s becomes a fraction 2/7

T = 338 = 33B8 = 28B58 = 28B56 + 2; so T /7 = 4B8 + 2/7 = 48 2/7

Introduction to the Chapters

Chapter 01, From Sticks to Icons, shows how rearranging four sticks creates a 4-icon with as many sticks as it represents; likewise with the other icons until ten having a name but no icon.

Chapter 02, Counting-sequences in Icons, shows that when counting by bundling, the bundle-icon is not used. Hence, when counting in tens, ten does not need an icon. A natural counting sequence will report both the bundles and the unbundled: 01, 02, …, 10, 11; or 0.1, 0.2, …, 1.0, 1.1 always including the bundle-name as the unit. Each bundle-size has its own counting sequence, but the standard is ten-counting in a sloppy version leaving out the unit and misplacing the decimal point by saying 23 instead of 2.3 tens.

Chapter 03, BundleCount in Icons, shows how a total T can be recounted in icon-bundles. Thus a total of nine things, represented by a line of sticks or beads on an abacus, can be counted in fours by a counting sequence. Also, they can be represented by a stack of bundles placed with one stick per bundle in a bundle cup that can written as a bracket (bundle-writing) and reported as a decimal number with a unit where the decimal point separates the bundles from the unbundled singles, T = 9 = 2B1 4s = 2.1 4s. Alternatively, a calculator can be asked to predict the counting result. Entering '9/4', we ask 'from 9, taking away 4s can be done how many times?' The calculator answers '2.some' so by entering '9 – 2x4' we ask 'from 9, once taking away 2 4s leaves what?' The answer '1' gives the calculator prediction T = 9 = 2.1 4s. Thus also operations are icons: /4 shows the broom wiping away 4 many times, – 4 shows the trace left when dragging away 4 only once, 2x shows the lifting needed to create a stack of 2 bundles, and +3 shows the juxtaposition of 3 singles left next to a stack of bundles. Moving 1 stick outside the bundle cup creates an overload T = 1B5 4s; and moving an extra stick in gives an underload, a deficit, T = 3B-3 4s. Thus by recounting, a total T of nine can be recounted in 4 different ways: T = nine = 9 1s = 2B1 4s = 1B5 4s = 3B-3 4s. This comes in handy when totals are added, subtracted, multiplied or divided. A good calculator says 2+3*4 = 14; a bad says 2+3*4 = 20.

Chapter 04, BundleCount with dices, shows how a total T can be recounted in icon-bundles where the total is shown on two similar dices and the icon-number is shown on a third dice.

Chapter 05, ReCount in the same Unit, shows how to recount a total T in the same unit by unbundling a bundle to singles thus creating an overload, or by borrowing extra singles that then has be counted for as a deficit. Thus a total of 2.1 5s can be written with overload as T = 1B6 5s or as T = 1.6 5s, or with borrowing as T = 3B-4 5s or as T = 3.-4 5s

Chapter 06, ReCount in a new Unit, shows how once counted in one unit, a total T can be recounted in another unit. Thus a total of 2 9s can be recounted in 6s as in chapter 3, again by lining, counting, bundling, stacking, bundle-writing and answering; and again checked by a calculator prediction using two formulas. The ReCount formula **T = (T/B)*B** saying that 'from T, T/B times Bs can be taken away'; and the ReStack formula **T = (T–B)+B** saying that 'From T, T–B is left when B is placed next to'. To change a unit is also called **proportionality**.

Chapter 07, ReCount in BundleBundles, shows how an overload in a bundle-cup can be removed by an extra cup for bundles-of-bundles. Thus counting a total T of 4 8s in 5s gives T = 6B2 5s. However, with 5 as the bundle-size, 5 bundles can be recounted as 1 bundle-of-bundles of 5s so that
T = 6B2 5s = B1B2 5s = 1BB1B2 5s or T = 6.2 5s = 11.2 5s.

Chapter 08, ReCount in Tens on Squared Paper or an Abacus, shows how easy totals counted in icon-bundles can be recounted in tens since the calculator is programmed to give the answer directly in its sloppy version. Thus to recount 3 8s in tens we enter 3*8 and get the answer 24, so T = 3 8s = 2.4 tens. Recounting icon-numbers in tens systematically will provide the multiplication tables showing individual patterns in a ten by ten square or on an abacus.

Chapter 09, ReCount from Tens, shows, as in chapter 3, that we can get the answer through a calculator prediction or through lining, rebundling, and bundle-writing. Only this time we shorten the line by using Roman numbers as icons. Recounting large numbers from tens, we save time by using a multiplication table. Thus to recount a total T of 253 in 7s we use bundle-writing to create overloads guide by the table:
T = 253 = 25B3 = 21B43 = 21B 42+1 = 3B6 *7 +1, so T = 253 = 36 7s + 1.

Chapter 10, ReCount Large Numbers in Tens, show how bundle-writing may be used to create overloads later to be removed to get the final answer. Thus to recount 7 43s in tens gives a total
T = 7 43s = 7*43 = 7*4B3 = 28B21 = 30B1 = 301 as confirmed by a calculator.

Chapter 11, DoubleCount with PerNumbers, shows that counting a quantity in two different physical units will provide a per-number to be used as a bridge connecting the two units. Thus counting a quantity as 4$ and as 5 kg gives the per-number 4$/5kg or 4/5 $/kg. Asking '8$ = ? kg', the answer comes from recounting the 8s in 4s to be able to use the per-number as a bridge between the two units:
T = 8$ = (8/4)*4$ = (8/4)*5kg = 10kg. Likewise when asking e.g.'?$ = 12kg'
Chapter 12, DoubleCount with Fractions and Percentages, shows that fractions and percentages can be treated as per-numbers. Thus asking '3/5 of 200$' is the same as asking '3 per 5 of 200$ gives ?'. So we recount the 200 in 5s to get the answer: T = 200$ = (200/5)*5$ giving (200/5)*3$ = 120$. And asking '3% of 250$' is the same as asking '3 per 100 of 250$'. So we recount the 250 in 100s to get the answer: T = 250$ = (250/100)*100$ gives (250/100)*3$ = 7.5$ as confirmed by writing '3/100*250' on a calculator.
Chapter13, ReCount PerNumbers, Fractions, shows how changing unit transforms per-numbers.
Chapter 14, Adding OnTop, shows that to add two totals T1 and T2 OnTop the units must be the same so recounting may be needed to change a unit. Thus adding 2 3s and 4 5s as 3s, the 4 5s must be recounted as 3s to give a total of 8.2 3s as confirmed by a calculator.
Chapter 15, Reversed Adding OnTop, shows that to reverse OnTop addition, the known total must be taken away before counting the rest in the unit of the second total. Thus asking '2 3s + ? 5s total 5 3s, we take away the 2 3s from the 5 3s before recounting the rest, T – T1, in 5s by saying (T–T1)/5 = ΔT/5 = 1.4 5s as confirmed by a calculator. Subtraction followed by division is called differentiation.
Chapter 16, Adding NextTo, shows that adding two totals T1 and T2 NextTo means adding their areas, also called integration. Thus adding 2 3s and 4 5s NextTo each other as 8s on a ten by ten square or on an abacus gives 3.2 8s as confirmed by a calculator.
Chapter 17, Reversed Adding NextTo, shows that to reverse NextTo addition, the known total must be taken away before counting the rest in the unit of the second total. Thus asking '2 3s + ? 5s total 3 8s, we take away the 2 3s from the 3 8s before recounting the rest, T – T1, in 5s by saying (T–T1)/5 = ΔT/5 = 3.3 5s as confirmed by a calculator. Together, integration and differentiation is called **calculus**.
Chapter 18, Adding Tens, shows that when adding tens, bundle-writing can be used to create and remove overloads. Thus adding totals as 27 and 85 creates an overload that can be removed by bundle-writing,
T = 27 + 85 = 2B7 + 8B5 = 10B12 = 11B2 = 112 as confirmed by a calculator.
Chapter 19, Reversed Adding Tens, the number added must be taken away which might result in a deficit calling for a unbundling a bundle, unless this is done first resulting in an overload that allows taking the number away without creating a deficit. Thus asking '? + 27 = 85' or '85 – 27', bundle-writing is used to remove the deficit, 85 – 27 = 8B5 – 2B7 = 6B-2 = 5B8 = 58; or used to create an overload,
85 – 27 = 8B5 – 2B7 = 7B15 – 2B7 = 5B8 = 58, both confirmed by a calculator.
Chapter 20, Recounting Solves Equations, shows that equations expressing a reversed calculation can be solved by recounting and restacking. Thus to solve the equation u*2 = 8, 8 is recounted in 2s as 8 = (8/2)*2 = 4*2, so that u = 4, checked by a calculator by entering 4*2. With u*2 = 8 solved by u = 8/2 we get a shortcut for solving equations: *Move to the opposite side with the opposite sign.*

u*2 = 8 = (8/2)*2 = 4*2	Here we recount 8 in 2s as 8 = (8/2)*2 = 4*2	**u = 4**
u+2 = 9 = (9-2)+2 = 7+2	Here we restack 9 to 9-2+2 = 7+2	**u = 7**
u/3 = 2	Here we recount 2 in 3s as 2 = (2/3)*3 = 2*3/3 = 6/3	**u = 6**
u-2 = 6	Here we restack 6 to 6-2+2 = 6+2-2 = 8-2	**u = 8**
2*u+3 = 15	Here we restack 15 to 15-3+3 = 12+3, and 2*u = 12 = 12/2*2 = 6*2	**u = 6**
2*u-3 = 15	Here we restack 15 to 15-3+3 = 15+3-3 = 18-3, and 2*u = 18 = 18/2*2 = 9*2	**u = 9**
u/2+3 = 15	Here we restack 15 to 15-3+3 = 12+3, and u/2 = 12 = 12/2*2 = 12*2/2 = 24/2	**u = 24**
2/u-3 = 15	Here we restack 15 to 15-3+3 = 15+3-3 = 18-3, and 2/u = 18 = 18/2*2 = 18*2/2 = 36/2	**u = 36**

Count & Color Squares, Odd & Even

1		2			3			4			5			6			7			8			9		

9					8			7			6			5			4			3		2			1

2			4			6			8			1		3			5			7			9		

1		2			4			8			3			5			9				6			7	

3			6			9				8			5			4			7			1		2	

22. Migrant Math 01

From Sticks to Icons

I I I I → 4 → 4 → FOUR

Many sticks can be arranged in a row of for example four ones.
Four ones can be rearranged to 1 icon with four sticks.
Written sloppy, the icon becomes a digit.
Icons are created for all numbers until ten.
Ten, eleven, twelve etc. has no icon because we count in tens.
Ten is counted as 1 bundle and no unbundles, ten = 10
Eleven is counted as 1 bundle and 1 unbundled, eleven = 11
Twelve is counted as 1 bundle and 2 unbundled, twelve = 12
In Danish, eleven and twelve means one left and two left, understood that a bundle has already been counted.

Migrant Math: Core Math for Late Beginners

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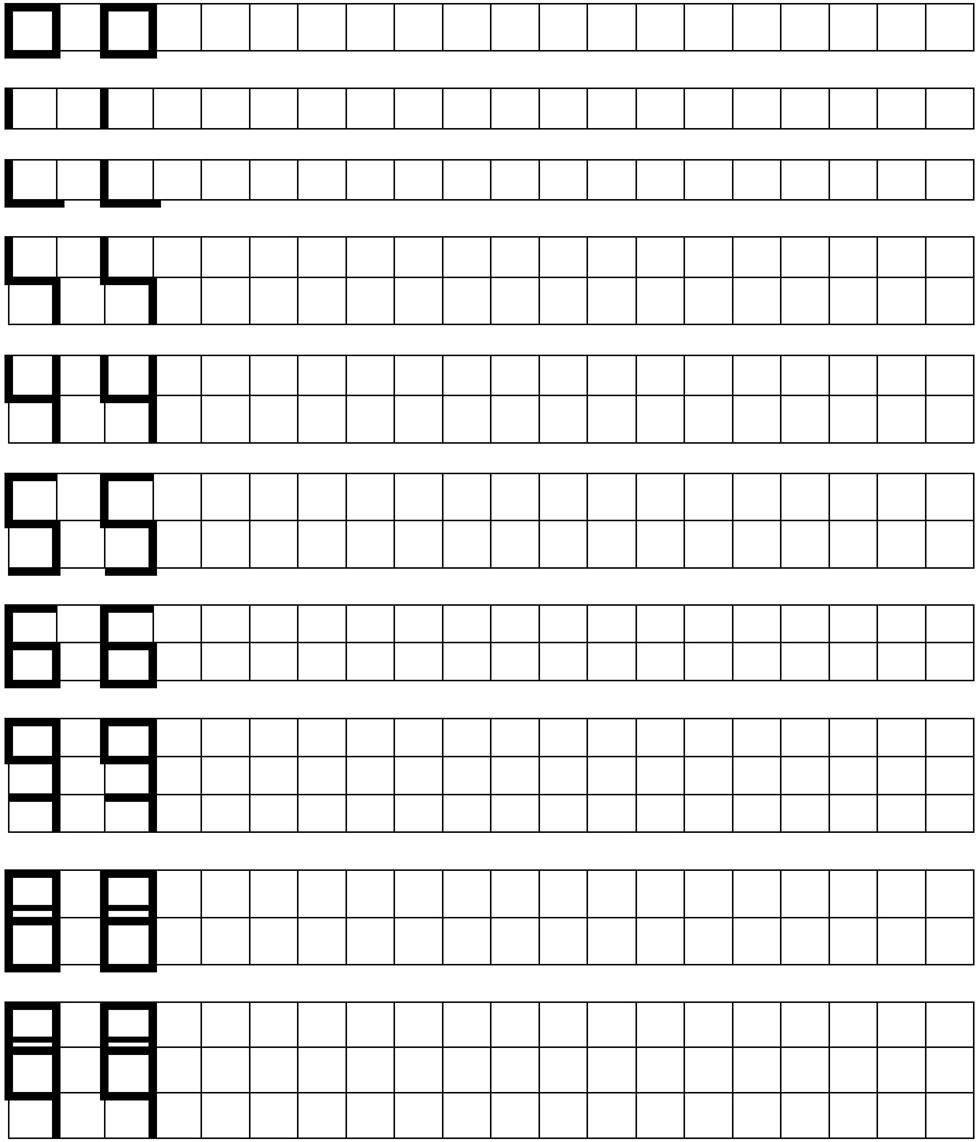
Mathematics as ManyMath
a Natural Science about Many

Job	Do
I	
II	
III	
IIII	
IIIII	
IIIII I	
IIIII II	
IIIII III	
IIIII IIII	



Write & Say

0											Zero
1											One
2											Two
3											Three
4											Four
5											Five
6											Six
7											Seven
8											Eight
9											Nine
10											Ten
11											Ten1
12											Ten2
13											Ten3
14											Ten4
15											Ten5
16											Ten6

Count 1s & 2s & 3s in Icons

	I	II	III	I	II	III	I	II	III	I	II	III
ten	01	03	06			1B2					2B1	
ten	.1	.3										
9												
9												
8												
8												
7												
7												
6												
6												
5												
5												
4												
3												
3												

Migrant Math 02

Counting-sequences in Icons

1, 2, 3, 4, 5, …

We count by bundling, so the bundle-icon is not used.

If we count in tens, ten does not need an icon.

A counting-sequence reports the bundles and the unbundled:

1, 2, 3, 4, …, 10, 11, 12

or 01, 02, …, 10, 11, 12

or 0.1, 0.2, …, 1.0, 1.1, 1.2 tens, with the bundle-name as the unit.

Each bundle-size has its own counting sequence.

The standard is ten-counting in a sloppy version leaving out the unit and misplacing the decimal point by saying 23 instead of 2.3 tens.

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Count in Icons

	I	I	I	I	I	I	I	I	I	I	I	I
ten	1	2	3	4	5	6	7	8	9	10	11	12
ten	01	02	03	04	05	06	07	08	09	1B	1B1	1B2
ten	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.	1.1	1.2
9	01	02	03	04	05	06	07	08	1B	1B1	1B2	1B3
9	.1	.2	.3	.4	.5	.6	.7	.8	1.	1.1	1.2	1.3
8												
8												
7												
7												
6												
6												
5												
5												
4												
4												

Count 2s in Icons

	II	II	II	II	II	II	II	II	II	II	II	II
ten	02				1B							2B2
ten												
9												
9												
8												
8												
7												
7												
6												
6												
5												
5												
4												
3												
3												

Job		Do	Calculator
7 in 4s	Line Count Bundle Cup Stack Answer		7 7
6 in 5s	Line Count Bundle Cup Stack Answer		6 6
5 in 4s	Line Count Bundle Cup Stack Answer		5 5
4 in 5s	Line Count Bundle Cup Stack Answer		4 4
3 in 5s	Line Count Bundle Cup Stack Answer		3 3

Migrant Math 03

BundleCount in Icons

T = 9 = IIIIIIIII = IIII IIII I = 2B1 **4s** = 2.1 **4s**

A total T is counted in icon-bundles that are stacked.
A total of nine sticks can be counted in fours by a counting sequence.
Also, we can place one stick per bundle in a bundle cup that can written as a bracket (bundle-writing) and reported as a decimal-number with a unit where the decimal point separates the bundles from the unbundled singles, T = 9 = 2B1 **4s** = 2.1 **4s**.
A calculator can predict the counting result.
With '9/4' we ask 'from 9, taking away 4s how many times?'
The answer is '2.some'
With '9 – 2x4' we ask 'from 9, taking away 2 4s leaves what?'
The answer '1' gives the calculator prediction T = 9 = 2.1 **4s**.
Moving 1 stick outside the bundle cup gives an overload, T = 1B5 **4s**.
Moving 1 stick inside gives an underload, a deficit, T = 3B-3 **4s**.
Thus a total T of nine can be recounted in 4 different ways:
T = nine = 9 1s = 2B1 **4s** = 1B5 **4s** = 3B-3 **4s**.
This is handy when totals are added, subtracted, multiplied or divided.
A good calculator says 2+3*4 = 14; a bad calculator says 2+3*4 = 20.

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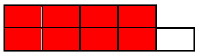
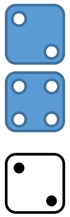
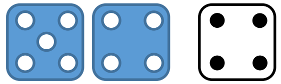
03. BundleCount in Icons

Job		Do	Calculator
9 in 5s	Line Count Bundle Stack Cup Answer	T = I I I I I I I I I 1, 2, 3, 4, B, 1B1, 1B2, 1B3, **1B4** T = ~~IIIII~~ I I I I  T = 1B4 5s = 0B9 5s = 2B-1 5s T = 9 = 1.4 5s	9/5 1.some 9 – 1*5 4 *9 – 0*5 9* *9 – 2*5 -1*
9 in 4s	Line Count Bundle Cup Stack Answer	T = I I I I I I I I I 1, 2, 3, B, 1B1, 1B2, 1B3, 2B, **2B1** T = ~~IIII~~ ~~IIII~~ I T = 2B1 4s = 1B5 4s = 3B-3 4s  T = 9 = 2.1 4s	9/4 2.some 9 – 2*4 1 *9 – 1*4 5* *9 – 3*4 -3*
9 in 3s	Line Count Bundle Cup Stack Answer		9/ 9 –
8 in 4s	Line Count Bundle Cup Stack Answer		8 8
8 in 3s	Line Count Bundle Cup Stack Answer		8 8

Job		Do	Calculator
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		
	Line Count Bundle Cup Stack Answer		

Migrant Math 04

BundleCount with Dices



T = 9 = ? **4s**

A total T can be recounted in icon-bundles.
The total is shown by two like dices.
The bundle-number is on a third dice where 1 counts as 7.

Calculator prediction:

9/4	2.some
9 – 2*4	1

Answer: T = 9 = 2.1 **4s**

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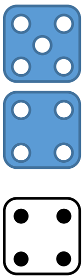
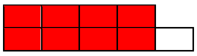
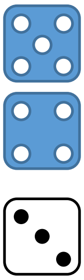
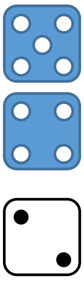
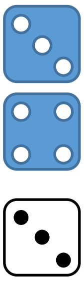
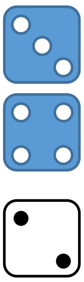
04. BundleCount with Dices

Job		Do	Calculator
	Line	T = I I I I I I I I I	
	Count	1, 2, 3, B, 1B1, 1B2, 1B3, 2B, <u>2B1</u>	9/4 2.some
	Bundle	T = ~~IIII~~ ~~IIII~~ I	9 – 2*4 1
	Cup	T = 2B1 4s = 1B5 4s = 3B-3 4s	*9 – 1*4* *5*
	Stack		*9 – 3*4* *-3*
9 i 4s	Answer	<u>T = 9 = 2.1 4s</u>	
	Line		
	Count		
	Bundle		9/
	Cup		9 –
	Stack		
	Answer		
	Line		
	Count		
	Bundle		9
	Cup		9
	Stack		
	Answer		
	Line		
	Count		
	Bundle		7
	Cup		7
	Stack		
	Answer		
	Line		
	Count		
	Bundle		7
	Cup		7
	Stack		
	Answer		

Job		Do	Cup	Answer
4.3 6s in 6s	Line UnBundle Borrow			
4.3 5s in 5s	Line UnBundle Borrow			
4.3 4s in 4s	Line UnBundle Borrow			
5 7s in 7s	Line UnBundle Borrow			
5 6s in 6s	Line UnBundle Borrow			
5 4s in 4s	Line UnBundle Borrow			
3 7s in 7s	Line UnBundle Borrow			
3 5s in 5s	Line UnBundle Borrow			
1.3 6s in 6s	Line UnBundle Borrow			
1.3 5s in 5s	Line UnBundle Borrow			

Migrant Math 05

ReCount in the same Unit

T = 2B1 **5s** = 1B6 **5s** = 3B-4 **5s**
T = 2.1 **5s** = 1.6 **5s** = 3.-4 **5s**

A total T is recounted in the same unit in two ways:

- create an overload: unbundle a bundle to singles
- create an underload:

borrow extra singles that becomes a deficit

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05. ReCount in the Same Unit

Job		Do	Cup	Answer
2.1 5s in 5s	Line UnBundle Borrow	T = IIIII IIIII I T = IIIII I I I I I I T = IIIII IIIII IIIII	2B1 1B6 3B-4	T = 2.1 **5s** T = 1.6 **5s** T = 3.-4 **5s**
2.1 4s in 4s	Line UnBundle Borrow			
2.1 3s in 3s	Line UnBundle Borrow			
2.1 6s in 6s	Line UnBundle Borrow			
2.1 7s in 7s	Line UnBundle Borrow			
3.2 7s in 7s	Line UnBundle Borrow			
3.2 6s in 6s	Line UnBundle Borrow			
3.2 5s in 5s	Line UnBundle Borrow			
3.2 4s in 4s	Line UnBundle Borrow			
3.2 3s in 3s	Line UnBundle Borrow			

Job		Do	Calculator
2 7s in 5s	Line Count Bundle Stack Cup Answer		2*7 2*7
2 6s in 5s	Line Count Bundle Stack Cup Answer		2*6 2*6
2 6s in 4s	Line Count Bundle Stack Cup Answer		2*6 2*6
2 6s in 3s	Line Count Bundle Stack Cup Answer		2*6 2*6
2 5s in 4s	Line Count Bundle Stack Cup Answer		5 5

Migrant Math 06

ReCount in a new Unit

T = 3 **5s** = ? **6s**

Once counted in one unit, a total T can be recounted in another unit.
A total of 3 **5s** can be recounted in **6s** as in chapter 04
- by lining, counting, bundling, stacking, bundle-writing and answering
- by asking a calculator to predict the result using two formulas:

The ReCount formula **T = (T/B)*B** saying that ‘from T, T/B times Bs can be taken away’
The ReStack formula **T = (T–B)+B** saying that ‘from T, T–B is left when B is placed next to’.

To change a unit is also called **proportionality**.

Calculator prediction:

3*5/6	2.some
3*5 – 2*6	3

Answer: T = 3 5s = 2.3 **6s**

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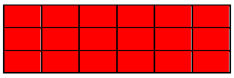
06. ReCount in a New Unit

Job		Do	Calculator
2 9s in 6s	Line Count Bundle Stack Cup Answer	T = I I I I I I I I I I I I I I I I I I 1, 2, 3, 4, B, 1B1, 1B2, 1B3, 1B4, …, **3B** T = ~~IIIIII~~ ~~IIIIII~~ ~~IIIIII~~  T = 3B T = 2 9s = 3 6s	2*9/6 3 2*9 – 3*6 0
2 9s in 5s	Line Count Bundle Stack Cup Answer		2*9/ 2*9 –
2 8s in 6s	Line Count Bundle Stack Cup Answer		2*8 2*8
2 8s in 5s	Line Count Bundle Stack Cup Answer		2*8 2*8
2 7s in 6s	Line Count Bundle Stack Cup Answer		2*7 2*7

Job		Do	Calculator
7	Cup	T = 7 = 3]1 = 1BB1B1	7/2 3.some
in 2s	Ans.	T = 7 = 3.1 **2s** = 11.1 **2s**	7 – 3*2 1
9	Cup	T = 9 = 4B1 = 2BB0B1 = 1BBB0BB0B1	9/2 4.some
in 2s	Ans.	T = 9 = 4.1 **2s** = 20.1 **2s** = 100.1 **2s**	9 – 4*2 1
3 4s	Cup		
in 2s	Ans.		
3 5s	Cup		
in 2s	Ans.		
5 4s	Cup		
in 2s	Ans.		
4 7s	Cup		
in 3s	Ans.		
4 8s	Cup		
in 3s	Ans.		
4 9s	Cup		
in 3s	Ans.		
5 7s	Cup		
in 3s	Ans.		
5 8s	Cup		
in 3s	Ans.		
5 9s	Cup		
in 3s	Ans.		
6 8s	Cup		
in 3s	Ans.		
7 8s	Cup		
in 3s	Ans.		

Migrant Math 07

ReCount in BundleBundles

T = 9.3 5s = 9B3 **5s** = 1BB4B3 **5s** = 14.3 **5s**

An overload in a bundle-cup can be removed
by an extra cup for bundles-of-bundles.
Counting a total T of 6 **8s** in **5s** gives T = 9.3 5s.
However, with 5 as the bundle-size,
5 bundles can be recounted as
1 bundle-of-bundles of **5s** so that
T = 6 **8s** = 9.3 **5s** = 14.3 **5s**.

Calculator prediction:

6*8/5	9.some	9/5	1.some
6*8 – 9*5	3	9 – 1*5	4

Answer: T = 6 **8s** = 9.3 **5s** = 14.3 **5s**

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07. Recount in BundleBundles

Job		Do	Calculator
4 8s in 5s	Cup Answer	T = 4 **8s** = 6B2 **5s** = 1BB1B2 **5s** T = 4 **8s** = 6.2 **5s** = 11.2 **5s** = 12.-3 **5s**	4*8/5 6.some 4*8 – 6*5 2
5 8s in 6s	Cup Answer		
6 9s in 7s	Cup Answer		
9 9s in 8s	Cup Answer		
3 9s in 4s	Cup Answer		
4 5s in 3s	Cup Answer		
6 8s in 5s	Cup Answer		
6 8s in 4s	Cup Answer		
7 8s in 5s	Cup Answer		
7 8s in 4s	Cup Answer		
8 8s in 5s	Cup Answer		
8 8s in 4s	Cup Answer		

Job		Do	Calculator
5s in tens			10 * 5 = 9 * 5 = 8 * 5 = 7 * 5 = 6 * 5 = 5 * 5 = 4 * 5 = 3 * 5 = 2 * 5 = 1 * 5 =
4s in tens			10 * 4 = 9 * 4 = 8 * 4 = 7 * 4 = 6 * 4 = 5 * 4 = 4 * 4 = 3 * 4 = 2 * 4 = 1 * 4 =
3s in tens			10 * 3 = 9 * 3 = 8 * 3 = 7 * 3 = 6 * 3 = 5 * 3 = 4 * 3 = 3 * 3 = 2 * 3 = 1 * 3 =

Migrant Math 08

ReCount in Tens on Squared Paper or an Abacus

T = 3 **8s** = ? **tens**   T = 3 **8s** = 3*8 = 24 = 2.4 **tens**

Totals counted in icon-bundles can easily be recounted in tens.
A calculator gives the answer directly in its sloppy version.
To recount 3 **8s** in tens, we enter 3*8 and get the answer 24.
So T = 3 **8s** = 24 = 2.4 **tens**.
Recounting icon-numbers in tens systematically gives
the **multiplication tables**, showing individual patterns
in a ten by ten square or on an abacus.

Calculator prediction:

3*8	24

Answer: T = 3 **8s** = 24 = 2.4 **tens**

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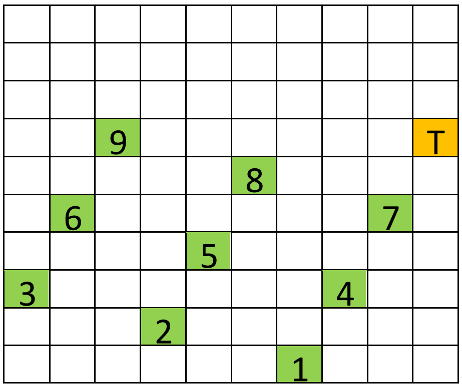
08. ReCount in Tens on Squared Paper or an Abacus

Job		Do	Calculator
7s **in tens**			10 * 7 = 70 9 * 7 = 63 8 * 7 = 56 7 * 7 = 49 6 * 7 = 42 5 * 7 = 35 4 * 7 = 28 3 * 7 = 21 2 * 7 = 14 1 * 7 = 7
8s **in tens**			10 * 8 = 9 * 8 = 8 * 8 = 7 * 8 = 6 * 8 = 5 * 8 = 4 * 8 = 3 * 8 = 2 * 8 = 1 * 8 =
9s **in tens**			10 * 9 = 9 * 9 = 8 * 9 = 7 * 9 = 6 * 9 = 5 * 9 = 4 * 9 = 3 * 9 = 2 * 9 = 1 * 9 =
6s **in tens**			10 * 6 = 9 * 6 = 8 * 6 = 7 * 6 = 6 * 6 = 5 * 6 = 4 * 6 = 3 * 6 = 2 * 6 = 1 * 6 =

Job		Do	Calculator
253 in 7s	Cup Ans.	T = 2BB5B3 = 25B3 = 21B43 = 21B42 + 1 T = 3B6 * 7 + 1 = 36 * 7 + 1 = <u>36 1/7 7s</u>	253/7 36.some 253 – 36*7 1
253 in 9s	Cup Ans.		
253 in 5s	Cup Ans.		
253 in 3s	Cup Ans.		
842 in 7s	Cup Ans.		
842 in 5s	Cup Ans.		
842 in 4s	Cup Ans.		
842 in 2s	Cup Ans.		
904 in 8s	Cup Ans.		
904 in 7s	Cup Ans.		
904 in 5s	Cup Ans.		
904 in 3s	Cup Ans.		
789 in 8s	Cup Ans.		
789 in 5s	Cup Ans.		
789 in 4s	Cup Ans.		

Migrant Math 09

ReCount from Tens

T = 3 **tens** = ? **7s**

A total of 3 **tens** can be recounted in **7s** as in chapter 06
- by lining (we shorten with Roman numbers as icons), counting, bundling, stacking, bundle-writing and answering
- by asking a calculator to predict the result using the two formulas

Calculator prediction:

30/7	4.some
30 – 4*7	2

Answer: T = 3 **tens** = 4.2 **7s** = <u>4 2/7 **7s**</u> *(fraction form)*

Recounting large numbers from tens, we save time using a multiplication table. So to recount a total T of 253 in 7s we use bundle-writing to create an overload guided by the table:

T = 253 = 25B3 = 21B43 = 21B 42 +1 = 3B6 *7 +1
T = 253 = 36 **7s** + 1 = 36 1/7 **7s**.

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09. Recount From Tens

Job		Do	Calculator
37 in 9s	Line	X X X V II	
	ReBundle	9I 9I 9I V II -> 9 9 9 X -> 9 9 9 9 1	37/9 4.some
	Cup	3B 7 = B37 = B36 + 1 = B4*9 + 1	37 – 4*9 1
	Answer	T = 37 = 4*9 + 1 = 4.1 **9s** = 4 1/9 **9s**	
37 in 7s	Line		
	ReBundle		
	Cup		
	Answer		
37 in 5s	Line		
	ReBundle		
	Cup		
	Answer		
42 in 7s	Line		
	ReBundle		
	Cup		
	Answer		
42 in 5s	Line		
	ReBundle		
	Cup		
	Answer		
26 in 7s	Line		
	ReBundle		
	Cup		
	Answer		
26 in 5s	Line		
	ReBundle		
	Cup		
	Answer		

Job		Do	Calculator
17 43s	Cup Ans.	T = 17 * 4B3 = 68B51 = 73B1 = 731 T = 17 **43s** = 73.1 **tens** = 731	17*43 731
27 43s	Cup Ans.		
37 43s	Cup Ans.		
47 43s	Cup Ans.		
57 43s	Cup Ans.		
67 43s	Cup Ans.		
77 43s	Cup Ans.		
87 43s	Cup Ans.		
32 243s	Cup Ans.	T = 32 * 2BB4B3 = 64BB128B96 = 64BB137B6 = 77BB7B6 = 777.6 **tens** = 7776	32*243 7776
35 413s	Cup Ans.		
43 343s	Cup Ans.		
56 453s	Cup Ans.		
62 637s	Cup Ans.		
74 843s	Cup Ans.		
87 543s	Cup Ans.		
92 493s	Cup Ans.		

Migrant Math 10

ReCount Large Numbers in Tens

T = 7 **43s** = 7*43 = 7*4B3 = 28B21 = 30B1 = 301

To reCount large numbers in Tens, bundle-writing is used to create an overload, later to be removed to get the final answer.
To recount 7 **43s** in tens gives a total
T = 7 **43s** = 7*43 = 7*4B3 = 28B21 = 30B1 = 301 = 30.1 **tens**
This makes sense: Shrinking the width of the stack from 43 to ten means increasing the height to keep the same total.

Calculator prediction:

7*43	301

Answer: T = 3 **8s** = 24 = 2.4 **tens**

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10. Recount Large Numbers in Tens

Job		Do	Calculator
7 43s	Cup Answer	T = 7 * 4B3 = 28B21 = 30B1 = 301 T = 7 **43s** = 30.1 **tens** = 301	7*43 301
8 43s	Cup Answer		
9 43s	Cup Answer		
6 43s	Cup Answer		
5 62s	Cup Answer		
4 62s	Cup Answer		
3 62s	Cup Answer		
2 62s	Cup Answer		
27 436s	Cup Answer		
3 436s	Cup Answer		
4 436s	Cup Answer		
5 436s	Cup Answer		
6 436s	Cup Answer		
7 436s	Cup Answer		
8 436s	Cup Answer		

Job	Do	Formula
With 4 $ per 5 kg 8$ = ?kg ?$ = 12 kg	8$ = (8/4)*4$ = (8/4)*5kg = 10kg 12kg = (12/5)*5kg = (12/5)*4$ = 9.6$	Kg = (kg/$)*$ Kg = (5/4)*8 = 10 $ = ($/kg)*kg $ = (4/5)*12 = 9.6
With 3 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 6 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 8 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 3 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 6 kg 8$ = ?kg ?$ = 12 kg		
With 4 $ per 8 kg 8$ = ?kg ?$ = 12 kg		
With 2 $ per 5 kg 8$ = ?kg ?$ = 12 kg		
With 2 $ per 7 kg 8$ = ?kg ?$ = 12 kg		

Migrant Math 11

DoubleCount with PerNumbers

With 4$/5kg, T = 8$ = (8/4)*4$ = (8/4)*5kg = 10kg

Counting a quantity in two different physical units provides a **per-number** to be used as a bridge connecting the two units.

Thus counting a quantity as 4$ and as 5 kg gives the per-number 4$/5kg or 4/5 $/kg.

Asking '8$ = ? kg', the answer comes from recounting the 8s in 4s to be able to use the per-number as a bridge between the two units:

T = 8$ = (8/4)*4$ = (8/4)*5kg = 10kg.

Likewise when asking e.g.'? $ = 12kg'

T = 12kg = (12/5)*5kg = (12/5)*4$ = 9.6$

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11. DoubleCount with PerNumbers

Job	Do	Formula
With 4 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**	8$ = (8/4)*4$ = (8/4)*5kg = 10kg 12kg = (12/5)*5kg = (12/5)*4$ = 9.6$	Kg = (kg/$)*$ Kg = (5/4)*8 = 10 $ = ($/kg)*kg $ = (4/5)*12 = 9.6
With 3 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 6 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 8 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 3 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 6 kg **8$ = ?kg** **?$ = 12 kg**		
With 4 $ per 8 kg **8$ = ?kg** **?$ = 12 kg**		
With 2 $ per 5 kg **8$ = ?kg** **?$ = 12 kg**		
With 2 $ per 7 kg **8$ = ?kg** **?$ = 12 kg**		

Job	Do	Calculator
3 per 5 of 200$	200$ = (200/5)*5$ Giving (200/5)*3$ = 120$	3/5*200 120
3 per 5 of 400$		
2 per 5 of 200$		
1 per 5 of 200$		
3 per 6 of 240$		
2 per 6 of 240$		
5 per 6 of 300$		
3 per 100 of 250$ or 3% of 250$	250$ = (250/100)*100$ Giving (250/100)*3$ = 7.5$	3/100*250 7.5
8 per 100 of 200$ or 8% of 200$		
20 per 100 of 200$ or 20% of 200$		
3 per 100 of 560$ or 3% of 560$		
8 per 100 of 560$ or 8% of 560$		
12 per 100 of 560$ or 12% of 560$		
20 per 100 of 560$ or 20% of 560$		
60 per 100 of 560$ or 60% of 560$		

Migrant Math 12

DoubleCount with Fractions & Percentages

With 4/5, T = 30$ = (30/5)*5$ gives (30/5)*4$ = 24$

Fractions and percentages can be treated as per-numbers.

Asking ‘3/5 of 200$’ is the same as asking ‘3 per 5 of 200$ gives ?’.

So we recount the 200 in 5s to get the answer:

T = 200$ = (200/5)*5$ giving (200/5)*3$ = 120$.

Asking ‘3% of 250$’ is the same as asking ‘3 per 100 of 250$’.

So we recount the 250 in 100s to get the answer:

T = 250$ = (250/100)*100$ gives (250/100)*3$ = 7.5$

as confirmed by writing ‘3/100*250’ on a calculator.

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12. DoubleCount with Fractions & Percentages

Job	Do	Calculator
3 per 5 of 200$	200$ = (200/5)*5$ Giving (200/5)*3$ = 120$	3/5*200 120
3 per 5 of 400$		
2 per 5 of 200$		
1 per 5 of 200$		
3 per 6 of 240$		
2 per 6 of 240$		
5 per 6 of 300$		
3 per 100 of 250$** **or 3% of 250$	250$ = (250/100)*100$ Giving (250/100)*3$ = 7.5$	3/100*250 7.5
8 per 100 of 200$** **or 8% of 200$		
20 per 100 of 200$** **or 20% of 200$		
3 per 100 of 560$** **or 3% of 560$		
8 per 100 of 560$** **or 8% of 560$		
12 per 100 of 560$** **or 12% of 560$		
20 per 100 of 560$** **or 20% of 560$		
60 per 100 of 560$** **or 60% of 560$		

Job	Do	Do	Calculator	Calculator
2/3 = ?	2/3 = 2 2s / 3 2s = 4/6 2/3 = 2 3s / 3 3s = 6/9	2/3 = 2 4s / 3 4s = 8/12 2/3 = 2 5s / 3 5s = 10/15	2/3 = 0.66.. 4/6 = 0.66..	8/12 = 0.66.. 10/15 = 0.66..
1/3 = ?				
1/5 = ?				
2/5 = ?				
3/5 = ?				
4/5 = ?				
4/6 2/6 6/8 2/8	4/6 = 2 2s / 3 2s = 2/3 2/6 = 1 2s / 3 2s = 1/3	6/8 = 3 2s / 4 2s = 3/4 2/8 = 1 2s / 4 2s = 1/4	4/6 = 0.66.. 2/3 = 0.66.. 2/6 = 0.33.. 1/3 = 0.33..	6/8 = 0.75 3/4 = 0.75 2/8 = 0.25 1/4 = 0.25
2/10 4/10 6/10 8/10				
2/12 4/12 6/12 8/12 10/12				
2/14 4/14 6/14 8/14 10/14 12/14				
2/16 4/16 6/16 8/16 10/16 12/16 14/16				

Migrant Math 13

ReCount PerNumbers & Fractions

$$\frac{2}{3} = \frac{2\ \mathbf{2s}}{3\ \mathbf{2s}} = \frac{2*2}{3*2} = \frac{4}{6}$$

Changing unit transforms per-numbers.
With 2 per 3, the per-number does not depend of the unit.
So we can always change unit to the same unit on both numbers.

2 per 3 = $\frac{2}{3} = \frac{2\ \mathbf{2s}}{3\ \mathbf{2s}} = \frac{2*2}{3*2} = \frac{4}{6}$ = 4 per 6

Or we can remove the same unit from both numbers

4 per 6 = $\frac{4}{6} = \frac{2*2}{3*2} = \frac{2\ \mathbf{2s}}{3\ \mathbf{2s}} = \frac{2}{3}$ = 2 per 3

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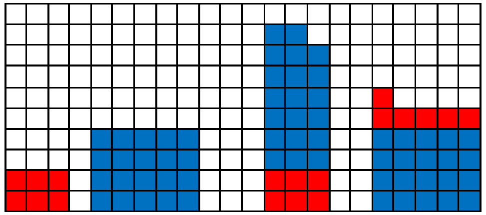
13. ReCount PerNumbers & Fractions

Job	Do	Do	Calculator	Calculator
2/3 **= ?**	2/3 = 2 2s / 3 2s = 4/6 2/3 = 2 3s / 3 3s = 6/9	2/3 = 2 4s / 3 4s = 8/12 2/3 = 2 5s / 3 5s = 10/15	2/3 = 0.66.. 4/6 = 0.66..	8/12 = 0.66.. 10/15 = 0.66..
1/3 **= ?**				
1/5 **= ?**				
2/5 **= ?**				
3/5 **= ?**				
4/5 **= ?**				
4/6 **2/6** **6/8** **2/8**	4/6 = 2 2s / 3 2s = 2/3 2/6 = 1 2s / 3 2s = 1/3	6/8 = 3 2s / 4 2s = 3/4 2/8 = 1 2s / 4 2s = 1/4	4/6 = 0.66.. 2/3 = 0.66.. 2/6 = 0.33.. 1/3 = 0.33..	6/8 = 0.75 3/4 = 0.75 2/8 = 0.25 1/4 = 0.25
2/10 **4/10** **6/10** **8/10**				
2/12 **4/12** **6/12** **8/12** **10/12**				
2/14 **4/14** **6/14** **8/14** **10/14** **12/14**				
2/16 **4/16** **6/16** **8/16** **10/16** **12/16** **14/16**				

Job	Do	Calculator
2 3s + 4 5s = ? 3s ? 5s		(2*3+4*5)/3 8.some (2*3+4*5) - 8*3 2 2 3s + 4 5s = 8.2 3s (2*3+4*5)/5 5.some (2*3+4*5) - 5*5 1 2 3s + 4 5s = 5.1 5s
2 4s + 3 5s = ? 4s ? 5s		
3 2s + 4 6s = ? 2s ? 6s		
2 5s + 4 3s = ? 5s ? 3s		
5 2s + 3 4s = ? 2s ? 4s		

Migrant Math 14

Add OnTop

T = 2 **3s** + 4 **5s** = ? **5s**

T = (2*3 + 4*5)/5 **5s** = 5.1 **5s**

To add two totals T1 and T2 OnTop, the units must be the same.
so recounting may be needed to change a unit.
To add 2 **3s** and 4 **5s** as **5s**,
the 2 **3s** must be recounted as 5s as (2*3)/5 **5s** = 1.1 **5s.**
T = 2 **3s** + 4 **5s** = 1.1 **5s** + 4 **5s** = 5.1 **5s**
as confirmed by a calculator.

Calculator prediction:

(2*3+4*5)/5	5.some
(2*3+4*5) – 5*5	1

Answer: T = 2 **3s** + 4 **5s** = 5.1 **5s**

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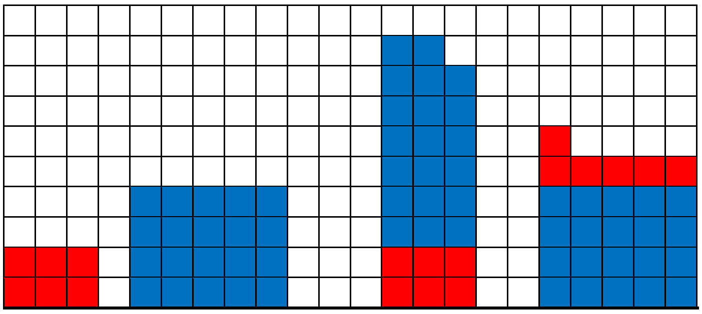
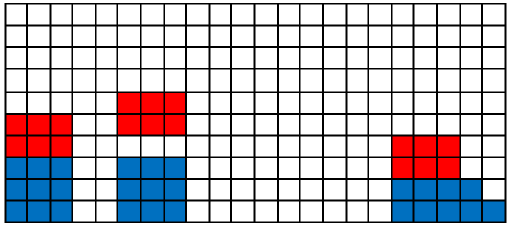
14. Add OnTop

Job	Do	Calculator
2 3s **+** **4 5s** **=** **? 3s** **? 5s**		(2*3+4*5)/3 8.some (2*3+4*5) - 8*3 2 <u>2 3s + 4 5s = 8.2 3s</u> (2*3+4*5)/5 5.some (2*3+4*5) - 5*5 1 <u>2 3s + 4 5s = 5.1 5s</u>
2 4s **+** **3 5s** **=** **? 4s** **? 5s**		
3 2s **+** **4 6s** **=** **? 2s** **? 6s**		
2 5s **+** **4 3s** **=** **? 5s** **? 3s**		
5 2s **+** **3 4s** **=** **? 2s** **? 4s**		

2 3s + ? 5s = 5 3s		(5*3-2*3)/5 1.some (5*3-2*3) - 1*5 4 2 3s + 1.4 5s = 5 3s
2 4s + ? 5s = 5 4s		
2 6s + ? 5s = 4 6s		
2 7s + ? 5s = 6 5s		
2 6s + ? 5s = 4 5s		

Migrant Math 15

Reversed Adding OnTop

T = 2 **5s** + ? **3s** = 6 **3s**

T = (6*3 – 2*5)/3 **3s** = 2.2 **3s**

To reverse OnTop addition, the known total must be taken away before counting the rest in the unit of the second total.
Asking '2 **5s** + ? **3s** total 6 **3s**, we take away the 2 **5s** from the 6 **3s** before recounting the rest, T – T1, in 3s by saying (T–T1)/3 **3s** = ΔT/3 **3s** = 2.2 **3s** as confirmed by a calculator.
Subtraction followed by division is differentiation, part of calculus.

Calculator prediction:

(6*3 – 2*5)/3	2.some
(6*3 – 2*5) – 1*3	2

Answer: T = 2 **5s** + 2.2 **3s** = 6 **3s**

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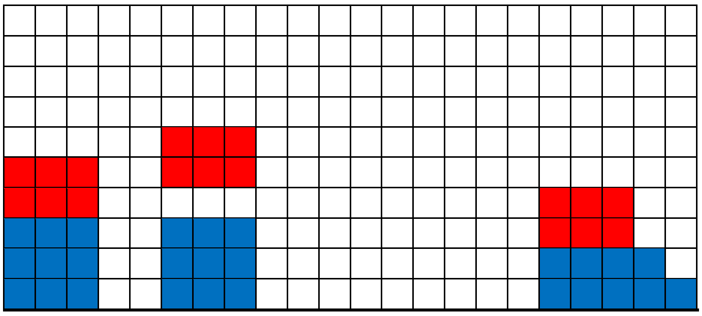
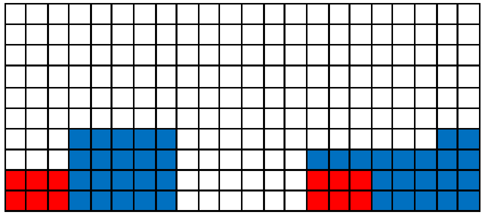
15. Reversed Adding OnTop

2 3s **+** **? 5s** **=** **5 3s**		(5*3-2*3)/5 1.some (5*3-2*3) - 1*5 4 2 3s + **1.4 5s** = 5 3s
2 4s **+** **? 5s** **=** **5 4s**		
2 6s **+** **? 5s** **=** **4 6s**		
2 7s **+** **? 5s** **=** **6 5s**		
2 6s **+** **? 5s** **=** **4 5s**		

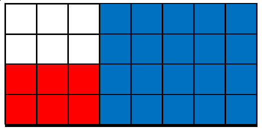
Job	Do	Calculator
2 3s + 4 5s = ? 8s		(2*3+4*5)/8 3.some (2*3+4*5) - 8*3 2 2 3s + 4 5s = 3.2 8s
3 2s + 4 5s = ? 7s		
2 3s + 4 6s = ? 9s		
2 4s + 4 5s = ? 9s		
4 3s + 2 4s = ? 6s		

Migrant Math 16

Add NextTo

T = 2 **3s** + 4 **5s** = ? **8s**
T = (2*3 + 4*5)/8 **8s** = 3.2 **8s**

To add two totals T1 and T2 NextTo means adding their areas.
Adding areas is called integration, a part of calculus.
To add 2 **3s** and 4 **5s** next to each other as **8s** on a ten by ten square or on an abacus gives 3.2 **8s** as confirmed by a calculator.



Calculator prediction:

(2*3+4*5)/8	3.some
(2*3+4*5) – 3*8	2

Answer: T = 2 **3s** + 4 **5s** = 3.2 **8s**

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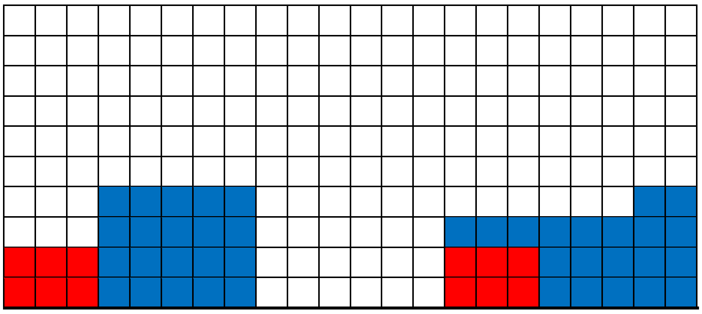
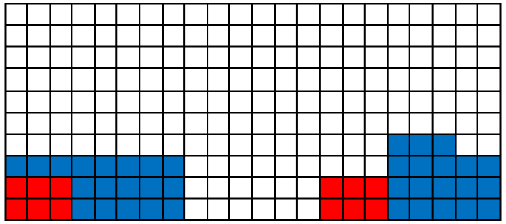
16. Add NextTo

Job	Do	Calculator
2 3s + 4 5s = ? 8s		(2*3+4*5)/8 3.some (2*3+4*5) - 8*3 2 <u>2 3s + 4 5s = 3.2 8s</u>
3 2s + 4 5s = ? 7s		
2 3s + 4 6s = ? 9s		
2 4s + 4 5s = ? 9s		
4 3s + 2 4s = ? 6s		

2 3s + ? 5s = 3 8s		(3*8-2*3)/5 3.some (3*8-2*3) - 3*5 3 2 3s + 3.3 5s = 3 8s
2 4s + ? 5s = 3 9s		
2 3s + ? 4s = 3 7s		
4 3s + ? 5s = 3 8s		
5 2s + ? 5s = 3 7s		

Migrant Math 17

Reversed Adding NextTo

T = 2 **3s** + ? **5s** = 3 **8s**

T = (3*8 – 2*3)/5 **5s** = 3.3 **5s**

To reverse NextTo addition, the known total must be taken away before counting the rest in the unit of the second total.
Asking ‘2 **3s** + ? **5s** total 3 **8s**, we take away the 2 **3s** from the 3 **8s** before recounting the rest, T – T1, in 5s:
(T–T1)/5 **5s** = ΔT/5 **5s** = 3.3 **5s** as confirmed by a calculator.
Subtraction followed by division is differentiation, part of calculus.

Calculator prediction:

(3*8 – 2*3)/5	3.some
(3*8 – 2*3) – 3*5	3

Answer: T = 2 **3s** + 3.3 **5s** = 3 **8s**

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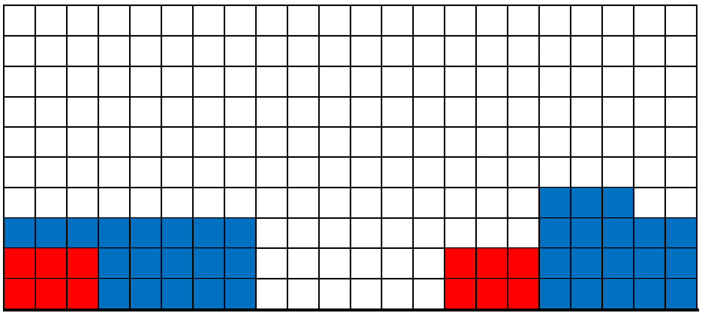
17. Reversed Adding NextTo

2 3s **+** **? 5s** **=** **3 8s**		(3*8-2*3)/5 3.some (3*8-2*3) - 3*5 3 <u>2 3s + **3.3 5s** = 3 8s</u>
2 4s **+** **? 5s** **=** **3 9s**		
2 3s **+** **? 4s** **=** **3 7s**		
4 3s **+** **? 5s** **=** **3 8s**		
5 2s **+** **? 5s** **=** **3 7s**		

Job		Do	Calculator
27 + 85	Cup	T = 2]7 + 8]5 = 10]12 = 11]2 = 112	27+85 112
	Answer	T = 27 + 85 = 11.2 tens = 112	
27 + 85	Cup		
	Answer		
33 + 78	Cup		
	Answer		
39 + 71	Cup		
	Answer		
45 + 67	Cup		
	Answer		
58 + 57	Cup		
	Answer		
57 + 49	Cup		
	Answer		
27 + 205	Cup		
	Answer		
33 + 198	Cup		
	Answer		
39 + 191	Cup		
	Answer		
45 + 187	Cup		
	Answer		
58 + 177	Cup		
	Answer		
57 + 169	Cup		
	Answer		
127 + 385	Cup		
	Answer		
433 + 578	Cup		
	Answer		

Migrant Math 18

Add tens

T = 27 + 85 = 2B7 + 8B5 = 10B12 = 11B2 = 112

Adding tens might create an overload in a bundle-cup or outside.
Bundle-writing is used to remove overloads.
Adding 27 and 85 creates an overload outside the bundle-cup.
The overload is removed by bundle-writing moving bundles inside.
T = 27 + 85 = 2B7 + 8B5 = 10B12 = 11B2 = 112
as confirmed by a calculator.

Calculator prediction:

27+85 112

Answer: T = 27 + 85 = 112

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18. Add Tens

Job		Do	Calculator
27 + 85	Cup Answer	T = 2B7 + 8B5 = 10B12 = 11B2 = 112 <u>T = 27 + 85 = 11.2 tens = 112</u>	27+85 112
27 + 85	Cup Answer		
33 + 78	Cup Answer		
39 + 71	Cup Answer		
45 + 67	Cup Answer		
58 + 57	Cup Answer		
57 + 49	Cup Answer		
27 + 205	Cup Answer		
33 + 198	Cup Answer		
39 + 191	Cup Answer		
45 + 187	Cup Answer		
58 + 177	Cup Answer		
57 + 169	Cup Answer		
127 + 385	Cup Answer		
433 + 578	Cup Answer		

Job		Do	Calculator
27 + ? = 85 85 – 27	Cup Answer	D = 8]5 – 2]7 = 6]-2 = 5]8 = 58 D = 8]5 – 2]7 = 7]15 – 2]7 = 5]8 = 58 T = 85 – 27 = 5.8 tens = 58	85 – 27 58
63 – 17	Cup Answer		
55 – 36	Cup Answer		
35 – 17	Cup Answer		
185 – 27	Cup Answer		
235 – 128	Cup Answer		
242 – 128	Cup Answer		
245 – 167	Cup Answer		
312 – 159	Cup Answer		
421 – 268	Cup Answer		

Migrant Math 19

Reversed adding tens

T = 85 – 27 = 8B5 – 2B7 = 6B-2 = 5B8 = 58

Reversing adding tens, the known number must be taken away.
This might give a deficit calling for unbundling a bundle.
Unless this is done first to create an overload that allows
taking the number away without creating a deficit.
Thus asking '? + 27 = 85' or '85 – 27', bundle-writing is used
to remove the deficit, or to create an overload
T = 85 – 27 = 8B5 – 2B7 = 6B-2 = 5B8 = 58
T = 85 – 27 = 8B5 – 2B7 = 7B15 – 2B7 = 5B8 = 58
both confirmed by a calculator.

Calculator prediction:

85 – 27	58

Answer: T = 85 – 27 = 58

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19. Reversed Adding Tens

Job		Do	Calculator
27 + ? = 85 **85 – 27**	Cup Answer	D = 8B5 – 2B7 = 6B-2 = 5B8 = 58 D = 8B5 – 2B7 = 7B15 – 2B7 = 5B8 = 58 T = 85 – 27 = 5.8 **tens** = 58	85 – 27 58
63 – 17	Cup Answer		
55 – 36	Cup Answer		
35 – 17	Cup Answer		
185 – 27	Cup Answer		
235 – 128	Cup Answer		
242 – 128	Cup Answer		
245 – 167	Cup Answer		
312 – 159	Cup Answer		
421 – 268	Cup Answer		

Do	Equation	Calculator
ReCount ReCount Answer	**2*u+3 = 15** = (15-3)+3 = 12 + 3 2*u = 12 = (12/2)*2 = 6*2 u = 6	2*6+3 15
ReCount ReCount Answer	**3*u+4 = 19**	
ReCount ReCount Answer	**4*u+6 = 38**	
ReCount ReCount Answer	**2*u-3 = 15** = (15-3)+3 = 15+3-3 = 18 - 3 2*u = 18 = (18/2)*2 = 9*2 u = 9	2*9-3 15
ReCount ReCount Answer	**3*u-4 = 8**	
ReCount ReCount Answer	**4*u-5 = 23**	
ReCount ReCount Answer	**u/2+3 = 15** = (15-3)+3 = 12 + 3 u/2 = 12 = (12/2)*2 = (12*2)/2 = 24/2 u = 24	24/2+3 15
ReCount ReCount Answer	**u/3+4= 12**	
ReCount ReCount Answer	**u/2-3 = 15** = (15-3)+3 = (15+3)-3 = 18 - 3 u/2 = 18 = (18/2)*2 = (18*2)/2 = 36*2 u = 36	36/2-3 15
ReCount ReCount Answer	**u/4-7 = 5**	
ReCount ReCount Answer	**u/5-8 = 2**	

Migrant Math 20

Recounting Solves Equations

u*2 = 8 = (8/2)*2 so u = 8/2 = 4

A reversed calculation is called an equation.
An equation can be solved by recounting and restacking.
In both cases an equation is solved by a moving-method:
Move to the opposite side with the opposite sign
In the end, the solution is tested.

To solve the equation u*2 = 8 8 is recounted as 8 = (8/2)*2	To solve the equation u+2 = 8 8 is restacked as 8 = (8–2)+2
u*2 = 8 = (8/2)*2 u = 8/2 = 4 Test: 4*2 = 8 ☺	u + 2 = 8 = (8–2)+2 u = 8–2= 6 Test: 6+2 = 8 ☺

A calculator with a solver will confirm the answer:

Solve(u*2 = 8)	4

Answer: u*2 = 8 is solved by u = 4

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20. ReCounting solves Equations

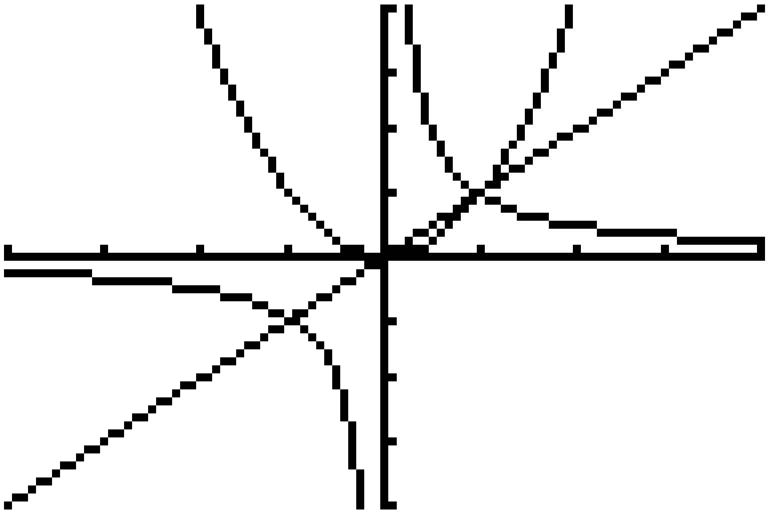
Do	Equation	Calculator
ReCount Answer	**u*2 = 30** = (30/2)*2 = 15*2 u = 15	15*2 30
ReCount Answer	**u*3 = 15**	
ReCount Answer	**u*4 = 32**	
ReCount Answer	**u*5 = 40**	
ReCount Answer	**u/3 = 12** = (12/3)*3 = 12*3/3 = 36/3 u = 36	36/3 12
ReCount Answer	**u/3 = 10**	
ReCount Answer	**u/4 = 8**	
ReCount Answer	**u/5 = 6**	
ReCount Answer	**u+2 = 30** = (30-2)+2 = 28 + 2 u = 28	28+2 30
ReCount Answer	**u+3 = 24**	
ReCount Answer	**u+4 = 20**	
ReCount Answer	**u+5 = 12**	
ReCount Answer	**u-2 = 30** = (30-2)+2 = 30+2-2 = 32- 2 u = 32	32-2 30
ReCount Answer	**u-3 = 20**	
ReCount Answer	**u-5 = 10**	

23. Mathematics Predicts, PreCalculus with a TI-82 or TI-84

Compendium & Projects

by Allan.Tarp@MATHeCADEMY.net

Version 1709



y = 1*x , y = x*x , y = 1/x

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Mathematics Predicts

Mathematics
    Algebra
        Uniting and Splitting Numbers
    Geometry
        Measuring Earth
    Statistics
        Accounting Many
	Mathematics contains Algebra, Geometry and Statistics
	Algebra (calculation) can predict counting processes, both the end result and the parts.
	Geometry (earth measuring) can be used to calculate plane figures and spatial forms.
	Statistics (counting) is used for counting the actual size of different quantities.

Mathematics has two main fields, Algebra and Geometry, as well as Statistics.

Geometry means 'earth-measuring' in Greek. Algebra means 'reuniting' in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus together algebra and geometry give an answer to the fundamental human question: how do we split the earth on which we live and what it produces?

Originally human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus trade was only possible with those highlanders that had silver in their mountains.

The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire. The dark Middle Ages came when the Greek silver mines were emptied and the Arabs conquered the Spanish mines.

German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number system and algebra.

The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: A+B+C = 180 and a side law: a^2+b^2=c^2. Pythagoras generalized this findings by claiming: All is numbers.

This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don't know Geometry. However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation.

The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has a n acceleration g; and the distance s and the time t are connected by the formula: s=½*g*t^2. However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middle men. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine.

The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move?

The church said 'among the stars'. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth.

But why do things fall? The church said: everything follows the unpredictable will of our metaphysical lord only attainable through belief prayers and church attendance. Newton objected: It follows its own will, a force called gravity that can be predicted by a formula telling how a force changes the motion, which made Newton develop change-calculations, calculus. So instead of obeying the church, people should enlighten themselves by knowledge, calculations and school attendance.

Brahe used his life to write down the positions of the planets among the stars. Kepler used these data to suggest that the sun is the center of the solar system, but could not prove it without sending up new planets. Newton, however, could validate his theory by different examples of falling and swinging bodies.

Newton's discoveries laid the foundation of the Enlightenment period realizing that when an apple follows its own will and not that of a metaphysical patronizor, humans could do the same. Thus by enlightening themselves people could replace the double patronization of the church and the prince with democracy, which lead to two democracies, one in The US and one in France. Also formulas could be used to predict and therefore gain control over nature, using this knowledge to build an industrial welfare society based upon a silver-free economy emerging when the English replaced the import silk and pepper from the Far East with production of cotton in the US creating the triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton.

Calculations Predict

Calculations predict the total T. 2*4 calculation types are used to unite and split into four different types of numbers:

Uniting or splitting	Variable	Constant
Unit-numbers $, kg, s	Plus + Minus -	Multiplication * Division /
Per-numbers $/kg, $/100$, %	Integration ∑ ∫ Differentiation Δ	Power ^ Log or root √

a $ and n $ total T $:	a+n = T
a $ n times total T $:	n*a = T
r % n times total T%:	(1+r)^n = 1+T
a1 kg at p1 $/kg + a2 kg at p2 $/kg total T $: p1*a1+ p2*a2 = T:	∑ p*a = T

Algebra means re-uniting in Arabic and can be translated to predictions. Algebra thus predicts the result of uniting singles into totals or splitting totals into singles.

There are four ways of uniting numbers: addition (+), multiplication (*), power (^) and integration (∑ or ∫).

Addition + predicts the result of uniting variable singles:

2$ and 3 $ and 4 $ total T $: 2+3+4 = T

Multiplication * predicts the result of uniting constant singles:

2$ + 2$ + 2$ + 2$ + 2$ = 5 times 2$ = T, 5*2 = T

Job	Count	Predict
32$ and 63 $	1,2,…,95	T = 32+63
2$ 36 times	2,4, …,72	T=72*2
20% 5 times	120, 144...	T=120%^5

Power ^ predicts the result of uniting constant percentages: 5 times 2% totals T%, 102%^5 = 1+T

Integration ∑ or ∫ predicts the result of uniting constant per-numbers:

2kg at 7$/kg + 3kg at 8$/kg totals T $: 7*2 + 8*3 = T , ∑ $/kg * kg = T , ∫ p*dx = T

Inverse or backward calculations predicts the result of splitting a Total into singles.

x+3 = 15	Question: Which number added to 3 gives 15?
x = 15-3	Prediction: 15-3 is the number that added to 3 gives 15. Test: 3+(15-3) = 15
Rule	**Plus-numbers move across as minus-numbers, and vice versa**

x*3 = 15	Question: Which number multiplied with 3 gives 15?
$x = \frac{15}{3}$	Prediction: $\frac{15}{3}$ is the number that added to 3 gives 15. Test: $3*\frac{15}{3} = 15$
Rule	**Multiplication-numbers move across as minus-numbers, and vice versa**

x^3 = 125	Question: Which number raised to power 3 gives 125?
$x = \sqrt[3]{125}$	Prediction: $\sqrt[3]{125}$ is the number that raised to power 3 gives 125. Test: $(\sqrt[3]{125})$^3= 125
Rule	**Exponent-numbers move across as reciprocal exponent-numbers, and vice versa**

3^x = 243	Question: 3 raised to which power gives 243?
$x = \frac{\log 243}{\log 3}$	Prediction: 3 raised to power $\frac{\log 243}{\log 3}$ gives 243. Test: 3^$\frac{\log 243}{\log 3}$ = 243. Notice: $\log_3(243) = \frac{\log 243}{\log 3}$
Rule	**Base-numbers move across as logarithm-numbers, and vice versa**

A mixed calculation containing more calculations can be reduced to a single calculation by bracketing the stronger one.

T = 2+3*4 = 2+(3*4) , T = 2+3^4 = 2+(3^4) , T = 2*3^4 = 2*(3^4) Priority: 1. (), 2.^, 3. *, 4. +

A formula-table can be used to document the solving of an equation.

The unknown number	c = ?	T = a+b*c	*The formula*
The known numbers	a = 2	14 = 2+(3*c)	*From a mixed to a single calculation by bracketing the stronger*
	b = 3	$\frac{(14-2)}{3} = c$	*+ moves across as the opposite -, and * moves across as /*
	T = 14		*Bracket the calculation already present*
		4 = c	*Perform the calculation*
Tests	Test	14 = 2+3*4	'MATHSolver 0 = -14 + 2+3*x' gives 'x = 4'
		14 = 14 ☺	

Tasks

Find the unknown number in the formula. Make more with randM (3,1)

1. T = a+b*c
2. T = a+b/c
3. T = a*b^c
4. T = a+b^c
5. T = a-b*c
6. T = a-b/c
7. T = a/b^c
8. T = a-b^c

	T	b	a	c
I	60		12	20
II	60	1.5		20
III	60	1.5	12	

Formulas Predict

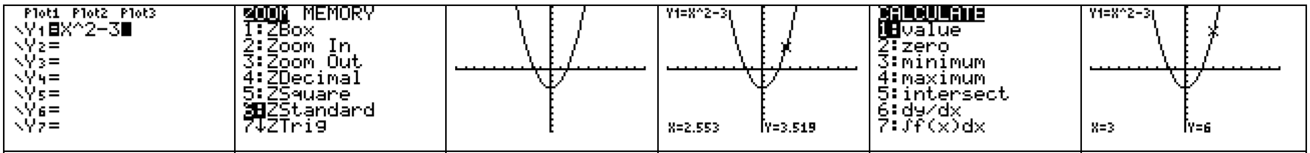
A formula contains a quantity y and a its calculation f, y = f(x,z,t) **An equation** is a formula with 1 unknown. An equation can be calculated or solved by finding the unknown. **A function** is a formula with 2 unknowns. A function can be tabled or graphed showing different scenarios: If x = a then y = f(a).	Purchase-formula: b \$ + x kg at a \$/kg totals y \$: b + x * a = y Sharing-formula: b \$ + a \$ shared between x persons totals y \$: b + a / x = y

A formula contains a quantity y and a its calculation f, y = f(x,z,t). Thus a formula might contain 2, 3, 4 or more variables. If the variables are replaced by fixed numbers, a formula is transformed into an equation or a function.

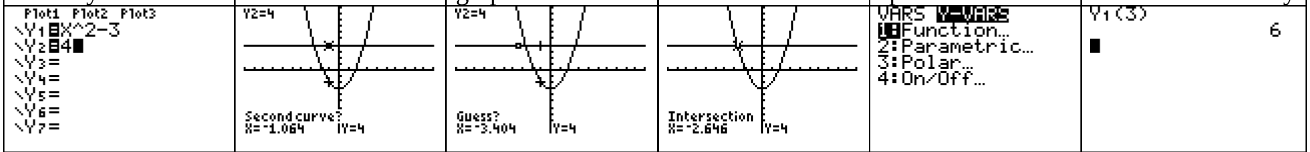
An equation is a formula with 1 unknown: y = 10 + 2*3, or 16 = b + 2*3, or 16 = 10 + a*3, or 16 = 10 + 2*x

An equation can be solved manually or by a calculator using MATHSolver. After using 'solve' the solution is tested by inserting all known numbers: 16 = 10 + 2*3 gives 16 = 16

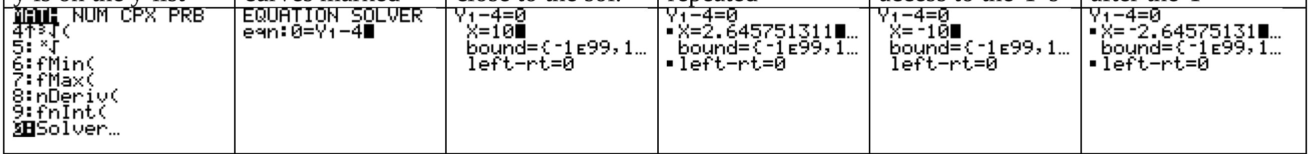
A function is a formula with 2 (or more) unknowns: y = b + 2*3, or y = 10 + 2*x, or 16 = b + 2*x, or 16 = 10 + a*x.
In a function one of the unknowns is isolated and entered on the calculators y-list. Thus x^2-y+3=0 gives y=x^2-3.



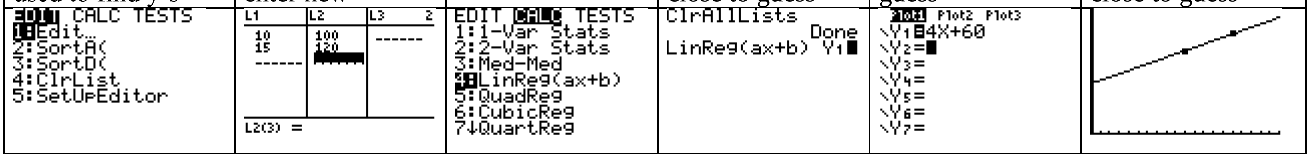
Formulas are put on the y-list	Always start with Standard Zoom	Choose Graph to graph	Choose Trace to see scenarios	Calc Value gives specific values	And is used for knownx/unknowny



Knowny/unknownx y is on the y-list	The intersecting curves marked	The cursor is close to the sol.	The procedure is repeated	VARS gives access to the Y-s	The known x is put after the Y



MATHSolver is used to find y's	CLEAR old and enter new	Enter a guess	Read the solution close to guess	Enter a new guess	Read the solution close to guess



From table to formula use STAT	Enter the table as lists	Choose a formula type	Add Y1 to bring formulas to y-list	Add Plot for visual control	Adjust window before graphing

Tasks: Find the question marks in three different ways: manually in a formula table, using graphs and using calculation.

1

x	y=3+2*x
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.7
?	-2.4
?	3.1
?	4.5

2

x	y=3-2*x
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.6
?	-2.5
?	3.2
?	4.6

3

x	y=x^2-4
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.8
?	-2.2
?	3.7
?	4.7

4

x	y=-x^2+5
-3.7	?
-2.4	?
3.1	?
4.5	?
?	-3.2
?	-2.6
?	3.3
?	4.3

5	x	10	20	30	
	y	30	50		80
6	x	10	15	25	
	y	100	130		180
7	x	10	20	40	
	y	100	70		10

Ans:	a	b	Formula	y	x	T
lin	2	10	y = 10+2*x	70	35	
exp	1,052	18	y = 18*1,052^x	83,33	29,2	13,6
pow	0,737	5,5	y = 5,5*x^0,737	67,41	37,84	
lin	6	40	y = 40+6*x	190	23,33	
exp	1,054	59,17	y = 59,17*1,054^x	219,7	21,2	13,2
pow	0,647	22,54	y = 22,54*x^0,647	180,92	24,8	
lin	-3	130	y = 130+-3*x	10	40	
exp	0,965	142,86	y = 142,86*0,965^x	34,3	74,56	-19,4
pow	-0,515	327,02	y = 327,02*x^-0,515	49	877,72	

Constant Change: Linear, exponential and power

1. The Capital changes by c

$Capital_{TERM}$	= $Capital_{START}$	+ change-number
y	= b	+ c

y = ?	**y = b+c**
b = 20 c = 5	y = 20+5 y = 25

problems A1-A4

b = ?	**y = b+c**
y = 20 c = 5	y–c = b 20–5 = b **15 = b**
test	*20 = 15+5* *20 = 20 OK*

problems A9-A12

c = ?	**y = b+c**
y = 30 b = 23	y–b = c 30–23 = c **7 = c**
test	*30 = 23+7* *30 = 30 OK*

problems A5-A8

c = ?	**b = y–c**
y = 21 b = 7	b + c = y c = y–b c = 21-7 **c = 14**
test	*7 = 21–14* *7 = 7 OK*

problems C1-C12

2. The Capital changes by c x times (linear change)

$Capital_{END}$	= $Capital_{START}$	+ change-number * x
y	= b	+ c * x

y = ?	**y = b+c*x**
b = 20 c = 5 x = 8	y = 20+5*8 y = 60

problems A13-A16

b = ?	**y = b+(c*x)**
y = 60 c = 5 x = 8	y–(c*x) = b 60–(5*8) = b **20 = b**
test	*60 = 20+5*8* *60 = 60 OK*

problems A17-A20

c = ?	**y = b+(c*x)**
y = 60 b = 20 x = 4	y–b = c*x $\frac{(y-b)}{x}$ = c (60-20)/4 = c **10 = c**
test	*60 = 20+10*4* *60 = 60 OK*

problems A21-A24

x = ?	**y = b+(c*x)**
y = 60 b = 20 c = 8	y–b = c*x $\frac{(y-b)}{a}$ = x (60-20)/8 = x **5 = x**
test	*60 = 20+8*5* *60 = 60 OK*

problems A25-A28

3. The Capital changes by a factor c

$Capital_{END}$	= $Capital_{START}$	* change-factor
y	= b	* c

y = ?	**y = b*c**
b = 20 c = 1.23	y = 20*1.23 y = 24.6

problems B1-B4

b = ?	**y = b*c**
y = 20 c = 1.45	y/c = b 20/1.45 = b **13.793 = b**
test	*20 = 13.793*1.45* *20 = 20.000 OK*

problems B9-B12

c = ?	**y = b*c**
y = 30 b = 23	y/b = c 30/23 = c **1.304 = c**
test	*30 = 23*1.304* *30 = 29.992OK*

problems B5-B8

c = ?	**b = y/c**
y = 21 b = 17	b * c = y c = y/b c = 21/17 **c = 1.235**
test	*17 = 21/1.235* *17 = 17.004 OK*

problems D1-D12

4. The Capital changes by a factor c (r%) x times (exponential)

$Capital_{END}$	= $Capital_{START}$	* change-factor x times
y	= b	* c ^ x = b* (1+r) ^ x

y = ?	**y = b*(1+r)^x**
b = 20 r = 20% = 0.20 x = 8	y = 20*1.2^8 y = 85.996

problems A&B29-32

b = ?	**y = b*((1+r)^x)**
y = 60 r = 20% = 0.20 x = 8	y/((1+r)^x) = b 60/(1.2^8) = b **13.954 = b**
test	*60 = 13.954*1.2^8* *60 = 60.000 OK*

problems A&B33-36

r = ?	**y = b*((1+r)^x)**
y = 30 b = 20 x = 5	y/b = (1+r)^x $\sqrt[x]{(y/b)}$ = 1+r $\sqrt[5]{(30/20)}$ – 1 = r 0.084 = **r= 8.4%**
test	*30 = 20*1.084^5* *30 = 29.935 OK*

problems A&B37-40

x = ?	**y = b*((1+r)^x)**
y = 70 b = 20 r = 30% = 0.30	y/b = (1+r)^x $\frac{\log (y/b)}{\log (1+r)}$ = x $\frac{\log (70/20)}{\log 1.3}$ = x **4.775 = x**
test	*70 = 20*1.3^4.775* *70 = 70.002 OK*

problems B41-44

4. The Capital changes by a factor x c times (power):

$Capital_{END}$	= $Capital_{START}$	* change-factor x times
y	= b	* x ^ c

y = ?	**y = b*(x^c)**
b = 20 c = 1.2 x = 8	y = 20*8^1.2 y = 242.515

problems B13-B16

b = ?	**y = b*(x^c)**
y = 60 c = 1.2 x = 8	y/(x^c) = b 60/(8^1.2) = b **4.948 = b**
test	*60 = 4.948*8^1.2* *60 = 59.998 OK*

problems B17-B20

x = ?	**y = b*(x^c)**
y = 30 b = 20 c = 5	y/b = x^c $\sqrt[c]{(y/b)}$ = x $\sqrt[5]{(30/20)}$ = x **1.084 = x**
test	*30 = 20*1.084^5* *30 = 29.935 OK*

problems B25-B28

c = ?	**y = b*(x^c)**
y = 70 b = 20 x = 1.2	y/b = x^c $\frac{\log (y/b)}{\log x}$ = c $\frac{\log (70/20)}{\log 1.2}$ = c **6.871 = c**
test	*70=20*1.2^6.871* *70 = 69.998 OK*

problems B21-B24

Straight lines. Linear change on (++) paper, exponential change on (+*) paper, power change on (**) paper.

Tables for Regression Formulas

A table allows creating a formula manually or using regression on a calculator.

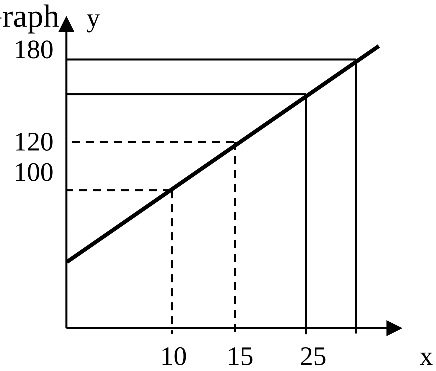
Table

x	10	15	25	?
y	100	120	?	180

Linear change +1 day , +5 $	y = b+c*x	x:+1, y:+c	++ change
Exponential change +1 day , +5 %	y = b*c^x	x:+1, y:+r% c = 1+r	+* change T: doubling
Power change +1 % , +5 %	y = b*x^c	x: +1%, y: +c%	** change

The numbers c and b can also be found using regression or an Excel spreadsheet

Graph



Linear change

c = ? | $c = \frac{y2-y1}{x2-x1}$

y2 = 120, y1 = 100, x2 = 15, x1 = 10 | $c = \frac{120-100}{15-10}$ | **c = 4**

b = ? | y = b+(c*x)

y = 100, c = 4, x = 10 | y–(c*x) = b | 100–(4*10) = b | **60 = b**

Test | 100 = 60+4*10 | 100 = 100 ☺

Formula | y = b+c*x

b = 60, c = 4 | **y = 60+4*x**

y = ? | y = 60+4*x

x = 25 | y = 60+4*25 | **y = 160**

x = ? | y = 60+(4*x)

y = 180 | y–60 = 4*x | $\frac{y-60}{4} = x$ | $\frac{180-60}{4} = x$ | **30 = x**

Exponential change

c = ? | $c = \sqrt[x2-x1]{\frac{y2}{y1}}$

y2 = 120, y1 = 100, x2 = 15, x1 = 10 | $c = \sqrt[15-10]{\frac{120}{100}}$

c = 1.037 = 1+r

r = c-1 = 1.037-1

r = 0,037 = **3.7%**

T = log2/logc = **19.1**

b = ? | y = b*(c^x)

y = 100, c = 1.037, x = 10 | $\frac{y}{c\text{^}x} = b$ | $\frac{100}{1.037\text{^}10} = b$ | **69.44 = b**

Formula | y = b*c^x

b = 69.44, c = 1.037 | **y = 69.44*1.037^x**

y = ? | **y = 69.44*1.037^x**

x = 25 | y = 69.44*1.037^25 | **y = 172.80**

x = ? | y = 69.44*(1.037^x)

y = 180 | $\frac{y}{69.44} = 1.037\text{^}x$ | $\frac{\log(\frac{180}{69.44})}{\log(1.037)} = x$ | **26.21 = x**

Test | 180 = 69.44*1.037^26.21 | 180 = 179.958 ☺

Power change

c = ? | $c = \frac{\log(\frac{y2}{y1})}{\log(\frac{x2}{x1})}$

y2 = 120, y1 = 100, x2 = 15, x1 = 10 | $c = \frac{\log(\frac{120}{100})}{\log(\frac{15}{10})}$ | **c = 0.450**

b = ? | y = b*(x^c)

y = 100, c = 0.450, x = 10 | $\frac{y}{x\text{^}c} = b$ | $\frac{100}{10\text{^}0.450} = b$ | **35.48 = b**

Formula | y = b*x^c

b = 35.48, c = 0.450 | **y = 35.48*x^0.450**

y = ? | y = 35.48*x^0.450

x = 25 | y = 35.48*25^0.450 | **y = 151.03**

x = ? | y = 35.48*x^0.450

y = 180 | $\frac{y}{35.48} = x\text{^}0.450$ | $\sqrt[0.450]{\frac{180}{35.48}} = x$ | **36.92 = x**

Test | 180 = 35.48*36.92^0.450 | 180 = 179.991 ☺

Pr.

Pr.					
1	x	10	20	30	
	y	30	50		80
2	x	10	15	25	
	y	100	130		180
3	x	10	20	35	
	y	60	40		10
4	x	10	20	40	
	y	100	70		10

Answers:

	c	b	Formula	y	x	T
lin	2	10	y = 10+2*x	70	35	
exp	1,052	18	y = 18*1,052^x	83,33	29,2	13,6
pot	0,737	5,5	y = 5,5*x^0,737	67,41	37,84	
lin	6	40	y = 40+6*x	190	23,33	
exp	1,054	59,17	y = 59,17*1,054^x	219,7	21,2	13,2
pot	0,647	22,54	y = 22,54*x^0,647	180,92	24,8	
lin	-2	80	y = 80+-2*x	10	35	
exp	0,96	90	y = 90*0,96^x	21,77	54,19	-17,1
pot	-0,585	230,74	y = 230,74*x^-0,585	28,83	213,92	
lin	-3	130	y = 130+-3*x	10	40	
exp	0,965	142,86	y = 142,86*0,965^x	34,3	74,56	-19,4
pot	-0,515	327,02	y = 327,02*x^-0,515	49	877,72	

PerNumber Problems

In a per-number problem, the per-number must be multiplied to a unit-number to set up an equation.

Type2.1 Traveling

Problem21: Train1 travels from A to B with the speed 40 km/t. Two hours later train2 leaves A for B with the speed 60 km/t. When and where will they meet?

Text Hours	*Per-Number*	*Unit-Number* x = ?	*ANSWER* *4*	*Equation* 40*(x+2) = 60*x
Speed1	40 km/h			40*x + 80 = 60*x
Speed2	60 km/h			80 = 60*x – 40*x = 20*x
Km1		40*(x+2) km	*240*	80/20 = x
Km2		60*x km	*240*	4 = x

Problem22: Train1 travels from A to B with the speed 40 km/t. At the same time train2 travels from B to A with the speed 60 km/t. The distance from A to B is 300 km? When and where do they meet?

Text Hours	*Per-Number*	*Unit-Number* x = ?	*ANSWER* *4*	*Equation* 40*x + 60*x = 300
Speed1	40 km/h			100*x = 300*x
Speed2	60 km/h			x = 300/100
Km1		40*x km	*120*	x = 3
Km2		60*x km	*180*	

Problem23: A motorboat travels the same distance in 3 hours upstream as 2 hours downstream. The stream has the speed 5 km/t . What is the speed of the boat?

Text Speed	*Per-Number* x = ? km/h	*Unit-Number*	*ANSWER* *25*	*Equation* km = km/h*h = (x-5)*3 = (x+5)*2
Speed up	x – 5 km/h		*20*	3*x - 15 = 2*x+10
Speed down	x + 5 km/h		*30*	3*x - 2*x = 10+15
Hours		3 hours		x = 25

Type2.2 Mixture problems

? Liter 40% alcohol + 3 liter 20% alcohol gives ? liter 32% alcohol

Text Liter-number	*Per-Number*	*Unit-Number* x = ? liter	*ANSWER* *4.5*	*Equation* 0.4*x + 0.2*3 = 0.32*(x+3)
Liter-number		x+3 liter	*7.5*	0.4*x + 0.6 = 0.32*x + 0.96
Alcohol1	40%	0.4*x liter		0.4*x - 0.32*x = 0.96- 0.6
Alcohol2	20%	0.2*3 liter		0.08*x = 0.36
Alcohol3	32%	0.32*(x+3)	liter	x = 0.36/0.08
				x = 4.5

Type2.3 Finance

400.000$. giving a yearly yield at 20.000$ is invested in the following way: One part goes to a bank paying an interest at 3% p.a., the rest goes to 8% bonds. How much goes to the bank?

Text Bank in thousands	*Per-Number*	*Unit-Number* x = ? $	*ANSWER* *240*	*Equation* 3%*x + 8%*(400-x) = 20
Bonds in thousands		x+3 $	*160*	0.03*x + 32 – 0.08*x = 20
Interest rate in bank	3%			32 - 20 = 0.08*x – 0.03*x
Interest rate on bonds	8%			12 = 0.05*x
Bank part		3%*x $		12/0.05 = x
Bond part		8%*(400-x)$		240 = x

Type2.4 Work problems

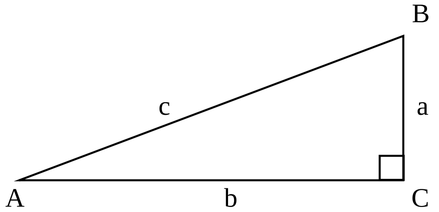
A can dig a ditch in 4 hours. B can dig the same ditch in 3 hours. How many hours if working together?

Text Hours	*Per-Number*	*Unit-Number* x = ? hours	*ANSWER* *12/7*	*Equation* ¼*x + 1/3*x = 1
A's speed	1/4 ditch/t			(¼ + 1/3)*x = 1
B's speed	1/3 ditch/t			7/12*x = 1
A's part		¼*x		x = 12/7
B's part		1/3*x		

Trigonometry

Any land can be divided in triangles Any triangle can be divided into right-angled triangles	Two Greek Formula: A+B+C = 180 a^2 + b^2 = c^2 Three Arabic Formula: $\sin A = \frac{a}{c}$ $\cos A = \frac{b}{c}$ $\tan A = \frac{a}{b}$

A triangle is defines by 3 pieces. The rest can be predicted by formulas. The Greeks only found two formulas, so trigonometry first was developed when the Arabs added three extra formulas.

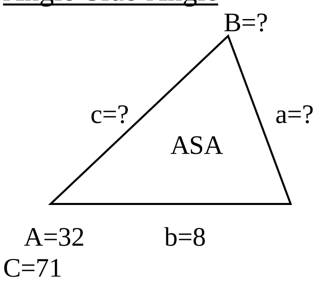


Greek formulas
A+B+C = 180 a^2 + b^2 = c^2 (Pyhtagoras)
Arabic formulas:
$\sin A = \frac{a}{c}$ (height in % of c) $\cos A = \frac{b}{c}$ (base in % af c) $\tan A = \frac{a}{b}$
A right-angled triangle can be seen as a rectangle divided by a diagonal.

In a non right-angled triangle the sine and cosine formulas have to be extended:

The Sine Rule	The Cosine Rule (The Extended Pythagoras)
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	a^2 = b^2 + c^2 - 2*b*c*cosA b^2 = a^2 + c^2 - 2*a*c*cosB c^2 = a^2 + b^2 - 2*a*b*cosC

<u>Angle-Side-Angle</u>

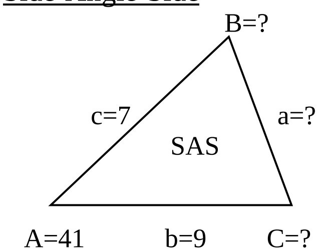


B = ?	A+B+C=180
A=32 C=71	B=180-A-C B=180-32-71 B=77

a = ?	$\frac{a}{\sin A} = \frac{b}{\sin B}$
b=8 A=32 B=77	$a = \frac{b*\sin A}{\sin B}$ $a = \frac{8*\sin 32}{\sin 77}$ a = 4.351

c = ?	$\frac{c}{\sin C} = \frac{b}{\sin B}$
b=8 C=71 B=77	$c = \frac{b*\sin C}{\sin B}$ $c = \frac{8*\sin 71}{\sin 77}$ c = 7.763

<u>Side-Angle-Side</u>

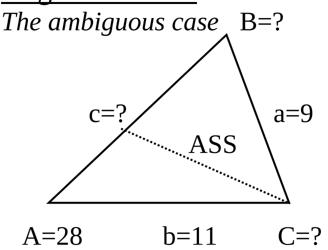


a = ?	a^2=b^2+c^2 -2*b*c*cosA
b=9 c=7 A=41	a^2=9^2+7^2 -2*9*7*cos41 a = √34.907 a = 5.908

B = ?	b^2=a^2+c^2 -2*a*c*cosB
b=9 c=7 a=5.9	$\cos B = \frac{a^2+c^2-b^2}{2*a*c}$ $\cos B = \frac{5.9^2+7^2-9^2}{2*5.9*7}$ B=cos-1(0.035)=88.0

C = ?	A+B+C=180
A=41 B=88	C=180-A-B B=180-41-88 B=51

<u>Angle-Side-Side</u>
The ambiguous case

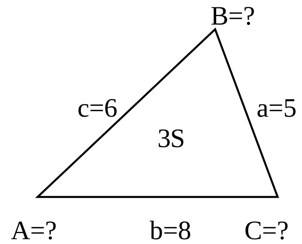


B = ?	$\frac{\sin B}{b} = \frac{\sin A}{a}$
b=11 A=28 a=9	$\sin B = \frac{b*\sin A}{a}$ $\sin B = \frac{11*\sin 28}{9}$ B=sin-1(0.574) $B = \left[\begin{matrix}35\\145\end{matrix}\right.$

C = ?	A+B+C=180
A=28 B=35 or B=145	C=180-A-B C=180-28-35 C=117 or C=180-28-145 C=7

c = ?	$\frac{c}{\sin C} = \frac{a}{\sin A}$
a=9 A=28 C=117 or C=7	$c = \frac{a*\sin C}{\sin A}$ $c = \frac{9*\sin 117}{\sin 28}$ c = 17.081 or c = 2.336

<u>Side-Side-Side</u>



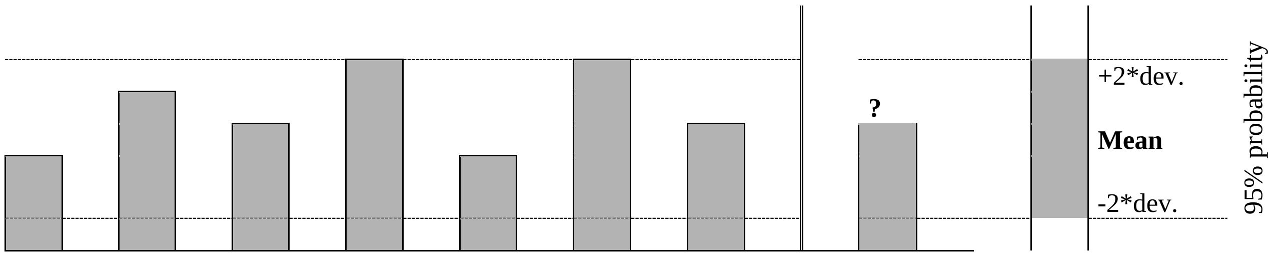
a=?	a^2=b^2+c^2 - 2*b*c*cosA
a=5 b=8 c=6	$\cos A = \frac{b^2+c^2-a^2}{2*b*c}$ $\cos A = \frac{8^2+6^2-5^2}{2*8*6}$ A=cos-1(0.781) A=38.6

b=?	b^2=a^2+c^2 - 2*a*c*cosB
a=5 b=8 c=6	$\cos B = \frac{a^2+c^2-b^2}{2*a*c}$ $\cos B = \frac{5^2+6^2-8^2}{2*5*6}$ B=cos-1(-0.05) B=92.9

C = ?	A+B+C=180
A=38.6 B=92.9	C=180-A-B C=180-38.8-92.9 C=48.5

Statistics, Stochastic Variation

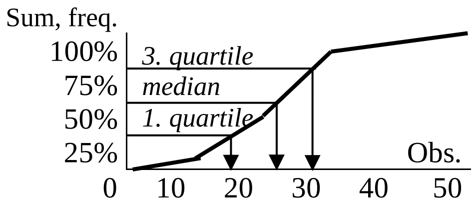
Numbers may be predictable or unpredictable. Unpredictable numbers are also called random or stochastic numbers. Numbers that cannot be pre-dicted can often be post-dicted by setting up a statistics on their former behavior. A statistical table contains two columns, one with the numbers and one with their frequencies.
If arranged in increasing order:
The median = the middle observation, 1. (3.) quartile = the middle observation in the 1. (2.) half.
A histogram shows the frequencies
An ogive shows the cumulated frequencies from which the three quartiles can be read and reported on a box-plot



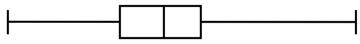
Statistics *Probability*, Post-dicts mean and deviation
pre-dicts x = Mean ± 2* deviation

1. Observations
x: 10, 12, 22, 12, 15, ..., **2. Grouping and counting frequencies,** Observations Frequency Rel. Freq.
Sum. freq. **Ogive**

x	h	p	∑p
0-10	3	3/40=0.075	0.075
10-20	12	0.300	0.375
20-30	18	0.450	0.825
30-50	7	0.175	1.000
Total	40	1.000	

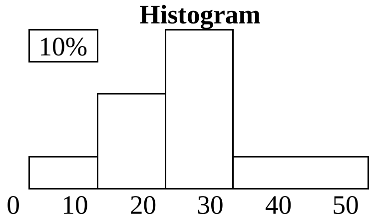


A Boxplot contains the median and two quartiles and the least and greatest observation



3. Mean or average: IF all the observations were the same ... however, the deviate
Observations Frequency Rel. Freq. Sum. freq. Mean **Histogram**

x	h	p	∑p	μ = ∑xi*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375
10-20	12	0.300	0.375	4.5
20-30	18	0.450	0.825	11.25
30-50	7	0.175	1.000	7
Total	40	1.000		23.1



4. Variance, deviation: IF all the deviations were the same ...
Observations Frequency Rel. Freq. Sum. freq. Mean Distance Variance

x	h	p	∑p	μ = ∑xi*pi	|xi - μ|	v = ∑(xi-μ)^2*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375	|5-23.1|=18.13	18.13^2*0.075=24.64
10-20	12	0.300	0.375	4.5	8.13	19.80
20-30	18	0.450	0.825	11.25	1.88	1.58
30-50	7	0.175	1.000	7	16.88	49.83
Total	40	1.000		23.1		1 s^2 = 95.86

Deviation s = √95.86 = 9.8

5. Prediction: x = Mean ± 2*deviation =μ ± 2*s = 23.1 ± 19.6 *Confidence-interval* = [3.5 ; 42.7]

6. Using technology

On a GDC the interval midpoints are entered under STAT. Rel. frequency = freq/sum(freq). CumFreq = cumsum(freq).

Obs.	Freq	Rel.freq	CumFreq
0	2	.05	.050
1	5	.125	.175
2	9	.225	.400
3	12	.300	.700
4	8	.200	.900
5	4	.100	1.000

The different numbers can be calculated using 1-var statistics:
Mean m = 2.8
Standard deviation, s = 1.3
Confidence-interval = m ± 2*s = [0.2;5.4]
1. quartile = 2
Median = 3
3. quartile = 4

Polynomials and Calculus

0. degree polynomial tells the (initial) point 1. degree polynomial tells the (initial) gradient or steepness, 2. degree polynomial tells the (initial) bending, 3. degree polynomial tells the (initial) counter-bending, 4. degree polynomial tells the (initial) counter-counter-bending	y = 5 y = 5 + 2*x y = 5 + 2*x + 0.3*x^2 y = 5 + 2*x + 0.7*x^2 – 0.2*x^3 y = 5 + 2*x + 0.7*x^2 – 0.2*x^3 + 0.3*x^4

Arabic numbers are polynomials: 4352 = 4*10^3 + 3*10^2 + 5*10 + 2. General form: y = 4*x^3 + 3*x^2 + 5*x + 2

Polynomials with bending graphs (degree over 1) have some interesting points:

Turning points, either top-points (maximum) or bottom-points (minimum).

Intersections with the x-axis (zeros), with the y-axis (y-intercept), or with other graphs.

Intersecting other graphs (equations graphically), Intersecting vertical lines (tracing values).

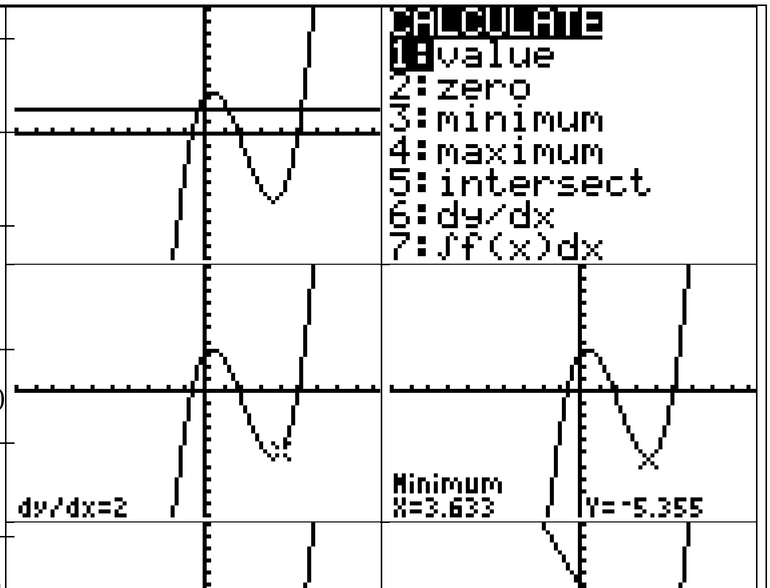
Shifting bending or curvature, where the bending changes its sign.

Tangent-point. A tangent is a straight line practically coinciding with the graph around the contact point, thus showing a scenario: this is how the graph would look like if the steepness stays constant.

If the curve graphs per-numbers, the total is found as the area under the per-number graph, i.e. by integration
If the curve graphs a Total, the per-numbers are found as the steepness of the total graph, i.e. by differentiation
Differentiation twice gives the bending, being positive when bending upwards and negative when bending downwards.
Finding the steepness (gradient, slope) formula is called differentiation. Finding the area under a curve is called integration. Together differentiation and integration are inverse operations called Calculus

Example: y = 0.5x^3-3x^2+2x+3

	Graphics	Formula
Intersecting the y-axis CALC value	y = 3	y1(0)
Intersecting the x-axis CALC zero	x = -0.694 x = 1.748 x = 4.946	Solve(0=Y1)
Intersecting y = 2 CALC intersection	x = -0.329 x = 1.181 x = 5.147	Solve(0=Y1-2)
Top CALC Maximum	x = 0.367 y = 3.355	MATH fMax(Y1,x,0,7)
Bottom CALC Minimum	x = 3.633 y = -5.355	MATH fMin(Y1,x,0,7)
Steepness in x = 4 CALC dy/dx	2	MATH nDeriv(Y1,x,4)
Area from 3 to 4 CALC Sf(x)dx	-5.125	MATH fnInt(Y1,x,3,4)
Tangent in x = 1 DRAW tangent x = 1	y=-2.5x+5	



Tasks

1. Repeat as above with y = 0.7x^3-4x^2+3x+4.
2. Repeat as above with y = -0.4x^3+2x^2-0.5x-3.
3. Produce your own polynomials using randM(4,1).
4. Produce your own polynomials using regression
5. Find the cheapest box without lid containing 1 liter.
6. Find the cheapest pipe without lid containing 1 liter.
7. Find the cheapest box with lid containing 1 liter.
8. Find the cheapest pipe with lid containing 1 liter.
9. Find the cheapest box with double lid containing 1 liter.
10. Find the cheapest pipe with double lid contain. 1 liter.
11. Find the cheapest cone without lid containing 1 liter.
12. Find the cheapest cone with lid containing 1 liter.
13. Find the cheapest cone with double lid contain. 1 liter.
14. y1 is a polynomial of degree 0. If y1 is a Total, what is its per-number? If y1 is a per-number, what is its total?
15. As 14 with polynomials of degree 1.
16. As 14 with polynomials of degree 2.
17. As 14 with polynomials of degree 3.

Two Equations with Two Unknowns; and Three

Two equations with two unknowns are solved manually, by intersection or by matrices	b \$ + 5kg at a \$/kg = 25 \$ b \$ + 8kg at a \$/kg = 34 \$	x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1&5\\1&8\end{pmatrix}*\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$

2 equations with 2 unknowns: The formula b \$ + 5kg at a \$/kg = 25\$ contains 2 unknowns and cannot be solved, unless we know another example of the same formula as e.g. b \$ + 8kg at a \$/kg = 34 \$.

Written as an equation system	Written as a matrix equation
x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1&5\\1&8\end{pmatrix}*\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$

Manually one variable is isolated in the first and inserted in the second equation: x=25-5*y, 25-5*y+8*y=34, y=3 & x=10.

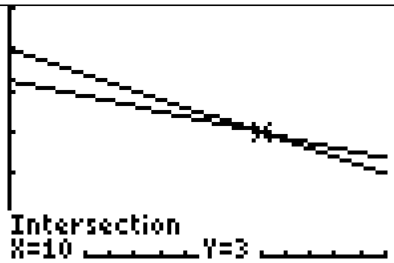
Using graphs, the y's are isolated and inserted into the y- editor.

x + 5*y = 25 gives y = (25-x)/5 ,

x + 8*y = 34 gives y = (34-x)/8

The intersection point is found by 'Calc Intersection' to x = 10 and y = 3.

Also we can use Math Solver 0 = Y1 –Y2.



Matrix-solutions is found by entering the matrices into the matrix-editor as ml and mr (matrix-left & -right):

$\underline{V}=\begin{pmatrix}x\\y\end{pmatrix}$ = ?	$\underline{\underline{ml}}*\underline{V}=\underline{mr}$
$\underline{\underline{ml}}=\begin{pmatrix}1&5\\1&8\end{pmatrix}$ $\underline{mr}=\begin{pmatrix}25\\34\end{pmatrix}$	$\underline{V}=\underline{\underline{ml}}$^-1 * $\underline{mr}$ $\underline{V}=\begin{pmatrix}1&5\\1&8\end{pmatrix}$ ^-1 * $\begin{pmatrix}25\\34\end{pmatrix}$ $\underline{V}=\begin{pmatrix}10\\3\end{pmatrix}$
Test	$\begin{pmatrix}1&5\\1&8\end{pmatrix}*\begin{pmatrix}10\\3\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$ $\begin{pmatrix}25\\34\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$

$\underline{V}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ = ?	$\underline{\underline{ml}}*\underline{V}=\underline{mr}$
$\underline{\underline{ml}}=\begin{pmatrix}2&5&2\\1&0&-1\\4&-3&6\end{pmatrix}$ $\underline{mr}=\begin{pmatrix}18\\-2\\16\end{pmatrix}$	$\underline{V}=\underline{\underline{ml}}$^-1 * $\underline{mr}$ $\underline{V}=\begin{pmatrix}2&5&2\\1&0&-1\\4&-3&6\end{pmatrix}$ ^-1 * $\begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\underline{V}=\begin{pmatrix}1\\2\\3\end{pmatrix}$
Test	$\begin{pmatrix}2&5&2\\1&0&-1\\4&-3&6\end{pmatrix}*\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\begin{pmatrix}18\\-2\\16\end{pmatrix}=\begin{pmatrix}18\\-2\\16\end{pmatrix}$

3 equations with 3 unknowns cannot be solved graphically, but manually and by using matrices:

Written as an equation system	Written as a matrix equation
3*x + 5*y + 2*z = 19 x – z = -2 4*x – 3*y + 6*z = 16	$\begin{pmatrix}3&5&2\\1&0&-1\\4&-3&6\end{pmatrix}*\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}19\\-2\\16\end{pmatrix}$

A matrix-solution is found by entering the matrices into the matrix-editor as ml and mr.

4 equations with 4 unknowns, 5 equations with 5 unknowns etc. Like 3 equations with 3 unknowns.

Equation systems for skill building can be generated by 'randM(3,3)' and 'randM(3,1).

Tasks. Solve the equation systems

1. 4x - 1*y = -9 4x - 4*y = 0 2. 4x + 2*y = 16 5x - 3*y = -2 3. 7x + 4*y = -1 -3x + 2*y = 19 4. 2x - 5*y = 16 3x - 4*y = 17	5. -7*x - 3*y - 7*z = 3 -1*x - 5*y + 1*z = -13 9*y - 5*z = 36 6. 4*x + 3*y + 7*z = 81 5*x + 3*y + 1*z = 54 2*x + 9*y + 5*z = 57 7. 2*x + 3*y - 1*z = - 6 5*x + 3*y - 4*z = -15 2*x – 2*y + 5*z = 40	8. 2*x + 5*y - 1*z + 9t = 118 1*x + 1*y - 9*z - 5t = -88 - 3*y + 7*z + 5t = -51 -3*x + 5*y + 2*z - 5t = -10 9. -6*x - 1*y + 8*z + 8t = 129 -2*x + 2*y - 5*z + 7t = 60 8*x + 6*y + 3*z + 3t = -40 -7*x - 4*y - 8*z - 4t = 12

Letter Calculation, Transposing Formulas

Change the T-formulas to a-formulas, b-formulas and c-formulas, and vice versa.

	T		a	b	c
1	$T = a + b\cdot c$		$a = T - b\cdot c$	$b = \frac{T-a}{c}$	$c = \frac{T-a}{b}$
2	$T = a - b\cdot c$		$a = T + b\cdot c$	$b = \frac{a-T}{c}$	$c = \frac{a-T}{b}$
3	$T = a + \frac{b}{c}$		$a = T - \frac{b}{c}$	$b = (T-a)\cdot c$	$c = \frac{b}{T-a}$
4	$T = a - \frac{b}{c}$		$a = T + \frac{b}{c}$	$b = (a-T)\cdot c$	$c = \frac{b}{a-T}$
5	$T = (a + b)\cdot c$		$a = \frac{T}{c} - b$	$b = \frac{T}{c} - a$	$c = \frac{T}{a+b}$
6	$T = (a - b)\cdot c$		$a = \frac{T}{c} + b$	$b = a - \frac{T}{c}$	$c = \frac{T}{a-b}$
7	$T = \frac{a+b}{c}$		$a = T\cdot c - b$	$b = T\cdot c - a$	$c = \frac{a+b}{T}$
8	$T = \frac{a-b}{c}$		$a = T\cdot c + b$	$b = a - T\cdot c$	$c = \frac{a-b}{T}$
9	$T = \frac{a}{b+c}$		$a = T\cdot (b+c)$	$b = \frac{a}{T} - c$	$c = \frac{a}{T} - b$
10	$T = \frac{a}{b-c}$		$a = T\cdot (b-c)$	$b = \frac{a}{T} + c$	$c = b - \frac{a}{T}$
11	$T = \frac{a}{b} + c$		$a = (T-c)\cdot b$	$b = \frac{a}{T-c}$	$c = T - \frac{a}{b}$
12	$T = \frac{a}{b} - c$		$a = (T+c)\cdot b$	$b = \frac{a}{T+c}$	$c = \frac{a}{b} - T$
13	$T = a\cdot b^{c}$		$a = \frac{T}{b^{c}}$	$b = \sqrt[c]{\frac{T}{a}}$	$c = \frac{\log(\frac{T}{a})}{\log b}$
14	$T = \frac{a}{b^{c}}$		$a = T\cdot b^{c}$	$b = \sqrt[c]{\frac{a}{T}}$	$c = \frac{\log(\frac{a}{T})}{\log b}$
15	$T = (a\cdot b)^{c}$		$a = \frac{\sqrt[c]{T}}{b}$	$b = \frac{\sqrt[c]{T}}{a}$	$c = \frac{\log T}{\log(a\cdot b)}$
16	$T = (\frac{a}{b})^{c}$		$a = \sqrt[c]{T}\cdot b$	$b = \frac{a}{\sqrt[c]{T}}$	$c = \frac{\log T}{\log(\frac{a}{b})}$
17	$T = (a + b)^{c}$		$a = \sqrt[c]{T} - b$	$b = \sqrt[c]{T} - a$	$c = \frac{\log T}{\log(a+b)}$
18	$T = (a - b)^{c}$		$a = \sqrt[c]{T} + b$	$b = a - \sqrt[c]{T}$	$c = \frac{\log T}{\log(a-b)}$
19	$T = a + b^{c}$		$a = T - b^{c}$	$b = \sqrt[c]{T-a}$	$c = \frac{\log(T-a)}{\log b}$
20	$T = a - b^{c}$		$a = T + b^{c}$	$b = \sqrt[c]{a-T}$	$c = \frac{\log(a-T)}{\log b}$
21	$T = a^{(b+c)}$		$a = \sqrt[(b+c)]{T}$	$b = \frac{\log T}{\log a} - c$	$c = \frac{\log T}{\log a} - b$
22	$T = a^{(b-c)}$		$a = \sqrt[(b-c)]{T}$	$b = \frac{\log T}{\log a} + c$	$c = b - \frac{\log T}{\log a}$

Constant Change, Routine Problems

Please use a formula table for the problems 1-40:

	y	b	c	*Answer* A: y = b+c	B: y = b*c
1		4	9	13	36
2		5	4	9	20
3		3	5	8	15
4		2	7	9	14
5	13	3		10	4.333
6	14	6		8	2.333
7	15	9		6	1.667
8	16	6		10	2.667
9	17		7	10	2.429
10	14		3	11	4.667
11	15		9	6	1.667
12	16		6	10	2.667

	y	b	c	*Answer* C: b = y–c	D: b = y/c
1		6	4	10	24
2		7	8	15	56
3		3	3	6	9
4		4	5	9	20
5	12	7		5	1.714
6	14	6		8	2.333
7	19	9		10	2.111
8	13	6		7	2.167
9	24		7	17	3.429
10	26		3	23	8.667
11	27		9	18	3.000
12	29		6	23	4.833

A **y = b+c*x**

	y	b	c	x	*Answer*
13		120	4	5	140
14		140	12	16	332
15		160	-8	7	104
16		180	-12	12	36
17	230		5	6	200
18	441		13	17	220
19	184		-7	8	240
20	117		-11	13	260
21	322	280		7	6
22	552	300		18	14
23	266	320		9	-6
24	200	340		14	-10
25	416	360	7		8
26	665	380	15		19
27	350	400	-5		10
28	285	420	-9		15

B **y = b*x^c**

	y	b	c	x	*Answer*
13	633,36	120,00	1,20	4,00	633,36
14	2753,14	140,00	1,10	15,00	2753,14
15	560,82	160,00	0,70	6,00	560,82
16	1817,57	180,00	0,80	18,00	1817,57
17	1430,29		1,27	5,00	185,24
18	723,06		1,15	16,00	29,82
19	109,62		0,75	7,00	25,47
20	4822,30		0,85	19,00	394,74
21	2814,60	280,00		6,00	1,29
22	9013,44	300,00		17,00	1,20
23	1854,65	320,00		8,00	0,85
24	5630,46	340,00		20,00	0,94
25	3920,50	360,00	1,36		5,79
26	7092,74	380,00	1,25		10,40
27	3113,64	400,00	0,85		11,18
28	6950,91	420,00	0,95		29,18

A **y = b*(1+r)**

	y	b	r	*Answer*	*Doubling*
29		120	4.3%	125.160	16.5
30		140	24.0%	173.600	3.2
31		160	-7.4%	148.160	-9.0
32		180	-11.7%	158.940	-5.6
33	230		5.2%	218.631	13.7
34	441		13.3%	389.232	5.6
35	184		-6.1%	195.953	-11.0
36	117		-10.7%	131.019	-6.1
37	322	281		14.6%	5.1
38	552	312		76.9%	1.2
39	266	320		-16.9%	-3.7
40	200	340		-41.2%	-1.3

B **y = b*(1+r)^x**

	y	b	r	x	*Answer*	*Doubling*
29		130	4.3%	4	153.844	16.5
30		140	24.0%	6	508.930	3.2
31		160	-7.4%	9	80.097	-9.0
32	120		20.5%	3	68.584	3.7
33	140		10.2%	4	94.930	7.1
34	160		-8.7%	5	252.212	-7.6
35	612	416		7	5.7%	12.5
36	274	165		8	6.5%	11.0
37	36	92		9	-9.9%	-6.6
38	892	280	27.6%		4.754	2.8
39	674	300	15.3%		5.686	4.9
40	83	320	-25.5%		4.584	-2.4

45	2^5	32	$x^5 = 30$	1.974	$5^x = 30$	2.113	$\sqrt[4]{(120/30)}$	1.414	$\frac{\log 12}{\log 1.04}$	63.357	$\frac{\log(120/30)}{\log 1.04}$	35.346
46	$3^{1.7}$	6.473	$x^7 = 40$	1.694	$3^x = 90$	4.096	$\sqrt[5]{(130/20)}$	1.454	$\frac{\log 130}{\log 1.5}$	12.005	$\frac{\log(130/40)}{\log 1.05}$	24.158
47	6^{-2}	0.028	$x^4=120$	3.310	$2^x = 70$	6.129	$\sqrt[6]{(140/12)}$	1.506	$\frac{\log 0.23}{\log 0.96}$	36.002	$\frac{\log(40/50)}{\log 0.96}$	5.466
48	$5^{-1.3}$	0.123	$x^6=140$	2.647	$4^x = 80$	3.161	$\sqrt[7]{(150/30)}$	1.258	$\frac{\log 0.15}{\log 0.87}$	13.623	$\frac{\log(50/130)}{\log 0.87}$	6.861

Forecast Problems

Linear Change

Answer

		Table		*Forecast*			*Forecast*		*Formula*
	x	*0*	*?*	*?*	*?*		*12*	*30*	
1	**Year**	*1978*	*1983*	*1990*	*?*		1990	**2008**	
	y	120	130	?	180	y=?	**144**	180	y=120+2*x
		+c=?					**c=**	**2**	
	x	*0*	*?*	*?*	*?*		*12*	*32*	
2	**Year**	*1980*	*1984*	*1992*	*?*		1992	**2012**	
	y	240	260	?	400	y=?	**300**	400	y=240+5*x
		+c=?					**c=**	**5**	
	x	*0*	*?*	*?*	*?*		*14*	*15*	
3	**Year**	*1985*	*1990*	*1999*	*?*		1999	**2000**	
	y	170	140	?	80	y=?	**86**	80	y=170-6*x
		+c=?					**c=**	**-6**	
	x	*0*	*?*	*?*	*?*		*12*	*22*	
4	**Year**	*1978*	*1982*	*1990*	*?*		1990	**2000**	
	y	260	240	?	150	y=?	**200**	150	y=260-5*x
		+c=?					**c=**	**-5**	

5 The price was 320$ in 1987, and 460$ in 1994.
What is the total and per-year increase?
With a constant change per year:
What is the price in 1992? When will the price be 400$?

5	*4*	
1992	**1991**	
420	400	y=320+20*x
c=	**20**	

6 A population was 520 in 1987 and 400 in 1995.
What is the total and per-year increase?
With a constant change per year:
What is the level in 1990? When will the level be 130?

3	*26*	
1990	**2013**	
475	130	y=520-15*x
c=	**-15**	

, **Exponential Change**

Answer

		Table		*Forecast*			*Forecast*		*Formula*
	n	*0*	*?*	*?*	*?*		*12*	*13.2*	*Doubling T*
7	**Year**	*1978*	*1983*	*1990*	*?*		1990	**1991.2**	
	y	120	140	?	180	y=?	**174**	180	y=120*1.031^x
		+r%=?		*T = ?*			**r%=**	**3.1%**	T = 22.7
	n	*0*	*?*	*?*	*?*		*12*	*25.5*	
8	**Year**	*1980*	*1984*	*1992*	*?*		1992	**2005.5**	
	y	240	260	?	400	y=?	**305**	400	y=240*1.02^x
		+r%=?		*T = ?*			**r%=**	**2.0%**	T = 35.0
	n	*0*	*?*	*?*	*?*		*14*	*19.4*	
9	**Year**	*1985*	*1990*	*1999*	*?*		1999	**2004.4**	
	y	170	140	?	80	y=?	**99**	80	y=170*0.962^x
		+r%=?		*T = ?*			**r%=**	**-3.8%**	T = -17.9
	n	*0*	*?*	*?*	*?*		*12*	*10.3*	
10	**Year**	*1978*	*1982*	*1990*	*?*		1990	**1988.3**	
	y	260	210	?	150	y=?	**137**	150	y=260*0.948^x
		+r%=?		*T = ?*			**r%=**	**-5.2%**	T = -13.0

11 The price was 320$ in 1987, and 460$ in 1994.
What is the total and per-year increase in percent?
With a constant change percent per year:
What is the price in 1992? When will the price be 400$?

5	*4.3*	
1992	**1991.3**	
415	400	y=320*1.053^x
r%=	**5.3%**	T = 13.4

12 A population was 520 in 1987 and 400 in 1994.
What is the total and per-year increase in percent?
With a constant change percent per year:
What is the level in 1990? When will the level be 150?

3	*71.0*	
1990	**2058.0**	
493	150	y=520*0.983^x
r%=	**-1.7%**	T = -40.4

Homework

1. In the triangle ABC, C is 90, A=42, c=5. Find the rest.
2. In the triangle ABC, C is 90, A=34, a=6. Find the rest.
3. In the triangle ABC, C is 90, A=28, b=7. Find the rest.
4. In the triangle ABC, C is 90, a=5, c=7. Find the rest.
5. In the triangle ABC, C is 90, b=4, c=7. Find the rest.
6. In the triangle ABC, C is 90, a=4, b=5. Find the rest.
7. In the triangle ABC, A is 32.6, b=4.6, c=5.2. Find the rest.
8. In the triangle ABC, A is 34.8, b=5.6, a=7.2. Find the rest.
8. In the triangle ABC, A is 42.6, B=74.6, c=6.2. Find the rest.
10. In the triangle ABC, A is 34.8, C=54.6, a=5.2. Find the rest.

11. (all lin, exp & pow)

x	y
2	10
7	15
9	?
?	30

12

x	y
3	8
7	12
9	?
?	28

13

x	y
1	20
5	30
9	?
?	80

14

x	y
10	80
20	62
30	?
?	30

15

x	y
12	64
18	42
25	?
?	24

16

x	y
3	50
12	28
20	?
?	10

17. In 1993 there was 420 $. In 1998 there was 630 $. In 2005 there was ? $. In ? there was 950 $. Linear and exponential and power change.
18. In 1994 there was 520 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 1250 $. Linear and exponential and power change.
19. In 1992 there was 920 $. In 1996 there was 730 $. In 2005 there was ? $. In ? there was 450 $. Linear and exponential and power change.
20. In 1994 there was 720 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 250 $. Linear and exponential and power change.
21. A capital had 753 $. increased with 20% 4 times and became ? $. What is the doubling-time?
22. A capital had 956 $. decreased with 25% 5 times and became ? $. What is the half-time?
23. A capital had 486 $. increased with 30% ? times and became 2345.83 $. What is the doubling-time?
24. A capital had 324 $. decreased with 35% ? times and became 25.88 $. What is the half-time?
25. A capital had 743 $. increased with ?% 4 times and became 2854.32 $. What is the doubling-time?
26. A capital had 896 $. decreased with ?% 5 times and became 45.09 $. What is the half-time?
27. A capital had ? $. increased with 50% 6 times and became 2423.83 $. What is the doubling-time?
28. A capital had ? $. decreased with 55% 7 times and became 2.45 $. What is the half-time?

31. (Polynomial regr.)

x	y
2	10
7	30
9	35
12	?
?	30
?	turn

32

x	y
3	8
7	5
11	12
9	?
?	28
?	turn

33

x	y
1	20
5	30
7	35
9	?
?	10
?	turn

34

x	y
10	60
20	120
30	30
40	70
50	?
?	80
?	turn

35

x	y
12	74
18	22
20	43
25	41
30	?
?	34
?	turn

36

x	y
3	9
12	28
15	8
17	14
20	?
?	10
?	turn

41. (Mean, ogive. & boxplot)

Obs	Freq
0-10	6
10-20	9
20-30	12
30-40	15
40-50	6

42

Obs	Freq
0-10	50
10-20	20
20-30	10
30-40	20
40-50	30

43

Obs	Freq
0-10	16
10-20	29
20-30	52
30-40	25
40-50	16

44

Obs	Freq
0-10	16
10-20	29
20-30	32
30-40	45
40-50	56
50-60	66

45

Obs	Freq
0-10	12
10-20	56
20-30	42
30-40	13
40-50	73
50-60	25

46

Obs	Freq
0-10	23
10-20	45
20-30	25
30-40	12
40-50	86
50-60	23
60-70	45

51. Solve the equation 2+3*(1+x)^4 = 20
52. Solve the equation 4+5*(1+x)^6 = 30
53. Solve the equation 40-3*(1-x)^4 = 20
54. Solve the equation 50-4*(1-x)^5 = 10
55. Transpose the formulas $T = d - e$, $T = d - \frac{e}{f}$, $T = d - \frac{e-f}{g}$
56. Transpose the formulas $T = \frac{d}{e}$, $T = \frac{d}{e} - f$, $T = \frac{d-e}{f} - g$

Project Forecasting

Problem: How to set up a forecast assuming constant growth?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real-World Problem
A capital is assumed to change constantly. From two data sets we would like to establish a forecast predicting the capital at a certain time and when a certain level will be reached.

2. The Mathematical Problem
We set up a table showing the capital to two different times. X are years, y is counted in 1000 $

x	y = ?
2	10
5	30
8	?
?	60

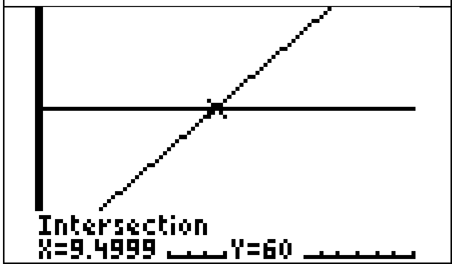
1. Linear ++ change y = c*x + b	x: +1, y: +c (gradient, slope)
2. Exponential +* change y = c*b^x = c*(1+r)^x	x: +1, y: + r% (interest rate, b=1+r)
3. Power ** change y = c*x^b	x: +1%, y: + r% (elasticity)

3. Solution to the Mathematical Problem
First we find the y-formulas using regression. We enter the table as lists L1 and L2 und STAT.
’LinReg Y1’ produces a linear model transferred to the y-list as Y1
’ExpReg Y1’ produces an exponential model transferred to the y-list as Y1
’PowerReg Y1’ produces a power model transferred to the y-list as Y1

Linear change

y = ?	y = 6.667*x – 3.333
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 6.667*8 – 3.333 = 50
Test	Trace x = 8 gives y = 50

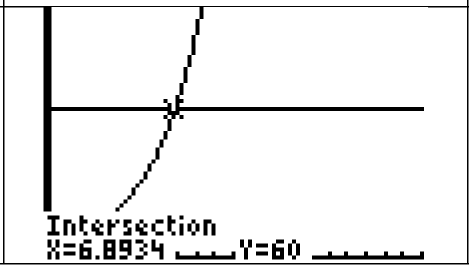
x = ?	y = 6.667*x – 3.333
y = 60	60 = (6.667*x)– 3.333 60 + 3.333 = 6.667*x 63.333/6.667 = x 9.5 = x
Test1	60 = 6.667*9.5– 3.333 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 9.5
Test3	CALC Intersection with y2=60 gives x = 9.5



Exponential change

y = ?	y = 4.807 * 1.442^x
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.807 * 1.442^8 = 89.9
Test	Trace x = 8 gives y = 89.9

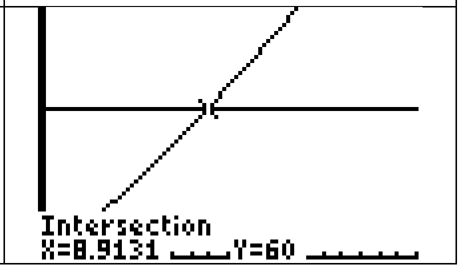
x = ?	y = 4.807 * 1.442^x
y = 60	60 = 4.807 * (1.442^x) 60/4.807 = 1.442^x log(60/4.807)/log1.142 = x 6.89 = x
Test1	60 = 4.807 * 1.442^6.89 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 6.89
Test3	CALC Intersection with y2=60 gives x = 6.89



Power change

y = ?	y = 4.356*x^1.199
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.356*8^1.199 = 52.7
Test	Trace x = 8 gives y = 52.7

x = ?	y = 4.356*x^1.199
y = 60	60 = 4.356*(x^1.199) 60/4.356 = x^1.199 1.199√(60/4.356) = x 8.91 = x
Test1	60 = 4.356*8.91^1.199 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 8.91
Test3	CALC Intersection with y2=60 gives x = 8.91



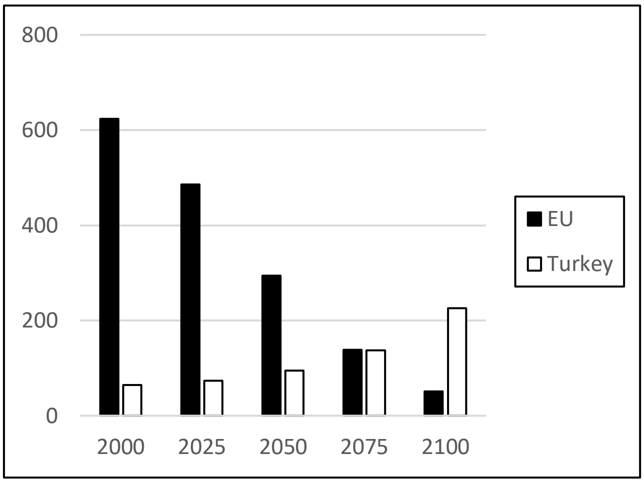
4. Solution to the Real-World Problem
We see that forecast can be made by using technology’s regression lines.
The forecasts give different answers to the same questions since different forms of change is assumed.
Linear change assumes that the gradient is constant
Exponential change assumes that the interest rate is constant
Power change assumes that the elasticity is constant

Project Population Forecast

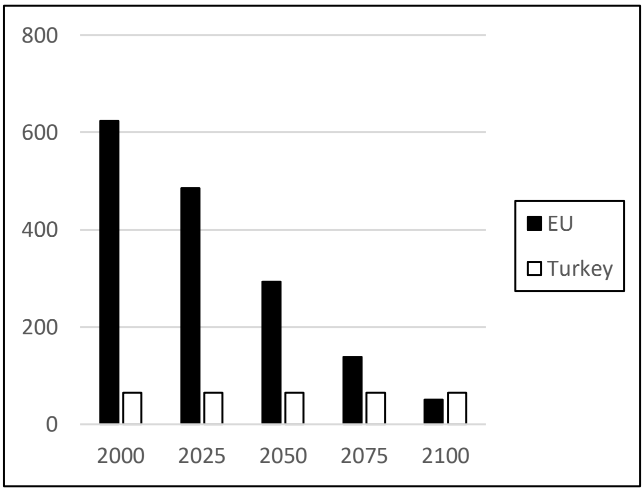
Problem:

From year 2000, find a 100 year forecast for the populations in the EU and Turkey, building on the assumption that the yearly change percent in the EU is -1% if on average each woman gives birth to 1.5 child.
Please, construct three scenarios: , 1. In Turkey, the yearly change percent in Turkey is 0,5% if on average each woman gives birth to 2.2 child
2. In Turkey, the yearly change percent in Turkey is 0% if production causes women to have fewer children
3. In Turkey, the yearly change percent in Turkey is 1% if religion causes women to give birth to 4.4 children

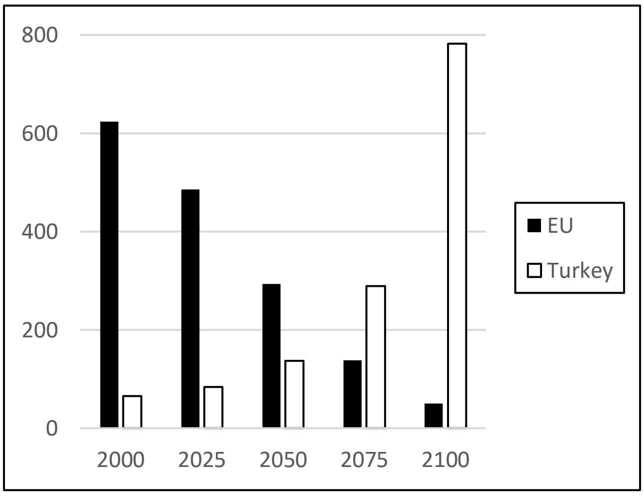
Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	74	94	137	226



Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	65	65	65	65



Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	83	137	289	782



Answer: In all three scenarios Turkey will exceede EU in year 2100

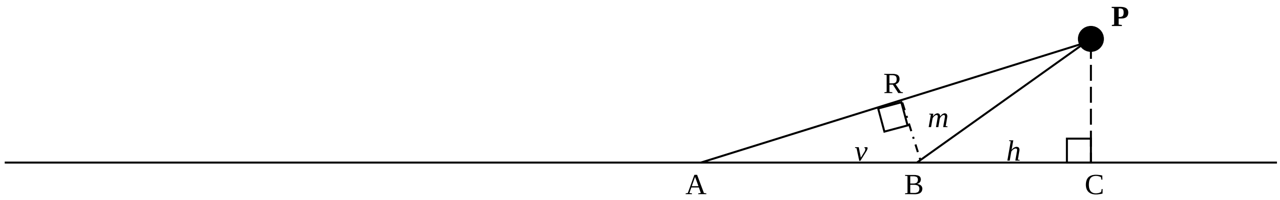
Project Distance to a Far-away Point, **Problem: How to determine the distance to an inaccessible distant point??**

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real-World Problem

From a given baseline we want to determine the distance to a far-away inaccessible point P.

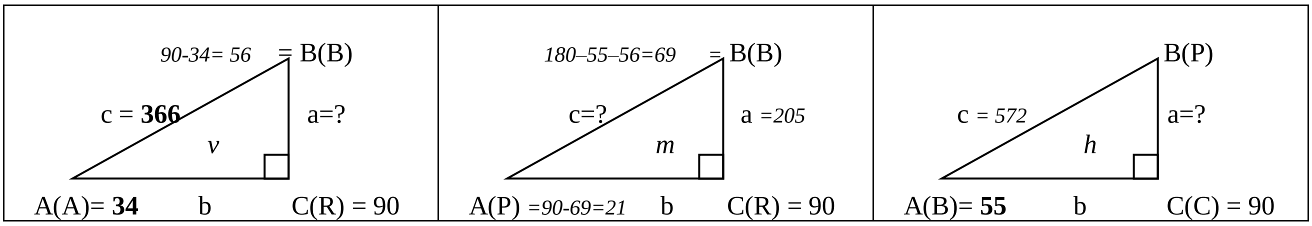


2. The Mathematical Problem

From a known baseline AB we measure the angles A and B to the inaccessible point P.

From the three right angled triangles ABR, BRP and BCP we calculate RB, BP as well as the distance PC.

Measurements: AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees



3. Solution to the Mathematical Problem

We set up three formula tables

Triangle ABR

a = ?	$\sin A = \frac{a}{c}$
A = 34 c = 366	$\sin 34 = \frac{a}{366}$ sin34 * 366 = a 205 = a
Test1 ☺	$\sin 34 = \frac{205}{366}$ 0.559 = 0.560
Test2 ☺	Math Solver $0 = \frac{x}{366} - \sin 34$ gives x = 205

Triangle PBR

c = ?	$\sin A = \frac{a}{c}$
A = 21 a = 205	$\sin 21 = \frac{205}{c}$ c * sin21 = 205 $c = \frac{205}{\sin 21}$ c = 572
Test1 ☺	$\sin 21 = \frac{205}{572}$ 0.358 = 0.358
Test2 ☺	Math Solver $0 = \frac{205}{x} - \sin 21$ gives x = 572

Triangle PBC

a = ?	$\sin A = \frac{a}{c}$
A = 55 c = 572	$\sin 55 = \frac{a}{572}$ sin55 * 572 = a 469 = a
Test1 ☺	$\sin 55 = \frac{469}{572}$ 0.819 = 0.820
Test2 ☺	Math Solver $0 = \frac{x}{572} - \sin 55$ gives x = 469

4. Solution to the Real-World Problem

By using trigonometry we are able to determine the distance to the inaccessible point P to 469 cm.

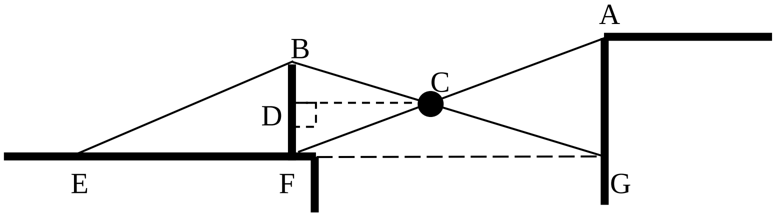
Project the Bridge, **Problem: How to determine the dimensions of a bridge?**

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

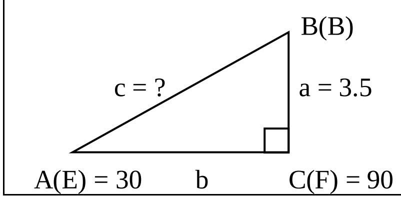
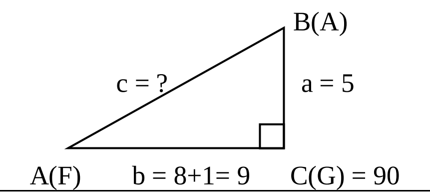
1. The Real-World Problem

Over a canyon a suspension bridge made of steel i fastened to the cliff and to a vertical upright. We want to determine the length of the 3 beams as well as the welding point. The left fixing angle must be 30 degrees.



2. The Mathematical Problem

From the right angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between the lines BG and FA.
Measurements: angle FEB = 30 degrees, FB = 3.5m, FG = 8m + 1m = 9m and AG = 5m.

		I a coordinate system with F as zero the following coordinates emerge: F: (0,0) and A: (9,5), as well as B: (0,3.5) and G: (9,0). Usinmg linear regression we determine the equations for the lines FA and BG.

3. Solution to the Mathematical Problem

We set up formula tables

Triangle EFB

c = ?	sinA = $\frac{a}{c}$
A = 30 a = 3.5	sin30 = $\frac{3.5}{c}$ sin30*c = 3.5 c = 3.5/sin30 c = 7.0
Test1 ☺	sin30 = $\frac{3.5}{7}$ 0.5 = 0.5
Test2 ☺	Math Solver 0 = $\frac{3.5}{x}$ - sin30 gives x = 7

Triangle FGA og GFB

c = ?	$a^2 + b^2 = c^2$
a = 5 b = 9	$5^2 + 9^2 = c^2$ √(106)= c 10.30 = c
Test1 & Test2	
c = ?	$a^2 + b^2 = c^2$
a = 3.5 b = 9	$3.5^2 + 9^2 = c^2$ √(93.25) = c 9.66 = c

Lines BG and FA

BG: ?	y = ax + b
Test	y = – 0.389x + 3.5 Found by LinReg L1, L2, Y1 Trace x=0 gives 3.5 Trace x=9 gives 0 StatPlot fits

Likewise we find
FA: ? y = 0.556x

Calc Intersection gives
x = 3.71 and y = 2.06

In the triangle FDC, DC = 3.71 and FD = 2.06
Pythagoras gives: FC = √(3.71^2 + 2.06^2) = 4.24

In the triangle BDC, DC = 3.71 and FD = 3.6-2.06 = 1.54
Pythagoras gives: BC = √(3.71^2 + 1.54^2) = 4.02

4. Solution to the Real-World Problem

Using trigonometry we have found the lengths of the three steel beams as EB = 7.00 m, FA = 10.30 m and BG = 9.66. The welding point is determined by FC = 4.24 m and BC = 4.02 m.

As an extra control the bridge can be drawn and build by pipe cleaners in the ration 1:100.

Project Golf, **Problem: How to hit a golf hole behind a hedge?**

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real-World Problem

From a position on a 2 meter high flat hill we want to send a golf ball over a 3 meter hedge 2 meter away on the hill to hit a hole situated 12 meters away at level zero.
What is the orbit of the ball? How high is the ball at the distance 10 meters? When does the ball have a height of 6 meters? How high does the ball go? What is the direction of the ball in the beginning, at 10 meters distance and at the impact?

2. The Mathematical Problem

We set up a table with the length x and the height y having the domain 0<x<12.

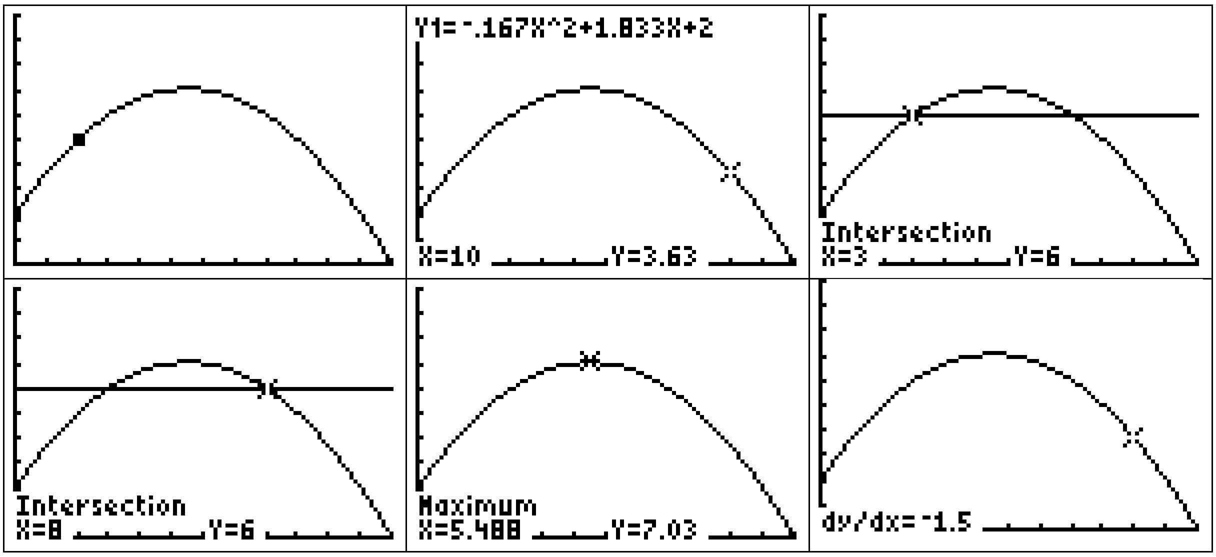
Length x	Height y	Direction v
0	2	?
2	5	
12	0	?
10	?	?
?	6	

3. Solution to the Mathematical Problem

We insert the table as lists L1 and L2. # data-sets allows a 2nd degree polynomial, quadratic regression, which produces the formula y= -0.167x^2 + 1.833x + 2, which is transferred to Y1.

Now the questions asked can be answered using formula tables and a calculator for graphing or calculating. The Y-number can be found by CALC Value, the x-number by CALC Intersection, the maximum by CALC Maximum, and the gradient by CALC dy/dx.

To determine the direction angle v we use the gradient formula: Tan v = gradient = dy/dx.



y = ?	y = y1
x=10	y = y1(10) = 3.667
Test	Trace x=10 gives y=3.67

x = ?	y = y1
y = 6	Math solver 0 = y1 - 6 gives x =3 & x = 8
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC Intersection gives x = 3 and 8

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx ≈ 0 at x = 5.5

v = ?	tan v = dy/dx
x = 12	tan v = -2.167 v=tan-1(-2.167) v =- 65.2

v = ?	tan v = dy/dx
x = 0	tan v = 1.833 v = tan-1(1.833) v = 61.4

v = ?	tan v = dy/dx
x =10	tan v = -1.5 v = tan-1(-1.5) v = -56.3

4. Solution to the Real-World Problem

The orbit of the ball is a parabola. The height of the ball at the distance 10 meters is 3.67 meters? At the distances 3 meters and 8 meters the ball has a height of 6 meters. The ball goes to the maximum height 7.04 meters? The direction of the ball in the beginning, at 10 meters distance and at the impact are 61.4 degrees, -65.2 degrees and -56.3 degrees.

Project Driving, **Problem: How far and how did Peter drive?**

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real-World Problem

When driving, the velocity 100 km/t is 100*1000/(60*60) = 27.8 m/s. A camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s og 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

2. The Mathematical Problem

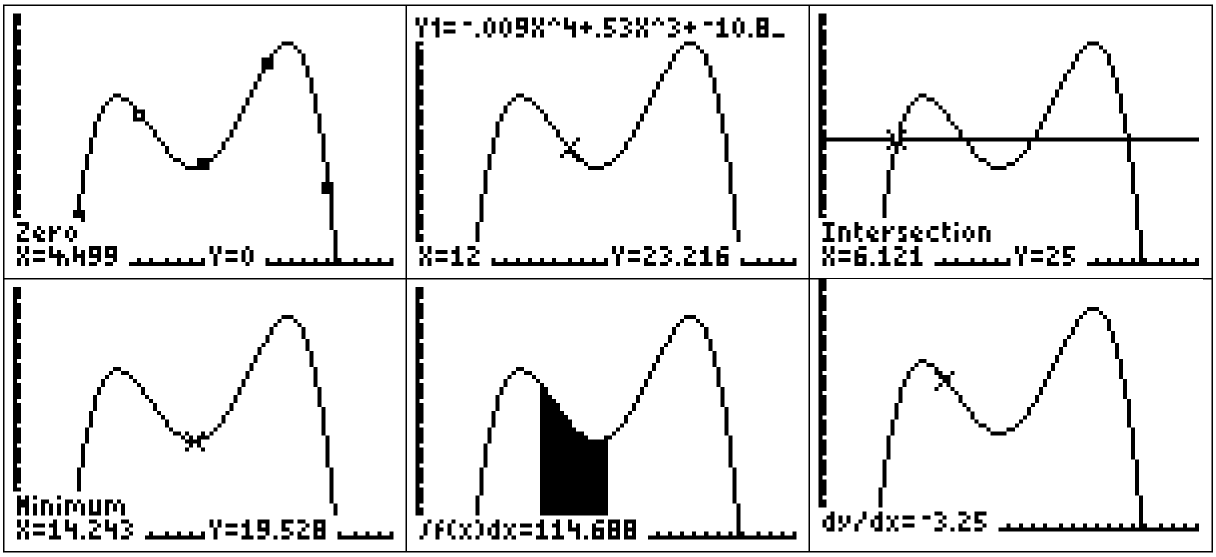
We set up a table showing time x and velocity y. The domain of the table is taken to be 0<x<30.

Time x sec	Velocity y m/s	Accel. dy/dx
5	10	?
10	30	?
15	20	?
20	40	?
25	15	?

3. Solution to the Mathematical Problem

On TI-84 the table is entered as the lists L1 og L2. 5 data sets allow quartic regression (a 4. degree polynomial with a 3-fold parabola) providing the formula y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235 placed as Y1. No the question asked can be answered using formula tables, or using technology, i.e. graphical readings or calculations.

Starting and ending points are found using 'CALC Zero'. Y-numbers are found using 'TRACE'. X-numbers are found using 'CALC Intersection'. Maximum and minimum are found with 'CALC Maximum/Minimum'. The total meter-number is obtained by summing up the m/s*s = ∫Y1dx. Acceleration is found by the gradient 'CALC dy/dx'.



y = ?	y = y1
x=12	y = y1(12) = 3.667
Test	TRACE x = 12 gives y = 23.216

x = ?	y = y1
y = 25	MATH Solver 0 = y1 - 25 gives x =6.12 and …
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC intersection gives x = 6.12, 11.44, 16.86 and 24.47

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx = 0 at x = 5.5

4. Solution to the Real-World Problem

The driving began after 4.50 sec. and ended after 25.62 sec. After 12 sec the velocity was 23.2 m/s. And it was 25m/s after 6.12 sec, 11.44 sec, 16.86 sec and 24.47 sec. Acceleration took place in the time-intervals (4.50; 8.19) and (14.24; 21.74). Deceleration in the intervals (8.19;14.24) and (21.74;25.62). Max-velocity was 44.28 m/s = 159 km/t. after 21.7 sec. In the time-intervals (5;10), (10;15), (15;20) and (20;25) the distance traveled was 142.8 m, 114.7 m, 142.8 m and 189.7 m. The acceleration in the beginning of these time-intervals were 17.75, -3.25, 1.25, 4.25, -21.25 m/s^2. The total distance traveled was 597.4 m.

Project Vine Box

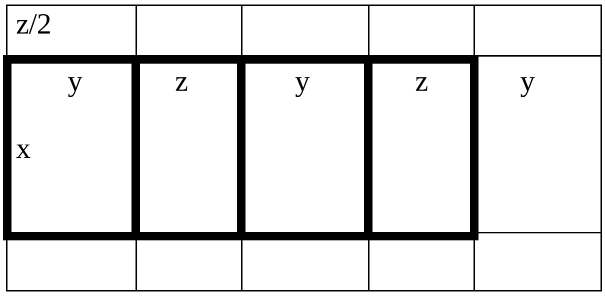
Problem: What are the dimensions of a 3 liters vine bag with the least surface area?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The Real-World Problem

Vine is sold in bottles or in boxes. A 3 liter bag will be constructed by cutting out a piece of cardboard.



2. The Mathematical Problem

The cardboard dimensions are called x, y & z all in dm. We express the volume V and the Surface S as formulas:

$V = x*y*z = 3$, $S = x*(3y+2z) + 2*z/2*(3y+2z)$

3. Solution to the Mathematical Problem

We expand the S-formula: $S = x*(3y+2z) + 2*z/2*(3y+2z) = 3xy + 2xz + 3yz + 2z^2$

We now insert $z = 3/(x*y)$ so that S only depends on two variables x and y:

$S = 3xy + 2xz + 3yz + 2z^2$ and $z = 3/(x*y)$ gives $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2*y^2}$

Scenario A. We assume that y should be half the length of x: $y = 0.5*x$. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2*y^2} = 1.5x^2 + \frac{21}{x} + \frac{72}{x^4}$, which gives $\frac{dS}{dx} = 3x - \frac{21}{x^2} - \frac{288}{x^5} = 0$ for $x = 2.4$

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

x = 2.4 and S = 19.56, so y = 0.5*x = 0.5*2.4 = 1.2, and z = 3/(2.4*1.2) = 1.0

Scenario B. We assume that y should be the same length of x: y = x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2*y^2} = 3x^2 + \frac{15}{x} + \frac{18}{x^4}$, which gives $\frac{dS}{dx} = 6x - \frac{15}{x^2} - \frac{72}{x^5} = 0$ for $x = 1.7$

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

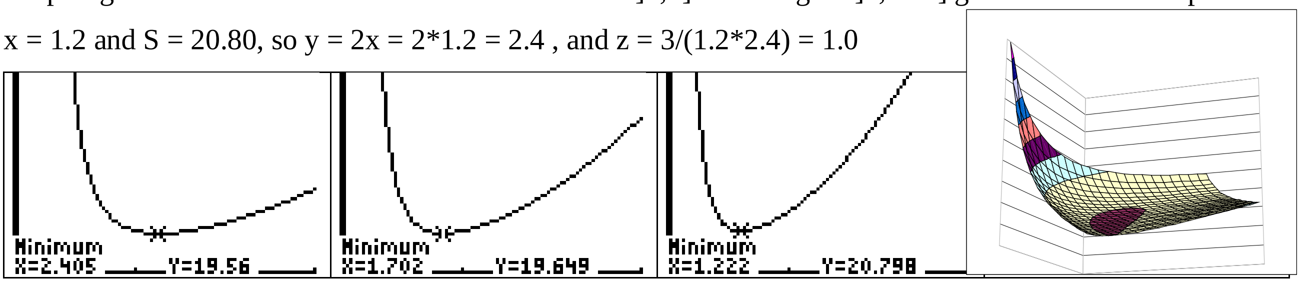
x = 1.7 and S = 19.65, so y = x = 1.7, and z = 3/(1.7*1.7) = 1.0

Scenario C. We assume that y should be double the length of x: $y = 2*x$. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2*y^2} = 6x^2 + \frac{12}{x} + \frac{4.5}{x^4}$, which gives $\frac{dS}{dx} = 12x - \frac{12}{x^2} - \frac{18}{x^5} = 0$ for $x = 1.2$

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

x = 1.2 and S = 20.80, so y = 2x = 2*1.2 = 2.4 , and z = 3/(1.2*2.4) = 1.0



Scenario A | Scenario B | Scenario C | 3 dim. in Excel

4. Solution to the Real-World Problem

We see that the minimum surface area is a little above 19 dm^2. Using an Excel-spreadsheet we can determine the optimal solution to be x = 2.1 and y = 1.4 and z = 1.0, giving a minimum surface area at 19.47 dm^3.

(Graphing $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2*y^2}$ does not give a curve but a surface as shown on the above Excel-file.)

Revision Problems Using TI-84

<table>
<tr><td>1.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>?</td></tr><tr><td>?</td><td>40</td></tr></table></td><td>Answer the question marks in case of a linear model.
Answer the question marks in case of an exponential model. What is the doubling time?
Answer the question marks in case of a power model.</td></tr>
<tr><td>2.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>18</td></tr><tr><td>15</td><td>?</td></tr><tr><td>?</td><td>10</td></tr></table></td><td>Answer the question marks in case of a quadratic model.
Find maxima or minima.
Find the equation for the tangent line in x = 2.
Find the gradient formula.
Find the gradient number in x = 5
Find the area formula
Find the area number from x = 1 to x = 6
Find the intersection points with the line y = 3 + 2x</td></tr>
<tr><td>3.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>14</td></tr><tr><td>12</td><td>18</td></tr><tr><td>15</td><td>?</td></tr><tr><td>?</td><td>30</td></tr></table></td><td>Answer the question marks in case of a cubic model.
Find maxima and minima.
Find the equation for the tangent line in x = 2.
Find the gradient formula.
Find the gradient number in x = 5
Find the area formula
Find the area number from x = 1 to x = 6
Find the intersection points with the line y = 3 + 2x</td></tr>
<tr><td>4.</td><td>3x + 4y = 15 & 5x – 6y = 12</td><td>Solve the simultaneous equations</td></tr>
<tr><td>5.</td><td>Given two points in a coordinate system P(2,4) and Q(6,10)</td><td>Find the midpoint of the line PQ.
Find the equation for the line through P and Q
Find the equation for the normal line to PQ passing through P
Find the angle between PQ and the x-axis.
Find the distance between P and Q
Find the distance from the line PQ to the point S(8,1)
Find the equation for the circle through P and Q and with the midpoint of PQ as centre.
Find the intersection point between the circle and the line y = 12-2x</td></tr>
<tr><td>6.</td><td>Let X be a normal random variable with mean m = 100 and standard deviation d = 12</td><td>$P(X<89) = ?$
$P(X>108) = ?$
$P(93<X<109) = ?$</td></tr>
<tr><td>7.</td><td>X counts the numbers of wins in 100 repetitions of a game with 65% winning chance.</td><td>$P(X<70) = ?$
$P(X\leq 60) = ?$
$P(X\geq 58) = ?$
$P(63<X\leq 72) = ?$</td></tr>
<tr><td>8.</td><td>sin(3x) = 0.4, $0\leq x\leq 2\pi$
con(½x) = -0.3, $0\leq x\leq 2\pi$
tan(2x) = 0.7, $0\leq x\leq 2\pi$</td><td>Find the solutions: Remember to adjust the window
Find the solutions:
Find the solutions:</td></tr>
<tr><td>9.</td><td>A = 40, b = 7, C = 90</td><td>Find a, B and c.</td></tr>
<tr><td>10.</td><td>a = 4, c = 7, C = 90</td><td>Find A, B and b.</td></tr>
<tr><td>11.</td><td>A = 40, b = 7, C = 68</td><td>Find a, B and c.</td></tr>
<tr><td>12.</td><td>A = 40, b = 7, c = 6.8</td><td>Find a, B and C.</td></tr>
<tr><td>13.</td><td>A = 40, b = 7, a = 6.2</td><td>Find c, B and C.</td></tr>
<tr><td>14.</td><td>a = 4, b = 7, c = 6.8</td><td>Find A, B and C.</td></tr>
<tr><td>15.</td><td>$T = \frac{d}{e-f} + g$</td><td>Transpose the T-formula to a d-, e-, f-, and g-formula</td></tr>
<tr><td>16.</td><td>The capital 785 increased with 2.7% 5 times and became ?</td><td>Find the answer
Find the corresponding doubling time.</td></tr>
<tr><td>17.</td><td>The capital 785 increased with 2.7% ? times and became 980</td><td>Find the answer
Find the corresponding doubling time.</td></tr>
<tr><td>18.</td><td>The capital 785 increased with ?% 5 times and became 980</td><td>Find the answer
Find the corresponding doubling time.</td></tr>
<tr><td>19.</td><td>-21</td><td>As 16-18, but with $ instead of %</td></tr>
</table>

Problem 1. *Linear model*

Equation:	y = ax+b
	y = x+9, found by Stat, Calc, LinReg
Test	y1(3) = 12 ☺

y = ?	y = x+9
x = 10	y = 19 found by y1(10)
Test	y = 19 found by CalcValue ☺

x = ?	y = x+9
y = 40	x = 31, found by Math, Solver 0 = y1-40
Test	y1(31) = 40 ☺

Exponential model

Equation:	y = a*b^x
	y = 9.671*1.075^x, found by Stat, Calc, ExpReg
Test	y1(3) = 12 ☺

y = ?	y = 9.671*1.075^x
x = 10	y = 19.853 found by y1(10)
Test	y = 19.853 found by CalcValue ☺

x = ?	y = 9.671*1.075^x
y = 40	x = 19.740, found by Math, Solver 0 = y1-40
Test	y1(19.740) = 40 ☺

Doubling time T = log2/logb = log2/log1.075 = 9.6

Power model

Equation:	y = a*x^b
	y = 8.264*x^0.340 found by Stat, Calc, PwrReg
Test	y1(3) = 12 ☺

y = ?	y = 8.264*x^0.340
x = 10	y = 18.060 found by y1(10)
Test	y = 18.060 found by CalcValue ☺

x = ?	y = 8.264*x^0.340
y = 40	x = 104.024 found by Math, Solver 0 = y1-40
Test	y1(104.024) = 40 ☺

Problem 2. *Quadratic model*

Equation:	y = a*x^2+b*x+c
	y = -0.048x^2+1.476x+8 found by Stat, Calc, QuadReg
Test	y1(3) = 12 ☺

y = ?	y = -0.048x^2+1.476x+8
x = 15	y = 19.429 found by y1(15)
Test	y = 19.429 found by Graph, Calc, Value ☺

x = ?	y = -0.048x^2+1.476x+8
y = 10	x = 1.420 or 29.580 found by Math, Solver 0 = y1-10
Test	y1(1.420) = 10 y1(29.580) = 10 ☺

Maximum:	y = -0.048x^2+1.476x+8
	(x,y) = (15.500,19.140) found by Graph, Calc, Maximum
Test	dy/dx = 0 for x = 15.5 y1(15.5) = 19.14 ☺

Tangent in x = 2	y = -0.048x^2+1.476x+8
x = 2	y = 1.286x + 8.190 found by Graph, Draw, Tangent

Gradient formula	y = -0.048x^2+1.476x+8
	y' = -0.095*x + 1.476, found by TI89
Test	∫y'dx = -0.048x^2+1.476x found by TI89 ☺

Gradient number:	y = -0.048x^2+1.476x+8
x = 5	dy/dx = 1 for x = 5 found by Graph, Calc, dy/dx
Test	1, found by Math, nDeriv ☺

Area formula:	y = -0.048x^2+1.476x+8
x = 2	∫ydx = -0.016*x^3 + 0.738*x^2 + 8.000*x found by TI89
Test	d(∫ydx)/dx = -0.048x^2+1.476x+8 found by TI89 ☺

Area number:	y = -0.048x^2+1.476x+8
	$\int_1^6 y\,dx$ = 62.421, found by Graph, Calc, ∫f(x)dx
Test	62.421, found by Math, fnInt ☺

Intersection points	y = -0.048x^2+1.476x+8 and y = 3+2x (y1 = y3)
	(x,y) = (-17.130,-31.260) and (x,y) = (6.130, 15. 260), found by Math, Solver 0 = y1-y3 and y1(-17.130) = -31.260 etc.
Test	tested by Graph, Calc, Intersect ☺

Problem 3. *Cubic model*

Equation:	y = a*x^3+b*x^2+c*x+d
	y = 0.086x^3-1.952x^2 +13.752x-14, found by Stat, Calc, CubicReg
Test	y1(3) = 12 ☺

y = ?	y = 0.086x^3-1.952x^2 +13.752x-14
x = 15	y = 42.286 found by y(15)
Test	y = 42.286 found by Graph, Calc, Value ☺

x = ?	y = 0.086x^3-1.952x^2 +13.752x-14
y = 30	x = 13.885 found by Math, Solver 0 = y1-30
Test	y1(13.885) = 30 ☺

Maximum Minimum:	y = 0.086x^3-1.952x^2+13.752x-14
	Max: (x,y) = (5.552, 16.841) found by Graph, Calc, Maximum Min: (x,y) = (9.634, 13.925) found by Graph, Calc, Minimum
Test	dy/dx = 0 for x = 5.552 y1(5.552) = 16.841 dy/dx = 0 for x = 9.643 y1(9.643) = 13.925 ☺

Tangent in x = 2	y = 0.086x^3-1.952x^2+13.752x-14
x = 2	y = 6.971x – 7.562 found by Graph, Draw, Tangent

Gradient formula	y = 0.086x^3-1.952x^2+13.752x-14
	y' = 0.257*x^2 – 3.905*x + 13.752, found by TI89
Test	∫y'dx = 0.086x^3-1.952x^2+13.752x found by TI89 ☺

Gradient number:	y = 0.086x^3-1.952x^2+13.752x-14
x = 5	y'(5) = 0.657 found by Graph, Calc, dy/dx
Test	0.657, 1 found by Math, nDeriv ☺

Area formula:	y = 0.086x^3-1.952x^2+13.752x-14
x = 2	∫ydx = 0.021*x^4 – 0.651*x^3+6.876*x^2+14*x found by TI89
Test	d(∫ydx)/dx = 0.086x^3-1.952x^2+13.752x-14 found by TI89 ☺

Area number:	y = 0.086x^3-1.952x^2+13.752x-14
	$\int_1^6 ydx$ = 58.496, found by Graph, Calc, ∫f(x)dx
Test	58.496, 62.421 found by Math, fnInt ☺

Intersection points with y = 3+2x: (x,y) = (2.129,-7.259) and (x,y) = (6.657, 16.315) and (x,y) = (13.991, 30.981) found by Math, Solver 0 = y1-y3 , tested by Graph, Calc, Intersect.

Problem 4

Solutions: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix}$, found by A*B = C, B = A^-1*C, where A = $\begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix}$ and B = $\begin{pmatrix} x \\ y \end{pmatrix}$ and C = $\begin{pmatrix} 15 \\ 12 \end{pmatrix}$

Tested by A*B = C : A*B = $\begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix} * \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$ = C ☺

Problem 5

Midpoint:	$(x,y) = (\frac{x1+x2}{2}, \frac{y1+y2}{2})$
x1 = 2 x2 = 6 y1 = 4 y2 = 10	$(x,y) = (\frac{2+6}{2}, \frac{4+10}{2})$ (x,y) = (4,7)
Test	Tested geometrically ☺

Gradient PQ:	$a = \frac{y2-y1}{x2-x1}$
x1 = 2 x2 = 6 y1 = 4 y2 = 10	$a = \frac{10-4}{6-2}$ a = 3/2 a = 1.5
Test	Tested geometrically ☺

Line PQ:	y = y1 + a*(x – x1)
a = 1.5 x1 = 2 y1 = 4	y = 4 + 1.5*(x – 2) y = 1.5*x + 1
Test	Tested geometrically ☺

Gradient perpend.:	c*a = -1
a = 3/2	c = -2/3 found by Math, Solver 0 = c*3/2+1
Test	Tested geometrically ☺

Normal:	y = y1 + a*(x – x1)
a = -2/3 x1 = 2 y1 = 4	y = 4 + -2/3*(x – 2) y = -2/3*x + 5.333
Test	Tested geometrically ☺

Distance PQ	$d = \sqrt{(x2-x1)^2 + (y2-y1)^2}$
x1 = 2 x2 = 6 y1 = 4 y2 = 10	$d = \sqrt{(6-2)^2 + (10-4)^2}$ d = 7.21
Test	Tested geometrically ☺

Distance point-line	$d = \frac{\lvert y1 - a*x1 - b\rvert}{\sqrt{1 + a^2}}$
a = 1.5 b = 1 x1 = 8 y1 = 1	$d = \frac{\lvert 1 - 1.5*8 - 1\rvert}{\sqrt{1 + 1.5^2}}$ d = 6.66
Test	Tested geometrically ☺

Circle equation	$(x-c1)^2 + (y - c2)^2 = r^2$
r = ½*7.21 r = 3.61 c1 = 4 c2 = 7	$(x -4)^2+(y - 7)^2 = 3.61^2$ $(x -4)^2+(y - 7)^2 = 13.03$
Test	Tested geometrically ☺

Intersection	$(x-c1)^2 + (y - c2)^2 = r^2$ and y = 12-2x
r = ½*7.21 = 3.61 c1 = 4 c2 = 7	(x,y) = (1.30,9.40) and (4.30,3.40) found by Math, Solver $0 = (x-4)^2 + (12-2x-7)^2 – 3.61^2$
Test	Tested geometrically ☺

Angle: tan(v) = a, a = 3/2 ; v = 56.31 found by Math, Solver 0 = tanv - 3/2, v>0 and v<90. Tested geometrically ☺

Problem 6	Problem 7
p(X<115) = 0.894, found by normalCdf(1EE-99,115,100,12)	p(X<70) = 0.827, found by binomCdf(100,0.65,0,69)
p(X<89) = 0.180, found by normalCdf(1EE-99,89,100,12)	p(X≤60) = 0.172, found by binomCdf(100,0.65,0,60)
p(X>108) = 0.253, found by normalCdf(108,1EE99,100,12)	p(X≥58) = 0.941, found by binomCdf(100,0.65,58,100)
p(93<X<109) = 0.494, found by normalCdf(93,109,100,12)	p(63<X≤72) = 0.571, found by binomCdf(100,0.65,64,72)

Problem 8

x = ?	sin(3x) = 0.4
	x = 0.137, or 0.910, or 2.232 or 3.004 or 4.326 or 5.099 found by Math, Solver 0 = y1-0.4
Test	tested by Graph, Calc, Intersect ☺

x = ?	cos(½x) = -0.3
	x = 3.745 found by Math, Solver 0 = y1+0.3
Test	tested by Graph, Calc, Intersect ☺

x = ?	tan(2x) = 0.7
	x = 0.305, or 1.876, or 3.447 or 5.018 found by Math, Solver 0 = y1-0.7
Test	tested by Graph, Calc, Intersect ☺

Problem 9

a = ?	tan A = a/b
A = 40 b = 7	a = 5.874 found by Math, Solver 0 = a/7-tan40
Test	tan40 = 5.874/7 0.839 = 0.839 ☺

c = ?	cos A = b/c
A = 40 b = 7	c = 9.138 found by Math, Solver 0 = 7/c-cos40
Test	cos40 = 7/9.138 0.766 = 0.766 ☺

B = ?	A + B = 90
A = 40	B = 50 found by Math, Solver 0 = 40+B-90
Test	50+40 = 90 90 = 90 ☺

Problem 10

b = ?	a^2 + b^2 = c^2
a = 4 c = 7	b = 5.745 found by Math, Solver 0 = 4^2+b^2-7^2
Test	4^2 + 5.745^2 = 7^2 49 = 49 ☺

A = ?	sinA = a/c
a = 4 c = 7	A = 34.85 found by Math, Solver 0 = 4^2+b^2-7^2
Test	Sin34.85 = 4/7 0.571 = 0.571 ☺

B = ?	A + B = 90
A = 34.85	B = 55.15 found by Math, Solver 0 = 34.85+B-90
Test	34.85+55.15 = 90 90 = 90 ☺

Problem 11

B = ?	A+B+C = 180
A = 40 C = 68	B = 72 found by Math, Solver 0 = 40+B+68-180
Test	40+72+68 = 180 180 = 180 ☺

a = ?	a/sinA = b/sinB
A = 40 B = 72 b = 7	a = 4.731 found by Math, Solver 0 = a/sin40- 7/sin72
Test	4.731/sin40 = 7/sin72 7.360 = 7.360 ☺

c = ?	c/sinC = b/sinB
C = 68 B = 72 b = 7	c = 6.824 Math, Solver 0 = c/sin68 - 7/sin72
Test	6.824/sin68 = 7/sin72 7.360 = 7.360 ☺

Problem 12

a = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
A = 40 c = 6.8 b = 7	a = 4.724 found by Math, Solver $0 = a^2-6.8^2-7^2+2*6.8*7*\cos 40$
Test	$4.724^2 = 6.8^2+7^2–2*6.8*7*\cos 40$ 22.316 = 22.316 ☺

B = ?	a/sinA = b/sinB
A = 40 b = 7 a = 4.72	B = 72.3 found by Math, Solver 0 = 4.724/sin40- 7/sinB
Test	4.724/sin40 = 7/sin72.3 7.348 = 7.348 ☺

C = ?	A+B+C = 180
A = 40 B = 72.3	C = 67.7 found by Math, Solver 0 = 40+72.3+C-180
Test	40+72.3+67.7 = 180 180 = 180 ☺

Problem 13

B = ?	a/sinA = b/sinB
A = 40 a = 6.2 b = 7	B = 46.53 or B = 133.47 found by Math, Solver 0 = 6.2/sin40 - 7/sinB
Test	6.2/sin40 = 7/sin46.53 = 7/sin133.47 9.645 = 9.645 = 9.645 ☺

C = ?	A+B+C = 180
A = 40 B = 46.5 or B = 133.5	C = 93.47or C = 6.53 found by Math, Solver 0 = 40+B+C-180
Test	40+46.53+93.47 = 180 180 = 180 ☺

c = ?	a/sinA = c/sinC
A = 40 a = 6.2 C = 93.47, C = 6.53	c = 9.628 or C = 1.097 found by Math, Solver 0 = 6.2/sin40 - c/sinC
Test	6.2/sin40 = 9.628 /sin93.47 = 9.628 /sin6.53 9.645 = 9.645 = 9.645 ☺

Problem 14

A = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
a = 4 c = 6.8 b = 7	A = 33.66 found by Math, Solver $0 = 4^2-6.8^2-7^2+ 2*6.8*7*\cos A$
Test	$4^2 = 6.8^2+7^2–2*6.8*7*\cos 33.66$ 16 = 16 ☺

B = ?	$b^2 = a^2+c^2–2*a*c*\cos B$
a = 4 c = 6.8 b = 7	B = 75.91 found by Math, Solver $0 = 7^2-4^2-6.8^2+2*6.8*4*\cos B$
Test	$7^2 = 4^2+6.8^2–2*6.8*4*\cos 75.91$ 49 = 49 ☺

C = ?	A+B+C = 180
A = 33.66 B = 75.91	C = 70.43 found by Math, Solver 0 = 33.66+75.91+C-180
Test	33.66+75.91+70.43 = 180 180 = 180 ☺

Problem 15

d = ?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$ d = (e-f)*(T-g)
Test	$T = \frac{(e-f)*(T-g)}{e-f} + g = T$

e = ?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$ (T-g)(e-f) = d $e = \frac{d}{T-g} + f$
Test	$T = \frac{d}{\frac{d}{T-g} + f - f} + g = T$

f = ?	$T = (\frac{d}{(e-f)}) + g$
	(T-g)(e-f) = d $e = \frac{d}{T-g} + f$ $e - \frac{d}{T-g} = f$
Test	$T = \frac{d}{e - e - \frac{d}{T-g}} + g = T$

g = ?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$ $T - (\frac{d}{(e-f)}) = g$
Test	$T = \frac{d}{e-f} + T - (\frac{d}{(e-f)}) = T$

Problems 16-18

y = ?	y = a*b^x
a = 785 b = 1.027 x = 5	y = 785*1.027^5 y = 896.85

T = log(2)/log(1.027) = 26.0

x = ?	y = a*b^x
a = 785 b = 1.027 y = 980	x = 8.3 found by Math, Solver 0 = 785*1.027^x - 980
Test	980 = 785*1.027^8.3 980 = 980 ☺

T = log(2)/log(1.027) = 26.0

b = ?	y = a*b^x
a = 785 y = 980 x = 5	b = 1.045 = 1 + 4.5% found by Math, Solver 0 = 785*b^5 - 980
Test	980 = 785*1.045^5 980 = 980 ☺

T = log(2)/log(1.045) = 15.7

Problems 19-21

y = ?	y = a*x + b
b = 785 a = 2.7 x = 5	y = 2.7*5 + 785 y = 798.5

x = ?	y = a*x + b
b = 785 a = 2.7 y = 980	x = 72.2 found by Math, Solver 0 = 2.7*x+785 - 980
Test	980 = 2.7*72.2 + 785 = 980 ☺

a = ?	y = a*x + b
b = 785 y = 980 x = 5	a = 39 found by Math, Solver 0 = a*5+785 - 980
Test	980 = 39*5 + 785 = 980 ☺

Math Ed & Research 2018

Final Version

Good Math & Goofy Math, which is Truer?

Conflicting Theories in Mathematics Education

Rethinking Line-Number Arithmetic as Block-Number Algebra

Addition Free Mathematics Rooted in STEM Re-Counting Formulas

Mathematics: Useful Abstractions or an Undiagnosed Compulsory Cure?

Fifty Years of Ineffective Math Education Research, Why? Oops, Wrong Numbers, Sorry

Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum

Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions

A Twin Curriculum Since Contemporary Mathematics May Block

the Road to its Educational Goal, Mastery of Many

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Preface

Mathematics Biennial 2018. Teachers, school administrators, teacher trainers, researchers from all over Sweden attended the mathematics biennial at KARLSTAD University on 25-26/1. It is a recurring Conference carried out every two years at different sites in Sweden. Participants have the opportunity to choose among 200 different program points and five of the lecturers are international researchers.

01. At my stand I presented 19 posters.

The 25th Adults Learning Mathematics international conference, ALM25, was held in London, in July, 2018 with the theme: Boundaries and Bridges: adults learning mathematics in a fractured world. The conference asked for 1000word submissions together with a 200word extended abstract. I sent in 4 proposals.

02-03. Migrants Master Many by Re-counting in Block- & Per-numbers, having the abstract: Children show a surprising mastery of Many when re-counting totals in the same or in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Solving equations when re-counting from tens to blocks, and practicing proportionality and calculus when adding on-top and next-to, block-numbers offer a direct way to a quantitative competence that allows migrants help rebuild their original country.

04-05. Math: Useful Abstractions or an Undiagnosed Compulsory Cure? Conflicting Theories in Mathematics Education. The chapters are: Philosophical Controversies, Psychological Controversies, Sociological Controversies, and Mathematical Controversies.

06-07. A STEM-based Mathematics without Addition, having the abstract: Its many applications make mathematics useful. But of course, it must be learned before it can be applied. And since it is hard, there is no alternative to hard work? Well, observing the quantitative competence children bring to school, we discover as an alternative to the present set-based mathe-matics a Many-based 'many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount formula occurring all over the STEM subjects.

08-09. Cure Math Dislike with 1 Cup & 5 Sticks: New Meanings to Numbers, Counting, Operations & Fractions

10-11. Good Math & Goofy Math, which is Truer? The chapters are Good and Goofy Statements, Good and Goofy Concepts Good and Goofy Textbooks Good and Goofy Math

12-13. A Core STEM Curriculum for Young Migrants, having the abstract: Observing children's quantitative competence uncovers as an alternative to the present set-derived mathe-matics a Many-based 'many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. Results are predicted by a recount formula occurring all over the STEM subjects.

To the 42th PME Conference in Sweden I sent in a short oral communication and a poster.

14. Fifty Years of Ineffective Math Education Research, Why? Oops, Wrong Numbers, Sorry

15. Rethinking Line-Number Arithmetic as Block-Number Algebra

I sent in three proposals for the Mathematics Education in the Digital Age Conference 5-7 September 2018 - University of Copenhagen

16. A Count-before-Adding Curriculum for Preschool and Migrants, having the abstract: Children show a surprising mastery of Many with a quantitative competence where totals are re-counted in the same and in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional

LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Allowed to keep their block-numbers, children and migrants will be practising proportionality and calculus when adding block-number on-top and next-to; and will be solving equations when re-counting from tens to blocks.

17. Difference-research Saving Dropout Ryan with a TI-82 Calculator, having the abstract: At principal asked for ideas to lower the number of dropouts in pre-calculus classes. The author proposed using a cheap TI-82, but the teachers rejected saying students weren't even able to use a TI-30. Still the principal allowed buying one for a class. A compendium called 'Formula Predict' replaced the textbook. A formula's left- and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects.

18. Conflicting Theories Help Teachers Improve Mathematics Education, having the abstract: Traditionally, education is seen as teachers transferring institutionalized knowledge to individual learners. As such, education involves several choices. Shall teachers teach or guide? Is mathematics an eternal truth or a social construction? Is it knowledge about, or knowing how to? How to motivate learning? Should a class be optional or mandatory? To answer, teacher education refers to theory from philosophy, psychology and sociology. Including the existence of conflicting theories will allow teachers try out alternatives if wanting to improve mathematics education.

To the 2018 CTRAS Conference in China I sent three proposals

19. Addition-free Core STEM Curriculum for Late Learners along the Silk Road, having the abstract: Its many applications make mathematics useful. But to solve core STEM tasks you need no addition, thus calling for an addition-free curriculum. Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount-formula being a core in all STEM subjects.

20. Good, Bad & Evil Mathematics - Sociological Imagination in Math Education

21. Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum, having the abstract: Its many applications make mathematics useful; and of course, it must be learned before applied. Or, can it be learned through its original roots? Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. Asking 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a recount-formula occurring all over mathematics. Once counted, totals can be added next-to or on-top rooting calculus and proportionality. From this 'Count-before-Adding' curriculum, Many-matics offers remedial micro-curricula for classes stuck in a traditional curriculum.

The next article was published in Journal of Mathematics Education, March 2018, Vol. 11, No. 1,

22. Mastering Many by Counting, Re-counting and Double-counting before Adding On-top and Next-to, having the abstract: Observing the quantitative competence children bring to school, and by using difference-research searching for differences making a difference, we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

At the EARCOME8 conference in Taiwan I presented a lecture and a paper

23-24. Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions, plus a PPP

25-26. The Simplicity of Math reveals a Core Curriculum, plus a PPP

The next paper addresses the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11), the Thematic Working Group 26, Mathematics in the Context of STEM Education. The paper was rejected for presentation.

27. Addition Free Migrant-Math Rooted in STEM Re-Counting Formulas, having the abstract: STEM typically contain multiplication formulas expressing re-counting in different units, thus calling for an addition-free curriculum. The mastery of Many children bring to school uncovers a Many-based ‘Many-matics’ as an alternative to the present self-referring set-based mathe-matics. To answer the question ‘How many in total?’ we count and re-count totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a re-count formula as a core formula in all STEM subjects.

The next two paper address the ICMI Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, Tsukuba, 26-30 November 2018. It was accepted for presentation.

28-29. A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many, having the abstract: Mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum. Plus a PPP with 77 slides.

30. A New Curriculum - But for Which of the 3x2 Kinds of Mathematics Education, an essay on observations and reflections at the ICMI study 24 curriculum conference, having the abstract: As part of institutionalized education, mathematics needs a curriculum describing goals and means. There are however three kinds of mathematics: pre-, present and post-‘setcentric’ mathematics; and there are two kinds of education: multi-year lines and half-year blocks. Thus, there are six kinds of mathematics education to choose from before deciding on a specific curriculum; and if changing, shall the curriculum stay within the actual kind or change to a different kind? The absence of federal states from the conference suggests that curricula should change from national multi-year macro-curricula to local half-year micro-curricula; and maybe change to post-setcentric mathematics.

Allan Tarp, Aarhus Denmark, December 2018

01. Posters at the 2018 Swedish Biennale Stand

●Math Dislike Cured by 1 Cup & 5 Sticks

●Migrant Math for STEM Teachers/Engineers

INTRO: Saving the Princess with BundleNumbers

LEFT

Good Math: MANY-Math, Tales about Totals
Bad Math: SET-Math, Tales about LineNumbers
Evil MATH: Fraction-Math, Tales about Operators
Good Math: Icons, Bundling, ReCounting & PerNumbers
Core Math from Childhood
Grand Theory in Math Ed Research & Difference Research

MIDDLE

Math Dislike CURED by 1 Cup & 5 Sticks
Improving Schools in Sweden
Migrant-Math making migrants STEM-Teachers or Engineers
Count before you Add
Kids own Math
Activities

RIGHT

1Year online CATS-Course
1Week STEM-Course
Is Math True always or sometimes? Is Mathematics well-defined?
PYRAMIDeDUCATION & Material
Beware of Institutions & Teachers & Research & Forced Classes
Good & Bad & Evil Education
Rejected Research Papers

Rejected Research Papers: “Math Competenc(i)es- Catholic or Protestant.”
“The Simplicity of Mathematics Designing a STEM-based Core Mathematics Curriculum for Young Male Migrants.”

(STEM: Science + Technology + Engineering + Mathematics)

MATHeCADEMY.net

Saving the Princess with BundleNumbers

Once upon a time, a Princess was stuck in division. She simply could not do 336/7 and locked herself in behind a bush of thorns. The King summoned all the Wise who agreed that the Princess should be motivated by reformulating the task to split 336 among 7. Only a newcomer objected that the task was to recount 336 in 7s. “Here we all count in tens, so please wait at the lawn outside.” To solve the disagreement whether 7 should be above or below or to the right or left of 336, the Wise recommended all methods tried out together with an alternative method saying no method at all allowing the Princess to invent her own method. But nothing helped.

“Are there no other methods? Who is out on the lawn?” the King asked. “Just a new-comer with crazy ideas”. But in spite of strong protests from the Wise the King let him in.

“You also want to teach me division?”, the Princess asked. “No, I bring you a cup with 5 sticks that we will count.” “But they are already counted?” “ We will count them in bundles of 2s. As we see on our hand, this can be done in three ways: as 1 bundle & 3, as 2 bundles & 1, and as 3 bundles less 1. Using the cup for the bundles, we see that all numbers have inside bundles and outside singles; and that a total can be counted in the standard way or with an outside overload or an underload.”

“But isn’t 336 a name for a point far out on a number-line?” the Princess asked.

“No. 336 is not a line-number as everyone claims, it is a bundle-number. Asking 3year-olds “How old next time?” they say 4; but object to 4 fingers held together 2 by 2: “That is not 4 that is 2 2s.”

Children both see and count the bundles; and come to school with 2dimensional bundle-numbers or block-numbers with the core of mathematics inside them: 3 2s may be added to 1 4s in two ways; on-top, the units must be the same, and changing units is just another word for proportionality; and next-to means adding areas which is just another word for integral calculus.

So 336 is a bundle-number with 33 bundles inside and 6 singles outside. Wanting 28 bundles inside we move 5 bundles outside; so 33B6 and 28B56 is the same number just recounted with an overload. Counted in 7s we have 4 inside and 8 outside.

Consequently, a block of 33 tens & 6, or 336, can be recounted as a block of 48 7s; which makes sense since a shorter width calls for a larger height.”

All of a sudden, the thorns changed to roses. The Newcomer got the Princess and half the kingdom where they lived happily ever after.

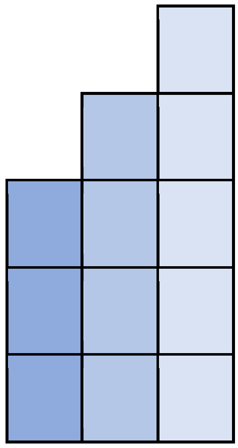
Good Math: MANY-Math

Tales about Totals

MANY-matics: A Natural Science about MANY

Meeting MANY we ask:

"How Many in Total?".



To answer, we Bundle and Stack in Blocks

T = 345 = 3*BB + 4*B + 5*1, or

T = 345 = 3*B^2 + 4*B + 5*1,

The SIMPLICITY of MANY-Math

First Iconize & Count & ReCount,

then Add OnTop & NextTo

4 ways to add: + , * , ^ , ∫

Algebra unite/*split*	Variable	Constant
Unit-numbers	$T = a+n$ $T-a = n$	$T = a*n$ $T/a = n$
Per-numbers	$T = \int a\,dn$ $dT/dn = a$	$T = a^n$ $\log_a(T)=n$, $\sqrt[n]{T}=a$

Bad Math: SET-Math

Tales about Numbers

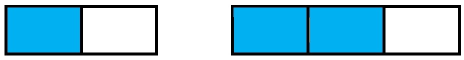
META-matics: concepts as examples of abstractions, not as abstractions of examples

00. Digits are symbols, not icons
01. Numbers are 1dimensional linear names, not 2dimensional blocks
02. Only ten-counting, no icon-counting
03. No ‘T=‘, only 42; not ‘T = 4.2 tens’
04. Add & Subtract before Multiply & Divide
05. Only OnTop addition - no NextTo addtion
06. 6*7 IS 42 – not 6 7s or 4.2 tens
07. 8/4 IS 8 split by 4; not 8 counted in 4s
08. No recounting to create or remove over- or underloads when operating on numbers
09. Solving equations by neutralizing; not by recounting in icons or reversed operations
10. Functions as set-relations; not as number-language sentences about the Total
11. Plane before coordinate geometry; not trigonometry before coordinate geometry
12. Differential before Integral Calculus.

Evil MATH: Fraction-Math

Tales about Operators

Mathema-TISM: True inside, but seldom outside



Claim: 1/2+2/3 IS 7/6

But 1 blue of 2 + 2 of 3 is 3 blues of 5,
and not 7 blues of 6?

Claim: 2+3 IS 5

But 2weeks + 3days is 17days?

Never ADD without units

00. Fractions are numbers; not operators needing numbers to become numbers

01. Fractions add without units; not with units making it integral calculus

02. Fractions before percentages and decimals; not the inverse order

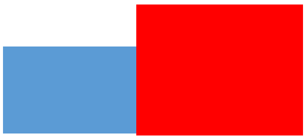
03. Fractions are equivalence classes in a set product; not per-numbers from double-counting in the same unit

Core Math from Childhood

Proportionality in Primary & Middle School

- Recount to change unit: 2 **3s** = (2*3/5)*5 = 1.1 **5s**
- With 2**$**/5**kg**, 20**kg** = (20/5)*5**kg** = (20/5)*2**$** = 8**$**
- Adding OnTop, 2 **3s** + 4 **5s** = ? **5s**

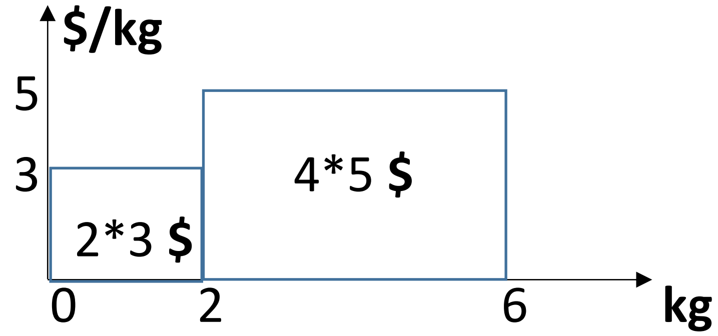
Calculus in Primary & Middle School



- Adding NextTo 2 **3s** + 4 **5s** = ? **8s**

	2 **kg**	at	3 **$/kg**
+	4 **kg**	at	5 **$/kg**
	(2+4)**kg**	at	~~(3+5)~~ **$/kg**

Unit-numbers add on-top.
Per-numbers add next-to as areas under the per-number graph.



Formulas in Primary & Middle School

- NumberLanguage Sentences about the Total, T = 4.2 tens = 42
- The general NumberFormula T = 3***B**^2+4***B**+5 with its examples
- T = a*x^2 + b*x + c; T = m*x, T = m*x + c, T = a*x^n, T = a*n^x

Equations in Primary & Middle School

Recount icons ← tens: x*7 = 42 = (42/7)*7, x = 42/7 (***opposite side & sign***)

The ReCount Formula is all over Mathematics

ReCount-formula: <u>T = (T/B)*B</u> 'from T, T/B times, B is taken away'

ReCounting	T = (T/B)*B	T = 8 = (8/2)*2 = 4*2 = 4 2s
Proportionality	$ = ($/kg)*kg	$ = price*kg
Coordinate Geometry	Δy = (Δy/Δx)*Δx	Δy = m*Δx
Differential Calculus	dy = (dy/dx)*dx	dy = y' * dx
Trigonometry	a = (a/c)*c = sinA*c:	a = (a/b)*b = tanA*b
Linearity	y = k*x	F = m*a, dist = vel*time, ...
Eigenvalues	Hψ = E*ψ	Schroedinger Equation

Grand Theory in Math Ed Res.

BAUMAN & WEBER: ***Beware of Goal-Means exchanges***
ARENDT: ***- and of the Banality of Evil***
HEIDEGGER: ***Respect the Subject & question the Predicate***
FOUCAULT: ***- also question Cures and Institutions***

Difference Research

Finding Differences making a Difference

Almost Everything can be Different

DIGITS:	**Icons vs. symbols**
NUMBERS:	**2D blocks vs. 1D lines**
OPERATIONS:	**Icons vs. inter-set mappings**
ADDITION:	**OnTop/NextTo vs. after**
MULTIPLICATION:	**ReCounting to tens vs. tables**
DIVISION:	**ReCounting from tens vs. splitting**
RE-COUNTING:	**Changing units vs. neglect**
DOUBLE-COUNTING:	**Proportionality vs. neglect**
PER-NUMBERS:	**Core numbers vs. neglect**
FRACTIONS:	**PerNumbers vs. rational numbers**
FRACTIONS:	**Operators vs. rational numbers**
FORMULAS:	**Total statements vs. inter-set relations**
EQUATIONS:	**ReCount in icons vs. open statements**
GEOMETRY:	**Trigonometry before coord. geometry vs. plane geometry first**
POLYNOMIALS:	**Number-formulas vs. functions**
CALCULUS:	**Integrate bef. differentiate vs. inverse**

1Year online CATS-Course

CATS: Count & Add in Time & Space

	Self Instructing QUESTIONS 1: Primary, 2: Secondary School
C1 COUNT	How to count Many? How to recount 8 in **3s**: T= 8 = ? **3s** How to count in standard bundles? How to recount 6**kg** in **$ with 2$/4kg**: T = 6**kg** = ?**$**
C2 COUNT	How can we count possibilities? How can we predict unpredictable numbers?
A1 ADD	How to add blocks concretely? T = 27 + 16 = 2**ten**7 + 1**ten**6 = 3**ten**13 = ? How to add blocks abstractly?
A2 ADD	What is a prime and a folding number? What is a per-number? How to add per-numbers?
T1 TIME	How can counting & adding be reversed ? Counting ? **3s** and adding 2 gave 14. Can all calculations be reversed?
T2 TIME	How to predict the terminal number • If the change is constant? • If the change is variable, but predictable?
S1 SPACE	How to count plane and spatial properties of blocks and round objects?
S2 SPACE	How to predict the position of points and lines? How to use the new calculation technology?
QL	What is quantitative literature? Does it also have the 3 different genres: fact, fiction and fiddle?

1Week STEM-Course

The Simplicity of Math:

First Count & ReCount - then Add OnTop & NextTo

Day 01. Good & Bad & Evil Math in General

The root of math: MANY or SET

Day 02. Good & Bad & Evil Math in Primary School

Iconize & Count & ReCount before you ADD

Day 03. Good & Bad & Evil Math in Middle School

DoubleCounting and PerNumbers vs. Fractions

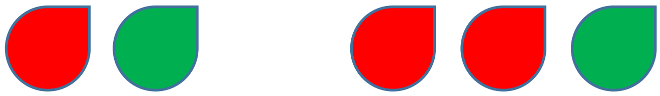
Day 04. Good & Bad & Evil Math in High School

Calculus: adding locally constant PerNumbers

Day 05. Good Math in a STEM setting

PerNumbers Predicting Matter in Time and Space

Is Math True always or sometimes?

<table>
<tr><th>Is this True</th><th>Always</th><th>Never</th><th>Sometimes</th></tr>
<tr><td>$2 + 3 = 5$</td><td></td><td></td><td>X</td></tr>
<tr><td></td><td colspan="3">2weeks + 3days = 17days; only with the same unit</td></tr>
<tr><td>$2 \times 3 = 6$</td><td>X</td><td></td><td></td></tr>
<tr><td></td><td colspan="3">2x3 is 2 3s III III that can always be recounted as 6 1s</td></tr>
<tr><td>$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$</td><td></td><td></td><td>X</td></tr>
<tr><td></td><td colspan="3">
1 red of 2 apples + 2 of 3 is 3 of 5, and not 7 of 6</td></tr>
<tr><td>$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$</td><td></td><td></td><td>X</td></tr>
<tr><td></td><td colspan="3">Only if taken of the same total</td></tr>
<tr><td>A FUNCTION is</td><td colspan="3">● <u>for example 2+x, but not 2+3</u>; i.e. a name for a calculation with an unspecified number (before SET, 1750-1900)
● <u>an example of a set relation</u>, where first component identity implies second component identity (after SET, 1900)</td></tr>
</table>

Is Mathematics Well Defined?

Ancient Greece

A common **LABEL** for **Quadrivium**: Arithmetic, Geometry, Music & Astronomy (Many by itself and in space & time). **Trivium**: Grammar, Logic, Rhetoric

PreModern

A common **LABEL** for Arithmetic & Geometry; different from “Rechnung”.

Modern

A self-referring **SET** of <u>Proofs</u> about **SET**-derived <u>Concepts</u>.

PostModern

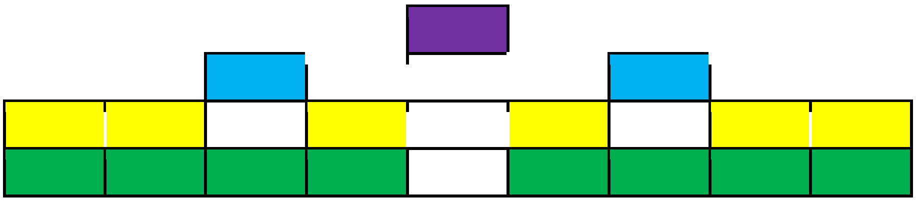
Many-math: A **Natural SCIENCE** Counting & Adding & Predicting Many.

PYRAMIDeDUCATION

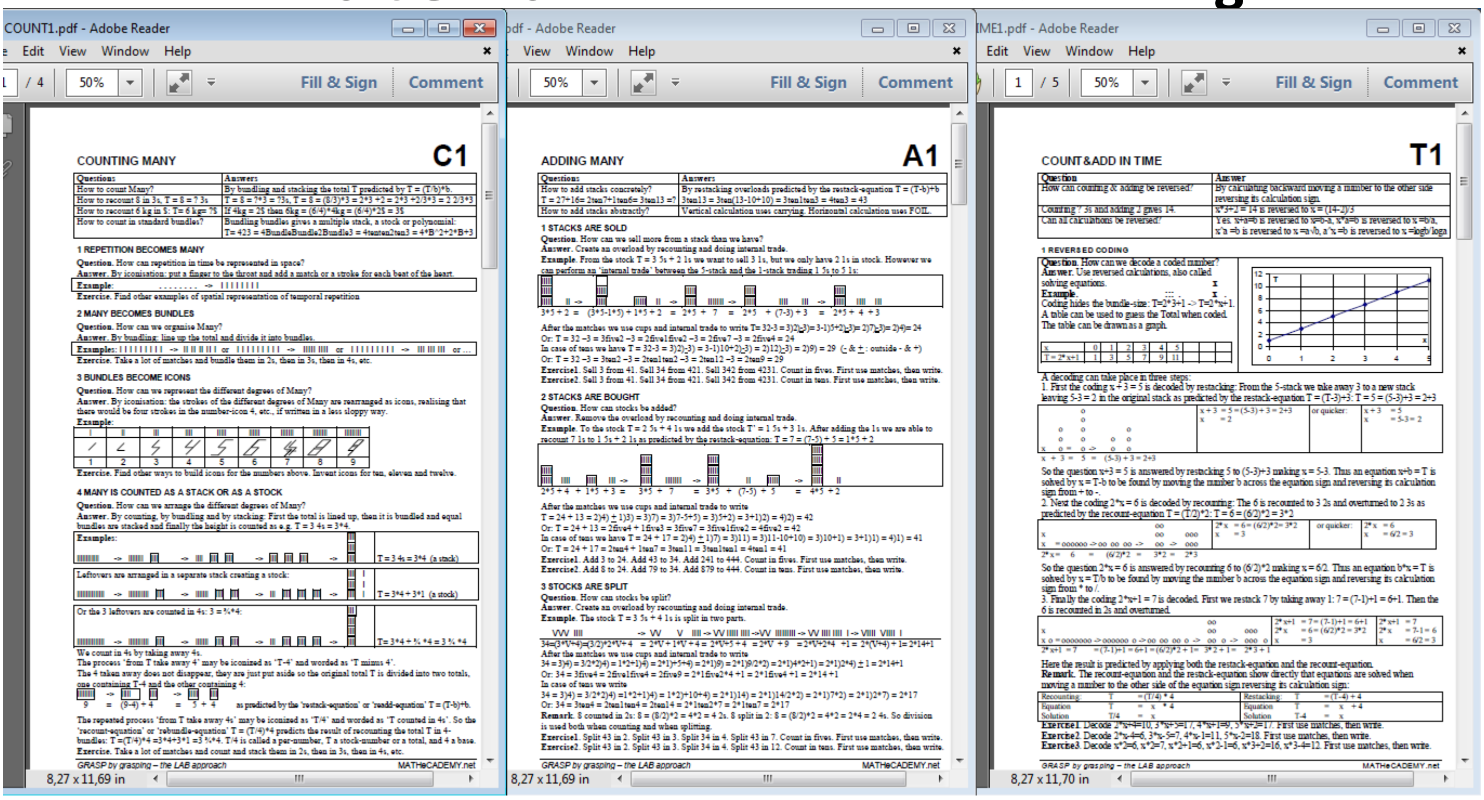
8 learners organized in 2 teams with 2 instructors and 3 pairs by turn.

- Each pair works together to solve **C**ount&**A**dd problems.
- The coach assists the instructors when instructing their team and when correcting the **C**ount&**A**dd assignments.
- Each learner pays by coaching a new group of 8 learners.

1 Coach
2 Instructors
3 Pairs
2 Teams

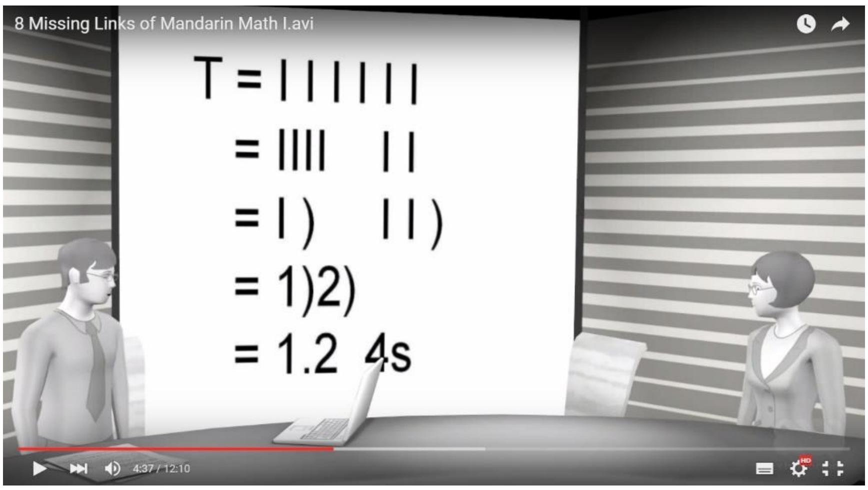


Material: short & self-instructing



YouTube Videos

- Postmodern Mathematics Debate
- CupCounting removes Math Dislike
- IconCounting & NextTo-Addition
- PreSchool Mathematics
- Fractions
- PreCalculus & Calculus
- Mandarin Mathematics
- World History



Beware of

INSTITUTIONS exchanging goals and means

- The goal of math education is to ~~learn math~~ master Many with quantitative competence

TEACHERS teaching Bad & Evil math

- LINE-numbers instead of BLOCK-numbers
- Addition before Counting & Multiplication
- Adding Fractions without units
- Differential before Integral Calculus

RESEARCH uncritically

- researching itself instead of math education
- exemplifying instead of questioning itself
- accepting math as self-referring MetaMath
- accepting 50 years of unsuccessful research

FORCED CLASSES

- Constraining young people to stay with their age-group for several years - instead of choosing their own daily ½year blocks in order to uncover and develop their personal talent

Rejected Research Papers

Allan Tarp, MATHeCADEMY.net, the 2018 MADIF Conference

The Simplicity of Mathematics Designing a STEM-based Core Math Curriculum for Young Male Migrants

Educational shortages described in the OECD report 'Improving Schools in Sweden' challenge traditional math education offered to young male migrants wanting a more civilized education to return help develop and rebuild their own country. Research offers little help as witnessed by continuing low PISA scores despite 50 years of mathematics education research. Can this be different? Can mathematics and education and research be different allowing migrants to succeed instead of fail? A different research, difference-research finding differences making a difference, shows it can. STEM-based, mathematics becomes Many-based bottom-up Many-matics instead of Set-based top-down Meta-matics.

REJECTED

Math Competenc(i)es - Catholic or Protestant?

Introduced at the beginning of the century, competencies should solve poor math performance. Adopted in Sweden together with increased math education research mediated through a well-funded centre, the decreasing Swedish PISA result came as a surprise, as did the critical 2015 OECD-report 'Improving Schools in Sweden'. But why did math competencies not work? A sociological view looking for a goal displacement gives an answer: Math competencies sees mathematics as a goal and not as one of many means, to be replaced by other means if not leading to the outside goal. Only the set-based university version is accepted as mathematics to be mediated by teachers through eight competencies, where only two are needed to master the outside goal of mathematics education, Many.

REJECTED

02. Migrants Master Many by Re-counting in Block- & Per-numbers - Short

Observing the quantitative competence children bring to school, and using difference-research, finding differences making a difference, we discover as a difference to the present set-based mathematics a Many-based: ‘many-matics’.

Here digits are icons with as many sticks as they represent. As are operations where division, multiplication and subtraction allow a total of seven to be bundle-counted as T = 2B1 3s, i.e. as a number-language sentence with a subject, a verb and a predicate as in the word-language; thus producing two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads or underloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations.

Here double-counting in two units creates per-numbers, becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers.

Here addition occurs on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes plane and coordinate geometry.

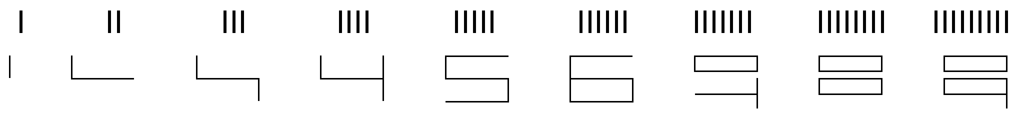
Solving equations when re-counting from tens to blocks, and practising proportionality and calculus when adding on-top and next-to, block-numbers offer a direct way to a quantitative competence that allows migrants help rebuild their original country.

03. Migrants Master Many by Re-counting in Block- & Per-numbers - Long

Children show a surprising mastery of Many when re-counting totals in the same or in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Solving equations when re-counting from tens to blocks, and practising proportionality and calculus when adding on-top and next-to, block-numbers offer a direct way to a quantitative competence that allows migrants help rebuild their original country.

Meeting Many

Difference-research, finding differences making a difference (Tarp, 2017), uncovers a different ‘Many-matics’, mastering Many by bundling and stacking. To count, first we rearrange sticks into icons with e.g. five sticks in the 5-icon if written less sloppy. Counted as ‘one bundle’, ten needs no icon.



1 2 3 4 5 6 7 8 9

Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, a number of bundles that may or may not be re-counted as ones. This inspires ‘bundle-counting’, re-counting a total in icon-bundles. Thus, a total T of 5 1s is re-counted in 2s as T = 2 2s & 1. Here the bundles can be placed inside a bundle-cup with a stick for each bundle, leaving the unbundled singles outside; and described by ‘bundle-writing’, T = 2B1 2s, or ‘decimal-writing’, T = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup:

T = 5 = I I I I I → II II I → $\boxed{\text{II}}$ I → 2B1 2s = 2.1 2s

Entering ‘5/2’, we ask a calculator ‘from 5 we take away 2s’ The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking ‘5 – 2*2’. The answer, 1, predicts that 5 = 2B1 2s = 2.1 2s as indirectly predicted on the bottom line.

5 / 2	2.some
5 – 2 * 2	1

We see that also operations are icons: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away bundles, and subtraction shows the trace left when taking away a stack only once.

A calculator thus uses a 're-count formula', *T = (T/B)*B*, to predict that 'from *T*, *T/B* times, *B*s can be taken away'. This re-count formula occurs all over mathematics: when relating proportional quantities as *y = c*x*; in trigonometry as *sine* and *cosine* and *tangent*, e.g. *a = (a/c)*c = sinA*c*; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, *dy = (dy/dx)*dx = y'*dx*.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 2*B*1 2s as 1*B*3 2s with an outside 'overload'; or as 3*B*-1 2s with an outside 'underload' thus rooting negative numbers.

Re-counting in a different unit means changing unit, also called proportionality or linearity. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2*B*2 5s.

Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s' The answer, 2.some, predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer, 2, predicts that 3 4s can be re-counted in 5s as 2*B*2 5s or 2.2 5s.

Re-counting to and from tens

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

Re-counting from tens to icons by asking '38 = ? 7s' is called an equation *x**7 = 38. It is easily solved by re-counting 38 in 7s: *x**7 = 38 = (38/7)*7. So *x* = 38/7 = 5 & 3/7 as predicted by a calculator showing that 38 = 5.3 7s = 5*7 + 3.

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg.

To answer the question '*T* = 6$ = ?kg', we re-count 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa: Asking 'T = 12kg = ?$ ', the answer is 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

Once counted, totals can be added on-top or next-to

Adding on-top in 5s, '3 5s + 2 3s = ? 5s?', re-counting must make the units the same. Asking a calculator, the two answers, '4.some' and '1', predict the result as 4*B*1 5s. Since 3*5 is an area, adding next-to in 8s, '3 5s + 2 3s = ? 8s?', means adding areas, called integral calculus. Asking a calculator, the two answers, '2.some' and '5', predict the result as 2*B*5 8s.

Reversing adding on-top and next-to

Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking '3 5s and how many 3s total 2*B*6 8s?', using sticks will give the answer 2*B*1 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula $y' = \Delta y/t = (y2 - y1)/t$.

References

Tarp, A. (2017). *Math ed & research 2017.* Retrieved from http://mathecademy.net/ 2017-math-articles/.

04. Math: Useful Abstractions or an Undiagnosed Compulsory Cure? Conflicting Theories in Mathematics Education - Short

Many countries face poor PISA results despite 50 years of research. Is this caused by conflicting theories?

Within philosophy, ancient Greece saw a controversy between the sophists, warning against choice presented as nature, and the philosophers seeing choice as an illusion since the physical is but examples of metaphysical forms. The natural science reinvention of scepticism inspired the two Enlightenment republics, the American and the French, transforming scepticism into pragmatism and post-structuralism.

Within psychology, a controversy exists between Vygotsky and Piaget recommending teaching as much and as little as possible.

Within sociology, a controversy exists between a structure and an actor viewpoint, reflecting societies with high or low degree of institutionalization. Thus, Foucault sees knowledge as socially constructed discourses; and describes schools as ‘prispitals’ mixing power techniques from a prison and a hospital: the learners are fixed in classrooms and diagnosed as ignorant to be cured.

Today two mathematics discourses exist. One is the institutionalized presentation of a body of self-referring knowledge being useful through its many applications.

The other is a silenced discourse seeing mathematics as a natural science about Many expressing itself in number-language sentences with a subject and a verb and a predicate as in the word-language.

05. Math: Useful Abstractions or an Undiagnosed Compulsory Cure? Conflicting Theories in Mathematics Education - Long

Philosophical Controversies

Ancient Greece saw two forms of knowledge, called ‘sophy’. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. The Christian Church eagerly took over a metaphysical patronage and changed the academies into monasteries, until the Reformation changed some back again.

By letting the laboratory precede the library, natural science reinvented scepticism. Newton discovered that falling objects obey their own will instead of that of a patronizor. This inspired the Enlightenment Century and it two republics, the American and the French, transforming scepticism into pragmatism and post-structuralism, based upon existentialism defined by Sartre as ‘Existence preceding essence’; and with the Heidegger warning: in a sentence, respect the subject, but question the predicate since it might be gossip. Thus, post-structuralism deconstructs ungrounded diagnoses forcing humans to accept unfounded patronization.

Psychological Controversies

As to how learners acquire knowledge, several constructivist theories exist among which are Vygotskian and Piagetian social and radical constructivism disagreeing by recommending teaching as much and as little as possible.

Vygotsky sees knowledge as essence to be transferred by good teaching. However, a learner can only take in unknown sentences about subjects already known, so the teacher must know the individual ‘zone of proximal development’ in order to successfully connect it to the institutionalized knowledge by scaffolding.

Whereas Piaget recommends meeting existence directly to allow learners form individual concepts and sentences to be negotiated and accommodated socially.

Sociological Controversies

As a social institution, education can be seen from a structure or an actor viewpoint, reflecting societies with high or low degree of institutionalization. Being highly institutionalized, continental Europa has developed a structure-based sociology seeing humans as bound by social structures. Thus, Foucault sees knowledge as socially constructed discourses; and describes a school as a 'prispital' mixing the power techniques of a prison and a hospital: the learners are fixed in classrooms and diagnosed as ignorant to be cured by discourses institutionalized as truth but instead exerting 'pastoral power'. Whereas their escape from Europe made Americans actors developing grounded theory, lifting Piagetian accommodation to a social level.

Mathematical Controversies

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about Many by itself, in space, in time, and in space and time. With astronomy and music gone, today mathematics should be a common label for geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about fifty years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics'. Here the invention of the concept Set created a 'meta-matics' defining concepts as examples of abstractions instead of as abstractions from examples. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: In the set M of sets not belonging to themselves, M belongs only if it does not.

So today two mathematics discourses exist. One discourse is the institutionalized presenting mathematics as a body of self-referring knowledge that shows its usefulness through its many applications; but of course, to be applied, first it must be learned, even if this may be hard as shown by poor PISA results.

The other is a silenced discourse seeing mathematics as a natural science about Many expressing itself in number-language sentences with a subject and a verb and a predicate as in the word-language. And showing the silenced differences:

Numbers could be icons & predicates in Tales of Many, T = 2 3s = 2*3. Instead they are changed from predicates to subjects by silencing the real subject, the total; and place-values hide the bundle structure.

Operations could be icons for the counting process as predicted by the re-count formula T = (T/B)*B, 'from T, T/B times, B can be taken away'. Instead they hide their icon-nature and their role in counting; and they are presented in the opposite order.

Addition could wait to after counting & recounting & double-counting have produced unit- and per-numbers. Instead it silences counting and next-to addition; and silences bundling; and uses carry instead of overloads; and assumes numbers as ten-based.

Fractions could be per-numbers coming from double-counting in the same unit and added with units by areas (integration) since they are, not numbers, but operators needing numbers to become numbers. Instead they are defined as rational numbers that can be added without units.

Equations could be re-counting from tens to icons and reversing on-top and next-to addition. Instead they are defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra.

Proportionality could be re-counting in another unit when adding on-top; or double-counting producing per-numbers and fractions. Instead it is defined as multiplicative thinking.

Trigonometry could be mutual recounting of the sides in a block halved by its diagonal. Instead it is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.

Functions could be number-language's sentences or formulas, T = 2*3, with subject & verb & predicate. Instead they are set-relations where first-component identity implies second-component identity.

Calculus could occur in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers. Instead differential calculus precedes integral calculus, presented as anti-differentiation.

References

Foucault, M. (1995). *Discipline & punish.* New York: Vintage Books.
Marino, G. (2004). *Basic writings of existentialism.* New York: Modern Library.
Russell, B. (1945). *A history of western philosophy.* New York: Touchstone Book.
Tarp, A. (2017). *Math ed & research 2017.* Retrieved from http://mathecademy.net/.

06. A STEM-based Mathematics without Addition - Short

Its many applications make mathematics useful. But of course, it must be learned before it can be applied. And since it is hard, there is no alternative to hard work.

Observing the quantitative competence children bring to school, we discover as an alternative to the present set-based mathe-matics a Many-based 'many-matics'.

with the number of sticks they represent, digits become icons. As do operations where division, multiplication and subtraction allow a total of seven to be bundle-counted as T = 2B1 3s, i.e. as a number-language sentence with a subject, a verb and a predicate as in the word-language. These two-dimensional block-numbers are ready to be re-counted in the same unit to remove or create overloads or underloads to ease operations; or in a new unit, called proportionality; or to and from tens rooting tables and solving equations.

Here double-counting in two units creates per-numbers, becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers.

Counting involves division and multiplication, predicted by a 'recount formula' T = (T/B)*B, occurring in all the STEM subjects science, technology, engineering and math, e.g. meter = (meter/sec)*sec = speed*sec.

So recounting and STEM tasks should precede addition.

07. A STEM-based Mathematics without Addition - Long

Its many applications make mathematics useful. But of course, it must be learned before it can be applied. And since it is hard, there is no alternative to hard work? Well, observing the quantitative competence children bring to school, we discover as an alternative to the present set-based mathematics a Many-based 'many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount formula occurring all over the STEM subjects.

Meeting many

Meeting many, we ask 'How many in total?' To answer, we count. Holding 4 fingers together 2 by 2, a 3year-old will say 'That is not 4, that is 2 2s', thus describing what exists, a number of bundles that may or may not be recounted as ones.

This inspires 'bundle-counting', recounting a total in icon-bundles. Thus, a total *T* of 5 1s is recounted in 2s as *T* = 2 2s & 1; and is described by 'bundle-writing', *T* = 2*B*1 2s, or 'decimal-writing', *T* = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

Entering '5/2', we ask a calculator 'from 5 we take away 2s'. The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking '5 – 2*2'. The answer, 1, predicts that 5 = 2*B*1 2s = 2.1 2s as indirectly predicted on the bottom line.

5 / 2	2.some
5 – 2 * 2	1

We see that also operations are icons: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away bundles, and subtraction shows the trace left when taking away a stack only once.

A calculator thus uses a 'recount formula', $T = (T/B)*B$, to predict that 'from *T*, *T/B* times, *B*s can be taken away'. This recount formula occurs all over mathematics: when relating proportional quantities as $y = c*x$; in trigonometry as *sine* and *cosine* and *tangent*, e.g. $a = (a/c)*c = sinA*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/ \Delta x)* \Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

Recounting in the same unit and in a different unit

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2*B*1 2s as 1*B*3 2s with an outside 'overload'; or as 3*B*-1 2s with an outside 'underload' thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48.

Recounting in a different unit means changing unit, also called proportionality or linearity. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2*B*2 5s.

Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s' The answer, 2.some, predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer, 2, predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

Recounting to and from tens

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation x*7 = 35. It is easily solved by recounting 35 in 7s: x*7 = 35 = (35/7)*7. So x = 35/7, showing that equations are solved by moving to opposite side with opposite calculation sign.

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question 'T = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

Mathematics in STEM subjects

STEM (Science, Technology, Engineering and Mathematics) combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots will help transforming nature into human necessities.

Nature consists of things in motion, combined in the momentum = mass*velocity. Things contain mass and molecules and electric charge. Thus, nature is counted in meter and second and kilogram and mole and coulomb.

Looking at the list of formulas we see that nature is predictable by recounting & per-numbers. Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus (Tarp, 2017).

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
meter = (meter/second) * second = velocity * second
Δ momentum = (Δ momentum/second) * second = force * seconds
Δ energy= (Δ energy/meter) * meter = force * meter = work
energy = (energy/kg/degree) * kg * degree = heat * kg * degree
force = (force/square-meter) * square-meter = pressure * square-meter
gram = (gram/mole) * mole = molar mass * mole
mole = (mole/litre) * litre = molarity * litre

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08. Cure Math Dislike with 1 Cup & 5 Sticks: New Meanings to Numbers, Counting, Operations & Fractions - Short

With a cup for the bundles, 5 sticks are 'bundle-counted' in 2s as 2B1 2s.

Moving bundles outside or inside creates 'overload' or 'underload': 5 = 2B1 = 1B3 = 3B-1 2s.

This eases operations: 336/7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

With bundle-counting, numbers are 2D block-numbers with units, instead of 1D line-numbers.

Counting 5 in 2s, division takes away 2s, multiplication stacks the bundles, and subtraction takes away the stack to find unbundled singles.

Operations are icons, as are digits, containing as many sticks as they represent.

A calculator predicts with a 're-count formula' T = (T/B)*B' saying that 'from T, T/B times we take B away from T'; and occurring all over mathematics.

Re-counting in a new unit means changing units, called proportionality.

Re-counting from icons to tens, multiplication predicts the result directly.

Re-counting from tens to icons is called an equation.

Double-counting in different units creates per-numbers as 4$/5kg, becoming fractions with like units.

Once counted, totals add on-top or next-to, rooting proportionality and integral calculus.

Reversing on-top or next-to addition roots solving equations and differential calculus.

Thus, calculus also appears as next-to addition of block-numbers and as adding fractions with units.

09. Curing Math Dislike with 1 Cup & 5 Sticks: New Meanings to Numbers, Counting, Operations & Fractions - Long

Division seems hard, but not with 1cup & 5sticks. Using a cup for the bundles, a total T of 5 sticks is 'BundleCounted' in 2s as T = 5 = I I I I I = II II I = 2)1 2s = 2B1 2s = 2.1 2s using a decimal point to separate the bundles from the singles. A total thus has an inside number of bundles and an outside number of singles.

Bundle-counting 7 in 3s thus leads to a 2dimensinal LEGO-like block-numbers T = 2B1 3s, different from the 1dimensional line-numbers taught in school; and containing three digits: the size, the inside bundles and the outside singles.

Including bundles in the counting sequence, we count 0Bundle1, 0B2, …, 0B9, 1B0, 1B1. Or 1B-1, 1B0, 1B1.

To ease operations, we re-count in the same unit by moving a bundle outside or inside the bundle-cup to create an 'overload' or 'underload': T = 5 = 2)1 2s = 1)3 2s = 3)-1 2s. This also applies when counting in tens: T = 42 = 4B2 = 3B12 = 5B-8.

Adding: T = 35 + 47 = 3B5 + 4B7 = 7B12 = 8B2 = 82

Subtracting: T = 75 – 47 = 7B5 – 4B7 = 3B-2 = 2B8 = 28

Multiplying: T = 7 x 48 = 7 x 4B8 = 28B56 = 33B6 = 336.

Dividing: T = 336/7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

Sticks or a folding ruler show digits as icons with as many sticks as they represent if written les sloppy. Operations are icons also: To count 7 in 3s we take away 3 many times iconized by a broom wiping away the 3s. Showing 7/3 = 2.some, a calculator predicts that 2 times we can take 3 away from 7. To stack the 2 3s we use multiplication iconizing a lift, 2x3 or 2*3. To look for unbundled singles, we drag away the stack of 2 3s iconized by a horizontal trace: 7 – 2*3 = 1.

A calculator thus predicts results by a 're-count formula' T = (T/B)*B' saying that 'from T, T/B times we can take B away from T', as e.g. T = 8 = (8/4)*4 = 2*4 = 2 4s. It occurs all over mathematics: as y = c*x; as a = (a/c)*c = sinA*c; as lie gradients, delta-y = (delta-y/delta-x)* delta-x = c* delta-x; and as derivatives, dy = (dy/dx)*dx = y'*dx.

Re-counting in a new unit means changing units, called proportionality. Again the re-count formula predicts the result: with T = 4 5s = ? 6s; first (4*5)/6 = 3.some; then (4*5) – (3*6) = 2. So: T = 4 5s = 3.2 6s.

Re-counting from icons to tens, multiplication predicts the result directly. Only, the calculator leaves out the unit and misplaces the decimal point: Asking T = 3 7s = ? tens, the answer is 3x7 = 21 i.e. 2.1 tens. Geometrically it makes sense that increasing the width of a block from 7 to ten means decreasing its height from 3 to 2.1 to keep the total unchanged.

Re-counting from tens to icons is called an equation, e.g. T = 35 = ? 7s. Using u for the unknown number, we solve the equation by re-counting 35 in 7s: u*7 = 35 = (35/7)*7, so u = 35/7 = 5. Geometrically it makes sense that decreasing the width of the block from ten to 7 means increasing its height from 3 to 5 to keep the total unchanged.

The 'move to opposite side with opposite sign' method applies to all equations: u+2 = 8 is solved by u = 8-2; u^3 = 20 is solved by u = 3√20, and 3^u = 20 is solved by u = log3(20), where root and logarithm is introduced as a factor-finder and a factor-counter.

Double-counting in different units creates per-numbers as 4$/5kg or 4/5 $/kg. With 20 kg, we recount 20 in 5s: T = 20kg = (20/5)*5kg = (20/5)*4$ = 16$. With 60$, we recount 60 in 4s: T = 60$ = (60/4)*4$ = (60/4)*5kg = 75kg.

Double-counting in the same unit creates fractions and percentages as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%.

We see that per-numbers and fractions are not numbers, but operators needing a number to become a number.

Counting thus involves dividing and multiplying and subtracting to predict that 7 = 2B1 3s = 2.1 3s. Geometrically, placing the unbundled single next-to the block of 2 3s makes it 0.1 3s, whereas counting it in 3s by placing it on-top of the block makes it 1/3 3s, so 1/3 3s = 0.1 3s.

Once counted, totals can be added on-top or next-to. To add on-top, two totals T1 = 2 3s and T2 = 4 5s must be re-counted to have the same unit, e.g. as T1 + T2 = 2 3s + 4 5s = 1.1 5s + 4 5s = 5.1 5s, thus using proportionality. To add next-to, the united total must be recounted in 8s: T1 + T2 = 2 3s + 4 5s = (2*3 + 4*5)/8 * 8 = 3.2 8s. Thus next-to addition geometrically means adding areas called integral calculus.

Reversing on-top or next-to roots solving equations and differential calculus.

Adding 3kg at 4$/kg and 5kg at 6$/kg, the unit-numbers 3 and 5 add directly but the per-numbers 4 and 6 add by their areas 3*4 and 5*6, i.e. by integration. Likewise with adding fractions.

Thus, calculus appears at all school levels: at primary, at lower and at upper secondary and at tertiary level.

References

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10. Good Math & Goofy Math, which is Truer? - Short

Looking at four statements we ask if they are true always, sometimes or never: 2+3 = 5, 2*3 = 6, 1/2+2/3 = 3/5, 1/2+2/3 = 7/6.

Looking at concepts we ask which definition is correct: A function is an example of a set-product where first-component identity implies second-component identity; or a function is e.g. y = 2+x but not y = 2+3, i.e. a name for a calculation with an unspecified number.

Looking at textbooks we ask: should we write '3*4' only, or 'T = 3*4' as a full sentence with a subject and a verb and a predicate to show the similarity of the number-language and the word-language. We then discuss how to apply the definitions below to numbers, operations, equations, functions, trigonometry, and calculus.

In good mathematics, concepts are defined bottom-up as abstractions from examples; and statements are always true inside and outside classrooms. In goofy mathematics, concepts are defined top-down as examples from abstractions; and statements are always true inside and sometimes outside classrooms. Thus, fractions are goofy mathematics if treated as numbers without units; but good mathematics if treated as per-numbers, i.e. as operators, needing a number to become a number.

11. Good Math & Goofy Math, which is Truer? - Long

To make true statements about the outside world, we must distinguish between good and goofy mathematics being true always and sometimes.

Good and Goofy Statements

Asking if they are true always, sometimes or never, we look at four statements: 2*3 = 6, and 2+3 = 5, and 1/2+2/3 = 3/5, and 1/2+2/3 = 7/6.

Saying '2*3 = 6' is stating that 2 3s can be recounted as 6 1s, which is always true. Saying '2+3 = 5' is true if the unit is the same; but may be false with different units, e.g. 2 weeks + 3 days = 17 days. So to be always true, addition must include the units as exemplified by the Mars Orbiter that crashed because of adding cm and inches. Multiplying, the units need not be the same as exemplified by physics where e.g. 2 Newton*3 meter = 6 Newton-meter = 6 Joule.

Adding 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and not 7 reds of 6 apples as taught in school; and true only if taken of the same total. So, depending on the units, 1/2+2/3 can take on many different values. Where goofy fractions neglect units, good fractions accept being operators needing numbers to become numbers, thus added by area as integration.

Good and Goofy Concepts

Looking at concepts we ask which definition is correct: A function is an example of a set-product where first-component identity implies second-component identity; or, a function is e.g. y = 2+x but not y = 2+3, i.e. a name for a calculation with an unspecified number.

To answer, we look at the history of mathematics. In ancient Greece the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas. With astronomy and music as independent knowledge areas, today mathematics is a common label for the two remaining activities, geometry and algebra both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite Many' in Arabic.

Then the invention of the set-concept transformed mathe-matics to 'meta-matics' with 'well-defined' self-referring concepts defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. However, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false; and to goofy mathematics.

Good and Goofy Textbooks

Looking at textbooks we ask: should we write ‘3*4’ only, or ‘T = 3*4’ as a full sentence with a subject and a verb and a predicate to show the similarity of the number- and the word-language.

The word-language assigns words to things through sentences, ‘This is a chair’. Asking ‘How many in total?’ we use the number-language to assign numbers to like things in sentences as ‘The total of legs on three chairs is 3 fours’, abbreviated to ‘T = 3 4s’ or ‘T = 3*4’.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ‘this is a chair’ leads to a meta-sentence ‘’is’ is a verb’. Likewise, the sentence ‘T = 3*4’ leads to a meta-sentence ‘’*’ is an operation’. And since the meta-language speaks about the language, the language should be taught and learned before the meta-language. Which is the case with the word-language, but not with the number-language.

With the total we include both what exist and what essence we claim, in accordance with the existentialist philosophy saying that existence should precede essence. Thus, the fingers on the left hand exist, but how they are counted can vary: T = 5 1s = 2 2s & 1 = 1 3s &2 = 2 3s less 1.

Good and Goofy Math

Good numbers are two-dimensional block-numbers carrying units as seen when writing out fully a total T of 345 as T = 3*B*B + 4*B + 5*1, also showing the four ways to unite numbers: multiplication, power and on-top and next-to block addition, also called integration. Including the unit when counting by bundling, a natural number as T = 2B3 4s has 3 digits: a size-number 4, a bundle-number 2 and a single-number 3, which add in different ways.

Goofy numbers are one-dimensional line-numbers silencing the unit and misplacing the decimal point by writing 23 instead of 2.3 tens.

Good addition asks for the units before adding on-top or next-to. And waits until bundle-counting and re-counting and double-counting has introduced division and multiplication and subtraction as well as the ‘recount formula’ T = (T/B)*B, saying that ‘T/B times B can be taken away from T’, as e.g. 8 = (8/2)*2 = 4*2 = 4 2s. Goofy addition add digit without considering the units.

Good division sees 8/2 as 8 split in 2s where goofy division sees it as 8 split in 2.

Good multiplication sees 3*4 as 3 4s that may or may not be re-counted in tens. Goofy multiplication sees 3*4 as 1.2 tens.

Good fractions are per-numbers coming from double-counting in the same unit. Goofy fractions neglect units.

Good functions names a difference between examples. Goofy functions state a banality: of course, measuring produces on number only.

Good equations are reversed calculations solved by moving to opposite side with opposite sign. Goofy equations use neutralizing performing identical operations to both sides; but silencing the group definition of abstract algebra applied.

Good trigonometry occurs before plane and coordinate geometry as mutual re-counting of the sides in a block halved by its diagonal. Goofy trigonometry occurs after.

Good calculus occurs in primary school as next-to addition of block-numbers, and in middle and high school as adding piecewise and locally constant per-numbers. Goofy calculus is postponed to the end of high school and lets differential calculus precede integral calculus.

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12. A Core STEM Curriculum for Young Migrants - Short

Seeing mathematics and education both as social institutions we can ask: is poor PISA performance and widespread learner dislike caused by a Baumanian goal displacement making a monopolized means, mathematics, the goal even if not leading to the original goal, to master Many with number-language sentences about how totals are counted, united and changed.

To master Many, the Foucauldian truth regime of 1D line-numbers should give way to what children bring to school, 2D block-numbers as 2 3s or 4 5s that when added on-top and next-to root core mathematics as proportionality to make the units like, and calculus to integrate the areas.

Block-numbers include decimals and negative numbers when recounting a total of 5 in 2s in three ways, overload and standard and underload: T = 5 = 1.3 2s = 2.1 2s = 3.-1 2s.

With addition postponed, division and multiplication and subtraction will allow counting, re-counting and double-counting totals in STEM multiplication formulas.

In the next number of the Philosophy of Mathematics Education Journal, my article will show the full potential of a STEM based 'recount before adding', curriculum allowing young migrants to become pre-STEM teachers or engineers in half a year starting from scratch.

13. A Core STEM Curriculum for Young Migrants - Long

Observing children's quantitative competence uncovers as an alternative to the present set-derived mathe-matics a Many-based 'many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. Results are predicted by a recount formula occurring all over the STEM subjects.

Meeting many

Meeting many, we ask 'How many in total?' To answer, we count. Holding 4 fingers together 2 by 2, a 3year-old will say 'That is not 4, that is 2 2s', thus describing what exists, a number of bundles forming a 2D 'block-number' that may or may not be recounted as 1D line-numbers.

Using 'bundle-counting', a total *T* of 5 1s is recounted in 2s as *T* = 2 2s & 1; and is described by 'bundle-writing', *T* = 2*B*1 2s, or 'decimal-writing', *T* = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

Entering '5/2', we ask a calculator 'from 5 we take away 2s'. The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking '5 – 2*2'. The answer, 1, predicts that 5 = 2*B*1 2s = 2.1 2s.

Operations thus are icons also: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away bundles, and subtraction shows the trace left when taking away a stack only once.

To changes units (called proportionality), a calculator uses a 'recount formula', $T = (T/B)*B$, to predict that 'from *T*, *T/B* times, *B*s can be taken away'. It occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as *sine* and *cosine* and *tangent*, e.g. $a = (a/c)*c = sinA*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/ \Delta x)* \Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

Recounting in the same unit and in a different unit

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2*B*1 2s as 1*B*3 2s with an outside 'overload'; or as 3*B*-1 2s with an outside 'underload' thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48; as well as multiplication, subtraction and addition.

Recounting to and from tens

Asking '3 4s = ? tens' is called times tables to be learned by heart. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right. Multiplication thus is a special form of division.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation, $x*7 = 35$. It is easily solved by recounting 35 in 7s: $x*7 = 35 = (35/7)*7$. So $x = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question '*T* = 6$ = ?kg', we recount 6 in 2s since the per-number is 2$/3kg: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

Mathematics in STEM subjects

STEM (Science, Technology, Engineering and Mathematics) allow humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots will help transforming nature into human necessities. Nature consists of things in motion, combined in the momentum = mass*velocity. Things contain mass and molecules and electric charge. Thus, nature is counted in meter and second and kilogram and mole and coulomb. Looking at the list of formulas we see that nature is predictable by recounting & per-numbers. Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus (Tarp, 2018).

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
meter = (meter/second) * second = velocity * second
Δ momentum = (Δ momentum/second) * second = force * seconds
Δ energy= (Δ energy/meter) * meter = force * meter = work
energy = (energy/kg/degree) * kg * degree = heat * kg * degree
force = (force/square-meter) * square-meter = pressure * square-meter
gram = (gram/mole) * mole = molar mass * mole

Social theory looking at math education

In social theory, Bauman (1990) warns against the danger of goal displacement where 'The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84).'

Foucault (1995) talks about 'truth regimes': 'A corpus of knowledge, techniques, 'scientific' discourses is formed and becomes entangled with the practice of the power to punish (p. 23).'

Let's accept mastery of Many as the real educational goal; and let's replace the truth regime of 1D line-numbers with 2D block-numbers. This will allow all young migrants develop a STEM-based quantitative competence benefitting both themselves and the society.

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14. Fifty Years of Ineffective Math Education Research, Why? Oops, Wrong Numbers, Sorry

Many countries face poor PISA results. Does its nature make mathematics so hard to learn despite 50 years of research? We need to read again the two founding fathers.

Freudenthal sees set-based university mathematics as so important to the outside world that it must be taught in schools. Skemp sees a true understanding of mathematics as based upon sets compared as to cardinality by, not counting them, but by establishing a correspondence between them.

Sociology points to a different explanation: maybe mathematics education has a goal displacement where it sees itself as the goal, and its outside root, Many, as a means.

So we may ask: as an alternative to the set-based tradition, is there is a different way to the outside goal of mathematics education, mastery of Many?

By observing the quantitative competences children bring to school, and by using difference-research searching for differences making a difference, we discover an alternative to the present set-based mathematics that was introduced some 50 years ago as ‘New Math’: a ‘many-matics’ seeing mathematics as a natural science about Many.

Here digits are icons with as many sticks as they represent. Also operations are icons where bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads or underloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations.

Here double-counting in two units creates per-numbers, becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers.

Addition here occurs on-top and next-to rooting proportionality, and integral calculus by adding areas; and here trigonometry precedes plane and coordinate geometry.

So, we need to research what happens if two-dimensional block-numbers replace one-dimensional line-numbers; if the order of operations is reversed; if bundle- counting, re-counting and double-counting precedes adding next-to and on-top; and, if using full sentences about the total in the number-language, as T = 2.1 3s with a subject, a verb and a predicate as in the word-language.

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15. Rethinking Line-Number Arithmetic as Block-Number Algebra

'Early Algebra' recommends further research in seven areas. Apart from rethinking the examples included, this poster addresses area 2 and 4, curricula activity and theorizing numbers and operations. Using Difference-research, searching for differences making a difference, the poster asks: Will a different block-number algebra allow rethinking traditional line-number arithmetic?

As sceptical thinking from the French and American Enlightenment republics, Foucault Concept Archaeology and Existentialism and Grounded Theory is used to look at the roots of Algebra.

In Arabic, Algebra means to reunite. Numbers as T = 345 = 3*B^2+4*B+5*1 show the four ways to unite a total: addition, multiplication, power and integration of juxtaposed blocks. They also show that asking 'How many in total?', the answer is expressed as a 'number-language' sentence containing, as does a word-language sentence, a subject and a verb and a predicate, thus rooting a formula with an equation sign.

Using grounded theory we observe that, before receiving formal education, preschool children use 2dimensional Bundle-numbers or Block-numbers as T = 2 5s & 1. Re-counting a total into a new unit by asking T = 3 4s = ? 5s, children quickly accept division as an icon for a broom wiping away 5-bundles, and multiplication as an icon for stacking the bundles into a block, and subtraction for the trace left when dragging away the block to look for unbundled leftover singles.

Likewise, children find it natural to formulate the recounting process as 'from a total T, T divided by the bundle B gives the number of times Bs can be taken away', shortened to a 'recount-formula' T = (T/B)*B. This formula allows using a calculator to predict the result of a re-counting process: Recounting 7 in 3s, we enter 7/3. The answer '2.some' predicts it can be done 2 times. Taking away the stack of 2 3s, the answer '7-2x3 = 1' shows the prediction: 7 can be re-counted as 2 3s & 1.

The recount-formula T = (T/B)*B leads directly to the heart of Algebra by allowing children to use formulas as a natural way to communicate in math education. The commutative, associative and distributive laws follow directly from watching 2- and 3-dimensinal blocks.

And solving equations takes place when re-counting from tens to icons: asking T = 40 = ? 8s the solution is found by re-counting 40 in 8s as 40 = (40/8)*8 = 5*8 = 5 8s.

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16. A Count-before-Adding Curriculum for Preschool and Migrants

Children show a surprising mastery of Many with a quantitative competence where totals are re-counted in the same and in a different unit, as well as to and from tens. And children enjoy using a calculator and a re-count formula to predict re-counting results. Thus, children bring to school two-dimensional LEGO-like block-numbers that are different from the one-dimensional line-numbers taught in school, seeing cardinality as linear. Allowed to keep their block-numbers, children and migrants will be practising proportionality and calculus when adding block-number on-top and next-to; and will be solving equations when re-counting from tens to blocks.

Keywords: preschool, add, equation, proportional, calculus.

Decreased PISA performance despite increased research

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life' (p. 3).

The ineffectiveness of mathematics education research may prove that, by its very nature, mathematics is indeed hard to learn. On the other hand, since mathematics education is a social institution, social theory may provide a different reason.

Social Theory Looking at Mathematics Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'.

However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right (p. 84).

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference. So, inspired by sociology we can ask the 'Cinderella question': 'as an alternative to the tradition, is there is a different way to the outside goal of mathematics education, mastery of Many?'

How well-defined is mathematics?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. Algebra thus contains the four ways to unite as shown when writing out fully the total T = 342 = 3*B^2 + 4*B + 2*1 = 3 bundles of bundles and 4 bundles and 2 unbundled singles = 3 blocks. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reverse splitting operation.

So, as a label, mathematics has no existence itself, only its content has, algebra and geometry; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek ‘many-matics’ rooted in Many was replaced by the ‘New Mathematics’.

Here the invention of the concept Set created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meets counter-examples as 2weeks + 3days is 17days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists that wanting to avoid being patronized by choices presented as nature (Russell, 1945), ‘difference-research’ is searching for hidden differences making a difference (Tarp, 2017). So, to avoid a goal displacement in mathematics education, difference-research asks the grounded theory question: How will mathematics look like if grounded in its outside root, the physical fact Many?

To answer, we will use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1970) from a personal to a social level, to allow Many to open itself for us and create its own categories and properties. In short, we will search for an alternative to the ruling tradition by returning to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space.

Meeting many

We live in space and in time. To include both when counting Many, we use two different ways of counting. Counting in space, we count blocks and report the result with LEGO-blocks or on a ten-by-ten abacus in ‘geometry-mode’. Counting in time, we count bundles and report the result on a ten-by-ten abacus in ‘algebra-mode’.

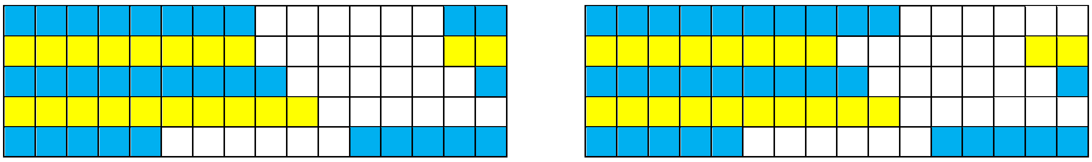
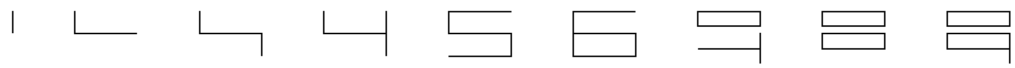


Table 1: 5 re-counted as 2 2s & 1 on an abacus in geometry- and in algebra-mode

Thus, to master Many, we count by bundling and stacking. But first we rearrange sticks into icons with e.g. five sticks in the five-icon 5 if written less sloppy. Counted as ‘1 bundle’, ten does not need an icon when used as the bundle-size.

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



1 2 3 4 5 6 7 8 9

Table 2: Sticks rearranged as icons with as many sticks as they represent

Holding 4 fingers together 2 by 2, a 3year old will say ‘That is not 4, that is 2 2s’, thus describing how Many presents itself, as a number of bundles that may or may not be re-counted as four ones. This inspires ‘bundle-counting’ bundling a total in icon-bundles. Thus, a total *T* of 5 1s is bundled in 2s as *T* = 2 2s and 1 where the bundles are placed inside a bundle-cup with a stick for each bundle, leaving the unbundled outside; and described by ‘bundle-writing’, *T* = 2*B*1, or ‘decimal-writing’, *T* = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

T = 5 = I I I I I → II II I → $\boxed{\text{II}}$ I → 2*B*1 2s = 2.1 2s

Entering ‘5/2’ we ask a calculator ‘from 5 we can take away 2s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 2s by asking ‘5 – 2*2’. From the answer ‘1’ we conclude that 5 = 2*B*1 2s. Likewise, showing ‘5 – 2*2 = 1’, a display indirectly predicts that 5 can be re-counted as 2 2s and 1, or as 2*B*1 2s.

5 / 2	2.some
5 – 2 * 2	1

Table 3: A calculator predicts that 5 re-counts in 2s as 2.1 2s

We see that also operations are icons: a stack of 2 3s is iconized as 2*3, or 2x3 showing a lift used 2 times to stack the 3s; division shows the broom wiping away 2s several times, and subtraction shows the trace left when taking away the stack once.

A calculator thus uses a ‘re-count formula‘, *T = (T/B)*B*, saying that ‘from *T*, *T/B* times *B*s can be taken away’. This re-count formula occurs all over mathematics: when relating proportional quantities as *y = c*x*; in trigonometry as sine and cosine and tangent, e.g. *a = (a/c)*c = sinA*c*; in coordinate geometry as line gradients, *Δy = (Δy/Δx)*Δx = c*Δx*; and in calculus as the derivative, *dy = (dy/dx)*dx = y’*dx*.

Re-counting in the same unit and in a different unit

Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 5 as 1*B*3 2s with an outside ‘overload’; or as 3*B*-1 2s with an outside ‘underload’ thus leading to negative numbers:

T = 5 = I I I I I I → II I I I → 1*B*3 2s, or *T* = 5 = I I I I I I → II II II → 3*B*-1 2s

To re-count in a different unit means changing unit, also called proportionality or linearity. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2B2 5s.

T = 3 4s = IIII IIII IIII → IIIII IIIII I I → 2*B*2 5s.

A calculator can predict the result. Entering ‘3*4/5’ we ask ‘from 3 4s we take away 5s how many times?’ The answer is ‘2.some’. To find the leftovers we take away the 2 5s and ask ‘3*4 – 2*5’. Receiving the answer ‘2’ we conclude that 3 4s can be re-counted in 5s as 2 5s and 2, or as 2*B*2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Table 4: A calculator predicts that 3 4s re-counts in 5s as 2.2 5s

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘6$ = ?kg’ we use the per-number to re-count 6 in 2s: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa: Asking ‘?$ = 12kg’, the answer is 12kg = (12/3)*3kg = (12/3)*2$

= 8\$. Double-counting in the same unit creates fractions and percentages: 2\$/3\$ = 2/3, and 2\$/100\$ = 2/100 = 2%.

Re-counting to and from tens

Asking ‘3 4s = ? tens’ is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, i.e. in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

Re-counting from tens to icons by asking ‘38 = ? 7s’ is called an equation *x**7 = 38. It is easily solved by re-counting 38 in 7s: *x**7 = 38 = (38/7)*7. So *x* = 38/7 = 5 & 3/7 as predicted by a calculator showing that 38 = 5.3 7s = 5*7 + 3.

38 / 7	5.some
38 – 5 * 7	3

Table 5: A calculator predicts that 38 re-counts in 7s as 5.3 7s

Once counted, totals can be added on-top or next-to

Asking ‘3 5s and 2 3s total how many 5s?’ we see that to add on-top, the units must be the same, so the 2 3s must be re-counted in 5s as 1*B*1 5s that added to the 3 5s gives a total of 4*B*1 5s.

IIII IIII IIII & III III → IIII IIII IIII IIII I → 4*B*1 5s.

For a calculator prediction, we use a bracket before counting in 5s: Asking ‘(3*5 + 2*3)/5’, the answer is 4.some. Taking away 4 5s leaves 1. Thus, we get 4*B*1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Table 6: A calculator predicts that the sum of 3 5s and 2 3s re-counts in 5s as 4.1 5s

Since 3*5 is an area, adding next-to means adding areas, called integration. Asking ‘3 5s and 2 3s total how many 8s?’ we use sticks to get the answer 2*B*5 8s.

IIII IIII IIII & III III → IIIII III IIIII III I I I I I → 2*B*5 8s = 2.5 8s

For a calculator prediction, we include the two totals in a bracket before counting in 8s: Asking ‘(3*5 + 2*3)/8’, the answer is 2.some. Taking away the 2 8s leaves 5. Thus we get 2*B*5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Table 7: A calculator predicts that the sum of 3 5s and 2 3s re-counts in 8s as 2.5 8s

Reversing adding on-top and next-to

Reversed addition is called backward calculation or solving equations. Reversing next-to addition is called reversed integration or differentiation. Asking ‘3 5s and how many 3s total 2*B*6 8s?’, using sticks will give the answer 2*B*1 3s:

IIII IIII IIII III III I ← IIIII III IIIII III I I I I I I ← 2*B*6 8s

For a calculator prediction, the remaining is bracketed before being counted in 3s.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Table 8: A calculator predicts that 2.6 8s comes from next-to addition of 2.1 3s to 3 5s

Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula

y' = Δy/t = (y2 – y1)/t.

Designing a count-before-adding curriculum

To study the effect of re-counting, an eight-part 'count-before-adding' curriculum can be designed where bundle-counting involves division, multiplication, subtraction before next-to and on-top addition to respect that totals must be counted before being added; in contrast to school that turns this order around and insists numbers be counted in tens only to be added on-top only, using carrying instead of overloads.

In the first micro-curriculum, the learner uses sticks and a folding ruler to build the number-icons up to nine; and uses strokes to draw the icons thus realizing there are as many sticks or strokes in the icon as it represents if written less sloppy. In the second, the learner counts a given total in icons by bundling sticks and using a cup for the bundles; and reports, first with bundle-writing and decimal-writing with a unit; then by using an abacus in geometry- and algebra-mode; always using a calculator to predict the counting result. In the third, the learner re-counts a total in the same unit thus creating or removing overloads and underloads.

In the fourth micro-curriculum, the learner re-counts a total in a different unit. In the fifth, the learner adds two block-numbers on-top of each other. In the sixth, the learner adds two block-numbers next-to each other. In the seventh, the learner reverses on-top addition. And in the eighth, the learner reverses next-to addition. Finally, the learner sees how double-counting creates per-numbers.

	Examples	Calculator prediction	
M2	7 1s is how many 3s? I I I I I I I → III III I → 2B1 3s So 7 1s = 2*B*1 3s = 2.1 3s	7/3 7 – 2*3	2.some 1
M3	'2.7 5s is also how many 5s?' V V IIIIIII = V V V V II = V V V IIIII 2*B*7 = 2+1*B*7–5 = 3*B*2 = 3+1*B*2–5 = 4*B*-3 So 2.7 5s = 3.2 5s = 4.-3 5s	(2*5+7)/5 (2*5+7) – 3*5 (2*5+7) – 4*5	3.some 2 -3
M4	2 5s is how many 4s?' V V = IIIII IIIII = IIII IIII I I So 2 5s = 2.2 4s	2*5 / 4 2*5 – 2*4	2.some 2
M5	'2 5s and 4 3s total how many 5s?' V V III III III III = V V V V I I So 2 5s + 4 3s = 4.2 5s	(2*5+4*3) /5 (2*5+4*3) – 4*5	4.some 2
M6	'2 5s and 4 3s total how many 8s?' V V III III III III = IIIIIIII IIIIIIII III III So 2 5s + 4 3s = 2.6 8s	(2*5+4*3) /8 (2*5+4*3) – 2*8	2.some 6
M7	'2 5s and ? 3s total 4 5s?' V V V V = V V III III III I So 2 5s + 3.1 3s = 4 5s	(4*5 – 2*5)/3 (4*5 – 2*5) – 3*3	3.some 1
M8	'2 5s and ? 3s total 2.1 8s?' IIIIIIII IIIIIIII I = IIIII III IIIII III I So 2 5s + 2.1 3s = 2.1 8s	(2*8+1 – 2*5)/3 (2*8+1–2*5) – 2*3	2.some 1

Table 9: A 'count-before-adding' curriculum shown as micro-curricula

To be tested in special education, an additional '1 cup & 5 sticks' micro-curriculum was added. Here using a cup for the bundles allows 5 sticks to be bundle-counted in 2s as 1*B*3 or 2*B*1 or 3*B*-1 to show that a total can be counted in three ways, overload and normal and underload, with an inside and an outside number for the bundles and the singles. So, to divide 336 by 7, 5 bundles move outside as 50 singles to re-count 336 with an overload: 336 = 33*B*6 = 28*B*56, which divided by 7 gives 4*B*8 or 48.

When tested, one curriculum used silent education where the teacher demonstrates and guides by actions only, not using words; and in one curriculum the teacher spoke a foreign language not understood by the learners. In both cases the abacus and the calculator quickly took over the communication. For further details watch the video www.youtube.com/watch?v=IE5nk2YEQIA.

After the micro-curricula a learner went back to her grade 6 class where proportionality created learning problems. The learner suggested renaming it to double-counting but the teacher insisted on following the textbook. However, observing that the class took over the double-counting method, the teacher gave in and allowed proportionality to be renamed and treated as double-counting. When asked what she had learned besides double-counting both learners and the teacher were amazed when hearing about next-to addition as integral calculus.

Thus bundle-counting together with a calculator for predicting re-counting results allowed the learner to reach the outside goal, mastering Many, by following an alternative to the institutionalized means that because of a goal displacement had become a stumbling block to her; and performing and reversing next-to addition introduced her to and prepared her for later calculus classes.

Literature on bundle-counting and block-numbers

No research literature on bundle-counting and block-numbers was found. Nor is it mentioned by Dienes (1964).

Conclusion and recommendations

To avoid a goal displacement in mathematics education, this paper searched for a different way to the goal of mathematics education, mastery of Many. Difference-research and Grounded Theory showed how mathematics looks like if grounded in its physical root, Many. To tell the difference, two names were coined, 'many-matics' versus 'meta-matism' mixing 'meta-matics', defining concepts as examples of abstractions instead of as abstractions from examples, with 'mathe-matism' valid only inside classrooms. To validate its findings, the paper includes a classroom test of a 'Count-before-adding' curriculum described in detail to allow it to be tested in other classrooms also. To improve mathematics education, the curriculum can be used in early childhood and in special education to give a physical understanding of how numbers come from counting by bundling and stacking before using the short way of writing numbers counted in tens without decimal points and units.

Likewise, it can be used for young migrants wanting to help rebuild their country as STEM-teachers or STEM-engineers (Science, Technology, Engineering, Math). A six months STEM-based many-matics curriculum has been designed allowing migrants use the first month to learn to reunite constant and variable unit- and per-numbers by addition and multiplication and integration and power, and by their reverse operations subtraction, division, differentiation and root/logarithm. The next five months then consist of modelling situations in science, technology, and engineering (Tarp, 2017). Finally, as an alternative to line-numbers, block-numbers opens up a completely new research paradigm (Kuhn, 1962).

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17. Difference-research Saving Dropout Ryan with a TI-82 Calculator

At principal asked for ideas to lower the number of dropouts in pre-calculus classes. The author proposed using a cheap TI-82, but the teachers rejected saying students weren't even able to use a TI-30. Still the principal allowed buying one for a class. A compendium called 'Formula Predict' replaced the textbook. A formula's left- and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects.

Keywords: precalculus, calculus, calculator, model, dropout.

The Task

The headmaster asked the mathematics teachers: "We have too many pre-calculus dropouts. What can we do?" I proposed buying the cheap TI-82 graphing calculator, but the other teachers rejected this proposal arguing that students weren't even able to use a simple TI-30. Still I was allowed to buy this calculator for my class allowing me to replace the textbook with a compendium emphasizing modeling with TI-82.

How Well-Defined is Mathematics After All?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

So, as a label, mathematics has no existence itself, only its content has, algebra and geometry; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-matics' rooted in Many was replaced by the 'New Mathematics' wanting concepts to be well-defined by being derived from the set-concept. This despite that Russell, by looking at the set of sets not belonging to itself, showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

Thus, the set-concept turned mathematics upside down to a 'meta-matism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathe-matism' true in the library, but not in the laboratory, as e.g. 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest that when counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

Despite being neither well-defined nor well-proved, mathematics still teaches meta-matism, thus creating huge problems to mathematics education.

Difference Research

Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists emphasized telling choice from nature to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should become the natural patronizing rulers (Russell, 1945).

Inspired by the ancient Greek sophists, 'difference-research' is searching for hidden differences making a difference (Tarp, 2017). So, it asks the grounded theory question: How will mathematics look like if grounded in its outside root, the physical fact Many? To answer, we will use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1970) from a personal to a social level, to allow Many to open itself for us and create its own categories and properties. In short, we will search for an alternative to the ruling tradition by returning to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space.

The Case of Teaching Math Dropouts

Being our language about quantities, mathematics is a core part of education in both primary and secondary school. Students generally accept the importance of learning mathematics, but many fail to see the meaning of doing so. Consequently, special core courses for math dropouts are developed.

Traditions of Core Precalculus Courses for Dropouts

A typical core course for math dropouts is halving the content and doubling the text volume. So, in a slow pace the students work their way through a textbook once more presenting mathematics as a subject about numbers, operations, equations and functions applied to space, time, mass and money. To prevent spending time on basic arithmetic, a TI-30 calculator is handed out, typically without instruction.

As to numbers, the tradition focuses on fractions and how to add fractions.

Then solving equations is introduced using the traditional balancing method, isolating the unknown by performing identical operations to both sides of the equation. Typically, the unknown occurs in fractions as $5 = 40/x$; or on both sides of the equation as $2*x + 3 = 4*x – 5$

Then relations between variables are introduced using tables, graphs and functions with special emphasize on the linear function $y = f(x) = a*x + b$.

In a traditional curriculum, a linear function is followed by the quadratic function. But a core course might instead go on to the exponential and power functions $y = b*a^\wedge x$ and $y = b*x^\wedge a$. To avoid solving its equations, the solutions are given as formulas.

Problems in Traditional Core Courses

A traditional core course wants to give a second chance to learners having dropped out of the traditional math course. However, from a sceptical viewpoint trying to avoid presenting choice as nature, several questions can be raised.

As to numbers, are fractions numbers or calculations that can be expressed with as many decimals as we want, typical asking for three significant figures? Is it meaningful to add fractions without units as shown by the fraction-paradox above?

As to equations, is the balancing method nature or choice presented as nature? The number $x = 8 – 3$ is defined as the number that added to 3 gives 8, $x + 3 = 8$. This can be restated as saying that the equation $x + 3 = 8$ has the solution $x = 8 – 3$; suggesting that the natural way to solve equations is the 'move to opposite side with opposite sign' method. This method applies to all reversed calculation, defining a root as a factor-finder, and a logarithm as a factor-counter.

$x + 3 = 15$ $x = 15 – 3$	$x * 3 = 15$ $x = 15/3$	$x^\wedge 3 = 125$ $x = \sqrt[3]{125}$	$3^\wedge x = 243$ $x = \ln 243/\ln 3$

Table 1: the basic equations solved by the 'opposite side & sign' method

As to relations between variables, is the function nature or choice presented as nature? A basic calculation as 3 + 5 = 8 contains three numbers. If one of these is unknown we have an equation to be solved, e.g. $3 + x = 5$, if not already solved, $3 + 5 = x$. With two unknown we have a formula as in $3 + x = y$, or a relation as in $x + y = 3$ that can be changed into the formula $y = 3 - x$. So, the natural relation between two unknown variables seems to be a formula.

As to solving exponential equations, is presenting solution formulas nature or choice presented as nature? Solving basic equations is just another way of defining inverse operations, so it is a natural thing to define a root and a logarithm as solutions to the basic equations involving power.

Desiging a Grounded Core Course

So, a traditional core course seems to be filled with examples of choices presented as nature. This leads to the question: is it possible to design a different core course based upon nature instead of choices presented as nature? In other words, what would be the content of a core course in pre-calculus grounded in the root of mathematics, the natural fact Many?

Mathematics as a Number-language

As to the nature of the subject itself, mathematics is a number-language that together with the word-language allows users to describe quantities and qualities in everyday life. Thus, a calculator is a typewriter using numbers instead of letters. A typewriter combines letters to words and sentences. A calculator combines figures to numbers that combined with operations becomes formulas. Thus, formulas are the sentences of the number-language.

The Number Formula shows the four Ways to Unite

The four Algebra ways to unite is seen when writing out fully the total *T* = 542 = 5**B*^2 + 4**B* + 2*1, i.e. as three blocks: 5 bundles of bundles and 4 bundles and 2 unbundled singles. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reverse splitting operation: subtraction, division, root and logarithm and differentiation.

Furthermore, we see that the number-formula, has as special cases the formulas for constant linear, exponential, elastic, and accelerated change:

T = *b***x*+*c*, *T* = *a***n*^*x*, *T* = *a***x*^*n*, and *T* = *a***x*^2 + *b***x* + *c*.

Formulas Predict

One difference between the word- and the number-language is that sentences describe whereas formulas predict the four different ways of uniting numbers:

Addition predicts the result of uniting unlike unit-numbers: uniting 2$ and 3$ gives a total that is predicted by the formula *T* = *a*+*b* = 2+3 = 5

Multiplication predicts the result of uniting like unit-numbers: uniting 2$ 5 times gives a total that is predicted by the formula *T* = *a***b* = 5*2 = 10.

Power predicts the result of uniting like per-numbers: uniting 2% 5 times gives a total that is predicted by the formula 1+*T* = *a*^*b* = 1.02^5, i.e. *T* = 0.104 = 10.4%.

Integration predicts the result of uniting unlike per-numbers: uniting 2kg at 7$/kg and 3kg at 8$/kg gives 5 kg at *T*$/5kg where *T* = 7*2 + 8*3 is the area under the per-number graph, *T* = ∫ *p***dx*.

Solving Equations with Solver

Thus, inverse operations solve equations; as do the TI-82 using a solver. An equation as $2 + x = 6$ always has a left-hand side and a right-hand side that can be entered on the calculator's Y-list as *Y1* and *Y2*. So, any equation has the form *Y1* = *Y2*, or *Y1* – *Y2* = 0 that only has to be entered to the solver once. After that, solving equations just means entering its two sides as *Y1* and *Y2*. Using graphs, *Y1* and *Y2* have the intersection points as solutions to the equation *Y1 = Y2*.

If one of the numbers in a calculation is unknown, then so is the result. A formula with two unknowns can be described by a table answering the question 'if *x* is this, then what is *y*?' Graphing a table allows the inverse question to be addressed by reading from the *y*-axis.

Producing Formulas with Regression

Once a formula is known, it produces answers by being solved or graphed. Real world data often come as tables, so to model real world problems we need to be able to set up formulas from tables. Simple formulas describe levels as e.g. cost = price*volume. Calculus formulas describe change and pre-calculus describes constant change.

If a variable y begins with the value b and changes by a number a x times, the *y = b + a*x*. This is called linear change and occurs in everyday trade and in interest-free saving.

If a variable y begins with the value b and changes by a percentage r x times, the *y = b*(1+r)^x* since adding 5% means multiplying with 105% = 1 + 5%. This is called exponential change and occurs when saving money in a bank and when populations grow or decay.

Combining linear and exponential change by depositing *a*$ *n* times to an interest rate *r*%, we get a saving *A*$ predicted by the formula *A/a = R/r* where the total interest rate *R* is predicted by the formula *1+R = (1+r)^n.*

The proof: from an account with *a/r*$ the interest is moved to another account together with its interest, thus containing *a/r*R* as a saving *A, which* gives *A/a = R/r*.

Thus, instalments can be studied as a race between a debt *D* growing with an interest rate *r*%, *T = D*(1+r)^n*, and a saving *A* growing from fixed deposits and interest rates, *A = a/r*R*. From this, the debt to be found be the formula *D*(1+r)^n = a/r*R.*

In the case of linear and exponential and power change, a two-line table allows us to find the two constants *b* and *a* using regression on a TI-82.

Multi-line tables can be modelled with polynomials. Thus, a three-line table might be modelled with a quadratic formula *y = b + a*x + c*x^2* including also a bending-number *c*; and a four-line table by a cubic formula *y = b + a*x + c*x^2 + d*x^3* including also a counter-bending-number *d*, etc.

Graphically, a second-degree polynomial is a bending ling, a parabola; and a third- degree polynomial is a double parabola. The top and the bottom of a bending curve as well as its zeros can be found directly by graphing methods on a TI-82.

Fractions as Per-numbers

Fractions are rooted in per-numbers: 3$ per 5 kg = 3$/5kg = 3/5 $/kg. To add per-numbers they first must be changed to unit numbers by being multiplied with their units: 3 kg at 4 $/kg + 5 kg at 6 $/kg = 8 kg at (3*4 + 5*6)/8 $/kg

Geometrically this means that the areas under its graph add per-numbers. Here again TI-82 comes in handy when calculating areas under graphs; also in the case where the graph is not horizontal but a bending line, representing the case when the per-number is changing continuously as e.g. in a falling body: 3 seconds at 4 m/s increasing to 6 m/s totals 15 m in the case of a constant acceleration.

Models as Quantitative Literature

With the ability to use TI-82 as a quantitative typewriter able to set up formulas from tables and to answer both x- and y-questions, it becomes possible to include models as quantitative literature.

All models share the same structure: A real-world problem is translated into a mathematical problem that is solved and translated back into a real-world solution.

	Problem		Solution
Mathematics		→	
	↑		↓
Real world		←	

Table 2: The four steps in modelling real-word situations

The project ‘Population versus food’ looks at the Malthusian warning: If population changes in a linear and food in an exponential way, hunger will eventually occur.

The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period.

From this two-line table regression can produce two formulas: with x counting years after 1850, the population is modeled by *Y1*= 815*1.013^*x* and the food by *Y2*= 300 + 30**x*. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

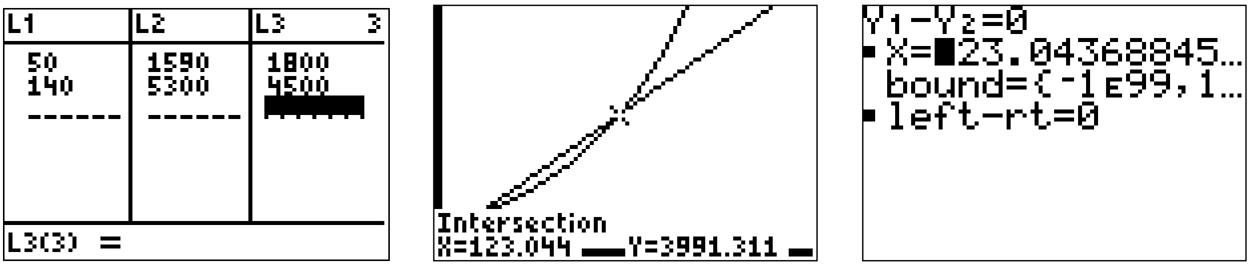

Table 3: Two two-line table regressions allow graphical and algebraic solutions

The project ‘Fundraising’ finds the revenue of a fundraising show assuming that all students will accept a free ticket, that 100 students will buy a 20$ ticket and that no one will buy a 40$ ticket.

From this three-line table the demand can be modeled by the quadratic formula *Y1* = .375**x*^2 – 27.5**x* + 500. Thus, the revenue formula is the product of the price and the demand, *Y2 = x*Y1*. Graphing methods shows that the maximum revenue will be 2688 $ at a ticket price of 12$.

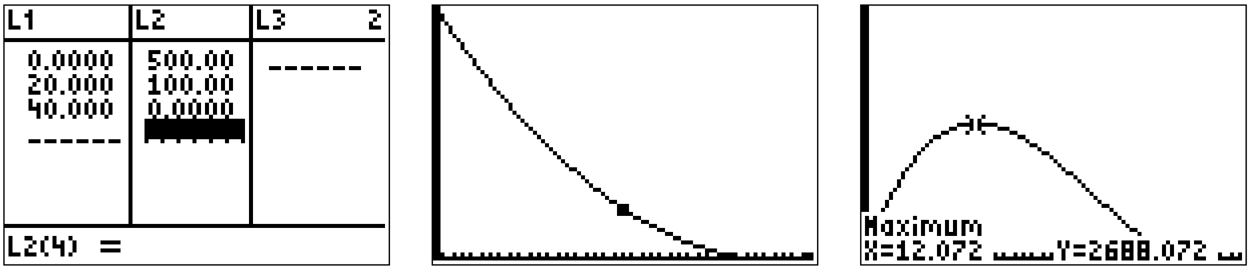

Table 4: A three-line table regression offers a parabola to be studied graphically

In the project ‘Driving with Peter’ his velocity is measured five times. A model answers two questions: When is Peter accelerating? And what distance did Peter travel in a given time interval?

From a five-line table the speed can be modeled by the 4th degree polynomial *Y1* = -0.009**x*^4 + 0.53**x*^3 - 10.875**x*^2 + 91.25**x* - 235. Visually, the triple parabola fits the data points. Graphing methods shows that a minimum speed is attained after 14.2 seconds; and that Peter traveled 115 meters from the 10th to the 15th second.

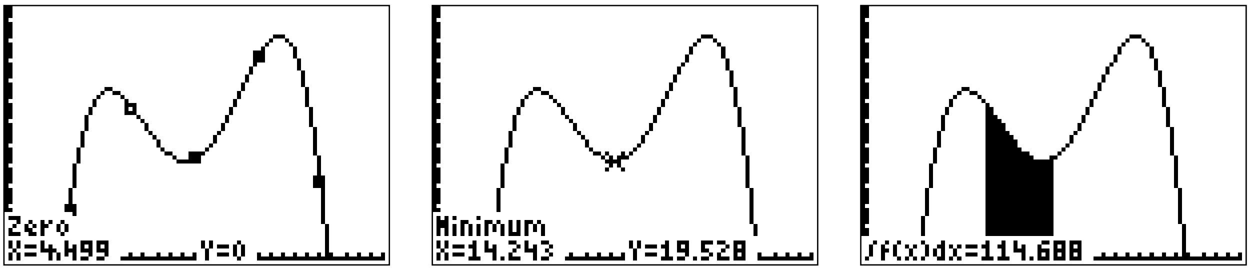

Table 5: A five-line table regression offers a triple-parabola to be studied graphically

Six other projects were included in the course.

The project 'Forecasts' modeled a capital growing constantly in three different ways: linear, exponential and potential. The project 'Determining a Distance' uses trigonometry to predict the distance to an inaccessible point on the other side of a river. The project 'The Bridge' uses trigonometry to predict the dimensions of a simple expansion bridge over a canyon.

The project 'Playing Golf' asks to predict the formula for the orbit of a ball that has to pass three given points: a starting point, an ending point and the top of a hedge. The project 'Saving and Pension' asks about the size of a ten years monthly pension created by a thirty years monthly payment of 1000$ at an interest rate of 0.4% per month. And the project 'The Takeover Try' asks how much company A has spent buying stocks in company B given a course described by a four-line table.

Testing the Core Course

The students expressed surprise and content with the course. Their hand-in was delivered on time. And they course finished before time allowing the inclusion of additional models from classical physics: vertical falling balls, projectile orbits, colliding balls, circular motion, pendulums, gravity points, drying wasted whisky with ice cubes, and supplying bulbs with energy.

At the written and the oral exam, for the first time at the school, all the students passed. Some student wanted to move on to a calculus class, other were reluctant arguing that they had already learned the core of calculus.

Reporting Back to the Headmaster

The headmaster expressed satisfaction, but the teachers didn't like the textbook and its traditional mathematics to be set aside. To encourage the teachers, the headmaster ordered the TI-82 to be bought to all pre-calculus classes.

Conclusion: Make Losers Users

Using difference-research, this action research project showed that dropout students get an extra chance by boiling mathematics down to its core grounded in its roots, the natural fact Many. Here numbers are polynomials showing the operations that allows totals to be united from or split into constant or variable unit- or per-numbers according to Algebra's reuniting project. In this way the core of algebra is solving equations with the 'opposite side & sign' method or with the solver on a graphing calculator. And the core of pre-calculus is using regression to translate tables to formulas that can be processed both geometrically and algebraically when entered into the Y-list of the TI-82. Thus, grounding mathematics in its root Many will allow all students to use a graphing calculator to predict the behavior of real-world quantities, thus reconquering the number-language taken from them by meta-matism.

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18. Conflicting Theories Help Teachers Improve Mathematics Education

Traditionally, education is seen as teachers transferring institutionalized knowledge to individual learners. As such, education involves several choices. Shall teachers teach or guide? Is mathematics an eternal truth or a social construction? Is it knowledge about, or knowing how to? How to motivate learning? Should a class be optional or mandatory? To answer, teacher education refers to theory from philosophy, psychology and sociology. Including the existence of conflicting theories will allow teachers try out alternatives if wanting to improve mathematics education.

Keywords: teacher education, mathematics, philosophy, psychology, sociology.

Philosophical Controversies

Ancient Greece saw two forms of knowledge, called 'sophy'. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. The Christian Church eagerly took over a metaphysical patronage and changed the academies into monasteries, until the Reformation changed some back again.

By letting the laboratory precede the library, natural science reinvented scepticism. Newton discovered that falling objects obey their own will instead of that of a patronizor. This inspired the Enlightenment Century and it two republics, the American and the French, transforming scepticism into pragmatism and post-structuralism, based upon existentialism defined by Sartre as 'Existence preceding essence'; and with the Heidegger warning: in a sentence, respect the subject, but question the predicate since it might be gossip. Thus, post-structuralism deconstructs ungrounded diagnoses forcing humans to accept unfounded patronization.

Psychological Controversies

As to how learners acquire knowledge, several constructivist theories exist among which are Vygotskian and Piagetian social and radical constructivism disagreeing by recommending teaching as much and as little as possible.

Vygotsky sees knowledge as true sentences to be transferred by good teaching. However, a learner can only take in unknown sentences about subjects already known, so the teacher must know the individual 'zone of proximal development' in order to successfully connect it to the institutionalized knowledge by scaffolding.

Whereas Piaget recommends meeting the new subjects directly to allow learners form individual concepts and sentences to be negotiated and accommodated socially.

Sociological Controversies

As a social institution, education can be seen from a structure or an actor viewpoint, reflecting societies with high or low degree of institutionalization. Being highly institutionalized, continental Europa has developed a structure-based sociology seeing humans as bound by social structures. Thus, Foucault sees knowledge as socially constructed discourses; and describes a school as a 'prispital' mixing the power techniques of a prison and a hospital: the learners are fixed in classrooms and diagnosed as ignorant to be cured by discourses institutionalized as truth but instead exerting 'pastoral power'. Whereas their escape from Europe made Americans actors developing grounded theory, lifting Piagetian accommodation to a social level.

Mathematical Controversies

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy, seen by the Greeks as knowledge about Many by itself, in space, in time, and in space and time. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric. With astronomy and music gone, today mathematics should be a common label for geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about fifty years ago when the Greek ‘many-matics’ rooted in Many was replaced by the ‘New Mathematics’. Here the invention of the concept Set created a ‘meta-matics’ defining concepts as examples of abstractions instead of as abstractions from examples. But, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

So today two mathematics discourses exist. One is the institutionalized self-referring ‘meta-matics’ presenting digits and fractions as numbers that can be added without units, and still producing poor PISA results despite fifty years of research. The other is a natural science about Many, a ‘many-matics’ presenting digits and fractions as operators or factors needing a number to become a number, and where adding number-blocks on-top and next-to leads directly to core mathematics as linearity and calculus. And where calculators predict the result of re-counting, preceding addition.

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19. Addition-free Core STEM Curriculum for Late Learners along the Silk Road

Its many applications make mathematics useful. But to solve core STEM tasks you need no addition, thus calling for an addition-free curriculum. Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. To answer the question 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two different units to create per-numbers, becoming fractions with like units. To predict a recounting result, we use a recount-formula being a core in all STEM subjects.

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015a) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change'.

To find an unorthodox solution we pretend that a university in China arranges a curriculum architect competition: 'Theorize the low success of 50 years of mathematics education research; and derive from this theory a STEM-based core curriculum that can be used for late learners along the coming new silk road, One Belt and One Road (OBOR).'

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to improve schools in Sweden and elsewhere.

Social theory looking at mathematics education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET changed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to ‘2x3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

Difference research looking at mathematics education

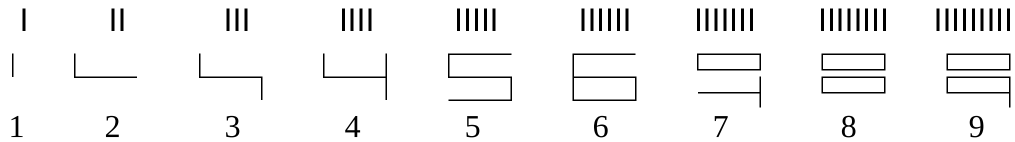
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ (Tarp, 2017) is searching for hidden differences making a difference. So, to avoid a goal displacement in math education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014). So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many creates a ‘count-before-adding’ curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence (Tarp, 2018b).

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:



Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, a number of bundles that may or may not be recounted as ones.

This inspires ‘bundle-counting’, recounting a total in icon-bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted before being processed by on-top and next-to addition, direct or reversed.

Thus, a total T of 5 1s is recounted in 2s as T = 2 2s & 1; and is described by ‘bundle-writing’, T = 2B1 2s, or ‘decimal-writing’, T = 2.1 2s, where a decimal point separates the inside bundles from the unbundled singles outside the bundle-cup.

So, to count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away).

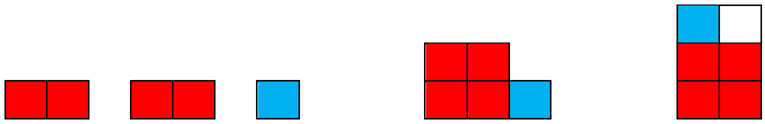
Entering ‘5/2’, we ask a calculator ‘from 5 we take away 2s’. The answer, 2.some, predicts that the singles come by taking away 2 2s, thus asking ‘5 – 2x2’. The answer, 1, predicts that 5 = 2*B*1 2s = 2.1 2s as indirectly predicted on the bottom line.

5 / 2	2.some
5 – 2 x 2	1

A calculator thus predicts the result by a recount-formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: 5/2 gives 2.some, and 5 – 2x2 gives 1, so T = 7 = 2B1 3s.

This recount-formula occurs all over mathematics: when relating proportional quantities: $y = c*x$; in trigonometry as sine, cosine and tangent: $a = (a/c)*c = \sin A*c$ and $b = (b/c)*c = \cos A*c$ and $a = (a/b)*b = \tan A*b$; in coordinate geometry as line gradients: $\Delta y = \Delta y/\Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In economics, the recount-formula becomes a price-formula: \$ = (\$/kg)*kg, \$ = (\$/day)*day, etc.

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 5 = 2 2s & 1 = 2B1 2s = 2.1 2s = 2 1/2 2s

Recounting in the same unit and in a different unit

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2*B*1 2s as 1*B*3 2s with an outside ‘overload’; or as 3*B*-1 2s with an outside ‘underload’ thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48.

Recounting in a different unit means changing unit, also called proportionality or linearity. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2*B*2 5s.

Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’ The answer, ‘2.some’, predicts that the singles come by taking away 2 5s, thus asking ‘3*4 – 2*5’. The answer, ‘2’, predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

Recounting to and from tens

Asking ‘3 4s = ? tens’ is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a ‘natural’ number.

Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $x*7 = 35$. It is easily solved by recounting 35 in 7s: $x*7 = 35 = (35/7)*7$. So $x = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

Double-counting creates proportionality as per-numbers

Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2\$ per 3kg, or 2\$/3kg. To answer the question ‘*T* = 6\$ = ?kg’, we recount 6 in 2s since the per-number is 2\$/3kg: 6\$ = (6/2)*2\$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2\$/3\$ = 2/3, and 2\$/100\$ = 2/100 = 2%.

A short curriculum in addition-free mathematics

01. To stress the importance of bundling, the counting sequence can be: 01, 02, …, 09, 10, 11 etc. And 01, 02, 03, 04, 05, Ten less 4, T-3, T-2, T-1, Ten, Ten and 1, T and 2, etc.

02. Ten fingers can be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers can be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables can be formulated as re-counting from icon-bundles to tens and use underload counting after 5: T = 4*7 = 4 7s = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 can be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

06. Solving proportional equations as 3*x = 12 can be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks can be done by re-counting in the per-number: With 3$/4kg, 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages can be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$ = (60/3)*3$ giving (60/3)*2$ = 40$

09. Trigonometry can precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sinA*c.

10. Counting by stacking bundles into adjacent blocks leads to the number-formula or bundle-formula called a polynomial: T = 456 = 4*BundleBundle + 5*Bundle + 6*single = 4*B^2 + 5*B + 6*1. In its general form, the number-formula T = a*x^2 + b*x + c contains the different formulas for constant change: T = a*x (proportionality), T = a*x^2 (acceleration), T = a*x^c (elasticity) and T = a*c^x (interest rate); as well as T = a*x+b (linearity).

11. Predictable change roots pre-calculus (if constant) and calculus (if changing). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

12. Integral can precede differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers. So both per-numbers and fractions are added with units as the area under the per-number graph.

Meeting Many in a STEM context

Having met Many by itself, we now meet Many in time and space in the present culture based upon STEM, described by OECD (2015b) as follows: 'In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.'

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we must build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

Nature consists of things in motion, combined in the momentum = mass*velocity. Things contain mass and molecules and electric charge. Thus, nature is counted in meter and second and kilogram and mole and coulomb. STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), $/kg (price), $/hour (wages), etc.

A list of core formulas shows that nature is predictable by recounting & per-numbers.:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/litre) * litre = molarity * litre
- momentum = (Δ momentum/second) * second = force * seconds
- Δ energy= (Δ energy/meter) * meter = force * meter = work
- Δ energy = (energy/kg/degree) * kg * degree = heat * kg * degree

Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

SCIENCE: counting and double-counting time, space, matter, force and energy

Counting time, the unit is seconds. A bundle of 60 seconds is called a minute; a bundle of 60 minutes is called an hour, and a bundle of 24 hours is called a day, of which a bundle of 7 is called a week. A year contains 365 or 366 days, and a month from 28 to 31 days.

Counting space, the international unit is meter, of which a bundle of 1000 is called a kilometer; and if split becomes a bundle of 1000 millimeters, 100 centimeters and 10 decimeters. Counting squares, the unit is 1 square-meter. Counting cubes, the unit is 1 cubic-meter, that is a bundle of 1000 cubic-decimeters, also called liters, that split up as a bundle of 1000 milliliters.

Counting matter, the international unit is gram that splits up into a bundle of 1000 milligrams and that unites in a bundle of 1000 to 1 kilogram, of which a bundle of 1000 is called 1 tons.

Counting force and energy, a force of 9.8 Newton will lift 1 kilogram, that will release an energy of 9.8 Joule when falling 1 meter.

Cutting up a stick in unequal lengths allows the pieces to be double-counted in liters and in kilograms giving a per-number around 0.7 kg/liter, also called the density.

A walk can be double-counted in meters and seconds giving a per-number at e.g. 3 meter/second, called the speed. When running, the speed might be around 10 meter/second. Since an hour is a bundle of 60 bundles of 60 seconds this would be 60*60 meters per hour or 3.6 kilometers per hour, or 3.6 km/h.

A pressure from a force applied to a surface can be double-counted in Newton and in square meters giving a per-number Newton per square-meter, also called Pascal.

Motion can be double-counted in Joules and seconds producing the per-number Joule/second called Watt. To run properly, a bulb needs 60 Watt, a human needs 110 Watt, and a kettle needs 2000 Watt, or 2 kiloWatt. From the Sun the Earth receives 1370 Watt per square meter.

Warming and boiling water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

Heating 1000 gram water 80 degrees in 167 seconds in a 2000 Watt kettle, the per-number will be 2000*167/80 Joule/degree, creating a double per-number 2000*167/80/ 1000 Joule/degree/gram or 4.18 Joule/degree/gram, called the specific heat of water.

Producing 100 gram steam in 113 seconds, the per-number is 2000*113/100 Joule/gram or 2260 J/g, called the heat of evaporation for water.

Dissolving material in water

In the sea, salt is dissolved in water. The tradition describes the solution as the number of moles per liter. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59gram = (100/59)*1mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

An electrical circuit

To work properly, a 2000Watt water kettle needs 2000Joules per second. The socket delivers 220Volts, a per-number double-counting the number of Joules per charge-unit.

Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm.

Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So, the 120Watt bulb only receives half of the energy of the 60Watt bulb.

How high up and how far out

A ping-pong ball is sent upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second.

To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the speed is split into a vertical and a horizontal part, both reducing it with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed by the formula: Horizontal distance to the top = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

The vertical distance is halved, but the vertical speed changes from 9.92 to 9.92*0.707 = 7.01 only. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the distance four times.

TECHNOLOGY: letting steam work

A water molecule contains two Hydrogen and one Oxygen atom weighing 2*1+16 units. A collection of a million billion billion molecules is called a mole; a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. The volume should increase accordingly. But, if kept constant, instead the inside pressure will increase.

Inside a cylinder, the ideal gas law, p*V = n*R*T, combines the pressure, p, and the volume, V, with the number of moles, n, and the absolute temperature, T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

So, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn.

In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus decreasing and increasing the pressure by letting steam in and out of the two holes. The leaving steam the is visible on steam locomotives.

In the third generation used in power plants, two cylinders, a hot and a cold, connect with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back to the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical power to industries and homes.

ENGINEERING: how many turns on a steep hill

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2.

In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2.

In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

MATHEMATICS: the simplicity of counting before adding next-to and on-top

Meeting Many, we ask 'How many in total?' To answer, we count and add. To count means to use division, multiplication and subtraction as icons for bundling, stacking and removing stacks to predict unit-numbers as blocks of stacked bundles; but also, to recount to change unit, and to double-count to get per-numbers bridging the units, both rooting proportionality.

Once counted and recounted and double-counted, totals can be added next-to or on-top, rooting integral calculus and proportionality; and that, if reversed, roots differential calculus and solving equations. Adding thus means uniting unit-numbers and per-numbers, but both can be constant or changing, so to predict, we need four uniting operations: addition and multiplication unite changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. As well as four splitting operations: subtraction and division split into changing and constant unit-numbers; and differentiation and root/logarithm split into changing and constant per-numbers. This resonates with the Arabic meaning of algebra, to reunite. And it appears in Arabic numbers written out fully as T = 456 = 4 bundles-of-bundles & 5 bundles & 6 unbundled, showing all four uniting operations, addition and multiplication and power and next-to addition of stacks; and showing that the word-language and the number-language share the same sentence form with a subject and a verb and a predicate or object. Finally, shapes can split into right-angled triangles, where the sides can be mutually recounted in three per-numbers, sine and cosine and tangent.

So, by its simplicity, mathematics is easy and quick to learn if education wants to do so.

Adding addition to the curriculum

Once counted, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being re-counted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Uniting constant and changing unit-numbers and per-numbers

The number-formula also shows the four ways to unite numbers offered by algebra meaning 'reuniting' in Arabic: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers:

Uniting/*splitting into*	Changing	Constant
Unit-numbers m, s, kg, $	T = a + n *T – a = n*	T = a * n *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	T = ∫ a dn *dT/dn = a*	T = a^n *loga(T) = n n√T = a*

Conclusion and recommendation

This paper argues that the low success of 50 years of mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views offer different kinds of mathematics: a set-based top-down ‘meta-matics’ that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based ‘Many-matics’ simply saying ‘To master Many, counting and recounting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating.’ A proposal for two separate a twin-curricula in counting and adding is found in Tarp (2018a).

Thus, this simplicity of mathematics as expressed in a Count-before-Adding curriculum allows bundle-numbers to replace line-numbers, and to learn core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM-teachers or STEM-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

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20. Good, Bad & Evil Mathematics - Sociological Imagination in Math Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now." (p. 16). As to organizations, he warns against a so-called goal displacement: "The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right." (p. 84)

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference. So, using sociological imagination we can ask the 'Cinderella question': 'as an alternative to the tradition, is there is a different way to the goal of mathematics education, mastery of Many?' In short, could there be different kinds of mathematics? And, could it be that among them, one is good, and one is bad, and one is evil?

Meaning 'Earth measuring' in Greek and 'Reuniting numbers' in Arabic, geometry and algebra clearly was grounded in the physical fact Many. But, around 1900 the set-concept transformed grounded mathematics into today's self-referring 'meta-matism'. Which is a mixture of 'meta-matics', defining its concepts by self-reference, i.e. top-down as examples of abstractions instead of bottom-up as abstractions from examples; and of 'mathe-matism' true inside but seldom outside classrooms. Here adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

The existence of three different versions of mathematics, 'many-math' and 'meta-matics' and 'mathe-matism', allows formulating the following definitions:

Good mathematics is absolute truths about things rooted in the outside world, e.g. *T* = 2*3 = 6. So good mathematics is tales about how totals are counted, united and changed; described by a number-language sentence with a subject, T, and a verb, is, and a predicate, 2*3.

Bad mathematics is relative truths about things rooted in the outside world. An example is claiming unconditionally that 2+3 = 5. So bad mathematics is tales about numbers without units.

Evil mathematics talks about something existing only inside classrooms. An example is claiming that fractions are numbers, and that they can be added without units as e.g. 1/2 + 2/3 = 7/6 even if 1 red of 2 apples plus 2 reds of 3 apples total 3 reds of 5 apples, and certainly not 7 reds of 6 apples. So bad mathematics is tales about fractions as numbers.

On this background, the paper will outline a grounded curriculum in Good Mathematics, free of self-reference and goal-displacement (Tarp, 2018).

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21. Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum

Its many applications make mathematics useful; and of course, it must be learned before applied. Or, can it be learned through its original roots? Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. Asking 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a recount-formula occurring all over mathematics. Once counted, totals can be added next-to or on-top rooting calculus and proportionality. From this 'Count-before-Adding' curriculum, Many-matics offers remedial micro-curricula for classes stuck in a traditional curriculum.

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change'. Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

Mathematics, before and now

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET changed grounded mathematics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to ‘2x3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ (Tarp, 2017) is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

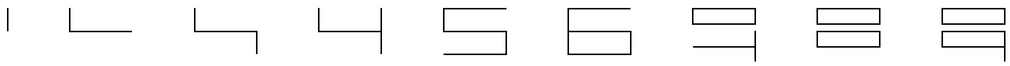
To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a list of remedial micro-curricula for classes stuck in a traditional mathematics curriculum. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a ‘Count-before-Adding’ Curriculum

Meeting Many, we ask ‘How many in Total?’ To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence (Tarp, 2018b).

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



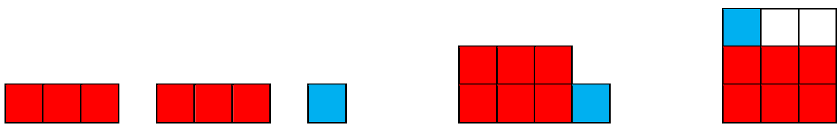
1 2 3 4 5 6 7 8 9

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T, we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: 7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

7 / 3	2.some
7 – 2 x 3	1

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations and differential calculus:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A typical mathematics curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed. Adding without units always precede bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, let us use difference research and imagination to uncover or invent remedial micro-curricula for classes stuck in the tradition.

Remedial micro-curricula for classes stuck in the tradition

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to use two-dimensional block-numbers and bundle-counting, recounting in the same and in a different unit, and calculator prediction before next-to and on-top addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to Here a difference is to talk about 8/2 as ‘8 counted in 2s’ instead of as ‘8 divided between 2’; and to rewrite the number as ‘10 or 5 times less something’ and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of ‘mathe-matism’ true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes ‘middle school integration’ later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of

reverse operations to establish the basic rule for solving equations ‘move to the opposite side with the opposite sign’: In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn’t have a reverse sign. This gives 3*u = 14 – 2. Since u doesn’t have a reverse sign, 3 moves to the other side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. ‘transform the formula T = a/(b-c) so that all letters become subjects.’ A hymn can be created: “Equations are the best we know / they’re solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don’t want to stop.”

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to ‘pre-dict’ a number, instead statistics can ‘post-dict’ its previous behavior. This allows predicting an interval that will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains

several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5^{th} root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them 'local constancy' and 'local linearity'. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children's own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Inspirational purposes have led to the creation of several DrAlTarp YouKu.com, SoKu.com videos, and MrAlTarp YouTube videso: Deconstructing Fractions, Deconstructing Calculus, Deconstructing PreCalculus Mathematics, Missing Links in Primary Mathematics, Missing Links in Secondary Mathematics, Postmodern Mathematics, PreSchool Math.

Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, one alternative presents itself directly for future studies creating a paradigm shift (Kuhn, 1962): to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining the misery coming from teaching meta-matism. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research should search for differences and test if they make a difference. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a is-sentence, trust the subject since it exists, but doubt the predicate, it is a verdict that might be gossip. So, maybe we should stop teaching essence and instead start letting learners meet and experience existence.

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22. Mastering Many by Counting, Re-counting and Double-counting before Adding On-top and Next-to

Observing the quantitative competence children bring to school, and by using difference-research searching for differences making a difference, we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

Keywords: numbers, operations, proportionality, calculus, early childhood

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed e.g. by the International Congress on Mathematics Education taking place each four year since 1969. However, despite 50 years of research, many countries still experience poor results in the Programme for International Student Assessment (PISA). In the former model country Sweden this caused the Organisation for Economic Co-operation and Development (OECD) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change' since 'more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.' (OECD, 2015, p. 3).

Mathematics thus seems to be hard by nature. But, with mathematics and education as social institutions, a different answer, by choice, may come from sociology, having imagination as a core part as pointed out by Mills (1959). Bauman (1990) agrees when talking about organizations:

> Sociological thinking is, one may say, a power in its own right, an *anti-fixating* power. It renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. (p.16) **Rational** action (..) is one in which the *end* to be achieved is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical. (p.79) Last but not least, the ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right: the new end against which the organization tends to measure the rationality of its performance (p.84).

It is a general opinion that the goal of mathematics education is to learn mathematics. However, this goal is self-referring. So maybe traditional mathematics has a goal displacement hiding a different more fruitful way to the outside goal, to master Many as it occurs in space and time?

Difference-research

To find differences we use 'Difference-research' (Tarp, 2018a) searching for differences making a difference, thus containing two parts: finding a difference, and testing it to see if it makes a difference. This paper focuses on the first part in order to find differences that can be tested to create a background for a possible paradigm shift (Kuhn, 1959).

Difference-research builds on sociological imagination; and on the skeptical thinking of the ancient Greek sophists warning against choice presented as nature. Thus disagreeing with Plato seeing choice as an illusion since the physical is but examples of meta-physical forms visible only to philosophers educated at his academy, later by Christianity turned into monasteries before being changed back again by the Reformation. In the Renaissance, this created the skeptical thinking of natural science, which rooted the Enlightenment century with its two republics, the American and the French (Russell, 1945).

Where France now has its fifth republic, the USA still has its first with skepticism as pragmatism and symbolic interactionism and grounded theory. To protect its republic, France has developed a skepticism inspired by the German thinker Heidegger, seen by Bauman as starting 'the second Copernican revolution' by asking: What is 'is'? (Bauman, 1992, p. ix).

Heidegger (1962) sees three of our seven basic is-statements as describing the core of Being: 'I am' and 'it is' and 'they are'; or, I exist in a world together with It and with They, with Things and with Others. To have real existence, the 'I' must create an authentic relationship to the 'It'. However, this is made difficult by the 'dictatorship' of the 'They', shutting the 'It' up in a predicate-prison of idle talk, gossip.

Heidegger thus uses existentialist thinking, described by Sartre (Marino, 2004) as holding that 'existence precedes essence' (p. 22). In France, Heidegger inspired the poststructuralist thinking pointing out that society forces words upon you to diagnose you so it can offer cures including one you cannot refuse, education, that forces words upon the things around you, thus forcing you into an unauthentic relationship to yourself and your world (Foucault, 1995; Lyotard, 1984; Tarp, 2016).

Difference-research tells what can be different from what cannot. From a Heidegger view, an is-sentence contains two things: a subject that exists and cannot be different, and a predicate that can and that may be gossip masked as essence, provoking 'the banality of Evil' (Arendt, 1963) if institutionalized. So, to discover its true nature, we need to meet the subject, Many, outside the predicate-prison of traditional mathematics. We will use Grounded Theory (Glaser and Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1970) from a personal to a social level, to allow Many create its own categories and properties. In this way, we can see if our observations can be assimilated to traditional mathematics or will suggest it be accommodated.

Our Two Languages with Word- and Number-Sentences

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: 'This is a chair'. As does the number-language assigning numbers instead: 'the 3 chairs each have 4 legs', abbreviated to 'the total is 3 fours', or 'T = 3 4s' or 'T = 3*4'. Unfortunately, the tradition hides the similarity between word- and number-sentences by leaving out the subject and the verb by just saying '3*4 = 12'.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence 'this is a chair' leads to a meta-sentence ''is' is an auxiliary verb'. Likewise, the sentence 'T = 3*4' leads to a meta-sentence ''*' is a commutative operation'.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead its self-referring set-based form has turned mathematics into a grammar labeling its outside roots as 'applications', used as means to dim the impeding consequences of teaching a grammar before its language.

So, using full sentences including the subject and the verb in number-language sentences is a difference to the tradition; as is teaching language before grammar.

Mathematics, Rooted in Many, or in Itself

The Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

However, 50 years ago the set-concept created a self-referring ‘New Math’ or ‘meta-matics’ with concepts defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. And neglecting that Russell, by looking at the set of sets not belonging to itself, showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

So, to find a difference we now return to the Greek origin to meet Many openly to uncover a ‘Many-matics’ as a natural science about Many.

Meeting Many, Children use Block-numbers to Count and Share

How to master Many can be observed from preschool children. Asked ‘How old next time?’, a 3year-old child will say ‘Four’ and show 4 fingers; but will react strongly if held together 2 by 2, ‘That is not 4, that is 2 2s.’

Children thus describes what exists in the world: bundles of 2s, and 2 of them. So, what children bring to school is 2-dimensional block-numbers, illustrated geometrically by LEGO blocks, together with some quantitative competence. Children thus love re-counting 5 sticks in 2s in various ways as 1 2s & 3, as 2 2s & 1, and as 3 2s less 1.

Sharing nine cakes, four children take one by turn saying ‘I take 1 of each 4’. With 1 left they might say ‘let’s count it as 4’. Thus, children share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts.

Children quickly observe the difference between a ‘stack-number’ as 6 = 3 2s or 2 3s, and a prime number as 3, serving only as a bundle-number by always leaving singles if stacked.

Finally, by turning and splitting 2-dimensional or 3-dimensional blocks, children see their commutative, distributive and associative properties as self-evident: of course, 2 3s is the same as 3 2s; and 6 3s can be split in 4 3s and 2 3s; and 2 3*4s is the same as 2*3 4s.

Meeting Many Openly

Many exists in space and time as multiplicity and repetition. Meeting Many we ask: ‘how many in total?’ To answer, we count and add. We count by bundling and stacking as seen when writing out fully the total T = 456 = 4*B^2 + 5*B + 6*1 showing three stacks or blocks added next-to each other: one with 4 bundles of bundles, one with 5 bundles, and one with 6 unbundled singles. Typically, we use ten as the bundle-size, formally called a base.

Digits occur by uniting e.g. five ones to one fives, rearranged as an icon with five strokes if written less sloppy. As the bundle-size, ten needs no icon when counted as 10, one bundle and no unbundled. Then follow eleven and twelve coming from Danish Vikings counting ‘one left’ and ‘two left’.

1	2	3	4	5	6	7	8	9
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII

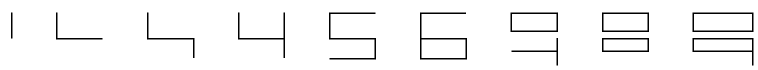

Figure 1. Digits as icons with as many sticks as they represent.

Counting by Bundling

We count in several ways. Some gather-hunter cultures count ‘one, two, many’. Agriculture needs to differentiate degrees of Many and typically bundles in tens. To include the bundle, we can count ‘0Bundle1, 0B2, 0B3,…, 1B, 1B1, 1B2’, etc.; or ‘0.1 tens, 0.2 tens’, etc., using a decimal point to separate the bundles from the unbundled singles. To signal nearness to the bundle we can count ‘1, 2, …, 7, bundle less 2, bundle less 1, bundle’, etc. Thus a number always contains three numbers: a number of bundles, a number of singles, and a number for the bundle-size.

Bundle-counting, we ask e.g. ‘A total of 7 is how many 3s?’ Using blocks, we stack the 3-bundles on-top of each other. The single can be placed next-to, or on-top counted in 3s. Thus, the result of counting 7 in 3s, T = 2 3s & 1, can be written as T = 2B1 3s using ‘bundle-writing’, and as T = 2.1 3s using ‘decimal-writing’, and as T = 2 1/3 3s using ‘fraction-writing’.

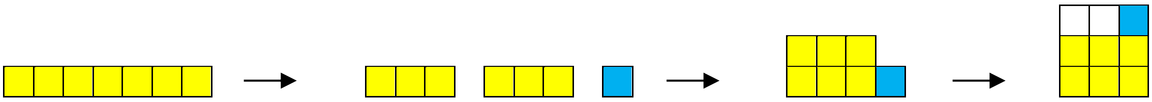

Figure 2. Seven bundle-counted as 2Bundle1 3s, as 2.1 3s, and as 2 1/3 3s.

Bundle-counting in Space and Time

We include space and time by using ‘geometry-counting’ in space, and ‘algebra-counting’ in time. Counting in space, we stack the bundles and report the result on an abacus in ‘geometry-mode’. Here the total 7 is on the below bar with 1 unbundled and a block with 2 bundles on the bars above.

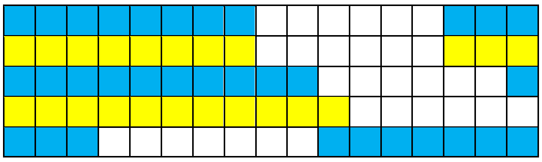

Figure 3. Seven bundle-counted as 2B1 3s on an abacus in geometry-mode.

Counting in time, we count the bundles and report the result on an abacus in ‘algebra-mode’. Here the total 7 is on the below bar with 1 unbundled and the number of bundles on the bars above.

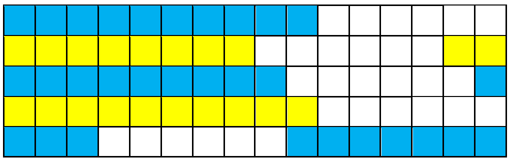

Figure 4. Seven bundle-counted as 2.1 3s on an abacus in algebra-mode.

A Calculator Predicts Counting-results

Iconizing the counting processes also, a calculator can predict a counting-result. A stack of 2 3s is iconized as 2x3 (or 2*3) showing a lift used 2 times to stack the 3s. As for taking away, subtraction shows the trace left when taking away just once, and division shows the broom wiping away several times.

So, entering ‘7/3’ we ask the calculator ‘from 7, 3s can be taken away how many times?’ The answer is ‘2. some’. To find the leftover singles we take away the stack of 2 3s by asking ‘7 – 2*3’. From the answer ‘1’ we conclude that 7 = 2B1 3s. Showing ‘7 – 2*3 = 1’, a display indirectly predicts that 7 can be re-counted as 2 3s and 1, or as 2B1 3s or 2.1 3s.

7 / 3	2.some
7 – 2 * 3	1

Figure 5. A calculator predicts how 7 re-counts in 3s as 2.1 3s.

A calculator thus uses a ‘re-count formula‘, $T = (T/B)*B$, saying that ‘from T, T/B times, Bs can be taken away’; and a ‘re-stack formula’, $T = (T–B)+B$, saying that ‘from T, $T–B$ is left, if B is taken away and placed next-to’. The formulas may be illustrated by LEGO blocks. The re-count formula introduces early algebra (Kieran, Pang, Schifter and Ng, 2016) from grade one; and it occurs all over mathematics and science as proportionality formulas. Likewise, the early use of a calculator shows the importance of mathematics as a language for prediction.

Cup-Counting Allows Re-Counting in the Same Unit

Cup-counting uses a cup when bundle-counting e.g. 7 in 3s. For each bundle we place a stick inside the cup, leaving the unbundled singles outside.

T = 7 = I I I I I I I → III III I → [▌▌] I → 2B1 3s = 2.1 3s

One stick moves outside the cup as a bundle of 1s, that moves back inside as 1 bundle. This will change the 'normal' form to an 'overload', or to an 'underload' leading to negative numbers that may be used freely in childhood even if adults abstain from doing so:

T = 7 = I I I I I I I → III I I I I → [▌] I I I I → 1*B*4 3s = 1.4 3s

T = 7 = I I I I I I I → III III III II → [▌▌▌] II → 3*B*-2 3s = 3.-2 3s

Re-Counting in a Different Unit

Re-counting in a different unit means changing units, also called proportionality. Re-counting 3 4s in 5s, the re-count formula and a calculator predict the result 2 5s & 2 by entering '3*4/5' and taking away the 2 5s.

3 * 4 / 5	2.some
3 * 4 – 2 * 5	2

Figure 6. A calculator predicts how 3 4s re-counts in 5s as 2.2 5s.

Re-Counting from Icons to Tens

A calculator has no ten-button. Instead, to re-count an icon-number as 3 4s in tens, it gives the result 1.2 tens directly in a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

3 * 4	12

Figure 7. A calculator predicts how 3 4s re-counts in tens as 1.2 tens.

Re-counting from icons to tens, 3 4s is a geometrical block that increases its base. Therefore, it must decrease its height to keep the total unchanged.

Re-counting in tens is called multiplication tables to be learned by heart. However, the ten-by-ten table can be reduced to a 4-by-4 table since 5 is half of ten and 6 is ten less 4, and 7 is ten less 3 etc. Thus *T* = 4*7 = 4 7s that re-counts in bundles of tens as

T = 4*7 = 4*1*B*-3 tens = 4*B*-12 tens = 3*B*-2 tens = 2*B*8 tens = 28

Such results generalize to algebraic formulas as $a*(b – c) = a*b – a*c$.

Re-Counting from Tens to Icons

Re-counting from tens to icons will decrease the base and increase the height. The question '38 is ? 7s' is called an equation '38 = *u**7', using the letter *u* for the unknown number. An equation is easily solved by recounting 38 in 7s, thus providing a natural 'to opposite side with opposite sign' method as a difference to the traditional 'do the same to both sides' method.

u*7 = 38 = (38/7)*7	so	u = 38/7 = 5 3/7

Figure 8. An equation solved by re-counting, the OppositeSide&Sign method.

Once Counted, Totals Can be Added On-Top or Next-To

To add on-top by asking '3 5s and 2 3s total how many 5s?', the units must be the same. So, 2 3s must be re-counted in 5s as 1*B*1 5s that added to the 3 5s gives 4*B*1 5s.

Using a calculator to predict the result, we use a bracket before counting in 5s: Asking '(3*5 + 2*3)/5', the answer is '4. Some'. Taking away 4 5s leaves 1. So again, we get the result 4*B*1 5s.

(3 * 5 + 2 * 3)/ 5	4.some
(3 * 5 + 2 * 3) – 4 * 5	1

Figure 9. A calculator predicts how 3 5s and 2 3s re-counts in 5s as 4.1 5s.

To add next-to by asking '3 5s and 2 3s total how many 8s?', we add by areas, called integral calculus. With blocks we get the answer 2*B*5 8s.

Using a calculator to predict the result, we use a bracket before counting in 8s: Asking '(3*5 + 2*3)/8', the answer is '2. Some'. Taking away 2 8s leaves 5. So again, we get the result 2*B*5 8s.

(3 * 5 + 2 * 3)/ 8	2.some
(4 * 5 + 2 * 3) – 2 * 8	5

Figure 10. A calculator predicts how 3 5s and 2 3s re-counts in 8s as 2.5 8s.

Reversing Adding On-Top and Next-To

Reversed addition may be called backward calculation or solving equations. Reversing next-to addition may be called reversed integration or differentiation. Asking '3 5s and how many 3s total 2*B*6 8s?', using blocks gives the answer 2*B*1 3s.

Using a calculator to predict the result, the remaining is bracketed before counting in 3s.

(2 * 8 + 6 – 3 * 5)/ 3	2
(2 * 8 + 6 – 3 * 5) – 2 * 3	1

Figure 11. A calculator predicts how 2.6 8s re-counts in 3 5s and 2.1 3s.

Adding or integrating two areas next-to each other means multiplying before adding. Reversed integration, i.e. differentiation, then means subtracting before dividing, as shown by the gradient formula $y' = \Delta y/t = (y2 - y1)/t$.

Double-Counting in Two Units Creates Per-Numbers and Proportionality

Double-counting the same total in two units is called proportionality, which produces 'per-numbers' as e.g. 2$ per 5kg, or 2$/5kg, or 2/5 $/kg.

To answer the question '*T* = 6$ = ?kg' we use the per-number to re-count 6 in 2s, that many times we have 5kg: *T* = 6$ = (6/2)*2$ = (6/2)*5kg = 3*5kg = 15kg. And vice versa: Asking '*T* = 20kg = ?$', the answer is *T* = 20kg = (20/5)*5kg = (20/5)*2$ = 4*2$ = 8$.

A total can be double-counted in colored blocks of different values, e.g. 1 red per 3 blues. Here, a total of 10 blues re-counts as *T* = 7*b* & 1*r* = 4*b* & 2*r* = 1*b* & 3*r*. Likewise, a total of 3 reds re-counts as *T* = 3*b* & 2*r* = 6*b* & 1*r* = 9*b*. Placed next to each other, this introduces a primitive coordinate system.

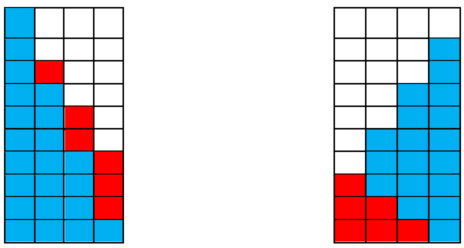

Figure 12. 10 blues left, and 3 reds right, re-counted in combinations.

Double-Counting in the Same Unit Creates Fractions as Per-Numbers

Double-counting a total in the same unit, per-numbers take the form of fractions, e.g. as 3$ per 5$ = 3/5; or percentages as 3$ per 100$ = 3/100 = 3%.

Thus, to find 3$ per 5$ of 20$, or 3/5 of 20, we re-count 20 in 5s as 20 = (20/5)*5 = 4*5. Now we have two options. Seeing 20 as 4 5s, 4 times we get 3, i.e. 4*3 = 12; and seeing 20 as 5 4s, we get 3 4s, i.e. 3*4 = 12.

Likewise, to find what 3$ per 5$ is in percent, i.e. per 100, we re-count 100 in 5s as 100 = (100/5)*5 = 20*5. Again, we have two options. Seeing 100 as 20 5s, 20 times we get 3, i.e. 20*3 = 60; and seeing 100 as 5 20s, we get 3 20s, i.e. 3*20 = 60. So, 3 per 5 gives 60 per 100 or 60%.

Including or removing units will enlarge or reduce fractions:

4/6 = 4 3s/6 3s = 4*3/6*3 = 12/18

4/6 = 2*2/3*2 = 2 2s/3 2s = 2/3

Adding Per-numbers Roots Integral Calculus before Differential Calculus

Adding 2kg at 3$/kg and 4kg at 5$/kg, the ‘unit-numbers’ 2 and 4 add directly, but the per-numbers 3 and 5 must be multiplied first, thus creating areas. So per-numbers and fractions are not numbers, but operators needing numbers to become numbers. Per-numbers thus add by the areas under the per-number graph, here being ‘piecewise constant’.

Asking ‘3 seconds at 4m/s increasing steadily to 5m/s’, the per-number is ‘locally constant’. This concept is formalized by an ‘*epsilon-delta* criterion’ seeing three forms of constancy: *y* is ‘globally constant’ *c* if, for any positive number *epsilon*, the difference between *y* and *c* is less than *epsilon.* And *y* is ‘piecewise constant’ *c* if an interval-width *delta* exists such that, for any positive number *epsilon*, the difference between *y* and *c* is less than *epsilon* in this interval. Interchanging *epsilon* and *delta* makes *y* ‘locally constant’ or continuous. Likewise, the change ratio $\Delta y/\Delta x$ can be globally, piecewise or locally constant, in the latter case written as *dy/dx = y’*.

With locally constant per-numbers, the area under the graph splits up into countless strips that add easily if written as differences since the middle terms then will disappear, leaving just the difference between the end- and start-values. Thus, adding areas precedes and motivates differential calculus.

Using Letters and Functions for Unspecified Numbers and Calculations

At the language level we can set up a calculation with an unspecified number *u*, e.g. *T* = 2 + ? = 2 + *u*. Also, at the meta-language level we can set up an unspecified formula with an unspecified number *u*, written as *T* = *f(u).*

With one unspecified number, a formula becomes an equation as 8 = 2**u*; with two, a formula becomes a function as T = 2**u*; and with three, a formula becomes a surface as T = 2**u* + 2**w*.

Although we can write it, *T* = *f*(2) is meaningless since 2 is not an unspecified number. When specified, a function can be linear or exponential, but it cannot be a number or increase. A total can increase, but the way it does so cannot. Mixing language and meta-language creates meaningless sentences as ‘the predicate ate the apple’.

A general number-formula as e.g. $T = a*x^2 + b*x + c$ is called a polynomial. It shows the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the algebra square’.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a*n}$ $T/n = a$
Per-numbers m/s, $/kg, $/100$ = %	$\boldsymbol{T = \int a*dn}$ $dT/dn = a$	$\boldsymbol{T = a\text{^}n}$ $n\sqrt{T} = a$ $\log_a T = n$

Figure 13. The ‘algebra-square’ shows the four ways to unite or split numbers.

The number-formula contains the formulas for constant change:

$T = b*x$ (proportional)

$T = b*x + c$ (linear)

$T = a * x\text{^}n$ (elastic)

$T = a * n\text{^}x$ (exponential)

$T = a*x\text{^}2 + b*x + c$ (accelerated)

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to ‘post-dict’ what we cannot ‘pre-dict’.

Combining linear and exponential change by *n* times depositing *a*$ to an interest rate *r*%, we get a saving *A*$ predicted by a simple formula, $A/a = R/r$, where the total interest rate *R* is predicted by the formula $1+R = (1+r)\text{^}n$.

The formula and the proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, $r*a/r = a$, and its own interest amount, thus containing a saving *A* that is the total interest amount $R*a/r$, which gives $A/a = R/r$.

Trigonometry before Geometry

The tradition introduces plane geometry before coordinate geometry and trigonometry. A difference is the opposite order with trigonometry first since halving a block by its diagonal allows the base and the height to be re-counted in the diagonal or in each other to create the per-numbers sine, cosine, tangent and gradient:

height = (height/base)*base = tangent*base = gradient*base.

This allows a calculator to find *pi* from a formula: $\pi = n*\tan(180/n)$ for *n* sufficiently large; and it allows to predict an angle *A* form its base *b* and height *a* by reversing the formula tan $A = a/b$.

Integrating plane and coordinate geometry allows geometry and algebra to always go hand in hand. In this way solving algebraic equations predicts intersection points in geometrical constructions, and vice versa.

Testing a Many-matics Micro-curriculum

A ‘1 cup and 5 sticks’ micro-curriculum can be designed to help a class stuck in division. The intervention begins by bundle-counting 5 sticks in 2s, using the cup for the bundles. The results, 1B3 2s and 2B1 2s and 3B-1 2s, show that a total can be counted as an inside number of bundles, and an outside number of singles; and written in three ways: overload and normal and underload.

So, to divide 336 by 7, we move 5 bundles outside as 50 singles to re-count 336 with an overload: 336 = 33B6 = 28B56, which divided by 7 gives 4B8 = 48. With multiplication, singles move inside as bundles: 7* 4B8 = 28B56 = 33B6 = 336. ‘Is it that easy?’ is a typical reaction.

Algebra before Arithmetic may now be Possible

Introducing algebra before arithmetic was central to the New Math idea and to the work of Davidov (Schmittau, 2004). Introducing algebra as generalized arithmetic, the book ‘Early Algebra’ describes how ‘a fourth-grade USA class is investigating what happens to the product of a multiplication expression when one factor is increased by a certain amount.’ (Kieran et al, 2016, p.17). The investigation begins with an example showing that 7*3 = 21, and 7*5 = 35, and 9*3 = 27.

In a first-grade class working with block-numbers with the bundle as the unit, the answer would be: 7*3 is 7 3s, and 7*5 is 7 5s, and 9*3 is 9 3s. So 7*5 means that 7 2s is added next-to 7 3s. Re-counted in tens this will increase the 2B1 tens with 1B4 tens to 3B5 tens. Likewise, 9*3 means that 2 3s is added on-top of 7 3s. Re-counted in tens this will increase the 2B1 tens with 0B6 tens to 2B7 tens.

Adding 2 to both numbers means adding additional 2 2s. Re-counted in tens this will increase the 2B1 tens with 1B4 tens and 0B6 tens and additional 0B4 tens to 4B5 tens.

Counting 7 as 9 less 2, and 3 as 5 less 2, will decrease the 9 5s with 2 5s and 2 9s. Only now we must add the 2 2s that was removed twice, so (9-2)*(5-2) = 9*5 - 9*2 - 2*5 + 2*2 as shown on a western ten by ten abacus as a 9 by 5 block. This roots the algebraic formula $(a – b)*(c – d) = a*c – a*d – b*c + b*d$.

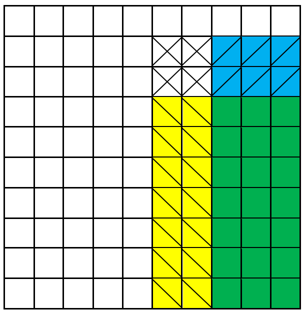

Figure 14. An abacus shows that 7*3 = (9-2)*(5-2) = 9*5 - 9*2 - 2*5 + 2*2.

Later follows a discussion on solving equations (pp. 25-29). In a first-grade class working with block-numbers with the bundle as the unit, solving the equation 3x + 9 = 5x + 1, the geometrical answer would be: to the left we have a block of 3B9 xs, and to the right we have a block of 5B1 xs. Removing 3 bundles and 1 single from both, we get 8 = 2x. Re-counting 8 in 2s we get 2*x = 8 = (8/2)*2, so x = 8/2 = 4.

The algebraic answer would be similar: to the left we have 3 bundles inside and 9 singles outside the bundle-cup, and to the right we have 5 bundles inside and 1 single outside. Removing 3 bundles from the inside and 1 single from the outside, we get 8 = 2x. Re-counting 8 in 2s we get 2*x = 8 = (8/2)*2, so x = 8/2 = 4.

Using block-numbers instead of line-numbers thus allows introducing algebra before arithmetic since with the re-count formula, counting and re-counting and double-counting precede addition.

Conclusion and Recommendation

Among the many research articles on counting and arithmetic, only few deal with block-numbers (Zybartas and Tarp, 2005). Dienes (2002), the inventor of Multi-base blocks, has similar ideas when saying (p. 1):

> The position of the written digits in a written number tells us whether they are counting singles or tens or hundreds or higher powers. (..) My contention has been, that in order to fully understand how the system works, we have to understand the concept of power. (..) In school, when young children learn how to write numbers, they use the base ten exclusively and they only use the exponents zero and one (namely denoting units and tens), since for some time they do not go beyond two digit numbers. So neither the base nor the exponent are varied, and it is a small wonder that children have trouble in understanding the place value convention.

Instead of talking about bases and higher powers, working with icon-bundles and bundles of bundles will avoid that 'neither the base nor the exponent are varied'. By seeing bundles as existence and bases as essence, block-numbers differ from Dienes' multi-base blocks that seem to have set-based mathematics as the goal, and blocks as a means.

Set, however, changed mathematics from a bottom-up Greek 'Many-matics' into today's self-referring top-down 'meta-matism', a mixture of 'meta-matics' with concepts defined top-down instead of bottom-up, and 'mathe-matism' with statements true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

So, mathematics is not hard by nature but by choice. And yes, a different way exists to its outside goal, mastery of Many. Still, it teaches line-numbers as essence to be added without units and without being first bundle-counted and re-counted and double-counted. By neglecting the existence of block-numbers and re-counting, it misses the golden learning opportunities from introducing formulas, proportionality, calculus and equations in early childhood education through its grounded alternative, Many-matics.

Consequently, let us welcome 'good' 2-dimensional block-numbers and drop 'bad' 1-dimensional line-numbers and 'evil' fractions (Tarp, 2018b). Let us bundle-count and re-count and double-count before adding on-top and next-to. Let us use full sentences about how to count and (re)unite totals. And, let difference-research use sociological imagination to design a diversity of micro-curricula (Tarp, 2017) to test if Many-matics makes a difference by fulfilling the 'Mathematics for All' dream.

Let existence precede essence in mathematics education also. So, think things.

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23. Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA result made OECD (2015) write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life.’ (p. 3).

This may prove that, by its very nature, mathematics is indeed hard to learn. On the other hand, since mathematics education is a social institution, social theory may provide a different reason.

Social Theory Looking at Mathematics Education

Mills (1959) describes imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking ‘renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now’ (p. 16).

Mathematics education is an example of ‘rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)’. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

Saying that the goal of mathematics education is to learn mathematics is one such goal displacement, made meaningless by its self-reference.

So, inspired by sociology we can ask the ‘Cinderella question’: ‘as an alternative to the tradition, is there is a different way to the goal of mathematics education, mastery of Many?’

In short, could there be different kinds of mathematics? And could it be that among them, one is good, and one is bad, and one is evil? In other words, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek ‘many-matics’ rooted in Many was replaced by the ‘New Mathematics’.

Here the invention of the concept Set created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’, being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

Good and Bad and Evil Mathematics

The existence of three different versions of mathematics, many-matics and meta-matics and mathe-matism, allows formulating the following definitions:

Good mathematics is absolute truths about things rooted in the outside world. An example is T = 2*3 = 6 stating that a total of 2 3s can be re-counted as 6 1s. So good mathematics is tales about totals, and how to count and unite them.

Bad mathematics is relative truths about things rooted in the outside world. An example is claiming that 2+3 = 5, only valid if the units are the same, else meeting contradictions as 2weeks + 3days = 17days. So bad mathematics is tales about numbers without units.

Evil mathematics talks about something existing only inside classrooms. An example is claiming that fractions are numbers, and that they can be added without units as claiming that 1/2 + 2/3 = 7/6 even if 1 red of 2 apples plus 2 reds of 3apples total 3reds of 5 apples and not 7reds of 6apples. So bad mathematics is tales about fractions as numbers.

Difference Research Looking at Mathematics Education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks the grounded theory question: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a 'Count-before-Adding' Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to use as units when counting:

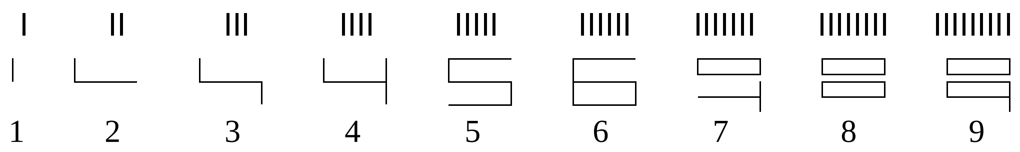


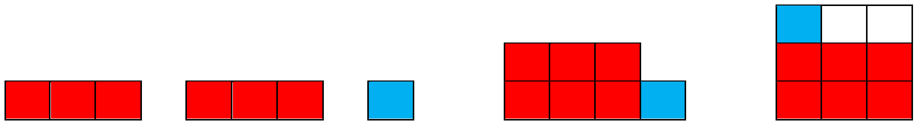
Figure 1. Digits as icons containing as many sticks as they represent

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T we take away bundles B thus rooting and iconizing division as a broom wiping away the bundles. Stacking the bundles roots and iconizes multiplication as a lift stacking the bundles into a block. Moving the stack away to look for unbundled singles roots and iconizes subtraction as a trace left when dragging the block away. A calculator predicts the counting result by a 're-count formula' $T = (T/B)*B$ saying that 'from T, T/B times, B can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

Placing the unbundled singles next-to or on-top of the stack of 3s roots decimals and fractions:



T = 7 = 2 3s & 1 = 2*B*1 3s = 2.1 3s = 2 1/3 3s

Figure 2. Re-counting a total of 7 in 3s, the unbundled single can be placed in three different ways

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, *T* = 42 = ? 7s = *u**7, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations, and differential calculus:

T = 2 3s + ? 4s = 5 7s gives differentiation: ? = (5*7 – 2*3)/4 = Δ*T*/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers *sine*, *cosine* and *tangent*. Traveling in a coordinate system, distances add directly when parallel, and by their squares when perpendicular. Re-counting the *y*-change in the *x*-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A Short Version of a Curriculum in Good Mathematics, Grounded Many-matics

01. To stress the importance of bundling, the counting sequence should be: 01, 02, …, 09, 10,11 etc.

02. The ten fingers should be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers should be re-counted in three ways (standard, and with over- and underload): *T* = 2*B*1 5s = 1*B*3 5s = 3*B*-1 5s = 3 bundles less 1 5s.

04. Multiplication tables should be formulated as re-counting from icon-bundles to tens and use overload counting after 5: *T* = 4 7s = 4*7 = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 should be formulated as re-counting from tens to 7s and use overload counting: *T* = 336 /7 = 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48

06. Solving proportional equations as 3**x* = 12 should be formulated as re-counting from tens to 3s: 3**x* = 12 = (12/3)*3 giving *x* = 12/3 illustrating the relevance of the ‘opposite side & sign’ method.

07. Proportional tasks should be done by re-counting in the per-number: With 3$/4kg, *T* = 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and *T* = 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages should be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$, so T = 60$ = (60/3)*3$ gives (60/3)*2$ = 40$

09. Integral should precede differential calculus and include adding both piecewise and locally constant per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at (2*3+4*5)$/(2+4)kg thus showing that per-numbers and fractions are added with their units as the area under the per-number graph.

10. Trigonometry should precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. $a = (a/c)*c = \sin A*c$.

Good and Bad Mathematics

Today's tradition begins with arithmetic telling about line-numbers, processed by four basic operations, later extended with negative numbers and rational numbers and reel numbers. Algebra then repeats it all with letters instead. Geometry begins with plane geometry followed by coordinate geometry and trigonometry later. Functions are special set-products, and differential calculus precedes integral calculus.

In general, we see mathematics as truths about well-defined concepts. So we begin by discussing what can be meant by good and bad concepts.

Good and Bad Concepts

As an example, let us look at a core concept in mathematics, a calculation. To differentiate between $y = 2*3$ and $y = 2*x$, around 1750 Euler defined the concept 'function' as a calculation containing unspecified numbers. Later, around 1900, set-based mathematics defined a function as an example of a set-product where first component identity implies second component identity.

So where the former is a bottom-up definition of a concept as an abstraction from examples, the latter is a top-down definition of a concept as an example of an abstraction.

Since examples are in the world and since Russell warned that by its self-reference the set-concept is meaningless, we can label bottom-up and top-down definitions good and bad concepts respectively.

Good and Bad Numbers

Good numbers should reflect that our number-language describes a total as counted in bundles and expressing the result in a full sentence with subject and verb and predicate as in the word-language, as e.g. T = 2 3s. These are the numbers that children bring to school, two-dimensional block-numbers that contain three different number-types: a 'unit-number' for the size, a 'bundle-number' and a 'single-number' for the number of bundles and unbundled singles. Totals then are written in bundle- form or in decimal-form with a unit where a bundle-B or a decimal point separates the inside bundles from the outside singles, as e.g. T = 3*B*2 tens = 3.2 tens.

Good numbers are flexible to allow a total to be re-counted in a different unit; or in the same unit to create an overload or underload to make calculations easier, as e.g. T = 3*B*2 tens = 2*B*12 tens = 4*B*-8 tens. Good numbers are shown in two ways: an algebraic with bundles, and a geometrical with blocks. Good numbers also tell that eleven and twelve come from the Vikings saying 'one left' and 'two left'.

Bad numbers do not respect the children's own two-dimensional block-numbers by insisting on one-dimensional line-numbers be introduced as names along a line without practicing bundling. Numbers follow a place value system with different places for the ones, tens, hundreds, and thousands; but seldom renaming them as bundles, bundle of bundles, and bundles of bundles of bundles.

Good and Bad Counting

A good counting sequence includes bundles in the names, as e.g. 01, 02, …, Bundle, 1*B*1, etc.; or 0Bundle1, 0*B*2, etc. Another sequence respects the nearness of a bundle by saying 0*B*6, 1*B*less3, 1*B*-2, etc.

Good counting lets counting and re-counting and double-counting precede addition; and allows the re-count formula to predict the counting-result; and it presents the symbols for division, multiplication and subtraction as icons coming from the counting process, thus introducing the operations in the opposite order.

Bad counting neglects the different forms of counting by going directly to adding, thus not respecting that totals must be counted before they can be added.

Bad counting treats numbers as names thus hiding their bundle nature by a place value system. This leads some to count ‘twenty-ten’ instead of ‘thirty’, and to confuse 23 and 32.

Good and Bad Addition

Good addition waits until after totals have been counted and re-counted in the same and in a different unit, to and from tens, and double-counted in two units to create per-numbers bridging the units. Likewise, good addition respects its two forms: on-top rooting proportionality since changing the units might be need; and next-to rooting integral calculus by being added by the areas.

Bad addition claims it priority as the fundamental operation defining the others: multiplication as repeated addition, and subtraction and division as reversed addition and multiplication. It insists on being the first operation being taught. Numbers must be counted in tens. Therefore there is no need to change or mention the unit; nor is there a need to add next-to as twenties.

Bad addition does not respect that in block-numbers as T = 2*B*3 4s, the three digits add differently. Unit-numbers, as 4, only add if adding next-to. Bundle-numbers, as 2, only add if the units are the same; else re-counting must make them so. Single-numbers, as 3, always add, but might be re-counted because of an overload.

Good and Bad Subtraction

Good subtraction sees its sign as iconizing the trace left when dragging away a stack to look for unbundled singles, thus leading on to division as repeated subtraction moving bundles away. It does not mind taking too much away and leaving an underload, as in 3*B*2 - 1*B*5 = 2*B*-3.

Bad subtraction sees its sign as a mere symbol; and sees itself as reversed addition; and doesn’t mind subtracting numbers without units.

Good and Bad Multiplication

Good multiplication sees its sign as iconizing a lift stacking bundles. It sees 5*7 as a block of 5 7s that may or may not be re-counted in tens as 3.5 tens or 35; and that has the width 7 and the height 5 that, if recounted in tens, must widen it width and consequently shorten its height. Thus, it always sees the last factor as the unit.

Good multiplication uses flexible numbers when re-counting in tens by multiplying, as e.g. *T* = 6*8 = 6*(ten-2) = (ten-4)*8 = (ten-4)*(ten-2). This allows reducing the ten by ten multiplication table to a five by five table.

Bad multiplication sees its sign as a mere symbol; and insists that all blocks must be re-counted in tens by saying that 5*7 IS 35. It insists that multiplication tables must be learned by heart.

Good and Bad Division

Good division sees its sign as iconizing a broom wiping away the 2s in *T* = 8/2. It sees 8/2 as 8 counted in 2s; and it finds it natural to be the first operation since when counting, bundling by division comes before stacking by multiplication and removing stacks by subtraction to look for unbundled singles.

Bad division sees its sign as a mere symbol; and teaches that 8/2 means 8 split between 2 instead of 8 counted in 2s. Bad division accepts to be last by saying that division is reversed multiplication; and insists that fractions cannot be introduced until after division.

Good and Bad Calculations

Good calculations use the re-count formula to allow a calculator to predict counting-results.

Bad calculations insist on using carrying so that the result comes out without overloads or underloads.

Good and Bad Proportionality

Good proportionality is introduced in grade 1 as re-counting in another unit predicted by the re-count formula. It is re-introduced when adding blocks on-top; and when double-counting in two units to create a per-number bridging the units by becoming a proportionality factor.

Bad proportionality is introduced in secondary school as an example of multiplicative thinking or of a linear function.

Good and Bad Equations

Good equations see equations as reversed calculations applying the opposite operations on the opposite side thus using the 'opposite side and sign' method in accordance with the definitions of opposite operations: 8-3 is the number *x* that added to 3 gives 8; thus if *x*+3 = 8 then *x* = 8-3. Likewise with the other operations.

Good equations sees equations as rooted in re-counting from tens to icons, as e.g. 40 = ? 8s, leading to an equation solved by re-counting 40 in 8s: *x**8 = 40 = (40/8)*8, thus *x* = 40/8 = 5.

Bad equations insist that the group definition of abstract algebra be used fully or partwise when solving an equation. It thus sees an equation as an open statement expressing identity between two number-names. The statements are transformed by identical operations aiming at neutralizing the numbers next to the unknown by applying commutative and associative laws.

2*x* = 8	an open statement about the identity of two number-names
(2*x*)*(1/2) = 8*(1/2)	½, the inverse element of 2, is multiplied to both names
(*x**2)*(1/2) = 4	since multiplication is commutative
*x**(2*(1/2)) = 4	since multiplication is associative
*x**1 = 4	by definition of an inverse element
x = 4	by definition of a neutral element

Figure 3. Solving an equation using the formal group definition from abstract algebra

Good and Bad Pre-calculus

Good pre-calculus shows that the number-formula, *T* = 345 = 3**BB* + 4**B* + 5*1 = 3**x*^2 + 4**x* + 5, has as special cases the formulas for constant linear, exponential, elastic, or accelerated change: *T = b*x+c*, *T = a*n^x*, *T = a*x^n*, and *T = a*x^2 + b*x + c*. It uses 'parallel wording' by calling root and logarithm 'factor-finder' and 'factor-counter' also. It introduces integral calculus with blending problems adding piecewise constant per-numbers, as e.g. 2kg at 3 $/kg plus 4kg at 5$/kg. It includes modeling examples from STEM areas (Science, Technology, Engineering, Mathematics)

Bad pre-calculus introduces linear and exponential functions as examples of a homomorphism satisfying the condition f(*x*#*y*) = f(*x*)$f(*y*). It includes modeling from classical word problems only.

Good and Bad Calculus

Good calculus begins with primary school calculus, adding two blocks next-to each other. It also includes middle school calculus adding piecewise constant per-numbers, to be carried on as high school calculus adding locally constant per-numbers.

It motivates the epsilon-delta definition of constancy as a way to formalize the three forms of constancy: global, piecewise and locally. It shows series with single changes and total changes calculated to realize that many single changes sum up as one single change, calculated as the difference between the end- and start-values since all the middle terms disappear.

This motivates the introduction of differential calculus as the ability to rewrite a block *h*dx* as a difference *dy, dy/dx = h*; and where the changes of block with sides f and g leads on to the fundamental formula of differential calculus, (f*g)'/(f*g)= f'/f + g'/g, giving (x^n)'/x^n = n*1/x, or (x^n)' = n*x^(n-1).

Bad calculus introduces differential calculus before integral calculus that is defined as anti-differentiation where the area under *h* is a primitive to *h*; and it introduces the epsilon-delta criterion without grounding it in different kinds of constancy.

Good and Bad Modeling

Good modeling is quantitative literature or number-stories coming in three genres as in word stories: Fact, fiction and fiddle. Fact and fiction are stories about factual and fictional things and actions. Fiddle is nonsense like 'This sentence is false' that is true if false, and vice versa.

Fact models, also called 'since-then' or 'room' models, quantify quantities and predict predictable quantities: "What is the area of the walls in this room?". Since the prediction is what is observed, fact models can be trusted. Fiction models, also called 'if-then' or 'rate' models, quantify quantities but predict unpredictable quantities: "My debt is gone in 5 years at this rate!". Fiction models are based upon assumptions and produce fictional numbers to be supplemented with parallel scenarios based on alternative assumptions. Fiddle models, also called 'then-what' or 'risk' models, quantify qualities that cannot be quantified: "Is the risk of this road high enough to cost a bridge?" Fiddle models should be rejected asking for a word description instead of a number description. (Tarp, 2017).

Bad modeling does not distinguish between the three genres but sees all models as approximations.

Good and Bad Geometry

Good geometry lets trigonometry precede plane geometry that is integrated with coordinate geometry to let algebra and geometry go hand in hand to allow formulas predict geometrical intersection points.

Bad geometry lets plane geometry precede coordinate geometry that precedes trigonometry.

Evil Mathematics

Evil mathematics talks about something existing only inside classrooms. Fractions as numbers and adding fractions without units are two examples. The tradition presents fractions as rational numbers, defined as equivalence classes in a set product created by the equivalence relation R, where (a,b) R (c,d) if $a*d = b*c$.

Grounded in double-counting in two units, fractions are per-numbers double-counted in the same unit, as e.g. 3$ per 5$ or 3 per 5 or 3/5. Both are operators needing a number to become a number. Both must be multiplied to unit-numbers before adding, i.e. they add by their areas as in integral calculus.

Shortening or enlarging fractions is not evil mathematics. They could be called 'footnote mathematics' since they deal with operator algebra seldom appearing outside classrooms. They deal with re-counting numbers by adding or removing common units: to shorten, 4/6 it is re-counted as 2 2s over 3 2s giving 2/3. To be enlarged, both take on the same unit so that 2/3 = 2 4s over 3 4s = 8/12.

Educating teachers, it is evil to silence the choices made in mathematics education. Instead, teachers should be informed about the available alternatives without hiding them in an orthodox tradition. Especially the difference between good and bad mathematics should be part of a teacher education.

Good and Bad Education

When children become teenagers, their identity work begins: ‘Who am I; and what can I do?’ So good education sees its goal as allowing teenagers to uncover end develop their personal talent through daily lessons in self-chosen practical or theoretical half-year blocks with teachers having only one subject; and praising the students for their talent or for their courage to try out something unknown.

Bad education sees its goal as selecting the best students for offices in the private or public sector. It uses fixed classes forcing teenagers to follow their age-group despite the biological fact that girls are two years ahead in mental development.

Good and Bad Research

Good research searches for truth about things that exist. It poses a question and chooses a methodology to transform reliable data into valid statements. Or it uses methodical skepticism to unmask choice masked as nature.

Bad research is e.g. master level work applying instead of questioning existing research. Or journalism describing something without being guided by a question.

With these three research genres, peer-review only works inside the same genre.

Conclusion and Recommendation

This paper used difference-research to look for different ways to the outside goal of mathematics education, mastery of Many. By meeting Many outside the present self-referring set-based tradition three ways were found, a good, and a bad, and an evil. Good mathematics respects the original tasks in Algebra and Geometry, to reunite Many and to measure earth. By identifying a hidden alternative, good mathematics creates a paradigm shift (Kuhn, 1962) that opens up a vast field for new research seeing mathematics as a many-matics, i.e. as a natural science about Many (cf. Tarp, 2018).

In short, we need to examine what happens if we allow children to keep and develop the quantitative competence they bring to school, two-dimensional block-numbers to be recounted and double-counted before being added on-top or next-to; and reported with full number-language sentences including both a subject that exists, and a verb, and a predicate that may be different.

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24. Good, Bad & Evil Mathematics - Tales of Totals, Numbers & Fractions, PPP

Allan.Tarp

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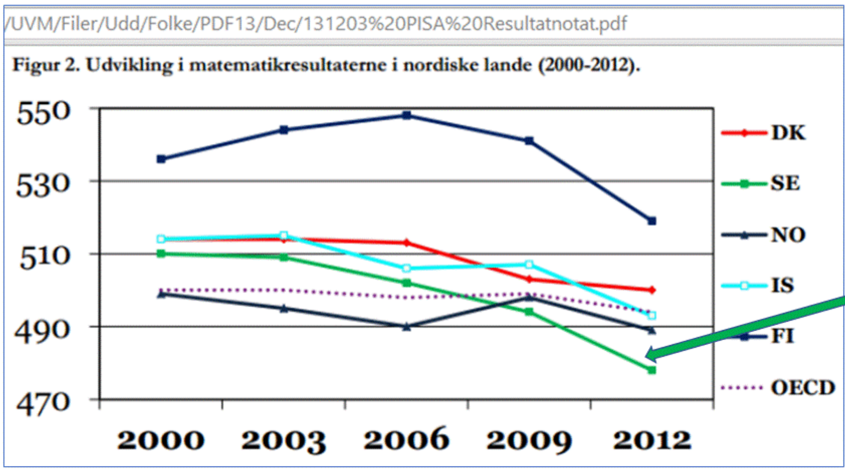
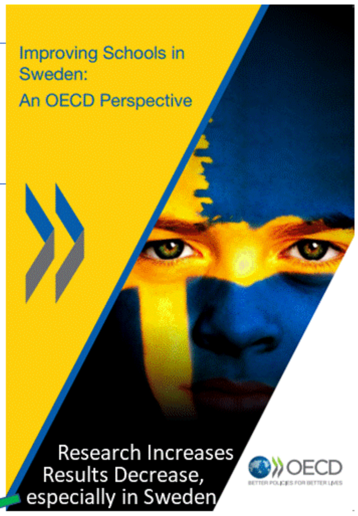
Teaching Teachers to Teach Mathe-Matics as ~~SET~~ MANY-Math

Problem: Poor PISA Performance
& Poor Research Results after 50 years





Negative Correlation among Research and Performance

Why?

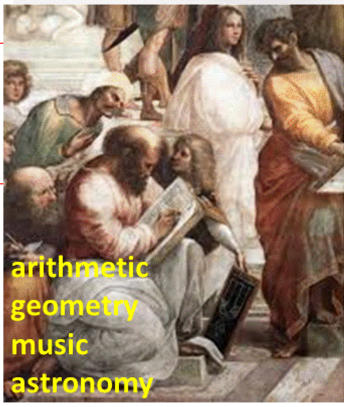
Is it Really Math we Teach?
Can Math be Different?

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY

Solution in a Nutshell:
From BAD to GOOD Math

1) All teach numbers. Don't. Tell tales about how Totals unite and change
2) All use 1D line-numbers. Don't. Use 2D block-numbers
3) All begin with addition. Don't. Begin with counting and division, multiplication and subtraction before adding next-to and on-top
4) All add fractions without units. Don't. Use units as in integral calculus
5) All include only the predicate (3*5). Don't. Use full language sentences with a subject, a verb and a predicate (T = 3*5)
6) All call it MatheMatics. Don't. It is MetaMatism, derived from SET, and falsified by e.g. 2+3 is 17 and not 5 in the case of weeks and days. Real MatheMatics is rooted in MANY.

One Definition of Mathematics



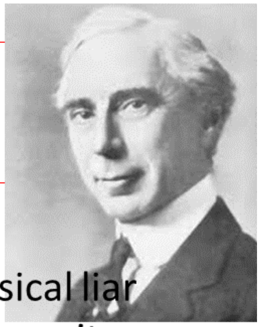
Pythagoras: mathematics, meaning knowledge, is a common label for 4 areas describing Many by itself and in space & time.

Together they formed the '**quadrivium**' recommended by Plato as a general curriculum after the '**trivium**' consisting of grammar & logic & rhetoric.

Grounded in Many as shown by names:

Geometry means to measure earth in Greek
Algebra means to reunite numbers in Arabic

Another Definition of Mathematics



Around 1900, **SET** made mathematics self-referring.

However, Russell said: Self-reference leads to the classical liar paradox 'this sentence is false', being false if true & opposite.

Let M be the set of sets not belonging to itself, $M = \{A \mid A \notin A\}$.

Then $M \in M \Leftrightarrow M \notin M$. Forget about sets. Use type theory instead.
So, by self-reference, fractions cannot be numbers.

Mathematics: Forget about Russell, he is not a mathematician.

Of course fractions are numbers, they are rational numbers.

Two Different Mathematics

The Ruling Set-based Top-Down Meta-matics from above

- Mathematics exists by itself as a collection of well-proven statements about well-defined concepts
- Concepts are defined from above as **examples from abstractions**
- Mathematics has many applications; and of course it must be taught and learned before it can be applied

a FUNCTION is an example of a set relation where component1-identity implies component2-identity

The Silenced Many-based Bottom-Up Many-matics from below

- Many exists all over the outside world, that schools prepare children and teenagers and adults for
- Concepts are defined from below as **abstractions from examples**
- Mathematics has many roots; but teaching it before applied is like teaching a grammar before its language

a FUNCTION is for example 2+x, but not 2+3; i.e. a name for a calculation with an unspecified number

How to Define Good & Bad & Evil Math: Four Questions to Answer (please discuss)

This is true	always	never	sometimes
2 + 3 = 5			
2 x 3 = 6			
$\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$			
$\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$			

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Four Questions Answered

This is true	always	never	sometimes
2 + 3 = 5			X Only with the same unit: 2weeks + 3days = 17days
2 x 3 = 6	X 2x3 is 2 **3s** III III that exist and may be recounted as 6 **1s** I I I I I I		
$\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$			X Depends on the units 1 red of 2 apples + 2 of 3 apples is 3 of 5 apples, and not 7 of 6
$\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$			X Only if taken of the same total

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Defining Good & Bad & Evil Mathematics

Good mathematics is absolute truths about outside existing things

- T = 2*3 = 6 stating that a total of 2 **3s** can be re-counted as 6 **1s**: III III = I I I I I I
- So good mathematics is tales about how to count and unite and change totals

Bad mathematics ('mathe-matism') is relative truths about outside existing things

- 2+3 = 5, valid with like units, else falsified by e.g. 2weeks + 3days = 17days
- So bad mathematics is tales about numbers without units

Evil mathematics is about what exists only inside classrooms

- 1/2 + 2/3 = 7/6, but 1red of 2 + 2reds of 3 = 3reds of 5, and not 7reds of 6
- So bad mathematics is tales about fractions as numbers.
  Fractions are not numbers, but operators, needing numbers to become numbers.

Today's **BAD** MatheMatics = **MetaMatism** = MetaMatics + MatheMatism

*What is **GOOD** MatheMatics =* ***ManyMatics**?*

Difference-Research finds Differences making a Difference, inspired by

Philosopy

- The ancient Greek sophists: To unmask choice masked as nature, find a difference
- In existentialism, Sartre said: EXISTENCE precedes ESSENCE
- Heidegger said: In a sentence, the SUBJECT exists, the PREDICATE is essence that can be different

Sociology (Bauman)

- Sociological imagination *"renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now. "*
- Goal Displacements: *"The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right."*

Psychology

- Don't teach about subjects, bring them to class to allow 'greifen vor begreifen' (Piaget, not Vygotsky)

So let us meet the existing subject **MANY** directly & outside its 'essence-prison'
so **MANY** can create its own categories using Grounded Theory

Our Two Language Houses

The **WORD language** assigns words in sentences with a subject, a verb, and a predicate.
The **NUMBER language** assigns numbers instead with a subject, a verb, and a predicate.
Both languages have a meta-language, a grammar, describing the language, describing the world.

The meta-language is about the language, so we should teach and learn language before grammar.
This is the case with the word-language only, since SET-math is a grammar of the number-language.
Mixing language levels creates nonsense: ‘The verb smiles’ & ‘The function increases’.

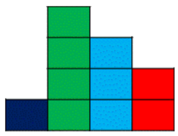
	WORD language	NUMBER language
Meta-language, grammar	‘is’ is a verb	‘*’ is an operation
Language	This is a chair	T = 3*4

WORLD

Children see Many as Bundles with Units

Asked ‘How old next time?’, a 3year-old says
4, but reacts when held together 2 by 2:
‘That is not 4, that is 2 **2s**’.

Seeing bundles as units, children use 2D LEGO-like **block-numbers**, not 1D **line-numbers**, taught in school, even if 2D Arabic block-numbers replaced 1D Roman line-numbers centuries ago.



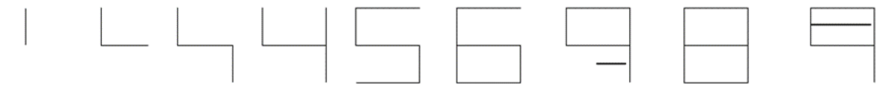
T = 1 4 3 2

T = MCCCCXXXII

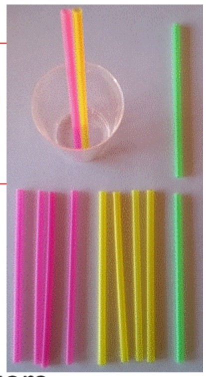
Many as Icons: IIII → ~~IIII~~ → 4

Meeting Many, we ask: "**How Many in Total**?"
To answer, we Math … oops sorry, it's a label, not an action word.
To answer, first we count, then we add. We name and iconize the degrees of Many until ten, that as 1 bundle has no icon or digit itself.

- Thus there are four sticks in a 4-icon, five in a 5-icon, etc.

one	two	three	four	five	six	seven	eight	nine
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
								
1	2	3	4	5	6	7	8	9

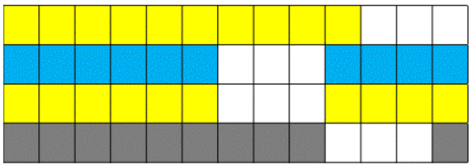
Cup- or BundleCounting in Icons: 9 = ? 4s

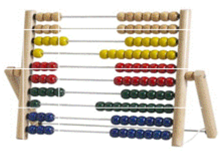


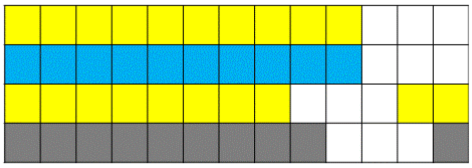
9 = IIIIIIIII = ~~IIII~~ ~~IIII~~ I = $\boxed{\begin{matrix}\text{~~IIII~~}\\ \text{~~IIII~~}\end{matrix}}$ I = 2B1 **4s** = 2.1 **4s**

To count, we bundle & use a bundle-cup with 1 stick per bundle.
We report with **bundle-writing** or **decimal-writing** where the decimal point separates inside bundles from outside single leftovers.
Shown on a western IKEA **ABACUS,** letting geometry & algebra go together.

Geometry/space mode — or — **A**lgebra/time mode



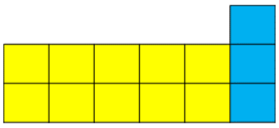




The UnBundled become Decimals or Fractions
0.3 **5s** or 3/5

When counting by bundling and stacking,
the unbundled single leftovers can be placed

NextTo the stack
counted as a stack of **1s**



T = 2**B**3 **5s** = 2.3 **5s**
A decimal number

OnTop of the stack
counted as a bundle

T = 2 3/5 **5s**
A fraction

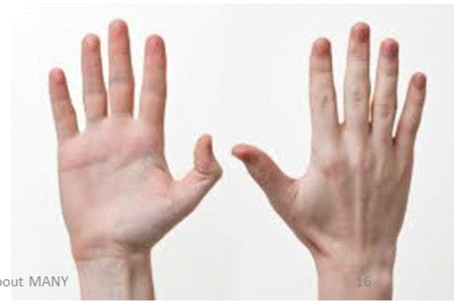
Counting Sequences

We may include bundling if saying ‘0**Bundle**3’ or ‘03’ instead of plain ‘3’

- ’0**Bundle**1, 0**B**2, 0**B**3, …, 0**B**8, 0**B**9, 1**B**0, 1**B**1, 1**B**2, … **tens**, or
- ‘01, 02, …, 1**Bundle** less 2, 1**B**-1, 1**B**0, 1**B**1(1left), 1**B**2, … **tens**

Counting fingers gives 1**B**0 **tens**, or

- 2**B**0 **5s**
- 2**B**2 **4s**
- 3**B**1 **3s** or 1**BB**1 **3s**



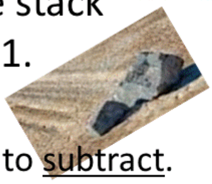
Operations as Icons

- To count 7 in **3s** we take away 3 many times, iconized by an uphill stroke, 7/3, showing the broom wiping away the **3s**.

7/3	**2.some**
7 – 2x3	**1**

- A calculator predicts: 3 can be taken away 2 times. Stacking the bundles is iconized as a lift, 2x3.

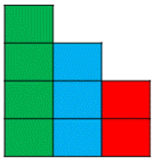
- To look for unbundled singles, we drag away the stack of 2 **3s,** iconized by a horizontal trace: 7 – 2x3 = 1.



Counting creates 3 operations: to <u>divide</u> & to <u>multiply</u> & to <u>subtract</u>.

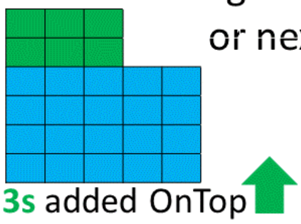
More Operations as Icons

- To bundle bundles also, **power** is iconized as a cap, 5^2, showing the number of times bundles are bundled.
- Counting a Total gives a **BundleFormula,** a polynomial:



T = 432 = 4***BundleBundle** + 3***Bundle** + 2*1 = 4***B**^2 + 3***B**^1 + 2***B**^0

- **Addition** is a cross + showing blocks placed on-top of or next-to each other.



4 **5s** & 2 **3s** added OnTop

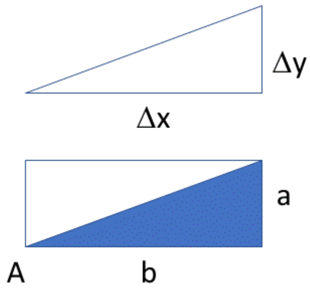
4 **5s** & 2 **3s** added NextTo

The ReCount Formula

7/3	2.some
7 – 2 * 3	1

Predicting T = 7 = 2.1 **3s**, the **ReCount formula T = (T/B)*B**
saying 'from T, T/B times, B can be taken away', is all over:

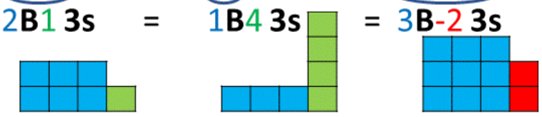
Proportionality	y = k * x
Linearity	Δy = (**Δy/Δx**) * Δx = **m** * Δx
Local linearity	dy = (**dy/dx**) * dx = **y'** * dx
Trigonometry	a = (**a/b**) * b = **tanA** * b
Trade	$ = (**$/kg**) * kg = **price** * kg
Science	meter = (**meter/second**) * second = **velocity** * second



ReCounting in the Same Unit gives Flexible Totals

A total can be counted in 3 ways: Normal Overload Underload

T = 7 = I I I I I I I = 2**B**1 **3s** = 1**B**4 **3s** = 3**B**-2 **3s**



Or, when counting in tens
T = 37 = 3**B**7 **tens** = 2**B**17 **tens** = 4**B**-3 **tens**

BundleWriting and flexible totals may cure Math Dislike in classes stuck in Division:
☹☹☹ T = 336 /7 = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48 ☺☺☺

Likewise with

Multiplication	T = 7* 48 = 7* 4**B**8 = 28**B**56 = 33**B**6 = 336
Subtraction	T = 53 – 29 = 5**B**3 – 2**B**9 = 3**B**-6 = 2**B**4 = 24
Addition	T = 53 + 29 = 5**B**3 + 2**B**9 = 7**B**12 = 8**B**2 = 82

ReCounting in a Different Unit creates Proportionality & Multiplication & Equations

4*5/6	3.some
4*5 – 3*6	2

ReCounting in different units changes units (**Proportionality**)
• T = 4 **5s** = ? **6s**. A calculator predicts with ReCount-formula: T = 3.2 **6s**
ReCounting from icons to tens gives **Multiplication**
• T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens**, predicted by multiplication
So multiplication is a special form of division
ReCounting from tens to icons creates **Equations** solved by recounting
• T = ? **7s** = 42 = (42/7)*7 with the solution ? = 42/7 = 6.
An equation is solved by moving to Opposite Side with Opposite Sign

u*7 = 42 = (42/7)*7
u = 42/7 = 6

Solving Equations by ReCounting, we may bracket Group Theory from Abstract Algebra

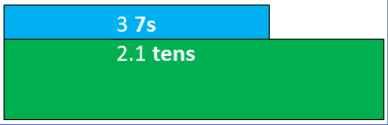
ManyMath

2 x *u* = 8 = (8/2) x 2	Solved by re-counting 8 in 2s
u = 8/2 = 4	Move: Opposite Side & Sign

SetMath (Don't test, but do remember bi-implication arrows)

	2 x *u* = 8	Multiplication has 1 as its **neutral element**, and 2 has ½as its **inverse element**
↕	(2 x *u*) x (½) = 8 x (½)	Multiplying 2's inverse element ½ to both number-names
↕	(*u* x 2) x (½) = 4	Applying the **commutative** law to u x 2; 4 is the short number-name for 8 x ½
↕	*u* x (2 x (½)) = 4	Applying the **associative** law
↕	*u* x 1 = 4	Applying the definition of an inverse element
↕	*u* = 4	Applying the definition of a neutral element. *With arrows a test is not needed.*

ReCounting Simplifies Multiplication Tables



Geometry: Multiplication means that, recounted in tens, a block increases its width and therefore must decrease its height to keep the total unchanged.

Thus T = 3*7 means 3 **7s** that may be recounted in tens as T = 2.1 **tens** = 21.

Algebra: The full ten-by-ten table can be reduced using that 6 is Bundle less 4, 7 is Bundle less 3, etc. This roots Early Algebra.

T = 2 **6s** = 2*6 = 2*(**B**-4) = 2**B**-8 = 2**B**-(1**B**-2) = 1**B**--2 = 1**B**+2 = 1**B**2 = 12

T = 4 **7s** = 4*7 = 4*(**B**-3) = 4**B** – 1**B**2 = 3**B**-2 = 2**B**8 = 28

T = 8 **7s** = 8*7 = (**B**-2)*(**B**-3) = **BB** – 2**B** – 3**B** + 6 = 10**B** – 2**B** – 3**B** + 6 = 5**B**6 = 56

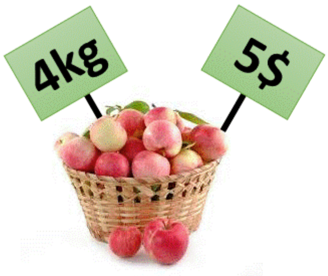
DoubleCounting in 2 Units creates PerNumbers

Apples are double-counted in kg and in $.

With **4kg = 5$** we have the **per-number** 4kg/5$ = 4/5 kg/$

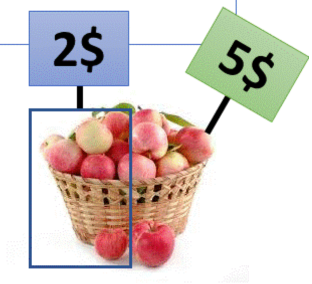
Questions:

12kg = ?$	**20$ = ?kg**
12kg = (12/4)*4kg = (12/4)*5$ = 15$	20$ = (20/5)*5$ = (20/5)*4kg = 16kg



Answer: Recount in the per-number

DoubleCounting in the Same Unit creates Fractions



The same unit: 2$ per 5$ = 2$/5$ = 2/5

• *Question: 2/5 = ? per 100; or 2$/5$ is ? per 100$*

Answer: recount 100 in 5s!

100$ = (100/5)*5$ gives (100/5)*2$ = 40$, so 2/5 = 40/100 = 40%

• *Question: 2/5 of 40 = ?; or with units: 2$ per 5$ of 40 $.*

Answer: recount 40 in 5s!

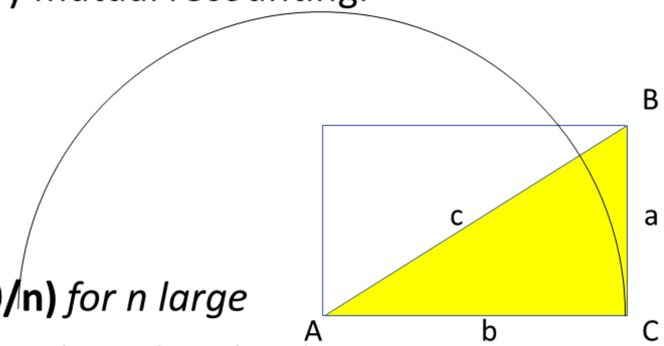
40$ = (40/5)*5$ gives (40/5)*2$ = 16$, so 2/5 of 40 = 16

Trigonometry ReCounts Sides in a HalfBlock

Halved by its diagonal, a block becomes a right angled triangle with three sides: the base b & the height a & the diagonal c, creating trigonometry by mutual recounting.

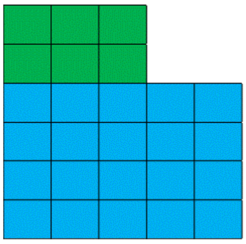
a = **(a/c)** * c = **sinA** * c
b = **(b/c)** * c = **cosA** * c
a = **(a/b)** * b = **tanA** * b

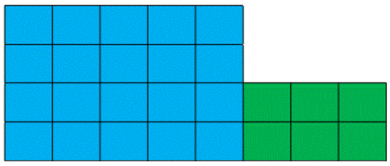
½Circle = π = **n*tan(180/n)** *for n large*



Once Counted & ReCounted, Totals can be Added

OnTop	NextTo
4 5s **+** 2 3s = 4 5s **+** 1**B**1 5s = 5**B**1 5s	4 5s + 2 3s = 3**B**2 8s
The units are changed to be the same *Change unit = Proportionality*	The areas are added *Adding areas = Integration*





The 3 Numbers in a Total add Differently

From totals as T1 = 2.3 **4s** and T2 = 3.4 **5s** we see that a Total has 3 numbers that add differently:

The bundle-size, the bundle-number, the single-number.

- Bundle-sizes stay unchanged unless the blocks are added next-to each other as in integration
- Bundle-numbers only add with like bundle-sizes.
- Singles always add.

Never add without units: Mars Climate Orbiter, planes?

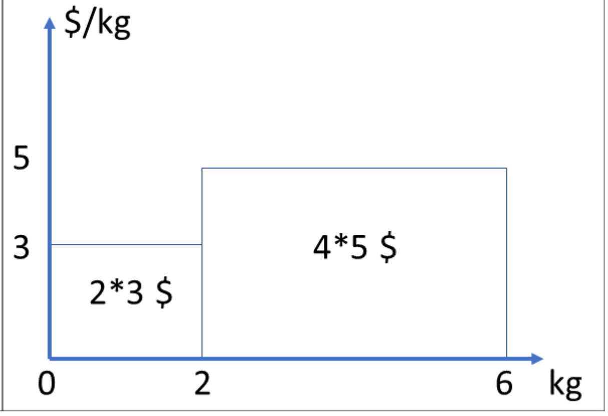
Adding PerNumbers as Areas (Integration)

2 kg at 3 $/kg
+ 4 kg at 5 $/kg

(2+4) kg at ? $/kg

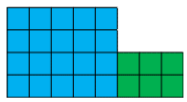
Unit-numbers add on-top.

Per-numbers add next-to as **areas** under the per-number graph.

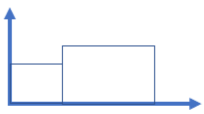


Primary & Middle & High School Calculus

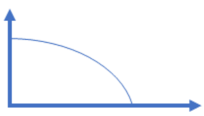
<u>Primary calculus</u>: Next-to addition of block-numbers



<u>Middle calculus</u>: Add piecewise constant per-numbers

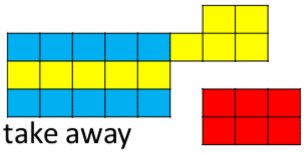


<u>High school calculus</u>: Add locally constant (continuous) per-numbers



Reversed Addition = Solving Equations

OppoSite Side with OppoSite Sign		NextTo
2 x ? = 8 = (8/2) x 2	**2 + ? = 8** = (8-2) + 2	**2 3s + ? 5s = 3.2 8s**
? = 8/2 *Solved by ReCounting*	? = 8-2 *Solved by ReStacking*	? = (3.2 8s – 2 3s)/5 *Solved by differentiation: (T-T1)/5 = ΔT/5*



Hymn to Equations

Equations are the best we know,
they are solved by isolation.
But first, the bracket must be placed
around multiplication.

We change the sign and take away
and only x itself will stay.
We just keep on moving, we never give up.
So feed us equations, we don't want to stop!

The Algebra project: How to re-unite

Concrete Algebra: 4 ways we Unite, + * ^ ∫ as shown by the Bundle Formula

T = 456 = 4*B^2 + 5*B^1 + 6*B^0

Totals exist as changing or constant **unit-numbers** or **per-numbers**

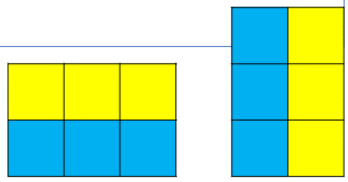
- Addition & Multiplication unite changing & constant unit-numbers
  - Subtraction & division split into changing & constant unit-numbers
- Integration & Power unite changing & constant per-numbers
  - Differentiation & root/logarithm (factor finder/counter) split into changing & constant per-numbers

Operations **unite** / *split into*	Changing	Constant
Unit-numbers *m, s, $, kg*	**T = a + n** *T – a = n*	**T = a*n** *T/n = a*
Per-numbers *m/s, $/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $log_a T = n$, $\sqrt[n]{T} = a$

The Simplicity of Mathematics

The Algebra project: How to re-unite

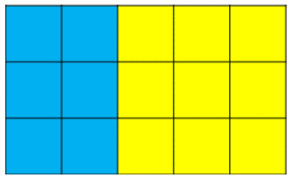
Abstract Algebra: (re)Uniting Units



- Turning a block will change the unit
  T = 2 3s = 2*3 -> T = 3 2s = 3*2, so T = 2*3 = 3*2

(The Commutative law)

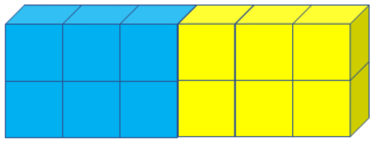
- A block may be split in two parts
  T = 3 5s = 3 2s + 3 3s or
  T = 3*5 = 3*(2+3) = 3*2 + 3*3



(The Distributive Law)

- A <u>united unit</u> as 6 that can be folded and fully stacked
  - a <u>prime unit</u> as 3 cannot.

  T = 2 6s = 2*(2*3) = (2*2)*3



(The Associative law)

T = 456 = 4*B^2 + 5*B + 6*1

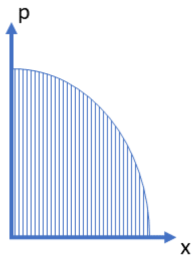
Bundle Formula: 5 ways of Constant Change

The number-formula contains formulas for constant change:

- T = b*x (proportional) *trade*
- T = b*x + c (linear) *trends*
- T = a * x^n (elastic) *science*
- T = a * n^x (exponential) *economy*
- T = a*x^2 + b*x + c (accelerated) *physics*

Two forms of NonConstant Change

Adding locally constant per-numbers means finding the area under the per-number graph as a sum of a large number of thin area-strips. But, if written as changes, this reduces to finding one total change since the middle terms cancel out. Writing p*dx = dF, or p = dF/dx motivates differential calculus, also useful to describe non-constant **predictable change**.



Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot predict.

Three forms of Constancy

A class is stuck in the **epsilon-delta** definition of continuity and differentiability. Here a difference is to rename them to 'local constancy' and 'local linearity'. As to constancy:

- y is globally constant c if for all positive numbers **epsilon**, the difference between y and c is less than epsilon.
- y is piecewise constant c if an interval-width **delta** exists such that for all positive numbers **epsilon**, the difference between y and c is less than epsilon, in this interval.
- y is locally constant c if for all positive numbers **epsilon**, an interval-width **delta** exists such that the difference between y and c is less than epsilon, in this interval.

Likewise, the change per-number **Δy/Δx** can be globally, piecewise or locally constant.
If locally constant , it is written as **dy/dx,** and y is called 'locally linear'.

Quantitative Literature or Modeling comes in 3 Genres also: Fact & Fiction & Fiddle

- Fact models or 'since-then' calculations use numbers and formulas to quantify and to predict predictable quantities as e.g. 'since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 = 20'. Fact models can be trusted once the numbers and the formulas and the calculation has been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars Climate Orbiter.
- Fiction models or 'if-then' calculations use numbers and formulas to quantify and to predict unpredictable quantities as e.g. 'if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20'. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made.
- Fiddle models or 'what-then' models use numbers and formulas to quantify and to predict unpredictable qualities as e.g. 'since a graveyard is cheaper than a hospital, then a bridge across the highway is too costly.' Fiddle models should be rejected and relegated to a qualitative description.

How Different is the Difference? I
SET Math versus MANY Math

	SET Math	**Many Math**
Goal/ Means	Learn Mathematics / Teach Mathematics	Learn to master Many / Tales of Many as counted, united, changed
Digits	Symbols like letters	Icons with as many sticks as they represent
Numbers	Line-numbers with place-value system Never with units	Block-numbers, stacking singles, bundles, bundle-bundles etc. Always with units
Number-types	Four types: Natural, Integers, Rational, Real	Positive & negative decimal numbers with units
Operations	Mapping from a set-product to the set. Order: Add, subtract, multiply, divide	Counting-icons: bundle /, stack x, remove -, unite on-top & next-to +). Opposite order
Division	8/2 means 8 split in 2	8/2 means 8 split in (counted in) 2s
ReCount PerNumber	Do not exist	Core concepts

How Different is the Difference? II
SET Math versus MANY Math

	SET Math	**Many Math**
Fractions	Rational numbers without units, and adding without units	Per-numbers, not numbers but operators needing a number to become a number, so added by integration
Equation	Statement about equivalent number-names	A recounting from tens to icons. Reversed operations
Function	A set relation where component1-identity implies comp.2-identity	A number-language sentence about the Total with a subject & a verb & a predicate
Propotio-nality	A linear function	A name for double-counting in two units
Calculus	Differentiation before integration (anti-differentiation)	Integration adds locally constant per-numbers. Integration before differentiation
Geometry	Plane before Coordinate before Trig.	Trigonometry before Coordinate Geometry

Difference-Research, Main Warning:
The 3x3 Goal Displacements in Math Education

Primary	Numbers	Could: be icons & predicates in Tales of Many, T = 2 3s = 2*3; show Bundles, T = 47 = 4B7 = 3B17 = 5B-3; T = 456 = 4*BB + 5*B + 6*1 Instead: are changed from predicates to subjects by silencing the real subject, the total. Place-values hide the bundle structure
	Operations	Could: be icons for the counting process as predicted by the RecountFormula T = (T/B)*B, from T pushing Bs away T/B times Instead: hide their icon-nature and their role in counting; are presented in the opposite order (+ - * /) of the natural order (/, *, -, +).
	Addition	Could: wait to after counting & recounting & double-counting have produced unit- and per-numbers; wait to after multiplication Instead: silences counting and next-to addition; silences bundling & uses carry instead of overloads; assumes numbers as ten-based
Middle	Fractions	Could: be per-numbers coming from double-counting in the same unit; be added by areas (integration) Instead: are defined as rational numbers that can be added without units (mathe-matism, true inside, seldom outside classrooms)
	Equations	Could: be introduced in primary as recounting from ten-bundles to icon-bundles; and as reversed on-top and next-to addition Instead: Defined as equivalence relations in a set of number-names to be neutralized by inverse elements using abstract algebra
	Proportionality	Could: be introduced in primary as recounting in another unit when adding on-top; be double-counting producing per-numbers Instead: defined as linear functions, or as multiplicative thinking supporting the claim that fractions and ratios are rational numbers
High	Trigonometry	Could: be introduced in primary as mutual recounting of the sides in a right-angled triangle, seen as a block halved by a diagonal Instead: is postponed till after geometry and coordinate geometry, thus splitting up geometry and algebra.
	Functions	Could: be introduced in primary as formulas, i.e. as the number-language's sentences, T = 2*3, with subject & verb & predicate Instead: are introduced as set-relations where first-component identity implies second-component identity
	Calculus	Could: be introduced in primary as next-to addition; and in middle & high as adding piecewise & locally-constant per-numbers Instead: differential calculus precedes integral calculus, presented as anti-differentiation

ReCounting looks like Dienes MultiBase Blocks

- "Dienes' name is synonymous with the Multi-base blocks (also known as Dienes blocks) which he invented for the teaching of place value.
- Dienes' place is unique in the field of mathematics education because of his theories on how mathematical structures can be taught from the early grades onwards using multiple embodiments through manipulatives, games, stories and dance."

(http://www.zoltandienes.com/about/)

Dienes on Numbers and MultiBase Blocks

"The position of the written digits in a written number tells us whether they are counting singles or tens or hundreds or higher powers. This is why our system of numbering, introduced in the middle ages by Arabs, is called the place value system. My contention has been, that in order to fully understand how the system works, we have to understand the concept of power. (..)

In school, when young children learn how to write numbers, they use the base ten exclusively and they only use the exponents zero and one (namely denoting units and tens) , since for some time they do not go beyond two digit numbers. So neither the base nor the exponent are varied, and it is a small wonder that children have trouble in understanding the place value convention. (..)

Educators today use the "multibase blocks", but most of them only use the base ten, yet they call the set "multibase". These educators miss the point of the material entirely."

(What is a base?, http://www.zoltandienes.com/academic-articles/)

Power & Base from Above, or Bundles from Below

Dienes teaches the 1D place value line-numbers with 2D & 3D blocks to show the importance of the power concept.

- ManyMatics teaches 2D block-numbers with units to show the importance of bundling singles, bundles & bundle-bundles.

base the base
bundle the bundle

Dienes sees numbers as examples of the abstract label **base**

- ManyMatics sees counting as an action with a concrete verb **bundle**

Dienes teaches top-down 'MetaMatics' derived from the concept Set

- ManyMatics teaches a bottom-up natural science about the fact Many; and sees Set as meaningless because of Russell's set-paradox.

Different Education

EU: Line-organized & Office-directed Schools

From secondary school, continental Europe uses **line-organized** education with forced classes and forced schedules, making teenagers stay together in age groups - even if boys are two years behind in mental development.

The classroom belongs to the class. This forces teachers to change room and (in lower secondary school) to teach several subjects outside their training.

Tertiary education is also **line-organized** preparing for offices in the public or private sector. This makes it difficult to change line in the case of unemployment.

This makes reproduction fall to 1.5 child/family, causing the European population to be halved each two generations since per female, (1.5/2)*(1.5/2) = .75*.75 ≈ 0.5.

Different Education

US: Block-organized & Talent-directed Schools

Alternatively, North America uses **block-organized** education saying to teenagers:

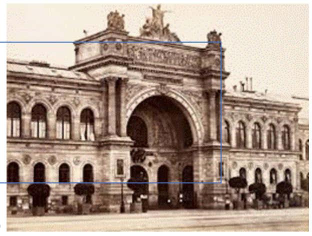
"Welcome, inside you carry a **talent**! Together we will uncover and develop your personal talent through self-chosen daily half-year blocks, academical or practical, together with 1subject teachers. If successful the school will say '**good job**, you have a **talent**, you need some more'. If not, the school will say '**good try**, you have **courage** to try out the unknown, now try something new'".

The classroom belongs to the teacher teaching one subject only.

Likewise, college is **block-organized** easy to supplement with additional blocks in the case of unemployment.

At the age of 25, most students have an education, a job and a family with three children, 1 for mother, 1 for father, and 1 for the state to secure reproduction.

Good & Bad Research



- Good research searches for truth about things that exist. It poses a question, and chooses a methodology to transform reliable data into valid statements. Or it uses methodic skepticism to unmask choice masked as nature.
- Bad research is e.g. master level work applying instead of questioning existing research. Or journalism describing something without being guided by a question.
- With these three research genres, peer-review only works inside the same genre.
- All conferences should have a **'salon des refusé'** to foster and boost new paradigms (Kuhn), as it does in art.

More Conflicting Theory in Math Ed Research

Philosophy

- Sophists: Unmask choice masked as nature by finding hidden differences
- Philosophy: All is nature and examples of meta-physical forms only visible to us

Sociology

- Structure: Institutions are good if rational and democratic
- Agent: Goal displacements in institutions lead to ‘the banality of evil’ (Arendt)

Psychology

- Piaget: Teach little, but allow the learner to meet the **existing** subject directly
- Vygotsky: We need good teaching to mediate institutionalized **essence**

More Enlightenment Sociology in Math Ed Research

Sociology can question institutions by asking: Offering education as a cure for the diagnose ‘uneducated’ is a self-referring irrationality. A power agenda behind?

Thus, inspired by Heidegger’s: ‘In sentences, trust the subject & doubt the predicate’, and wanting to protect its Enlightenment republic, French post-structuralism says:

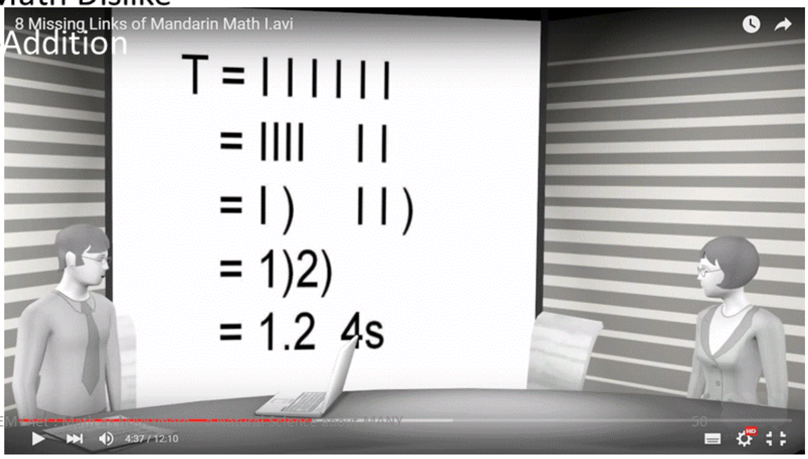
- Derrida: Words can be fake, and install instead of label (DeConstruction)
- Lyotard: Truth can be fake (PostModern skepticism towards meta-narratives)
- Foucault: Diagnoses and discourses can be fake, still allowing curing institutions to expand (a school is really a ‘pris-pital’ mixing power techniques from a prison and a hospital, and with learners as ‘patien-mates’)
- Bourdieu: Education is fake by using symbolic violence (and mathematics especially) to create a new knowledge-nobility

ManyMath is Different
But does it make a Difference? Try it out

- Watch some YouTube or YouKu videos (MrAlTarp/DrAlTarp)
- Try the **CupCount before you Add** Booklet
- Try a 1day free Skype seminar **How to Cure Math Dislike**
- Try Action Learning and Action Research, e.g. **1Cup & 5Sticks**
- Collect data and Report on 8 **MicroCurricula**, M1-M8
- Try a 1year online <u>InService TeacherTraining </u>at the MATH**e**CADEMY.net using PYRAMID**e**DUCATION to teach teachers to teach MatheMatics as **ManyMatics**, a Natural Science about the root of mathematics, **Many**

Some MrAlTarp YouTube Videos
Screens & Scripts on MATHeCADEMY.net

- Postmodern Mathematics Debate
- CupCounting removes Math Dislike
- IconCounting & NextTo-Addition
- PreSchool Mathematics
- Fractions
- PreCalculus
- Calculus
- Mandarin Mathematics
- World History



Action Learning & Action Research

Imagine a difference

Design → Report

Teach / Share

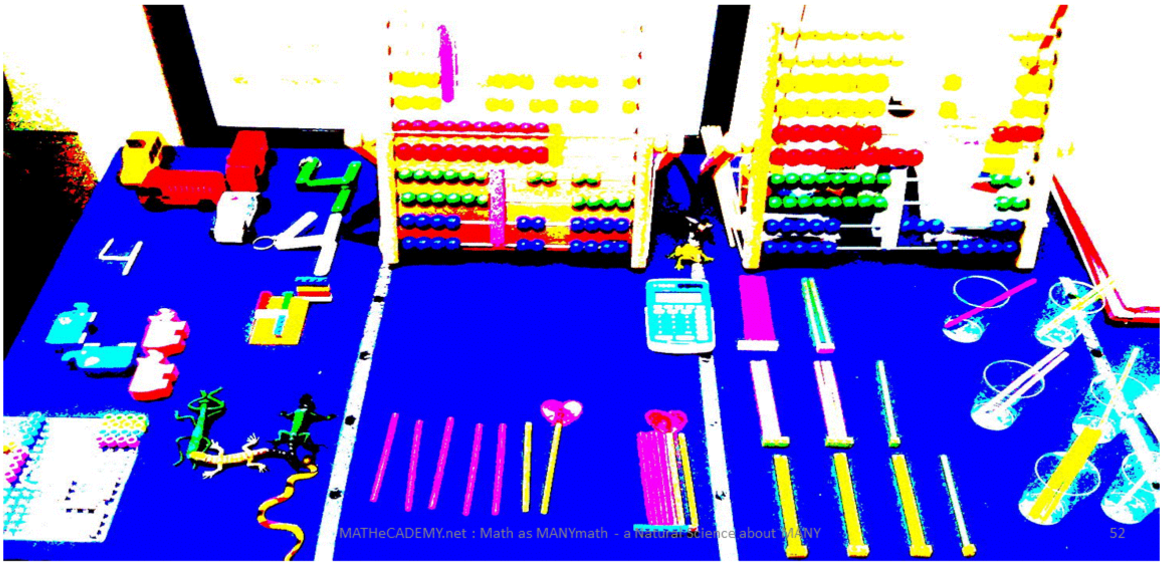
Observe

Lyotard dissenting Paralogy
Quality indicator:
Ungrounded rejection

Example
Calculators in PreSchool
and Special Needs education
Paper rejected at MADIF10

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 51

Numbers as Icons & ReCounting 7 in 5s & 3s & 2s



MatheMatics: Unmask Yourself, Please

- In Greek you mean 'knowledge'. You were chosen as a common label for 4 activities: Music, Astronomy, Geometry & Arithmetic. Later only 2 activities remained: Geometry and Algebra
- Then Set transformed you from a natural science about the physical fact Many to a metaphysical subject, MetaMatism, combining MetaMatics and MatheMatism
- So please, unmask your true identity, and tell us how you would like to be presented in education:
- MetaMatism for the few - or ManyMatics for the many.

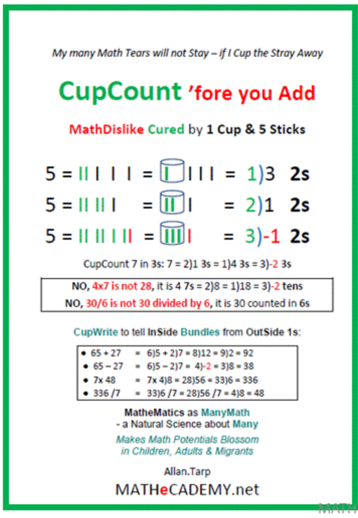
From Bad & Evil Math to Good Math:

1) Respect the Child's own 2D Block
2) Count, ReCount & DoubleCount before Adding OnTop & NextTo
3) Let Existence precede Essence: **Think Things**

Slides on MATHeCADEMY.net
Details in ***Journal of Mathematics Education***

Thank You for Listening

CupCount 'fore you Add Booklet, free to Download


My many Math Tears will not Stay – if I Cup the Stray Away

CupCount 'fore you Add

MathDislike Cured by 1 Cup & 5 Sticks

5 = IIIII = 1)3 2s

5 = IIIII = 2)1 2s

5 = IIIII = 3)-1 2s

CupCount 7 in 3s: 7 = 2)1 3s = 1)4 3s = 3)-2 3s

NO, 4x7 is not 28, it is 4 7s = 2)8 = 1)18 = 3)-2 tens
NO, 30/6 is not 30 divided by 6, it is 30 counted in 6s

CupWrite to tell InSide Bundles from OutSide 1s:

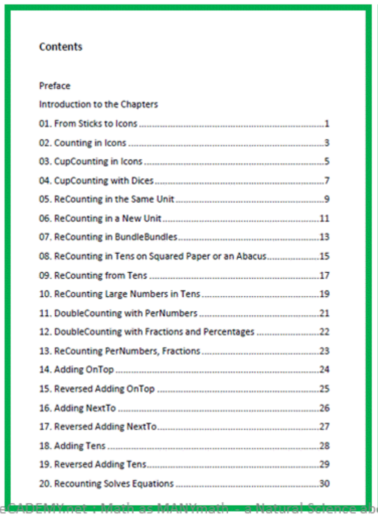
- 65 + 27 = 6)5 + 2)7 = 8)12 = 9)2 = 92
- 65 – 27 = 6)5 – 2)7 = 4)-2 = 3)8 = 38
- 7x 48 = 7x 4)8 = 28)56 = 33)6 = 336
- 336 /7 = 33)6 /7 = 28)56 /7 = 4)8 = 48

MatheMatics as ManyMath
- a Natural Science about Many

Makes Math Potentials Blossom in Children, Adults & Migrants

Allan.Tarp

MATHeCADEMY.net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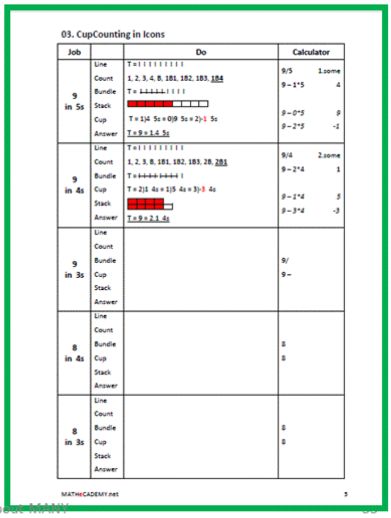


1day free Skype Seminar:
To Cure Math Dislike, CupCount before you Add

Action Learning based on the Child's own 2D NumberLanguage

09-11. Listen and Discuss the PowerPointPresentation

To Cure MathDislike, replace MetaMatism with ManyMath

- **MetaMatism** = MetaMatics + MatheMatism
- **MetaMatics** presents a concept TopDown as an example instead of BottomUp as an abstraction
- **MatheMatism** is true inside but rarely outside classrooms
- **ManyMath**, a natural science about Many mastering Many by BundleCounting & Adding NextTo and OnTop.

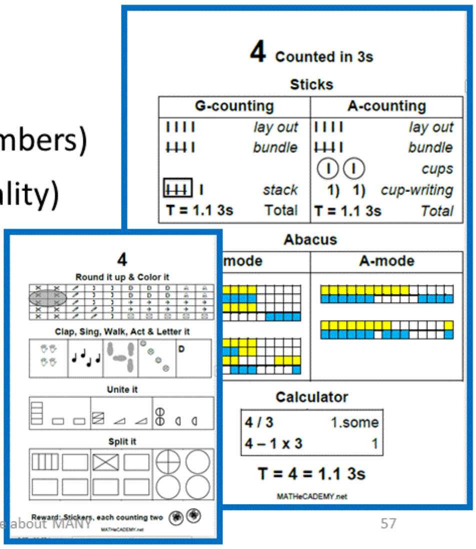
11-13. Skype Conference. Lunch.

13-15. Do: Try out the CupCount before you Add booklet to experience proportionality & calculus & solving equations as golden LearningOpportunities in BundleCounting & NextTo Addition.

15-16. Coffee. Skype Conference.

8 MicroCurricula for Action Learning & Research

C1. Create Icons
C2. Count in Icons
C3. ReCount in the Same Icon (Negative Numbers)
C4. ReCount in a Different Icon (Proportionality)
A1. Add OnTop (Proportionality)
A2. Add NextTo (Integrate)
A3. Reverse Adding OnTop (Solve Equations)
A4. Reverse Adding NextTo (Differentiate)



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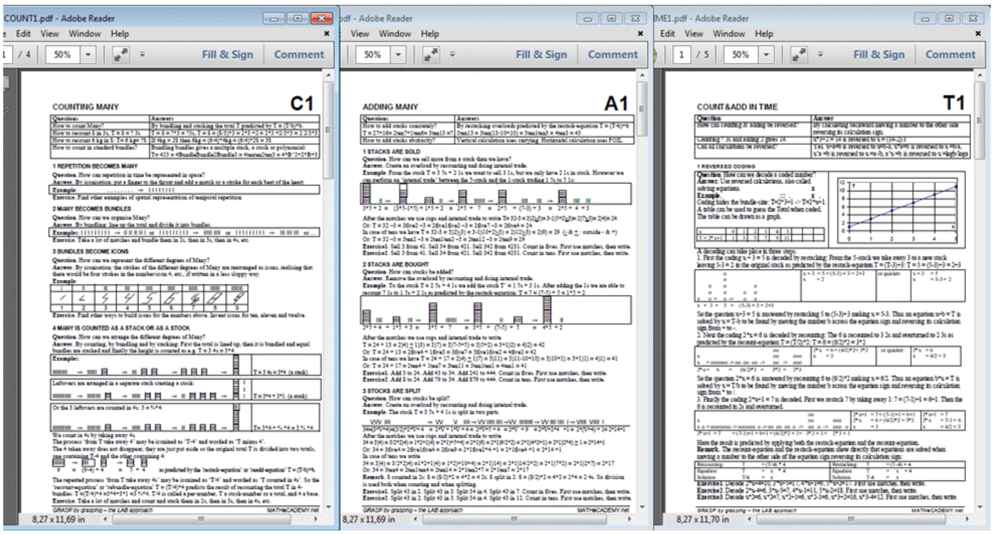
Teacher Training in CATS ManyMath
Count & Add in Time & Space



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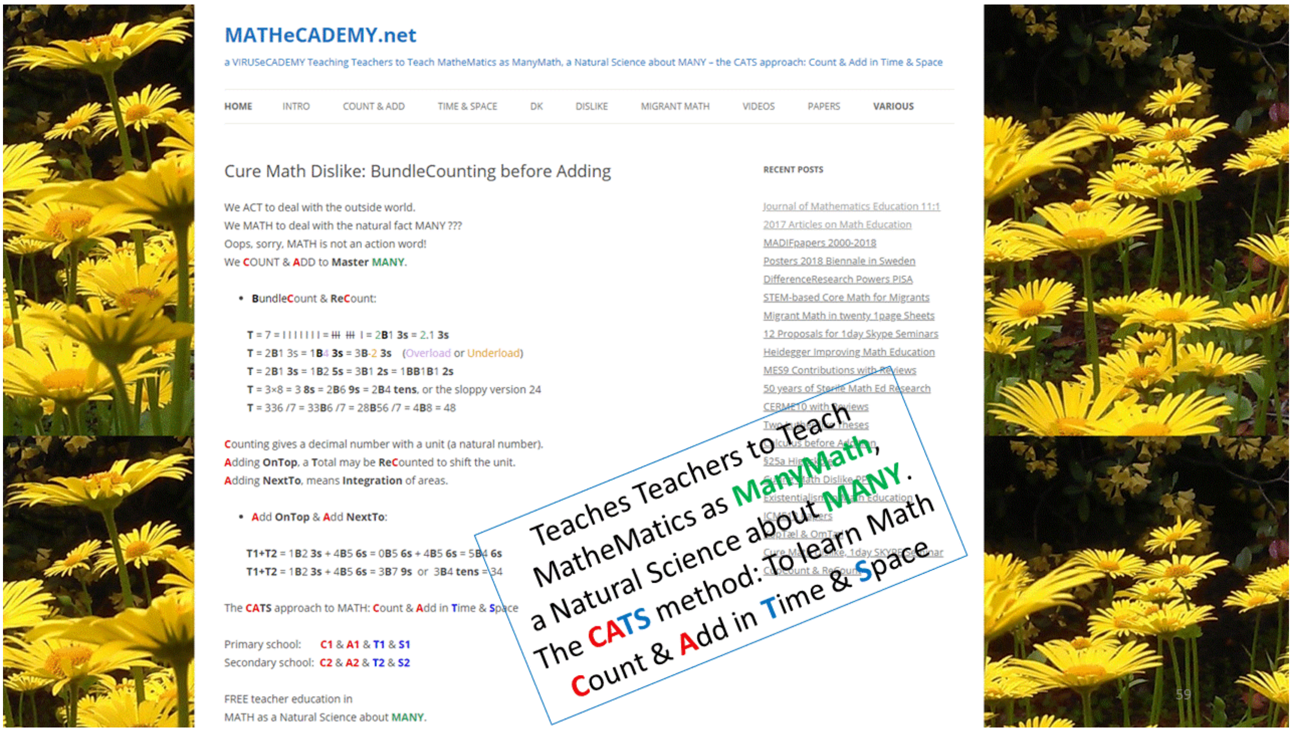
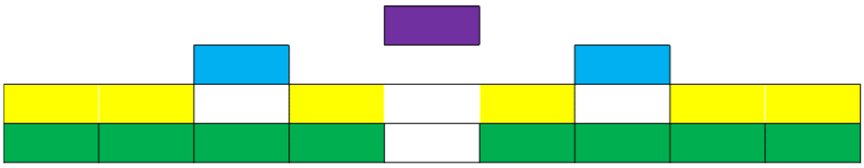


PYRAMIDeDUCATION	To learn MATH: Count&Add MANY *Always ask Many, not the Instructor* MATHeCADEMY.net - a VIRUSeCADEMY

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

- Each pair works together to solve Count&Add problems.
- The coach assists the instructors when instructing their team and when correcting the Count&Add assignments.
- Each teacher pays by coaching a new group of 8 teachers.

1 Coach
2 Instructors
3 Pairs
2 Teams



28b. Different Mathematics

Main Parts of a ManyMath Curriculum

Primary School – respecting and developing the Child's own 2D NumberLanguage

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- BundleCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: divide, multiply, subtract, add on-top & next-to

Middle school – integrating algebra and geometry, the content of the label math

- DoubleCounting produces PerNumbers as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always, so length becomes change and vv.

High School – integrating algebra and geometry to master CHANGE

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

Quadratic Equations with 3 Cards

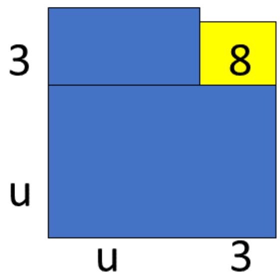
Solve the quadratic equation **u^2 + 6u + 8 = 0**

(u+3)^2 = u^2 + 6u + 8 + 1

(u+3)^2 = 0 + 1

u+3 = ± 1

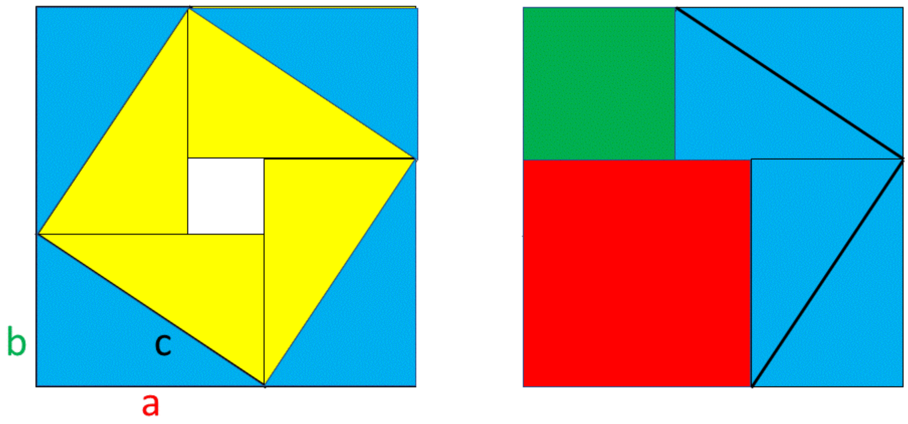
u = -3 ± 1



Solution: **u = -4, u = -2**

MATHeCDMY

Pythagoras shown by 4 Cards with Diagonals



c^2 + 4 ½cards **a^2** + **b^2** + 2 cards

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25. The Simplicity of Math reveals a Core Curriculum

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015a) to write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change’.

To find an unorthodox solution we pretend that a university in southern Sweden arranges a curriculum architect competition: ‘Theorize the low success of 50 years of mathematics education research; and derive a STEM-based core curriculum from this theory.’

Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking ‘renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now’ (p. 16).

Mathematics education is an example of ‘rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)’. However

> The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn’t the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, ‘to measure earth’ in Greek and ‘to reunite’ in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here the invention of the concept SET created a Set-based ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts. However, ‘well-defined’ meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2x3=6' stating that 2 3s can always be re-counted as 6 1s.

Difference Research Looking at Mathematics Education

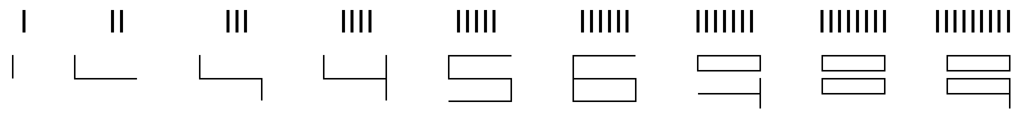
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a core mathematics curriculum based upon exemplary situations of Many in a STEM context, seen as having a positive effect on learners with a non-standard background (Han et al, 2014). So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a 'Count-before-Adding' Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence.

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:



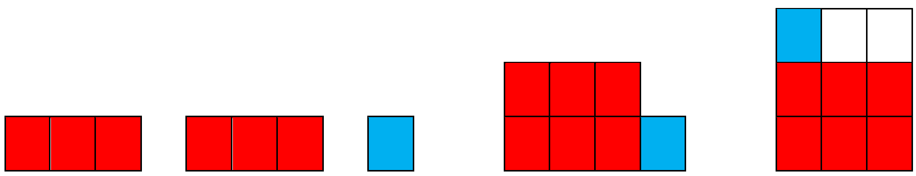
1 2 3 4 5 6 7 8 9

We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)*B saying that 'from T, T/B times, B can be taken away':

7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations and differential calculus:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the 'geometry & algebra, always together, never apart' principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Alternative Versions of Standard Mathematics

01. To stress the importance of bundling, the counting sequence should be: 01, 02, …, 09, 10,11 etc.

02. The ten fingers should be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers should be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables should be formulated as re-counting from icon-bundles to tens and use underload counting after 5: T = 4*7 = 4 7s = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 should be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

06. Solving proportional equations as 3*x = 12 should be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the 'opposite side & sign' method.

07. Proportional tasks should be done by re-counting in the per-number: With 3$/4kg, 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg

08. Fractions and percentages should be seen as per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$ = (60/3)*3$ giving (60/3)*2$ = 40$

09. Integral should precede differential calculus and include adding both piecewise and locally constant per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = (2+4)kg at (2*3+4*5)$/(2+4)kg thus showing that per-numbers and fractions are added with their units as the area under the per-number graph.

10. Trigonometry should precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sinA*c.

Level & Change Formulas

Re-counting and double-counting leads to the recount-formula T = (T/B)*B occurring all over mathematics: when re-counting or double-counting to change unit in proportional quantities; when re-counting to solve equations; in trigonometry to mutually re-count the sides in a right triangle; and in calculus to mutually re-count the changes as dy = (dy/dx)*dx = y'*dx. In economics, the recount-formula becomes a price-formula: $ = ($/kg)*kg, $ = ($/day)*day, etc.

Counting by stacking bundles into adjacent blocks leads to the number-formula called a polynomial:

T = 456 = 4*BundleBundle + 5*Bundle + 6*single = 4*B^2 + 5*B + 6*1.

In its general form, the number-formula T = a*x^2 + b*x + c contains the different formulas for constant change: T = a*x (proportionality), T = a*x+b (linearity), T = a*x^2 (acceleration), T = a*x^c (elasticity) and T = a*c^x (interest rate).

The number-formula also shows the four ways to unite numbers offered by algebra meaning 'reuniting' in Arabic: addition and multiplication add variable and constant unit-numbers; and integration and power unite variable and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into variable and constant unit-numbers; and differentiation and root & logarithm split a total in variable and constant per-numbers:

Uniting/*splitting into*	Variable	Constant
Unit-numbers m, s, kg, $	T = a + n *T – a = n*	T = a * n *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	T = ∫ a dn *dT/dn = a*	T = a^n, *loga(T) = n n√T = a*

Meeting Many in a STEM Context

Having met Many by itself, we now meet Many in time and space in the present culture based upon STEM, described by OECD (2015b) as follows: 'In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.'

STEM thus combines basic knowledge about how humans interact with nature to survive and prosper: Mathematics provides formulas predicting nature's physical and chemical behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

A falling ball introduces nature's three main actors, matter and force and motion, similar to the three social actors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is lying still on the ground. Is the motion gone? No, motion cannot disappear. Motion transfers through collisions, now present as increased motion in molecules; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant but its entropy increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water to electricity. To get to the dam, we must build roads along the hillside.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), $/kg (price), $/hour (wages), etc.

An Electrical Circuit

To work properly, a 2000Watt water kettle needs 2000Joules per second. The socket delivers 220Volts, a per-number double-counting the number of Joules per charge-unit.

Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere.

To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2Volt/Ampere called Ohm.

Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere.

Thus, with a 60Watt and a 120Watt bulb, the latter needs twice the current, and consequently half the resistance of the former.

Supplied next-to each other from the same source, the combined resistance R must be decreased as shown by reciprocal addition, 1/R = 1/R1 + 1/R2. But supplied after each other, the resistances add directly, R = R1 + R2. Since the current is the same, the Watt-consumption is proportional to the Volt-delivery, again proportional to the resistance. So, the 120Watt bulb only receives half of the energy of the 60Watt bulb.

Warming and Boiling Water

In a water kettle, a double-counting can take place between the time and the energy used to warm the water to boiling, and to transform the water to steam.

Heating 1000gram water 80degrees in 167seconds in a 2000Watt kettle, the per-number will be 2000*167/80Joule/degree, creating a double per-number 2000*167/80/1000Joule/degree/gram or 4.18Joule/degree/gram, called the specific heat of water.

Producing 100gram steam in 113seconds, the per-number is 2000*113/100Joule/gram or 2260Joule/g, called the heat of evaporation for water.

Conclusion and Recommendation

This paper argues that the low success of 50 years of mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views offer different kinds of mathematics: a set-based top-down ‘meta-matics’ that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based ‘Many-matics’ simply saying ‘To master Many, counting produces constant or variable unit-or per-numbers, uniting by adding or multiplying or powering or integrating.’

Thus, this simplicity of mathematics as expressed in a Count-before-Adding curriculum allows bundle-numbers to replace line-numbers, and to learn core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young male migrants learn core STEM subjects at the same time, thus allowing them to become STEM-teachers or STEM-engineers to return help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

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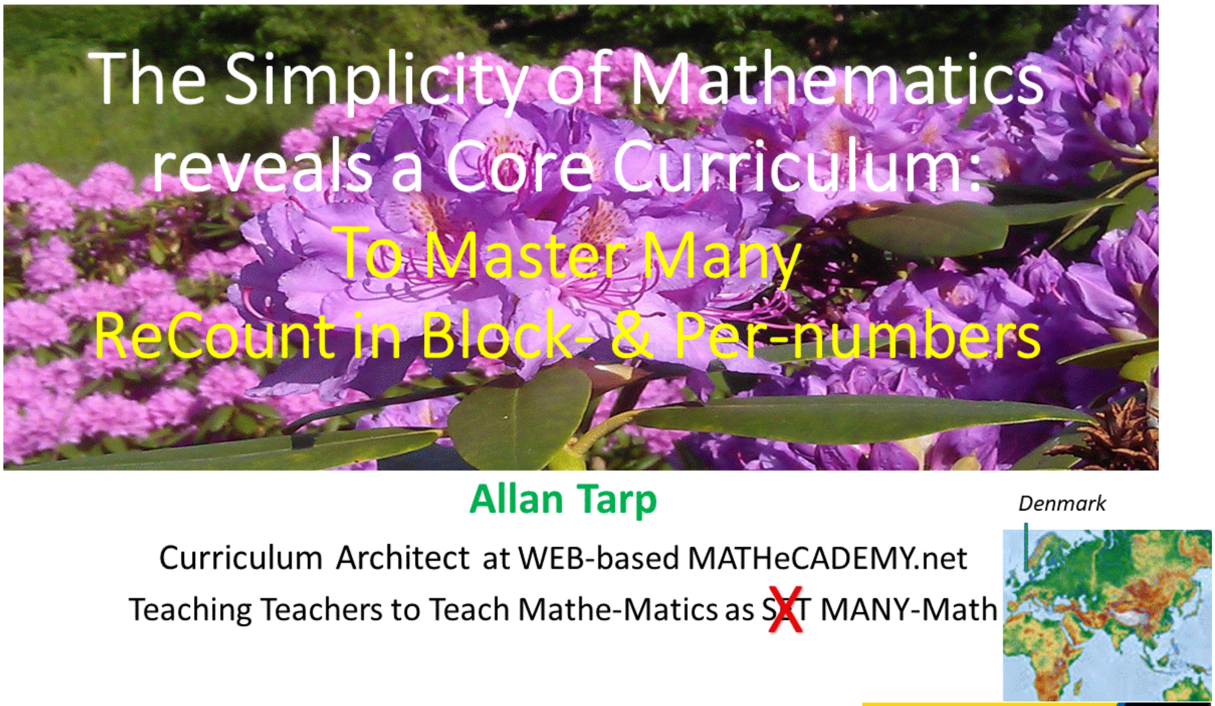
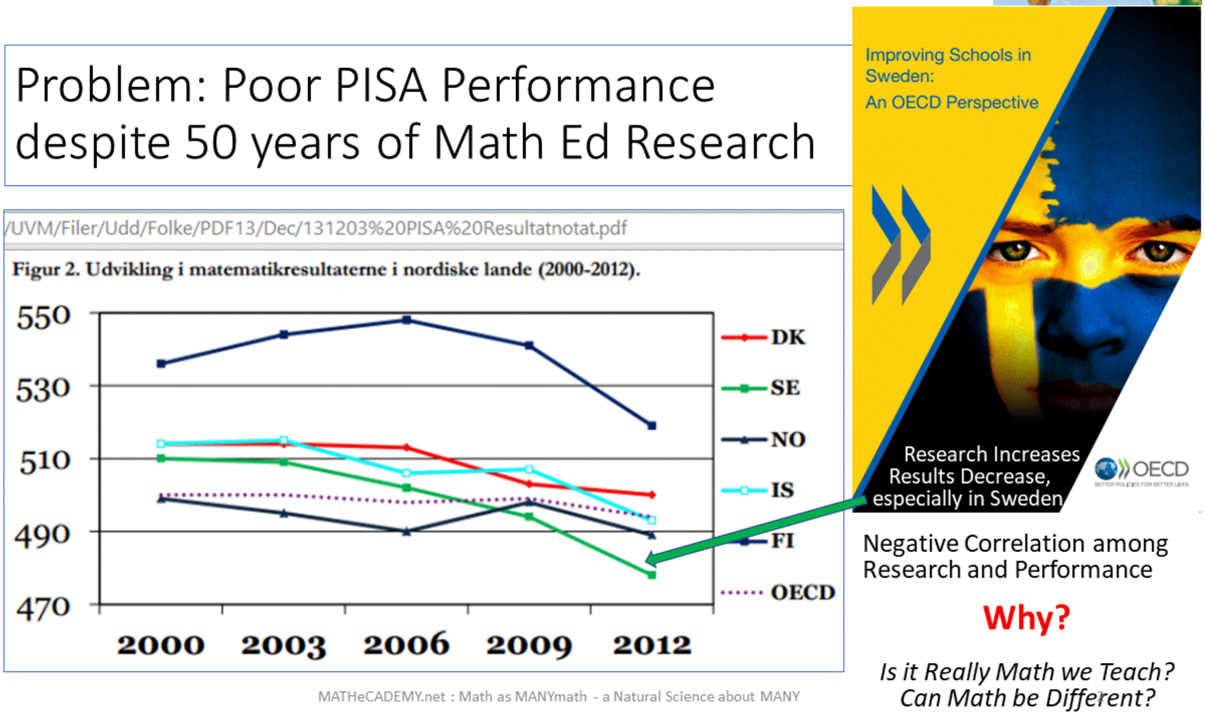
26. The Simplicity of Math reveals a Core Curriculum, PPP





Solution in a Nutshell
From BAD to GOOD Math

1) All teach numbers. Don’t. Tell tales about how Totals unite and change
2) All use 1D line-numbers. Don’t. Use 2D block-numbers
3) All begin with addition. Don’t. Begin with counting and division, multiplication and subtraction before adding next-to and on-top
4) All add fractions without units. Don’t. Use units as in integral calculus
5) All include only the predicate (3*5). Don’t. Use full language sentences with a subject, a verb and a predicate (T = 3*5)
6) All call it MatheMatics. Don’t. It is MetaMatism, derived from SET SET, and falsified by e.g. 2+3 is 17 and not 5 in the case of weeks and days. MatheMatics is rooted in MANY MANY.

A Call for Curriculum Architects

With many young male migrants in Sweden, a university may write out a competition:

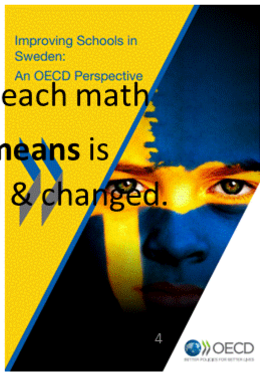
‘Theorize the poor PISA performance; and derive from this a STEM-based core curriculum for young male migrants.’

Didactics: Define one **goal** and several **means**.

The Tradition: The **goal** is to learn mathematics. The **means** is to teach math.

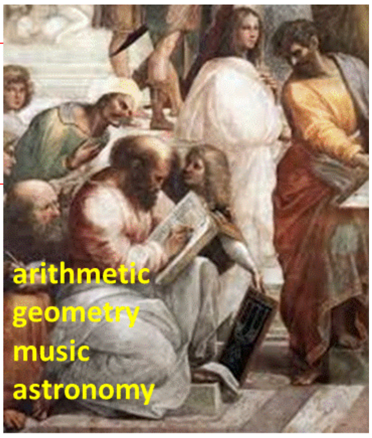
A Difference: The **goal** is to master Many in space and time. The **means** is number-language sentences about how Many is counted & added & changed.

Prerequisites: None, start from scratch.



MATHeCDMY

Definitions of MatheMatics



Pythagoras: mathematics, meaning knowledge, is a common label for 4 areas describing Many by itself and in space & time:

- **Geometry** means to measure earth in Greek
- **Algebra** means to reunite numbers in Arabic

Around 1900, **SET** made mathematics self-referring. However, Russell said:

Self-reference leads to the classical liar paradox 'this sentence is false', being false if true & opp.

Just look at the set of sets, not belonging to itself. If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

So, forget about sets, and forget about fractions as numbers, by self-reference they cannot be so.

Mathematics: Forget about Russell, he is not a mathematician. Of course fractions are numbers.

Two Different Mathematics

The ruling Set-based Top-Down **Meta-matics**

- Concepts are defined from above as **examples from abstractions**

a FUNCTION is an example of a set relation with component-1 identity implying component-2 identity

The silenced Many-based Bottom-Up **Many-math**

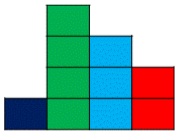
- Concepts are defined from below as **abstractions from examples**

a FUNCTION is for example 2+x, but not 2+3;
i.e. a name for a calculation with an unspecified number

Children see Many as Bundles with Units

Asked ‘How old next time?’, a 3year-old says 4, but reacts when held together 2 by 2:
‘That is not 4, that is 2 **2s**’.

Seeing bundles as units, children use 2D LEGO-like **block-numbers**, not 1D **line-numbers**, taught in school, even if 2D Arabic block-numbers replaced 1D Roman line-numbers centuries ago.



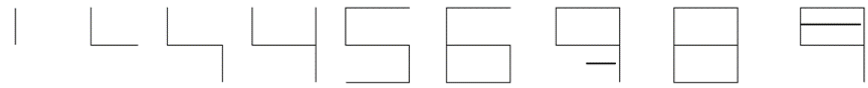
T = 1 4 3 2

T = MCCCCXXXII

Many as Icons: I I I I → ~~IIII~~ → 

Meeting Many, we ask: “**How Many in Total**?”
To answer, we Math … oops sorry, it’s a label, not an action word.
To answer, first we count, then we add. We name and iconize the degrees of Many until ten, that as 1 bundle has no icon or digit itself.
• Thus there are four sticks in a 4-icon, five in a 5-icon, etc.

one	two	three	four	five	six	seven	eight	nine
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
1	2	3	4	5	6	7	8	9

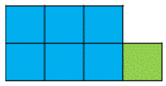


Operations as Icons also

A decimal point
parts inside bundles
from outside singles

We count by bundling & stacking:

T = 7 = I I I I I I I = III III I = $\boxed{\begin{matrix}III\\III\end{matrix}}$ I = 2**B**1 **3s** = 2.1 **3s**



- Thus, to count 7 in **3s** we take away 3 many times, iconized by an uphill stroke, 7/3, showing the broom wiping away the **3s**.

7/3	2.some
7 – 2x3	1

- A calculator predicts: 3 can be taken away 2 times. Stacking the bundles is iconized as a lift, 2x3.

- To look for unbundled singles, we drag away the stack of 2 **3s,** iconized by a horizontal trace: 7 – 2x3 = 1.

Counting creates 3 operations: to <u>divide</u> & to <u>multiply</u> & to <u>subtract</u>.

Totals as a Bundle Formula

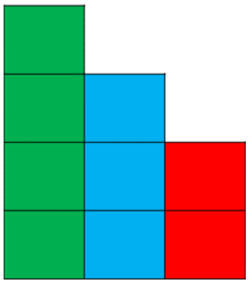
- To bundle bundles also, **power** is iconized as a cap, 5^2, showing the number of times bundles have been bundled.
- **Addition** is a cross + showing blocks juxtaposed next-to or on-top of each other.

Counting gives a Total as a **BundleFormula** called a polynomial.

Here all numbers have units:

T = 432 = 4***BundleBundle** + 3***Bundle** + 2*1

= 4***B^2** + 3***B +** 2*1



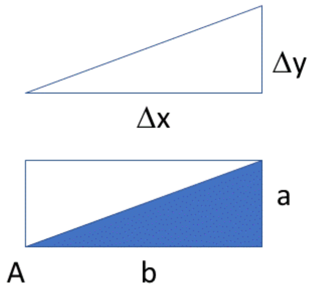
The ReCount Formula

7/3	2.some
7 – 2 * 3	1

ReCount formula T = (T/B)*B

Predicting T = 7 = 2.1 **3s**, the saying 'from T, T/B times, B can be taken away', is all over:

Proportionality	y = k * x
Linearity	Δy = (**Δy/Δx**) * Δx = **m** * Δx
Local linearity	dy = (**dy/dx**) * dx = **y'** * dx
Trigonometry	a = (**a/b**) * b = **tanA** * b
Trade	$ = (**$/kg**) * kg = **price** * kg
Science	meter = (**meter/second**) * second = **velocity** * second

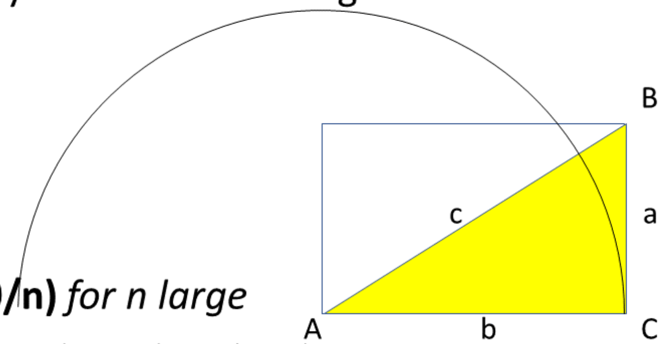


Trigonometry ReCounts Sides in a HalfBlock

Halved by its diagonal, a block becomes a right angled triangle with three sides: the base b & the height a & the diagonal c, creating trigonometry by mutual recounting.

a = (**a/c**) * c = **sinA** * c
b = (**b/c**) * c = **cosA** * c
a = (**a/b**) * b = **tanA** * b

½Circle = **π = n*tan(180/n)** *for n large*



ReCounting creates Proportionality and Overloads & Underloads

ReCounting in a **new unit** changes units (**proportionality**)
T = 4 **5s** = ? **6s**. The ReCount-formula predicts T = 3.2 **6s**

4*5/6	3.some
4*5 – 3*6	2

ReCounting in the **same unit** creates overloads & underloads

T = 7 = I I I I I I I = 𝍸 𝍸 I = 𝍸 I I I I = 𝍸 𝍸 𝍸 II

T = 7 = 2**B**1 **3s** = 1**B**4 **3s** = 3**B**-2 **3s**

BundleWriting may cure Math Dislike in classes stuck in Division:

T = 336 /7 = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

Likewise:

Multiplication	T = 7* 48 = 7* 4**B**8 = 28**B**56 = 33**B**6 = 336
Subtraction	T = 53 – 28 = 5**B**3 – 2**B**8 = 3**B**-5 = 2**B**5 = 25
Addition	T = 53 + 28 = 5**B**3 + 2**B**8 = 7**B**11 = 8**B**1 = 81

ReCounting creates Multiplication & Equations

ReCounting from icons to tens is predicted by **Multiplication**
T = 5 **7s** = ? **tens** = 5*7 = 35 = 3.5 **tens**

ReCounting from tens to icons is predicted by **Equations**
T = ? **7s** = 42 = (42/7)*7 recounting 42 in 7s, so ? = 42/7

u*7 = 42 = (42/7)*7	
u = 42/7	= 6

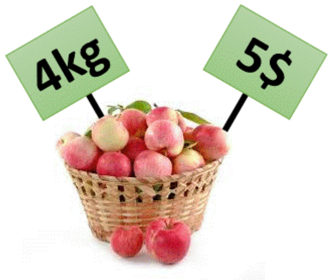
An equation is solved by moving to Opposite Side with opposite Sign

	7 x *u* = 42	Multiplication has 1 as its **neutral element**, and 7 has 1/7 as its **inverse element**
↕	(7 x *u*) x (1/7) = 42x(1/7)	Multiplying 7's inverse element 1/7 to both [illegible]her-names
↕	(*u* x 7) x (1/7) = 6	Applying the **commutative law** to u x 7; 6 is the s[illegible]umb[illegible]e for 42 x 1/7
↕	*u* x (7 x (1/7)) = 6	Applying the **associative law**
↕	*u* x 1 = 6	Applying the definition of an inverse element
↕	*u* = 6	Applying the definition of a neutral element[illegible]n arrows a test is [illegible] needed.

DoubleCounting in 2 units creates PerNumbers

Apples are double-counted in **kg** and in **$**.
With 4**kg = 5$** we have the **PerNumber** 4kg/5$ = 4/5 kg/$
Questions:

12**kg** = **?$**	20**$** = **?kg**
12kg = (12/4)*4kg	20$ = (20/5)*5$
= (12/4)*5$	= (20/5)*4kg
= 15$	= 16kg



Answer: Recount in the per-number

- With like units, per-numbers become fractions: 2$ per 5$ = 2$/5$ = 2/5

The BundleFormula T = 432 = 4*B^2 + 3*B + 2*1 shows the 4 ways, Many Unite (*the Simplicity of Math*)

Many exists as **changing** & **constant block-numbers** & **per-numbers**

- Addition & Multiplication unite changing & constant block-numbers
  Subtraction & Division split into changing & constant block-numbers
- Integration & Power unite changing & constant per-numbers
  Differentiation & Root/Logarithm split into changing & constant per-numbers

Operations **unite** / *split into*	**Changing**	**Constant**
Block-numbers *m, s, $, kg*	T = a + n *T – a = n*	T = a*n *T/n = a*
Per-numbers *m/s, $/kg, m/(100m) = %*	T = ∫ a dn *dT/dn = a*	T = a^n $\log_a T = n$, $\sqrt[n]{T} = a$

Theorizing Poor PISA Performance

Poor PISA performance is caused by 4 blind spots:

- Mathematics should respect its nature as a NumberLanguage with 3part sentences (subject-verb-predicate) and a grammar, as in the WordLanguage.
- Seen as a goal in itself, math hides its outside goal, to master Many, so we teach TopDown MetaMatics instead of BottomUp ManyMath
- We use 1D line-numbers instead of 2D block-numbers with 3 numbers: the size of the bundle & the number of bundles & the number of unbundled – and they add differently
- By this complexity, addition OnTop and NextTo should be postponed to after BundleCounting & ReCounting & DoubleCounting in STEM-tasks

1D

+ - x /

2D

/ x - +

STEM (ScienceTechEngMath) based Core Curriculum for Migrants

Nature consists of things in motion, combined in momentum = mass*velocity

Things contain **mass** & **molecules** & **eclectric charge**.

Nature is counted in **meter** & **second** & **kilogram** & **mole** & **coulomb**.

<u>Nature is predictable by ReCounting & PerNumbers:</u>

kilogram = (**kilogram/cubic-meter**) * cubic-meter = **density** * cubic-meter

meter = (**meter/second**) * second = **velocity** * second

Δ momentum = (Δ **momentum/second**) * second = **force** * seconds

Δ energy= (Δ **energy/meter**) * meter = **force** * meter = work

*Energy = ½*mass*velocity squared*

PerNumbers/ReCounting in ScienceTechEngMath II

Energy = (**energy/kg**) * kg = **melting/evaporation heat** * kg

Energy = (**energy/kg/degree**) * kg * degree = **heat** * kg * degree

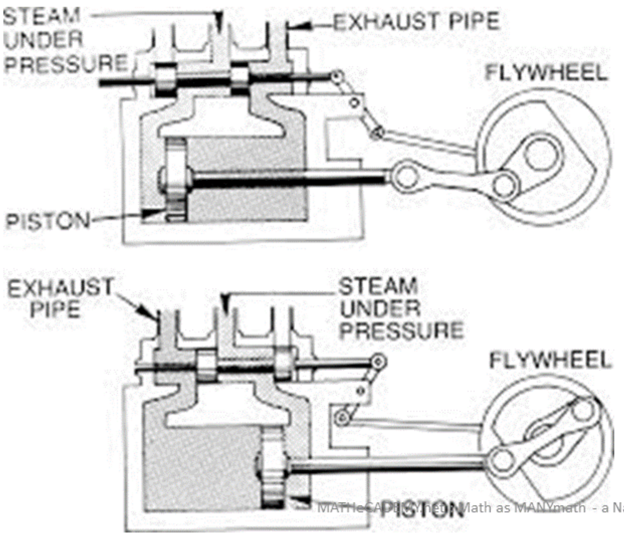
force = (**force/square-meter**) * square-meter = **pressure** * square-meter

gram = (**gram/mole**) * mole = **molar mass** * mole

mole = (**mole/liter**) * liter = **molarity** * liter

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY

Technology I
Steam at work: p*V = n*R*T



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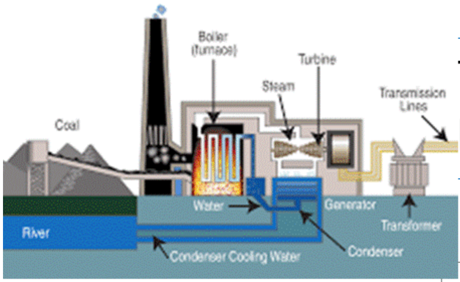


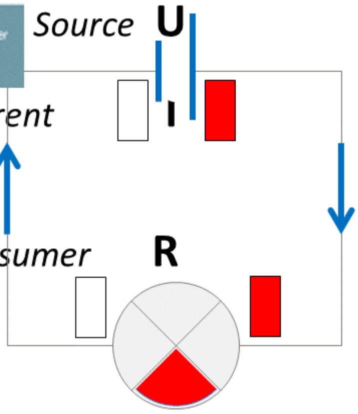
Technology II
Electrons at work: P= U*I & U = R*I

Source **U**

Current **I**

Consumer **R**

Consumption **P**



Volt = Energy/Coulomb

Ampere = Coulomb/second

Resistance in Ohm

Watt = Energy/second

Engineering
How many turns on a steep hill

On a 30 degree hillside, a 10 degree road is to be constructed.
How many turns will there be on a 1 x 1 km hillside?

- We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.
- In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2.
- In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2.
- In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).
- Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

The Simplicity of Mathematics reveals a Core Curriculum
To Master Many: ReCount in Block- & Per-numbers

Thank You for Listening

Slides & full paper on
MATH**e**CADEMY.net

Details in
Journal of Mathematics Education
vol. 11 #1

DifferenceResearch finds Differences making a Difference, inspired by

- The ancient Greek sophists:

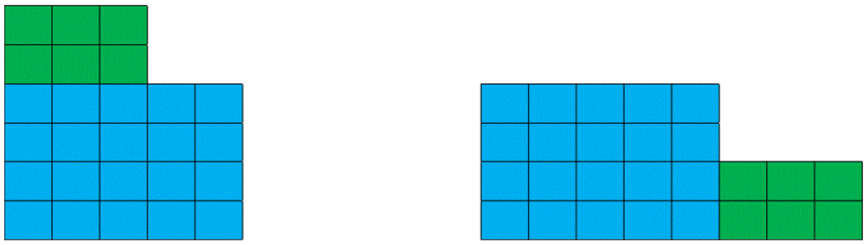
Differences unmask choice masked as nature

- In existentialism, Sartre: *EXISTENCE precedes ESSENCE.* Heidegger: *In sentences, the SUBJECT exists, but the PREDICATE is essence that often can be different.*

Let's meet the subject, **MANY**, directly &
outside its 'essence-prison'

Totals Add OnTop & NextTo

OnTop	NextTo
4 5s + 2 3s = 4 5s + 1B1 5s = 5B1 5s	4 5s + 2 3s = 3B2 8s
The units are changed to be the same *Change unit = Proportionality*	The areas are added *Adding areas = Integration*



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PerNumbers & Fractions add as **Integral Calculus**

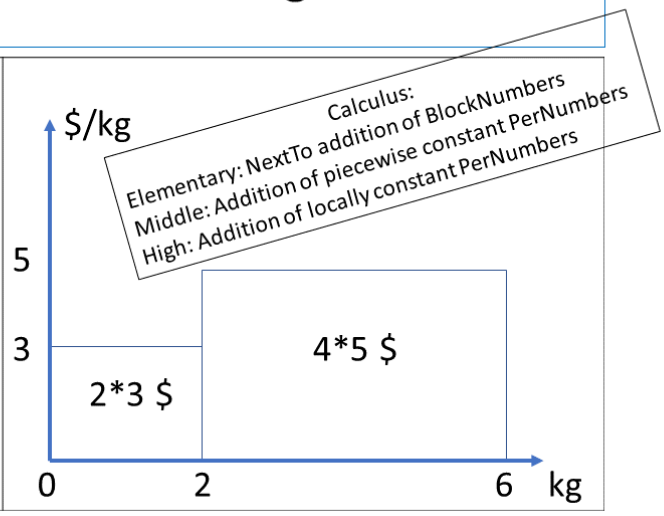
2 kg at 3 $/kg

+ 4 kg at 5 $/kg

(2+4) kg at ? $/kg

Unit-numbers add on-top.

Per-numbers add next-to as **areas** under the per-number graph, i.e. as **integral calculus**



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27. Addition-Free Migrant-Math Rooted in STEM Re-Counting Formulas

STEM typically contains multiplication formulas expressing re-counting in different units, thus calling for an addition-free curriculum. The mastery of Many children bring to school uncovers a Many-based 'Many-matics' as an alternative to the present self-referring set-based mathe-matics. To answer the question 'How many in total?' we count and re-count totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a re-count formula as a core formula in all STEM subjects.

Decreased PISA performance despite increased research

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish Centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD to write the report "Improving Schools in Sweden" (2015a) describing its school system as "in need of urgent change".

To find an unorthodox solution we pretend that a university in southern Sweden, challenged by numerous young male migrants, arranges a curriculum architect competition: "Theorize the low success of 50 years of mathematics education research; and derive from this theory a STEM based core curriculum allowing young migrants to return as STEM pre-teachers and pre-engineers."

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to help migrants and how to improve schools in Sweden and elsewhere.

Social theory looking at mathematics education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16).

As a institutions, of which mathematics education is an example, he talks about of rational action "in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)". He then points out that "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement (p. 84)."

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since such a goal statement is meaningless by its self-reference. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'. Here the invention of the concept Set created a Set-based 'meta-matics', self-referential defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. But, then Russell looked at the set of sets not belonging to itself. Here a set belongs only if it does not: if $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. Thus pointing out that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false.

In this way, Set changed grounded classical mathe-matics into today’s self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside classrooms where adding numbers without units as ‘2 + 3 IS 5’ meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to ‘2*3 = 6’ stating that 2 3s can always be re-counted as 6 1s.

Difference research looks at mathematics education

Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, ‘Difference-research’ (Tarp, 2017) is searching for hidden differences making a difference. An additional inspiration comes from existentialist philosophy described by Sartre (2007, p. 20) as holding that “Existences precedes essence”. So, to avoid a goal displacement in math education, difference-research asks: How will math look like if grounded in its outside root, Many?

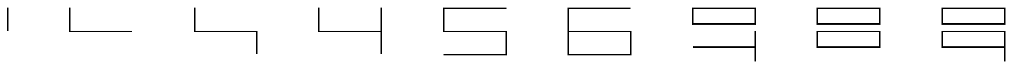
To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a mathematics core curriculum based upon examples of Many in a STEM context (Lawrenz et al, 2017). So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many creates a ‘count-before-adding’ curriculum

Meeting Many, we ask “How many in Total?” To answer, we total by counting to create number-language sentences as e.g. T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence; and connecting the outside total T with its inside predicate 2 3s (Tarp, 2018b).

Rearranging many 1s into one symbol with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII



1	2	3	4	5	6	7	8	9

Holding 4 fingers together 2 by 2, a 3year-old will say ‘That is not 4, that is 2 2s’, thus describing what exists, bundles of 2s and 2 of them. This inspires ‘bundle-counting’, re-counting a total in icon-bundles to be stacked as bundle- or block-numbers, which can be re-counted and double-counted before being processed by on-top and next-to addition, direct or reversed. Thus, a total T of 5 1s is re-counted in 2s as T = 2 2s & 1; described by ‘bundle-writing’, T = 2B1 2s; or by ‘decimal-writing’, T = 2.1 2s, where, with a bundle-cup, a decimal point separates the bundles inside from the outside unbundled singles; or by ‘deficit-writing’, T = 3B-1 2s = 3.-1 2s = 3 bundles less 1 2s.

So, to count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator thus predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: entering ‘5/2’ on a calculator gives ‘2.some’, and ‘5 – 2x2’ gives ‘1’, so T = 7 = 2B1 3s. The unbundled can be placed next-to or on-top the stack thus rooting decimals, fractions and negative numbers.

The re-count formula occurs all over. With proportionality: y = c*x; in trigonometry as sine, cosine and tangent: a = (a/c)*c = sinA*c and b = (b/c)*c = cosA*c and a = (a/b)*b = tanA*b; in coordinate geometry as line gradients: Δy = Δy/Δx = c* Δx; and in calculus as the derivative, dy = (dy/dx)*dx = y’*dx. In economics, the re-count formula is a price formula: $ = ($/kg)*kg, $ = ($/day)*day, etc.

Re-counting in the same unit or in a different unit
Once counted, totals can be re-counted in the same unit, or in a different unit. Re-counting in the same unit, changing a bundle to singles allows re-counting a total of 2B1 2s as 1B3 2s with an outside ‘overload’; or as 3B-1 2s with an outside ‘underload’ thus rooting negative numbers. This eases division: 336 = 33B6 = 28B56, so 336/7 = 4B8 = 48; or 336 = 35B-14, so 336/7 = 5B-2 = 48. Re-counting in a different unit means changing unit, also called proportionality. Asking ‘3 4s is how many 5s?’, sticks show that 3 4s becomes 2B2 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’ The answer, ‘2.some’, predicts that the singles come from taking away 2 5s, now asking ‘3*4 – 2*5’. The answer, ‘2’, predicts that 3 4s can be re-counted in 5s as 2B2 5s or 2.2 5s.

Re-counting to and from tens
Asking ‘3 4s = ? tens’ is called times tables to be learned by heart. Using sticks to de-bundle and re-bundle shows that 3 4s is 1.2 tens. Using the re-count formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right. Re-counting from tens to icons by asking ‘35 = ? 7s’ is called an equation x*7 = 35. It is easily solved by re-counting 35 in 7s: x*7 = 35 = (35/7)*7. So x = 35/7, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-counting creates proportionality as per-numbers
Counting a quantity in 2 different physical units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘T = 6$ = ?kg’, we re-count 6 in the per-number 2s: 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

A short curriculum in addition-free mathematics

01. To stress the importance of bundling, the counting sequence can be: 01, 02, …, 09, 10, 11 etc.; or 01, 02, 03, 04, 05, Ten less 4, T-3, T-2, T-1, Ten, Ten and 1, T and 2, etc.

02. Ten fingers can be counted also as 13 7s, 20 5s, 22 4s, 31 3s, 101 3s, 5 2s, and 1010 2s.

03. A Total of five fingers can be re-counted in three ways (standard and with over- and underload): T = 2B1 5s = 1B3 5s = 3B-1 5s = 3 bundles less 1 5s.

04. Multiplication tables can be formulated as re-counting from icon-bundles to tens and use underload counting after 5: T = 4*7 = 4 7s = 4*(ten less 3) = 40 less 12 = 30 less 2 = 28.

05. Dividing by 7 can be formulated as re-counting from tens to 7s and use overload counting: T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

06. Solving proportional equations as 3*x = 12 can be formulated as re-counting from tens to 3s: 3*x = 12 = (12/3)*3 giving x = 12/3 illustrating the relevance of the ‘opposite side & sign’ method.

07. Proportional tasks can be done by re-counting in the per-number: With 3$/4kg, 20kg = (20/4)*4kg = (20/4)*3$ = 15$; and 18$ = (18/3)*3$ = (18/3)*4kg = 24 kg.

08. Fractions and percentages are per-numbers coming from double-counting in the same unit, 2/3 = 2$/3$. So 2/3 of 60 = 2$/3$ of 60$ = (60/3)*3$ giving (60/3)*2$ = 40$

09. Trigonometry can precede plane and coordinate geometry to show how, in a box halved by its diagonal, the sides can be mutually re-counted as e.g. a = (a/c)*c = sinA*c.

10. Counting by stacking bundles into adjacent blocks leads to the number formula or bundle formula called a polynomial: T = 456 = 4*BundleBundle + 5*Bundle + 6*single = 4*B^2 + 5*B + 6*1. In its general form, the number formula T = a*x^2 + b*x + c contains the different formulas for constant change: T = a*x (proportionality), T = a*x^2 (acceleration), T = a*x^c (elasticity) and T = a*c^x (interest rate); as well as T = a*x+b (linearity or affinity, strictly).

11. Predictable change roots pre-calculus (if constant) and calculus (if changing). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

12. Integral can precede differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers. So, both per-numbers and fractions are added with units as the area under the per-number graph.

Meeting Many in a STEM context

OECD (2015b) says: ‘In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.’ STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature’s behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

Nature as heavy things in motion

To meet we must specify place and time in a nature consisting of heavy things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature’s three main factors, matter and force and motion, like the three social factors, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water into electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to dissolving matter in water; to the trajectory of a ball pulled down by gravity; to put steam and electrons to work in a power plant creating an electrical circuit transporting energy from a source to many consumers.

Heavy things in motion are combined by the momentum = mass*velocity, a multiplication formula as most STEM formulas expressing re-counting by per-numbers: kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter, where force is the per-number change in momentum per second. Thus, STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Warming and boiling water
In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 2000 Joule per second in 167 seconds will heat 1000 gram water 80 degrees we get a double per-number 2000*167/80/1000 Joule/degree/gram or 4.18 Joule/degree/gram, called the specific heat capacity of water. Producing 100 gram steam in 113 seconds, the per-number is 2000*113/100 Joule/gram or 2260 J/g, called the heat of evaporation for water.

Dissolving material in water
In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so re-counting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 moles/liters, or 0.676 moles/liter.

Building batteries with water
At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H2O, and carbon dioxide CO2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH4, the energy comes from using 4 Os to burn it.

Technology and engineering: letting steam and electrons produce and distribute energy
A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units. Thus a mole of water weighs 18 gram. Since the density of water is roughly 1000 gram/liter, the volume of 1000 moles is 18 liters. Transformed into steam, its volume increases to more than 22.4*1000 liters, or an increase factor of 22,400 liters per 18 liters = 1244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, p*V = n*R*T, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g. steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to industries and homes.

An electrical circuit
Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220

Volts, a per-number double-counting the number of Joules per charge-unit. Re-counting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 charge-units per second, which is called the electrical current counted in Ampere. To create this current, the kettle must have a resistance R according to a circuit law Volt = Resistance*Ampere, i.e., 220 = R*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per charge-unit)*(charge-unit per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120 Watt bulb, because of proportionality the latter needs twice the current, and consequently half the resistance of the former.

How high up and how far out

An inclined gun sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,71 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin45 = cos45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

Compared with the horizontal, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

Adding addition to the curriculum

Once counted as block-numbers, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being re-counted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations where the question 2 3s + ? 4s = 5 7s leads to differentiation: ? = (5*7 – 2*3)/4 = ΔT/4. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular.

The number formula T = 456 = 4*B^2 + 5*B + 6*1 shows there are four ways to unite numbers: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers (Tarp, 2018b).

Conclusion and recommendation

This paper argues that 50 years of unsuccessful mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views lead to different kinds of mathematics: a set-based top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying "To master Many, counting and re-counting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating." A proposal for two separate twin-curricula in counting and adding is found in Tarp (2018a). Thus, the simplicity of mathematics as expressed in a 'count-before-adding' curriculum allows replacing block-numbers with line-numbers, and learning core mathematics as proportionality, calculus, equations and per-numbers in early childhood. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM pre-teachers or pre-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017). Thus, it is possible to solve STEM problems without learning addition, that is not well-defined since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

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28. A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many

Mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory, we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum.

POOR PISA PERFORMANCE DESPITE FIFTY YEARS OF RESEARCH

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in math education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 years since 1969. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to significantly below the OECD average in 2012, causing OECD (2015) to write the report 'Improving Schools in Sweden'. Likewise, math dislike seems to be widespread in high performing countries also. With mathematics and education as social institutions, grand theory may explain this 'irrelevance paradox', the apparent negative correlation between research and performance.

GRAND THEORY

Ancient Greece saw two forms of knowledge, 'sophy'. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Christianity eagerly took over metaphysical patronage and changed the academies into monasteries. The sophist skepticism was revived by Brahe and Newton, insisting that knowledge about nature comes from laboratory observations, not from library books (Russell, 1945).

Newton's discovery of a non-metaphysical changing will spurred the Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. Two republics arose, in the United States and in France. The US still has its first Republic, France its fifth, since its German-speaking neighbors tried to overthrow the French Republic again and again.

In North America, the sophist warning against hidden patronization lives on in American pragmatism and symbolic interactionism; and in Grounded Theory, the method of natural research resonating with Piaget's principles of natural learning. In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructural thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnosed cures, and education; all may hide patronizing choices presented as nature (Lyotard, 1984).

Within philosophy itself, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that 'existence precedes essence', exemplified by the Heidegger-warning: In a sentence, trust the subject, it exists; doubt the predicate, it is essence coming from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber was the first to theorize the increasing goal-oriented rationalization that de-enchant the world and create an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16). But he also formulates a warning (p. 84): "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right". Which may lead to 'the banality of evil' (Arendt, 1963).

As to what we say about the world, Foucault (1995) focuses on discourses about humans that, if labeled scientific, establish a 'truth regime'. In the first part of his work, he shows how a discourse disciplines itself by only accepting comments to already accepted comments. In the second part he shows how a discourse disciplines also its subject by locking humans up in a predicate prison of abnormalities from which they can only escape by accepting the diagnose and cure offered by the 'pastoral power' of the truth regime. Foucault thus sees a school as a 'pris-pital' mixing the power techniques of a prison and a hospital: the 'pati-mates' must return to their cell daily and accept the diagnose 'un-educated' to be cured by, of course, education as defined by the ruling truth regime.

Mathematics, stable until the arrival of SET

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught 'reckoning' in primary school and 'arithmetic' and 'geometry' in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here a wish for exactness and unity created a SET-derived 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

SET thus transformed classical grounded 'many-matics' into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meets counter-examples as e.g. 1week + 2days is 9days.

Proportionality illustrates the variety of mastery of Many and of quantitative competence

Proportionality is rooted in questions as "2kg costs 5$, what does 7kg cost; and what does 12$ buy?"

Europe used the 'Regula de Tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, the divide. So first we ask, Q1: '2kg cost 5$, 7kg cost ?$' to get to the answer (7*5/2)$ = 17.5$. Then we ask, Q2: '5$ buys 2kg, 12$ buys ?kg' to get to the answer (12*2)/5$ = 4.8kg.

Then, two new methods appeared, 'find the unit', and cross multiplication in an equation expressing like proportions or ratios:

Q1: 1kg costs 5/2$, so 7kg cost 7*(5/2) = 17.5$. Q2: 1$ buys 2/5kg, so 12$ buys 12*(2/5) = 4.8kg.
Q1: $2/5 = 7/x$, so $2*x = 7*5$, $x = (7*5)/2 = 17.5$. Q2: $2/5 = x/12$, so $5*x = 12*2$, $x = (12*2)/5 = 4.8$.

SET chose modeling with linear functions to show the relevance of abstract algebra's group theory: Let us define a linear function $f(x) = c*x$ from the set of kg-numbers to the set of \$-numbers, having as domain $DM = \{x \in R \mid x>0\}$. Knowing that $f(2) = 5$, we set up the equation $f(2) = c*2 = 5$ to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: $c*2 = 5$, $(c*2)*½ = 5*½$, $c*(2*½) = 5/2$, $c*1 = 5/2$, $c = 5/2$. With $f(x) = 5/2*x$, the inverse function is $f\text{-}1(x) = 2/5*x$. So with 7kg, $f(7) = 5/2*7 = 17.5$\$; an with 12\$, $f\text{-}1(12) = 2/5*12 = 4.8$kg.

In the future, we simply 're-count' in the 'per-number' 2kg/5$ coming from 'double-counting' the total *T*. Q1: *T* = 7kg = (7/2)*2kg = (7/2)*5$ = 17.5$; Q2: *T* = 12$ = (12/5)*5$ = (12/5)*2kg = 4.8kg.

Grand theory looks at mathematics education

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologically, how we perceive and verbalize it. University mathematics holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB+4*B+5*1, resonating with what children bring to school, e.g. T = 2 5s.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? University mathematics holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian psychology.

Sociologically, a Weberian viewpoint would ask if SET is a rationalization of Many gone too far leaving Many de-enchanted and the learners in an iron cage. A Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, contemporary university mathematics has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word 'mathematics' must go from goal descriptions. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a pris-pital, a Foucaultian viewpoint, would ask, first which structure to choose, European line-organization forcing a return to the same cell after each hour, day and month for several years; or the North American block-organization changing cell each hour, and changing the daily schedule twice a year? Next, as prisoners of a 'the goal of math education is to learn math' discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g. by examining and developing the existing mastery children bring to school?

Meeting Many, children bundle in block-numbers to count and share

How to master Many can be learned from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists, and with units: bundles of 2s, and 2 of them.

Children also use block-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 extra'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 4s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as $T = 7 = 1B5 = 2B3 = 3B1 = 4B\cancel{1}$; or even as 1*BB*3 or 1*BB*1*B*1. Also, children love to count in 3s, 4s, and in hands.

Sharing 9 cakes, 4 children take one by turn saying they take 1 of each 4. Taking away 4s roots division as counting in 4s; and with 1 left they often say "let's count it as 4". Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. And they smile when seeing that entering '9/4' allows a calculator to predict the sharing result as 2 1/4; and when seeing that entering '2*5/3' will predict the result of sharing 2 5s between 3 children.

Children thus master sharing, taking parts and splitting into parts before division and counting- and splitting-fractions is taught; which they may like to learn before being forced to add without units.

So why not develop instead of rejecting the core mastery of Many that children bring to school?

A typical contemporary mathematics curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, $T = 2$ 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1*B*14 tens is not allowed. Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

A QUESTION GUIDED COUNTING CURRICULUM

The question guided re-enchantment curriculum in counting could be named 'Mastering Many by counting, recounting and double-counting'. The design is inspired by Tarp (2018). It accepts that while eight competencies might be needed to learn university mathematics (Niss, 2003), only two are needed to master Many (Tarp, 2002), counting and uniting, motivating a twin curriculum. The corresponding pre-service or in-service teacher education can be found at the MATHeCADEMY.net. Remedial curricula for classes stuck in contemporary mathematics can be found in Tarp (2017).

Q01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

Q02, counting sequences: "How to count fingers?" Here the learning opportunity is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H". Follow-up activities could be counting things.

Q03, icon-counting: "How to count fingers by bundling?" Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported in a number-language sentence with subject, verb and predicate: $T = 5 =$ 1Bundle3 2s = 2*B*1 2s = 3*B*-1 2s = 1*BB*1 2s, called an 'inside bundle-number' describing the 'outside block-number'. A western abacus shows this in 'outside geometry space-mode' with the 2 2s on the second and third bar and 1 on the first bar; or in 'inside algebra time-mode' with 2 on the second bar and 1 on the first bar. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, calculator-prediction: "How can a calculator predict a counting result?" Here the learning opportunity is to see the division sign as an icon for a broom wiping away bundles: 5/2 means 'from 5, wipe away bundles of 2s'. The calculator says '2.some', thus predicting it can be done 2 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes the trace left when dragging away the block to look for unbundled singles. By showing '5-2*2 = 1' the calculator indirectly predicts that a total of 5 can be recounted as 2B1 2s. An additional learning opportunity is to write and use the 'recount-formula' $T = (T/B)*B$ saying "From *T*, *T*/*B* times *B* can be taken away." This proportionality formula occurs all over mathematics and science. Follow-up activities could be counting cents: 7 2s is how many fives and tens? 8 5s is how many tens?

Q05, unbundled as decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the learning opportunity is to see that with blocks, the unbundled occur in three ways. Next-to the block as a block of its own, written as $T = 7 =$ 2.1 3s, where a decimal point separates the bundles from the singles. Or on-top as a part of the bundle, written as $T = 7 =$ 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q06, prime or foldable units: “Which blocks can be folded?” Here the learning opportunity is to examine the stability of a block. The block T = 2 4s = 2*4 has 4 as the unit. Turning over gives T = 4*2, now with 2 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a ‘prime unit’ and 4 a ‘foldable unit’, 4 = 2 2s. So, a block of 3 2s cannot be folded, whereas a block of 3 4s can: T = 3 4s = 3 * (2*2) = (3*2) * 2. A number is called even if it can be written with 2 as the unit, else odd.

Q07, finding units: “What are possible units in T = 12?”. Here the learning opportunity is that units come from factorizing in prime units, T = 12 = 2*2*3. Follow-up activities could be other examples.

Q08, recounting in another unit: “How to change a unit?” Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

Q09, recounting from tens to icons: “How to change unit from tens to icons?” Here the learning opportunity is that asking ‘T = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s as 24 = (24/8)*8 so that the unknown number is u = 24/8 attained by moving 8 to the opposite side with the opposite sign. Follow-up activities could be other examples of recounting from tens to icons.

Q10, recounting from icons to tens: “How to change unit from icons to tens?” Here the learning opportunity is that if asking ‘T = 3 7s = ? tens’, the recount-formula cannot be used since the calculator has no ten-button. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Follow-up activities could be other examples of recounting from icons to tens.

Q11, double-counting in two units: “How to double-count in two different units?” Here the learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To answer questions we just recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa, asking ‘?$ = 12kg’, the answer is T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Follow-up activities could be numerous other examples of double-counting in two different units since per-numbers and proportionality are core concepts.

Q12, double-counting in the same unit: “How to double-count in the same unit?” Here the learning opportunity is that when double-counted in the same unit, per-numbers take the form of fractions, 3$ per 5$ = 3/5; or percentages, 3 per hundred = 3/100 = 3%. Thus, to find a fraction or a percentage of a total, again we just recount in the per-number. Also, we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Follow-up activities could be other examples of double-counting in the same unit since fractions and percentages are core concepts.

Q13, recounting the sides in a block. “How to recount the sides of a block halved by its diagonal?” Here, in a block with base b, height a, and diagonal c, mutual recounting creates the trigonometric per-numbers: $a = (a/c)*c = \sin A*c$; $b = (b/c)*c = \cos A*c$; $a = (a/b)*b = \tan A*b$. Thus, rotating a line can be described by a per-number a/b, or as tan$A$ per 1, allowing angles to be found from per-numbers. Follow-up activities could be other blocks e.g. from a folding ruler.

Q14, double-counting in STEM (Science, Technology, Engineering, Math) multiplication formulas with per-numbers coming from double-counting. Examples: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter) * square-meter = pressure * square-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg) * kg = heat * kg; gram = (gram/mole) * mole = molar mass * mole; Δ momentum = (Δ momentum/sec) * sec = force * sec; Δ energy = (Δ energy/ meter) * meter = force * meter = work; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp; dollar = (dollar/hour)*hour = wage*hour.

Q15, navigating. “Avoid the rocks on a squared paper”. Four rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock? Predict and measure the angles on the journey.

A QUESTION GUIDED UNITING CURRICULUM

The question guided re-enchantment curriculum in uniting could be named ‘Mastering Many by uniting and splitting constant and changing unit-numbers and per-numbers’.

A general bundle-formula $T = a*x\text{^}2 + b*x + c$ is called a polynomial. It shows the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the algebra square’.

Q21, next-to addition: “With $T1$ = 2 3s and $T2$ = 4 5s, what is $T1$+$T2$ when added next-to as 8s?” Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus. Follow-up activities could be other examples of next-to addition.

Q22, reversed next-to addition: “If $T1$ = 2 3s and $T2$ add next-to as T = 4 7s, what is $T2$?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Follow-up activities could be other examples of reversed next-to addition.

Q23, on-top addition: “With $T1$ = 2 3s and $T2$ = 4 5s, what is $T1$+$T2$ when added on-top as 3s; and as 5s?” Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Follow-up activities could be other examples of on-top addition.

Q24, reversed on-top addition: “If $T1$ = 2 3s and $T2$ as some 5s add to T = 4 5s, what is $T2$?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again is called differential calculus. An underload is removed by recounting. Follow-up activities could be other examples of reversed on-top addition.

Q25, adding tens: “With $T1$ = 23 and $T2$ = 48, what is $T1$+$T2$ when added as tens?” Again, recounting removes an overload: $T1$+$T2$ = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71; or T = 236 + 487 = 2BB3B6 + 4BB8B7 = 6BB11B13 = 6BB12B3 = 7BB2B3 = 723.

Q26, subtracting tens: “If $T1$ = 23 and $T2$ add to T = 71, what is $T2$?” Again, recounting removes an underload: $T2$ = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or $T2$ = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, showing that -(-1) = +1.

Q27, multiplying tens: “What is 7 43s recounted in tens?” Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Q28, dividing tens: “What is 348 recounted in 6s?” Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 3BB4B8 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58.

Q29, adding per-numbers: “2kg of 3\$/kg + 4kg of 5\$/kg = 6kg of what?” Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform into unit-number by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining constancy by epsilon and delta.

Q30, subtracting per-numbers: "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a 'locally constant' situation, subtracting per-numbers is called differential calculus.

Q31, finding common units: "Only add with like units, so how to add *T* = 4ab^2 + 6abc?". Here units come from factorizing: *T* = 2*2**a***b***b* + 2*3**a***b***c* = 2**b**(2**a***b*) + 3**c**(2**a***b*) = 2*b*+3*c* 2*ab*s.

CONCLUSION

A curriculum wants to develop brains, and colonizing wants to develop countries. De-colonizing accepts that maybe countries and brains can develop themselves if helped by options instead of directions from developed countries and brains. Some prefer a direction-giving multi-year macro-curriculum; others prefer option-giving half-year micro-curricula. Some prefer a curriculum to be a cure prescribing mathematics competencies and literacy; others prefer developing the existing quantitative competence and numeracy, defined by dictionaries as the ability to use numbers and operations in everyday life, thus silencing the word 'mathematics' to avoid a hidden continuing colonization. In the transition period between colonizing and decolonizing brains, grand theory has an advice to the 'irrelevance paradox' of mathematics education research: accept the brain's own epistemology to avoid a goal displacement blocking the road to its educational goal, mastery of Many.

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29. Counting Before Adding, a PPP for the Article on a Twin Curriculum

COUNTING before ADDING

The Child's Own Twin Curriculum

Count & ReCount & DoubleCount

before Adding NextTo & OnTop

master many with manymath

Allan.Tarp@MATHeCADEMY.net, November 2018

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Background: Our two language houses

The **WORD language** assigns words in sentences with The **NUMBER language** assigns numbers instead with	• a subject • a verb • a predicate

Both languages have a META-language, a grammar, describing the language, that is learned before the grammar. But does mathematics respect teaching language before grammar?

	WORD language	NUMBER language
META-language, grammar	*'is' is a verb*	*'x' is an operation*
Language	This is a chair Mother tongue	T = 3x4 Mathematics

WORLD

MATHeCADEMY.net : Math as MANYmath - a Natural Science about MANY 2

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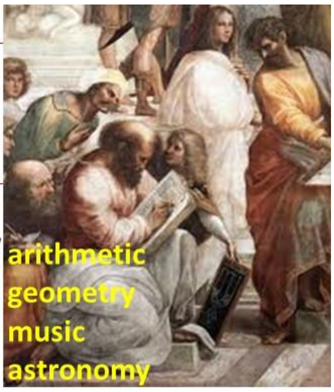
How well defined is mathematics?



In ancient Greece, Pythagoras used mathematics, meaning knowledge, as a common label for four descriptions of Many by itself & in space & time:

Together they formed the '**quadrivium**' recommended by Plato as a general curriculum after the '**trivium**' consisting of grammar & logic & rhetoric.
Geometry & algebra are both grounded in Many as shown by names:
Geometry means to measure earth in Greek
Algebra means to reunite numbers in Arabic

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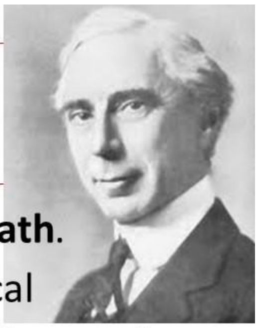
Modern mathematics, MetaMath



Around 1900, **SET** made math a self-referring **MetaMath**.
But Russell saw that self-reference leads to the classical liar paradox 'this sentence is false', being false if true & opposite:
"Let M be the set of sets not belonging to itself, $M = \{A \mid A \notin A\}$.
Then $M \in M \Leftrightarrow M \notin M$. Forget about sets. Use type theory instead.
So, by self-reference, fractions cannot be numbers."
Mathematics: "Forget about Russell, he is not a mathematician.
We just institutionalize fractions as so-called rational numbers."

MATHeCADEMY

Institutions as thorns protecting Sleeping Beauties

- Weber on institutionalization: Rationalized too far, mathematics may become an **iron cage** that **disenchants** its subject.
- Bauman on self-reference: „The ideal model of action subjected to rationality as the supreme criterion contains a danger of so-called **goal displacement**. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right“.
- Arendt: Just following orders may lead to '**the banality of evil**'.
- Sartre on existentialism: „**Existence precedes essence**“.

MATHeCADEMY

Three curriculum choices to make

What is the goal of mathematics education?

1. To learn institutionalized mathematics, or
2. To learn to master what exists, Many

What is the core means?

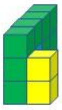
1. To learn about numbers without units, or
2. To learn how to number with units

1, -2, 3/4, √5, π, e

T = 3**B**2 = 3.2 = 4.-2 **4s**

What are numbers?

1. One-dimensional line-numbers without units, or
2. Two-dimensional block-numbers with units



Choosing 1 may have caused 50 years of less successful math education research.

MATHeCADEMY

Different curricula: MetaMath or ManyMath

What is the goal of mathematics education?

1. To learn mathematics (self reference pointing up, Vygotsky theory)
2. To learn to Master Many (external reference pointing down, Piaget theory)

What is a core means?

1. To learn about numbers (operations on specified and unspecified numbers)
2. To learn how to number (number-language sentences about counting & adding totals)

What are numbers?

1. 1D line-numbers (integer, natural, rational, real, place value system)
2. 2D bundle-numbers (constant & changing unit-numbers & per-numbers)

MATHeCADEMY

Why teach children if they already know?

With education curing un-educatedness, we ask:

To CURE, be SURE

1. The diagnosed is not already cured
2. The diagnose is not self-referring: *teach math to learn math*

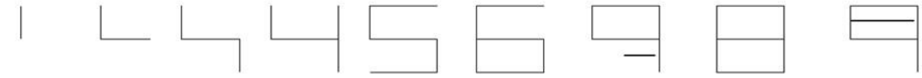
Core Questions:

- What Mastery of Many does the child have already?
- What could be a <u>ChildCenteredCurriculum</u> in Quantitative Competence?

Creating icons: I I I I → IIII → 4 →

Children love making number-icons of cars, dolls, spoons, sticks.
Counting in ones means naming the different degrees of Many.
Changing four ones to one fours creates a 4-icon with four sticks.
An icon contains as many sticks as it represents, if written less sloppy.
Once created, icons become units to use when counting in bundles.

one	two	three	four	five	six	seven	eight	nine
I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII

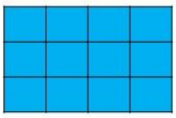


1	2	3	4	5	6	7	8	9

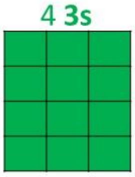
Children see Many as bundles with units

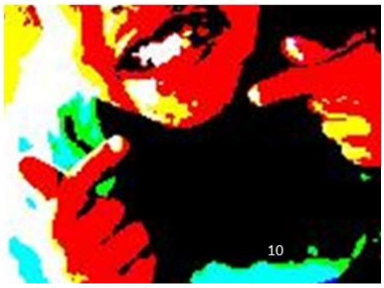
- “How old next time?” A 3year old says “Four” showing 4 fingers: I I I I
- But, the child reacts strongly to 4 fingers held together 2 by 2: II II
- “That is not four, that is two twos”
- The child describes what exists, and with units: bundles of 2s, and 2 of them
- The block 3 **4s** has two numbers: 3 (the counting-number) and **4** (the unit-number)
- Children also use bundle-numbers with Lego blocks:

3 **4s**



4 **3s**





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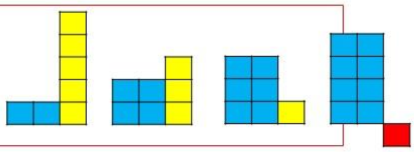
To count Many, children bundle 

- Children are flexible when re-counting a Total of 7 sticks in **2s**:

I I I I I I I ● ~~I I~~ I I I I I ● ~~I I~~ ~~I I~~ I I I ● ~~I I~~ ~~I I~~ ~~I I~~ I ● ~~I I~~ ~~I I~~ ~~I I~~ ~~I I~~

T = 7 = 1 **2s** &5 = 2 **2s** &3 = 3 **2s** &1 = 4 **2s** less 1

- And children don't mind writing a total of 7 using 'bundle-writing':

T = 7 = 1B5 = 2B3 = 3B1 = 4B ~~1~~, or even as

T = 7 = 1BB3 = 1BB1B1 = 2BB ~~1~~

- Also, children love to count in **3s**, **4s**, and in **hands**:

Thus, a number is a multi-counting of bundles as units *(…, bundles-of-bundles, bundles, unbundled)*

IIII I I
T = 7 = 1 **5s** & 2
T = 7 = 1B2 **5s**

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Counting bundles gives a number formula

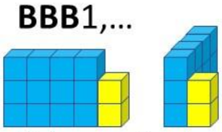
Children have fun when counting bundles, bundles of bundles, etc.:

With ten-bundles: 01, 02, …, 09, **Bundle**,

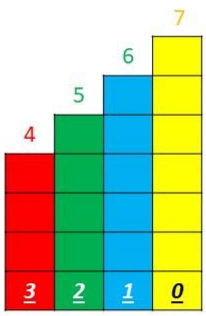
B1, **B**2,…, 9**B**8, 9**B**9, **BundleBundle**,

BB1,…, 2**BB**3**B**4,…, 9**BB**9**B**9, **BundleBundleBundle**,

BBB1,…

 With blocks turned to hide the units behind:

B is marked with 1, **BB** with 2, **BBB** with 3, etc., singles with 0.



Later, this is a number formula T = 4567 = 4**BBB**5**BB**6**B**7 = 4x**B**^3 + 5x**B**^2 + 6x**B** + 7

Counting ten fingers & counting in tens

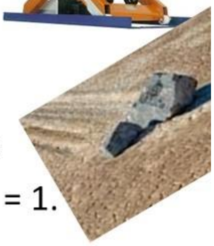
Children have fun when flexibly counting ten fingers in different ways:

- The Roman way: 01, 02, 03, HandLess1, **HAND**, Hand1, H2, H3, 2H-1, 2H, 2H1, 2H2
- The Viking way: 01, 02, 03, 04, HALF, 06, 07, less2, less1, **FULL**, 1left, 2left
- The modern way: 01, 02,…, 09, **ten**, ten1, ten2,…, 9ten8, 9ten9, **tenten**, tenten1,…, 2tenten3ten4,…, 9tenten9ten9, **tententen**, tententen1,…

Division, multiplication & subtraction as icons also

'From 9 take away **4s**' we write <u>9/4</u>
<u>iconizing</u> the sweeping away by a broom, called division.

'2 times stack **4s**' we write <u>2x4</u>
<u>iconizing</u> the stacking up by a lift called multiplication.



'From 9 take away 2 **4s**' to look for un-bundled we write <u>9 – 2x4</u>
<u>iconizing</u> the dragging away by a trace called subtraction.

So counting includes division and multiplication and subtraction:

Finding the bundles: 9 = 9/4 **4s**. Finding the un-bundled: 9 – 2x4 = 1.

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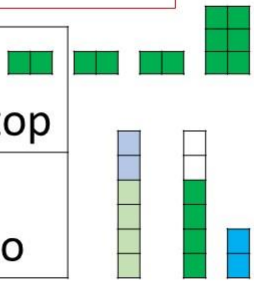
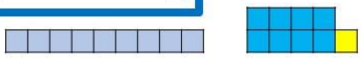
Counting creates two counting formulas

ReCount **T = (T/B) x B**	from a total **T**, **T/B** times, **Bs** is taken away and stacked on-top
ReStack **T = (T–B) + B**	from a stack **T**, **T–B** is left when **B** is taken away and placed next-to



With formulas, a calculator can **predict** the counting-result 9 = 2**B**1 **4s**

9/4	**2.some**
9 – 2x4	**1**



As sentences of the number language, ***formulas predict***

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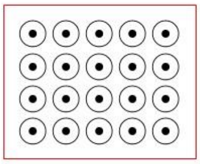
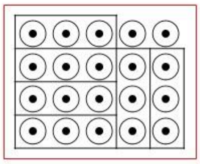
To share Many, children take away bundles predicted by division, multiplication and subtraction

2 preschoolers share 6 cakes by taking away **2s** from 6, thus rooting division as counting in **2s**. ⊙⊙⊙⊙⊙⊙ → ⊙⊙⊙⊙ ⊙⊙ → ⊙⊙ ⊙⊙ ⊙⊙

- They smile when seeing that entering '6/2' allows a calculator to predict that they can take cakes 3 times.
- And when seeing that '4x5/3' predicts that 3 children can take cakes 6 times (or 6 cakes 1 time) when sharing 4 rows of 5 cakes.
- And when seeing that '4x5-6x3' predicts that 2 will be left.

6/2	3
4x5/3	6.some
4x5-6x3	2





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Question Guided Counting Curriculum

A question guided re-enchanting COUNTING curriculum could be named <u>Mastering Many by counting, re-counting & double-counting</u>.

- The design accepts that while 8 competences might be needed to learn university mathematics, only 2 are needed to Master Many: COUNTING & ADDING, motivating a twin curriculum.
- The corresponding pre-service or in-service question guided teacher education can be found at the MATHeCADEMY.net.
- Remedial micro-curricula for classes stuck in traditional mathematics can be found there also.

MATHeCADEMY

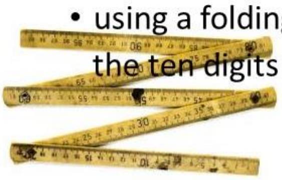
Q01, icon making IIII → IIII → 4

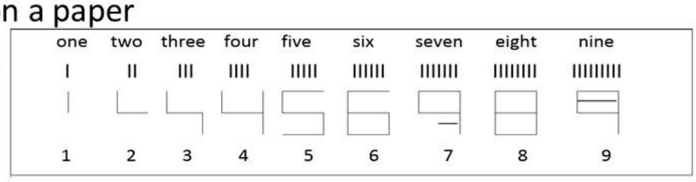
"The digit-icon 4 seems to be have four sticks. Does this apply to all digit-icons?"

We can change <u>many ones</u> to <u>one icon</u> with as many sticks or strokes as it represents, if written in a less sloppy way.

Follow-up activities could be:

- rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.
- rearranging sticks on a table or on a paper
- using a folding ruler to construct the ten digits as icons





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Q02, counting sequences I

"How to count fingers?"

- Using **5s** as the bundle-size, five fingers can be counted as
  "01, 02, 03, 04, **Bundle**"
- And ten fingers can be counted as
  "01, 02, **Bundle** less2, **Bundle** -1, **Bundle**"
  "**Bundle**&1, **B**&2, 2**B** less2, 2**B**-1, 2**B**".

Follow-up activities could be counting the fingers in **3s** and **4s** and **7s**:

T = ten = 1**B**3 **7s** = 2**B**2 **4s** = 3**B**1 **3s** = 1**BB**1 **3s**.

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Q02, counting sequences II

Counted as 1**B**, the bundle-number needs no icon. So counting a dozen cakes we say:

in	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙	⊙
4s	01	02	03	**B**	1**B**1	1**B**2	1**B**3	2**B**	2**B**1	2**B**2	2**B**3	3**B**
7s	01	02	03	04	05	06	**B**	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5
tens	01	02	03	04	05	06	07	08	09	**B**	1**B**1	1**B**2

The number names, eleven and twelve come from 'one left' and 'two left' in Danish, (en / twe levnet), again showing that counting takes place by taking away bundles.

Q03, bundle-counting in icon-units I

"How to count by bundling?"

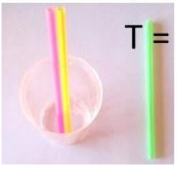
Five fingers can be bundle-counted in pairs or triplets, allowing both an overload and an underload; and reported in a number-language sentence with a subject & a verb & a predicate as e.g. T = 2 **3s.**

IIIII • ǂ III • ǂ ǂ I • ǂ ǂ ǂ • ǂ ǂ I

T = 5 = 1Bundle3 **2s** = 2B1 **2s** = 3B-1 **2s** = 1BB1 **2s**

- Cup- & decimal-writing separates inside bundles from outside singles:

T = 5 = 1]3 **2s** = 2]1 **2s** = 3]-1 **2s** = 1]]0]1 **2s**



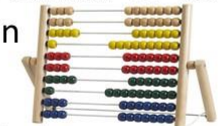
T = 5 = 1.3 **2s** = 2.1 **2s** = 3.-1 **2s** = 10.1 **2s**

Likewise, if counting in ten-bundles: T = 57 = 5**B**7 = 4**B**17 = 6**B**-3 **tens**

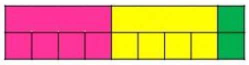
Q03, bundle-counting in icon-units II

To count 9 in **4s**, we may bundle in a cup with 1 stick per bundle.
9 = IIIIIIIII = ~~IIII~~ ~~IIII~~ I = **II]**I = 2]1 **4s** = 2B1 **4s** = 2.1 **4s**

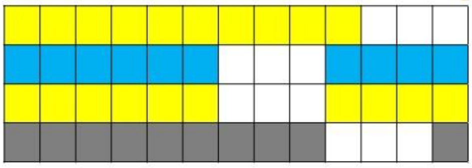
We may report with cup-, bundle- or decimal-writing,
or on a western **ABACUS** in

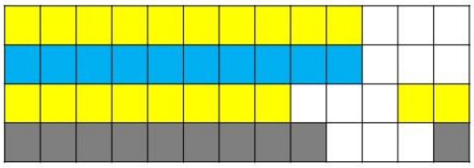


Lego blocks
or CentiCubes

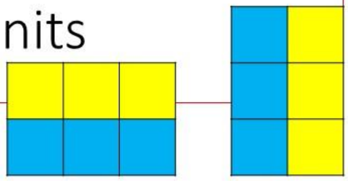


Outside geometry mode or Inside algebra mode

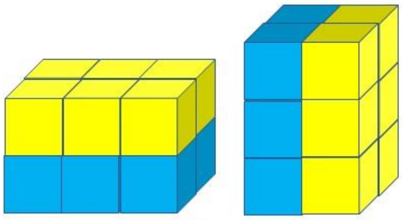




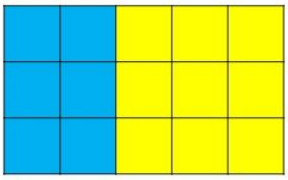
Switching & uniting & splitting units



- Turning a 2D block will change the unit
  T = 2 **3s** = 2x3 → T = 3 **2s** = 3x2,
  So T = 2x3 = 3x2 *(The Commutative law)*



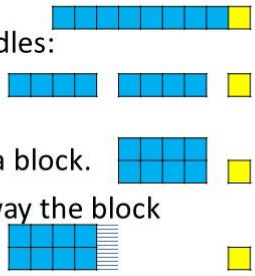
- Turning a 3D block will also change the unit
  So T = 2x(2x3) = (2x2)x3 *(The Associative law)*



- A block may split into two parts
  T = 3 **5s** = 3 **2s** + 3 **3s** or
  So T = 3x5 = 3x(2+3) = 3x2 + 3x3 *(The Distributive law)*

Q04, calculators predict

"Can a calculator predict a counting result?"



We may see the <u>division</u> sign as an icon for a broom wiping away bundles:
9/4 means 'from 9, wipe away bundles of 4s'.

- The calculator says '2.some', thus predicting it can be done 2 times.
  Now the <u>multiplication</u> sign iconizes a lift stacking the bundles into a block.
- Finally, the <u>subtraction</u> sign iconizes the trace left when dragging away the block to look for unbundled singles.
- With '9-2x4 = 1' the calculator predicts that 9 can be recounted as 2**B**1 **4s**.

9/4	**2.some**
9 – 2x4	**1**

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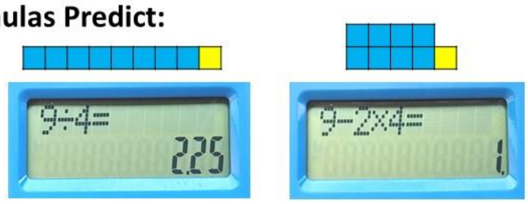
Q04, counting creates 2 counting formulas

ReCount **T = (T/B) x B**	from a total **T**, **T/B** times, **Bs** is taken away and stacked
ReStack **T = (T–B) + B**	from a total **T**, **T–B** is left, when **B** is taken away and placed next-to

As sentences of the number language, **Formulas Predict:**
Predicting that **T** = 9 = 2.1 **4s:**

9/4	**2.some**
9 – 2x4	**1**



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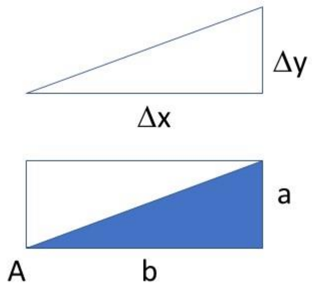
Q04, the recounting formula is a core formula

T = (T/B)*B saying 'from T, T/B times, Bs can be taken away', is all over:

Proportionality	y = k * x
Linearity	Δy = (**Δy/Δx**) * Δx = **m** * Δx
Local linearity	dy = (**dy/dx**) * dx = **y'** * dx
Trigonometry	a = (**a/b**) * b = **tanA** * b
Trade	$ = (**$/kg**) * kg = **price** * kg
Science	meter = (**meter/second**) * second = **velocity** * second



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Q05, unbundled as decimals or negatives or fractions

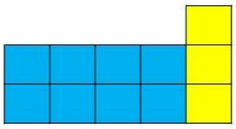
0.3 **4s** or 0.-1 **4s** or 3/4 **4s**

"Where to put the unbundled singles?"

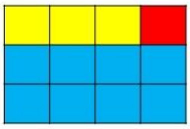
When counting by bundling, the unbundled singles can be placed

NextTo the block — counted as a block of **1s**

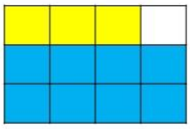
OnTop of the block — counted as a bundle / counted in bundles



T = 2B3 **4s** = 2.3 **4s**
A decimal number



T = 3B-1 **4s** = 3.-1 **4s**
A negative number



T = 2 3/4 **4s**
A fraction

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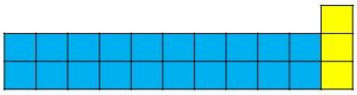
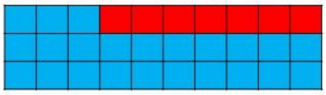
Q05, counting in tens

"Where to put the unbundled singles with tens?"

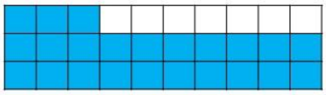
Counting in tens, an outside Total of 2 **tens** & 3 can be described inside as T = 23 if leaving out the unit, or as



T = 2.3 **tens**



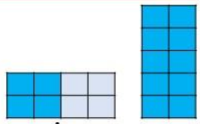
T = 3.-7 **tens**



T = 2 3/10 **tens**

Q06, prime & foldable bundle-units

"When can blocks be folded in like bundles?"



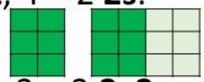
The block T = 2 **4s** = 2 x 4 has 4 as the bundle-unit.

Turning over gives T = 4 **2s** = 4 x 2, now with 2 as the bundle-unit.

4s can be folded in another bundle as 2 **2s**, whereas 2s cannot.

(1 is not a bundle, nor a unit since a bundle-of-bundles stays as 1).

We call 2 a **prime bundle-unit** and 4 a **foldable bundle-unit**, 4 = 2 **2s**.



A block of 3 **2s** cannot be folded.

A block of 3 **4s** can be folded: T = 3 **4s** = 3 x (2 x 2) = (3 x 2) x 2 = 2 **3x2s**.

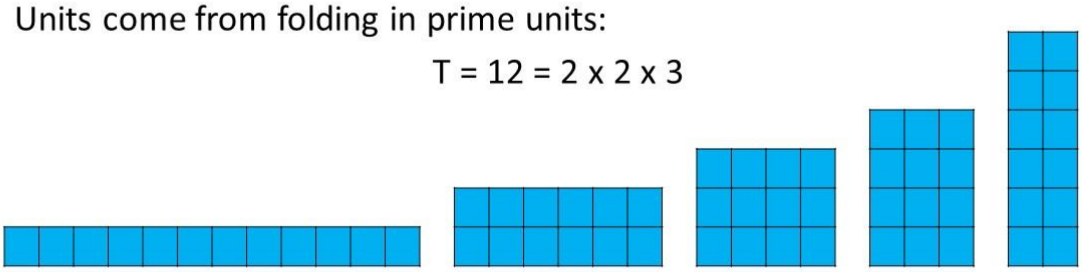
A number is called **even** if it can be written with 2 as the unit, else **odd**.

Q07, finding possible units

"What are possible units in T = 12?"

Units come from folding in prime units:

T = 12 = 2 x 2 x 3



T = 12 **1s** = 2 **2x3s** = 3 **2x2s** = 2x2 **3s** = 2x3 **2s**

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Q08, recounting in a different unit

“How to change a unit?”

The recount-formula allows changing the unit.

Asking T = 3 **4s** = ? **5s**, the recount-formula gives T = 3 **4s** = (3x4/5) **5s**.

Entering 3x4/5, the answer ‘2.some’ shows that a block of 2 **5s** can be taken away.

With 3x4–2x5, the answer ‘2’ shows that 3 **4s** can be recounted as 2B2 **5s** or 2.2 **5s**.

3 **4**s = IIII IIII IIII = IIII I III II II = IIIII IIIII II = 2B2 **5**s = 2.2 **5s**

3x4/5	**2.**some
3x4 – 2x5	**2**

Change Unit = Proportionality

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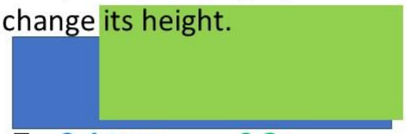
Q09, recounting from tens to icons

“How to change unit from tens to icons?”

Asking ‘T = 2.4 **tens** = 24 = ? **8s**’, we just recount 24 in **8s**:
T = 24 = (24/8)x8 = 3x8 = 3 **8s.**

Formulated as an **equation** we use ***u*** for the unknown number, ***u***x8 = 24. Recounting 24 in 8s shows that ***u*** is 24/8 attained by moving 8 **to opposite side - with opposite sign**	To keep its size, a block changing its unit must also change its height.  T = 2.4 **tens** = 3 **8s**

u x 8 = 24 = (24/8)x8
u = 24/8 = 3

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Q10, recounting from icons to tens (multiplication) 3 7s = ? tens

"How to change unit from icons to tens?"

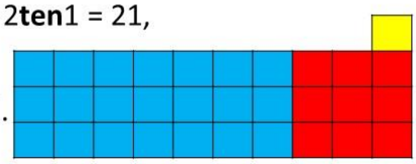
Asking 'T = 3 **7s** = ? **tens**', the recount-formula cannot be used since the calculator has no ten-button. However, it gives the answer directly by using multiplication alone: T = 3 **7s** = 3x7 = 21 = 2.1 **tens**, only it leaves out the unit and the decimal point.

Alternatively, we may use 'less-numbers', so 7 = **ten** less 3

T = 3x7 = 3 x (**ten** less 3) = 3 x **ten** less 3x3 = 3**ten** less 9 = 2**ten**1 = 21,

or with 9 = **ten** less 1:

T = 3**ten** less (**ten** less1) = 2**ten** lessless 1 = 2**ten** & 1 = 21.



showing that 'lessless' cancel out

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Recounting large numbers in or from tens: *same size, but new form*

Recounting 6 **47s** in **tens** Recounting 476 in **7s**

*BundleWriting seprates **INSIDE** bundles from **OUTSIDE** singles*

T = 6 x 47 = 6 x 4**B**7	T = 476 = 47.6 **tens**
= 24**B**42	= 47**B**6
= 28**B**2	= 42**B**56
= 28.2	= 6x7**B**8x7
tens	= 68 x 7

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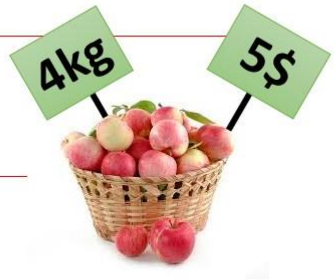
Q11, double-counting in two units creates bridging PerNumbers & proportionality



"How to double-count in two units?"

DoubleCounting in kg & $, we get **4kg = 5$** or

4kg **per** 5$ = 4kg/5$ = 4/5 kg/$ = a PerNumber.

With 4kg bridged to 5$ we answer questions by recounting in the per-number.

Questions:

7kg = ?$	**8$ = ?kg**
7kg = (7/4) x 4kg = (7/4) x 5$ = 8.75$	8$ = (8/5) x 5$ = (8/5) x 4kg = 6.4kg

Answer: *Recount in the PerNumber (Proportionality)*

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Q12, double-counting in the same unit creates fractions

"How to double-count in the same unit?"

Double-counted in the same unit, per-numbers are fractions, 2$ per 9$ = 2/9, or percentages, 2 per 100 = 2/100 = 2%.

To find a fraction or a percentage of a total, again we just recount in the per-number.

- **Taking 3 per 4 = taking ? per 100.** With 3 bridged to 4, we recount 100 in 4s:

100 = (100/4)*4 giving (100/4)*3 = 75, and 75 per 100 = 75%.

- **Taking 3 per 4 of 60 gives ?.** With 3 bridged to 4, we recount 60 in 4s:

60 = (60/4)*4 giving (60/4)*3 = 45.

- **Taking 20 per 100 of 60 gives ?.** With 20 bridged to 100, we recount 60 in 100s:

60 = (60/100)*100 giving (60/100)*20 = 12 .

We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number.

Q12, enlarging or shortening units

"How to enlarge or shorten units in fractions?"

Taking 2/3 of 12 means taking 2 per 3 of 12.

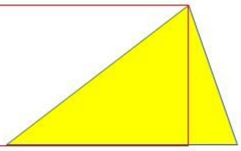
With 2 bridged to 3, we recount 12 in **3s**, 12 = (12/3)*3 = 4*3

So 4 times we can take 2, i.e. 8 of the 12. Thus 2 per 3 = 8 per 12.

This may be used for enlarging or shortening fractions by inserting or removing the same unit above and below the fraction line:

$\frac{2}{3} = \frac{2\ \mathbf{4s}}{3\ \mathbf{4s}} = \frac{2*4}{3*4} = \frac{8}{12}$ ● $\frac{8}{12} = \frac{2*4}{3*4} = \frac{2\ \mathbf{4s}}{3\ \mathbf{4s}} = \frac{2}{3}$ ● $\frac{12abc}{8a} = \frac{3*4*a*b}{2*4*a} = \frac{3*b\ \mathbf{4as}}{2\ \mathbf{4as}} = \frac{3b}{2}$

Q13, recounting the sides in a block



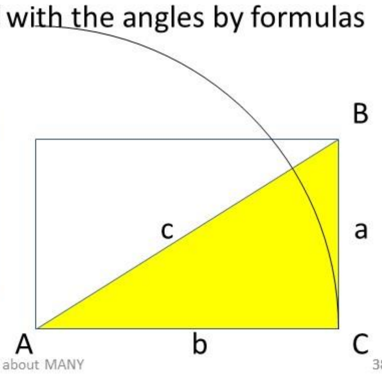
Geometry means to measure earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula. And connected with the angles by formulas recounting a side in the other side or in the diagonal:

A+B+C = 180

a*a + b*b = c*c (the Pythagoras formula)

sinA = a/c; cosA = b/c; tanA = a/b = Δy/Δx = gradient

Circle: circum./diam. = π **= n*tan(180/n)** for n large



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Q14, double-counting gives per-numbers in STEM multiplication formulas I

STEM (Science, Technology, Engineering, Math) typically contains multiplication formulas with per-numbers coming from double-counting.

Examples:

- kg = (**kg/cubic-meter**) x cubic-meter = **density** x cubic-meter
- force = (**force/square-meter**) x square-meter = **pressure** x square-meter
- meter = (**meter/sec**) x sec = **velocity** x sec
- energy = (**energy/sec**) x sec = **Watt** x sec
- energy = (**energy/kg**) x kg = **heat** x kg

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Q14, double-counting gives per-numbers in STEM multiplication formulas II

Extra STEM examples:

- gram = (**gram/mole**) x mole = **molar mass** x mole;
- Δ momentum = (Δ **momentum/sec**) x sec = **force** x sec;
- Δ energy = (Δ **energy/ meter**) x meter = **force** x meter = work;
- **energy/sec** = (energy/charge) x (charge/sec) or **Watt** = **Volt** x **Amp**;
- dollar = (**dollar/hour**) x hour = **wage** x hour;
- dollar = (**dollar/meter**) x meter = **rate** x meter
- dollar = (**dollar/kg**) x kg = **price** x kg.

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Q15, navigating on a squared paper

First steps into coordinate geometry, to always keep algebra and geometry together.

"Collect treasures on the rocks "	**"Plan a trip to treasure island"**
Three rocks are placed on a squared paper.	Departure point: 3cm out & 2cm up
The rocks have the values -1, 1, and 2.	Destination point: 7cm out & 4cm up.
A journey begins in the midpoint.	Plan a voyage with 1 out per day.
Two dices tell the out- and up- change,	How many days before reaching the island?
where odd numbers are negative.	What is your position after 2 days?
How many points before reaching the edge?	What is your position after n days?
Predict and measure angles on the journey.	What is the angle traveled?

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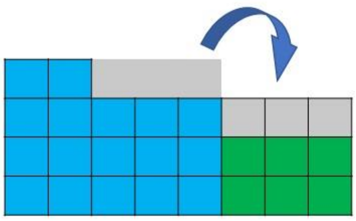
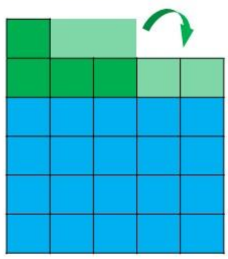
Counted & recounted, Totals can be added

BUT: NextTo	**or OnTop**
4 **5s** + 2 **3s** = 3**B**2 **8s**	4 **5s** + 2 **3s** = 4 **5s** + 1**B**1 **5s** = 5**B**1 **5s**
The areas are integrated *Adding areas = Integration*	The units are changed to be the same *Change unit = Proportionality*





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Four ways to unite into a Total

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2 + 4***B** + 5 (a polynomial) shows the four ways to add: +, *, ^, next-to block-addition (integration).
Addition and multiplication add changing and constant unit-numbers.
Integration and power add changing and constant per-numbers.
We might call this beautiful simplicity the 'Algebra Square'.

Operations unite	**changing**	**constant**
Unit-numbers *m, s, \$, kg*	**T = a + n**	**T = a * n**
Per-numbers *m/s, \$/kg, m/(100m) = %*	**T = ∫ a dn**	**T = a^n**

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Five ways to split a Total

The 4 uniting operations (+, *, ^, ∫) each has a reverse splitting operation:
Addition has subtraction (–), and multiplication has division (/).
Power has factor-finding (root, √) and factor-counting (logarithm, log).
Integration has per-number finding (differentiation dT/dn = T').

Reversing operations is solving equations, done by moving to **opposite side** with **opposite sign**.

Operations unite / *split into*	**changing**	**constant**
Unit-numbers *m, s, \$, kg*	**T = a + n** *T – a = n*	**T = a * n** *T/n = a*
Per-numbers *m/s, \$/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $\log_a T = n$, $\sqrt[n]{T} = a$

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Question Guided Adding Curriculum

A question guided re-enchanting ADDING curriculum could be named ‘<u>Mastering Many by uniting and splitting constant and changing unit-numbers and per-numbers</u>’.

- A corresponding pre-service and in-service question guided teacher education can be found at the MATHeCADEMY.net.
- Remedial curricula for classes stuck in traditional mathematics can be found there also.

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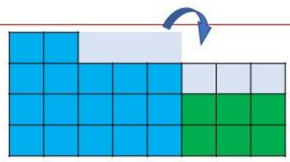
Q21, next-to addition



“With T1 = 4 **5s** and T2 = 2 **3s**, what is T1+T2 when added next-to as **8s**?”

<u>Outside</u>, next-to addition geometrically means adding areas.
Next-to addition is also called integral calculus.

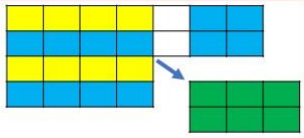
<u>Inside</u>, the recount formula algebraically predicts the result.
Here multiplication precedes addition.

T = (T/**B**) x **B**

= ((4x5 + 2x3)/8) x 8 = 3.2 **8s**

(4x5 + 2x3)/8	**3.some**
(4x5 + 2x3) – 3x8	**2**

Q22, reversed next-to addition



"If T1 = 2 **3s** and T2 add next-to as 4 **7s**, what is T2?"

<u>Outside</u>, we remove the initial block T1 and recount the rest in **4s**.

Thus reversed next-to addition geometrically means subtracting areas.

Reversed next-to addition is also called differential calculus.

<u>Inside</u>, the recount formula algebraically predicts the result.

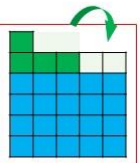
Here subtraction precedes division; which is natural as reversed integration.

T2 = (T2/**B**) x **B**

= ((4x7 – 2x3)/4) x 4 = 5.2 **4s**

(4x7 – 2x3)/4	**5.some**
(4x7 – 2x3) – 5x4	**2**

Q23, on-top addition

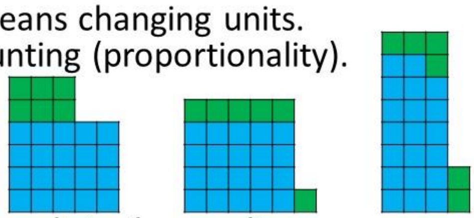


"With T1 = 4 **5s** and T2 = 2 **3s**, what is T1+T2 when added on-top?"

<u>Outside</u>, on-top addition geometrically means changing units.
On-top addition thus often involves recounting (proportionality).

T = 4 **5s** + 2 **3s** = 4 **5s** + 1.1 **5s** = 5.1 **5s**

T = 4 **5s** + 2 **3s** = 6.2 **3s** + 2 **3s** = 8.2 **3s**



<u>Inside</u>, the recount formula algebraically predicts the result.
Here again, multiplication precedes addition.

T = (T/**B**) x **B**

= ((4x5 + 2x3)/5) x 5 = 5.1 **5s**

(4x5 + 2x3)/5	**5.some**
(4x5 + 2x3) – 5x5	**1**

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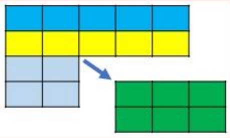
Q24, reversed on-top addition



“T1 = 2 **3s** and how many **5s** (T2) add on-top as 4 **5s**?”

Outside, we remove the initial block T1 and recount the rest in **5s**.

Thus reversed next-to addition geometrically means subtracting areas.

Reversed on-top addition is also called differential calculus.

Inside, the recount formula algebraically predicts the result.

Here again, subtraction precedes division.

T2 = (T2/**B**) x **B**

= ((4x5 – 2x3)/5) x 5 = 2.4 **5s**

(4x5 – 2x3)/5	**2.some**
(4x5 – 2x3) – 2x5	**4**

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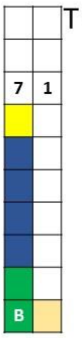
Q25, adding tens on-top



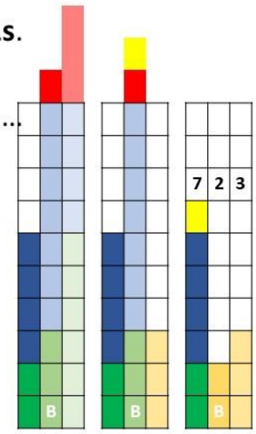
“If T1 = 23 and T2 = 48, what is T1+T2 as **tens**?”

Outside and inside, we recount overloads by changing 1 **tens** to 10 **1s**.

T = 23 + 48 = 2**ten**3 + 4**ten**8 = 6**ten**11 = 6**ten**1**ten**1 = 7**ten**1 =71

T = 236 + 487 = 2**tenten**3**ten**6 + 4**tenten**8**ten**7 = 6**tenten**11**ten**13 = …

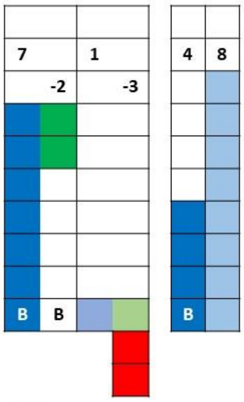
T1+T2 = 23 + 48	T = **2**36 + 487
= 2**B**3 + 4**B**8	= 2**BB**3**B**6 + 4**BB**8**B**7
= 6**B**11	= 6**BB**11**B**13
= 7**B**1	= 6**BB**12**B**3
= 71	= 7**BB**2**B**3
	= 723



Q26, subtracting tens on-top

"If T1 = 23 and T2 add to T = 71, what is T2 as **tens**?"

Outside and inside, we recount underloads by changing 1 **tens** to 10 **1s.**



T = 71 – 23 = 7**ten**1 – 2**ten**3 = 5**ten**-2 = 4**ten**8 = 48

T = 956 – 487 = 9**tenten**5**ten**6 – 4**tenten**8**ten**7 = 5**tenten**-3**ten**-1 = …

T2 = 71 – 23 = 7**B**1 – 2**B**3 = 5**B**-2 = 4**B**8 = 48	T2 = 956 – 487 = 9**BB**5**B**6 – 4**BB**8**B**7 = 5**BB**-3**B**-1 = 4**BB**7**B**-1 = 4**BB**6**B**9 = 469

Q27, from icons to tens, multiplication

A multiplication table recounts icon-blocks in ten-blocks: T = 7 **3s** = ? **Tens.**

To recount 7 **3s** in **tens** we can use that 7 is **ten** less3, and 3 is 5 less2:

From the 10 **5s** we remove 3 **5s** (/) and 2 **tens** (\).

But then we must add the 3 **2s** that was removed twice.

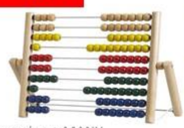
T = 7x3 = (ten – 3)x(5 – 2) = tenx5 – 3x5 – tenx2 + 3x2

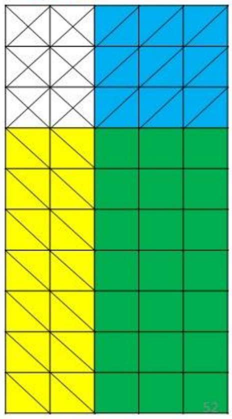
= 50 – 15 – 20 + 6 = 21.

Shown on a western ten by ten abacus as a 10 by 5 block.

This roots the algebra formula showing that – x – is +

(a – b) x (c – d) = a x c – a x d – b x c + b x d





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Q27, multiplication tables

"What is 7 **8s** recounted in **tens**?"

Using underload-numbers after 5, we recount to remove underloads:

T = 7 x 8 = 7 x **B**-2 = 7**B**-14 = 7**B** – 1**B**4 = 6**B**-4 = 5**B**6 = 56	T = 7 x 8 = **B**-3 x **B**-2 = 1**BB** – 3**B** – 2**B** + 6 = 10**B** – 3**B** – 2**B** + 6 = 5**B**6 = 56

	2	**3**	**4**	**5**	**B-4**	**B-3**	**B-2**	**B-1**
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

	2	**3**	**4**	**5**	**B-4**	**B-3**	**B-2**	**B-1**
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
B-4	12	18	24	30	36	42	48	54
B-3	14	21	28	35	42	49	56	63
B-2	16	24	32	40	48	56	64	72
B-1	18	27	36	45	54	63	72	81

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Q27, Recounting BundleBundles in tens (squares: ..., 4 **4s** = ? **tens,** 5 **5s** = ? **tens**, ...)

Using the multiplication table, we recount the different bundle-bundles (called squares) in **tens**:

S4 = 4 **4s** = 4x4 = 16

S5 = 5 **5s** = 5x5 = 25, etc.

We see that to get to the next square we add the sides twice, + 1:

5*5 = 4*4 + 2*4 + 1, or with 4 = n:

(n+1)*(n+1) = n*n + 2*n + 1, or

$(n+1)^2 = n^2 + 2*n + 1$

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	**1**									
2		**4**								
3			**9**							
4				**16**						
5					**25**					
6						**36**				
7							**49**			
8								**64**		
9									**81**	
10										100

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Q27, recounting from icons to tens (multiplication)

Recount 43 7s in tens:

T = 43x7 = 301 = 30.1 tens

Horizontally we write 43 as 4**ten**3 or 4**B**3.
Vertically, we write 7.
Multiplying, we get 28**B** and 21.
So, T = 43x7 = 28**B**21 = 30**B**1 = 301.

With underload, 43 is 5**ten**-7 or 5**B**-7.
Vertically, we write 7.
Multiplying, we get 35**B** and -49.
So, T = 43x7 = 35**B**-49 = 30**B**1 = 301.

overload

4**B**	3	43x
?	?	7
4**B**	3	43x
28**B**	21	7
30**B**	1	301

underload

5**B**	-7	43x
?	?	7
5**B**	-7	43x
35**B**	-49	7
30**B**	1	301

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Q27, recounting 27 43s in tens (multiplication)

Recounting 27 **43s** in **tens**: 27 x 43 = 1161 = 116.1 **tens**

overload

	2B	7	27x43
	?	?	4B
	?	?	3
?**BB**	?**B**	?	
	2B	7	27x43
	8BB	28B	4B
	6B	21	3
8**BB**	34**B**	21	
8**BB**	36**B**	1	
11**BB**	6**B**	1	1161

underload

	3B	-3	27x43
	?	?	4B
	?	?	3
?**BB**	?**B**	?	
	3B	-3	27x43
	12BB	-12B	4B
	9B	-9	3
12**BB**	-3**B**	-9	
12**BB**	-4**B**	1	
11**BB**	6**B**	1	1161

underload

	3B	-3	27x43
	?	?	5B
	?	?	-7
?**BB**	?**B**	?	
	3B	-3	27x43
	15BB	-15B	5B
	-21B	21	-7
15**BB**	-36**B**	21	
15**BB**	-34**B**	1	
11**BB**	6**B**	1	1161

MATHeCADEMY

Q28, recounting from tens to icons (division)

Recount 30.1 **tens** in **7s**: 301/7 = 43

Recount 30.6 **tens** in **7s**: 306/7 = 43 5/7

overload

4**B**	3	43x
28**B**	21	7
30**B**	1	301
?	?	?x
?	?	7
30**B**	1	301

underload

5**B**	-7	43x
35**B**	-49	7
30**B**	1	301
?	?	?x
?	?	7
30**B**	1	301

Multiplying is top-down; division is bottom-up.

Below, we write 301 = 30**B**1. Above we recount 301 as 28**B**21 to count in **7s**. So, T = 301 = 43x7.

Below, we write 306 as 28**B**26 first, then as 28**B**21 + 5 to count in **7s**. So, T = 306 = 43x7 +5.

4**B**	3 +5/7	43 5/7x
28**B**	21+5	7
30**B**	6	306
?	?	?x
?	?	7
30**B**	6	306

MATHeCADEMY

Q28, recounting 1161 in **43s** (division)

	2B	7	27x
	8BB	28B	4B
	6B	21	3
8**BB**	34**B**	21	
8**BB**	36**B**	1	
11**BB**	6**B**	1	1161
	? = 2B	?	?x
	8BB	30B	4B
	6B	1	3
?	?	?	
8**BB**	36**B**	1	
11**BB**	6**B**	1	1161

Overload

Underload

	3B	-3	27x
	8BB	-12B	4B
	9B	-9	3
12**BB**	-3**B**	-9	
12**BB**	-4**B**	1	
11**BB**	6**B**	1	1161
	? = 3B	?	?x
	12BB	-13B	4B
	9B	1	3
?	?	?	
12**BB**	-4**B**	1	
11**BB**	6**B**	1	1161

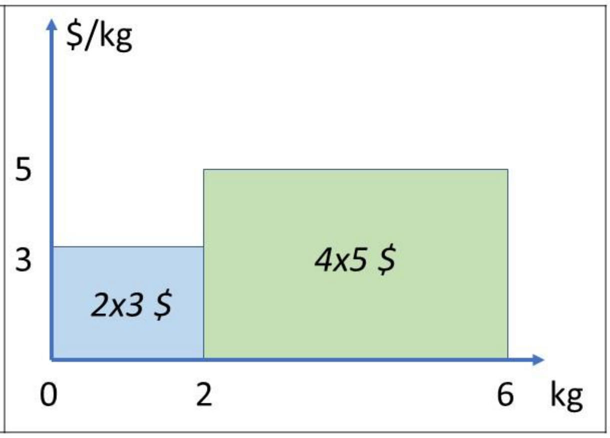
MATHeCADEMY

Q29, adding PerNumbers as areas (integration)

"2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg?"

	2 kg	at	3 $/kg
+	4 kg	at	5 $/kg
(2+4)	kg	at	? $/kg

- Unit-numbers add on-top.
- Per-numbers add next-to as **areas** under the per-number graph. Here multiplication precedes addition.



MATHeCADEMY

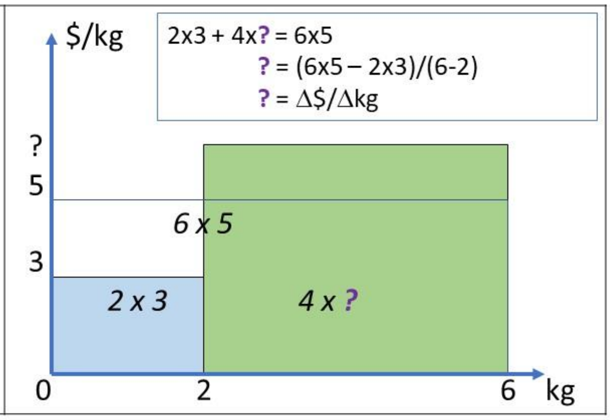
Q30, subtracting PerNumbers (differentiation)

"2kg at 3$/kg + 4kg at **what** = 6kg at 5$/kg?"

	2 kg	at	3 $/kg
+	4 kg	at	? $/kg
	6 kg	at	5 $/kg

Outside, we remove the initial 2x3 block and recount the rest in 4s. Geometrically, reversed per-number addition means subtracting areas to be reshaped, called differential calculus.

Inside, the recount-formula algebraically predicts the result. Here subtraction (giving a change, Δ) precedes division.

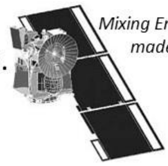


MATHeCADEMY

Never add without units, the fraction paradox

The Teacher	The Students
What is 1/2 + 2/3?	Well, 1/2 + 2/3 = (1+2)/(2+3) = 3/5
No! 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cakes + 2/3 of 3 cakes is 1+2 of 2+3 cakes, i.e. 3/5 of 5 cakes! How can it be 7 cakes out of 6 cakes?
Inside this classroom 1/2 + 2/3 **IS** 7/6 !	

Fractions are not numbers, but operators, needing numbers to become numbers.
2+3 IS 5! No, 2weeks + 3days is 17days; and 2m + 3cm = 203cm.
2*3 IS 6! Yes, since 3 is the unit, and 2 **3s** can be recounted to 6 1s.
Adding without units: MatheMatism.



Mixing English and metric units made NASA's Mars Climate Orbiter fail in 1999.

MATHeCADEMY

Q31, adding unspecified numbers

"Only add like units, so how to add $T = 4ab^2 + 6abc$?"

Here units come from folding (factoring):

$T = 4ab^2 + 6abc = T1 + T2$

$= 2 * 2 * a * b * b \quad + \quad 2 * 3 * a * b * c$

$= 2 * b * (2 * a * b) \quad + \quad 3 * c * (2 * a * b)$

$= (2b+3c) * \mathbf{2ab}$

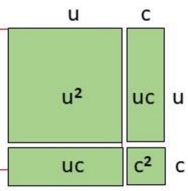
$= 2b+3c$ $\mathbf{2ab}$s

a factor-filter

$4ab^2$	2	2	*a*	*b*	*b*
$6abc$	2	3	*a*	*b*	*c*
unit	**2**		***a***	***b***	
T1:		2			*b*
T2:		3			*c*

MATHeCADEMY

Q31, multiplying unspecified numbers



"How to multiply unspecified two-digit numbers T1 and T2?"

T1*T2 = (2u+3)*(4u-5)

	2u	+3	T1*T2
	?	?	4u
	?	?	-5
?uu	?**u**	?	
	2u	+3	T1*T2
	8uu	+12**u**	4u
	-10**u**	-15	-5
8**u*u**	+2**u**	-15	
8**u²**	+2**u**	-15	

T1*T2 = (u+c)*(u-c)

	u	+c	T1*T2
	?	?	u
	?	?	-c
?**uu**	?**u**	?	
	u	+c	T1*T2
	uu	+cu	u
	-cu	-cc	-c
uu		-cc	
u²		**-c²**	

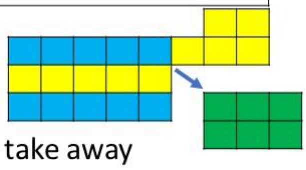
T1*T2 = (u+c)*(u+c) = $(u+c)^2$

	u	+c	T1*T2
	?	?	u
	?	?	+c
?**uu**	?**u**	?	
	u	+c	T1*T2
	uu	+cu	u
	+cu	+cc	+c
uu	**+2cu**	+cc	
u²	**+2cu**	**-c²**	

MATHeCADEMY

Reversed Addition = Solving Equations

OppoSite Side with OppoSite Sign		**NextTo**
2 x ? = 8 = (8/2) x 2	**2 + ? = 8** = (8-2) + 2	2 **3s** + ? **5s** = 3.2 **8s**
? = 8/2 *Solved by ReCounting*	? = 8-2 *Solved by ReStacking*	? = (3.2 **8s** – 2 **3s**)/5 *Solved by differentiation: (T-T1)/5 = ΔT/5*



Hymn to Equations

Equations are the best we know,
they are solved by isolation.
But first, the bracket must be placed
around multiplication.

We change the sign and take away
and only x itself will stay.
We just keep on moving, we never give up.
So feed us equations, we don't want to stop!

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Solving equations by recounting, we may bracket Group Theory from Abstract Algebra

ManyMath

2 x *u* = 8 = (8/2) x 2	Solved by re-counting 8 in 2s
u = 8/2 = 4	Move: **O**pposite **S**ide with OppoSite **S**ign

MetaMath (Don't test, but DO remember the bi-implication arrows)

	2 x *u* = 8	Multiplication has 1 as its **neutral element**, and 2 has ½ as its **inverse element**
↕	(2 x *u*) x (½) = 8 x (½)	Multiplying 2's inverse element ½ to both number-names
↕	(*u* x 2) x (½) = 4	Applying the **commutative** law to u x 2; 4 is the short number-name for 8 x ½
↕	*u* x (2 x (½)) = 4	Applying the **associative** law
↕	*u* x 1 = 4	Applying the definition of an inverse element
↕	*u* = 4	Applying the definition of a neutral element. *With arrows a test is not needed.*

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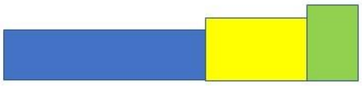
Conclusions

What Mastery of Many does the child have already?

- Children typically see Many as blocks with a number af bundles, and use flexible numbers with units and with over- or underloads

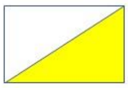
In ManyMath, BLOCKS are fundamental:

- in numbers: 456 = three blocks

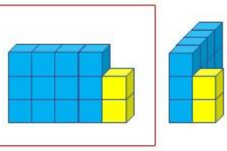


- in algebra: adding blocks next-to or on-top

- in geometry: recounting half-blocks



The child’s own twin math curriculum



1) Digits are (sloppy) icons, with as many sticks as they represent.

2) Totals are counted by bundling, giving <u>outside</u> geometrical multi-blocks, & (when turned to hide the units behind) <u>inside</u> algebraic bundle-numbers.

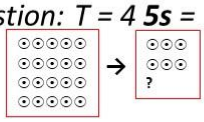
3) Operations are icons, showing the 3 counting steps: Removing bundles & stacking bundles & removing stacks to find the unbundled.

4) The operation order is division first, then multiplication, then subtraction. Addition next-to & on-top comes later after totals are counted & re-counted.

5) Counting & re-counting & double-counting is big fun, when predicted by a calculator with the recount formula: **T = (T/B)xB** (from T, T/B times, Bs can be taken away)

Question: T = 4 ***5s*** *= ?* ***3s*** ● *Answer: T = 4* ***5s*** *= 6****B****2* ***3s*** ● *Prediction:*

4x5/3	6.some
4x5 – 6x3	2



Comparing with a traditional math curriculum I

A traditional curriculum: operations on specified and unspecified numbers.

- Digits are given directly as symbols, without letting children discover digits as icons with as many strokes or sticks as they represent.
- Numbers are one-dimensional line-numbers with digits respecting a place value system, without letting children discover the thrill of two-dimensional bundling and stacking counting both <u>singles</u> and <u>bundles</u> and <u>bundles-of-bundles</u> etc., and that includes the unit.
- Seldom, if ever, 0 is included as ‘01, 02, 03’ in the counting sequence to show the importance of bundling.

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Comparing with a traditional math curriculum II

- Never children are told that eleven and twelve comes from the Vikings, counting '(ten and) 1 left', '(ten and) 2 left'.
- Never children use full number-language sentences, T = 2 **5s**, including both a subject & a verb & a predicate with a unit.
- Seldom children are asked to describe numbers after ten as 1**B**4 **tens** or 1**ten**4 or 1.4 **tens** with a unit and with a decimal point separating bundles and unbundled singles.
- Seldom 17 is recounted as 2**B**-3 or 2.-3 **tens**. Nor is 24 recounted as 1**B**14 **tens** or 3**B**-6 **tens**.

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Comparing with a traditional math curriculum III

- Never it respects the natural order of operations. Instead it turns the order around by giving addition without units priority over subtraction & multiplication & division.
- In short, children never experience the enchantment of counting, re-counting and double-counting Many before being forced to add on-top only, thus neglecting next-to addition.

So, re-enchanting Many is the goal of the twin curriculum in Mastery of Many through respecting and developing the children's existing mastery and quantitative competence.

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Proportionality shows the variety of mastery of Many I

Proportionality, **Q1**: "2kg costs 5\$, what does 7kg cost"; **Q2**: "What does 12\$ buy?"

1) Regula de Tri (rule of three)

Re-phrase with shifting units, the unknown at last. From behind, first multiply then divide.

Q1: '2kg cost 5\$, 7kg cost ?\$'. Multiply-then-divide gives the \$-number 7x5/2 = 17.5.

Q2: '5\$ buys 2kg, 12\$ buys ?kg'. Multiply-then-divide gives the kg-number 12x2/5 = 4.8.

2) Find the unit

Q1: 1kg costs 5/2\$, so 7kg cost 7x(5/2) = 17.5\$. **Q2**: 1\$ buys 2/5kg, so 12\$ buys 12x(2/5) = 4.8kg

3) Cross multiplication

Q1: 2/5 = 7/*u*, so 2**u* = 7*5, *u* = (7*5)/2 = 17.5. **Q2**: 2/5 = *u*/12, so 5**u* = 12*2, *u* = (12*2)/5 = 4.8

4) 'Re-counting' in the 'per-number' 2kg/5\$ coming from 'double-counting' the total T.

Q1: T = 7kg = (7/2)x2kg = (7/2)x5\$ = 17.5\$; **Q2**: T = 12\$ = (12/5)x5\$ = (12/5)x2kg = 4.8kg.

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Proportionality shows the variety of mastery of Many II

5) Modeling with linear functions using group theory from abstract algebra.

- A linear function $f(x) = c*x$ from the set of positive kg-numbers to the set of positive \$-numbers, has the domain $DM = \{x \in R \mid x>0\}$.
- Knowing that $f(2) = c*2 = 5$, this equation is solved by multiplying with the inverse element to 2 on both sides, and applying the associative law, and the definition of an inverse element, and of the neutral element under multiplication:
  $c*2 = 5$ ● $(c*2)*½ = 5*½$ ● $c*(2*½) = 5/2$ ● $c*1 = 5/2$ ● $c = 5/2$.
- With $f(x) = 5/2*x$, the inverse function is $f^{-1}(x) = 2/5*x$.
- With 7kg, the answer is $f(7) = 5/2*7 = 17.5$\$.
- With 12\$, the answer is $f^{-1}(12) = 2/5*12 = 4.8$kg.

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Main parts of a ManyMath curriculum

Primary School – respecting and developing the Child's own 2D NumberLanguage

- Digits are Icons and Natural numbers are 2dimensional block-numbers with units
- BundleCounting & ReCounting before Adding
- NextTo Addition (PreSchool Calculus) before OnTop Addition
- Natural order of operations: divide, multiply, subtract, add on-top & next-to

Middle school – integrating algebra and geometry, the content of the label 'math'

- DoubleCounting produces PerNumbers and fractions as operators needing numbers to become numbers, thus being added as areas (MiddleSchool Calculus)
- Geometry and Algebra go hand in hand always, so length becomes change and vv.

High School – integrating algebra and geometry to master CHANGE

- Change as the core concept: constant, predictable and unpredictable change
- Integral Calculus before Differential Calculus

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Question guided teacher education

MATHeCADEMY.net

Teaches Teachers to Teach MatheMatics as ManyMath, a Natural Science about MANY.

To learn Math, Count & Add MANY, using the CATS method:

Count & Add in Time & Space

- Primary: C1 & A1 & T1 & S1
- Secondary: C2 & A2 & T2 & S2

MATHeCADEMY.net
a VIRUSeCADEMY:
ask Many, not the Instructor

SUMMARY

	QUESTIONS	ANSWERS
C1 COUNT	How to count Many? How to recount 8 in 3s: T= 8 = ? 3s How to recount 6kg in $: T=6kg=?$ How to count in standard bundles?	By bundling and stacking the total T predicted by T = (T/b)*b T= 8= ?*3= ?3s, T= 8=(8/3)*3 = 2*3 +2 = 2*3 +2/3*3 = 2 2/3*3 If 4kg = 2$ then 6kg = (6/4)*4kg = (6/4)*2$ = 3$ Bundling bundles gives a multiple stack, a stock or polynomial: T = 423 = 4BundleBundle+2Bundle+3 = 4tenten2ten3 = 4*B^2+2*B+3
C2 COUNT	How can we count possibilities? How can we predict unpredictable numbers?	By using the numbers in Pascal's triangle We 'post-dict' that the average number is 8.2 with the deviation 2.3. We 'pre-dict' that the next number, with 95% probability, will fall in the confidence interval 8.2 ± 4.6 (average ± 2*deviation)
A1 ADD	How to add stacks concretely? T=27+16= 2ten7+1ten6= 3ten13=? How to add stacks abstractly?	By restacking overloads predicted by the restack-equation T= (T-b)+b T = 27+16 = 2 ten 7+1 ten 6 = 3 ten 13 = 3 ten 1 ten 3 = 4 ten 3 = 43 Vertical calculation uses carrying. Horizontal calculation uses FOIL
A2 ADD	What is a prime number? What is a per-number? How to add per-numbers?	Fold-numbers can be folded: 10=2fold5. Prime-numbers cannot: 5=1fold5 Per-numbers occur when counting, when pricing and when splitting. The $/day-number a is multiplied with the day-number b before added to the total $-number T: T2 = T1 + a*b
T1 TIME	How can counting & adding be reversed ? Counting ? 3s and adding 2 gave 14. Can all calculations be reversed?	By calculating backward, i.e. by moving a number to the other side of the equation sign and reversing its calculation sign. x*3+2=14 is reversed to x = (14-2)/3 Yes. x+a=b is reversed to x=b-a, x*a=b is reversed to x=b/a, x^a=b is reversed to x=a√b, a^x=b is reversed to x=logb/loga
T2 TIME	How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable?	By using constant change-equations: If Ko = 30 and ΔK/n = a = 2, then K7 = Ko+a*n = 30+2*7 = 44 If Ko = 30 and ΔK/K = r = 2%, then K7= Ko*(1+r)^n= 30*1.02^7= 34.46 By solving a variable change-equation: If Ko = 30 and dK/dx = K', then ΔK = K-Ko = ∫K'dx
S1 SPACE	How to count plane and spatial properties of stacks and boxes and round objects?	By using a ruler, a protractor and a triangular shape. By the 3 Greek Pythagoras', mini, midi & maxi By the 3 Arabic recount-equations: sinA=a/c, cosA=b/c, tanA=a/b
S2 SPACE	How to predict the position of points and lines? How to use the new calculation technology?	By using a coordinate-system: If Po(x,y) = (3,4) and if Δy/Δx = 2, then P1(8,y) = P1(x+Δx,y+Δy) = P1((8-3)+3,4+2*(8-3)) = (8,14) Computers can calculate a set of numbers (vectors) and a set of vectors (matrices)
QL	What is quantitative literature? Does quantitative literature also have the 3 different genres: fact, fiction and fiddle?	Quantitative literature tells about Many in time and space The word and the number language share genres: Fact is a since-so calculation or a room-calculation Fiction is an if-then calculation or a rate-calculation Fiddle is a so-what calculation or a risk-calculation

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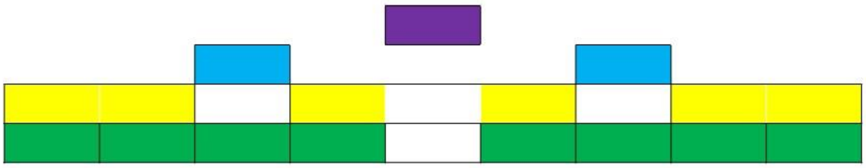
PYRAMIDeDUCATION

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

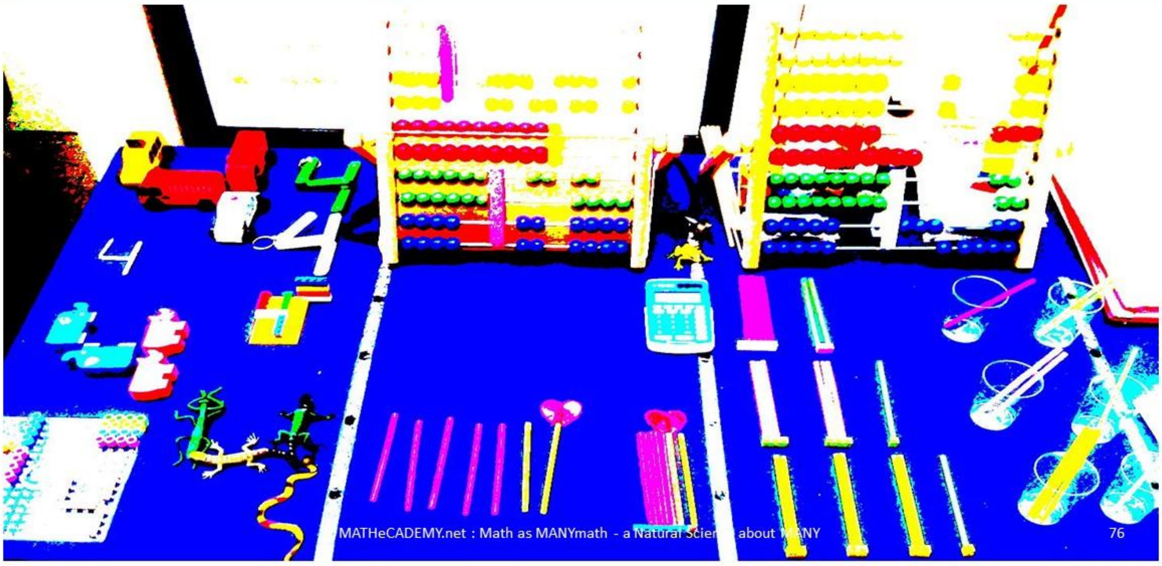
- Each pair works together to solve Count&Add problems.
- The coach assists the instructors when instructing their team and when correcting their Count&Add assignments.
- Each teacher pays by coaching a new group of 8 teachers.

1 Coach
2 Instructors
3 Pairs
2 Teams



Number Icons

ReCounting 7 in 5s & 3s & 2s



Theoretical background

Tarp, A. (2018). Mastering Many by counting, recounting and double-counting before adding on-top and next-to.
Journal of Mathematics Education, March 2018, 11(1), 103-117.

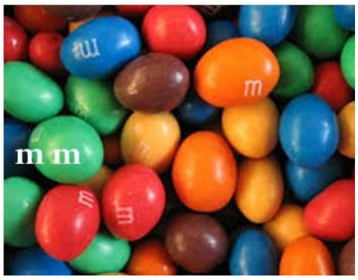
COUNTING before ADDING

The Child's Own Twin Curriculum

Count & ReCount & DoubleCount

before Adding NextTo & OnTop

master many
manymath



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30. A New Curriculum - But for Which of the 3x2 Kinds of Mathematics Education

An essay on observations and reflections at the ICMI study 24 curriculum conference

As part of institutionalized education, mathematics needs a curriculum describing goals and means. There are however three kinds of mathematics: pre-, present and post-'setcentric' mathematics; and there are two kinds of education: multi-year lines and half-year blocks. Thus, there are six kinds of mathematics education to choose from before deciding on a specific curriculum; and if changing, shall the curriculum stay within the actual kind or change to a different kind? The absence of federal states from the conference suggests that curricula should change from national multi-year macro-curricula to local half-year micro-curricula; and maybe change to post-setcentric mathematics.

COHERENCE AND RELEVANCE IN THE SCHOOL MATHEMATICS CURRICULUM

The International Commission on Mathematical Instruction, ICMI, has named its 24th study "School mathematics Curriculum Reforms: Challenges, Changes and Opportunities". Its discussion document has 5 themes among which theme B, "Analysing school mathematics curriculum for coherence and relevance" says that "All mathematics curricula set out the goals expected to be achieved in learning through the teaching of mathematics; and embed particular values, which may be explicit or implicit."

So, to analyze we use the verb 'cohere' and the predicate 'relevant' when asking: "to what does this curriculum cohere and to what is it relevant?" As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.'

We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that 'to cohere' means 'to hold together firmly as parts of the same mass'. As to synonyms for cohere, it lists: 'accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.' And as to antonyms, it lists: 'differ (from), disagree (with).'

In the same dictionary, the word 'relevant' means 'having significant and demonstrable bearing on the matter at hand'. As to synonyms for relevant, it lists: 'applicable, apposite, apropos, germane, material, pertinent, pointed, relative.' And as to antonyms, it lists: 'extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.'

If we accept the verb 'apply' as having a meaning close to the predicate 'relevant', we can rephrase the above analysis question using verbs only: "to what does this curriculum cohere and apply?"

Seeing education metaphorically as bridging an individual start level for skills and knowledge to a common end level described by goals and values, we may now give a first definition of an ideal curriculum: "To apply to a learning process as relevant and useable, a curriculum coheres to the start and end levels for skills and knowledge."

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?
- How to make the end level cohere to goals and values expressed by the society?
- How to make the end level cohere to goals and values expressed by the learners?

- How to make the bridge cohere to previous and following bridges?
- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

GOALS AND VALUES EXPRESSED BY THE SOCIETY

In her plenary address about the ‘OECD 2030 Learning Framework’, Taguma shared a vision:

> The members of the OECD Education 2030 Working Group are committed to helping every learner develop as a whole person, fulfil his or her potential and help shape a shared future built on the well-being of individuals, communities and the planet. (..) And in an era characterised by a new explosion of scientific knowledge and a growing array of complex societal problems, it is appropriate that curricula should continue to evolve, perhaps in radical ways (p. 10).

Talking about learner agency, Taguma said:

> Future-ready students need to exercise agency, in their own education and throughout life. (..) To help enable agency, educators must not only recognise learners’ individuality, (..) Two factors, in particular, help learners enable agency. The first is a personalised learning environment that supports and motivates each student to nurture his or her passions, make connections between different learning experiences and opportunities, and design their own learning projects and processes in collaboration with others. The second is building a solid foundation: literacy and numeracy remain crucial. (p. 11)

By emphasizing learner’s individual potentials, personalised learning environment and own learning projects and processes, Taguma seems to indicate that flexible half-year micro-curricula may cohere better with learners’ future needs than rigid multi-year macro-curricula. As to specifics, numeracy is mentioned as one of the two parts of a solid foundation helping learners enable agency.

DIFFERENT KINDS OF NUMERACY

Numeracy, however, is not that well defined. Oxford Dictionaries and Merriam-Webster agree on saying ‘ability to understand and work with numbers’; whereas the private organization National Numeracy (nationalnumeracy.org.uk) says ‘By numeracy we mean the ability to use mathematics in everyday life’.

The wish to show usage was also part of the Kilpatrick address, describing mathematics as bipolar:

> I want to stress that bipolarity because I think that’s an important quality of the school curriculum and every teacher and every country has to deal with: how much attention do we give to the purer side of mathematics. The New Math thought that it should be entire but that didn’t work really as well as people thought. So how much attention do we give to the pure part of mathematics and how much to the applications and how much do we engage together. Because it turns out if the applications are well-chosen and can be understood by the children then that helps them move toward the purer parts of the field. (p. 20)

After discussing some problems caused by applications in the curriculum, Kilpatrick concludes:

> If we stick with pure mathematics, with no application, what students cannot see, “when will I ever use this?”, it’s not surprising that they don’t go onto take more mathematics. So, I think for self-preservation, mathematicians and mathematics educators should work on the question of: how do we orchestrate the curriculum so that applications play a good role? There is even is even a problem with the word applications, because it implies first you do the mathematics, then you apply it. And actually, it can go the other way. (p. 22)

So, discussing what came first, the hen or the egg, applications or mathematics, makes it problematic to define numeracy as the ability to apply mathematics since it gives mathematics a primacy and a monopoly as a prerequisite for numeracy. At the plenary afterwards discussion, I suggested using the word ‘re-rooting’ instead of ‘applying’ to indicate that from the beginning, mathematics was rooted in the outside world as shown by the original meanings of geometry and algebra: ‘to measure earth’ in Greek and ‘to reunite’ in Arabic.

MATHEMATICS THROUHG HISTORY

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy, seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra. And in Europe, Germanic countries taught ‘reckoning’ in primary school and ‘arithmetic’ and ‘geometry’ in the lower secondary school until about 50 years ago when they all were replaced by the ‘New Mathematics’.

Here a wish for exactness and unity created a ‘setcentric’ ‘meta-matics’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox ‘this sentence is false’, being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

Setcentrism thus changed classical grounded ‘many-matics’ into a self-referring ‘meta-matism’, a mixture of meta-matics and ‘mathe-matism’ true inside but seldom outside a classroom where adding numbers without units as ‘1 + 2 IS 3’ meets counter-examples as e.g. 1week + 2days is 9days.

The introduction of the setcentric New Mathematics created different reactions. Inside the United States it was quickly abandoned with a ‘back-to-basics’ movement. Outside it was implemented at teacher education, and in schools where it gradually softened. However, it never retook its original form or name, despite, in contrast to ‘mathematics’, ‘reckon’ is an action-word better suited to the general aim of education, to teach humans to master the outside world through appropriate actions.

DIFFERENT KINDS OF MATHEMATICS

So, a curriculum must choose between a pre-, a present, and a post-setcentric mathematics as illustrated by an example from McCallum’s plenary talk. After noting that “a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships” (p. 4), McCallum asked the audience: “What is the difference between 5/3 and 5÷3”.

Pre-setcentric mathematics will say that 5/3 is a number on the number-line reached by taking 5 steps of the length coming from dividing the unit in 3 parts; and that 5÷3 means 5 items shared between 3.

Present setcentric mathematics will say that 5/3 is a rational number defined as an equivalence class in the product set of integers, created by the equivalence relation (a,b) eq. (c,d) if cross-multiplication holds, axd = bxc; and, with 1/3 as the inverse element to 3 under multiplication, 5÷3 should be written as 5x1/3, i.e. the as the solution to the equation 3xu = 5, found by applying and thus legitimizing abstract algebra and group theory; thus finally saying goodbye to the Renaissance use of a vertical line to separate addends from subtrahends, and a horizontal line to separate multipliers from divisors.

Post-setcentric mathematics (Tarp, 2018) sees setcentric mathematics as meta-matism hiding the original Greek meaning og mathematics as a science about Many. In this ‘Many-math’, 5/3 is a per-number coming from double-counting in different units (5\$/3kg), becoming a fraction with like units (5\$/3\$ = 5/3). Here per-numbers and fractions are not numbers but operators needing a number to become a number (5/3 of 3 is 5, 5/3 of 6 is 10); and 5÷3 means 5 counted in 3s occurring in the ‘recount-formula’ recounting a total T in bundles of 3s as T = (T/3)x3, saying ‘from T, T/3 times, 3 can be taken away'. This gives flexible numbers: T = 5 = 1B2 3s = 1.2 3s = 1 2/3 3s = 2B-1 3s = 2.-1 3s, introduced in grade one where bundle-counting and re-counting in another unit precedes adding, and where recounting from tens to icons, T = 2.4 tens = ? 6s, leads to the equation $T = u$x6 = 24 = (24/6)x6 solved by recounting. In post-setcentric mathematics, per-numbers, fractions, ratios and proportionality melt together since double-counting in two units gives per-numbers as ratios, becoming fractions with like units. And here proportionality means changing units using the recount-formula to recount in the per-number: With 5$/3kg, "how much for 20$?" is found by re-counting 20 in 5s: T = 20$ = (20/5)x5$ = (20/5)x3kg = 12 kg. Likewise if asking "how much for 15 kg?"

DIFFERENT KINDS OF EDUCATION

As to education, from secondary school there is a choice between multi-year lines and half-year blocks. At the discussion after the Kilpatrick plenary session I made a comment about these two educational systems, which mas a lady from the United States say I was misinforming since in the states Calculus required a full year block. Together with other comments in the break, this made me realize that internationally there is little awareness of these two different kinds of educational systems. So here is another example of what the Greek sophists warned against, choice masked as nature.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

At the conference, the almost total absence of federal states as Germany, Canada, the United States and Russia seems to indicate that the problems reside with multi-year national curricula, becoming rigid traditions difficult to change. While federal competition or half-year blocks creates flexibility through an opportunity to try out different curricula.

Moreover, as a social institution involving individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlightens teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching only one subject in their own classrooms.

To protect its republic against its German speaking neighbors, France created elite schools, criticized today for exerting hidden patronization. Bourdieu thus calls education 'symbolic violence', and Foucault points out that a school is really a 'pris-pital' mixing power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings.

DIFFERENT KINDS OF COMPETENCES

As to competences, new to many curricula, there are at least three alternatives to choose among. The European Union recommends two basic competences, acquiring and applying, when saying that "Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge."

At the conference two alternatives notions of competences were presented. In his plenary address, Niss recommended a matrix with 8 competences per concept (p. 73). In his paper, Tarp (pp. 317-324) acknowledged that 8 competences may be needed if the goal of mathematics education is to learn present setcentric university mathematics; but if the goal is to learn to master Many with post-setcentric mathematics, then only two competences are needed: counting and adding, rooting a twin curriculum teaching counting, recounting in different units and double-counting before adding.

MAKING THE LEARNING ROAD MORE PASSABLE

Once a curriculum is chosen, the next question is to make its bridge between the start and end levels for skills and knowledge more passable. Here didactics and pedagogy come in; didactics as the captain choosing the way from the start to the end, typically presented as a textbook leaving it to pedagogy, the lieutenants, to take the learners through the different stages.

The didactical choices must answer general questions from grand theory. Thus, philosophy will ask: shall the curriculum follow the existentialist recommendation, that existence precedes essence? And psychology will ask: shall the curriculum follow Vygotsky mediating institutionalized essence, or Piaget arranging learning meetings with what exists in the outside world? And sociology will ask: on which ethical grounds are children and teenagers retained to be cured by institutionalized education?

COLONIZING OR DECOLONIZING CURRICULA

The conference contained two plenary panels, the first with contributors from France, China, The Philippines and Denmark, almost all from the northern hemisphere; the second with contributors from Chile, Australia, Lebanon and South Africa, almost all from the southern hemisphere. Where the first panel talked more about solutions, the second panel talked more about problems.

In the first panel, France and Denmark represented some of the world's most centralized states with war-time educational systems dating back to the Napoleon era, which in France created elite-schools to protect the young republic from the Germans, and in Germany created the Humboldt Bildung schools to end the French occupation by mediating nationalism, and to sort out the population elite for jobs as civil servants in the new central administration; both just replacing the blood-nobility with a knowledge-nobility as noted by Bourdieu. The Bildung system latter spread to most of Europe.

Not surprisingly, both countries see university mathematics as the goal of mathematics education ('mathematics is what mathematicians do'), despite the obvious self-reference avoided by instead formulating the goal as e.g. learning numerical competence, mastery of Many or number-language. Seeing mathematics as the goal, makes mathematics education an example of a goal displacement (Bauman) where a monopoly transforms a means into a goal. A monopoly that makes setcentric mathematics an example of what Habermas and Derrida would call a 'center-periphery colonization', to be decentered and decolonized by deconstruction.

Artigue from France thus advocated an anthropological theory of the didactic, ATD, (p. 43-44), with a 'didactic transposition process' containing four parts: scholarly knowledge (institutions producing and using the knowledge), knowledge to be taught (educational system, 'noosphere'), taught knowledge (classroom), and learned available knowledge (community of study).

> The theory of didactic transposition developed in the early 1980s to overcome the limitation of the prevalent vision at the time, seeing in the development of taught knowledge a simple process of elementarization of scholarly knowledge (Chevallard 1985). Beyond the well-known succession offered by this theory, which goes from the reference knowledge to the knowledge actually taught in classrooms (..), ecological concepts such as those of niche, habitat and trophic chain (Artaud 1997) are also essential in it.

Niss from Denmark described the Danish 'KOM Project' leading to eight mathematical competencies per mathematical topic (pp. 71-72).

> The KOM Project took its point of departure in the need for creating and adopting a general conceptualisation of mathematics that goes across and beyond educational levels and institutions. (..) We therefore decided to base our work on an attempt to define and characterise mathematical competence in an overarching sense that would pertain to and make sense in any mathematical context. Focusing - as a consequence of this approach - first and foremost on the *enactment* of mathematics means attributing, at first, a secondary role to mathematical content. We then came up with the following definition of mathematical competence: Possessing *mathematical competence* – mastering mathematics – is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve actual or potential mathematical challenges of any kind. In order to identify and characterise the fundamental constituents in mathematical competence, we introduced the notion of mathematical competencies: A *mathematical competency* is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve a certain kind of mathematical challenge.

Some of the consequences by being colonized by setcentrism was described in the second panel.

In his paper 'School Mathematics Reform in South Africa: A Curriculum for All and by All?' Volmink from South Africa Volmink writes (pp. 106-107):

> At the same time the educational measurement industry both locally and internationally has, with its narrow focus, taken the attention away from the things that matter and has led to a traditional approach of raising the knowledge level. South Africa performs very poorly on the TIMSS study. In the 2015 study South Africa was ranked 38th out of 39 countries at Grade 9 level for mathematics and 47th out of 48 countries for Grade 5 level numeracy. Also in the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ), South Africa was placed 9th out of the 15 countries participating in Mathematics and Science – and these are countries which spend less on education and are not as wealthy as we are. South Africa has now developed its own Annual National Assessment (ANA) tests for Grades 3, 6 and 9. In the ANA of 2011 Grade 3 learners scored an average of 35% for literacy and 28% for numeracy while Grade 6 learners averaged 28% for literacy and 30% for numeracy.

After thanking for the opportunity to participate in a cooperative effort on the search of better education for boys, girls and young people around the world, Oteiza from Chile talked about 'The Gap Factor' creating social and economic differences. A slide with the distribution of raw scores at PSU mathematics by type of school roughly showed that out of 80 points, the median scores were 40 and 20 for private and public schools respectively. In his paper, Oteiza writes (pp. 81-83):

> Results, in national tests, show that students attending public schools, close to de 85% of school population, are not fulfilling those standards. How does mathematical school curriculum contribute to this gap? How might mathematical curriculum be a factor in the reduction of these differences? (..) There is tremendous and extremely valuable talent diversity. Can we justify the existence of only one curriculum and only one way to evaluate it through standardized tests? (..) There is a fundamental role played by researchers, and research and development centers and institutions. (..) How do the questions that originate in the classroom reach a research center or a graduate program? "*Publish or perish*" has led our researchers to publish in prestigious international journals, but, are the problems and local questions addressed by those publications?"

The Gap Factor is also addressed in a paper by Hoyos from Mexico (pp. 258-259):

> The PISA 2009 had 6 performance levels (from level 1 to level 6). In the global mathematics scale, level 6 is the highest and level 1 is the lowest. (..) It is to notice that, in PISA 2009, 21.8% of Mexican students do not reach level 1, and, in PISA 2015, the percentage of the same level is a little bit higher (25.6%). In other words, the percentage of Mexican students that in PISA 2009 are below level 2 (i.e., attaining the level 1 or zero) was 51%, and this percentage is 57% in PISA 2015, evidencing then an increment of Mexican students in the poor levels of performance. According to the INEE, students at levels 1 or cero are susceptible to experiment serious difficulties in using mathematics and benefiting from new educational opportunities throughout its life. Therefore, the challenges of an adequate educational attention to this population are huge, even more if it is also considered that approximately another fourth of the total Mexican population (33.3 million) are children under 15 years of age, a population in priority of attention".

As a comment to Volminks remark “Another reason for its lack of efficacy was the sense of scepticism and even distrust about the notion of People's Mathematics as a poor substitute for the “real mathematics”” (p. 104), and inspired by the sociological Centre–Periphery Model for colonizing, by post-colonial studies, and by Habermas’ notion of rationalization and colonization of the lifeworld by the instrumental rationality of bureaucracies, I formulated the following question in the afterwards discussion: “As former colonies you might ask: Has colonizing stopped, or is it still taking place? Is there an outside central mathematics that is still colonizing the mind? What happens to what could be called local math, street math, ethno-math or the child’s own math?”

CONCLUSION AND RECOMMENDATIONS

Designing a curriculum for mathematics education involves several choices. First pre-, present and post-setcentric mathematics together with multi-year lines and half-year blocks constitute 3x2 different kinds of mathematics education. Combined with three different ways of seeing competences, this offers a total of 18 different ways in which to perform mathematics education at each of the three educational levels, primary and secondary and tertiary, which may even be divided into parts.

Once chosen, institutional rigidity may hinder curriculum changes. So, to avoid the ethical issues of forcing cures from self-referring diagnoses upon children and teenagers in need of guidance instead of cures, the absence of participants from federal states might be taken as an advice to replace the national multi-year macro-curriculum with regional half-year micro-curricula. At the same time, adopting the post version of setcentric mathematics will make the curriculum coherent with the mastery of Many that children bring to school, and relevant to learning the quantitative competence and numeracy desired by society.

And, as Derrida says in an essay called ‘Ellipsis’ in ‘Writing and Difference’: “Why would one mourn for the centre? Is not the centre, the absence of play and difference, another name for death?”

POSTSCRIPT: MANY-MATH, A POST-SETCENTRIC MATHEMATICS FOR ALL

As post-setcentric mathematics, Many-math, can provide numeracy for all by celebrating the simplicity of mathematics occurring when recounting the ten fingers in bundles of 3s:

T = ten = 1B7 3s = 2B4 3s = 3B1 3s = 4B-2 3s. Or, if seeing 3 bundles of 3s as 1 bundle of bundles,

T = ten = 1BB0B1 3s = 1*B^2 + 0*B + 1 3s, or T = ten = 1BB1B-2 3s = 1*B^2 + 1*B – 2 3s.

This number-formula shows that a number is really a multi-numbering of singles, bundles, bundles of bundles etc. represented geometrically by parallel block-numbers with units. Also, it shows the four ways to unite: on-top addition, multiplication, power and next-to addition, also called integration. Which are precisely the four ways to unite constant and changing unit- and per-numbers numbers into totals as seen by including the units; each with a reverse way to split totals. Thus, addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the Algebra Square’, also showing that equations are solved by moving to the opposite side with opposite signs.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	***T = a + n*** *T – n = a*	***T = a*n*** *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	***T = ∫a*dn*** *dT/dn = a*	***T = a^n*** *n√T = a logaT = n*

An unbundled single can be placed on-top of the block counted in 3s as T = 1 = 1/3 3s, or next-to the block as a block of its own written as T = 1 = .1 3s Writing T = ten = 3 1/3 3s = 3.1 3s = 4.-2 3s thus introduces fractions and decimals and negative numbers together with counting.

The importance of bundling as the unit is emphasized by counting: 1, 2, 3, 4, 5, 6 or bundle less 4, 7 or B-3, 8 or B-2, 9 or B-1, ten or 1 bundle naught, 1B1, …, 1B5, 2B-4, 2B-3, 2B-2, 2B-1, 2B naught.

This resonates with 'Viking-counting': 1, 2, 3, 4, hand, and1, and2, and3, less2, less1, half, 1left, 2left. Here '1left' and '2left' still exist as 'eleven' and 'twelve', and 'half' when saying 'half-tree', 'half-four' and 'half-five' instead of 50, 70 and 90 in Danish, counting in scores; as did Lincoln in his Gettysburg address: "Four scores and seven years ago …"

Counting means wiping away bundles (called division iconized as a broom) to be stacked (called multiplication iconized as a lift) to be removed to find unbundled singles (called subtraction iconized as a horizontal trace). Thus, counting means postponing adding and introducing the operations in the opposite order of the tradition, and with new meanings: 7/3 means 7 counted in 3s, 2x3 means stacking 3s 2 times. Addition has two forms, on-top needing recounting to make the units like, and next-to adding areas, i.e. integral calculus. Reversed they create equations and differential calculus.

The recount-formula, T = (T/B)*B, appears all over mathematics and science as proportionality or linearity formula:

- Change unit, T = (T/B)*B, e.g. T = 8 = (8/2)*2 = 4*2 = 4 2s
- Proportionality, $ = ($/kg)*kg = price*kg
- Trigonometry, a = (a/c)*c = sinA*c, a = (a/b)*b = tanA*b, b = (b/c)*c = cosA*c
- STEM-formulas, meter = (meter/sec)*sec = speed*sec, kg = (kg/m^3)*m^3 = density*m^3
- Coordinate geometry, Δy = (Δy/Δx)*Δx = m*Δx
- Differential calculus, dy = (dy/dx)*dx = y' * dx

The number-formula also contains the formulas for constant change: *T = b*x* (proportional), *T = b*x + c* (linear), *T = a*x^n* (elastic), *T = a*n^x* (exponential), *T = a*x^2 + b*x + c* (accelerated).

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics 'post-dicting' what we cannot be 'pre-dicted'.

THE GENERAL CURRICULUM CHOICES OF POST-SETCENTRIC MATHEMATICS

Making the curriculum bridge cohere with the individual start levels in a class is obtained by always beginning with the number-formula, and with recounting tens in icons less than ten, e.g. T = 4.2 tens = ? 7s, or u*7 = 42 = (42/7)*7, thus solving equations by moving to opposite side with opposite sign. And by always using full number-language sentences with a subject, a verb and a predicate as in the word language, e.g. T = 2*3. This also makes the bridge cohere to previous and following bridges.

Making the end level cohere to goals and values expressed by the society and by the learners is obtained by choosing mastery as the end goal, not of the inside self-referring setcentric construction of contemporary university mathematics, but of the outside universal physical reality, Many.

Making the bridge passable is obtained by choosing Piagetian psychology instead of Vygotskyan.

FLEXIBLE NUMBERS MAKE TEACHERS FOLLOW

Changing a curriculum raises the question: will the teachers follow? Here, seeing the advantage of flexible numbers makes teachers interested in learning more about post-setcentric mathematics:

Typically, division creates problems to students, e.g. 336/7. With flexible numbers a total of 336 can be recounted with an overload as T = 336 = 33B6 = 28B56, so 336/7 = 28B56 /7 = 4B8 = 48; or with an underload as T = 336 = 33B6 = 35B-14, so 336/7 = 35B-14 /7 = 5B-2 = 48.

Flexible numbers ease all operations:

T = 48*7 = 4B8*7 = 28B56 = 33B6 = 336

T = 92 – 28 = 9B2 – 2B8 = 7B-6 = 6B4 = 64

T = 54 + 28 = 5B4 + 2B8 = 7B12 = 8B2 = 82

To learn more about flexible numbers, a group of teachers can go to the MATHeCADEMY.net designed to teach teachers to teach MatheMatics as ManyMatics, a natural science about Many, to watch some of its YouTube videos. Next, the group can try out the “Free 1day Skype Teacher Seminar: Cure Math Dislike by ReCounting” where, in the morning, a power point presentation ‘Curing Math Dislike’ is watched and discussed locally, and at a Skype conference with an instructor. After lunch the group tries out a ‘BundleCount before you Add booklet’ to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows after the coffee break.

To learn more, a group of eight teachers can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for secondary school. For modelling, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where the 8 teachers form 2 teams of 4, choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers. The material mediates learning by experimenting with the subject in number-language sentences, i.e. the total T. Thus, the material is self-instructing, saying “When in doubt, ask the subject, not the instructor”.

The material for primary and secondary school has a short question-and-answer format. The question could be: “How to count Many? How to recount 8 in 3s? How to count in standard bundles?” The corresponding answers would be: “By bundling and stacking the total T, predicted by T = (T/B)*B. So, T = 8 = (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s = 3.-1 3s. Bundling bundles gives multiple blocks, a polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4*B^2 + 5*B + 6*1.”

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The same Mathematics Curriculum for Different Students

Addition-free STEM-based Math for Migrants

Math Dislike Cured with Inside-Outside Deconstruction

Developing the Child's Own Mastery of Many

What is Math - and Why Learn it?

Flexible BundleNumbers

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Preface

● The texts 01 and 02 concern the ICMI Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, held in Tsukuba, Japan, 26-30 November 2018. My paper, 'A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many', is included in the 2018 Articles. The discussion document theme B, "Analysing school mathematics curriculum reforms for coherence and relevance" had five sub-questions, and I was asked to contribute to writing a chapter addressing key-question B2, "How are mathematics content and pedagogical approaches in reforms determined for different groups of students (for e.g. in different curriculum levels or tracks) and by whom? How do curriculum reforms establish new structures in content, stakeholders (e.g. students and teachers), and school organisations; and what are their effects?", in short called mathematics for all.

The first outline was scheduled to February 15, 2019. Having almost finished the outline, on February 12 we received a mail saying "The deadline for the chapter outlines has been extended until at least the end of February (from 15 February). I would like to open discussion between us on several matters." On February 20 I sent in my response (see 02).

I never got any reaction so on March 30 I sent a mail to the organizing committee saying "To stimulate our work, would it be an idea to send out a monthly or quarterly newsletter reporting on the progress and challenges being made and met?" No response came so I began writing a proposal for a contribution (se 01). Then on May 10 I got a mail saying "There will not be a separate chapter on key question 2", to which I responded "I think that the question 'math for all' as focused on in the question B2 is so important that it deserves an answer. When mentioned at the conference that no paper addressed this I objected since my paper is addressing the question if it with a different way of organizing math education will be able to include all. Furthermore, I have written a first draft that I send on February 12. Moreover, I have collected a substantial amount of material to include, just waiting for an answer to my mail on February 20 and March 28. So I am going to write a chapter with the focus originally decided upon since I think the research question deserves an answer as mentioned above. I will send it to you as planned before at the end of June. You might then decide not to include it, it will be your choice, then I will publish it elsewhere since the question is very important and since the material, I have collected shows that it is indeed possible to have mathematics for all in different ways."

I then sent in my proposal before July 1 but heard nothing then or after the time limit for a reaction on August 15. So apparently my contribution will be the chapter that was commissioned and rejected without even being read.

01. The same Mathematics curriculum for different students.

The paper has the following chapters: 01. A need for curricula for all students 02. Addressing the need 03. Coherence and relevance 04. Parallel tracks to the main curriculum, examples 05. Pre-calculus, typically the last mandatory curriculum 06. Precalculus in the Danish parallel high school, a case study 07. A refugee camp curriculum 08. Do we really need parallel curricula 09. Conclusion.

02. Comments to a discussion paper

● At the Genoa University April 8-11, Paolo Boero held an international workshop called 'Habermas' elaboration on rationality and mathematics education' over the Habermas text 'Some further clarifications of the concept of communicative rationality'. I was allowed to give a short presentation.

03. A Mathematics Teacher Using Communicative Rationality Towards Children

● At Beijing Normal University June 28-30 2019, the 2019 Classroom Teaching Research for All Students Conference (CTRAS) took place with the conference theme 'innovative practices and research initiatives in STEM integration that supports all students' mathematics learning (..) The conference will provide participants from around the world with the opportunity to share: 1) best practices of STEM integration; 2) the most contemporary STEM research initiatives; 3) innovative curriculum of STEM integration; and 4) professional development approaches for STEM educators.' I contributed with a paper with a power point presentation, a proposal for a short presentation, a poster and a workshop, 04-08

● At Paderborn University September 16-19 the third international conference on mathematics textbooks research and development, ICMT3, will take place. Invitations were sent out to contribute with oral communications, workshops, posters, papers and symposia. I sent in one oral presentation (09), one paper (14), one workshop (11) and three posters (10, 12, 13). All were rejected except for the paper that was accepted for a ten minutes oral presentation.

09. Developing the Child's Own Mastery of Many, oral presentation

10. Math Dislike Cured with Inside-Outside Deconstruction, poster

11. Learning from The Child's Own Mathematics, workshop

12. Five Alternative Ways to Teach Proportionality, poster

13. New Textbooks, but for Which of the 3x2 Kinds of Mathematics Education, poster

14. Developing the Child's Own Mastery of Many, paper. The abstract says: Sociological imagination sees continuing educational problems as possibly caused by a goal displacement making mathematics see itself as the goal instead of its outside root, mastery of Many. Typically, the number-language is taught inside-inside as examples of its meta-language. However, as the word-language, it can also be taught inside-outside, thus bridging it to the outside world it describes. So, textbooks should not reject, but further guide the mastery of Many that children bring to school.

The chapters are called 'is one curriculum and textbook for all students possible, meeting many, children bundle to count and share, textbooks for a question guided counting curriculum, textbook for a question guided adding curriculum, discussion and future research.'

15. The PowerPointPresentation is called 'The Child's Own Mastery of Many, Count & ReCount & DoubleCount, before Adding NextTo & OnTop' and contains 43 slides.

● At Freiburg Pädagogische Hochschule October 7-8 the third Educating the Educators International Conference on approaches to scaling-up professional development in maths and science education will take place. Invitations were sent out to contribute with oral presentation sessions in the three dimensions (personal, material and structural) to report on projects, approaches and research, workshop sessions actively involving all participants, discussion group sessions also actively involving all participants, poster sessions and materials market, allowing participants to exhibit interesting professional development materials (including classroom materials) and learn about other materials.

The conference focused on three topics wanting to 'serve as a lever and platform for international exchange about concepts and experiences. The aim is to present and discuss different approaches which ensure a high quality of the education of educators:

* Personal dimension: Which roles, contents and activities have to be considered in the professional development courses for PD course leaders and facilitators in professional learning?

* Material dimension: Which role can materials play in professional development for maths and science teachers (classroom materials, face-to-face PD materials and e-learning PD materials)?

* Structural dimension: How can projects or initiatives for scaling up professional development look like and how can they be evaluated?

I sent in four proposals. One was rejected (16, sent as a poster for topic 3), two were accepted as posters (17 sent as a presentation for topic 1, 18 sent as a workshop for topic 2), one was accepted for presentation (19 sent as a discussion group 3). The proposal for a material market (21) was accepted.

16. Addition-Free Math Make Migrants and Refugees Stem Educators

17. Recounting Before Adding Makes Teachers Course Leaders and Facilitators

18. Self-explanatory Learning Material to Improve your Mastery of Many

19. Can Grounded Math and Education and Research Become Relevant to Learners

20. The PowerPointPresentation is called ‘Can Grounded Mathematics & Education & Research become Relevant to Learners?’ and contains 54 slides.

21. Recounting in Icon-Units and in Tens Before Adding Totals Next-To and On-Top, together with the posters presented at the stand.

● The following note is handed out to students and to teacher to have a basic discussion of the need and form of mathematics education.

22. What is Math - and Why Learn it?

● This material is meant for high school to illustrate how algebra and geometry should be always together and never apart.

23. Mathematics with Playing Cards

● This math compendium is meant for a high school pre-calculus course to illustrate the point made in (01) that it is possible to start all over from the bottom in a pre-calculus course, and also to give an introduction to calculus presenting integral calculus before differential calculus. The compendium also includes several projects modeling real world problems.

24. Mathematics Predicts, PreCalculus

● In Växjö January 14-15, 2020, the Swedish Society for Research in Mathematics Education welcome to Madif 12, its twelfth research seminar in connection with the Matematikbiennalen 2020. The theme of the seminar is ‘Sustainable mathematics education in a digitalized world’. I sent in three papers inspired by (01), one on early childhood education (25), and one on middle school (26), and one on precalculus (27) as well as two proposals for a workshop (28, 29). All were rejected.

25. Sustainable Adaption to Quantity: From Number Sense to Many Sense

The abstract says: Their biological capacity to adapt to their environment make children develop a number-language based upon two-dimensional block- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report ‘Improving Schools in Sweden’. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language. And math education must accept that its goal is not to mediate the truth regime of setcentric university math, but to develop the child’s already existing adaption to Many.

The chapters are called: Decreased PISA Performance Despite Increased Research Mathematics and its Education, Biology Looks at Education, Philosophy Looks at Education, Psychology Looks at Education, Sociology Looks at Education, Meeting Many, Children Bundle to Count and Share, A Contemporary Mathematics Curriculum, The Difference to a Typical Contemporary Mathematics Curriculum, Mathematics as a Number-Language, Discussing Number Sense and Number Nonsense, Conclusion and Recommendation.

26. Per-numbers connect Fractions and Proportionality and Calculus and Equations Sustainable Adaption to Quantity: From Number Sense to Many Sense

The abstract says: In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word ‘percent’ to also talk about other ‘per-numbers’ as e.g. ‘per-five’ thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

The chapters are called: Mathematics is Hard, or is it, The ICMT3 Conference, Different Ways of Seeing Fractions, Ratios and Rates, Per-numbers Occur when Double-counting a Total in two Units, Fractions as Per-numbers, Expanding and Shortening Fractions, Taking Fractions of Fractions, the Per-number Way, Direct and Inverse Proportionality, Adding Fractions, the Per-number Way, Solving Proportionality Equations by Recounting , Seven Ways to Solve the two Proportionality Questions, A Case: Peter, about to Peter Out of Teaching, Discussion and Recommendation

27. Sustainable Adaption to Double-Quantity: From Pre-Calculus to Per-Number Calculations

The abstract says: Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

The chapters are called: A need for curricula for all students, A Traditional Precalculus Curriculum, A Different Precalculus Curriculum, Precalculus, building on or rebuilding?, Using Sociological Imagination to Create a Paradigm Shift, A Grounded Outside-Inside Fresh-start Precalculus from Scratch, Solving Equations by Moving to Opposite Side with Opposite Sign, Recounting Grounds Proportionality, Double-counting Grounds Per-numbers and Fractions, The Change Formulas, Precalculus Deals with Uniting Constant Per-Numbers as Factors, Calculus Deals with Uniting Changing Per-Numbers as Areas, Statistics Deals with Unpredictable Change, Modeling in Precalculus Exemplifies Quantitative Literature, A Literature Based Compendium, An Example of a Fresh/start Precalculus Curriculum, An Example of an Exam Question, Discussion and Conclusion.

28. A Lyotardian dissension to the early childhood consensus on numbers and operations. The chapter are called: Can sociological imagination improve mathematics education? Consensus and Dissension on Early Childhood Numbers & Operations. Time Table for the Workshop.

29. Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis? The chapter are called: Does Mathematics Education Research have an Irrelevance Paradox? The Replication Crisis in Science. Time Table for the Workshop.

30. Bundle Counting Table. A guide to bundle-counting in pre-school. Written for the stand at the Matematikbiennale.

31. Proposals for the 2020 Swedish Math Biennale. All were rejected.

● At the Ho Chi Minh City University of Education on December 7, a conference was held called ‘Psychology and Mathematics education’. I was invited to give the plenary talk Saturday, which I named after the paper I send in (32), together with a Power Point Presentation (33). Sunday, I gave a talk on modeling to a group of master students. Monday, I gave a talk to a class of senior students on a poster presentation from the ‘Educating the Educators’ conference in Freiburg, Germany, in October, and handed out the notes ‘What is Math - and Why Learn it?’ and ‘Bundle Counting

Table’. Tuesday, I gave a talk to the staff on research in mathematics education and networks to join and design research as a methodology to use when researching the implementation of the new activity-based curriculum inspired by Kolb’s experimental learning theory.

32. De-Modeling Numbers, Operations and Equations: From Inside-Inside to Outside-Inside Understanding

The abstract says: Adapting to the outside fact Many, children internalize social number-names, but how do they externalize them when communicating about outside numerosity? Mastering Many, children use bundle-numbers with units; and flexibly use fractions and decimals and negative numbers to account for the unbundled singles. This suggests designing a curriculum that by replacing abstract-based with concrete-based psychology mediates understanding through de-modeling core mathematics, thus allowing children to expand the number-language they bring to school.

The chapters are: 1. Introduction, 2. Materials/ Subjects and Methods, 2.1. Reflections on Different forms of Mathematics, 2.2. Reflections on Different forms of Psychology, 2.3. Merging Mathematics and Psychology, 2.4. De-modelling Digits, 2.4.1. Designing and Implementing a micro-curriculum, 2.5. Reflections on how to De-model Bundle-counting Sequences, 2.5.1. Designing and Implementing a micro-curriculum, 2.6. Reflections on how to De-model Operations, 2.7. Reflections on how to Recount into Tens, 2.7.1. Designing and Implementing a micro-curriculum, 2.8. Reflections on how to Model Double-counting with Per-numbers and Fractions, 2.9. Reflections on how to De-model Trigonometry, 3. Results and Discussion, 4. Conclusion.

33. De-Model Numbers, Operations and Equations, PPP.

34. Visit to Ho Chi Minh City University of Education December 7-13 2019.

● The ICMT3 and Educating Educators conferences used peer-reviews, and in the first you were allowed to comment on the reviews

35. Review 01 ICMT3

36. Review 02 ICMT3

37. Comments to ICMT3 Reviewers

38. Educating Educators Reviews

Aarhus, December 2019, Allan Tarp

01. The Same Mathematics Curriculum for Different Students

To offer mathematics to all students, parallel tracks often occur from the middle of secondary school. The main track continues with a full curriculum, while parallel tracks might use a reduced curriculum leaving out e.g. calculus; or they might contain a different kind of mathematics meant to be more relevant to students by including more applications. However, an opportunity here presents itself for designing the same curriculum for all students no matter which track they may choose. As number-language, why not let mathematics follow the communicative turn that took place in language education in the 1970s by prioritizing its connection to the outside world higher than its inside connection to its grammar? We will consider examples of all three curricula options.

01. A Need for Curricula for all Students

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing the Organisation for Economic Co-operation and Development (OECD, 2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

> PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)

Other countries also experience declining PISA results; and in high performing countries not all students are doing well.

02. Addressing the Need

By saying "All students should study mathematics in each of the four years that they are enrolled in high school." the US National Council of Teachers of Mathematics (2000, p. 18) has addressed the need for curricula for all students in their publication 'Principles and Standards for School Mathematics'. In the overview the Council writes

> We live in a mathematical world. Whenever we decide on a purchase, choose an insurance or health plan, or use a spreadsheet, we rely on mathematical understanding (..) In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors. (..) everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics. Students need to learn a new set of mathematics basics that enable them to compute fluently and to solve problems creatively and resourcefully. (p. 1)

In this way the Council points out that it is important to master 'mathematical competence', i.e. to understand and do mathematics to solve problems creatively and to compute fluently. This will benefit the personal life, the workplace, as well as further study leading to productive futures.

Consequently, the Council has included in the publication a curriculum that "is mathematically rich providing students with opportunities to learn important mathematical concepts and procedures with understanding". This in order to "provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals."

The publication includes a set of standards: "The Standards for school mathematics describe the mathematical understanding, knowledge, and skills that students should acquire from prekindergarten to grade 12." The five standards

> present goals in the mathematical content areas of number and operations, algebra , geometry, measurement and data analysis and probability. (..) Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century" (p. 2)

In the chapter 'Number and operations', the Council writes

> Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense. Students with number sense naturally decompose numbers (..) For example, children in the lower elementary grades can learn that numbers can be decomposed and thought about in many different ways - that 24 is 2 tens and 4 ones and also two sets of 12. (p. 7)

In the chapter 'The Curriculum Principle', the Council writes

> A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades (..) for teachers at each level to know what mathematics their students have already studied and will study in future grades. (p. 3, 4)

All in all, the Council points to the necessity of designing a curriculum that is relevant in students' 'personal life, in the workplace and in further study' and that is coherent at the same time to allow teachers to know 'what mathematics their students have already studied and will study in future grades'.

03. Coherence and Relevance

So, in their publication, the National Council of Teachers of Mathematics stresses the importance of coherence and relevance. To allow teachers follow a prescribed curriculum effectively, and to allow students build upon what they already know, it must be 'well articulated across the grades'. And, to have importance for students a curriculum must be relevant by supplying them with 'the basic skills and understandings that students will need to function effectively in the twenty-first century'.

With 'cohere' as a verb and 'relevant' as a predicate we can ask: "to what does this curriculum cohere, and to what is it relevant?" As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.'

We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that 'to cohere' means 'to hold together firmly as parts of the same mass'. As to synonyms for cohere, it lists: 'accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.' And as to antonyms, it lists: 'differ (from), disagree (with).'

In the same dictionary, the word 'relevant' means 'having significant and demonstrable bearing on the matter at hand'. As to synonyms for relevant, it lists: 'applicable, apposite, apropos, germane, material, pertinent, pointed, relative.' And as to antonyms, it lists: 'extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.'

If we accept the verb 'apply' as having a meaning close to the predicate 'relevant', we can rephrase the above analysis question using verbs only: "to what does this curriculum cohere and apply?"

Metaphorically, we may see education as increasing skills and knowledge by bridging individual start levels to a common end level described by institutional goals. So, we may now give a first definition of an ideal curriculum: "To apply to a learning process as relevant, a curriculum coheres to the individual start levels and to the end goal, which again coheres with the need expressed by the society funding the educational institution."

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

- How to make the bridge cohere with the individual start levels in a class?
- How to make the end level cohere to goals expressed by the society?
- How to make the end level cohere to goals expressed by the learners?
- How to make the bridge cohere to previous and following bridges?
- How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

04. Parallel Tracks to the Main Curriculum, Examples

In their publication chapter Grades 9 through 12, the National Council of Teachers of Mathematics discusses to the possibility to introduce parallel courses in the high school.

> In secondary school, all students should learn an ambitious common foundation of mathematical ideas and applications. This shared mathematical understanding is as important for students who will enter the workplace as it is for those who will pursue further study in mathematics and science. All students should study mathematics in each of the four years that they are enrolled in high school.
>
> Because students' interests and inspirations may change during and after high school, their mathematics education should guarantee access to a broad spectrum of careers and educational options. They should experience the interplay of algebra, geometry, statistics, probability and discrete mathematics.
>
> High school mathematics builds on the skills and understandings developed in the lower grades. (..) High school students can study mathematics that extends beyond the material expected of all students in at least three ways. One is to include in the curriculum material that extends the foundational material in depth or sophistication. Two other approaches make use of supplementary courses. In the first students enroll in additional courses concurrent with those expected of all students. In the second, students complete a three-year version of the shared material and take other mathematics courses. In both situations, students can choose from such courses as computer science, technical mathematics, statistics, and calculus. Each of these approaches has the essential property that all students learn the same foundation of mathematics but some, if they wish, can study additional mathematics. (p. 18-19)

The Council thus emphasizes the importance of studying 'mathematics in each of the four years that they are enrolled in high school'. This the council sees as feasible if implementing one or more of three options allowing students to 'study mathematics that extends beyond the material expected of all students'. Some students may want to study 'material that extends the foundational material in depth or sophistication'. Others may want to take additional courses cohering to the college level, especially calculus. Others may want to take additional courses relevant to their daily life or a workplace. We will now look at two examples of that both including examples of finite mathematics, a subject that is normally outside a standard high school curriculum.

For all Practical Purposes, Introduction to Contemporary Mathematics

In the US, the Consortium for Mathematics and its Applications (COMAP) has worked out a material called ' For all practical purposes' (COMAP, 1988). In its preface, the material presents itself as

> (..) an introductory mathematics course for students in the liberal arts or other nontechnical curricula. The course consists of twenty-six half-hour television shows, the textbook, and this Telecourse guide. This series shows mathematics at work in today's world. (..) For all practical purposes aims to develop conceptual understanding of the tools and language of mathematics and the ability to reason using them. We expect most students will have completed elementary algebra and some geometry in high school. We do not assume students will be pursuing additional courses in mathematics, at least none beyond the introductory level. (p. iii)

As to content, the material has five parts (p. v - vi)

Part one focuses on graph theory and linear programming illustrated with network as scheduling and planning. It includes an overview show and four additional shows called street smarts: street networks; trains, planes and critical paths; juggling machines: scheduling problems; juicy problems: linear programming.

Part two deals with statistics and probability illustrated with collecting and deducing from data. It includes an overview show and four additional shows called behind the headlines: collecting data; picture this: organizing data; place your bets: probability; confident conclusions: statistical inference.

Part three focuses on social choice, fair division and game theory illustrated by different voting systems and conflict handling. It includes an overview show and four additional shows called the impossible dream: election theory; more equal than others: weighted voting; zero-sum games: games of conflict; prisoner's dilemma: games of partial conflict.

Part four focuses on using geometry, the classical conic sections, shapes for tiling a surface, geometric growth in finance in and in population, and measurement. It includes an overview show and four additional shows called how big is too big: scale and form; it grows and grows: populations; stand up conic: conic sections; it started in Greece: measurement.

Part five focuses on computer algorithms. It includes an overview show and four additional shows called rules of the games: algorithms; counting by two's: numerical representation; creating a cde: encoding information; moving picture show: computer graphics.

The video sections are available on YouTube.

A Portuguese Parallel High School Curriculum

Portugal followed up on the COMAP initiative. In his paper called "Secondary mathematics for the social sciences" (Silva, 2018), Jaime Silva describes how the initiative inspired an innovative two-year curriculum for the Portuguese upper secondary school.

As to the background, Silva writes

> There are two recurring debates about the mathematics curriculum in secondary schools, especially in countries like Portugal where compulsory education goes till the 12th grade. First, should all students study mathematics (not necessarily the same) or should the curriculum leave a part of the students "happy" without the mathematics "torture"? Second, should all students study the same "classic" mathematics, around ideas from differential and integral calculus with some Geometry and some Statistics?
>
> When the 2001 revision (in great part in application today) of the Portuguese Secondary School curriculum was made (involving the 10th, 11th and 12th grades) different kinds of courses were introduced for the different tracks (but not for all of them) that traditionally existed. Mathematics A is for the Science and Technology track and for the Economics track and is a compulsory course. Mathematics B is for the Arts track and is an optional course. Mathematics Applied to the Social Sciences (MACS) is for the Social Sciences track and is an optional course. The Languages track was left without mathematics or science. Later the last two tracks were merged and the MACS course remained optional for the new merged track. The technological or professional tracks could have Mathematics B, Mathematics for the Arts or Modules of Mathematics (3 to 10 to be chosen from 16 different modules, depending on the professions). (p. 309)

As to the result of debating a reform in Portugal, Silva writes

> When, in 2001, there was a possibility to introduce a new Mathematics course for the "Social Sciences" track, for the 10th and 11th grade students, there were some discussions of what could be offered. The model chosen was inspired in the course "For All Practical Purposes" (COMAP, 2000) developed by COMAP because it "uses both contemporary and classic examples to help students appreciate the use of math in their everyday lives". As a consequence, a set of independent chapters, each one with some specific purpose, was chosen for this syllabus, that included 2 years of study, with 4.5 hours of classes per week (normally 3 classes of 90 minutes each). The topics chosen were: 10th grade Decision Methods:

> Election Methods, Apportionment, Fair Division; Mathematical Models: Financial models, Population models Statistics (regression); 11th grade Graph models, Probability models, Statistics (inference). (p. 310)

As to the goal of the curriculum, Silva writes

> The stated goal of this course is for the students to have *"significant mathematical experiences that allow them to appreciate adequately the importance of the mathematical approaches in their future activities"*. This means that the main goal is not to master specific mathematical concepts, but it is really to give students a new perspective on the real world with mathematics, and to change the students view of the importance that mathematical tools will have in their future life. In this course it is expected that the students study simple real situations in a form as complete as possible, and *"develop the skills to formulate and solve mathematically problems and develop the skill to communicate mathematical ideas (students should be able to write and read texts with mathematical content describing concrete situations)".* (p. 310)

As to the reception of the curriculum, Silva writes

> This was a huge challenge for the Portuguese educational system because most of these topics had never been covered before, and most teachers did not even study Graph Theory at University. Election Methods, Apportionment and Fair Division were of course completely new to everybody. The reception was good from the part of the Portuguese Math Teacher Association APM, as it considered that *"the methodologies and activities suggested in the MACS program promote the development of the skills of social intervention, of citizenship and others"*. The reception from the scientific society SPM was rather negative because they considered the syllabus did not have enough mathematical content. (p. 310-311)

As to the present state of the curriculum, Silva writes

> After 15 years there is no thorough evaluation of how the course is run in practice in the schools, or which is the real impact on the further education or activities of the students that studied "Mathematics Applied to the Social Sciences". In Portugal there is no institution in charge of this type of work and evaluations are done on a case by case basis. All Secondary Schools need to do selfevaluations but normally just compare internal statistics to national ones to see where they are in the national scene. In the reports consulted there was no special mention to the MACS course and so we have the impression that the MACS course entered the normal Portuguese routine in Secondary School. (p. 315)

So as to a parallel track to the traditional curriculum, the National Council of Teachers of Mathematics suggests that including a different kind of mathematics might be an option, e.g. finite mathematics. In the US this idea was taken up by the Consortium for Mathematics and its Applications (COMAP) working out a material including a textbook and a series of television shows to show 'mathematics at work in today's world'. Part of this material was also included in a parallel curriculum in Portugal called 'Mathematics Applied to the Social Sciences' (MACS) offering to Portuguese students also to study mathematics in each of their high school years, as the National Council of Teachers of Mathematics recommends.

05. Precalculus, Typically the last Mandatory Curriculum

This chapter looks at the part of a mathematics curriculum called precalculus, typically being the first part that is described in a parallel curriculum since it contains operations as root and logarithm that is not considered part of a basic mathematics algebra curriculum. First, we look at an example of a traditional precalculus curriculum. The we ask what could be an ideal precalculus curriculum, and illustrates it with two examples. In the next chapter, we look at a special case, a Danish precalculus curriculum that has served both as a parallel and a serial curriculum during the last 50 years.

A Traditional Precalculus Course

An example of a traditional precalculus course is found in the Research and Education Association book precalculus (Woodward, 2010). The book has ten chapters. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

An Ideal Precalculus Curriculum

In their publication, the National Council of Teachers of Mathematics writes "High school mathematics builds on the skills and understandings developed in the lower grades. (p. 19)"

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they have from the lower grades?

Mathe-matics, Meta-matics, and Mathe-matism

Furthermore, what kind of mathematics has been taught? Was it 'grounded mathematics' abstracted bottom-up from its outside roots, or 'ungrounded mathematics' or 'meta-matics' exemplified top-down from inside abstractions, maybe becoming 'meta-matism' if mixed with 'mathe-matism' (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept 'function', Euler saw it as a bottom-up abstracted name for 'standby calculations' containing specified and unspecified numbers. Later meta-matics defined a function top-down as an example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may be seen as a number-language sentence containing a subject, a verb and a predicate allowing its value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group concepts applying the associative and commutative law as well as the neutral element and inverse elements thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2.

2*u = 6, so (2*u)*½ = 6*½, so (u*2)*½ = 3, so u*(2*½) = 3, so u*1 = 3, so u = 3.

However the equation 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula 'T = (T/B)*B' present all over mathematics as the proportionality formula thus solved in one step:

2*u = 6 = (6/2)*2, giving u = 6/2 = 3.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Bauman (1990) as the core of sociology. Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction about an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

Restarting from Scratch with Grounded Outside-Inside Mathematics

Let students see how outside degrees of Many are iconized by inside digits with as many strokes as it represents, five strokes in the 5-icon etc. Let students see that after nine we count by bundling creating icons for the counting operations as well, where division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see that an outside block of 2 3s becomes an inside calculation 2*3 and vice versa. Let students see the commutative law by turning and a*b block, and see the distributive law by splitting a into c and d, and see the associative law by turning an a*b*c box.

Let students see that both the word- and the number-language use full sentences with a subject, a verb, and an object or predicate, abbreviating ‘the total is 2 3s’ to ‘T = 2*3’

Let students enjoy flexible bundle-numbers where decimals and fractions negative and numbers are created to describe the unbundle ones placed next-to or on-top of the block, thus allowing 5 to be recounted in 3s as T = 5 = 1B2 = 1.2 B = 1 2/3 B = 2B-1.

Let student see, that recounting in other units may be predicted by the recount-formula ‘T = (T/B)*B’ saying “From the total T, T/B times, B may be pushed away”. Let students see that where the recount-formula in primary school recounts 6 in 2s as 6 = (6/2)*2 = 3*2, in secondary school the same task is formulated as solving the equation u*2 = 6 as u*2 = 6 = (6/2)*2 giving u = 6/2, thus moving 2 to the opposite side with the opposite calculation sign.

Let students see the power of ‘flexible bundle-numbers’ when the inside multiplication 7*8 = (B-3)*(B-2) = BB-2B-3B+6 = 5B6 = 56 may be illustrated on an outside ten by ten block, thus showing that of course minus times minus must give plus since the 2*3 corner has been subtracted twice.

Let students see that double-counting in two units create per-numbers as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and asking ‘9kg = ?$’ we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

And, that double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Thus finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

And, that double-counting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA * c; b = (b/c)*c = cosA * c; a = (a/b)*b = tanA * b. With a circle filled from the inside by right triangles, this also allows phi to be found from a formula: circumference/diameter = π ≈ n*tan(180/n) for n large.

And, how recounting and double-counting physical units create per-numbers and proportionality all over STEM, Science, Technology, Engineering and mathematics: kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter.

Also, let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation formula. Writing 'y = f(x)' means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Algebra and Geometry, Always Together, Never Apart

Let students enjoy the power and beauty of integrating algebra and geometry.

First, let students enjoy seeing that multiplication creates blocks with areas where 3*7 is 3 7s that may be algebraically recounted in tens as 2.1 tens. Or, that may be geometrically transformed to a square u^2 giving the algebraic equation u^2 = 21, creating root as the reverse calculation to power, u = √21. Which may be found approximately by locating the nearest number p below u, here p = 4, so that u^2 = (4+t)^2 = 4^2 + 2*4*t + t^2 = 21.

Neglecting t^2 since t is less than 1, we get 4^2 + 2*4*t = 21, which gives t = $\frac{21 - 4^2}{4*2}$, or t = $\frac{N - p^2}{p*2}$, if p is the nearest number below u, where u^2 = N.

As an approximation, we then get √N ≈ p + t = p + $\frac{N - p^2}{p*2}$; or √N ≈ $\frac{N+ p^2}{p*2}$, if p^2 < N < (p+1)^2

Then let students enjoy the power and beauty of predicting where a line geometrically intersects lines or circles or parabolas by algebraically solving two equations with two unknowns, also predicted by a computer software.

A Number Seen as a multiple Numbering

Let students see the number 456 as what it really is, not three ordered digits obeying a place-value system, but three numberings of bundles-of-bundles, bundles, and unbundled ones as expressed in the number-formula, formally called a polynomial: T = 456 = 4*B^2 + 5*B + 6*1, with B = ten.

Let students see that a number-formula contains the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, students see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Figure 01. *The 'algebra-square' shows the four ways to unite or split numbers.*

Let students see calculations as predictions, where 2+3 predicts what happens when counting on 3 times from 2; where 2*5 predicts what happens when adding 2$ 5 times; where 1.02^5 predicts what happens when adding 2% 5 times; and where adding the areas 2*3 + 4*5 predicts how to add the per-numbers when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Solving Equations by Reversed Calculation Moving Numbers to Opposite Side

Let students see the subtraction '5-3' as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers 'to opposite sign with opposite calculation sign'; a rule that applies also to the other reversed operations:

- the division u = 5/3 is the number u that multiplied with 3 gives 5, u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, u^3 = 5, making root a 'factor-finder'
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, 3^u = 5, making logarithm a 'factor-counter'.

Let students see multiple calculations reduce to single calculations by unhiding 'hidden bracket' where 2+3*4 = 2+(3*4) since with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s. This will prevent solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5, by allowing the hidden bracket to be shown: 2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4 to be verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy singing a 'Hymn to Equations': "Equations are the best we know, they're solved by isolation. But first the bracket must be placed, around multiplication. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don't want to stop!"

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let student place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, the first two terms disappear by adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = 0 + b^2/4 – c = b^2/4 – c.

Now, moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c

u + b/2 = $\pm\sqrt{b\text{^}2/4 - c}$

u = -b/2 $\pm\sqrt{b\text{^}2/4 - c}$

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, containing also the formulas for constant change: T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change: constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to 'post-dict' what we cannot 'pre-dict'.

Combining linear and exponential change by n times depositing a$ to an interest rate r%, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest rate R is predicted by the formula 1+R = (1+r)^n. Such a saving may be used to neutralize a debt Do, that in the same period has changed to D = Do*(1+R).

The formula and the proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Constant Change

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers creating the operations integration and its reverse, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power and its two inverse operations, root and logarithm.

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$ changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, leading to the formula for change with a constant change-factor, also called exponential change, T = b*a^n. Reversing the question, this formula entails two equations.

The first equation asks about an unknown change-percent. Thus, we might want to find which percent that added ten times will give a total change-percent 70%, or, formulated with change-factors, what is the change-factor, a, that applied ten times gives the change-factor 1.70. So here the job is ‘factor-finding’ which leads to defining the tenth root of 1.70, i.e. 10√1.70, as predicting the factor, a, that applied 10 times gives 1.70: If a^10 = 1.70 then a = 10√1.70 = 1.054 = 105.4%. This is verified by testing: 1.054^10 = 1.692. Thus, the answer is “5.4% is the percent that added ten times will give a total change-percent 70%.”

We notice that 5.4% added ten times gives 54% only, so the 16% remaining to 70% is the effect of compound interest adding 5.4% also to the previous changes.

Here we solve the equation a^10 = 1.70 by moving the exponent to the opposite side with the opposite calculation sign, the tenth root, a = 10√1.70. This resonates with the ‘to opposite side with opposite calculation sign’ method that also solved the equations a+3 = 7 by a = 7-3, and a*3 = 20 by a = 20/3.

The second equation asks about a time-period. Thus, we might want to find how many times 7% must be added to give 70%, 1.07^n = 1.70. So here the job is factor-counting which leads to defining the logarithm log1.07(1.70) as the number of factors 1.07 that will give a total factor at 1.70: If 1.07^n = 1.70 then n = log1.07(1.70) = 7,84 verified by testing: 1.07^7.84 = 1.700.

We notice that simple addition of 7% ten times gives 70%, but with compound interest it gives a total change-factor 1.07^10 = 1.967, i.e. an additional change at 96.7%-70% = 26.7%, explaining why only 7.84 periods are needed instead of ten.

Here we solve the equation 1.07^n = 1.70 by moving the base to the opposite side with the opposite calculation sign, the base logarithm, n = log1.07(1.70). Again, this resonates with the ‘to opposite side with opposite calculation sign’ method.

An ideal precalculus curriculum could ‘de-model’ the constant percent change exponential formula T = b*a^n to outside real-world problems as a capital in a bank, or as a population increasing or radioactive atoms decreasing by a constant change-percent per year.

De-modeling may also lead to situations where the change-elasticity is constant as in science multiplication formulas wanting to relate a percent change in T with a percent change in n.

An example is the area of a square T = s^2 where a 1% change in the side s will give a 2% change in the square, approximately: With To = s^2, T1 = (s*1.01)^2 = s^2*1.01^2 = s^2* 1.0201 = To*1.0201.

Once mastery of constant change-percent is established, it is possible to look at time series in statistical tables asking e.g. “How has the unemployment changed over a ten-year period?” Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b+a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n. These average numbers would allow setting up a forecast predicting the yearly numbers in the ten-year period, if the numbers were predictable. However, they are not, so instead of predicting the number with a formula, we might 'post-dict' the numbers using statistics dealing with unpredictable numbers, but still trying to predict a plausible interval by describing the unpredictable random change by nonfictional numbers, median and quartiles, or by fictional numbers, mean and standard deviation.

Calculus Deals with Adding Per-Numbers by Their Areas

Likewise, real-world phenomena as unemployment occur in both time and space, so unemployment may also change in space, e.g. from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status, also called contingency tables or crosstabs. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add, they must be multiplied back to unit-numbers to find the total percent. However, once you multiply you create an area, and adding per-numbers by areas is what calculus is about, thus here introduced in a natural way through double-tables from statistical materials.

An example: in one region 10 persons have 50% unemployment, in another, 90 persons have 5% unemployment. To find the total, the unit-numbers can be added directly to 100 persons, but the percent-numbers must be multiplied back to numbers: 10 persons have 10*0,5 = 5 unemployed; and 90 persons have 90*0,05 = 4.5 unemployed, a total of 5+4.5 unemployed = 9.5 unemployed among 100 persons, i.e. a total of 9.5% unemployment, also called the weighted average. Later, this may be renamed to Bayes formula for conditional probability.

Calculus as adding per-numbers by their areas may also be introduced through mixture problems asking e.g. '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?' Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before being added, thus adding by their areas.

Modeling in Precalculus

Furthermore, the entry of graphing calculators allows authentic modeling to be included in a pre-calculus curriculum thus giving a natural introduction to the following calculus curriculum as well.

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A' = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A’ = x’ = 1; and that if A = ½*x^2, then y = A’ = (½*x^2)’ = x; and that if A = 1/3*x^3, then y = A’ = (1/3*x^3)’ = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to x = 3, instead of adding small strips we just calculate the change in the area over this distance.

This makes sense since adding many small strips means adding many small changes, which gives just one final change since all the in-between end- and start-values cancel out:

$\int_1^3 y * dx = \int_1^3 dA = \Delta_1^3\, A = \Delta_1^3\, (\frac{1}{3} * x^3) = \text{end} - \text{start} = \frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$

On the calculus course we just leave out the area by renaming it to a ‘primitive’ or an ‘antiderivative’ when writing

$\int_1^3 y * dx = \int_1^3 x^2 * dx = \Delta_1^3\, (\frac{1}{3} * x^3) = \text{end} - \text{start} = \frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$

A graphing calculator show that this suggestion holds. So, finding areas under per-number graphs not only allows adding per-numbers, it also gives a grounded and natural introduction to integral and differential calculus where integration precedes differentiation just as additions precedes subtraction.

From the outside, regression allows giving a practical introduction to calculus by analysing a road trip where the per-number speed is measured in five second intervals to respectively 10 m/s, 30 m/s, 20 m/s 40 m/s and 15 m/s. With a five data-set table we can choose to model with a degree4 polynomial found by regression. Within this model we can predict when the driving began and ended, what the speed and the acceleration was after 12 seconds, when the speed was 25m/s, when acceleration and braking took place, what the maximum speed was, and what distance is covered in total and in the different intervals.

Another example of regression is the project ‘Population versus food’ looking at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2- line table regression can produce two formulas: with x counting years after 1850, the population is modeled by Y1= 815*1.013^x and the food by Y2= 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e. from 1973.

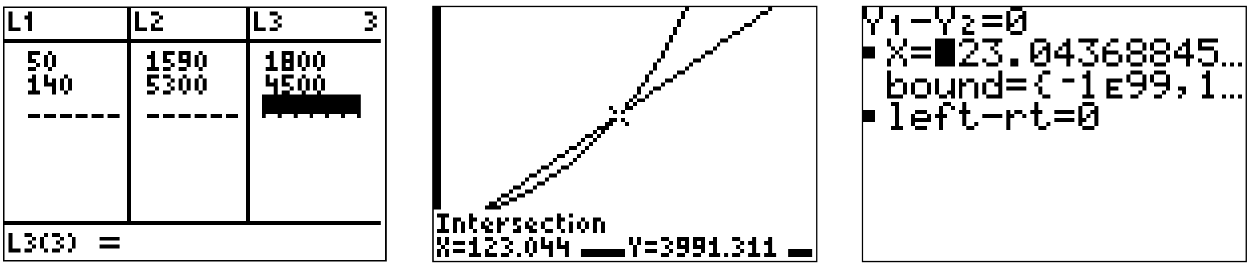

Figure 02. A Malthusian model of population and food levels

An example of an ideal precalculus curriculum is described in a paper called ‘Saving Dropout Ryan With a Ti-82’ (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren’t even able to use a TI-30. A compendium called ‘Formula Predict’ (Tarp, 2009) replaced the textbook. A formula’s left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by ‘solve Y1-Y2 = 0’. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

With the increased educational interest in STEM, modeling also allows including science-problems as e.g. the transfer of heat taking place when placing an ice cube in water or in a mixture of water and alcohol, or the transfer of energy taking place when connecting an energy source with energy consuming bulbs in series or parallel; as well as technology problems as how to send of a golf ball to hit a desired hole, or when to jump from a swing to maximize the jumping length; as well as engineering problems as how to build a road inclining 5% on a hillside inclining 10%.

Furthermore, precalculus allows students to play with change-equations, later called differential equations since change is calculated as a difference, $\Delta T = T2 – T1$. Using a spreadsheet, it is fun to see the behavior of a total that changes with a constant number or a constant percent, as well as with a decreasing number or a decreasing percent, as well as with half the distance to a maximum value or with a percent decreasing until disappearing at a maximum value. And to see the behavior of a total accelerating with a constant number as in the case of gravity, or with a number proportional to its distance to an equilibrium point as in the case of a spring.

So, by focusing on uniting and splitting into constant per-numbers, the ideal precalculus curriculum has constant change-percent as its core. This will cohere with a previous curriculum on constant change-number or linearity; as well as with the following curriculum on calculus focusing on uniting and splitting into locally constant per-numbers, thus dealing with local linearity. Likewise, such a precalculus curriculum is relevant to the workplace where forecasts based upon assumptions of a constant change-number or change-percent are frequent. This curriculum is also relevant to the students' daily life as participants in civil society where tables presented in the media are frequent.

Two Curriculum Examples Inspired by an Ideal Precalculus Curriculum

An example of a curriculum inspired by the above outline was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

Later, around year 2000, another version was designed but not tested. The curriculum goal was stated as: 'the students develop their number-language so they can participate in social practices involving quantitative descriptions of change and shape.' The curriculum means included

1. Numbers and calculations. Quantities and qualities. Number-language, word-language, meta-language. Unit-numbers and per-numbers, and how to calculate their totals. Equations as predicting statements. Forwards and reverse calculations.

2. Change calculations. Change measuring change with change-number and change-percent and index-number. Calculation rules for the change of a sum, a product and a ratio.

3. Constant change. Change with a constant change-number. Change with a constant change-percent. Change with both.

4. Unpredictable change. Fractals, mean and deviation, 95% confidence interval. Binomial distribution approximated by a normal distribution.

Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

5. Trigonometry. Dividing and measuring earth. Calculation the sides and angles in a triangle.

06. PRECALCULUS IN THE DANISH PARALLEL HIGH SCHOOL, A CASE STUDY

In the post-war era, the Organization for Economic Co-operation and Development (OECD) called for increasing the population knowledge level, e.g. by offering a second chance to take a high school degree giving entrance to tertiary education. In Denmark in 1966, this resulted in creating a two-year education called 'Higher preparation exam' as a parallel to the traditional high school. Two levels of two-years mathematics courses were included, a basic precalculus course for those who did not choose the calculus course.

The 1966 Curriculum

The precalculus curriculum came from leaving out small parts of the calculus curriculum, thus being an example of a reduced curriculum.

The goal of the calculus course stated it should 'supply students with knowledge about basic mathematical thinking and about applications in other subject areas, thus providing them with prerequisites for carrying through tertiary education needing mathematics.'

The goal of the precalculus course was reduced to 'supplying students with an impression of mathematical thinking and method and to mediate mathematical knowledge useful also to other subject areas.'

So, where the calculus curriculum has to cohere and be relevant to tertiary education needing mathematics, the precalculus course is a parallel curriculum meant to be relevant to the students themselves and to other high school subjects.

The content of the precalculus curriculum had five sections.

The first section contained basic concepts from set theory as sets, subsets, complementary set, union, intersection, product, difference. The function concept. Mapping into an on a different set, one-to one mapping, inverse mapping (inverse function), composite mappings. The calculus curriculum added nothing here.

Section two contained concepts from abstract algebra: Composition rules. The associative law. The commutative law. Neutral element. Inverse element. The group concept with examples. Rules for operations on real numbers. Numeric value. Here the calculus curriculum added the distributive law, the concept of a ring and a field, the ring of whole numbers as well as quotient classes. The calculus curriculum added nothing here.

Section three contained equations and inequalities. Examples on open statements in one or two variables. Equations and inequalities of degree one and two with one unknown. Equations and inequalities with the unknown placed inside a square root or a numeric sign. Simple examples of Equations and inequalities of degree one and two with two unknowns. Graphical illustration. The calculus curriculum added nothing here.

Section four contained basic functions. The linear function in one variable. A piecewise linear function. The second-degree polynomial. The logarithm function with base ten, the logarithmic scale, the calculator stick, the use of logarithm tables. Trigonometric functions, tables with functions values. Calculations on a right-angled triangle using trigonometric functions. Here the calculus curriculum added rational functions in one variable, exponential functions, and the addition formulas and logarithmic formulas in trigonometry.

Section five contained combinatorics. The multiplication principle. Permutations and combinations. Here the calculus curriculum added probability theory, probability field, and examples of probability based upon combinatorics.

Finally, the calculus curriculum added a section about calculus.

The new set-based mathematics coming into education around 1960 inspired the 1966 precalculus curriculum thus cohering with the university mathematics at that time, but it was not especially relevant to the students. Many had difficulties understanding it and they often complained about seeing no reason for learning it or why it was taught.

In my own class, I presented it as a legal game where we were educating us to become lawyers that could convince a jury that we were using lawful methods to solving equations in one of two different methods by referring to the relevant paragraphs in the law. The first method was the traditional one used at that time way by moving numbers to the opposite side with opposite calculation sign, now legitimized by the theorem that in a group the equation a*u = b has as a solution a^-1*b. The second method was a new way with many small steps where, for each step, you have to refer to laws for associativity, and commutativity etc.; and, where a group contained exactly the paragraphs needed to use this method. Once seen that way, the students found it easy but boring. However, they accepted since they needed the exam to go on, and we typically finished the course in half time allowing time for writing a script for a movie to be presented at the annual gala party.

So, all in all, the 1963 curriculum was coherent with the next step, calculus, and with the university math view at that time, set-based; but it was mostly irrelevant to the students.

The 1974 Curriculum

The student rebellion in 1968 asked for relevance in education, which led to a second precalculus in 1974 revision. Here the goal was stated as ‘giving the students a mathematical knowledge that could be useful to other subjects and to their daily life, as well as an impression of mathematical methods thinking’. Now the curriculum structure was changed from a parallel one to a serial one where all students took the precalculus course and some chose to continue with the calculus course afterwards just specifying in its curriculum what was needed to be added.

The 1974 precalculus curriculum now had four sections.

The first section contained concepts from set theory and logic and combinatorics. Set, subset; solution set to an open statement, examples on solving simple equations and inequalities in one variable; the multiplication principle, combinations.

Section two contained the function concepts: Domain, function value, range; injective function; monotony intervals; inverse function, composite function.

Section three contained special functions; graphical illustration. A linear function, a piecewise linear function, an exponential function; examples of functions defined by tables; coordinate system, logarithmic paper.

Section four contained descriptive statistics. Observations described by numbers; frequency and their distribution and cumulated distribution; graphical illustration; statistical descriptors.

Section five described probability and statistics. A random experiment, outcome space, probability function, probability field; sampling; binomial distribution; binomial testing with zero hypothesis, critical set, significance level, single and double-sided test, failure of first degree.

Section six was called ‘Free lessons’. 20m lessons are to be used for studying details in one of the above sections, or together with one or more other school subjects to work with an area applying mathematics.

The second 1974 curriculum thus maintains a basis of set-theory but leaves out the abstract algebra. As to functions, it replaces the second-degree polynomial with the exponential function. Here trigonometry is excluded to be included in the calculus curriculum.

The combinatorics section is to great extent replaced by descriptive statics.

Finally, the section has been added with quite detailed probability theory and testing theory within statistics.

All in all, the coherence with the university set-based mathematics has been softened by leaving out abstract algebra and second-degree polynomial. Instead of introducing a first-degree polynomial together with a second-degree polynomial, the former now is introduced as a linear function together with the exponential function allowing modelling outside change with both a constant change-number and a constant change-percent. This makes the curriculum more relevant to the students individually as well as to other high school subjects as required by the goal statement.

The quite detailed section on testing theory was supposed to make the curriculum more relevant to students but the degree of detail make it fail to do so by drowning in quite abstract concepts.

The 1990 Curriculum

As the years passed on it was observed that the free hours were used on trigonometry, and on savings and instalments, the first cohering with the following calculus course, the latter highly relevant to many students, and at the same time combining linear and exponential change, the core of the curriculum. This let to designing an alternative curriculum around 1990 to choose instead of the standard curriculum if wanted.

The 1990 curriculum did not change the goal but included the following subjects

1) Numbers, integers, rational and real numbers together with their calculation rules. Number sets. Calculations with power and root.

2) Calculations including percent and interest rates: Average percent, index number, weighed average. Simple and compound interest, saving and installments.

3) Geometry and trigonometry. Similar triangles. Right triangles. Calculations on sides and angles.

4) Functions. The function concept, domain, functional values, range, monotony. Various ways to define a function. Elementary functions as linear, piecewise linear and exponential growth and decay. Coordinate system. Examples of simple equations and inequalities including the functions mentioned above.

5) Probability and statistics. A stochastic experiment. Discrete stochastic variables, probability distribution, mean value, binomial distribution, observation sets described graphically, representation by statistical descriptors, examples of a normal distribution, normal distribution paper.

6) Calculation aids. Pocket calculator, formulas, tables, semi logarithmic paper, normal distribution paper.

The 2005 Curriculum

Then a major reform of the Danish upper secondary high school was planned for 2005. As to precalculus, it was inspired by the entry of graphing calculators and computer assisted systems allowing regression to transform tables into formulas, thus allowing realistic modeling to be included.

Now the goal defined the competences students should acquire:

The students can

- handle simple formulas and translate between symbolic and natural language and use symbolic language to solve simple problems with a mathematical content.
- apply simple statistical models for describing a given data set, pose questions based upon the model and sense what kind of answers are to be expected and knows how to formulate conclusion in a clear language.
- apply relations between variables to model a given data set, can make forecasts, and can reflect on them and their domain of relevance
- describe geometrical models and solve geometrical problems
- produce simple mathematical reasoning
- demonstrate knowledge about mathematical methods, applications of mathematics, and examples of cooperation between mathematics and other sciences, as well as its cultural and historical development
- apply information technology for solving mathematical problems

The means include

- The hierarchy of operations, solving equations graphically and with simple analytical methods, calculating percent and interest rates, absolute and relative change
- Formulas describing direct and inverse proportionality as well as linear, exponential ad power relations between variables
- Simple statistical methods for handling data sets, graphical representation of statistical materials, simple statistical descriptors
- Ratios in similar triangles and trigonometry used for calculations in arbitrary triangles.
- xy-plot of data sets together with characteristics of linear, exponential and power relations, the use of regression.
- Additional activities for 25 lessons are examples of mathematical reasoning and proofs, modeling authentic data sets, examples of historical mathematics.

The 2017 Curriculum

Then in 2017 e new reform was made to inspire more students to continue with the calculus level by moving some subjects to the precalculus level:

- interpreting the slope of a tangent as a growth rate in a mathematical model
- combinatorics, basic probability theory and symmetrical probability space
- the function concept and characteristics of linear, exponential and power functions and their graphs
- graphical handling of a quadratic function, and the logarithm functions and their characteristics
- graphical determination of a tangent, and monotony intervals, as well as finding extrema values in a closed interval
- prime characteristics at mathematical models and simple modelling using the functions above alone or in combination.

Relevance and Coherence

The 1966 had internal coherence with the previous and following curriculum, but with the emphasis on abstract algebra, there was little external coherence. It was indirectly relevant to students wanting later to take a calculus course but only little relevant to the daily life of students

The 1972 curriculum took the consequence and changed from a parallel curriculum to a serial curriculum so that it had internal coherence to the calculus curriculum, and by replacing quadratics with exponential functions, it obtained an external relevance to change calculations with a constant change-number or a constant change-percent. Also, including a considerable amount of probability gave coherence to eternal testing situations, however these were not part of student daily life, so they didn't add to the relevance for students. However, including the free lessons allowed the students to choose areas that they found relevant, in this case interest rates and saving and installment calculations as well as trigonometry.

The 1990 curriculum was inspired by this and re-included trigonometry and interest rates while at the same time reducing probability a little.

The 2005 reform was informed by the occurrence of competence concept as well as the advances in calculation technology. Her the function concept was replaced by variables to make it cohere more with external applications in science and economics and daily life. Now the probability was gone, so this curriculum showed coherence and relevance to external appliers and to the student's daily life as well for other school subjects. It was close to the ideal precalculus curriculum.

The 2017 reform was inspired by the wish to motive more to continue with a calculus course, so part of this was moved down to the precalculus level, making the two levels cohere better, however the things imported had little relevance to the students' daily life.

07. A Refugee Camp Curriculum

The name 'refugee camp curriculum' is a metaphor for a situation where mathematics is taught from the beginning and with simple manipulatives. Thus, it is also a proposal for a curriculum for early childhood education, for adult education, for educating immigrants and for learning mathematics outside institutionalized education. It considers mathematics a number-language parallel to our word-language, both describing the outside world in full sentences, typically containing a subject and a verb and a predicate. The task of the number-language is to describe the natural fact Many in space and time, first by counting and recounting and double-counting to transform outside examples of Many to inside sentences about the total; then by adding to unite (or split) inside totals in different ways depending on their units and on them being constant or changing. This allows designing a curriculum for all students inspired by Tarp (2018) that focuses on proportionality, solving equations and calculus from the beginning, since proportionality occurs when recounting in a different unit, equations occur when recounting from tens to icons, and calculus occurs when adding block-numbers next-to and when adding per-numbers coming from double-counting in two units.

Talking about 'refugee camp mathematics' thus allows locating a setting where children do not have access to normal education, thus raising the question 'What kind and how much mathematics can children learn outside normal education especially when residing outside normal housing conditions and without access to traditional leaning materials?'. This motivates another question 'How much mathematics can be learned as 'finger-math' using the examples of Many coming from the body as fingers, arms, toes and legs?'

So the goal of 'refugee camp mathematics' is to learn core mathematics through 'Finger-math' disclosing how much math comes from counting the fingers.

Focus 01. Digits as Icons with as Many Outside Sticks and Inside Strokes as They Present

Activity 01. With outside things (sticks, cars, dolls, animals), many ones are rearranged into one many-icon with as many things as it represents. Inside, we write the icon with as many strokes as it represents. Observe that the actual digits from 1 to 9 are icons with as many strokes as they represent if written less sloppy. A discovery glass showing nothing is an icon for zero. When counting by bundling in tens, ten become '1 Bundle, 0 unbundled' or 1B0 or just 10, thus needing no icon since after nine, a double-counting takes place of bundles and unbundled.

Focus 02. Counting Ten Fingers in Various Ways

Activity 01. Double-count ten fingers in bundles of 5s and in singles

● Outside, lift the finger to be counted; inside say “0 bundle 1, 0B2, 0B3, 0B4, 0B5 or 1B0. Then continue with saying “1B1, …, 1B5 or 2B”. ● Outside, look at the fingers not yet counted; inside say “1 bundle less4, 1B-3, 1B-2, 1B-1, 1B or 1B0. Then continue with saying “2B-4, …, 2B or 2B0”. ● Outside, show the fingers as ten ones. ● Outside, show ten fingers as 1 5s and 5 1s; inside say “The total is 1Bundle5 5s” and write ‘T = 1B5 5s’. ● Outside, show ten fingers as 2 5s; inside say “The total is 2Bundle0 5s” and write ‘T = 2B0 5s’.

Activity 02. Double-count ten fingers in bundles of tens and in singles

● Outside, lift the finger to be counted; inside say “0 bundle 1, 0B2, 0B3, …, 0B9, 0Bten, or 1B0”. ● Outside, look at the fingers not yet counted; inside say “1 bundle less9, 1B-8, …, 1B-2, 1B-1, 1B or 1B0.

Activity 03. Counting ten fingers in bundles of 4s using ‘flexible bundle-numbers’.

● Outside, show the fingers as ten ones, then as one tens. ● Outside, show ten fingers as 1 4s and 6 1s; inside say “The total is 1Bundle6 4s, an overload” and write ‘T = 1B6 4s’ ● Outside, show ten fingers as 2 4s and 2 1s; inside say “The total is 2Bundle2 4s, a standard form” and write ‘T = 2B2 4s’. ● Outside, show ten fingers as 3 4s less 2; inside say “The total is 3Bundle, less2, 4s, an underload” and write ‘T = 3B-2 4s’.

Activity 04. Counting ten fingers in bundles of 3s using ‘flexible bundle-numbers’.

● Outside, show ten fingers as 1 3s and 7 1s; inside say “The total is 1Bundle7 3s, an overload” and write ‘T = 1B7 3s’. ● Outside, show ten fingers as 2 3s and 4 1s; inside say “The total is 2Bundle4 3s, an overload” and write ‘T = 2B4 3s’. ● Outside, show ten fingers as 3 3s and 1 1s; inside say “The total is 3Bundle1 3s, a standard form” and write ‘T = 3B1 3s’. ● Outside, show ten fingers as 4 3s less 2; inside say “The total is 4Bundle, less2, 3s, an underload” and write ‘T = 4B-2 3s’.

Activity 05. Counting ten fingers in bundles of 3s, now also using bundles of bundles.

● Outside, show ten fingers as 3 3s (a bundle of bundles) and 1 1s; inside say “The total is 1BundleBundle1 3s” and write ‘T = 1BB1 3s’. Now, inside say “The total is 1BundleBundle 0 Bundle 1 3s” and write ‘T = 1BB 0B 1 3s’. Now, inside say “The total is 1BundleBundle 1 Bundle, less2, 3s” and write ‘T = 1BB 1B -2 3s’.

Focus 03. Counting Ten Sticks in Various Ways

The same as Focus 02, but now with sticks instead of fingers.

Focus 04. Counting Ten Cubes in Various Ways

The same as Focus 02, but now with cubes, e.g. centi-cubes or Lego Bricks, instead of fingers. When possible, transform multiple bundles into 1 block, e.g. 2 4s = 1 2x4 block; inside say “The total is 1 2x4 block” and write ‘T = 2B0 4s.’

Focus 05. Counting a Dozen Finger-parts in Various Ways

Except for the thumps, our fingers all have three parts. So, four fingers have three parts four times, i.e. a total of T = 4 3s = 1 dozen finger-parts.

Focus 05 is the same as focus 02, but now with a dozen finger-parts instead of ten fingers.

Focus 06. Counting a Dozen Sticks in Various Ways

Focus 06 is the same as focus 03, but now with a dozen sticks instead of ten.

Focus 07. Counting a Dozen Cubes in Various Ways

Focus 07 is the same as focus 04, but now with a dozen cubes instead of ten.

Focus 08. Counting Numbers with Underloads and Overloads.

Activity 01. Totals counted in tens may also be recounted in under- or overloads.

● Inside, rewrite T = 23 as T = 2B3 tens, then as 1B13 tens, then as 3B-7tens. ● Try other two-digit numbers as well. ● Inside, rewrite T = 234 as T = 2BB3B4 tens, then as T = 2BB 2B14, then as T = 2BB 4B-6. Now rewrite T = 234 as T = 23B4, then as 22B14, then as 24B-6. Now rewrite T = 234 as T = 3BB-7B4, then as 3BB-6B-6. ● Try other three-digit numbers as well.

Focus 09. Operations as Icons Showing Pushing, Lifting and Pulling

Activity 01. Transform the three outside counting operations (push, lift and pull) into three inside operation-icons: division, multiplication and subtraction.

● Outside, place five sticks as 5 1s. ● Outside, push away 2s with a hand or a sheet; inside say "The total 5 is counted in 2s by pushing away 2s with a broom iconized as an uphill stroke" and write 'T = 5 = 5/2 2s'. ● Outside, rearrange the 2 2s into 1 2x2 block by lifting up the bundles into a stack; inside say "The bundles are stacked into a 2x2 block by lifting up bundles iconized as a lift" and write 'T = 2 2s = 2x2'. ● Outside, pull away the 2x2 block to locate unbundled 1s; inside say "The 2x2 block is pulled away, iconized as a rope" and write 'T = 5 – 2x2 = 1'.

Five counted in 2s:

I I I I I (push away 2s) II II I (lift to stack) II II I (pull to find unbundles ones) II II I

Focus 10. The Inside Recount-Formula T = (T/B)xB Predicts Outside Bundlecounting Results

Activity 01. Use a calculator to predict a bundle-counting result by a recount-formula T = (T/B)xB, saying "from T, T/B times, B is pushed away", thus using a full number-language sentence with a subject, a verb and a predicate.

● Outside, place five cubes as 5 1s. ● Outside, push away 2s with a 'broom'; inside say "Asked '5/2', a calculator answers '2.some', meaning that 2 times we can push ways bundles of 2s. ● Outside , stack the 2s into one 2x2 stack by lifting; inside say "We lift the 2 bundles into one 2x2 stack, and we write T = 2 2s = 2x2 ● Outside, we locate the unbundled by, from 5 pulling away the 2x2 block; inside we say "Asked '5-2x2', a calculator answers '1'. We write T = 2B1 2s and say "The recount-formula predicts that 5 recounts in 2s as T = 2B1 2s, which is tested by recounting five sticks manually outside."

Activity 02. The same as activity 01, but now with 4 3s counted in 5s, 4s and 3s.

Focus 11. Discovering Decimals, Fractions and Negative Numbers.

Activity 01. When bundle-counting a total, the unbundled can be placed next-to or on-top.

● Outside, chose seven cubes to be counted in 3s. ● Outside, push away 3s to be lifted into a 2x3 stack to be pulled away to locate one unbundled single. Inside use the recount-formula to predict the result, and say "seven ones recounts as 2B1 3s" and write T = 2B1 3s. ● Outside, place the single next-to the stack. Inside say "Placed next-to the stack the single becomes a decimal-fraction '.1' so now seven recounts as 2.1 3s" and write T = 2.1 3s. ● Outside, place the single on-top of the stack. Inside say "Placed on-top of the stack the single becomes a fraction-part 1 of 3, so now seven recounts as 2 1/3 3s" and write T = 2 1/3 3s. Now, inside say "Placed on-top of the stack the single becomes a full bundle less 2, so now seven recounts as 3.-2 3s" and write T = 3.-2 3s. Finally, inside say "With 3 3s as 1 bundle-bundle of 3s, seven recounts as 1BB-2 3s."

Activity 02. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 3s.

Activity 03. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 4s.

Activity 04. The same as activity 01, but now with first 2 then 3 etc. until a dozen counted in 5s.

Focus 12. Recount in a New Unit to Change Units, Predicted by the Recount-Formula

Activity 01. When bundle-counting, all numbers have units that may be changed into a new unit by recounting predicted by the recount-formula.

● Outside, chose 3 4s to be recounted in 5s. ● Outside, rearrange the block in 5s to find the answer T = 3 4s = 2B2 5s. Inside use the recount-formula to predict the result, and say "three fours recounts as 2B2 5s" and write T = 3 4s = 2B2 5s = 3B-3 5s = 2 2/5 5s. Repeat with other examples as e.g. 4 5s recounted in 6s.

Focus 13. Recount from Tens to Icons

Activity 01. A total counted in tens may be recounted in icons, traditionally called division.

● Outside, chose 29 or 2B9 tens to be recounted in 8s. ● Outside, rearrange the block in 8s to find the answer T = 29 = 3B5 8s and notice that a block that decreases its base must increase its height to keep the total the same. Inside use the recount-formula to predict the result, and say "With the recount-formula, a calculator predicts that 2 bundle 9 tens recounts as 3B5 8s" and write T = 29 = 2B9 tens = 3B 5 8s = 4B-3 8s = 3 5/8 8s. Repeat with other examples as e.g. 27 recounted in 6s.

* Now, inside reformulate the outside question 'T = 29 = ? 8s' as an equation using the letter u for the unknown number, u*8 = 24, to be solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign. Use an outside ten-by-ten abacus to see that when a block decreases its base from ten to 8, it must increase its height from 2.4 to 3. Repeat with other examples as e.g. 17 = ? 3s.

Focus 14. Recount from Icons to Tens

Activity 01. Oops, without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. Use an outside ten-by-ten abacus to see that when a block increases its base from 7 to ten, it must decrease its height from 3 to 2.1.

Activity 02. Use 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently 'less less 1' means adding 1.

Focus 15. Double-Counting in Two Physical Units

Activity 01. We observe that double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$. Repeat with other examples as e.g. 4$ per 5days.

Focus 16. Double-Counting in the Same Unit Creates Fractions

Activity 01. Double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%. We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Repeat with other examples as e.g. 2$/5$.

Focus 17. Mutually Double-Counting the Sides in a Block Halved by its Diagonal

Activity 01. Recount sides in a block halved by its diagonal? Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. Use these formulas to predict the sides in a half-block with base 6 and angle 30 degrees. Use these formulas to predict the angles and side in a half-block with base 6 and height 4.

Focus 18. Adding Next-to

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Since 3*5 is an area, adding next-to in 8s means adding areas, called integral calculus. Asking a calculator, the two answers, '2.some' and '5', predict the result as 2B5 8s.

Focus 19. Reversed Adding Next-to

Activity 01. With T1 = 2 3s and T2 adding next-to as T = 4 7s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Asking '3 5s and how many 3s total 2B6 8s?', using sticks will give the answer 2B1 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula

y' = Δy/t = (y2 – y1)/t.

Focus 20. Adding On-top

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Adding on-top in 5s, '3 5s + 2 3s = ? 5s?', re-counting must make the units the same. Asking a calculator, the two answers, '4.some' and '1', predict the result as 4B1 5s.

Focus 21. Reversed Adding On-top

Activity 01. With T1 = 2 3s and T2 as some 5s adding to T = 4 5s, what is T2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting. Reversed addition is called backward calculation or solving equations.

Focus 22. Adding Tens

Activity 01. With T1 = 23 and T2 = 48, what is T1+T2 id added as tens?" Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Focus 23. Subtracting Tens

Activity 01. "If T1 = 23 and T2 add to T = 71, what is T2?" Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Focus 24. Multiplying Tens

Activity 01. "What is 7 43s recounted in tens?" Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Focus 25. Dividing Tens

Activity 01. "What is 348 recounted in 6s?" Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Focus 26. Adding Per-Numbers

Activity 01. “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining local constancy by epsilon and delta.

Activity 02. Two groups of voters have a different positive attitude to a proposal. How to find the total positive attitude?

● Asking “20 voters with 30% positive + 60 voters with 10% positive = 80 voters with ? positive.” Here we see that the unit-numbers 20 and 40 add directly whereas the per-numbers 30% and 10% add by areas since they must first transform to unit-numbers by multiplication, creating the areas.

Focus 27. Subtracting Per-Numbers

Activity 01. “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Focus 28. Adding Differences

Activity 01. Adding many numbers is time-consuming, but not if the numbers are changes, then the sum is simply calculated as the change from the start to the end-number.

● Write down ten numbers vertically. The first number must be 3 and the last 5, the rest can be any numbers between 1 and 9. In the next column write down the individual changes ‘end-start’. In the third column add up the individual changes along the way. Try to explain why the result must be 5-3 regardless of the in-between numbers.

● Draw a square with side n. Let n have a small positive change t. Show that the square will change with two next blocks when disregarding the small txt square. This shows that the change in an n*n square is 2*n*t, so if we want to add arears under a y = 2*n curve we must add very many small areas y*t = 2*n*t. However, since each may be written as a change in a square, we just have to find the change of the square from the start-point to the end-point. That is how integral calculus works.

Focus 29. Finding Common Units

Activity 01. “Only add with like units, so how add T = 4ab^2 + 6abc?”. Here units come from factorizing: T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b).

Focus 30. Finding Square Roots

Activity 01. A 7x7 square can be recounted in tens as 4.9 tens. The inverse question is how to transform a 6x7 block into a square, or in other words, to find the square root of 4.2 tens. A quick way to approach a relevant number is to first find two consecutive numbers, p and p+1, that squared are too low and too high. Then the an approximate value for the square root may be calculated by using that if p^2 < N < (p+1)^2, then $\sqrt{N} \approx \frac{N + p^{\wedge}2}{p*2}$.

Conclusion

A curriculum for a refugee camp assumes that the learners have only the knowledge they acquire outside traditional education. The same is the case for street children living outside traditional homes; and for nomadic people always moving around.

However, a refugee camp curriculum might also be applied in a traditional school setting allowing the children to keep on to the two-dimensional block numbers they bring to school allowing them to learn core mathematics as proportionality, equations, functions and calculus in the first grade, thus not needing parallel curricula later on.

So, the need for parallel curricula after grade 9 is not there by nature, but by choice. It is the result of disrespecting the mastery of many children bring to school and force them to adopts numbers as names along a number line, and force them to add numbers that are given to them without allowing them to find them themselves by counting, recounting and double-counting.

08. Do We Really Need Parallel Curricula?

Why do we need different curricula for different groups of students? Why can’t all students have the same curriculum? After all, the word-language does not need different curricula for different groups, so why does the number-language?

Both languages have two levels, a language level describing the ‘outside’ world, and a grammar level describing the ‘inside’ language. In the word-language, the language level is for all students and includes many examples of real-world descriptions, both fact and fiction. Whereas grammar level details are reserved for special students. Could it be the same with the number-language, teaching the language level to all students including many examples of fact and fiction? And reserving grammar level details to special students?

Before 1970, schools taught language as an example of its grammar (Chomsky, 1965). Then a reaction emerged in the so-called ‘communicative turn’ in language education. In his book ‘Explorations in the function of language’ Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a “communicative approach to the teaching of language (p. ix)” allowing more students to learn a less correct language to be used for communication about outside things and actions.

Thus, in language teaching the communicative turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language. So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use
- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

After all, the word language seems more voluminous with its many letters, words and sentence rules. In contrast, a pocket calculator shows that the number language contains ten digits together with a minor number of operations and an equal sign. And, where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. (Tarp, 2018)

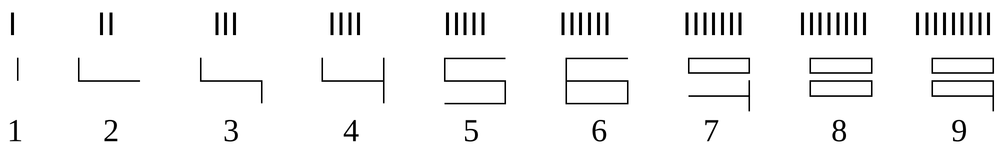

Figure 03. Digits as icons with as many sticks as they represent.

Furthermore, also the operations are icons describing how we total by counting unbundled, bundles, bundles of bundles etc. Here division iconizes pushing away bundles to be stacked, iconized by a multiplication lift, again to be pulled away, iconized by a subtraction rope, to identify unbundled singles that are placed next-to the stack iconized by an addition cross, or by a decimal point; or on-top iconized by a fraction or a negative number.

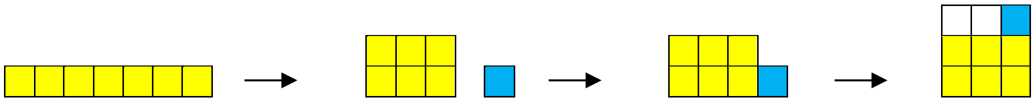

Figure 04. Seven counted as 2 3s & 1 or 2*B*1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

The operations allow predicting counting by a recount-sentence or formula ‘T = (T/B)*B’ saying that ‘from T, T/B times, B can be taken away’, making natural numbers as bundle- or block numbers as e.g. T = 3B2 4s or T = 3*4+2. And, using proportionality to change the unit when two blocks need the same unit to be added on-top, or next-to in a combined unit called integral calculus.

So, it seems as if early childhood education may introduce core mathematics as proportionality, solving equations, and integral calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction. Thus, there is indeed an opportunity to design a core curriculum in mathematics for all students without splitting it up in tracks. But, only if the word- and the number-language are taught and learned in the same way by describing outside things and actions in words and in numbers coming from counting and adding.

So, why not introduce a paradigm shift by teaching the number-language and the word-language in the same way through its use, and not before, thus allowing both languages being taught in the space between the inside language and the outside world.

Why keep on teaching the number-language in the space between the language and its meta-language or grammar, which makes the number-language more abstract, leaving many educational challenges unsolved despite close to half a century of mathematics education research.

Why not begin teaching children how to number, and stop teaching children about numbers and operation to be explained and learned before they can be applied to the outside world.

Why not accept and develop children’s already existing ‘many-sense’, instead of teaching them the eight different aspects of what is called ‘number-sense’ described by Sayers and Andrews (2015) that after reviewing research in the Whole Number Arithmetic domain created a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

And, why not simply let children talk about counting and adding constant and changing unit-numbers and per-numbers using full sentences with a subject, a verb, and a predicate; instead of teaching them the eight different components of what is called ‘mathematical competencies’ (Niss, 2003), thus reducing their numbers from eight to two: count and add (Tarp, 2002)?

So maybe we should go back to the mother Humboldt university in Berlin and reflect on Karl Marx thesis 11 written on the staircase: ”Die Philosophen haben die Welt nur verschieden interpretiert; es kömmt drauf an, sie zu verändern.” (The philosophers have only interpreted the world, in various ways. The point, however, is to change it.)

09. Conclusion

Let us return to the dream of the National Council of Teachers of Mathematics, to “provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals." Consequently, "everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics." Furthermore, let us also accept what the council write about numbers: "Number pervades all areas of mathematics."

So let us look for a curriculum that allows the students to understand and use and numbers, and see how far such a curriculum can carry all students without splitting into parallel tracks.

Now, what does it mean to understand a number like 456?

Is the ability to say that the three digits obey a place-value system where, from right to left, the first digits is ones, then tens, then hundred, then thousands, then, oops no-name unless we use the Chinese name wan, then no-name, then million, then no-name, then no-name, then billions or milliards, etc. Names and lack of names that give little meaning to children where only few understand why ten has its own name but not its own icon but has two digits as 10.

On the other hand, is it the ability to understand that of course ten becomes 10 since it is short for '1 bundle and no singles'? And, that it would have been 20 had we counted in bundles of 5s instead as they do on an eastern abacus, where the two digits 10 then would be used for the bundle size 5.

And that ten is just another word for bundle, and hundred for bundle-bundle, i.e. 2 times bundling; and thousand for bundle-bundle-bundle, i.e. or bundling 3 times, etc. where we never end in a situation with no name. Isn't it both power and beauty to transform an unorganized total into a repeated bundling with the ability that only the decimal point moves if you change the number of bundling, T = 32.1 tens = 3.21 tentens, which is not the case with romans bundling where 3 tens is 6 fives. The romans didn't stick to bundling bundles since they bundled in both fives and tens and fifties but not in 5 5s, i.e. in 25s. Power and beauty comes from bundle bundles only.

Consequently, to understand the number 456 is to see it, not as one number, but as three numberings of a total that has been bundled 0 times, bundled 1 times, bundled 2 times, etc. And to read the total as 4 bundled 2 times and 5 bundled once and 6 not bundled, or as 4 bundle-bundles and 5 bundles and 6 unbundles singles. And to write the total as T = 4BB 5B 6. And to allow the same total to be recounted with an underload as T = 4BB 6B -4, or with an overload as T = 45B 6 = 4BB 56; or as T = 45B -4 if combining overload and underload.

This understanding allows an existing unorganized total become a number-language sentence connecting the outside subject T to an inside calculation, T = 4*B^2 + 5*B^1 + 6*B^0. Which again is an example, or specification, of an unspecified number-formula or polynomial T = a*x^2 + 5*x + 6.

The power and beauty of the number-formula is manifold. It shows four ways to unite: power, multiplication, addition and next-to block addition also called integration. By including the units, we realize that there are only four types of numbers in the world as shown in the algebra-square above, constant and changing unit-numbers and per-numbers, united by precisely these four ways: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers.

Furthermore, we observe that splitting a total into parts will reverse uniting parts into a total, meaning that all uniting operations have reverse operations: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers. This makes root a factor-finder, and logarithm a factor-counter, and differentiation a finder of per-numbers.

And, if we use the word 'equation' for the need to split instead of unite, we observe that solving an equation means isolating the unknown by moving numbers to the opposite side with opposite calculation sign. Furthermore, using variables instead of digits we observe that the number-formula contains the different formulas for constant change as shown above.

As to a non-constant change, there are two kinds. Predictable change roots calculus as shown by the algebra-square; and unpredictable change roots statistics to instead 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for unpredictable numbers.

Thus the 'power and beauty' of mathematics resides in the number-formula, as does the ability 'to use mathematics in students' personal life, in the workplace and in further study'. So, designing a curriculum based upon the number-formula will 'provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.'

Furthermore, a number-formula based curriculum need not split into parallel curricula until after calculus, i.e. until after secondary education.

So, one number-language curriculum for all is possible, as it is for the word-language. Thus, it is possible to allow all students to learn about the four ways to unite and the five ways to split a total.

The most effective way to design a curriculum for all students is to adopt the curriculum designed refugee camp from the beginning since it accepts and develops the number-language children bring to school. Presenting figures and operations as icons, it bridges outside existence with inside essence. All four uniting methods occur in grade one when counting and recounting in different units, and when adding totals next-to and on-top. It respects the natural order of operations by letting division precede multiplication and subtraction, thus postponing addition until after counting, recounting and double-counting have taken place. It introduces the core recounting-formula expressing proportionality when changing units from the beginning, which allows a calculator to predict inside an outside recounting result. By connecting outside blocks with inside bundle-writing, geometry and algebra are introduced as Siamese twins never to part. Using flexible bundle-numbers connects inside decimals, fractions and negative numbers to unbundled leftovers placed next-to or on-top the outside block. It introduces solving equations when recounting from tens to icons. It introduces per-numbers and fractions when double counting in units that may be the same or different. And, it introduces trigonometry before geometry when double-counting sides in a block halved by its diagonal.

Another option is to integrate calculus in a precalculus course by presenting integral calculus before differential calculus, which makes sense since until now inverse operations are always taught after the operation, subtraction after addition etc. Consequently, differential calculus should wait until after it has been motivated by integral calculus that is motivated by adding changing per-numbers in trade and physics, and by adding percent in statistical double-tables.

In their publication, the National Council of Teachers of Mathematics writes "High school mathematics builds on the skills and understandings developed in the lower grades. (p. 19)" If this has to be like that then high school education will suffer from lack of student skills and misunderstandings; and often teachers say that precalculus is the hardest course to teach because of a poor student knowledge background.

So, we have to ask: Can we design a fresh-start curriculum for high school that integrates precalculus and calculus? And indeed, it is possible to go back to the power and beauty of the number-formula as described above, and build a curriculum based upon the algebra-square. It gives an overview of the four kinds of numbers that exist in the outside world, and how to unite or split them. It shows a direct way to solve equations based upon the definitions of the reverse operations: move to opposite side with opposite calculation sign.

Furthermore, it provides 2x2 guiding questions: how to unite or split into constant per-numbers, as needed outside when facing change with a constant change-factor? And how to unite or split into changing per-numbers that are piecewise or locally constant, as needed outside when describing e.g. the motion with a changing velocity of a falling object.

As a reverse operation, differential calculus is a quick way to deliver the change-formula that solve the integration problem of adding the many area-strips coming from transforming locally constant per-numbers to unit-numbers by multiplication. Also, by providing change-formulas, differential calculus can extend the formulas for constant change coming from the number-formula. An additional extension comes from combining constant change-number and change-percent to one of the most beautiful formulas in mathematics that is too often ignored, the saving-formula, A/a = R/r, a formula that is highly applicable in individual and social financial decisions.

Working with constant and changing change also raises the question what to do about unpredictable change, which leads directly into statistics and probability.

So designing and implementing a fresh-start integrated precalculus and calculus curriculum will allow the National Council of Teachers of Mathematics to have their dream come through, so that in the future high schools can provide all students "with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals."

As a number-language, mathematics is placed between its outside roots and its inside meta-language or grammar. So, institutionalized education must make a choice: should the number-language be learned through its grammar before being applied to outside descriptions; or should it as the word-language be learned through its use to describe the outside world? I short, shall mathematics education tach about numbers and operations and postpone applications till after this has been taught? Or shall mathematics education teach how to number and how to use operations to predict a numbering result thus teaching rooting instead of applications?

Choosing the first 'inside-inside' option means connecting mathematics to its grammar as a 'meta-matics' defining concepts 'from above' as top-down examples from abstractions instead of 'from below' as bottom-up abstractions from examples. This is illustrated by the function concept that can be defined from above as an example of a set-product relation where first component identity implies second-component identity, or from below as a common name for 'stand-by' calculations containing unspecified numbers.

Choosing the inside-inside 'mathematics-as-metamatics' option means teaching about numbers and operations before applying them. Here numbers never carry units but become names on a number-line; here numbers are added by counting on; and the other operations are presented as inside means to inside tasks: multiplication as repeated addition, power as repeated multiplication, subtraction as inverse addition, and division as inverse multiplication. Here fractions are numbers instead of operators needing numbers to become numbers. Here adding numbers and fractions without units leads to 'mathe-matism', true inside classrooms where 2+3 is 5 unconditionally, but seldom outside classrooms where counterexamples exist as e.g. 2weeks + 3days is 17days or 2 3/7 weeks. Here geometry and algebra occur independently and before trigonometry. Here primary and lower secondary school focus on addition, subtraction, multiplication and division with power and root present as squaring and square roots, thus leaving general roots and logarithm and trigonometry to the different tracks in upper secondary school where differential calculus is introduced before integral calculus, if at all.

Choosing the inside-outside 'mathematics-as-manymath' option means to teach digits as icons with as many strokes as they represent. And to also teach operations as icons, rooted in the counting process where division wipes away bundles to be stacked by multiplication, again to be removed by subtraction to identify unbundled singles. This will allow giving a final description of the total using a full sentence with a subject, a verb and a predicate predicted by the recount-formula T = (T/B)*B, e.g. T = 2Bundle 1 3s = 2.1 3s = 2 1/3 3s thus including decimal numbers and fractions in a natural number. Here a double description of Many as an outside block and an inside bundle-number allows outside geometry and inside algebra to be united from the start. Once counted, totals can be recounted. First in the same unit to create overloads and underloads introducing negative numbers. Then between icon- and ten-bundles introducing the multiplication table and solving equations. Then double-counting in two units creates per-numbers becoming fractions with like units. Finally, recounting the sides in a block halved by its diagonal will root trigonometry before geometry, that integrated with algebra can predict intersection points. Then follows addition and reversed addition in its two versions, on-top or next-to. On-top addition calls for recounting the totals in the same unit, thus rooting proportionality. And next-to addition means adding blocks as areas, thus rooting integral calculus. Reversed addition roots equations and differential calculus. Per-numbers are added as operators including the units, thus rooting integral calculus, later defined as adding locally constant per-numbers. Thus, this option means that the core of mathematics is learned in primary school allowing ample of time in secondary school to enjoy the number-language literature by examining existing models or producing models yourself. And it means that only one curriculum is needed for all students as in the word-language.

Furthermore, the root and use of calculus to add changing per-numbers is easily introduced at the precalculus level when adding ingredients with different per-numbers and when adding categories in statistics with different percent.

And, the fact that the difficulty by adding many numbers disappears when the numbers can be written as change-numbers since adding up any number of small changes total just one change from the start- to the end-number. Which of course motivates differential calculus.

Consequently, there is no need for a parallel curriculum to the traditional, since everybody can learn calculus in a communicative way. Of course, one additional optional course may be given to look at all the theoretical footnotes.

To offer a completely different kind of mathematics as graph theory and game theory and voting theory risks depriving the students of the understanding that mathematics is put in the world as a number-language that use operations to predict the result of counting, recounting and double-counting. A language that only needs four operations to unite parts into a total, and only five operations to split a total into parts.

Without calculus in the final high school curriculum, students may not understand how to add per-numbers and might add them as unit-numbers instead of as areas; and this will close many ‘doors to productive futures’ as the US National Council of Teachers of Mathematics talks about.

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02. COMMENTS TO A DISCUSSION PAPER

Dear Colleagues. First some information. The deadline for the chapter outlines has been extended until at least the end of February (from 15 February). I would like to open discussion between us on several matters. Please "Reply All" if you have any thoughts to add.

1. Recap conversations and conclusions from the conference.

Since neither Jennie nor I were with you two at the conference, it would be helpful if Allan and Jaime could summarise their work at the conference for us. This will provide a common starting point for us. Can you please outline (summary dot points will be fine) the key matters you discussed, and any conclusions you reached.

<u>Comments to question one</u>

The last day at the conference was for the most part used to discuss the meaning of the two terms 'coherence and relevance'. That left only very little time for us to discuss more than a general structure of our contribution.

We decided to use a format focussing on case studies of curricula from upper secondary school since this is where the side-curricula to the main curriculum occur; and to include our own contributions as exemplary cases.

Silva: a curriculum that introduces untraditional mathematics more relevant to students

Tarp: an untraditional curriculum for starters as e.g. children or migrants

- ("But we should be thinking outside the box, and beginning with the child, rather than the discipline", Discussion summary Session 1 Group B ICMI Study 24 Conference, Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Working\Records of conversations.)
- ("(..) it is appropriate that curricula should continue to evolve, perhaps in radical ways.", Taguma, p. 10)
- ("Coherence with what children bring to school. We colonize the "mastery of many" that children have when they bring to school. What kind of mathematics would we have if we built on it? In Europe we are still stuck with set-based mathematics. Be careful about adding without units. This leads to incoherence with what is inside the classroom and what is outside.", Wednesday session notes McCallum, Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Working\Records of conversations)

As well as a Danish end of high school pre-calculus curriculum that I had described in my working group

- ("Denmark: they tried to make a different sort of coherence between mathematics and the outside world with precalculus course. The used different names for linear and exponential functions; growth by adding and growth by percentage. Each was connected with a context from the real world (buying, bank accounts). Again, this is an example of coherence through relevance.", Session_2_McCallum, Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Working\Records of conversations)

The February draft follows these lines and includes also a section for contributions from other papers and sources.

2. Definitions of coherence and relevance

These two terms will need to be defined and used consistently in all the writing in this Theme. The discussion of the terms and their definitions will be an important part of the introduction to the section that contains the chapters for the Key Questions.

I would like to make the following distinction between the two terms as they apply to the intended curriculum:

coherence is 'internal' – the alignment (or otherwise) within and between various aspects of the curriculum

relevance is 'external' – the interaction between the curriculum and the needs and aspirations (of students/young people, the workplace, universities, society) that often drive reforms

Is this distinction sound? Please add your thoughts below.

Comments to question two

Thank you very much indeed for making this very clear, and for choosing definitions cohering with the dictionary definitions of the two terms. This is really very helpful.

At my short initial presentation at the conference, I included these two definitions in my presentation as seen in the Dropbox version (Dropbox\icmi_stady_24\Theme_B\2. Conference\Theme B Presentations). On slide 11 I wrote "MerriamWebster: History and Etymology for cohere, borrowed from Latin cohaerēre "to stick together, be in contact with, be connected," from co- CO- + haerēre "to be closely attached, stick,". And on slide 12 I wrote about relevance "Oxford Dict.: Closely connected or appropriate to what is being done or considered."

When we began discussing the meaning of the two terms the last day at the conference, I suggested that instead of trying to find a common understanding in the group we should accept and respect the general meaning of the two terms as described in dictionaries. However, the discussion between a minority of the persons continued almost all of the time where many were silently waiting for the work in the five writing groups to begin. Of course, it is a little difficult to have a common discussion among peoples with so many different cultural backgrounds and with so different access to the common language used. But the question could be raised if the time could have been spent in a different more productive way by accepting that day what now has been accepted, the dictionary meanings of the two words.

Also, in my post-conference essay "A New Curriculum - But For Which Of The 3x2 Kinds Of Mathematics Education" that I have sent to the members of theme B, I devoted the first chapter "Coherence And Relevance In The School Mathematics Curriculum" to discuss the meaning of these two terms, again arguing that we should stay with the dictionary definitions since they seem to agree.

So again, thank you very much for allowing us to do that.

3. The focus for our work

If we broadly accept the distinction above, it seems that the focus for the work on KQ2 will be mostly on 'relevance' of the reforms to the groups of students for whom it is intended. The coherence of the reforms may also need to be analysed, but my feeling is that the emphasis must be on relevance.

Do you agree? Please add your thoughts below.

Comments to question three

Again, thank you very much for making this clear. I could not agree more. As a social institution, education also has a social responsibility to educate all children and teenagers. So in the case where a structure has been chosen that keeps children and teenagers together in multi-year classes and tracks, the question about curricula for the side-track to the main track becomes an important one, that need to be addressed and that should be the main if not the full focus in this key question. The alternative is to allow teenagers to choose among a list of half-year or full-year blocks as in North America, practically absent at the conference, which seems to indicate that the curriculum reform problems reside with multi-year classes and tracks.

4. A lack of papers directly relating to KQ2

This is a known and difficult matter:

I have attached the spreadsheet of the other links for KQ2 from the survey

KQ1 and KQ3 (especially) would seem to have papers about reforms that are directed towards particular needs, you may be aware of other research (not part of Study 24) that is particularly relevant to our work

Please share your responses to these suggestions and any cases that you think will be worth analysing from these other sources.

<u>Comments to question four</u>

Again, thank you very much for giving us the opportunity to share inputs with the other groups, and to receive input from them.

Once the draft structure is discussed and adapted and accepted, contributors will be asked to elaborate on their suggestions as to specific questions asked, and to identify relevant references.

The questions, of course, will be inspired by the title of the ICMI study 24: "School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities." Auguste Comte invented the term 'sociology' and developed 'positivism' as the idea that the dynamic laws of physics also applies to social conditions. Although questioned, inspiration may still be found in this idea, also in the case of social curricula.

The fundamental law of physical dynamics is Newton's second law originally saying, "The rate of change of the momentum of a body is directly proportional to the net force acting on it." Today it is formulated as "A force's impulse gives a change in momentum", or as an equation "the product of the force and its period is proportional to the product of the inertial mass and the change affected."

This means that a change depends upon three factors: The change increases with the net force applied and with the period it is applied, and it decreases with the size of the inertia of the system. And here, the net force is the combination of all the forces applied.

Based upon this, two questions will be answered in the cases and asked specifically to contributors:

> In the latest reform for the side-tracks in upper secondary school:
>
> 01) Which outside and inside forces were acting and with what strength and over which period?
>
> 02) What was the resistance against the change exerted by systemic inertia?

5. A policy dimension

In my experience reforms can often be initiated in response to policy directions outside of mathematics education. For example, in my country policies relating to gender and mathematics, or improving educational outcomes of students from particular ethnic backgrounds (including indigenous students) tend to drive the goals and some of the substance of the intended mathematics curriculum.

Is this common? Do you think it is important to discuss this policy dimension in relation to any reforms we analyse for relevance (and coherence)?

<u>Comments to question five</u>

Thank you very much for asking this question that coheres with what is called the outside force in the above description of Newton's law exported to social dynamics. Interesting examples hopefully will occur describing what happens when outside forces pointing to increased relevance meet inside forces pointing to the importance of keeping inside coherence unchanged.

03. A Mathematics Teacher Using Communicative Rationality Towards Children

Defining, as Habermas, communicative rationality as ‘wanting to reach understanding to secure the participant speakers an intersubjectively shared lifeworld, thereby securing the horizon within which everyone can refer to one and the same objective world’; and defining the objective world as ‘the totality of entities concerning which true propositions are possible’ (thus, to avoid self-reference, not seeing propositions as part of the objective world); and seeing a speech act as ‘a speaker pursuing the aim of reaching understanding with a hearer about something’, we might ask:

How can a math teacher use communicative rationality to establish a non-patronizing power-free rational dialogue with grade one children about the objective fact Many, present in both the children and the teacher’s life-world; thus accepting four fingers held together two by two being rationalized as (as do children) ‘the total is two twos’ and not just as ‘four’?

It turns out, that accepting the children’s 2dimensional block-numbers instead of letting the system-world colonize their lifeworld by enforcing upon them 1dimensional line-numbers, will allow co-creating and co-developing a mastery of Many (a post-setcentric ‘ManyMath’) where digits are icons with as many strokes as they represent (5 strokes in the 5-icon); and where also operations are icons for the counting process (division is a broom sweeping away bundles, multiplication is a lift stacking bundles into a block, subtraction is a rope drawing away the block to look for unbundle singles, placed next to the block as decimals or on-top of the block counted in bundles as fractions or negative numbers).

Once counted, a total can be recounted in the same unit to create underload and overload (T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s); or in another unit predicted by a calculator with the recount formula ‘T = (T/B)*B’ saying ‘From T, T/B times, B can be taken away’; or from tens to icons rooting equations solved by recounting (? 7s = u*7 = 42 = (42/7)*7, so ? = u = 42/7); or from icons to tens rooting multiplication tables (T = 7 8s = ?tens); or in a different units creating per-numbers used to bridge the unites by recounting (with T = 2kg = 3$ we have the per-number 2kg/3$ = 2/3 kg/$, and T = 6kg = (6/2)*2kg = (6/2)*3$ = 9$).

Once counted and recounted, totals can be added on-top needing recounting (proportionality) to make the units like, or next-to that by adding areas is integral calculus, that leads to differential calculus when reversed.

In short, having as a dream to establish third generation Enlightenment republics in Europe, Habermas uses Weber’s warning against rationalization taken too far to become an iron cage to, in Habermas’ version, warn against a colonization of the lifeworld by systems.

Thus, in the case of mathematics education, the institutionalized system wants to colonize the children’s own Many-math by forcing upon them, not mathematics, but ‘meta-matism’, a mixture of ‘meta-matics’ defining concepts as examples of abstractions instead of as abstractions from examples; and ‘mathe-matism’ true inside itself where 2+3 IS 5 unconditionally, but seldom outside in the objective world where adding numbers without units creates counter-examples as for example 2weeks + 3days = 17 days = 2 3/7 weeks.

Maybe Marx has a point in his Feuerbach Thesis 11: “Philosophers have hitherto only interpreted the world in various ways; the point is to change it.”

Tarp, A. (2018). Mastering Many. *Journal of Mathematics Education 11*(1), 103-117.

04. ADDITION-FREE STEM-BASED MATH FOR MIGRANTS

A curriculum architect is asked to avoid traditional mistakes when designing a curriculum for young migrants that will allow them to soon become STEM pre-teachers and pre-engineers. Multiplication formulas expressing recounting in different units suggest an addition-free curriculum. To answer the question 'How many in total?' we count and recount totals by bundling in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. A recount formula that expresses proportionality when changing units is a core prediction formula in all STEM subjects.

DECREASED PISA PERFORMANCE DESPITE INCREASED RESEARCH

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise has funding, see e.g. the Swedish National Centre for Mathematics Education. Yet, despite extra research and funding, and despite being warned against the possible irrelevance of a growing research industry (Tarp, 2004), decreasing Swedish PISA results caused OECD to write the report "Improving Schools in Sweden" (2015a) describing its school system as "in need of urgent change" since "more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life (p. 3)."

To find an unorthodox solution to poor PISA performance we pretend that a university in southern Sweden, challenged by numerous young male migrants, arranges a curriculum architect competition: "Theorize the low success of 50 years of mathematics education research; and derive from this a STEM based core curriculum allowing young migrants to soon become STEM pre-teachers and pre-engineers."

Since mathematics education is a social institution, social theory may give a clue to the lacking research success and how to improve schools in Sweden and elsewhere.

SOCIAL THEORY LOOKING AT MATHEMATICS EDUCATION

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now (p. 16)."

As to institutions, of which mathematics education is an example, Bauman talks about rational action "in which the *end* is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such *means* to the end as promise to be most effective and economical (p. 79)". He then points out that "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called **goal displacement** (p. 84)." Of which one example is saying that the goal of mathematics education is to learn mathematics since such a goal statement is obviously made meaningless by its self-reference.

The link between a goal and its means is also present in existentialist philosophy described by Sartre (2007) as holding that "Existences precedes essence (p. 20)". Likewise, Arendt (1963) points out that practicing a means blindly without reflecting on its goal might lead to practicing "the banality of evil". Which makes Bourdieu (1977) says that "All pedagogic action is, objectively, symbolic violence insofar as it is the imposition of a cultural arbitrary by an arbitrary power (p. 5)". This raises the question if mathematics and education is universal or chosen, more or less arbitrarily.

DIFFERENT KINDS OF EDUCATION

The International Commission on Mathematical Instruction, ICMI, named its 24th study "School mathematics Curriculum Reforms: Challenges, Changes and Opportunities". At its conference in Tsukuba, Japan, in November 2018 it became clear during plenary discussions that internationally there is little awareness of two different kinds of educational systems practiced from secondary school.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

Moreover, as a social institution involving monopolizing and individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlightens teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching only one subject, and in their own classrooms.

To protect its republic against attack from its German speaking neighbors, France created elite schools with multi-year forced classes, called 'pris-pitals' by Foucault (1995) pointing out that it mixes power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic.

DIFFERENT KINDS OF MATHEMATICS

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, rhetoric and logic (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both being action-words rooted in the physical fact Many through their original meanings. This resonates with the primary goal of knowledge seeking and education, to be able to master the outside world through proper actions. And in Europe, Germanic countries taught counting and reckoning in primary school and algebra and geometry in the lower secondary school until about 50 years ago when they all were replaced by the setbased 'New Math' even if mathematics is a mere label and not an action-word. But the point was that by being setbased mathematics could become a self-referential 'meta-matics' needing no outside root. Instead it could define concepts top-down as examples of inside abstractions instead of bottom-up as abstractions from outside examples.

Russell objected by pointing to the set of sets not belonging to itself. Here a set belongs only if it does not: if $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. In this way Russell shows that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false. Instead Russell proposed a type theory banning self-reference. However, mathematics ignored Russell's paradox and his type theory since it prevented fraction from being numbers by being defined from numbers.

Instead, setbased mathematics changed classical grounded mathe-matics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2*3 = 6' stating that 2 3s can always be recounted as 6 1s.

Although spreading around the world, the United States rejected the New Math by going 'back to basics'. So today three kinds of mathematics may be taught: a pre-setbased, a present setbased and a post-setbased version (Tarp, 2017).

THE TRADITION OF MATHEMATICS EDUCATION

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5. This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations having addition as the base with subtraction as reversed operation, where multiplication is repeated addition with division as reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction creates negative numbers, where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2. Then halving a block by its diagonal creates a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

Then calculations with unspecified numbers leads to creating additional education about the results of operating on such numbers, e.g. that (*a*+*b*) * (*a*-*b*) = *a*^2 - *b*^2.

In a calculation, changing the input will change the output. Relating the changes creates an operation on the calculation called differentiation, also creating additional education about the results of operating on calculations, e.g. that (*f**g*)'/(*f**g*) = *f'*/*f* + *g'*/*g*. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6*x^2dx = 2*x^3.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modeling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

THEORIZING THE SUCCESS OF MATH EDUCATION RESEARCH

When trying to theorize the low success of 50 years of mathematics education research, the first question must be what we mean by mathematics and education and research.

As to education, who needs it if they already know? So, we must ask: what is it that students do not know and must be educated in? Or in other words: what is the goal of mathematics education? Two answers present themselves, one pointing to on the outside existence rooting mathematics, the other to its inside institutionalized essence.

Giving precedence to inside essence over outside existence the answer is: of course, the goal of mathematics education is to teach mathematics as defined by mathematicians at the universities. Modern societies institutionalize the creation and mediation of knowledge as universities and schools. Here priority should be given to useful knowledge as mathematics; and of course, mathematics must be taught before it can be applied, else there is nothing to apply! However, although very useful, mathematics is at the same time very hard to learn as witnessed again and again by research, carefully and in detail describing students' learning problems. So, 50 years of mathematics education research has not been unsuccessful, on the contrary, it has been extremely successful in proving that, by its very nature, mathematics is indeed difficult. The 'essence precedes existence' stance is typically argued by university scholars as e.g. Bruner (1962), Skemp (1971), Freudenthal (1973), and Niss (1994).

Giving precedence to outside existence over inside essence the answer is: It is correct that research has demonstrated many learning difficulties. However, what has been taught is not an outside rooted mathe-matics, but an inside self-referring meta-matism as defined above. And, until now research has primarily studied the two contemporary versions of mathematics, the pre-setbased and the present setbased version whereas very little if any research has studied the post-setbased mathematics that gives precedence to existence over essence by accepting and developing the mastery of Many in the number-language that children develop before school.

Giving precedence to essence or existence makes a difference to math education.

In its pre-setbased version, mathematics presents digits as symbols, and numbers as a sequence of digits obeying a place value system. Once a counting sequence is established, addition is defined as counting on, after which the other operations are defined from addition. Fractions are seen as numbers.

In its present setbased version, mathematics uses the inside concept set for deriving other concepts. Here numbers describe the cardinality of a set, and an operation is a function from a set product into a set. Again, addition is taught as the first operation.

In its post-setbased version, mathematics presents digits as icons with as many sticks as they represent; and numbers always carry units as part of number-language sentences bridging the outside existence with inside essence, thus connecting outside blocks with inside bundles, T = 2 3s = 2B0 3s. Here operations are icons also, and here counting comes before adding to respect that counting involves taking away bundles by division to be stacked by multiplication, to be pulled away by subtraction to find unbundled ones. And here counting and recounting and double-counting precedes the two forms of addition, on-top and next-to. And here fractions are per-numbers, both being operators needing numbers to become numbers.

Likewise, the core concept ‘function’ is treated differently. Pre-setbased mathematics sees a function as a calculation containing specified and unspecified numbers. Present setbased mathematics sees a function as a subset of set product where first-component identity implies second-component identity. Post-setbased mathematics sees a function as a number-language sentence T = 2*3 relating an outside existing total with an inside chosen essence.

Choosing an ‘inside-outside’ view will make mathematics self-referring and difficult by its missing link to its outside roots. Whereas choosing an ‘outside-inside’ view will allow mathematics develop the language children use to assign numbers to outside things and actions, i.e. a number-language similar to the word-language.

Mathematics as the Grammar of the Number-Language

To communicate we have two languages, a word-language and a number-language. The word-language assigns words to things in sentences with a subject, a verb, and an object or predicate: ”This is a chair”. As does the number-language assigning numbers instead: ”The 3 chairs each have 4 legs”, abbreviated to ”The total is 3 fours”, or ”T = 3 4s” or ”T = 3*4”. Unfortunately, the tradition hides the similarity between word- and number-sentences by leaving out the subject and the verb and just saying ”3*4 = 12”.

Both languages have a meta-language, a grammar, describing the language, describing the world. Thus, the sentence ”This is a chair” leads to a meta-sentence ”The word ‘is’ is an auxiliary verb”. Likewise, the sentence ”T = 3*4” leads to a meta-sentence ”The operation ‘*’ is commutative”.

Since the meta-language speaks about the language, we should teach and learn the language before the meta-language. This is the case with the word-language only. Instead its self-referring setbased form has turned mathematics into a grammar labeling its outside roots as ‘applications’, used as means to dim the impeding consequences of teaching a grammar before its language.

Before 1970, language was taught as an example of its grammar (Chomsky, 1965). Then a reaction emerged. In his book ‘Explorations in the function of language’ Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to

> explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a “communicative approach to the teaching of language (p. ix)” allowing more students to learn a less correct language to be used for communication about outside things and actions.

Time for a Linguistic Turn in the Number-Language also

Thus, in language teaching a new version of the linguistic turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language.

So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use
- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

Maybe the time has come to realize that the two statements ‘2+3 = 5’ and ‘2*3 = 6’ have a different truth status.

The former is a conditional truth depending on the units. But, with 3 as the unit, the latter is an unconditional truth since 2 3s may always be recounted as 6 1s.

In short, maybe it is time to look for a different outside-inside mathematics to replace the present tradition, inside-outside meta-matism? And to ask what kind of math grows from the mastery of Many that children develop through use and before school?

DIFFERENCE RESEARCH LOOKS AT MATHEMATICS EDUCATION

To answer, we let Many open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a mathematics core curriculum based upon examples of Many in a STEM context (Lawrenz et al, 2017). Using ‘Difference-research’ (Tarp, 2017) searching for hidden differences making a difference, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

MEETING MANY CREATES A ‘COUNT-BEFORE-ADD’ CURRICULUM

Meeting Many, we ask “How many in Total?” To answer, we count by bundling to create a number-language sentence as e.g. T = 2 3s that contains a subject and a verb and a predicate as in a word-language sentence; and that connects the outside total T with its inside predicate 2 3s (Tarp, 2018b). Rearranging many 1s into one symbol with as many sticks or strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting by bundling and stacking:

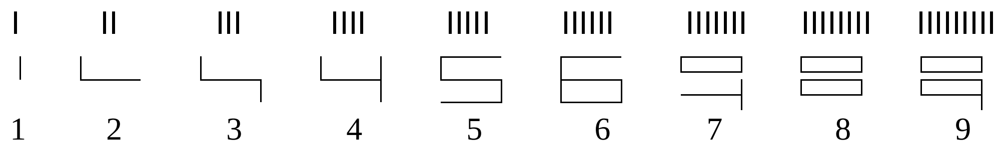

Figure 1: Digits as icons with as many sticks as they represent.

Holding 4 fingers together 2 by 2, a 3year-old will say ‘This is not 4, this is 2 2s’, thus describing what exists, bundles of 2s and 2 of them. This inspires ‘bundle-counting’, counting a total in icon-bundles to be stacked as bundle- or block-numbers, which can be recounted and double-counted before being processed by next-to and on-top addition, direct or reversed. Thus, a total T of 5 1s is recounted in 2s as T = 2 2s & 1; described by 'bundle-writing' as T = 2*B*1 2s; or by 'decimal-writing', T = 2.1 2s, where, with a bundle-cup, a decimal point separates the bundles inside from the outside unbundled singles; or by 'deficit-writing', T = 3*B*-1 2s = 3.-1 2s = 3 bundles less 1 2s.

To bundle-count a total T we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a rope pulling the block away).

A calculator thus predicts the result by a 'recount formula' $T = (T/B)*B$ saying that 'from T, T/B times, Bs can be taken away': entering '5/2' on a calculator gives '2.some', and '5 – 2x2' gives '1', so T = 5 = 2*B*1 2s. The unbundled can be placed next-to the stack as .1 or on-top as ½ counted in 2s, thus rooting decimals and fractions.

The recount formula occurs all over science. With proportionality: $y = c*x$; in trigonometry as sine, cosine and tangent: $a = (a/c)*c = \sin A*c$ and $b = (b/c)*c = \cos A*c$ and $a = (a/b)*b = \tan A*b$; in coordinate geometry as line gradients: $\Delta y = \Delta y/\Delta x = c* \Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In economics, the recount formula is a price formula: $ = ($/kg)*kg = price*kg, $ = ($/day)*day = price*day, etc.

Recounting in the Same Unit or in a Different Unit

Once counted, totals can be recounted in the same unit, or in a different unit. Recounting in the same unit, changing a bundle to singles allows recounting a total of 2*B*1 2s as 1*B*3 2s with an outside 'overload'; or as 3*B*-1 2s with an outside 'underload' thus rooting negative numbers. This eases division: 336 = 33*B*6 = 28*B*56, so 336/7 = 4*B*8 = 48; or 336 = 35*B*-14, so 336/7 = 5*B*-2 = 48.

Recounting in a different unit means changing unit, also called proportionality. Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2*B*2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer, '2.some', predicts that the unbundled singles come from taking away 2 5s, now asking '3*4 – 2*5'. The answer, '2', predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s or 2 2/5 5s.

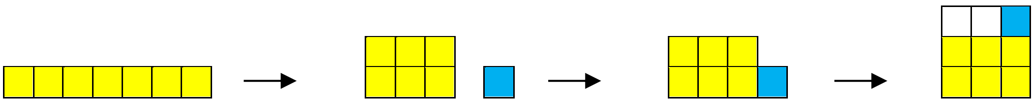

Figure 2: Seven counted as 2 3s & 1 or 2*B*1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

Recounting from Icons to and from Tens

Recounting from icons to tens by asking e.g. '2 7s = ? tens' is eased by using underloads: T = 2 7s = 2*7 = 2*(B-3) = 20-6 = 14; and T = 6 8s = 6*8 = (B-4)*(B-2) = BB - 4B - 2B -- 4*2 = 10B – 4B – 2B + 8 = 4*B*8 = 48. This makes sense since widening the base form t7 to ten will shorten the height from 6 to 4.8.

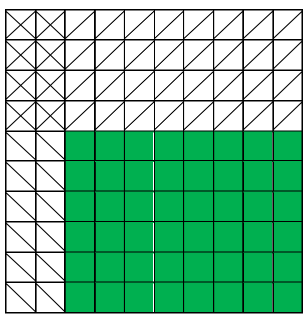

Figure 3: On an abacus 6 8s = 6*8 = (B-4)*(B-2) = 10B - 4B - 2B + 4 2s = 4*B*8 = 48.

Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer directly as 2*7 = 14, thus using a short form that leaves out the unit and misplaces the decimal point one place to the right.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation *u**7 = 35. It is easily solved by recounting 35 in 7s: *u**7 = 35 = (35/7)*7. So *u* = 35/7, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-Counting Creates Proportionality as Per-Numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To answer the question '*T* = 6$ = ?kg', we recount 6 in the per-number 2 and use the per-number to bridge 2$ and 3kg: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and vice versa: *T* = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Double-counting in the same unit creates fractions: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%. So 2/3 of 60 = 2$/3$ of 60$, where 60$ = (60/3)*3$ then gives (60/3)*2$ = 40$.

Double-Counting the Sides in a Block Creates Trigonometry

Halving a block by its diagonal allows mutual recounting of the sides, which creates trigonometry to precede plane and coordinate geometry: $a = (a/c)*c = \sin A*c$, and $a = (a/b)*b = \tan A*b$. Filling a circle with blocks shows that $\pi = n*\tan(180/n)$ for n large.

A SHORT CURRICULUM IN ADDITION-FREE MATHEMATICS

00. Playing with '1digit math' (Zybartas et al, 2005): Rearranging 3 cars into one 3-icon, etc. Recounting a total of ten fingers in bundles of e.g. 3s: T = 1Bundle7 = 2B4 = 3B1 = 4Bundle less 2 or 4B-2, and using both fingers and sticks and centi-cubes or LEGO bricks to experience algebra and geometry as always together, never apart. Recounting in a different unit when asking e.g. T = 2 3s = ?4s. Recounting to and from tens when asking e.g. T = 5 6s = ? tens, and T = 4B2 tens = ? 7s. Uniting blocks next-to and on-top when asking e.g. T = 2 3s & 4 5s = ? 8s; and T = 2 3s & 4 5s = ? 3s; and T = 2 3s & 4 5s = ? 5s. Splitting blocks next-to and on-top when asking e.g. T = 2 3s & ? 5s = 3 8s; and T = 2 3s & ? 5s = 7 3s; and T = 2 3s & ? 5s = 4 5s.

01. Until nine, many ones may be rearranged into one icon with as many sticks or strokes as it represents. As one bundle, ten needs no icon. So, a total typically consists of several countings: of unbundled ones, of bundles, of bundles of bundles, etc.

02. Parallel counting sequence stress the importance of bundling: 0Bundle1, 0*B*2, …, 0*B*9, 1*B*0, 1*B*1 etc.; or 0*B*1, 0*B*2, 0*B*3, 0*B*4, 0*B*5 or half Bundle, Bundle less 4, *B*-3, *B*-2, *B*-1, Bundle or 1*B*0, Bundle and 1 or 1*B*1, Bundle and 2 or 1*B*2, etc., thus rooting negative numbers. Here we mention that the Vikings used the words 'eleven' and twelve' as short for 'one-left' and 'two-left'. Using other bundles as units, ten fingers may be counted as 1*B*3 7s, 2*B*0 5s, 2*B*2 4s, 3*B*1 3s, 1*BB*0*B*1 3s, 5*B*0 2s, and 1*BBB*0*BB*1*B*0 2s. A Total of five fingers can be recounted in 2s in three ways, standard or with overload or underload: *T* = 2*B*1 2s = 1*B*3 2s = 3*B*-1 2s = 3 bundles less 1 2s.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. For prediction, a calculator uses a 'recount formula', *T* = (*T/B*)**B*, saying that 'from *T*, *T/B* times, *B*s can be taken away'.

04. Recounting in a different unit is called proportionality. Asking '3 4s = ? 5s', we enter '3*4/5' to ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s can be recounted in 5s as 2*B*2 5s. The unbundled can be placed next-to the bundles separated by a decimal point, or on-top counted in bundles, thus rooting decimals and fractions, *T* = 3 4s = 2*B*2 5s = 2.2 5s = 2 2/5 5s.

05. Recounting from tens to icons by asking '35 = ? 7s' is called an equation *u**7 = 35, solved by recounting 35 in 7s: *u**7 = 35 = (35/7)*7. So *u* = 35/7, showing that equations are solved by moving

to opposite side with opposite calculation sign. Division is eased by using overloads or underloads: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48. As is multiplication: T = 4*78 = 4* 7*B*8 = 28*B*32 = 31*B*2 = 312.

06. Recounting from icons to tens by asking ‘2 7s = ? tens’ is eased by underloads: *T* = 2*7 = 2*(*B*-3) = 20-6 = 14; 6*8 = (*B*-4)*(*B*-2) = *BB*-4*B*-2*B*--8 = 100 – 60 + 8 = 48.

07. Double-counting a quantity in two units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘*T* = 6$ = ?kg’, we recount 6 in the per-number: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Trigonometry can precede plane and coordinate geometry to show how, in a block halved by its diagonal, the sides can be mutually recounted as e.g. $a = (a/c)*c = \sin A*c$.

MEETING MANY IN A STEM CONTEXT

OECD (2015b) says: “In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.” STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature’s behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e. tools, motors and computers, that combined to robots help transforming nature into human necessities.

Nature as Things in Motion

To meet, we must specify space and time in a nature consisting of things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature’s three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

In nature, we count matter in kilograms, space in meters and time in seconds. Things in motion have a momentum = mass * velocity, a multiplication formula as most STEM formulas expressing recounting by per-numbers:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/liter) * liter = molarity * liter
- energy = (energy/kg/degree) * kg * degree = heat * kg * degree
- Δ momentum = (Δ momentum/second) * second = force * seconds
- Δ energy = (Δ energy/meter) * meter = force * meter = work
- energy/sec = (energy/charge) * (charge/sec) or Watt = Volt * Amp.

Thus, STEM-subjects swarm with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Warming and Boiling Water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number 410/70/1.4 Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is 316/0.14 kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving Material in Water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 mole/liter, or 0.676 mole/liter.

Building Batteries with Water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H2O, and carbon dioxide CO2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH4, the energy comes from using 4 oxygen atoms to burn it.

Technology & Engineering: Steam and Electrons Produce and Distribute Energy

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units making a mole of water weigh 18 gram. Since the density of water is roughly 1 kilogram/liter, the volume of 1000 moles is 18 liters. With about 22.4 liter per mole, its volume increases to about 22.4*1000 liters if transformed into steam, which is an increase factor of 22,400 liters per 18 liters = 1,244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, $p*V = n*R*T$, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used. The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g. steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to consumers.

An Electrical Circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per ‘carrier’ (charge-unit). Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 carriers per second, which is called the electrical current counted in Ampere, a per-number double-counting the number of carriers per second. To create this current, the kettle must have a resistance R according to a circuit law ‘Volt = Resistance*Ampere’, i.e., 220 = Resistance*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per carrier)*(carrier per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120 Watt bulb, the latter needs twice the energy and current, and consequently has half the resistance of the former, making the latter receive half the energy if connected in series.

How High Up and How Far Out

A spring sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, e.g. 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the spring 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, e.g. 2.5 meters, 5 meters and 0,71 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin45 = cos45 = 0,707. The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed *u* by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (*u**0,707)*0,71, solved as *u* = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app. Compared with the horizontal distance, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2. So horizontally, the distance and the speed are proportional. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

ADDING ADDITION TO THE CURRICULUM

Once counted as block-numbers, totals can be added next-to as areas, thus rooting integral calculus; or on-top after being recounted in the same unit, thus rooting proportionality. And both next-to and on-top addition can be reversed, thus rooting differential calculus and equations where the question 2 3s + ? 4s = 5 7s leads to differentiation: ? = (5*7 – 2*3)/4 = $\Delta T/4$.

Integral calculus thus precedes differential calculus and include adding both piecewise and locally constant (continuous) per-numbers. Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 3 add directly, but the per-numbers must be multiplied into unit-numbers. So, both per-numbers and fractions must be multiplied by the units before being added as the area under the per-number graph.

Using overloads and underloads eases addition and subtraction: $T = 23 + 49 = 2B3 + 4B9 = 6B12 = 7B2 = 72$; and T = 56 – 27 = 5*B*6 – 2*B*7 = 3*B*-1 = 2*B*9 = 29.

Moving in a coordinate system, distances add directly when parallel; and by squares when perpendicular. Re-counting the y-change in the x-change creates a linear change formula $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$, algebraically predicting geometrical intersection points, thus observing a 'geometry & algebra, always together, never apart' principle.

The number-formula $T = 456 = 4*B$^2 + 5*B + 6*1 shows the four ways to unite numbers offered by algebra meaning 'reuniting' in Arabic: addition and multiplication add changing and constant unit-numbers; and integration and power unite changing and constant per-numbers. And since any operation can be reversed: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers (Tarp, 2018b):

Uniting/splitting into	Changing	Constant
Unit-numbers m, s, kg, $	***T = a + n*** *T – a = n*	***T = a * n*** *T/n = a*
Per-numbers m/s, $/kg, $/100$ = %	***T = ∫ a dn*** *dT/dn = a*	***T = a^n*** *loga(T) = n* *n√T = a*

Figure 4: An 'Algebra-Square' with the 4 and 5 ways to unite and split totals.

In its general form, the number formula $T = a*x$^2 + $b*x$ + c contains the different formulas for constant change: $T = a*x$ (proportionality), $T = a*x$^2 (acceleration), *T = a*x^c* (elasticity) and *T = a*c^x* (interest rate); as well as $T = a*x+b$ (linearity, or affinity, strictly).

As constant/changing, predictable change roots pre-calculus/calculus. Unpredictable change roots statistics to 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

Engineering: How Many Turns on a Steep Hill

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let *A* and *B* label the ground corners of the hillside. *C* labels the point where a road from *A* meets the edge for the first time, and *D* is vertically below *C* on ground level. We want to find the distance $BC = u$.

In the triangle *BCD*, the angle *B* is 30 degrees, and *BD* = *u**cos(30). With Pythagoras we get *u*^2 = *CD*^2 + *BD*^2 = *CD*^2 + *u*^2*cos(30)^2, or *CD*^2 = *u*^2(1-cos(30)^2) = *u*^2*sin(30)^2. In the triangle *ACD*, the angle *A* is 10 degrees, and *AD* = *AC**cos(10). With Pythagoras we get *AC^2 = CD^2 + AD^2 = CD^2 + AC*^2*cos(10)^2, or *CD*^2 = *AC*^2(1-cos(10)^2) = *AC*^2*sin(10)^2. In the triangle *ACB*, *AB* = 1 and *BC = u*, so with Pythagoras we get *AC*^2 = 1^2 + *u*^2, or *AC* = √(1+*u*^2).

Consequently, *u*^2*sin(30)^2 = *AC*^2*sin(10)^2, or *u* = *AC**sin(10)/sin(30) = *AC*r*,
or *u* = √(1+*u*^2)**r*, or *u*^2 = (1+*u*^2)**r*^2, or *u*^2*(1-*r*^2) = *r*^2, or *u*^2 = *r*^2/(1-*r*^2) = 0.137, giving the distance *BC* = *u* = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

CONCLUSION AND RECOMMENDATION

This paper argues that 50 years of unsuccessful mathematics education research may be caused by a goal displacement seeing mathematics as the goal instead of as an inside means to the outside goal, mastery of Many in time and space. The two views lead to different kinds of mathematics: a setbased top-down 'meta-matics' that by its self-reference is indeed hard to teach and learn; and a bottom-up Many-based 'Many-matics' simply saying "To master Many, counting and recounting and double-counting produces constant or changing unit-numbers or per-numbers, uniting by adding or multiplying or powering or integrating." A proposal for two separate twin-curricula in counting and adding is found in Tarp (2018a).

Thus, the simplicity of mathematics as expressed in a 'count-before-adding' curriculum allows replacing line-numbers with block-numbers. Imbedded in STEM-examples, young migrants learn core STEM subjects at the same time, thus allowing them to become STEM pre-teachers or pre-engineers to help develop or rebuild their own country. The full curriculum can be found in a 27-page paper (Tarp, 2017).

Thus, it is possible to solve core STEM problems without learning addition, that later should be introduced in both versions since blocks can be added both on-top using proportionality to make the units the same, and next-to by areas as integral calculus.

So, as with another foreign language, why not learn the number-language through its use. And celebrate that core mathematics as proportionality, equations, per-numbers and calculus grow directly from the mastery of Many that children develop through use and before school? Let us see math, not as a goal in itself, but as an inside means to an outside goal that is reached better and by more with quantitative communication.

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05. BUNDLE-COUNTING PREVENTS & CURES MATH DISLIKE

Inside-Outside Mathematics

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5. This offers a 'natural' curriculum with multidigit numbers obeying a place-value system; and with operations having addition as the base with subtraction as reversed operation, where multiplication is repeated addition with division as reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction creates negative numbers, where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2. Then halving a block by its diagonal creates a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2, and where π = n*sin(180/n) for n large.

Then calculations with unspecified numbers leads to creating additional education about the results of operating on such numbers, e.g. that (a+b) * (a-b) = a^2 - b^2.

In a calculation, changing the input will change the output. Relating the changes creates an operation on the calculation called differentiation, also creating additional education about the results of operating on calculations, e.g. that (f*g)/(f*g) = f/f + g/g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6*x^2dx = 2*x^3.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modeling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature.

Outside-Inside Mathematics

But, as with another foreign language, why not learn the number-language through its use? Is the goal of mathematics education to learn mathematics – or to learn how to master Many? Is math a goal in itself, or an inside means to an outside goal, that may be reached better and by more through quantitative communication? What math grows from the mastery of Many that children develop through use and before school?

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total *T* by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. *T* = 2 5s.

02. Icons thus inspires 'bundle-counting' and 'bundle-writing' where a total *T* of 5 1s is recounted in 2s as *T* = 1*B*3 2s = 2*B*1 2s = 3*B*-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, *T* = 5 = 2*B*1 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a 'recount formula', *T* = (*T/B*)**B*, to predict that 'from *T*, *T/B* times, *B*s can be taken away'. This recount

formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

04. Recounting in a different unit is called proportionality. Asking ‘3 4s = ? 5s’, sticks say 2*B*2 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’. The answer ‘2.some’ predicts that the singles come by taking away 2 5s, thus asking ‘3*4 – 2*5’. The answer ‘2’ predicts that 3 4s can be recounted in 5s as 2*B*2 5s or 2.2 5s.

05. Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking ‘2 7s = ? tens’ is eased by using underloads: $T = 2*7 = 2*(B-3) = 20-6 = 14$; and $6*8 = (B-4)*(B-2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$.

07. Double-counting a quantity in units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘*T* = 6$ = ?kg’, we recount 6 in 2s since the per-number is 2$/3kg: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

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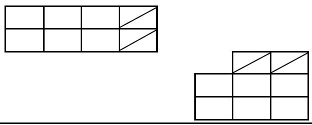
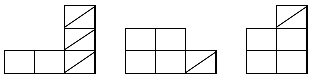
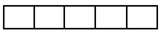
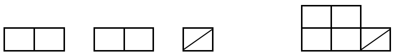
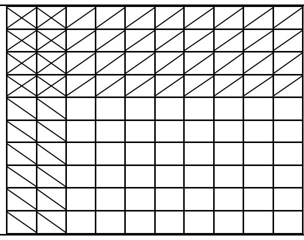
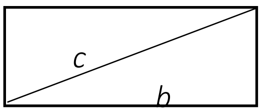
06. Flexible Bundle-Numbers

respect & develop **Kids Own Math** *Outside & Inside Math*

Digits as ICONS III IIII IIIII		3 4 5
Operations as ICONS	Push Lift Pull	/ X –
Count Fingers in 5s using BundleCounting & BundleNumbers	I●●●● II●●● III●● IIII● IIIII IIIII I●●●●	T = 0*B*1 = 1*B*-4 5s T = 0*B*2 = 1*B*-3 5s T = 0*B*3 = 1*B*-2 5s T = 0*B*4 = 1*B*-1 5s T = 1*B*0 = 5 T = 1*B*1 = 2*B*-4 5s
Unbundled creates Decimals & Fractions & Negative Numbers IIIIIIII → IIII IIII II		T = 2*B*2 **3s** = 2.2 **3s** T = 2 2/3 **3s** T = 3*B*-1 3s = 3.-1 **3s** T = 1*BB* 0*B* -1 (T = *p***x*^2 + *q***x* + *r*)
ReCount in Same Unit creates Flexible Numbers IIIII III → 53	5: IIII III IIII II IIII IIII 	T = 1*B*3 *Overload* T = 2*B*1 *Standard* T = 3*B*-1 *Underload* T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 **tens**
Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7* 48 = ? = 336 /7 = ? =	6*B*5 + 2*B*7 = 8*B*12 = 9*B*2 = 92 6*B*5 – 2*B*7 = 4*B*-2 = 3*B*8 = 38 7* 4*B*8 = 28*B*56 = 33*B*6 = 336 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48
ReCount in New Unit  *ReCount-Formula:*	5 = ? **2s**  T = (T/*B*) * *B*	T = 5 = (5/2)*2 = ? = 2*B*1 **2s** 5/2 2.some 5 - 2*2 1
ReCount: Tens to Icons III IIIII = ? **7s**	3*B*5 **tens** = *u**7	*u**7 = 35 = (35/7)*7 so *u* = 35/7
ReCount: Icons to Tens 6 **8s** = ? **tens** 		T = 6 **8s** = 6*8 = (*B*-4)*(*B*-2) = *BB* - 4*B* - 2*B* - - 8 = 10*B* - 6*B* + 8 = 4*B*8 = 4.8 **tens** = 48
DoubleCount gives PerNumbers	2$ per 3kg = 2$/3kg	T = 6$ = (6/2) * 2$ = (6/2) * 3kg = 9kg
Like Units: Fractions 5% of 40	5$/100$ of 40$	T = 40$ = (40/100)*100$ gives (40/100)*5$ = 2$
DoubleCount a Block halved by its Diagonal		*a* = (*a*/*c*)**c* = sin *A***c* *a* = (*a*/*b*)**b* = tan *A***b* π= *n**tan(180/*n*) *for n large*

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Add NextTo OnTop	T = 2 **3s** + 4 **5s** = 3*B*2 **8s** *Integration* T = 2 **3s** + 4 **5s** = 1*B*1 **5s** + 4 **5s** = 5*B*1 **5s** *Proportionality*
MatheMatism	ADDING WITHOUT UNITS Digits or Fractions or Per-numbers

Flexible Bundle-Numbers Respect & Develop Kids Own Math

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total T by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. T = 2 5s.

02. Icons thus inspires ‘bundle-counting’ and ‘bundle-writing’ where a total T of 5 1s is recounted in 2s as T = 1B3 2s = 2B1 2s = 3B-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, T = 5 = 2B1 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: T = 336 = 33B6 = 28B56 = 35B-14, so 336/7 = 4B8 = 5B-2 = 48.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a ‘recount formula‘, $T = (T/B)*B$, to predict that ‘from T, T/B times, Bs can be taken away’. This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

04. Recounting in a different unit is called proportionality. Asking ‘3 4s = ? 5s’, sticks say 2B2 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’. The answer ‘2.some’ predicts that the singles come by taking away 2 5s, thus asking ‘3*4 – 2*5’. The answer ‘2’ predicts that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

05. Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking ‘2 7s = ? tens’ is eased by using underloads: $T = 2*7 = 2*(B-3) = 20-6 = 14$; and $6*8 = (B-4)*(B-2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$.

07. Double-counting a quantity in units gives a ‘per-number’ as e.g. 2\$ per 3kg, or 2\$/3kg. To answer the question ‘T = 6\$ = ?kg’, we recount 6 in 2s since the per-number is 2\$/3kg: T = 6\$ = (6/2)*2\$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2\$/3\$ = 2/3, and 2\$/100\$ = 2/100 = 2%.

08. Next-to addition geometrically means adding by areas, so multiplication precedes addition. Next-to addition is also called integral calculus, or differential if reversed.

09. On-top addition means using the recount-formula to get like units. Changing units is also called proportionality, or solving equations if reversed.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education*, *11*(1), 103-117.

07. WORKSHOP IN ADDITION-FREE STEM-BASED MATH

Nature as Heavy Things in Motion in Time and Space

A falling ball introduces nature's three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

Matter and force and motion all represent different degrees of Many, thus calling for a science about Many. This is how mathematics arose in ancient Greece, so it should respect its root as a natural science by letting multiplication precede addition since the basic science formulas are multiplication formulas expressing 'per-numbers' coming from double-counting: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter)*sq.-meter = pressure*sq.-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg)*kg = heat*kg; Δ momentum = (Δ momentum/sec)*sec = force*sec = impulse; Δ energy = (Δ energy/meter)*meter = force*meter = work; gram = (gram/mole)*mole = molar mass*mole; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp.

Counting in Icon-Bundles Allows Recounting in the Same and in a Different Unit

Meeting many, we observe that five ones may be recounted as one five-icon. Likewise, with the other digits; thus being, not symbols, but icons with as many strokes or sticks as they represent. 'Bundle-counting' in icon-bundles allows 'bundle-writing' where a total *T* of 5 1s is recounted in 2s as *T* = 1*B*3 2s = 2*B*1 2s = 3*B*-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, *T* = 5 = 2*B*1 2s = 2.1 2s = 2 ½ 2s.

Recounting in the same unit to create or remove over- or underloads eases operations. Example: *T* = 336 = 33*B*6 = 28*B*56 = 35*B*-14, so 336/7 = 4*B*8 = 5*B*-2 = 48.

Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross.

This creates a 'recount formula', *T* = (*T/B*)**B*, saying that 'from *T*, *T/B* times, *B*s can be taken away'. This formula predicts the result of recounting in another unit, called proportionality: Asking '3 4s is how many 5s?', sticks show that 3 4s becomes 2*B*2 5s. Entering '3*4/5' we ask a calculator 'from 3 4s we take away 5s'. The answer '2.some' predicts that the unbundled singles come by taking away 2 5s, thus asking '3*4 – 2*5'. The answer '2' predicts that 3 4s recount in 5s as 2*B*2 5s or 2.2 5s or 2 2/5 5s.

This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$.

Recounting to and from Tens

Times tables ask '2 7s = ? tens', eased by using underloads: $T = 2*7 = 2*(B\text{-}3) = 20\text{-}6 = 14$; and $6*8 = (B\text{-}4)*(B\text{-}2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$. Using the recount formula is impossible since the calculator has no ten-button. Instead it is programmed to give the answer as 3*4 = 12, using a short form that leaves out the unit and misplaces the decimal point one place to the right, strangely enough called a 'natural' number.

Recounting from tens to icons by asking '35 = ? 7s' is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to the opposite side with the opposite calculation sign.

Double-counting Creates Proportionality as Per-Numbers

Counting a quantity in 2 different physical units gives a 'per-number' as e.g. 2\$ per 3kg, or 2\$/3kg. To answer the question '*T* = 6\$ = ?kg', we recount 6 in 2s since the per-number is 2\$/3kg: *T* = 6\$ = (6/2)*2\$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2\$/3\$ = 2/3, and 2\$/100\$ = 2/100 = 2%.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Workshop exercises in addition-free STEM-based math

STEM: Mathematics as one of the natural Sciences, applied in Technology and Engineering

Using 'Outside-Inside Math' allows outside science problems to be solved by inside math formulas.

Outside degrees of Many create inside number-icons with the number of strokes they represent. Outside counting-operations, occurring when bundles are pushed away, lifted and pulled away to find unbundled ones, create the operation-icons division, /, and multiplication, x, and subtraction -.

Once bundle-counted, recounting in different units (called proportionality) create a 'recount-formula', $T = (T/B)*B$, saying that 'from *T*, *T/B* times, *B*s can be taken away'; occurring all over math and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c$; in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$; in science as speed: $\Delta s = (\Delta s/\Delta t)*\Delta t = v*\Delta t$.

Asking '3 **4s** is how many **5s**?', outside sticks show that 3 **4s** becomes 2*B*2 **5s**: IIII IIII IIII -> VV II.

To predict inside, we enter '3*4/5' to ask a calculator 'from 3 **4s** we take away **5s'**. The answer '2.some' predicts that the unbundled ones come by taking away 2 **5s**. Now, asking '3*4 – 2*5' gives '2'. So, 3 **4s** = 2*B*2 **5s = 2.2 5s**.

3*4/5	2.some
3*4 – 2*5	2

Recounting a quantity in two different physical units gives a 'per-number' as e.g. 2m per 3sec, or 2m/3sec. To answer the question '*T* = 6m = ?sec', we recount 6 in **2s** since the per-number is 2m/3sec: *T* = 6m = (6/2)*2m = (6/2)*3sec = 9sec. Double-counting in the same unit creates fractions and %: 2\$/3\$ = 2/3, and 2\$/100\$ = 2/100 = 2%. 5% of 40 = ?; T = 40 = (40/100)*100 gives (40/100)*5 = 2.

kg = (kg/cubic-meter)*cubic-meter = density*cub.-meter force = (force/square-meter)*sq.-meter = press.*sq.-meter meter = (meter/sec)*sec = velocity*sec energy = (energy/sec)*sec = Watt*sec energy = (energy/kg)*kg = heat*kg	Δ momentum = (Δ mom./sec)*sec = force*sec = impulse Δ energy = (Δ energy/meter)*meter = force*meter = work gram = (gram/mole)*mole = molar mass*mole energy/sec = (energy/charge)*(charge/sec), or Watt = Volt*Amp.

Science multiplication formulas expressing 'per-numbers' coming from double-counting

Five Ways to Solve Proportionality Questions

Inside recounting solves outside questions as “If 2m need 5sec, then 7m need ?sec; and 12sec gives ?m”

● Europe used the ‘Regula de Tri’ (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: ‘2m takes 5s, 7m takes ?s’ to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: ‘5s gives 2m, 12s gives ?m’ to get to the answer (12*2)/5s = 4.8m.

Then, two new methods appeared, ‘find the unit’, and cross multiplication in an equation expressing like proportions or ratios:

● Q1: 1m takes 5/2s, so 7m takes 7*(5/2) = 17.5s. Q2: 1s gives 2/5m, so 12s gives 12*(2/5) = 4.8m.

● Q1: 2/5 = 7/*x*, so 2**x* = 7*5, *x* = (7*5)/2 = 17.5. Q2: 2/5 = *x*/12, so 5**x* = 12*2, *x* = (12*2)/5 = 4.8.

● Alternatively, we may recount in the ‘per-number’ 2m/5s coming from ‘double-counting’ the total *T*. Q1: *T* = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: *T* = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● SET introduced modeling with linear functions to show the relevance of abstract algebra’s group theory: Let us define a linear function *f*(*x*) = *c***x* from the set of m-numbers to the set of s-numbers, having as domain *DM* = {*x*∈R l *x*>0}. Knowing that *f*(2) = 5, we set up the equation *f*(2) = *c**2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: *c**2 = 5, (*c**2)*½ = 5*½, *c**(2*½) = 5/2, *c**1 = 5/2, *c* = 5/2. With *f*(*x*) = 5/2**x*, the inverse function is *f*-1(*x*) = 2/5**x*. So with 7m, *f*(7) = 5/2*7 = 17.5s; and with 12s, *f*-1(12) = 2/5*12 = 4.8m.

Three different kinds of mathematics answering the question: What is a function?

pre-setcentric:	*a function is a calculation with specified and unspecified numbers.*
present setcentric:	*a function is a subset of a set-product where component identity transfers*.
post-setcentric:	*a function is a number-language sentence with a subject, a verb and a predicate.*

EXERCICES

E01. With sticks, transform many OUTSIDE ones into one INSIDE many-icon with as many strokes as it represents.

E02. Name fingers as 5s using BundleCounting & BundleNumbers: 0**B**1 = 1**B**-4, 0**B**2 = 1**B**-3, …**5s**

E03. Count 5 fingers in 2s using flexible bundle-numbers: T = 5 = 1**B**3 = 2**B**1 = 3**B**-1 **2s** (overload, standard, underload)

E04. Recount ten fingers in **4s**, **3s** and **2s**: T = ten = 1**B**6 = 2**B**2 = 3**B**-2 **4s**; T = 3**B**1 = 1**BB**1 = 1**BB** 0**B** 1 = 10.1 **3s**; T = 5**B**0 = 4**B**2 = 2**BB** 1**B** 0 = 1**BBB** 0**BB** 1**B** 0 **2s**. ReCount 7 fingers in **3s**: T = 7 = 2B1= 1BB-2 = 2.1 = 3.-2 = 2 1/3.

E05. Write traditional numbers as flexible BundleNumbers: T = 53 = 5**B**3 = 4**B**13 = 6**B**-7 tens

E06. Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7* 48 = ? = 336 /7 = ? =	6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92 6**B**5 – 2**B**7 = 4**B**-2 = 3**B**8 = 38 7* 4**B**8 = 28**B**56 = 33**B**6 = 336 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

E07. With cubes, transform the three OUTSIDE parts of a counting process, PUSH & LIFT & PULL, into three INSIDE operation-icons: division & multiplication & subtraction.

Five counted in **2s**: I I I I I (push away **2s**) II II I (lift to stack) II II I (pull to find unbundles ones) II II I.

E08. OUTSIDE BundleCounting with icons as units may be predicted INSIDE by a **recount-formula T = (T/B)*B**, (from T, T/B times, take Bs away) using a full number-language sentence with a subject, a verb and a predicate.

OUTSIDE: T = I I I I I; T counted in **2s**: II II I; T – 2x2: = II II I ; INSIDE:

5/2	2.some
5 – 2x2	1

E09. Recount in a new unit to change units, predicted by the recount-formula

OUTSIDE, use sticks or cubes to answer 3 **4s** = ? **5s**. INSIDE, the recount-formula predicts 3x4/5

E10. Recount from tens to icons

OUTSIDE, to answer the question ‘40 = ? **5s’**, on squared paper transform the block 4.0 **tens** to **5s**.

INSIDE, formulate an equation to be solved by recounting 40 in **5s**:

u*5 = 40 = (40/5)*5, so u = 40/5.

Notice that recounting gives the solution rule ‘move to opposite side with opposite calculation sign’.

E11. Recount from icons to tens

OUTSIDE, to answer ‘3 **7s** = ? **tens’** on squared paper transform the block 3 **7s** to **tens**.

INSIDE: oops, with no ten-button on a calculator we can’t use the recount-formula? Oh, we just multiply!

E12. ReCounting in two physical units

Recounting in two physical units gives a ‘per-number’ as e.g. 2m per 3sec, or 2m/3sec.

To answer the question ‘T = 6m = ?sec’, we just recount 6 in the per-number **2s**: T = 6m = (6/2)*2m = (6/2)*3sec = 9sec.

E13. Solving STEM proportionality heating problems with recounting

With a heater giving 20 J in 30 sec, what does 40 sec give, and how many seconds is needed for 50J?

With 40 Joules melting 5kg, what will 60 Joules melt and what will 7 kg need?

With 3 degrees needs 50 Joules, what does 7 degrees need; and what does 70 Joules give?

With 4 deg. in 20kg needing 50 Joules, what does 9 deg. in 30 kg need? What does 70 Joules give in 40 kg?

E14. Mutual ReCounting the sides in a block halved by its diagonal creates trigonometry:

a = (a/b)*b = tanA*b

Draw a vertical tangent to a circle with radius r. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/radius with tangent of the angle on a calculator.

E15. Engineering

A 12x12 square ABCD has AB on the ground and is inclined 20 degrees. From B, a straight road is to be constructed intersecting the borderline AD in the point E, inclined 5 degrees. Find the length DE.

Hint: Show that if DE = 2, then the incline of the road is 3.2 degrees.

E16. Traveling

With 4 meters taking 5 seconds, what does 6 meters take; and what does 7 seconds give?

With distance d and speed v and time t related as d = v*t, what time is needed to go 20m with velocity 4m/s?

With distance d and time related as d = 5*t^2, what time is needed to go 30m?

Hint: Use that if p^2 < N < $(p+1)^2$, then √N ≈ $\frac{N+p^2}{2p}$

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**10~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	~~10**B**10~~
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

09. DEVELOPING THE CHILD'S OWN MASTERY OF MANY

Oral Presentation Outline

Present setcentric and pre-setcentric math are challenged by post-setcentric math seeing math, not as a goal, but as a means to develop the mastery of Many children bring to school.

Asked "How old next time?", a 3year old says "Four" showing 4 fingers; but protests if held together 2 by 2: "That is not four, that is two 2s", thus describing what exists, and with units: the total is bundles of 2s, and 2 of them. Children thus develop both word- and number-sentences with a subject, a verb and a predicate. The outside total exists as a natural fact, but the inside predication it chosen and can be changed: T = 4 1s = 2 2s = 1Bundle 1 3s = 0Bundle less 1 5s, etc.

Post-setcentric textbooks allow children to develop further their mastery of Many by counting and recounting totals before adding them; and to number instead of being taught about numbers. A textbook thus can be based upon the following 'research' questions.

- The digit 5 is an icon with five sticks. Does this apply to all digits?
- How to count fingers in different sequences and bundles?
- How can a calculator predict a recounting result?
- What to do with the unbundled singles?
- How to recount in the same or in another unit?
- How to recount between tens and icons?
- How to recount the sides in a block halved by its diagonal?
- How to perform and reverse next-to and on-top addition?
- How to perform and reverse adding per-numbers and fractions?

Tarp, A. (2018). Mastering Many. *Journal of Mathematics Education 11*(1), 103-117.

10. MATH DISLIKE CURED WITH INSIDE-OUTSIDE DECONSTRUCTION

Poster

In math, division often creates dislike. Sociological imagination asks: Could math teach how to number instead of about numbers? Here existentialism and deconstruction ask: can the outside existence be predicated differently?

Outside totals are predicated more naturally by digits as icons with as many strokes as they represent.

Outside operations are predicated more naturally by division wiping away bundles to be stacked by multiplication, to be taken away to look for singles by subtraction.

Once reported by a full number-sentence with an outside subject and an inside predicate, a total can be recounted in the same unit with overloads or underloads:

T = 47 = 4Bundle7 tens = 3B17 tens = 5B-3 tens.

Such flexible numbers ease operations, e.g.:

T = 245 /7 = 24B5 /7 = 21B35 /7 = 3B5 = 35.

Recounting in another unit uses a ‘recount formula‘, T = (T/B)xB, saying ‘from T, T/B times, Bs can be taken away’. (4x5/6)x6 thus predicts that 4 5s recounts as 3B2 6s. Recounting 3 in 5s as T = 3 = (3/5) 5s creates fractions. Recounting a total in 2kg and 3L gives a ‘per-number’ 2kg per 3L = 2/3 kg/L, bridging the units by recounting in the per-number: T = 6kg = (6/2)x2kg = (6/2)x3L = 9L.

Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation ux7 = 35 (= (35/7)x7), giving u = 35/7 by recounting.

Recounting from icons to tens by asking ‘3 4s = ? tens’ is called times tables eased by flexible numbers.

Tarp, A. (2018). Mastering Many. *Journal of Mathematics Education 11*(1), 103-117.

11. LEARNING FROM THE CHILD'S OWN MATHEMATICS

Workshop

Post-setcentric math (Tarp, 2018) allows children to develop their mastery of Many by counting and re-counting totals before adding them; and to number instead of being taught about numbers. A textbook is based upon the following 'research' questions.

"The digit 5 is an icon with five sticks, does this apply to all digits?" Many ones change to one icon with as many sticks as it represents, shown on a folding ruler.

"How to number fingers in different sequences?" Five fingers: '01, 02, 03, 04, Hand' to include the bundle; or '01, 02, Hand less2, H-1, Hand. Ten fingers: …, H&1, H&2, (ten) less 2, (ten) less one, ten, one left, two left (en levnet, to levnet in Viking counting).

"How to number fingers by bundle-counting?" Ten fingers count in 5s as T = ten = 2Bundle0 5s = 2B0 5s (= 2.0 5s later), using a number-sentence with a subject, a verb and a predicate, and called an inside flexible 'bundle-number' describing the outside Lego 'block-number'. In 4s as T = 2B2 4s; in 3s as T = 3B1 3s = 1BB1 3s = 1BB0B1 3s, or T = 1xB^2 + 0xB + 1, showing the four ways to unite: on-top addition, multiplication, power and next-to block addition called calculus.

"How to number fingers with overloads or underloads?" Ten fingers bundle-count in 3s as T = ten = 1B7 3s = 2B4 3s = 3B1 3s = 4B-2 3s. Over- and underloads ease operations: T = 225 /3 = 22B5 /3 = 21B15 /3 = 7B5 = 75.

"How can a calculator predict a recounting result?" Division iconizes wiping away bundles. Multiplication iconizes stacking bundles into a block. Subtraction iconizes dragging it away to look for unbundled singles. Showing '7/2' as '3.some', and '7-3x2' as '1' predicts that 7 recounts in 2s as 3B1 2s, using a 'recount-formula' T = (T/B)xB, saying "From T, T/B times B can be taken away", occurring all over math and science.

"How to recount in another unit?" Asking T = 4 5s = ? 6s, the recount-formula says T = 4 5s = (4x5/6) 6s. The answer '3.some' suggests entering '4x5–3x6'. The answer '2' predicts that 4 5s recount in 6s as 3B2 6s. Asking T = 3 = ? 5s, the recount-formula says T = 3 = (3/5) 5s thus creating fractions. Recounting a total in 2kg and 3L gives a 'per-number' 2kg per 3L = 2/3 kg/L, bridging the units by recounting in the per-number: T = 6kg = (6/2)x2kg = (6/2)x3L = 9L; and T = 12L = (12/3)x3L = (12/3)x2kg = 8kg. Likewise, a fraction of a total is found by recounting in the per-number.

"How to recount tens into icons?" Asking 'T = 2.4 tens = 24 = ? 8s' is an equation, ux8 = 24 (= (24/8)x8), giving u = 24/8 by recounting.

"How to recount icons into tens?" Asking 'T = 3 8s = ? tens creates the multiplication tables eased by flexible numbers: 3 8s = 3x(B less 2) = 3B-6 = 2B4 = 24.

"How to recount the sides in a block halved by its diagonal?" With base b, height a, and diagonal c, mutual recounting gives the trigonometric per-numbers: a = (a/c)xc = sinAx c, etc.

Tarp, A. (2018). Mastering Many by Counting. *Journal of Mathematics Education 11*(1), 103-117.

12. FIVE ALTERNATIVE WAYS TO TEACH PROPORTIONALITY

Poster

Proportionality is rooted in questions as "If 2kg costs 5$, what does 7kg cost; and what does 12$ buy?"

A1) Europe used the 'Regula de Tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide.

So first we ask, Q1: '2kg cost 5$, 7kg cost ?$' to get to the answer (7*5/2)$ = 17.5$.

Then we ask, Q2: '5$ buys 2kg, 12$ buys ?kg' to get to the answer (12*2)/5$ = 4.8kg.

A2) Then, two new methods appeared, 'find the unit':

Q1: 1kg costs 5/2$, so 7kg cost 7*(5/2) = 17.5$.

Q2: 1$ buys 2/5kg, so 12$ buys 12*(2/5) = 4.8kg.

A3) And cross multiplication in an equation e*pressing like proportions or ratios:

Q1: 2/5 = 7/u, so 2*u = 7*5, u = (7*5)/2 = 17.5.

Q2: 2/5 = u/12, so 5*u = 12*2, u = (12*2)/5 = 4.8.

A4) Set-based New Math chose modelling with linear functions to show the relevance of abstract algebra's group theory: Let us define a linear function f(u) = c*u. Knowing that f(2) = 5, we set up the equation f(2) = c*2 = 5 to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law:

c*2 = 5, (c*2)*½ = 5*½, c*(2*½) = 5/2, c*1 = 5/2, c = 5/2.

With f(u) = 5/2*u, the inverse function is f-1(u) = 2/5*u.

So with 7kg, f(7) = 5/2*7 = 17.5$

And with 12$, f-1(12) = 2/5*12 = 4.8kg.

A5) Recounting in two units we get a 'per-number' 2kg/5$ to bridge the units.

Q1: T = 7kg = (7/2)*2kg = (7/2)*5$ = 17.5$;

Q2: T = 12$ = (12/5)*5$ = (12/5)*2kg = 4.8kg.

13. NEW TEXTBOOKS, BUT FOR WHICH OF THE 3X2 KINDS OF MAT EDUCATION

Poster

A curriculum must choose between a pre-, a present, and a post-setcentric mathematics, differing in answers to the question: “What is the difference between 5/3 and 5÷3”.

Pre-setcentric math sees 5/3 as a number on the number-line reached by taking 5 steps of the length coming from dividing one in 3; and sees 5÷3 as 5 shared between 3.

Present setcentric mathematics sees 5/3 as an equivalence class a/b created by cross-multiplication, a x d = b x c. With 1/3 as the inverse element to 3 under multiplication, 5÷3 should be written as 5 x 1/3, i.e. the solution to the equation 3 x u = 5, found by applying and thus legitimizing abstract algebra and group theory.

Post-setcentric mathematics sees 5/3 as a per-number coming from recounting the same total in different units (5£/3kg), becoming a fraction with like units (5£/3£ = 5/3), used to bridge the two units: T = 20£ = (20/5)x5£ = (20/5)x3kg = 12 kg. And sees 5÷3 as 5 counted in 3s occurring in the ‘recount-formula’ recounting a total T in bundles of 3s as T = (T/3)x3, saying ‘from T, T/3 times, 3s can be taken away’. This gives flexible numbers: T = 5 = 1B2 3s = 1.2 3s = 1 2/3 3s = 2B-1 3s = 2.-1 3s

Unitary states typically have one multi-year curriculum for primary and lower secondary education, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even self-chosen half-year curricula from secondary school as in North America.

14. DEVELOPING THE CHILD'S OWN MASTERY OF MANY

Sociological imagination sees continuing educational problems as possibly caused by a goal displacement making mathematics see itself as the goal instead of its outside root, mastery of Many. Typically, the number-language is taught inside-inside as examples of its meta-language. However, as the word-language, it can also be taught inside-outside, thus bridging it to the outside world it describes. So, textbooks should not reject, but further guide the mastery of Many that children bring to school.

IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Yet, despite increased research and funding, decreasing Swedish PISA results made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life' (p. 3). Research thus still leaves many issues unsolved after half a century. Inspired by Sartre (2007, p. 20) saying that in existentialism 'existence precedes essence' and by Bauman's (1990, p. 84) sociological imagination, we can ask if mathematics education has a 'goal displacement' seeing its present essence as a goal instead of as an inside means to its outside existing root and goal, mastery of Many?

Mathematics education is based upon textbooks that again are based upon a curriculum for primary and lower secondary school supplemented with side-curricula for upper secondary school. But why can't all students have the same curriculum? After all, the word-language does not need different curricula for different groups of students, so why does the number-language?

Both languages have two levels, a language level describing the outside world, and a grammar level describing the inside language. In the word-language, the language level is for all students and includes examples of real-world descriptions, both fact and fiction, whereas grammar level details are reserved for special students. Could it be the same with the number-language, learned by all students through describing fact and fiction? And where grammar level details are reserved to special students?

Also, in contrast to the many letters, words and sentence rules in word-language, a pocket calculator shows that the number-language contains only ten digits and a few operations. And where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5 icon etc. And so are the operations describing counting unbundled, bundles, bundles of bundles where division iconizes wiping away bundles to be stacked, iconized by a multiplication lift, again to be drawn away, iconized by a subtraction rope, to identify unbundled singles that may be placed next-to the stack iconized by an addition cross.

Could it be that the numbering competence children bring to school contain core mathematics as proportionality and calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction? This would allow designing a curriculum for all students without splitting it up into tracks. And allow the word-language and the number-language to be taught and learned in the same way by describing outside things and actions with inside words and numbers coming from counting and adding.

However, instead of teaching children how to number, the tradition teaches children about numbers, and about operations, both to be learned before being applied to the outside world (Bussi and Sun, 2018). Thus, where word-language is taught in the space between the inside language and the outside world, the number-language is taught in the inside space between the language and its meta-language or grammar, which makes the number-language more abstract and difficult to learn and to apply.

So maybe research should go back to the mother Humboldt university in Berlin and reflect on the Karl Marx thesis 11 written on the staircase: “The philosophers have only interpreted the world, in various ways. The point, however, is to change it.”

MEETING MANY, CHILDREN BUNDLE TO COUNT AND SHARE

How to master Many can be observed in a power-free dialogue (Habermas, 1981) with preschool children. Asked “How old next time?”, a 3year old will say “Four” and show 4 fingers; but will react strongly if held together 2 by 2: ‘That is not four, that is two twos’, thus insisting that the outside existing bundles should inside be predicated by a ‘bundle-number’ including the unit. Children also use bundle-numbers when talking about Lego blocks as ‘2 3s’ or ‘3 4s’. When asked “How many 3s when united?” they typically say ‘5 3s and 3’; and when asked “How many 4s?” they may say ‘5 4s less 2’; and, integrating them next-to each other, they typically say ‘2 7s and 4’.

Children have fun ‘bundle-counting’ their fingers in 3s in various ways: as 1Bundle7 3s, ‘bundle-written’ as T = 1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Sharing 9 cakes, 4 children take one by turn, and they smile when seeing that ‘9/4’ predicts that they can take a cake twice, thus seeing division by 4 as taking away 4s.

Children thus master numbering and sharing before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the core mastery of Many that children bring to school?

Numeracy as ‘the ability to understand and work with numbers’ (Oxford Dictionary) thus has an outside interpretation by the child’s own mastery of Many that contrasts the inside interpretations seeing numeracy as applying institutionalized mathematics.

TEXTBOOKS FOR A QUESTION GUIDED COUNTING CURRICULUM

Typically, a mediating curriculum sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a developing curriculum sees mathematics as an exoteric means to develop the children’s existing ability to master Many by numbering outside totals and blocks with inside two-dimensional bundle-numbers. This calls for different textbooks from grade 1 that don’t mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Tarp (2018) holding that only two competences are needed to master Many, counting and adding. The corresponding pre-service and in-service teacher education may be found at the MATHeCADEMY.net.

Q01, icon-making: “The digit 5 seems to be an icon with five sticks. Does this apply to all digits?” Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons; and by comparing with Roman numbers.

Q02, counting sequences: “How to count fingers?” Here the learning opportunity is that five fingers can also be counted “01, 02, 03, 04, Hand” to include the bundle; and ten fingers as “01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H”.

Q03, icon-counting: “How to count fingers by bundling?” Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate: T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s, called an ‘inside bundle-number’ describing the ‘outside block-number’. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, calculator-prediction: “How can a calculator predict a counting result?” Here the learning opportunity is to see the division sign as an icon for a broom pushing away bundles: 7/2 means ‘from 7, push away bundles of 2s’. The calculator says ‘3.some’, thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes a rope pulling away the block to look for unbundled singles. By showing ‘7-3*2 = 1’ the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional learning opportunity is to write 8 = (8/2)*2 as a ‘recount-formula’ T = (T/B)*B, saying “From T, T/B times B can be taken away”, to predict counting and recounting examples.

Q05, recounting in another unit: “How to change a unit?” Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer ‘2.some’ shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer ‘2’ shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives fractions: T = 3 = (3/5)*5. Another learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Q06, unbundled as decimals, fractions or negative numbers: “Where to put the unbundled singles?” Here the learning opportunity is to see that the unbundled occur in three ways: Next-to the block as a block of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q07, prime or foldable units: “Which blocks can be folded?” Here the learning opportunity is to examine the symmetry of a block. The block T = 2 4s = 2*4 has 4 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a ‘prime unit’ and 4 a ‘foldable unit’, 4 = 2 2s. A number is called even or symmetrical if it can be folded in 2s, else the number is called odd.

Q08, finding units: “What are possible units in T = 12?”. Here the learning opportunity is that units come from factoring in prime units, 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

Q09, recounting from tens to icons: “How to change unit from tens to icons?” Here the learning opportunity is that asking ‘T = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

Q10, recounting from icons to tens: “How to change unit from icons to tens?” Here the learning opportunity is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking ‘T = 3 7s = ? tens’. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently ‘less less 1’ means adding 1.

Q11, recounting block-sides. “How to recount sides in a block halved by its diagonal?” Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

Q12, navigating. “Avoid the rocks on a squared paper”. Rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock?

TEXTBOOK FOR A QUESTION GUIDED ADDING CURRICULUM

Counting ten fingers in 3s gives T = 1BundleBundle1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the algebra square’.

Q13, next-to addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?” Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

Q14, reversed next-to addition: “If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

Q15, on-top addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?” Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

Q16, reversed on-top addition: “If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?” Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

Q17, adding tens: “With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?” Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Q18, subtracting tens: “If T1 = 23 and T2 add to T = 71, what is T2?” Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Q19, multiplying tens: “What is 7 43s recounted in tens?” Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Q20, dividing tens: “What is 348 recounted in 6s?” Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Q21, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining local constancy by epsilon and delta.

Q22, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Q23, finding common units: “Only add with like units, so how add T = 4ab^2 + 6abc?”. Here units come from factorizing: T = 2*2*a*b*b + 2*3*a*b*c = (2b+3c) *2ab.

DISCUSSION AND FUTURE RESEARCH

So yes, a curriculum for all students is possible without splitting it up into tracks. For the mastery of Many that children bring to school contains core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social ‘colonization’ of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child’s existing mastery of Many inspired by, and not repressed by, the present of many historically versions of mathematics.

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16. ADDITION-FREE MATH MAKE MIGRANTS AND REFUGEES STEM EDUCATORS

TOPIC 03, research-based poster

Abstract

This poster presents a design for migrants and refugees as STEM educators, relating to the topic 3 question 1 about projects aiming at a widespread implementation of innovative teaching.

As an integrated subject in a STEM package, mathematics should respect its historic Greek roots as a common name for four studies of Many in time and space, arithmetic and geometry and music and astronomy. And it should respect the mastery of Many, children bring to school which includes using full number-language sentences with a subject and a verb and a predicate as in the word-language, and using numbers with units as in T = 3*4 = 3 4s, coming from taking away bundles by division to be stacked by multiplication. With units as multipliers, the recount-formula T = (T/B)*B, saying ‘From T, T/B times, B can be taken away’, has the same proportionality form as most STEM formulas. So, a different Many-based mathematics should be designed to practice recounting before adding. This ManyMath presents a shortcut to the core of mathematics, proportionality and equations and calculus, that allow migrants and refugees to return to help develop or rebuild their countries as STEM educators.

The future need for education as described by the UN and OECD

Among the 17 ‘UN Sustainable Development Goals’, goal 4, quality education, states that ‘obtaining a quality education is the foundation to improving people’s lives and sustainable development. (..) The reasons for lack of quality education are due to lack of adequately trained teachers (..) For quality education to be provided to the children of impoverished families, investment is needed in educational scholarships, teacher training workshops (..) More than half of children that have not enrolled in school live in sub-Saharan Africa. An estimated 50 per cent of out-of-school children of primary school age live in conflict-affected areas. 617 million youth worldwide lack basic mathematics and literacy skills.’

The UN states that by 2030 the goal 4 targets, will ‘substantially increase the number of youth and adults who have relevant skills, including technical and vocational skills, for employment, decent jobs and entrepreneurship (..) ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy (..) substantially increase the supply of qualified teachers, including through international cooperation for teacher training in developing countries’.

The ‘OECD 2030 Learning Framework’, states that ‘The members of the OECD Education 2030 Working Group are committed to helping every learner develop as a whole person, fulfil his or her potential and help shape a shared future built on the well-being of individuals, communities and the planet. (..) Future-ready students need to exercise agency, in their own education and throughout life. (..) To help enable agency, educators must not only recognise learners’ individuality, (..) Two factors, in particular, help learners enable agency. The first is a personalised learning environment that supports and motivates each student to nurture his or her passions, make connections between different learning experiences and opportunities, and design their own learning projects and processes in collaboration with others. The second is building a solid foundation: literacy and numeracy remain crucial.’

An answer: addition-free STEM-based recounting math for migrants and refugees

A01. Counting fingers shows the existence of a number-language with sentences containing a subject and a verb and a predicate as in the word-language: the total is two fives, or T = 2 5s = 2*5. Here the outside total T exists unchanged while its inside predication changes with how it is bundled. Five fingers thus may recount in 2s as T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s, using an overload or a normal or an underload form, rooting negative numbers. The unbundled singles can be placed next-to the stack as decimals, or on-top counted as a fraction: T = 5 = 2.1 2s = 2 1/2 2s. Counting ten fingers in 3s show that also bundles can be bundled: T = ten = 1B7 = 2B4 = 3B1 =

4B-2 = 1BB1 = 1BB0B1 3s, or T = 1*B^2 + 0*B + 1 showing directly the four ways to unite numbers, on-top addition and multiplication and power and next-to block-addition called calculus, corresponding to the Arabic meaning of the word algebra, to reunite. Other units are tens, dozens, scores, meters, seconds, dollars, etc. Bundling bundles create names as hundred for ten tens, gros for a dozen dozens, a minute for 60 seconds, wan for BBBB tens, etc.

A02. With bundles we only need ten digits, each an inside icon bridging to the outside quantity it represents if written less sloppy, and with zero as a looking glass finding nothing.

A03. Likewise, operators are inside icons reflecting outside actions: a division broom wipes away bundles, to be stacked by a multiplication lift, to be removed by a subtraction rope, to identify unbundles singles, to be placed next-to the stack as decimals or on-top as fractions.

A04. Recounting in another unit, we ask 'T = 3 4s = ? 5s'. The recount-formula allows a calculator predict the answer. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s recounts in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives a fraction: T = 3 = (3/5)*5. Recounting in physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg, bridging the units by recounting in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and vice versa.

A05. Recounting from tens to icons, we ask 'T = 2.4 tens = 24 = ? 8s'. This is an equation, u*8 = 24, that is easily solved by recounting 24 in 8s as 24 = (24/8)*8. Thus, the unknown number u = 24/8 is found by moving 8 to the opposite side with the opposite calculation sign.

A06. Recounting from icons to tens, we ask 'T = 3 7s = ? tens'. With no ten-button, a calculator cannot use the recount-formula to predict the answer. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. The multiplication tables may use 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten, less 3) = 3 * ten, less 3*3 = 3ten, less 9 = 3ten, less (ten less1) = 2ten, less less 1 = 2ten & 1 = 21. And, 7*9 = (ten, less 3)*(ten, less 1) = tenten, less 3ten, less 1ten, lessless3 = 6ten & 3 = 63.

A07. Recounting an axb block halved by its diagonal c, will create per-numbers:

a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b; and $\pi \approx$ n*sin(180/n).

A08. Trying to stay inside a squared paper when navigating from the middle, and with two dices telling the horizontal and vertical change where odd numbers are negative, roots statistics.

A09. STEM contains multiplication formulas with per-numbers: meter = (meter/sec)*sec = velocity*sec, kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter)*square-meter = pressure*square-meter; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg)*kg = heat * kg. Lego-bricks: number = (number/meter)*meter = density*meter.

A10. Adding 2 3s and 4 5s on-top, the units must be harmonized by recounting. Adding next-to means adding areas, called integral calculus, as when adding per-numbers and fractions that must change to unit-numbers by multiplication, thus creating areas to be added.

A11. Reversing addition, asking '2 3s and ? 5s total 4 5s or 2 8s' will become equations, 2*3 + u*5 = 4*5 and 2*3 + u*5 = 2*8, solved by moving to opposite side with opposite sign.

References

Tarp, A, 2018, "Mastering Many", *Journal of Mathematics Education, vol 11*(1), pp. 103-117.

17. RECOUNTING BEFORE ADDING MAKES TEACHERS COURSE LEADERS AND FACILITATORS

TOPIC 01, Research-based poster

Abstract

This Poster presents innovative perspectives in educating math educators from the child's perspective, relating to the topic 1 question 2 about successful programs and essential contents, and to question 6 how primary teachers can become learning community facilitators.

Recounting in bundle-numbers allows teachers to secure that no students are left behind.

The three learning levels at the MATHeCADEMY.net allows a teacher to become both a facilitator for a professional learning community and a course leader initiating pyramid-organized professional development locally or globally on the internet. The open source inquiry-based material is organized as individual inspiration, group reflection and school development, thus creating self-sustaining learning communities that ensures sustainability. The learning levels are research based; and by seeing mathematics as a natural science about the physical fact Many, they develop the quantitative competence children bring to school, thus including all students despite diversity as to gender or ethnicity or social or cultural background.

Peter, stuck in division, until learning about recounting in flexible bundle-numbers

Being a mathematics teacher in an ordinary class and in an adult class, both showing severe dislike towards division and fractions, Peter is about to give up teaching when he hears about a one-day workshop on curing math dislike by recounting totals in flexible bundle-numbers.

Here 5 sticks are recounted in 2s in three different ways, overload and standard and underload, occurring as outside blocks, and inside bundle-formulas: T = 5 = I I I I I = II I I I = 1*B*3 2s = II II I = 2*B*1 2s = II II Iǂ = 3*B*-1 2s. Likewise, if using ten-bundling: T = 57 = 5*B*7 tens = 4*B*17 tens = 6*B*-3 tens; or T = 567 = 56*B*7 = 50*B*67 = 60*B*-33 = 5*BB*6*B*7 tens.

Operations are eased by recounting in over- or underloads:

When dividing 336/7, 336 is bundle-written as 33*B*6. This is recounted as 28*B*56 that divided by 7 gives 4*B*8 or 48; or as 35*B*-14 that divided by 7 gives 5*B*-2 or 4*B*8 or 48.

Likewise, with subtraction: T = 65 – 48 = 6*B*5 – 4*B*8 = 2*B*-3 = 1*B*7 = 17; or T = 65 – 48 = 6*B*5 – 4*B*8 = 5*B*15 – 4*B*8 = 1*B*7 = 17.

Likewise, with multiplication: T = 7 x 48 = 7 x 4*B*8 = 28*B*56 = 33*B*6 = 336.

Likewise, with addition: T = 17 + 48 = 1*B*7 + 4*B*8 = 5*B*15 = 6*B*5 = 65.

A chatroom recommends watching the video 'CupCount and ReCount before you Add' (https://goo.gl/eBRFTy), and to download a 'CupCount & ReCount Booklet' for self-testing. Realizing its innovative potentials, he gives a copy to his colleagues, and they ask the school to arrange a free 1day Skype seminar in curing math dislike by recounting in bundle-numbers.

In the morning they watch the PowerPoint presentation 'Curing Math Dislike' confronting the three forms of mathematics, a pre- and a present and a post-setcentric version.

Present setcentric mathematics is called 'MetaMatism' as a mixture of 'MatheMatism', true inside a classroom but rarely outside where '2+3 = 5' is contradicted by e.g. 2weeks+3days = 17days, and 'MetaMatics', presenting a concept top-down as an example of an abstraction instead of bottom-up as an abstraction from many examples: 'A function IS an example of a set-product', instead of 'a function is a name for a formula with some unspecified numbers.'

The post-setcentric version is called 'ManyMath' by seeing mathematics as a natural science about the physical fact Many, to be counted and recounted in bundle-units before being added (or split) next-to or on-top. Here digits are icons with as many sticks as they represent. Likewise, operations iconize bundle-counting: a division broom pushes away bundles, to be stacked by a multiplication lift, to be pulled away by a subtraction rope, to look for unbundled singles, to be placed in a separate stack as decimals, or on-top counted as a fraction of a bundle. A 'recount-formula', T = (T/B)xB, allowing a calculator predict that 'from T, T/B times, B can be taken away', occurs as proportionality all over mathematics and science.

The recounting seminar includes two Skype sessions with an external course leader.

Observing ManyMath curing math dislike, the school asks Peter to take in 1year e-learning course at the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMath. Peter here experiences PYRAMIDeDUCATION where 8 are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. An external coach assists the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays by coaching a new group of 8 teachers.

The four e-learning courses for primary and for secondary school are called CATS, inspired by the fact that, to deal with Many, we Count & Add in Time & Space.

Primary school mathematics is learned through educational sentence-free meetings with the sentence subject, thus developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic 'grasp-to-grasp' learning. Thus, learning means asking, not the instructor but the subject talked about. Using full number-language sentences with a subject and a verb and a predicate as in the word-language allows modelling from the beginning by recounting both bundles, distances, time periods, money etc.

Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic 'gossip-learning': Thank you for telling me something new about something I already knew.

The material is inquiry-based with guiding questions. In primary school, the four sets of questions are as follows. COUNT: How to count Many? How to recount 8 in 3s? How to recount 6kg in $ with 2$ per 4kg? How to count in standard bundles? ADD: How to add stacks concretely? How to add stacks abstractly? TIME: How can counting & adding be reversed? How many 3s plus 2 gives 14? Can all operations be reversed? SPACE: How to count plane and spatial properties of stacks and boxes and round objects?

In secondary school, the four sets of questions are as follows. COUNT: How to count possibilities? How to predict unpredictable numbers? ADD: What is a prime number? What is a per-number? How to add per-numbers? TIME: How to predict the terminal number when the change is constant? How to predict the terminal number when the change is variable, but predictable? SPACE: How to predict the position of points and lines? How to use the new calculation technology? QUANTITATIVE LITERATURE, what is that? Does it also have the 3 different genres: fact, fiction and fiddle?

The three MATHeCADEMY.net learning level thus allows Peter to become a facilitator for a local learning community and a course leader initiating pyramid-organized professional development locally or globally on the internet. After coaching a learning pyramid at the school to allow eight other teachers to be trained as facilitators and course leaders, the school may ask Peter to take the secondary school course also so the school can become as a local center for curing math dislike, thus allowing students to excel in both primary and secondary mathematics.

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18. SELF-EXPLANATORY LEARNING MATERIAL TO IMPROVE YOUR MASTERY OF MANY

TOPIC 02, Research-based poster or oral or workshop

Abstract

This poster illustrates a hands-on experience with educating math educators from the child's perspective, relating to the topic 2 question 4 about designing innovative self-explanatory material that has large potentials for scaling-up.

It is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but will protest when held together two by two by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. Meeting Many, children develop a number-language with full sentences including a subject and a verb and a predicate as in the word-language, as well as 2-dimensional block-numbers with units, neglected by the school's 1-dimensional line-names, making some children count-over by saying 'twenty-ten'. So, the goal of the workshop is to inquire into the mastery of Many children bring to school to see what kind of mathematics occur if allowing the children to develop their already existing quantitative competence under proper guidance.

Digits and operations as icons bridging inside signs and outside existence

Matches and a folding ruler show that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent: five sticks in the 5-icon, etc.

Operations also are inside icons reflecting outside actions: a division broom pushes away bundles, to be stacked by a multiplication lift, to be pulled by a subtraction rope to identify unbundles ones, to be placed next-to the stack as decimals, or on-top as fractions or negatives, predicted by a 'recount-formula', $T = (T/B)*B$, saying 'from T, T/B times, B is taken away'.

Bundle-counting fingers roots negative numbers and polynomials

To emphasize bundles, the fingers may be bundle-counted as: 0Bundle1, 0*B*2, 0*B*3, 0*B*4, ½*B*, 0*B*6, 0*B*7, and then 0*B*8, 0*B*9, 1*B*0; or 1*B* less2, 1*B*-1, 1*B*0, continuing with 'Viking-counting' one-left (eleven), two left (twelve), and finally BundleBundle as 100. Two-digit numbers are named by their two neighbours: T = 68 = 6*B*8 tens = 7*B*-2 tens = 6ten8 = 7ten-2.

Counting ten fingers in 3s introduces bundles of bundles: T = ten = 3*B*1 3s = 1*BB*1 3s, leading on to the general number-formula or polynomial T = ten = 1**B*^2 + 0**B* + 1*1 3s. Likewise counting in tens, T = 345 = 3**BB* + 4**B* + 5*1 = 3**B*^2 + 4**B* + 5*1, showing the four ways to unite numbers (the Arabic meaning of Algebra): on-top addition, multiplication, power and next-to block-addition called integration, all with reverse splitting operations: subtraction, division, factor-finding (root), factor-counting (logarithm), and differentiation.

Block-counting cubes roots decimals, fractions and negative numbers

Block-counting 8 cubes in 5s gives 1 5s and 3 unbundled 1s as predicted: T = 8 = (8/5)*5 = 1*5 & 3. Placing the 3 1s after the 1 5s roots decimal-writing, T = 1.3 5s = 2.-2 5s. Placing the unbundled instead on-top of the block of bundles roots fractions and decimal numbers, T = 8 = 1 3/5 5s = 2 - 2/5 5s = 2 5s less 2. Counting in tens, T = 68 = 6 8/10 tens = 6.8 tens = 7.-2 tens.

Recounting roots flexible numbers and proportionality and per-numbers

Recounting in the same unit creates flexible numbers: T = 68 = 6.8 tens = 7.-2 tens

Recounting in another unit by asking e.g. 'T = 3 4s = ? 5s', the recount-formula allows a calculator to predict the answer. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s recounts in 5s as 2*B*2 5s or 2.2 5s. Counting 3 in 5s gives a fraction: T = 3 = (3/5)*5. Recounting in physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg, bridging the units by recounting in the per-number: Asking '6$ = ?kg', we recount 6 in 2s: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and vice versa.

Recounting from tens and to tens roots equations and multiplication tables

Recounting from tens to icons by asking e.g. '*T* = 2.4 tens = 24 = ? 8s' becomes an equation, u*8 = 24, that is easily solved by recounting 24 in 8s as 24 = (24/8)*8 so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite calculation sign.

Recounting from icons to tens by asking e.g. '*T* = 3 7s = ? tens' we notice that with no ten-button on a calculator, the recount-formula cannot predict the answer. But, it is programmed to give the answer directly by using multiplication alone: *T* = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. The multiplication tables may use 'less-numbers', geometrically on an abacus, or algebraically with brackets: *T* = 3*7 = 3 * (ten, less 3) = 3 * ten, less 3*3 = 3ten, less 9 = 3ten, less (ten less1) = 2ten, less less 1 = 2ten & 1 = 21. And, 7*9 = (ten, less 3)*(ten, less 1) = ten ten, less 3ten, less 1ten, lessless3 = 6ten & 3 = 63.

Recounting is exemplified in STEM-formulas

STEM contains multiplication formulas with per-numbers: meter = (meter/sec)*sec = velocity*sec, kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter)*square-meter = pressure*square-meter; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg)*kg = heat * kg. Lego-bricks: number = (number/meter)*meter = density*meter.

Recounting sides in a block halved by its diagonal roots angles, trigonometry and pi

Recounting a block with base b and height a, halved by its diagonal c, creates per-numbers:

$a = (a/c)*c = \sin A*c$; $b = (b/c)*c = \cos A*c$; $a = (a/b)*b = \tan A*b$; and $\pi \approx n*\sin(180/n)$.

Adding totals on-top and its reverse roots proportionality and differential calculus

Adding 2 3s and 4 5s on-top, the units must be harmonized by recounting. Adding next-to means adding areas, called integral calculus; as when adding per-numbers and fraction that must change to unit-numbers by multiplication, thus creating areas to be added.

Reversing addition by asking e.g. '2 3s and ? 5s total 4 5s or 2 8s' will become equations, 2*3 + u*5 = 4*5 or 2*3 + u*5 = 2*8, solved by moving to opposite side with opposite sign.

Grand theory holds conflicting conceptions on concepts

Within philosophy, Platonism and Existentialism argue if concepts are examples of abstractions or abstractions from examples. Within psychology, Vygotsky and Piaget argue if concepts are constructions mediated socially or experienced individually. Within sociology, the agent-structure debate is about establishing inclusion by accepting the agent's own concepts or establishing exclusion by insisting on teaching and learning institutionalized concepts.

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19. CAN GROUNDED MATH AND EDUCATION AND RESEARCH BECOME RELEVANT TO LEARNERS

TOPIC 03, presentation

Abstract

This presentation relates to the topic 3 question 3 about challenges to be overcome even if innovative teaching has been designed and initiated. The main question about mathematics education and its research is: 'If 50 years of research fails to solve the problems of math education, then what can?' The presentation allows the audience to give comments to the five section questions that are inspired by the Chomsky-Foucault debate on Human Nature.

Humans communicate in languages, a word-language and a number-language. We learn to speak the word-language in the family, and we are taught to read and write in institutionalized education, also mediating the number-language under the name mathematics, thus emphasizing the three r's: Reading, Writing and Arithmetic. Despite intensive research, international tests show that the learning of the number-language is deteriorating in many countries.

This raises two questions: May a change in mathematics, education and research make more learners reach the goal of math education? Is the goal of mathematics education to echo an inside university truth regime labelled mathematics, or to master the outside fact Many?

Education in general

On our planet, life takes the form of single black cells, or green or grey cells combined as plants or animals. Humans only need a few children in their lifetime, since transforming the forelegs to hands and fingers allows humans to grasp the food, and to share information through communication and education by developing a language when associating sounds to what they grasp. Where food must be split in portions, information can be shared through education.

Education takes place in the family and in the workplace; and in institutions with primary, secondary and tertiary education for children, for teenagers and for adults. English language does not have continental Europe's words for education using Plato's cave to picture learners as unformed and living below: Bildung, Unterricht, Erziehung, Didactics, etc. Likewise, Europe still holds on to the multi-year line-organized office preparing education that was created by the German autocracy shortly after 1800 to mobilize the population against the French democracy, whereas the North American republics use self-chosen half-year block-organized talent developing education from secondary school. So, how well-defined is 'education'?

Mathematics and its education

The Pythagoreans used the word 'mathematics' as a common label for their knowledge about Many by itself and in space and time, arithmetic and geometry and music and astronomy. Without the two latter, mathematics later became a common label for arithmetic, algebra and geometry, which may be called pre-setcentric math, challenged by the present setcentric 'New Math' appearing in the 1960s, again challenged by a post-setcentric math seeing math as a natural science about its outside roots, Many, since setcentric mathematics never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we master the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon. Continental Europe taught reckoning, called 'Rechnung' in German, until the arrival of the New Math. When opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only offers classes in mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. So, how well-defined is mathematics and its education?

The teacher and the learner

It seems natural to say that the job of a teacher is to teach learners so that learning takes place, checked by written tests. However, continental Europe calls a teacher a ‘Lehrer’ thus using the same word as for learning. In addition, a Lehrer is supposed to facilitate Bildung, Unterricht and Erziehung and to foster competences. In teacher education, the subject didactics, meant to determine the content of Bildung, is unknown outside the continent. In the American high school, teachers have their own classroom to teach one subject; outside teachers must teach several subjects to students forced to stay in the same class for several years.

As to learners, the tradition sees learning taking place when learners follow external instructions from the teacher in class and from the textbook at home. Then constructivism came along suggesting that instead learning mostly takes place through internal construction when working with peers or with manipulatives. So how well-defined is a ‘teacher’ and a ‘learner’?

Research and conflicting theories

Typically, research is seen as a search for laws predicating essence to an existent subject. But, is the subject the root or an example of its predication? Holding that existence precedes essence, Existentialism has no doubt, but what about other philosophical observations?

Using the word sophy for knowledge, the ancient Greek sophists warned against choice masked as nature whereas the philosophers saw choice as an illusion since the physical is but examples of metaphysical forms only visible to them when educated at the Plato academy as scholastic ‘late opponents’ defending their comments to an already defended comment against three opponents. Newton’s natural science installed validation by unfalsified predictions instead, which inspired the 18th century Enlightenment period, which again created counter-enlightenment, so today research still uses Plato scholasticism outside the natural sciences.

Using classrooms to gather data, math education research could be a grounded natural science, but seems to prefer scholastics by researching, not math education itself, but theories on math education instead. But this raises questions about what to do with conflicting theories:

Within philosophy the Greek controversy between sophists and philosophers is revived today between structuralism on one side and French post-structuralism and American pragmatism on the other side. Within Psychology, Vygotsky sees education as building ladders from the present theory regime to the learner’s learning zone, where Piaget replaces this top-down view with a bottom-up view inspired by American Grounded Theory allowing inside categories to grow from concrete outside experiences and observations. And Sociology fiercely discusses who constructs who in the relation between individual agency and social structure.

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21. RECOUNTING IN ICON-UNITS AND IN TENS BEFORE ADDING TOTALS NEXT-TO AND ON-TOP

Material presentation

At the conference the plan is to exhibit the MATHeCADEMY.net's three levels for developing mastery of Many individually, school- and nation-wise, and globally. The focus is on a teacher being emotionally touched by the student's learning problems and wanting to cure math dislike. The background is the math dislike often occurs when teaching division.

The aim is to, by seeing how the CATS approach cures math dislike, a teacher is mobilized to be a facilitator in a one-day Skype seminar as well as a course leader by taking a one-year online training in the CATS approach to math education: to master Many, we Count and Add in Time and Space on a primary and on a secondary level. Accepting and developing the mastery of Many children bring to school, the material emphasizes bundle- and block-counting and recounting totals in icon-units and in tens before adding them next-to and on-top.

The school subjects concerned are mathematics and STEM subjects. The language available is English. The material was designed by Allan Tarp as part of a phd project and was applied and developed when hired as a web-based distance education educator at pre-service education in Denmark, and draws upon experiences from a period as a visiting professor at an in-service and pre-service teacher training academy in South Africa.

The material as well as the CATS teacher training program for primary and for secondary teachers is available at the MATHeCADEMY.net website which may be franchised freely by any university, together with MrAlTarp YouTube videos.

Bedsides allowing educators to be educated as facilitators and course leaders, the material also allows migrants and refugees to be educated as STEM educators able to return to help develop or rebuild their countries.

References

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Wrong Numbers

~~LineNumbers~~
~~with place values~~ ☹

IconNumbers
BundleNumbers
PerNumbers ☺

Respect & Develop
Kids' own Flexible BundleNumbers with Units

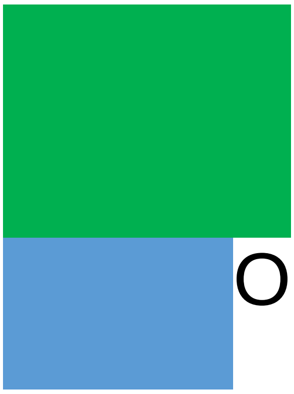
~~T is 48~~ No:
T is **4*B*8 = 3*B*18 = 5*B*-2**

Wrong Operations

~~8/2 is 8 split by 2~~ NO:
8/2 is 8 counted in **2s**

~~5x8 is 40~~ NO:
5x8 is 5 **8s**

2 **3s** + 4 **5s** = ???

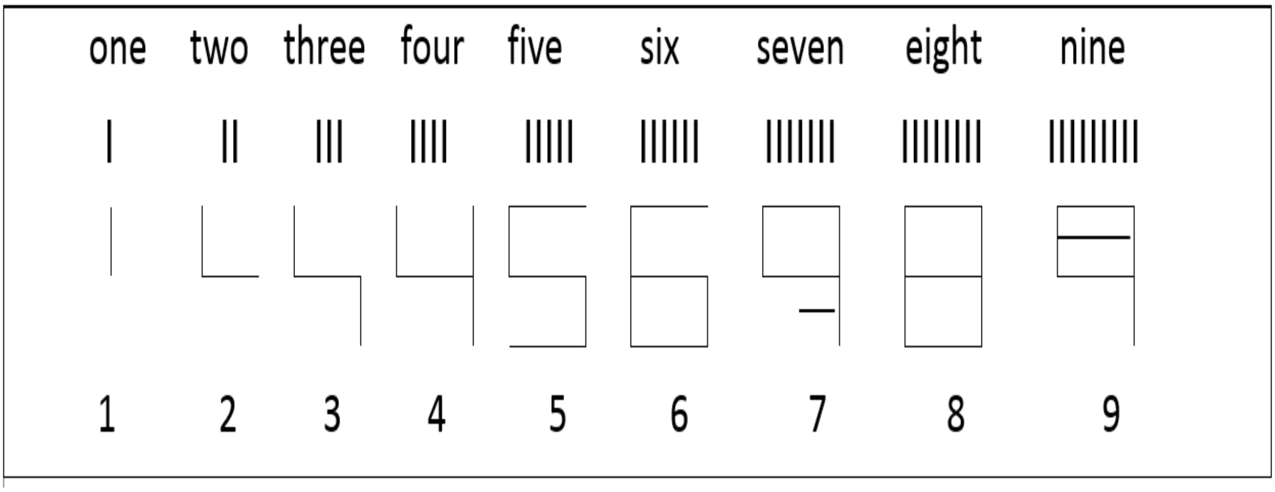


OnTop or NextTo

Wrong Math = **Dislike**

Numbers are Icons

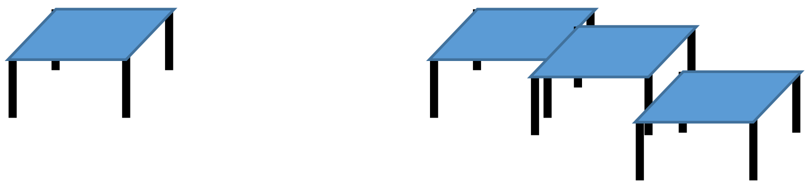
5 sticks in the 5-icon etc.



Our two Language Houses have two Floors

The WORD-language assigns words in sentences with a subject, a verb & a predicate.
The NUMBER-language assigns numbers instead.
Both languages have a META-language, a grammar, describing the language, that is learned before the grammar in the word-language, but not in the number-language.

	WORD language	NUMBER language
META-language grammar	***'is' is a verb***	***'x' is an operation***
Language	**This is a table**	**T = 3x4**



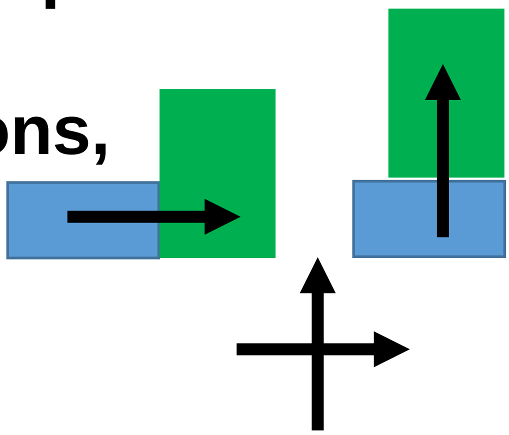
Operations are Icons

From 9 PUSH away 2s we write <u>9/2</u> iconized by a broom, called *division*.

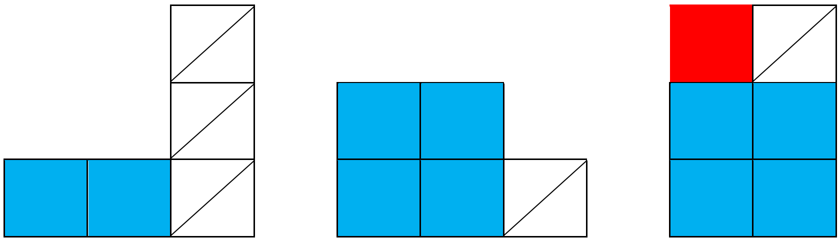
4 times LIFTING 2s to a stack we write <u>4x2</u> iconized by a lift called *multiplication*.

From 9 PULL away 4 2s' to find un-bundled we write <u>9 – 4x2</u> iconized by a rope, called *subtraction*.

UNITING next-to or on-top we write A+C <u>iconized</u> by two directions, called *addition.*



Flexible Bundle-Numbers



Overload *Standard* *Underload*

I I I I I = ǂ I I I = ǂ ǂ I = ǂ ǂ ǂ

5 = 1*B*3 = 2*B*1 = 3*B*-1 **2s**

5 = 1.3 = 2.1 = 3.-1 **2s**

= 2 ½ **2s**

48 = 4*B*8 = 3*B*18 = 5*B*-2

T = 65 + 27 = ? = 6***B***5 + 2***B***7 = 8***B***12 = 9***B***2 = 92

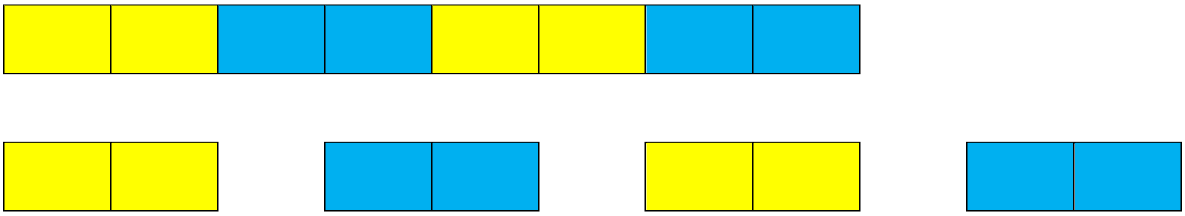
T = 65 – 27 = ? = 6***B***5 – 2***B***7 = 4***B***-2 = 3***B***8 = 38

T = 7* 48 = ? = 7* 4***B***8 = 28***B***56 = 33***B***6 = 336

T = 336 /7 = ? = 33***B***6 /7 = 28***B***56 /7 = 4***B***8 = 48

The RecountFormula

Recounting a total T in B-bundles



8 = (8/2)*2   = 4*2

T = (T/B)*B

From T, T/B times, push B away

Solves equations:

u*2 = 8   *= (8/2)*2*

u   = 8/2 (opposite side & sign)

$u + 2 = 8$	$u*2 = 8$	u^$8 = 2$	2^$u = 8$
$u = 8 - 2$	$u = 8/2$	$u = \sqrt[8]{2}$	$u = \log_2(8)$

Root: factor-finder & log: factor-counter

Used in STEM-formulas

m = (m/sec)*sec = speed*sec

$ = ($/hour)*hour = rate*hour

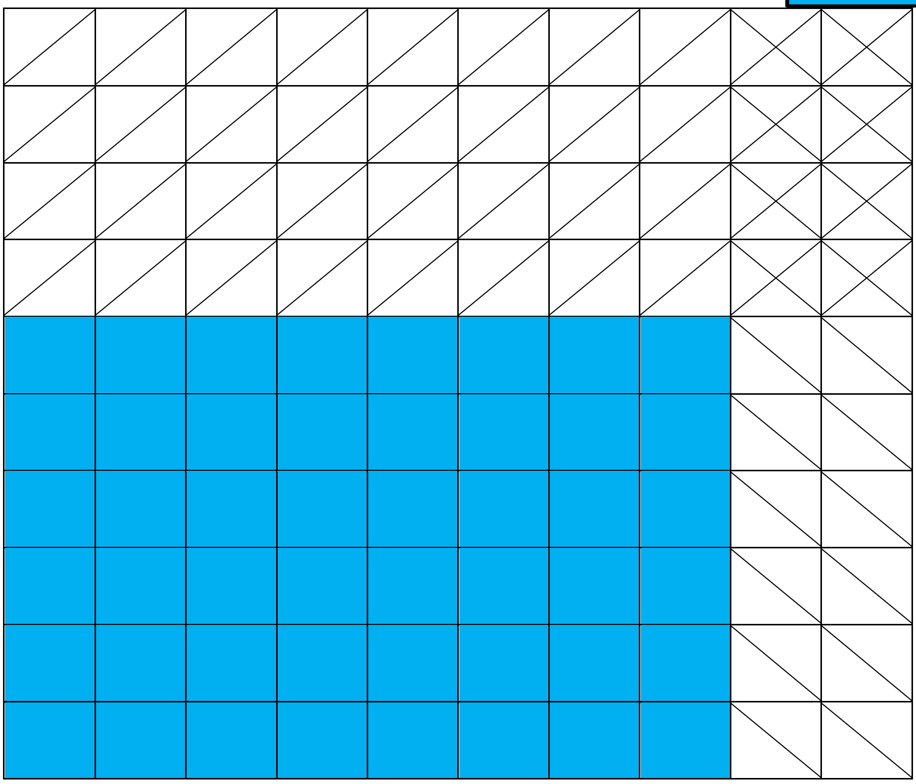
Joule = Joule/sec)*sec = Watt*sec

Ten-numbers

Tables: recount to tens

6 **8s** = ? **tens**

longer base - shorter height:



T = 6 **8s** = 6*8

= (***B***-4)*(***B***-2)

= ***BB*** - 4***B*** - 2***B*** - - 8

= 10***B*** - 6***B*** + 8

= 4***B***8 = 4.8 **tens** = 48

Per-numbers

DoubleCounting in kg & $ gives a **Per-number 2$/3kg**

<u>8$ = ?kg</u>

8$ = (8/2) x 2$
= (8/2) x 3kg = 12kg

<u>9kg = ?$</u>

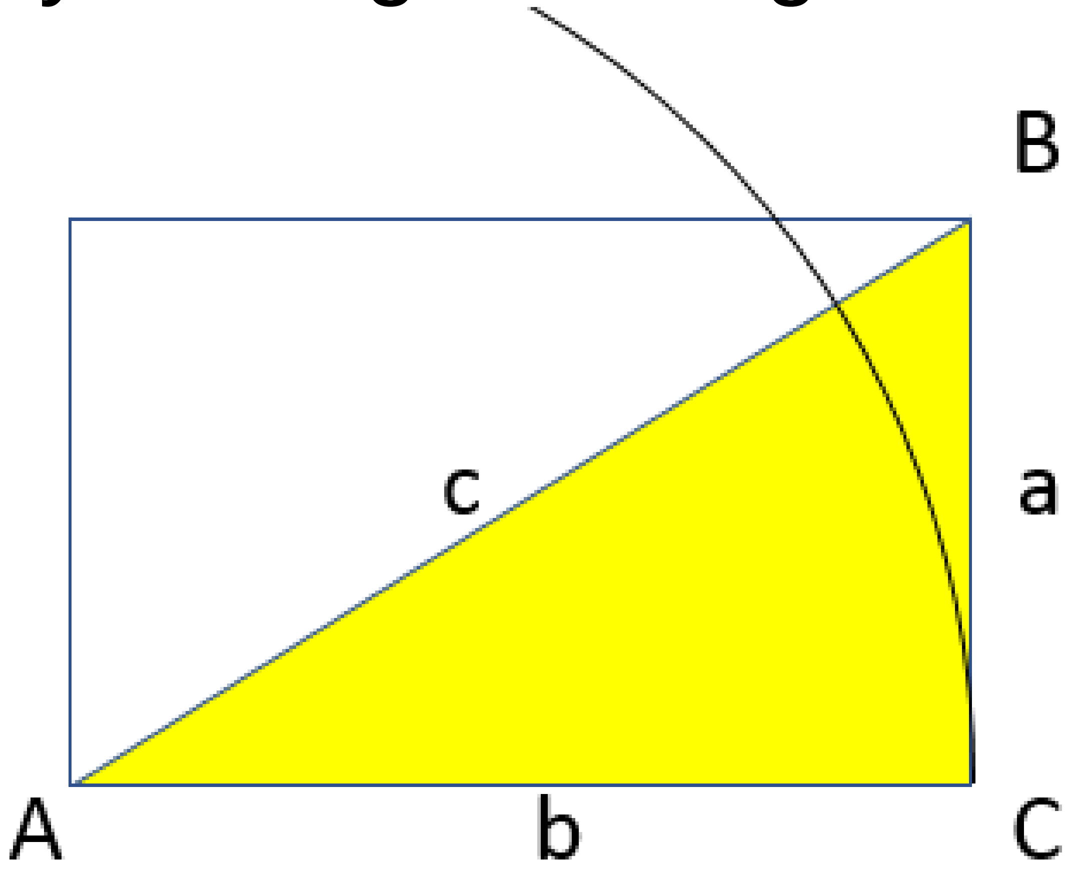
9kg = (9/3) x 3kg
= (9/3) x 2$ = 6$

With like units, per-numbers are fractions: **2$/3$ = 2/3**

STEM-formulas contain per-numbers coming from double-counting:
m = (m/sec) * sec = speed * sec
kg = (kg/m^3) * m^3 = density * m^3

Side-numbers

Recount sides in a box halved by its diagonal: Trigonometry



T = (T/B)*B

a = (a/c)*c = **sinA***c

a = (a/b)*b = **tanA***b

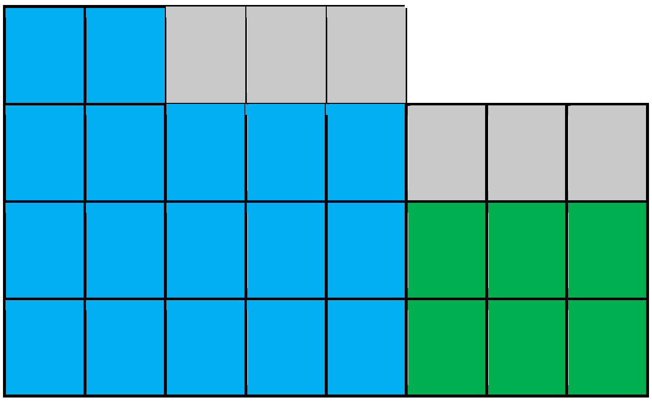
π = n*sin(180/n) for n large

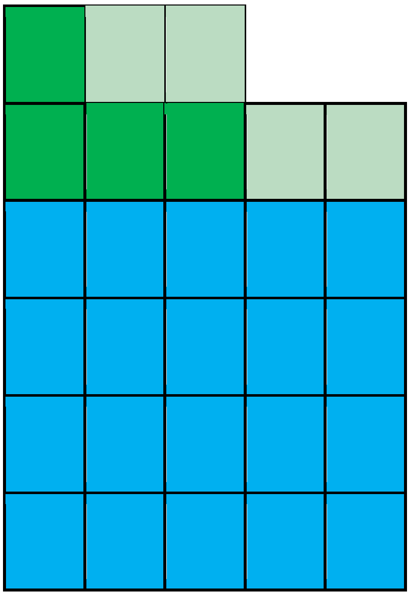
c*c = a*a + b*b

Addition is not Well Defined

Counted & Recounted, Totals may Add

BUT: NextTo	or OnTop
4 **5s** + 2 **3s** = 3**B**2 **8s**	4 **5s** + 2 **3s** = 5**B**1 **5s**
The areas are integrated *Adding areas = Integration*	Units changed to the same *Change unit = Proportionality*

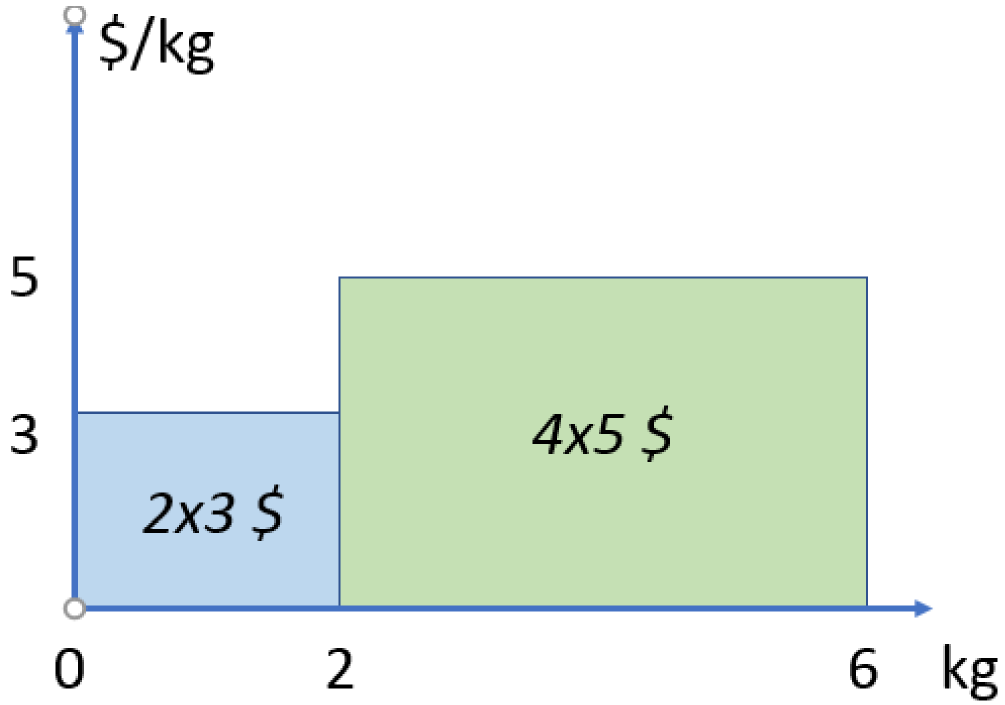




Adding fractions and per-numbers: Calculus

2 kg	at	3 $/kg
+ 4 kg	at	5 $/kg
(2+4) kg	at	? $/kg

Unit-numbers add on-top.
Per-numbers add next-to as areas under the per-number graph:



4 Ways to Unite & Split

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2+4***B**+5 (a polynomial) shows the four ways to add:

+, *, ^, next-to block-addition (integration)

<u>Add</u> & <u>multiply</u> add changing and constant unit-numbers.
<u>Integrate</u> & <u>power</u> add changing and constant per-numbers.

The 4 uniting operations have a reverse splitting operation:

Add has <u>subtract</u> (–), and multiply has <u>divide</u> (/).

Power has factor-find (<u>root</u>, √) and factor-count (<u>logarithm</u>, log).

Integrate has per-number find (<u>differentiate</u> dT/dn = T‘).

Reversing operations solve equations by moving to **opposite side** with **opposite sign**.

Operations unite/ *split into*	**changing**	**constant**
Unit-numbers *m, s, $, kg*	**T = a + n** *T – a = n*	**T = a * n** *T/n = a*
Per-numbers *m/s, $/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $\log_a T = n$, $\sqrt[n]{T} = a$

We call this beautiful simplicity the ‘**Algebra Square**’ since in Arabic, algebra means to reunite.

Solving Equations

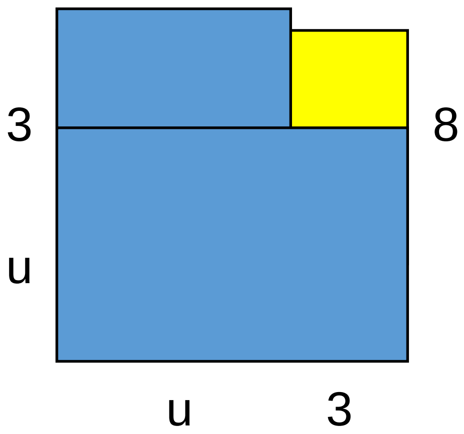
ManyMath: Recount

2 x u = 6 = (6/2) x 2	Solved by recounting 6
u = 6/2 **= 3**	Test: 2 x 3 = 6 OK

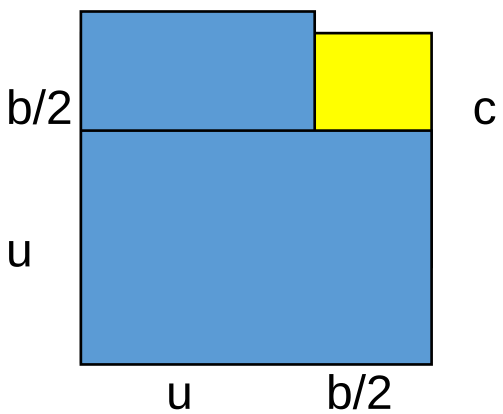
MatheMatics: Neutralize with Abstract Algebra

	2 x u = 6	Multiply has 1 as neutral element, and 2 has ½ as inverse element
↕	(2 x u)x½ = 6x½	Multiply 2's inverse element to both number-names
↕	(u x 2)x½ = 3	Apply the commutative law to ux2, 3 is the short number-name for 6x½
↕	u x (2x½) = 3	Apply the associative law
↕	u x 1 = 3	Apply the definition of an inverse element
↕	**u = 3**	Apply definition of a neutral element *With arrows, a test is not needed*

Quadratic Equations
with 3 Cards

<table>
<tr>
<td>Solve the quadratic equation
</td>
<td>u^2 + 6u + 8 = 0
(u+3)^2 = u^2 + 6u + 8 + 1
(u+3)^2 = 0 + 1
u+3 = ± 1
u = -3 ± 1
Solution: u = -4, u = -2</td>
</tr>
</table>

With unspecified numbers:

<table>
<tr>
<td>Solve the quadratic equation
</td>
<td>u^2 + b*u + c = 0
(u+b/2)^2 = u^2 + b*u + c + (b/2)^2-c
(u+b/2)^2 = 0 + D/4
u+b/2 = ± √D/2 , D = b^2-4c
u = -b/2 ± √D/2
Solution: u = (-b ± √D)/2</td>
</tr>
</table>

MATHeCADEMY.net

- Teaches Teachers to Teach MatheMatics as **Many**Math, a natural science about **Many**
- Cures **Math Dislike** when counting fingers in flexible bundle-numbers
- YouTube videos
- Free 1day Skype Seminars

IconNumbers • ReCounting 7 in **5s** & **3s** & **2s**

22. WHAT IS MATH - AND WHY LEARN IT?

”What is math - and why learn it?” Two questions you want me to answer, my dear nephew.

0. What does the word mathematics mean?

In Greek, ‘mathematics’ means ’knowledge’. The Pythagoreans used it as a common label for their four knowledge areas: Stars, music, forms and numbers. Later stars and music left, so today it only includes the study of forms, in Greek called geometry meaning earth-measuring; and the study of numbers, in Arabic called algebra, meaning to reunite. With a coordinate-system coordinating the two, algebra is now the important part giving us a number-language, which together with our word-language allows us to assign numbers and words to things and actions by using sentences with a subject, a verb and a predicate or object:

“The table is green” and “The total is 3 4s” or “T = 3*4”. Our number-language thus describes Many by numbers and operations.

1. Numbers and operations are icons picturing how we transform Many into symbols

The first ten degrees of Many we unite: five sticks into one 5-icon, etc. The icons become units when counting Many by uniting unbundles singles, bundles, bundles of bundles. Operations are icons also:

Counting 8 in **2s** can be predicted by <u>division</u>, iconized by a broom pushing away **2s**: 8/2 = 4, so 8 = 4 **2s**.

Stacking the **2s** into a block can be predicted by <u>multiplication</u>, iconized by a lift pushing up the **2s**: 8 = 4x2.

Looking for unbundled can be predicted by <u>subtraction</u>, iconized by a rope pulling away the 4 **2s**: 8 – 4x2.

Uniting bundles and singles is predicted by <u>addition</u>, iconized by a cross, +, placing blocks next-to or on-top.

Recounting a total T in B-bundles is predicted by a ‘recount-formula’: saying ‘From T, T/B times, B can be pushed away’. Recounting 9 in **2s**, the calculator predicts the result 9 = 4B1= 4.1 **2s** = 4 **2s** + 1	**T = (T/B)*B** 9/2 4.some 9 – 4x2 1

Now, let us write out the total 345 as we say it when bundling in ones, tens, and ten-tens, or hundreds, we get T = 3*B^2 + 4*B + 5*1.

This shows that uniting takes place with four operations: number-addition unite unlike numbers, multiplication unite like numbers, power unite like factors, and block-addition (integration) unite unlike areas. So, one number is really many numberings united by calculations.

Thus, mathematics may also be called calculation on specified and unspecified numbers and formulas.

2. Placeholders

A letter like x is a placeholder for an unspecified number. A letter like f is a placeholder for an unspecified calculation formula. Writing ‘y = f(x)’ means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Writing y = f(2) is meaningless, since 2 is not an unspecified number. The first letters of the alphabet are used for unspecified numbers that do not vary.

3. Calculation formula predict

The addition calculation T = 5+3 predicts the result without having to count on. So, instead of adding 5 and 3 by 3 times counting on from 5, we can predict the result by the calculation 5+3 = 8.

Likewise, with the other calculations:

- The multiplication calculation 5*3 predicts the result of 3 times adding 5 to itself.
- The power calculation 5^3 predicts the result of 3 times multiplying 5 with itself.

4. Reverse calculations may also be predicted

'5 + 3 = ?' is an example of a forward calculation. '5 + ? = 8' is an example of a reversed calculation, often written as 5 + x = 8, called an equation that asks: which is the number that added to 5 gives 8? An equation may be solved by guessing, or by inventing a reverse operation called subtraction, x = 8 – 5; so, by definition, 8-5 is the number x that added to 5 gives 8. The calculator says that 8-5 is 3. We now test to see if this is the solution by calculating separately the left and right side of the equation. The left side gives 5 + x = 5 + 3 = 8. The right side is already calculated as 8. When the left side is equal to the right side, the test shows that x = 3 is indeed a solution to the equation.

Likewise, with the other examples of reverse calculations:

- $\frac{8}{5}$ is the number x, that multiplied with 5 gives 8. So, it solves the equation 5*x = 8.
- $\sqrt[5]{8}$ is the number x, that multiplied with itself 5 times gives 8. So, it solves the equation x^5 = 8.
- $\log_5(8)$ is the number x of times to multiply 5 with itself to give 8. So, it solves the equation 5^x = 8.

Thus, where the root is a factor-finder, the logarithm is a factor-counter.

Together we see that an equation is solved by 'moving to <u>opposite side</u> with <u>opposite sign</u>'

5 + x = 8	5*x = 8	x^5 = 8	5^x = 8
x = 8 – 5	x = $\frac{8}{5}$	x = $\sqrt[5]{8}$	x = $\log_5(8)$

5. Double-counting creates per-numbers and fractions

Double-counting in two units creates per-numbers as e.g. 3$ per 4kg or 3$/4kg or ¾ $/kg.

To bridge the units, we just recount the per-number: 15$ = (15/3)*3$ = (15/3)*4kg = 20kg.

With the same unit, a per-number becomes a fractions or percent: 3$/4$ = ¾, 3$/100$ = 3%.

Again, the per-number bridges: To find ¾ of 20, we recount 20 in 4s. 20 = (20/4)*4 gives (20/4)*3 = 15.

6. Change formulas

The unspecified number-formula T = a*x^2 + c*x + d contains basic change-formulas:

- T = c*x; proportionality, linearity
- T = c*x+d; linear formula, change by adding, constant change-number, degree1 polynomial
- T = a*x^2 + c*x + d; parabola-formula, change by acceleration, constant changing change-number, degree2 polynomial
- T = a*b^x; exponential formula, change by multiplying, constant change-percent
- T = a*x^b; power formula, percent-percent change, constant elasticity

7. Use

- Asking ‘3kg at 5$ per kg gives what?’, the answer can be predicted by T = 3*5 = 15$.
- Asking ‘10 years at 5% per year gives what?’, the answer can be predicted by the formula T = 105%^10 – 100% = 62.9% = 50% in plain interest plus 12.9% in compound interest.
- Asking ‘If an x-change of 1% gives a y-change of 3%, what will an x-change of 7% give?’, the answer can be predicted by the approximate formula T = 1.07^3 – 100% = 22.5% = 21% plus 1.5% extra elasticity.
- Asking ’Will 2kg at 3$/kg plus 4kg at 5$/kg total (2+4)kg at (3+5)$/kg?’, the answer is ‘yes and no’.

The unit-numbers 2 and 4 can be added directly, whereas the per-numbers 3 and 5 must first be multiplied to unit-numbers 2*3 and 4*5 before they can be added as areas.

Thus, geometrically per-numbers add by the area below the per-number curve, also called by integral calculus.

A piecewise (or local) constant p-curve means adding many area strips, each seen as the change of the area, p*Δx = ΔA, which allows the area to be found from the equation A = Δp/Δx, or A = dp/dx in case of local constancy, called a differential equation since changes are found as differences. We therefore invent d/dx-calculation also called differential calculus.

Geometrically, dy/dx is the local slope of a locally linear y-curve. It can be used to calculate a curve's geometric top or bottom points where the curve and its tangent are horizontal with a zero slope.

8. Conclusion.

So, my dear Nephew, Mathematics is a foreign word for calculation, called algebra in Arabic. It allows us to unite and split totals into constant and changing unit- and per-numbers.

Love, your uncle Allan.

Algebra **unites**/*splits into*	**Changing**	**Constant**
Unit-numbers (meter, second, dollar)	**T = a + b** $T - b = a$	**T = a*b** $\frac{T}{b} = a$
Per-numbers (m/sec, m/100m = %)	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

23. Mathematics with Playing Cards

This booklet contains short articles, most of which have been printed in the LMFK member magazine for Danish upper secondary school math teachers. Thus, (2013.6) indicates that the article has been published in magazine nr. 6 from 2013. The goal is to show how mathematics formulas may be discovered by working with ordinary playing cards. Some formulas are limited by the fact that cards only have positive numbers, so the question if the formulas also apply to negative numbers may be partly answered by testing.

The article on Heron's formula is the only one not using playing cards.

Content

01. The little, medium and big Pythagoras with 3, 4 and 5 playing cards (2015, 2)

02. PI with three playing cards (2014, 6)

03. Proportionality with the 2 playing cards (2015, 1)

04. Product rules with 2-4 playing cards (2014, 6)

05. The quadratic equation with 2 playing cards (2014, 4)

06. Change by adding and multiplying with playing cards

07. The saving formula with 9 playing cards (2014, 2)

08. The change of a product with 3 playing cards (2013, 6)

09. Integral- and differential calculus with 2 playing cards (2015, 1)

10. Differentiating sine and cosine with 3 playing cards (2014, 4)

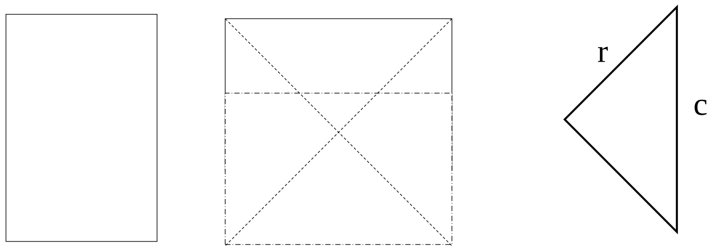
11. Topology with 6 playing cards

12. Heron's formula, triangle circles and Pythagoras in factor form (2011, 6)

1. The Little, Medium and Large Pythagoras with 3, 4 and 5 Playing Cards

I. The Little Pythagoras

A third playing card can show how much two others should be moved to form a square with the side length c. The two diagonals, each with length $2*r$, form four like isosceles right-angled triangles, each with the area $\frac{1}{2}*r^2$. So, $c^2 = 4*\frac{1}{2}*r^2 = r^2 + r^2$.

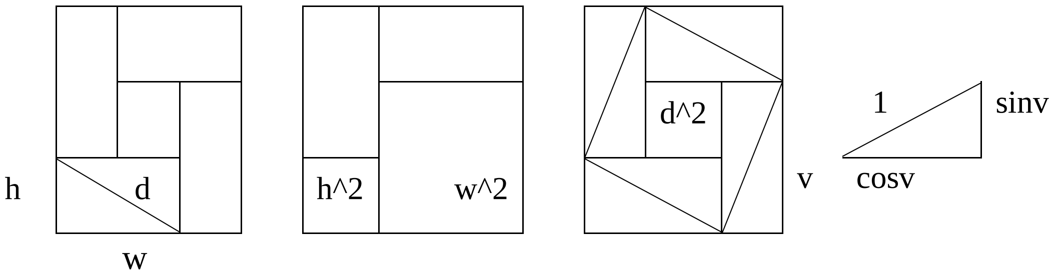


II. The Medium Pythagoras

Four playing cards have the width w and height h. The deck is rotated a quarter turn and placed to the right of the lower card, which remains unturned. This is repeated 3 times thus forming a shape that covers the area $h^2 + w^2$ + two cards.

During this process, also the diagonals with length d also make a quarter turn, and they now form the area d^2, which covers the shape above together with four half cards. Since four half-cards is the same as two whole cards, $d^2 = h^2 + w^2$.

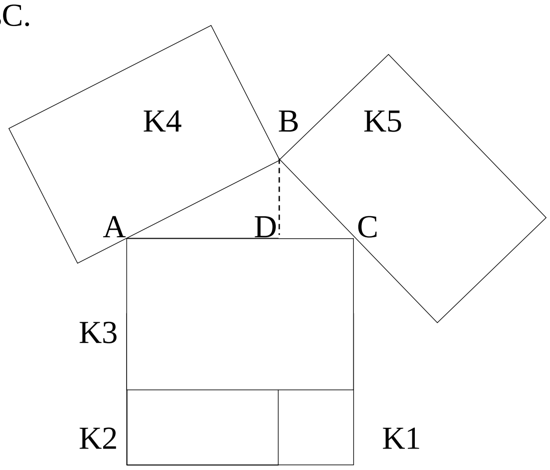
In particular, $\sin v^2 + \cos v^2 = 1$ in a right triangle with diagonal 1 and sides sin v and cos v.



III. The Large Pythagoras

The card K1 is placed horizontally. K2 is placed on-top and makes a quarter turn so the lower left corners are congruent. K3 is placed on-top so that K1 and K3 form a square. The cards K4 and K5 are used to generate the triangle ABC.

Here, the height BD is an extension of K2 's right side. Also, BD divides ABC into two right-angled triangles, BDA and BDC. In the right triangle BDC we see that $DB = a*\sin C$ and $DC = a*\cos C$.



AC's outer square consists of two squares formed by AD and DC, as well as two small strips. AD's square will be AC's square minus the two large strips, plus DC's square, which is deducted twice:

AD ^ 2 = AC^2 + DC ^ 2 – 2*DC*AC = b ^ 2 + (a*cosC) ^ 2 – 2*a*b*cosC

Using that

a^2*sinC^2 + a^2*cosC^2 = a^2*(sinC^2 + cosC^2) = a^2 *1 = a^2

We see that, in the left triangle ABD

AB^2 = DB^2 + AD^2.

= a^2*sinC^2 + a^2*cosC^2 + b^2 – 2*a*b*cos C

= a^2 + b^2 – 2*a*b*cos C

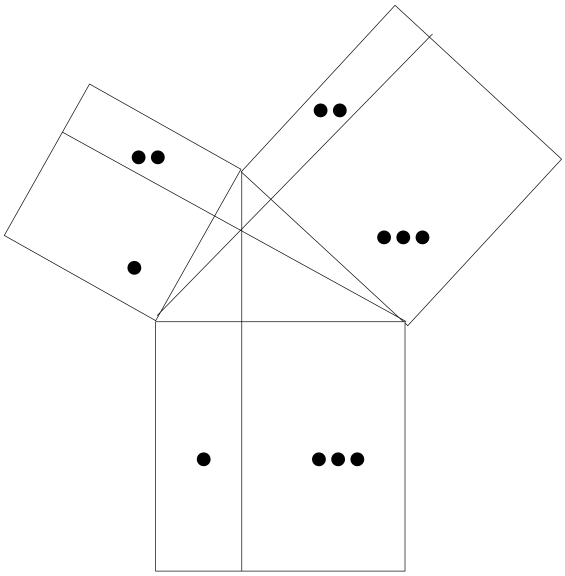
This implies that c^2 = a^2 + b^2 – 2*a*b*cosC (the large Pythagoras).

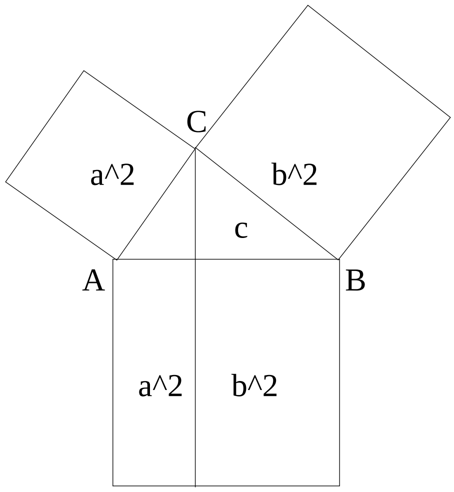
At the same time, we see that the height from B divides b's outer square in two parts with the areas b*c*cosA to the A side, and a*b*cosC to the C side. Likewise, we see that the height from A divides a's outer square in two parts with the areas a*c*cosB to B's side, and a*b*cosC to C's side. As well as to height from C divides c's outer square in two parts with the areas a*c*cosB to B's side and area b*c*cosA to a's side.

Consequently, the following two rules apply:

In a triangle without obtuse angles, the heights divide the opposite side’s outer squares in parts that are pairwise identical at the three angles, with a*b*cosC at angle C, and so on.

In a right triangle, the height divides the diagonal’s outer square into parts corresponding to the short side’s squares.





02. Pi with three playing cards

Three playing cards placed horizontally have the length b. The two top cards are rotated 90 degrees and placed so the three cards lower left corners are congruent. The top card is shifted to the right until it covers the lower card. The 2 top cards now form a square with side length b and diagonal d.

This square can be inscribed in a circle with the center at the intersection of diagonals and with the diagonal as diameter. Divided into 4 identical (red) isosceles triangles, their outer sides can be considered as a first approximation A1 to the circle circumference.

The outsides may be calculated by halving the center angles

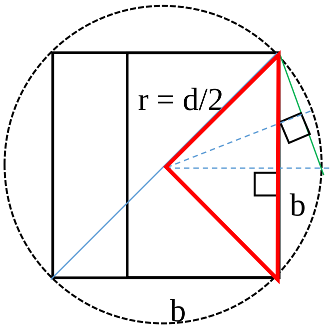
A1 = 4*2*d/2*sin (360/4/2) = d*4*sin (180/4).

The next approximation A2 is available as the outsides of the 8 (green) isosceles triangles, which comes when halving the center angles and keeping the radius r as inside:

A2 = 8*2*(d/2)*sin (180/4/2) = d*8*sin (180/8).

Continuing in this way, the approximation An = d*n*sin(180/n) will approach more and more to the circle circumference, which then can be written as d*pi where pi = n*sin(180/n) for n sufficiently large:

n	100	1000	10000	table value
n*sin(180/n) ≈ π	3.141076	3.141587	3.141593	3.141593…

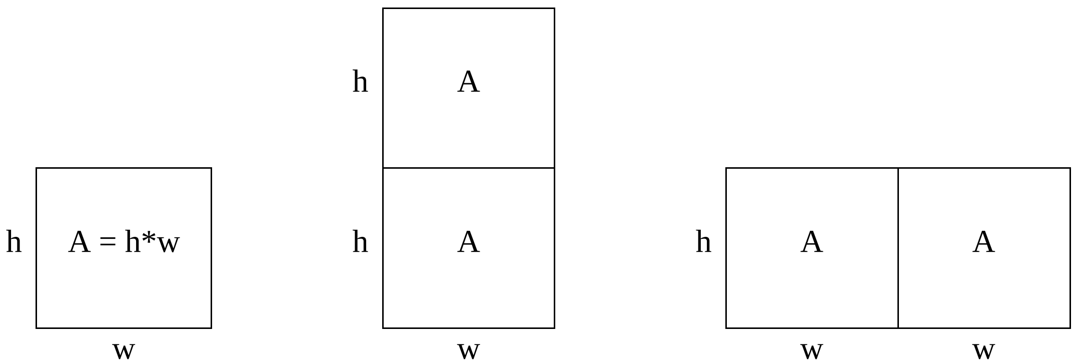


03. Proportionality with 2 playing cards

The product formulas A = h*w is very common, often called the proportionality formula or a bridging formula with the per-number h connecting the two unit-numbers A and w. The per-numbers may be $/kg, meter/second, or mol/liter in economics, physics or chemistry.

A product formula is illustrated by a playing card where the height h and width w will give the area A = h*w. Placed after the other vertically, the width w is constant, and doubling the height h implies doubling the area A. Thus, the height and the area are proportional. Likewise, we see that the width and the area are proportional when placing the cards after each other horizontally.

Moving the top card from vertical to horizontal position keeps the surface area A constant whereas the width w and height h will be doubled and halved respectively, thus being inverse proportional.



04. The Product Rules with 2-4 playing cards

A playing card with width a and height b has the area $a*b$.

A. Card 1 is placed horizontally with card 2 turned and placed on top so the bottom left corners are congruent. In this way, the upper-right corner becomes a square with the area $(a-b)^2$, obtained by removing two cards from the area a^2 and add b^2, since this area is removed twice:

$(a – b)^2 = a^2 – 2*a*b + b^2$

Card 1 divides card 2 in an upper part with the area $(a-b)*a$, and a lower part with the area b^2:

$(a – b)*b = a*b – b^2$

B. Card 2 moves vertically up until leaving card 1. Card 2 will then split card 1 into two parts, where the left part with the area b^2 together with card 2 covers the area $(a + b)*b$:

$(a + b)*b = a*b + b^2$

C. Card 3 is placed horizontally on top of card 2 so the upper left corners are congruent. Card 2 divides card 1 and card 3 into two parts, where the right parts are included in a rectangle with area $(a+b)*(a-b)$. And where card 1 's share and the visible part of card 2 both have the area $(a-b)*b$. The bent area then is the difference between a^2 and b^2:

$(a + b)*(a – b) = a^2 – b^2$

D. Card 3 is shifted to the right. A vertical card 4 is added below so that the four cards form a square with the side length $a + b$, combined to the left by the area a^2 at the bottom, and b^2 at the top, together with the two cards C3 and C4 to the right.

$(a + b)^2 = a^2 + b^2 + 2*a*b$

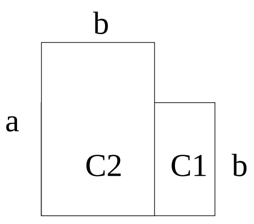


figure A

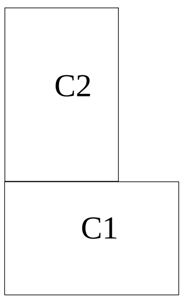


figure B

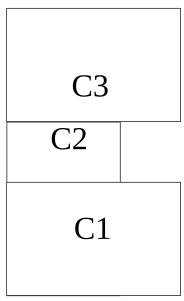


figure C

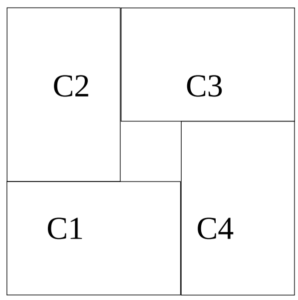


figure D

05. A quadratic equation solved with two playing cards

The equation $x + 2 = 8$ asks: what is the number x that added to 2 gives 8? To answer, we invent an opposite operation to addition, called subtraction, where the number $x = 8 – 2$ by definition is the number x that added to 2 gives 8. Thus, we see that an equation is solved when the unknown number is isolated by moving a number to the opposite side with the opposite calculation sign.

Likewise, by definition, the number $x = 8/2$ is the solution to the equation $x*2 = 8$ asking for the number x that multiplied with 2 gives 8. Likewise, the number $x = \pm \sqrt{8}$ is the solution to the equation $x^2 = 8$ asking for the number x that multiplied with itself gives 8. Again, we see that an equation is solved by moving a number to the opposite side with the opposite calculation sign.

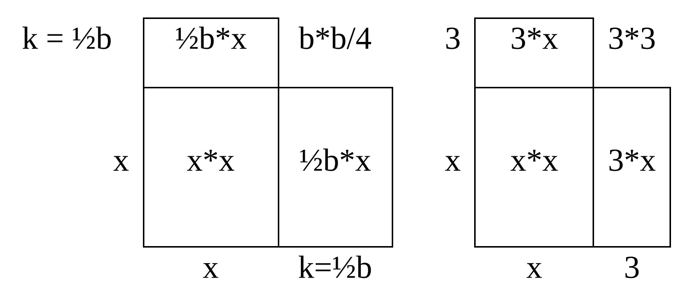
In the quadratic equation $x^2 + 6*x + 8 = 0$ there are two unknown x's so it needs to be rewritten, so there is only one x.

Two playing cards has the width k and the height x + k. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that

$(x+k)^2 = x^2 + 2*k*x + k^2$, or, have the unknown x only once on the right side:

$(x+k)^2 – k^2 = x^2 + 2*k*x$, or 'x plus k squared, minus k squared gives x squared + double-k x

We can now rewrite the equation $x^2 + 6x + 8 = 0$ first to $(x^2 + 2*3*x) + 8 = 0$, then to $(x+3)^2 – 3^2 + 8 = 0$, and finally to $(x+3)^2 – 1 = 0$ that is solved by three times moving to the opposite side:



$x^2 + 6x + 8 = 0$

$(x+6/2)^2 – (6/2)^2 + 8 = 0$

$(x+3)^2 – 1 = 0$;

Now 3 times, we move to opposite side:

$(x+3)^2 = 1$

$x+3 = \pm\sqrt{1}$

$x = -3+1 = -2$, and $x = -3-1 = -4$

With unspecified letter-numbers, the quadratic equation is solved in the same way:

$x^2 + b*x + c = 0$, giving

$(x^2 + 2*b/2*x) + c = 0$, giving

$(x + b/2)^2 – (b/2)^2 + c = 0$, giving

$(x + b/2)^2 = b^2/4 – c = D/4 = 0$, where $D = b^2 – 4*c$ is called the discriminant.

Thus, the quadratic equation $x^2 + b*x + c = 0$ has two solutions, or one, or none, depending on if the value of the discriminant D is positive, zero, or negative.

$(x + \frac{b}{2})^2 = \frac{D}{4}$; so $x + \frac{b}{2} = \pm \frac{\sqrt{D}}{2}$; so $x = \frac{-b}{2} \pm \frac{\sqrt{D}}{2}$; so $x = \frac{-b \pm \sqrt{D}}{2}$

06. Change by adding or multiplying with playing cards

Change by adding or multiplying occurs when x times an initial value b will be respectively added or multiplied by the same number a. This gives the terminal values respectively $y = b + a*x$ and $y = b*a ^ x$, also called linear and exponential growth. The two change forms may be illustrated with two decks of playing cards.

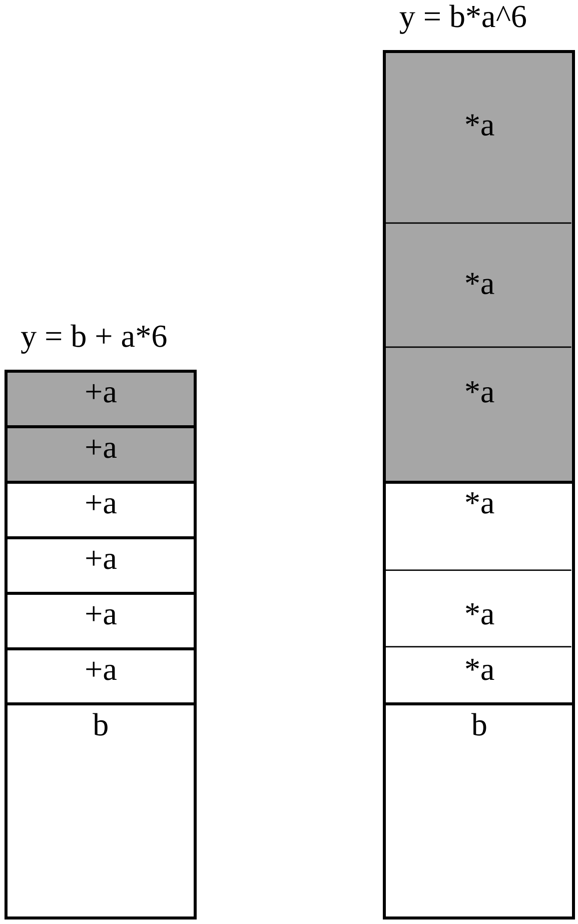
To the left is placed a deck of 7 cards. The top card stays. The next six cards are shifted upwards with a quarter of the length of the first card to show the terminal number after six times adding with the same change-number a.

Next to we place a deck of 6 cards. The top card stays.

The next three cards are shifted upwards with a quarter of the length of the previous stack to show that the terminal number after 3 changes is roughly the same as with four changes when changing by adding since as 125%*125*125%% = 195.3%, i.e. almost 200%. Thus, a change-percent at 25% means doubling after three changes.

So, after 6 changes, the initial number has been doubled twice, thus having 4 as the total change-factor or change-multiplier.

Linear change with b = 100$, and a = 25$. Exponential change with b = 100$, and a = 125%.



07. The saving formula with nine playing cards

With a monthly deposit a $ and interest percent, a saving combines change by adding and by multiplying. A saving is also called an annuity.

A saving occurs if a bank creates two accounts, K1 and K2. K2 receives the one-time deposit a/r $. Each month, K1 receives first the monthly interest percent r of its own amount, and then the monthly interest amount of the amount in K2, i.e. a fixed deposit of a/r $ * r = a $.

After n months, K1 will contain a saving A growing monthly from a deposit of a$ and an interest percent r. But at the same time K1 will contain the total interest percent R of the initial amount a/r $ in K2, so A = a/r*R or A/R = a/r, where 1+R = (1 + r) ^ n.

This can be illustrated with nine playing cards placed on a A4 paper divided into two with K1 to the left, and K2 to the right.

After the first month, K1 receives the interest percent r of its deposit, 0$; as well as a$ from K2, shown with a playing card placed horizontally with the backside up.

After the second month, K1 receives the interest percent r of its deposit, r*a $, shown with a vertical card push up a little; as well as a $ from K2, shown with a playing card placed horizontally with the backside up.

We will continue until the end of the fifth month. We gradually increase the pushing up of the vertical interest card because of the growing deposit in K1. Finally, we cover the right part of the two last cards with a white paper strip.

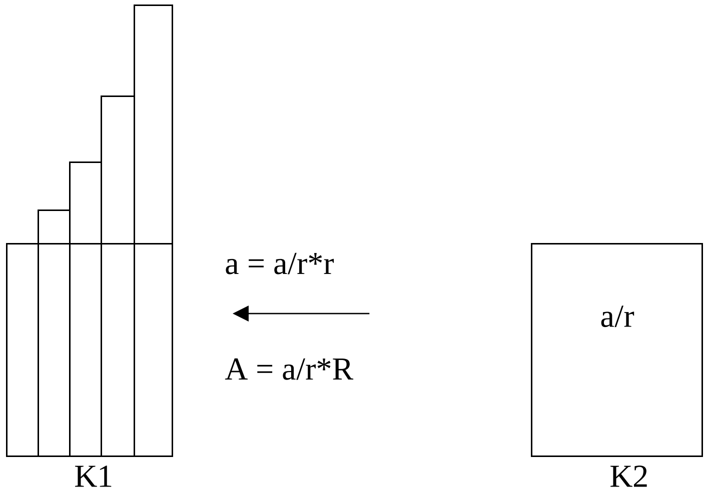
K1 now contains a saving, A, consisting of a series of constant $-deposits, the horizontal cards, and the interest on these amounts, the vertical cards.

At the same time, the horizontal cards represent the simple interest of K2, whereas the vertical cards represent the compound interest of K2.

So, again we have that the saving is A = a/r*R, giving A/R = a/r.

As to the relation between the total interest rate R, single interest rate r, and compound-interest rate RR, the cards show that R = n*r + RR or RR = R – n*r.

Thus, the simple and the compound interest might be taxed differently.



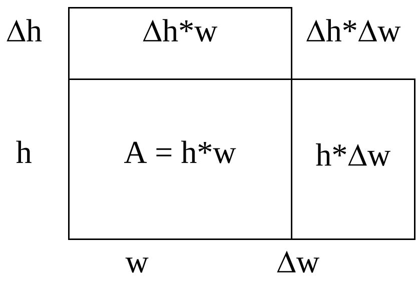
08. Product change with three playing cards

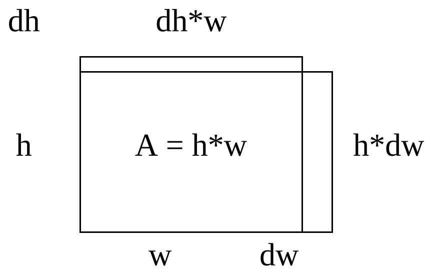
In geometry, the product of two numbers h and w occurs as the area A of a rectangle with height h and width w, A = h*w.

In economics the product occurs as the total $-number for w kg at h $/kg, T = h*w, or, more generally, each time a per-number is multiplied up to a quantity.

The question now is how changes in h and w will change the product.

Three playing cards has the height h and width w. The top card stays, the middle card is shifted off a piece Δw to the right, and the bottom card is shifted off a piece Δh upwards.





We see that the change in area, ΔA, consists of three parts, Δh*w and h*Δw and Δh*Δw.

With small changes, the last corner part may be neglected since the product of two small numbers gives a very small number: Assume that in the product 2*3 both numbers are changed with 0.01 to

(2+0.01)*(3+0.01) = 2*3 + 2*0.01 + 0.01*3 + 0.01*0.01
= 6 + 0.02 + 0.03 + 0.0001
= 6. 0501
= 6.05 with three significant figures.

Furthermore, the corner part has less and less influence, the smaller the change is.

Change t	0.1	0.01	0.001
h*w = (2+t)*(3+t)	6.51	6.0501	6.005001

Writing a small change as d instead of Δ will give the following rule for how a product is changed by small changes in its factors:

dA = d(h*w) = dh*w + h*dw, or as percentages:

dA/A = d(h*w)/(h*w) = dh/h + dw/w

Thus, with products, the change-percentages almost just add: Changing a kg-number with 3% and a $/kg-number with 5% will make the $-number change with approximately 3% + 5% = 8%. This rule applies to changes less than 10% with decreasing precision.

Since A = h*w, the per-number h = A/w.

Moving to opposite side with opposite calculation sign, we get

dh/h*w = dA/A – dw/w, or d(A/w)/(A/w) = dA/A – dw/w

Thus, with ratios, the change-percentages almost just subtract: Changing a $-number with 7% and a kg-number with 3% will make the $/kg-number change with approximately 7% – 3% = 4%. Again, this rule applies to changes less than 10% with decreasing precision.

So, with y = x^n we get that dy/y = n*dx/x, or dy/dx = n*y/x = n*x^n/x = n*x^(n-1)

09. Integral- and differential calculus with 2 playing cards

Where unit-numbers add directly, per-numbers add by their area: 2 kg at 6$/kg plus 3 kg at 4$/kg gives a total (2 + 3) kg at (6*2 + 4*3)/(2 + 3) $/kg.

This can be shown with two cards placed side by side, card1 placed vertically, and card2 placed horizontally. Card1 has the per-number 6$/kg vertically and the unit-number 2 kg horizontally, giving the area 12 $.

The unit-numbers add directly, the dollar-numbers to 12+12 = 24, and the kilo-numbers to 2+3 = 5.

The per-numbers, the $/kg-numbers 6 and 4, add by their area 24/5 = 4.8.

So Σ ($/kg) = Σ $ / Σ kg.

Graphing the cards in a coordinate system provides the rule: per-numbers add by the area under the per-number graph, i.e. by integration.

Integration uses multiplication before addition. In contrast, subtraction comes before division when reversing integration, also called differentiation:

the question “2 kg at 6$/kg plus 3 kg at u $/kg gives a total of 24$” is answered by first removing card1, and then recount the card2 area in 3s, thus applying subtraction before division:

u = (24 – C1)/3 = ΔC/3

Integration may be seen as a process of change: Card1 specifies a starting area. Placed next-to, card2 makes the area change to a new width w and the new area A. But now the variable height means a variable per-number, that might be found with differential calculus:

Card1 contains the initial numbers: the height ho, the width wo and area number Ao. Card2 contains the change-numbers

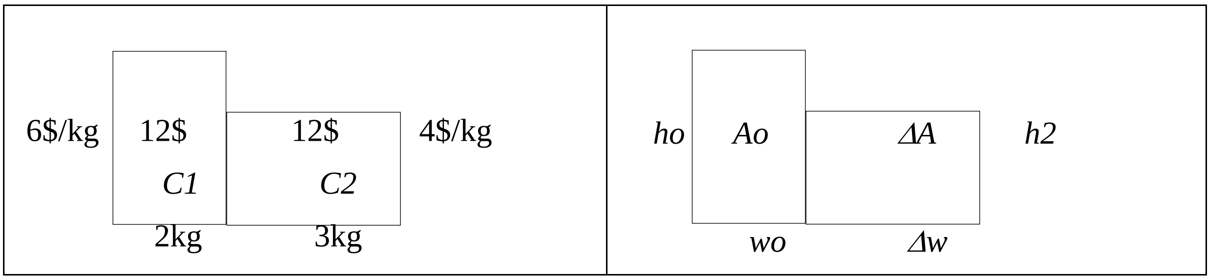
$\Delta w = w – wo$, and

$\Delta A = A – Ao$, which is also $h2*\Delta w$.

We then find the per-number for card2 as

$h2 = (A – Ao)/(w – wo) = \Delta A/\Delta w = \Delta y/\Delta x = \Delta f/\Delta x$

Here the width-number and the area-number are graphed as a line in an x-y coordinate system, where the per-number will be the slope of the area or unit-number curve y, typically given as A = y = f (x), i.e. as a formula with a variable number x.



10. How to differentiate sine and cosine with three playing cards

Three playing cards have the height h and width w. Card 1 rotates so w forms the angle v with the horizontal direction. Card 2 rotates 90 degrees and is placed at the end of card 1, so w here forms the angle v with the vertical direction. Card 3 remains horizontal and kicks under card 2 and over card 1 until forming a triangle with h as the long side.

As known, sine and cosine may be read as the first and second coordinate in a unit circle.

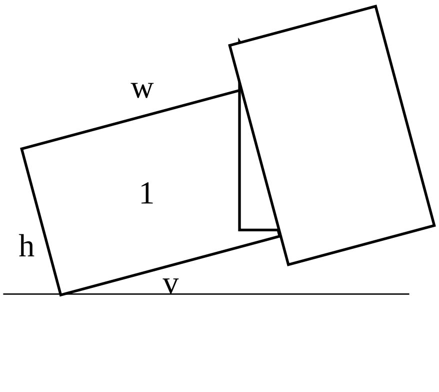
If the angle v gets a very small change dv, the circle (left w on card 2) is approximately linear.

Furthermore, the two left legs from v and v + dv (lower and upper w on card 1) are approximately parallel.

If v is measured in radian, the triangle's vertical, horizontal and long side will be three increment sides d(sin v) and -d (cos v) and dv. The long side forms the angle v with vertical, so

$$\cos v = \frac{d(\sin v)}{dv}, \text{ and } \sin v = -\frac{d(\cos v)}{dv},$$

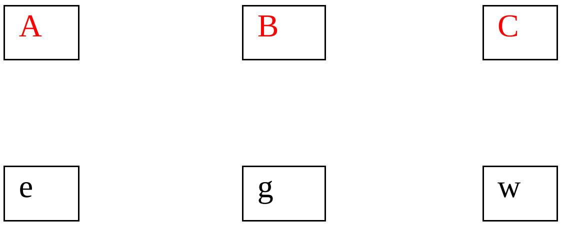
$$\text{or, } \frac{d(\sin v)}{dv} = (\sin v)' = \cos v, \text{ and } \frac{d(\cos v)}{dv} = (\cos v)' = -\sin v$$



11. Topology with six playing cards

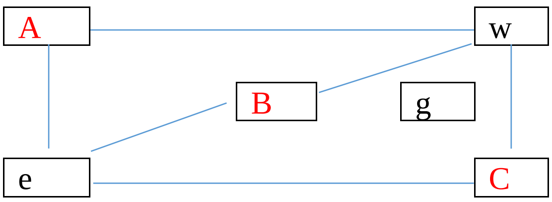
Six playing cards may illustrate a supply problem, a classic problem in topology, i.e. geometry, where neither distances nor angles, but only the relative positions between the points matter.

Problem: How can three houses A, B and C be provided with electricity, gas and water without the wires crossing?



We notice that the connection from house A to electricity to house C to water forms a closed ring that splits the plane in two areas, inside and outside. To be connected, house B and gas must be on the same side.

Suppose House B and gas is inside. Then the connection from electricity to house B to water will be splitting the inside in two closed areas with houses A and C in different areas. Located in one area, gas cannot be connected to the house located in the other area.



Now, suppose that house B and gas is outside. Then the connection from electricity to house B to water to another house back to electricity will enclose the third House, and the argument above can now be repeated.

Conclusion: the task cannot be solved unless we add a bridge whereby the plan changes its topology to a torus which is a plane with a handle.

Gluing a strip together at the ends, you can get from the outside to the inside in two ways: You can turn the strip a half turn (a Möbius strip), or you can punch a ‘wormhole’ (crosscap) in the strip.

In network analysis, topology is used to describe the number of bridges or handles in a given network.

12. Heron’s formula, a triangle’s circles, and Pythagoras on factor form

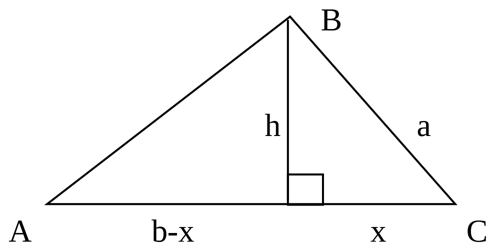
Heron’s area formula T = √(s*(s-a)*(s-b)*(s-c)) comes from repeating the factor formula

p^2 – q^2 = (p+q)*(p-q).

If 2*s = a+b+c, then 2*s – 2*a = -a+b+c, 2*s – 2*b = a-b+c and 2*s – 2*c = a+b-c

Proof.

Let B be the largest angle in the triangle ABC. Then the height h from B will divide the triangle into two right triangles, and it will divide the side b into two parts, x towards C and b – x towards A.



The lengths h and x are found from the two right triangles: x^2 + h^2 = a^2, so h^2 = a^2 – x^2.

Inserting this in (b-x)^2 + h^2 = c^2 gives c^2 = (b-x)^2 + a^2 – x^2.

Moving to opposite side with opposite sign gives

c^2 – a^2 = (b-x)^2 – x^2 = (b-x+x)*(b-x-x) = b*(b-2*x) = b^2 – 2*b*x.

Consequently, 2*b*x = a^2 + b^2 – c^2.

Now T = ½*h*b, so 16*T^2 = 4*h^2*b^2 = 4*(a^2 – x^2)*b^2 = 4*a^2*b^2 – 4*x^2*b^2 = (2*a*b)^2 – (2*x*b)^2 = (2*a*b + 2*b*x)* (2*a*b – 2*b*x)

Now we insert that 2*b*x = a^2 + b^2 – c^2:

16*T^2 = (2*a*b + a^2 + b^2 – c^2)* (2*a*b – a^2 – b^2 + c^2) = ((a+b)^2 - c^2)*(-(a-b)^2 + c^2) = ((a+b+c)*(a+b-c))*((a-b+c)*(-a+b+c)) = 2*s*(2s-2c)*(2s-2b)*(2s-2a) = 16*s*(s-a)*(s-b)*(s-c).

Consequently T^2 = s*(s-a)*(s-b)*(s-c).

Et viola: T = √(s*(s-a)*(s-b)*(s-c)).

Let the inscribed circle have the radius r. Then

T = ½*r*a + ½*r*b + ½*r*c = ½*r*2s = r*s, which gives

r = √((s-a)*(s-b)*(s-c)/s).

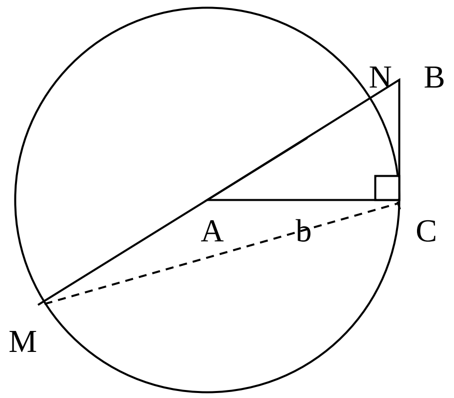
Let the circumscribed circle have the radius R. this allows extending the sinus relations

2*R = a/sinA.

Inserting this into the area formula T = ½*b*c*sinA gives the formula 4*R*T = a*b*c, “4 Round Turns teaches you abc”

The Pythagorean theorem can be found both on term form, a^2 + b^2 = c^2, and on factor form,   a^2 = (c + b)*(c-b).

The factor form is found by drawing a circle with center A and radius b. Let M and N be the points where c intersect the circle. Then BM = c + b and BN = c-b. The factor form follows from of the two similar triangles BCN and BMC, where the angles BCN and BMC are like, as they span the same arc; or from calculating the point B's ‘power-of-a-point theorem’.



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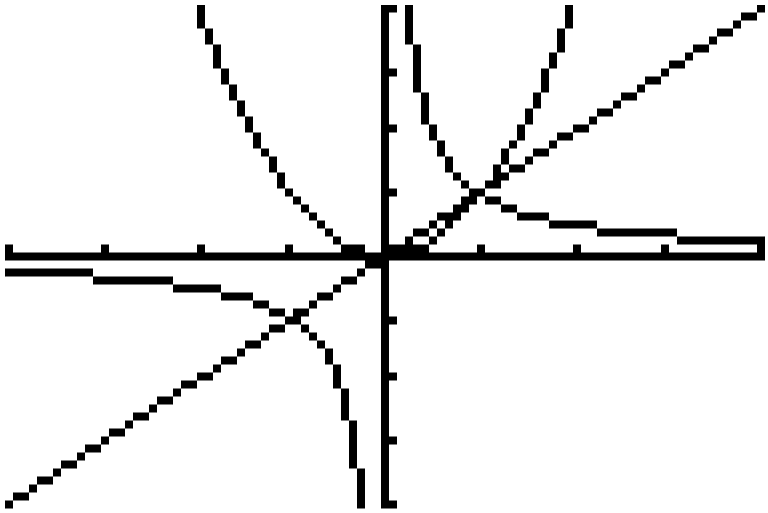
24. Mathematics Predicts
PreCalculus
How to add constant PerNumbers



y = 1*x , y = x*x , y = 1/x

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Compendium & Projects

by Allan.Tarp@MATHeCADEMY.net

Version 0719

Mathematics Predicts

Mathematics
    Algebra
        Uniting and Splitting Numbers
    Geometry
        Measuring Earth
    Statistics
        Accounting Many
	Mathematics contains Algebra, Geometry and Statistics
	Algebra (calculation) can predict counting processes, both the end result and the parts.
	Geometry (earth measuring) can be used to calculate plane figures and spatial forms.
	Statistics (counting) is used for counting the actual size of different quantities.

Mathematics has two main fields, Algebra and Geometry, as well as Statistics. Geometry means ’earth-measuring’ in Greek. Algebra means ’reuniting’ in Arabic thus giving an answer to the question: How to unite single numbers to totals, and how to split totals into single numbers? Thus, together algebra and geometry give an answer to the fundamental human question: how do we divide the earth and what it produces?

Originally human survived as other animals as gathers and hunters. The first culture change takes place in the warm rives-valleys where anything could grow, especially luxury goods as pepper and silk. Thus, trade was only possible with those highlanders that had silver in their mountains.

The silver mines outside Athens financed Greek culture and democracy. The silver mines in Spain financed the Roman empire. The dark Middle Ages came when the Greek silver mines were empty, and the Arabs conquered the Spanish mines. German silver is found in the Harz shortly after year 1000. This reopened the trade routes and financed the Italian Renaissance and the numerous German principalities. Italy became so rich that money could be lend out thus creating banks and interest calculations. The trade route passed through Arabia developing trigonometry, a new number-system and algebra.

The Greek geometry began within the Pythagorean closed church discovering formulas to predict sound harmony and triangular forms. To create harmonic sounds, the length out the vibrating string must have certain number proportions; and a triangle obeys two laws, and angle-law: A+B+C = 180 and a side law: a^2+b^2=c^2. Pythagoras generalized these findings by claiming: All is numbers.

This inspired Plato to install in Athens an Academy based on the belief that the physical is examples of metaphysical forms only visible to philosophers educated at the Academy. The prime example was Geometry and a sign above the entrance said: do not enter if you don’t know Geometry. However., Plato discovered no more formulas, and Christianity transformed his academies into cloisters, later to be transformed back into universities after the Reformation.

The next formula was found by Galileo in Renaissance Italy: A falling or rolling object has an acceleration g; and the distance s and the time t are connected by the formula: s = ½*g*t^2.

However, Italy went bankrupt when the pepper price fell to 1/3 in Lisbon after the Portuguese found the trade route around Africa to India thus avoiding Arabic middlemen. Spain tried to find a third way to India by sailing towards the west. Instead Spain discovered the West Indies. Here was neither silk or pepper, but a lot of silver, e.g. in the land of silver, Argentine.

The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move?

The church said ’among the stars’. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth.

But why do things fall? The church said: everything follows the unpredictable will of our metaphysical lord only attainable through belief prayers and church attendance. Newton objected: It follows its own will, a force called gravity that can be predicted by a formula telling how a force changes the motion, which made Newton develop change-calculations, calculus. So instead of obeying the church, people should enlighten themselves by knowledge, calculations and school attendance.

Brahe used his life to write down the positions of the planets among the stars. Kepler used these data to suggest that the sun is the center of the solar system but could not prove it without sending up new planets. Newton, however, could validate his theory by different examples of falling and swinging bodies.

Newton's discoveries laid the foundation of the Enlightenment period realizing that when an apple follows its own will and not that of a metaphysical patronizor, humans could do the same. Thus, by enlightening themselves people could replace the double patronization of the church and the prince with democracy, which lead to two democracies, one in The US and one in France. Also formulas could be used to predict and therefore gain control over nature, using this knowledge to build an industrial welfare society based upon a silver-free economy emerging when the English replaced the import silk and pepper from the Far East with production of cotton in the US creating the triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton.

Calculations Predict

Calculations predict the total T. 2*4 calculation types are used to unite and split into four different types of numbers:			a \$ and n \$ total T \$:	a+n = T
Uniting or splitting	Variable	Constant	a \$ n times total T \$:	n*a = T
Unit-numbers \$, kg, s	Plus + Minus -	Multiplication * Division /	r % n times total T%:	(1+r)^n = 1+T
Per-numbers \$/kg, \$/100\$, %	Integration ∑ ∫ Differentiation ∆	Power ^ Log or root √	a1 kg at p1 \$/kg + a2 kg at p2 \$/kg total T \$: p1*a1+ p2*a2 = T:	∑ p*a = T

Algebra means re-uniting in Arabic and can be translated to predictions. Algebra thus predicts the result of uniting singles into totals or splitting totals into singles.

There are four ways of uniting numbers: addition (+), multiplication (*), power (^) and integration (∑ or ∫).

Addition + predicts the result of uniting variable singles:
2\$ and 3 \$ and 4 \$ total T \$: 2+3+4 = T

Multiplication * predicts the result of uniting constant singles:
2\$ + 2\$ + 2\$ + 2\$ + 2\$ = 5 times 2\$ = T, 5*2 = T

Job	Count	Predict
32\$ and 63 \$	1,2,…,95	T = 32+63
2\$ 36 times	2,4, …,72	T = 72*2
20% 5 times	120, 144...	T = 120%^5

Power ^ predicts the result of uniting constant percentages: 5 times 2% totals T%, 102%^5 = 1+T

Integration ∑ or ∫ predicts the result of uniting constant per-numbers:
2kg at 7\$/kg + 3kg at 8\$/kg totals T \$: 7*2 + 8*3 = T , ∑ \$/kg * kg = T , ∫ p*dx = T

Inverse or backward calculations predicts the result of splitting a Total into singles.

x+3 = 15	Question: Which number added to 3 gives 15?
x = 15-3	Prediction: 15-3 is the number that added to 3 gives 15. Test: 3+(15-3) = 15
Rule	**Plus-numbers move across as minus-numbers, and vice versa**

x*3 = 15	Question: Which number multiplied with 3 gives 15?
$x = \frac{15}{3}$	Prediction: $\frac{15}{3}$ is the number that added to 3 gives 15. Test: $3*\frac{15}{3} = 15$
Rule	**Multiplication-numbers move across as minus-numbers, and vice versa**

x^3 = 125	Question: Which number raised to power 3 gives 125?
$x = \sqrt[3]{125}$	Prediction: $\sqrt[3]{125}$ is the number that raised to power 3 gives 125. Test: ($\sqrt[3]{125}$)^3= 125
Rule	**Exponent-numbers move across as reciprocal exponent-numbers, and vice versa**

3^x = 243	Question: 3 raised to which power gives 243?
$x = \frac{\log 243}{\log 3}$	Prediction: 3 raised to power $\frac{\log 243}{\log 3}$ gives 243. Test: 3^$\frac{\log 243}{\log 3}$ = 243
Rule	**Base-numbers move across as logarithm-numbers, and vice versa**

A multi-calculation containing more calculations reduce to a single calculation by bracketing the stronger one.

T = 2+3*4 = 2+(3*4), T = 2+3^4 = 2+(3^4) , T = 2*3^4 = 2*(3^4) <u>Priority: 1. (), 2.^, 3. *, 4. +</u>

A formula-table can be used to document the solving of an equation.

The unknown number	c = ?	T = a+b*c	*The formula*
The known numbers	a = 2 b = 3 T = 14	14 = 2+(3*c) $\frac{(14-2)}{3} = c$	*From a mixed to a single calculation by bracketing the stronger* *+ moves across as the opposite -, and * moves across as /* *Bracket the calculation already present*
		4 = c	*Perform the calculation*
Tests	Test	14 = 2+3*4 14 = 14 ☺	’MATHSolver 0 = -14 + 2+3*x’ gives ‘x = 4’

Exercises

Find the unknown number in the formula. Make more with randM (3,1)

1. T = a+b*c
2. T = a+b/c
3. T = a*b^c
4. T = a+b^c
5. T = a-b*c
6. T = a-b/c
7. T = a/b^c
8. T = a-b^c

	T	b	a	c
I	60		12	20
II	60	1.5		20
III	60	1.5	12	

Formulas Predict

A formula contains a quantity y and its calculation f, y = f(x,z,t) **An equation** is a formula with 1 unknown. An equation can be calculated or solved by finding the unknown. **A function** is a formula with 2 unknowns. A function can be tabled or graphed showing different scenarios: If x = a then y = f(a).	Purchase-formula: b \$ + x kg at a \$/kg totals y \$: b + x * a = y Sharing-formula: b \$ + a \$ shared between x persons totals y \$: b + a/x = y

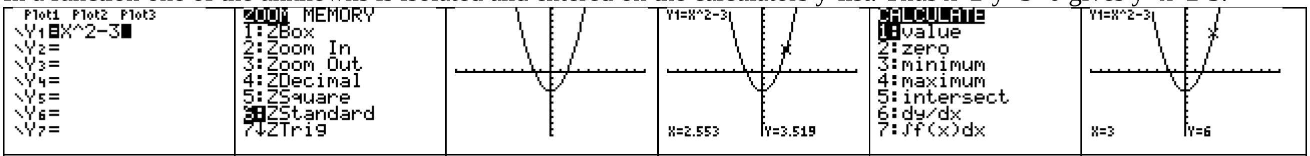
A formula contains a quantity y and a its calculation f, y = f(x,z,t). Thus a formula might contain 2, 3, 4 or more variables. If the variables are replaced by fixed numbers, a formula is transformed into an equation or a function.

An equation is a formula with 1 unknown: y = 10 + 2*3, or 16 = b + 2*3, or 16 = 10 + a*3, or 16 = 10 + 2*x

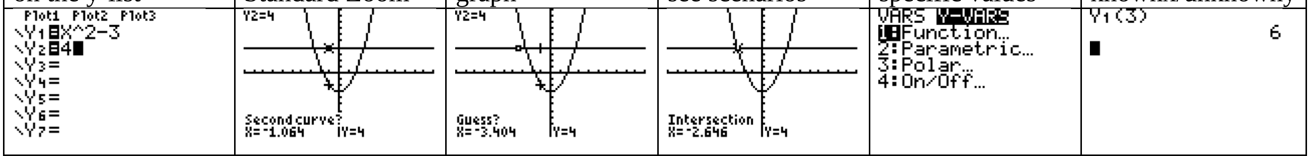
An equation can be solved manually or by a calculator using MATH-Solver. After using 'solve' the solution is tested by inserting all known numbers: 16 = 10 + 2*3 gives 16 = 16

A function is a formula with 2 (or more) unknowns: y = b + 2*3, or y = 10 + 2*x, or 16 = b + 2*x, or 16 = 10 + a*x.

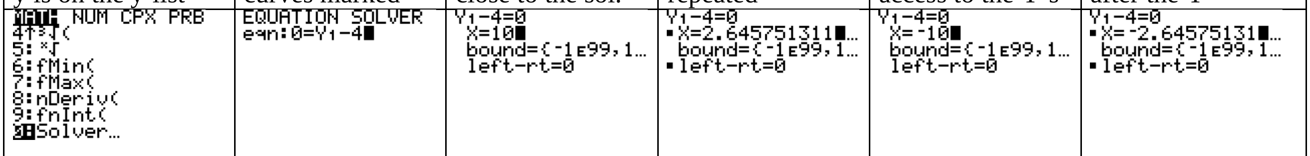
In a function one of the unknowns is isolated and entered on the calculators y-list. Thus x^2-y+3=0 gives y=x^2-3.



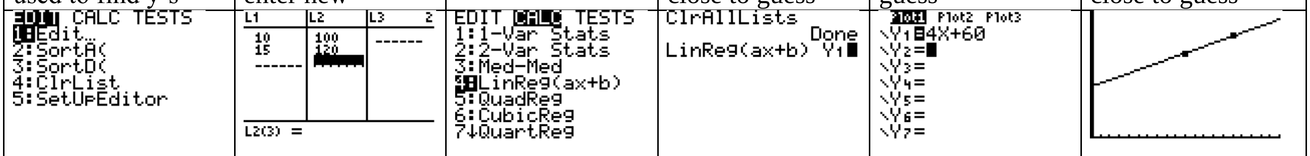
Formulas are put on the y-list	Always start with Standard Zoom	Choose Graph to graph	Choose Trace to see scenarios	Calc Value gives specific values	And is used for knownx/unknowny



Knowny/unknownx y is on the y-list	The intersecting curves marked	The cursor is close to the sol.	The procedure is repeated	VARS gives access to the Y-s	The known x is put after the Y



MATHSolver is used to find y's	CLEAR old and enter new	Enter a guess	Read the solution close to guess	Enter a new guess	Read the solution close to guess



From table to formula use STAT	Enter the table as lists	Choose a formula type	Add Y1 to bring formulas to y-list	Add Plot for visual control	Adjust window before graphing

Exercises: Find the question marks in 3 different ways: manually in a formula table, using graphs and using calculation.

1

x	y=3+2*x
-3.7	?
-2.4	?
3.1	?
4.5	?

2

x	y=3-2*x
-3.7	?
-2.4	?
3.1	?
4.5	?

3

x	y=x^2-4
-3.7	?
-2.4	?
3.1	?
4.5	?

4

x	y=-x^2+5
-3.7	?
-2.4	?
3.1	?
4.5	?

?	-3.7	?	-3.6	?	-3.8	?	-3.2
?	-2.4	?	-2.5	?	-2.2	?	-2.6
?	3.1	?	3.2	?	3.7	?	3.3
?	4.5	?	4.6	?	4.7	?	4.3

5	x	10	20	30	
	y	30	50		80
6	x	10	15	25	
	y	100	130		180
7	x	10	20	40	
	y	100	70		10

Ans:	a	b	Formula	y	x	T
lin	2	10	y = 10+2*x	70	35	
exp	1,052	18	y = 18*1,052^x	83,33	29,2	13,6
pow	0,737	5,5	y = 5,5*x^0,737	67,41	37,84	
lin	6	40	y = 40+6*x	190	23,33	
exp	1,054	59,17	y = 59,17*1,054^x	219,7	21,2	13,2
pow	0,647	22,54	y = 22,54*x^0,647	180,92	24,8	
lin	-3	130	y = 130+-3*x	10	40	
exp	0,965	142,86	y = 142,86*0,965^x	34,3	74,56	-19,4
pow	-0,515	327,02	y = 327,02*x^-0,515	49	877,72	

PerNumbers and Fractions

Unit-numbers have 1 unit, per-numbers have 2 or %. Per-numbers and fractions come from double-counting.	Per-numbers must convert to unit-numbers before being added (weighted average, conditional probability).

The ReCount-formula T = (T/B)*B predicts a counting result: 8 recounted in 2s gives: 8 = (8/2)*2 = 4*2.

The ReCount-formula solves equations: <u>u*5 = 40</u> = (40/5)*5, so u = 40/5 (move to opposite side with opposite sign).

Double-counting (proportionality) occurs in many places. A commodity may be double-counted as 4 kg and $5kr thus giving the per-number 4 kg per 5, or 4 kg/5 $, or 4/5 kg/$. Typical questions are: 60 kg = ? $ and 60$ =? kg

Method 1, recounting numbers	60 kg = (60/4) *4 kg = (60/4) *5 $ = 75 $	60$ = $(60/5)*5$r = 60/5*4kg = 48kg
Method 2, recounting units	$ = ($/kg)*kg = 5/4 *60 = 75	kg = kg/$*$ = 4/5*60 = 48

Per-numbers as fractions and percent

With like units, per-numbers become fractions or percentages: 4kg/5kg = 4/5, 4kg/100kg = 4/100 = 4%

40% of 30$ = u$	30$ is u% of 75$	30$ is 40% of u$
30 = (30/100)*100 giving (30/100)*40 = 12	100 = (100/75)*75 giving (100/75)*30 = 40	100% = (100/40)*40% giving (100/40)*30$ = 75$

Adding per-numbers and fractions

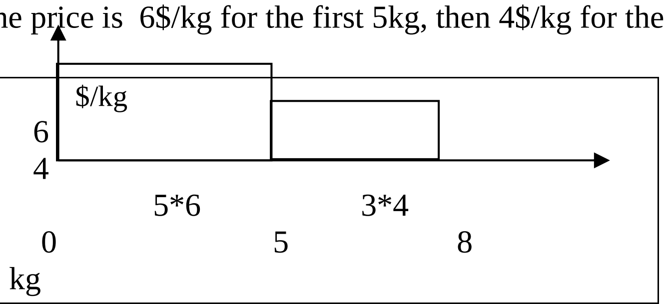
Example 1. Big volumes may give a discount: The price is 6$/kg for the first 5kg, then 4$/kg for the next 3kg.

T1 = 5kg at 6 $/kg = 5*6 $ = 30 $

T2 = 3kg at 4 $/kg = 3*4 $ = 12 $

T = 8kg at u $/kg = 8*u $ = 42 $

8*u = 42 = (42/8)*8, so u = 42/8 = 5.25 $\frac{kr}{kg}$



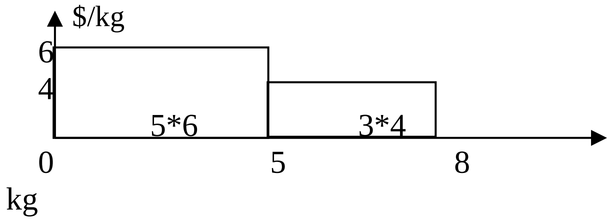
Example 2. Fractions are operators needing a number to become a number: 3/5 of 5kg added to 2/3 of 3 kg totals what?

T1 = 3/5 of 5 kg = 5*3/5*5 kg = 3 kg

T2 = 2/3 of 3 kg = 3*2/3*3 kg = 2 kg

T = u of 8 kg = u*8 kg = 5 kg

u*8 = 5 = (5/8)*8, so u = $\frac{5}{8}$ $\boxed{\text{so, } \frac{3}{5} + \frac{2}{3} = \frac{5}{8}}$



Notice: Per-numbers (and fractions) add as areas under the per-number graph, also called integral calculus.

Interest rate

250$ + 8% = ?$. $ + % doesn't work. But 100%+8% = 108%, and 108% of 250$ = 1.08*250$ = (1+0.08)*250$ = 270$.

Rate-formula: T = b*(1+r) , T: terminal number, b: beginning number, r: rent, 1+r: rate-factor = rate-multiplier.

The rate r% added n times: T = b*(1+r)*(1+r) *(1+r)* … = b*(1+r)^n = b*(1+R), R: total rate, 1+R = (1+r)^n

8% added 4 times gives R = 1.08^4 – 1 = 0.360 = 36.0% = 8%*4 + 4.0% = simple rate + compound rate (CR)

Continuous rate: 8% added continuously gives the rate-factor e^0,08 = 1.0833, i.e. a rate of 8.33% including 0.33% as compound rate. e = (1+1/n)^n for n very large.

Exercises

1. Density = 1.23 kg/l. 7.5 kg = ? l, ? kg = 34 l
2. Molar mass = 16 g/mol. 234 g = ? mol. ? g = 34.5 mol
3. Concentration = 2.4 mol/l. 34 mol = ? l. ? mol = 3.5 l
4. Price = 3.6 $/kg. 346 $ = ? kg. ? $ = 234 kg
5. Speed = 23.4 m/s. 34 m = ? s. ? m = 56 s
6. 20 is ?% of 30. ? is 20% of 30. 20 is 30% of ?.

7: Rent = 80 $/day. 200 $ = ? days. ? $ = 40 days

8: 40 kg = 60 l. 75 kg = ? l. ? kg = 200 l.

9: 40 $ = 82 days. 75 kg = ? days. ? kg = 200 days.

10: 30 m = 20 s. 65 m = ? s. ? m = 800 s.

11: 60 J = 90 s. 55 J = ? s. ? J = 400 s.

12. Calculate the empty fields when T = b*(1+r)^n

	T	b	r	n	R	n*r	CR
1		10	3.2%	20			
2	34		3.2%	20			
3	34	10		20			
4	34	10	3.2%				
5	34		3.2%		72%		
6	34			20		50%	
7		10	3.2%		68%		
8		10		20			12%
9	34		3.2%				23%

PerNumber Problems

In a per-number problem, the per-number must be multiplied to a unit-number to set up an equation.

Type2.1 Traveling

Problem21: Train1 travels from A to B with the speed 40 km/t. Two hours later train2 leaves A for B with the speed 60 km/t. When and where will they meet?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*
Hours		x = ?	*4*	40*(x+2) = 60*x
Speed1	40 km/h			40*x + 80 = 60*x
Speed2	60 km/h			80 = 60*x – 40*x = 20*x
Km1		40*(x+2) km	*240*	80/20 = x
Km2		60*x km	*240*	4 = x

Problem22: Train1 travels from A to B with the speed 40 km/t. At the same time train2 travels from B to A with the speed 60 km/t. The distance from A to B is 300 km? When and where do they meet?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*
Hours		x = ?	*4*	40*x + 60*x = 300
Speed1	40 km/h			100*x = 300*x
Speed2	60 km/h			x = 300/100
Km1		40*x km	*120*	x = 3
Km2		60*x km	*180*	

Problem23: A motorboat travels the same distance in 3 hours upstream as 2 hours downstream. The stream has the speed 5 km/t . What is the speed of the boat?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*
Speed	x = ? km/h		*25*	km = km/h*h = (x-5)*3 = (x+5)*2
Speed up	x – 5 km/h		*20*	3*x - 15 = 2*x+10
Speed down	x + 5 km/h		*30*	3*x - 2*x = 10+15
Hours		3 hours		x = 25

Type2.2 Mixture problems

? Liter 40% alcohol + 3 liter 20% alcohol gives ? liter 32% alcohol

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*
Liter-number		x = ? liter	*4.5*	0.4*x + 0.2*3 = 0.32*(x+3)
Liter-number		x+3 liter	*7.5*	0.4*x + 0.6 = 0.32*x + 0.96
Alcohol1	40%	0.4*x liter		0.4*x - 0.32*x = 0.96- 0.6
Alcohol2	20%	0.2*3 liter		0.08*x = 0.36
Alcohol3	32%	0.32*(x+3)	liter	x = 0.36/0.08
				x = 4.5

Type2.3 Finance

400.000$. giving a yearly yield at 20.000$ is invested in the following way: One part goes to a bank paying an interest at 3% p.a., the rest goes to 8% bonds. How much goes to the bank?

Text	*Per-Number*	*Unit-Number*	*ANSWER*	*Equation*
Bank in thousands		x = ? $	*240*	3%*x + 8%*(400-x) = 20
Bonds in thousands		x+3 $	*160*	0.03*x + 32 – 0.08*x = 20
Interest rate in bank	3%			32 - 20 = 0.08*x – 0.03*x
Interest rate on bonds	8%			12 = 0.05*x
Bank part		3%*x $		12/0.05 = x

Bond part		8%*(400-x)$	240 = x

Type2.4 Work problems

A can dig a ditch in 4 hours. B can dig the same ditch in 3 hours. How many hours if working together?

Text Hours	*Per-Number*	*Unit-Number* x = ? hours	*ANSWER* *12/7*	*Equation* ¼*x + 1/3*x = 1
A's speed B's speed A's part B's part	1/4 ditch/t 1/3 ditch/t	 ¼*x 1/3*x		(¼ + 1/3)*x = 1 7/12*x = 1 x = 12/7

Constant Change

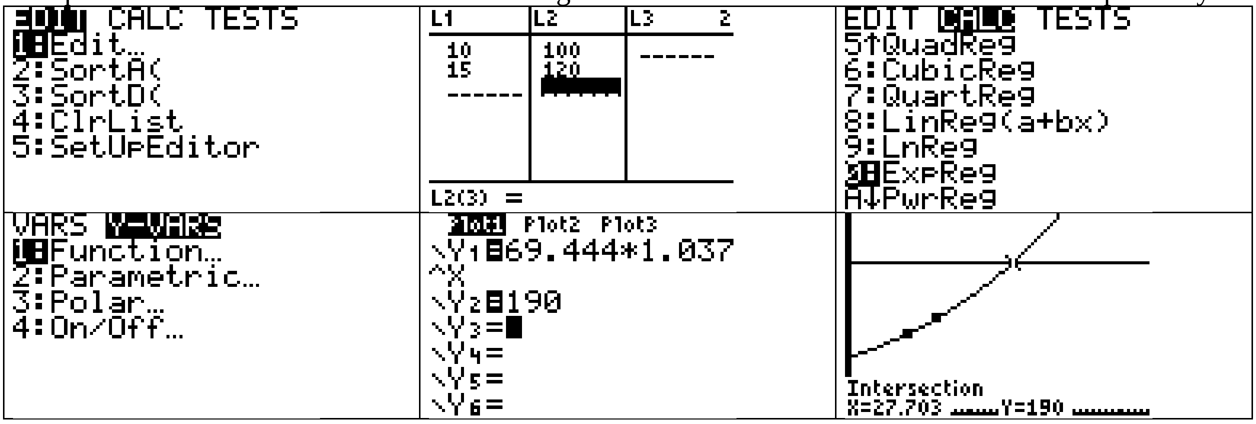
Table

x	y
10	100
15	120
25	?
?	190

Change forms

Linear ++ change	Exponential +* change	Power ** change
y = b+a*x	y = b*a^x	y = b*x^a
x: +1 day y: +a $ (change-number)	x: +1 day y: +r % (change-percent) x: + T, y: + 100% (or -50%) T = log2/loga = ln2/lna	x: +1 % y: +a % (elasticity)

To model and forecast we transform a table into a formula by using regression to find the numbers a and b. To use regression, first we select STATE EDIT. Then we enter the table. Now we select STATE CALC and e.g. ExpReg. Under the VARS Y-VARS we add Y1 to have the regression formula placed directly on the y-list. On the Y-list we add Y2 = 190 and turn on PLOT1 to see if the curve goes through the table points. The questions asked can now be answered using TRACE 25 and CALC INTERSECTION respectively.



The questions asked can also be answered using formula schemes:

y = ?	y = 60+4·x
x = 25	y = 60+4·25 y = 160
x = ? y = 180	y = 60+(4·x) y–60 = 4·x $\frac{y-60}{4}$ = x $\frac{180-60}{4}$ = x 30 = x

y = ?	y = 69.44·1.037^x
x = 25	y = 69.44·1.037^25 y = 172.80
x = ? y = 180	y = 69.44·(1.037^x) $\frac{y}{69.44}$ = 1.037^x $\frac{\log\left(\frac{180}{69.44}\right)}{\log(1.037)}$ = x 26.21 = x
Test	180 = 69.44*1.037^26.21 180 = 179.958 ☺

y = ?	y = 35.48·x^0.450
x = 25	y = 35.48·25^0.450 y = 151.03
x = ? y = 180	y = 35.48·x^0.450 $\frac{y}{35.48}$ = x^0.450 $\sqrt[0.450]{\frac{180}{35.48}}$ = x 36.92 = x
Test	180 = 35.48*36.92^0.450 180 = 179.991 ☺

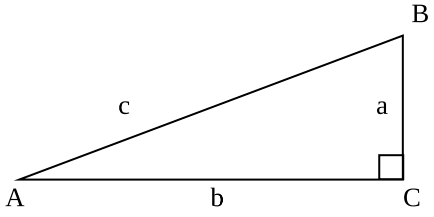
Exercises

						Ans:	a	b	Formula	y	x	Doubl. T
1	x	10	20	30		lin	2	10	y = 10+2*x	70	35	
	y	30	50		80	exp	1,052	18	y = 18*1,052^x	83,33	29,2	13,6
						pot	0,737	5,5	y = 5,5*x^0,737	67,41	37,84	
2	x	10	15	25		lin	6	40	y = 40+6*x	190	23,33	
	y	100	130		180	exp	1,054	59,17	y = 59,17*1,054^x	219,7	21,2	13,2
						pot	0,647	22,54	y = 22,54*x^0,647	180,92	24,8	
3	x	10	20	35		lin	-2	80	y = 80+-2*x	10	35	
	y	60	40		10	exp	0,96	90	y = 90*0,96^x	21,77	54,19	-17,1
						pot	-0,585	230,74	y = 230,74*x^-0,585	28,83	213,92	
4	x	10	20	40		lin	-3	130	y = 130+-3*x	10	40	
	y	100	70		10	exp	0,965	142,86	y = 142,86*0,965^x	34,3	74,56	-19,4
						pot	-0,515	327,02	y = 327,02*x^-0,515	49	877,72	

Trigonometry

Any land can be divided in triangles Any triangle can be divided into right-angled triangles	Two Greek Formula: A+B+C = 180 a^2 + b^2 = c^2 Three Arabic Formula: $\sin A = \frac{a}{c}$ $\cos A = \frac{b}{c}$ $\tan A = \frac{a}{b}$

A triangle is defined by 3 pieces. The rest can be predicted by formulas. The Greeks only found two formulas, so trigonometry first was developed when the Arabs added three extra formulas.



Greek formulas

A+B+C = 180 a^2 + b^2 = c^2 (Pythagoras)

Arabic formulas:

$\sin A = \frac{a}{c}$ (height in % of c) $\cos A = \frac{b}{c}$ (base in % of c)

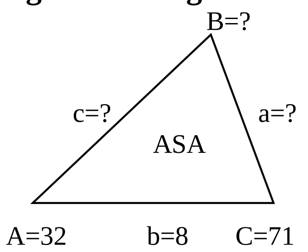
$\tan A = \frac{a}{b}$

A right triangle can be seen as a rectangle halved by a diagonal.

In a non right-angled triangle, the sine and cosine formulas have to be extended:

The Sine Rule	**The Cosine Rule (The Extended Pythagoras)**
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	a^2 = b^2 + c^2 - 2*b*c*cosA b^2 = a^2 + c^2 - 2*a*c*cosB c^2 = a^2 + b^2 - 2*a*b*cosC

Angle-Side-Angle

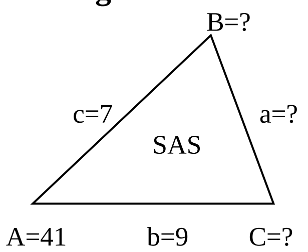


B = ?	A+B+C=180
A=32 C=71	B=180-A-C B=180-32-71 B=77

a = ?	$\frac{a}{\sin A} = \frac{b}{\sin B}$
b=8 A=32 B=77	$a = \frac{b \cdot \sin A}{\sin B}$ $a = \frac{8 \cdot \sin 32}{\sin 77}$ a = 4.351

c = ?	$\frac{c}{\sin C} = \frac{b}{\sin B}$
b=8 C=71 B=77	$c = \frac{b \cdot \sin C}{\sin B}$ $c = \frac{8 \cdot \sin 71}{\sin 77}$ c = 7.763

Side-Angle-Side



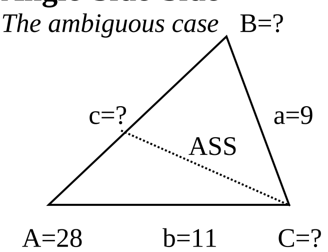
a = ?	a^2=b^2+c^2 -2*b*c*cosA
b=9 c=7 A=41	a^2=9^2+7^2 -2*9*7*cos41 a = √34.907 a = 5.908

B = ?	b^2=a^2+c^2 -2*a*c*cosB
b=9 c=7 a=5.9	$\cos B = \frac{a^2+c^2-b^2}{2 \cdot a \cdot c}$ $\cos B = \frac{5.9^2+7^2-9^2}{2 \cdot 5.9 \cdot 7}$ B=cos-1(0.035)=88.0

C = ?	A+B+C=180
A=41 B=88	C=180-A-B B=180-41-88 B=51

Angle-Side-Side

The ambiguous case

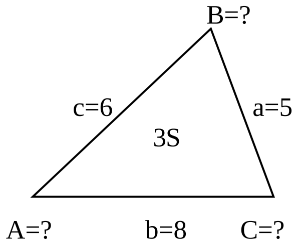


B = ?	$\frac{\sin B}{b} = \frac{\sin A}{a}$
b=11 A=28 a=9	$\sin B = \frac{b \cdot \sin A}{a}$ $\sin B = \frac{11 \cdot \sin 28}{9}$ B=sin-1(0.574) $B = \left[\begin{array}{l}35\\145\end{array}\right.$

C = ?	A+B+C=180
A=28 B=35 or B=145	C=180-A-B C=180-28-35 C=117 or C=180-28-145 C=7

c = ?	$\frac{c}{\sin C} = \frac{a}{\sin A}$
a=9 A=28 C=117 or C=7	$c = \frac{a \cdot \sin C}{\sin A}$ $c = \frac{9 \cdot \sin 117}{\sin 28}$ c = 17.081 or c = 2.336

Side-Side-Side



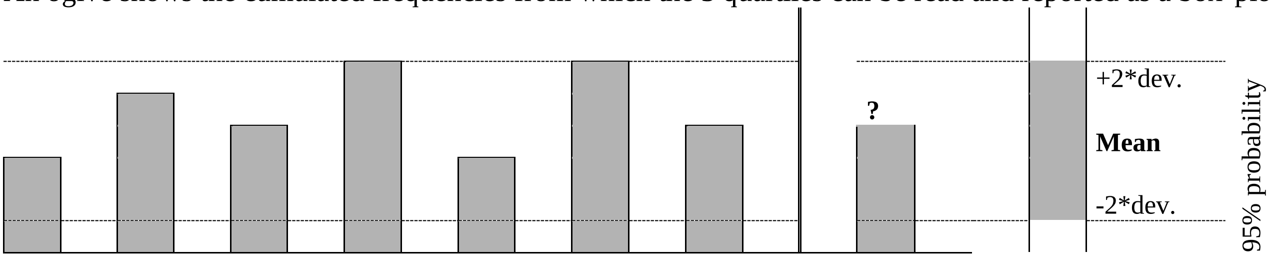
a=?	a^2=b^2+c^2 - 2*b*c*cosA
a=5 b=8 c=6	cosA=$\frac{b^2+c^2-a^2}{2*b*c}$ cosA =$\frac{8^2+6^2-5^2}{2*8*6}$ A = cos-1(0.781) A = 38.6

b=?	b^2=a^2+c^2 - 2*a*c*cosB
a=5 b=8 c=6	cosB=$\frac{a^2+c^2-b^2}{2*a*c}$ cosB=$\frac{5^2+6^2-8^2}{2*5*6}$ B = cos-1(-0.05) B = 92.9

C = ?	A+B+C=180
A=38.6 B=92.9	C=180-A-B C=180-38.8-92.9 C=48.5

Statistics, Stochastic Variation

Numbers may be predictable or unpredictable. Unpredictable numbers are also called random or stochastic numbers. Numbers that cannot be pre-dicted can often be post-dicted by setting up a statistics on their former behavior. A statistical table contains two columns, one with the numbers and one with their frequencies.
If arranged in increasing order:
The median = the middle observation, 1. (3.) quartile = the middle observation in the 1. (2.) half.
A histogram shows the frequencies
An ogive shows the cumulated frequencies from which the 3 quartiles can be read and reported as a box-plot.



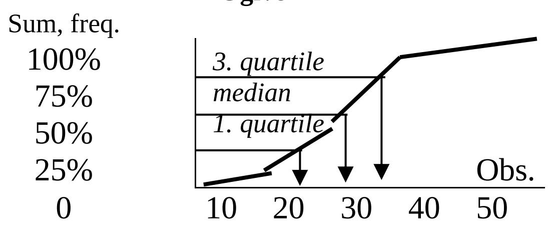
Statistics Post-dicts mean and deviation — *Probability* pre-dicts x = Mean ± 2* deviation

1. Observations
x: 10, 12, 22, 12, 15, ...

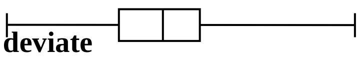
2. Grouping and counting frequencies

Observations x	Frequency h	Rel. Freq. p	Sum. freq. ∑p
0-10	3	3/40=0.075	0.075
10-20	12	0.300	0.375
20-30	18	0.450	0.825
30-50	7	0.175	1.000
Total	40	1.000	

Ogive



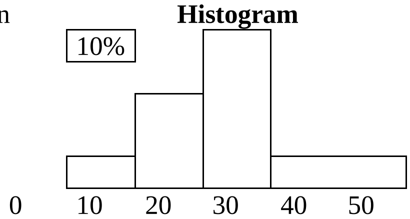
A Boxplot contains the median and two quartiles and the least and greatest observation



3. Mean or average: IF all the observations were the same ... however, the deviate

Observations x	Frequency h	Rel. Freq. p	Sum. freq. ∑p	Mean μ = ∑xi*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375
10-20	12	0.300	0.375	4.5
20-30	18	0.450	0.825	11.25
30-50	7	0.175	1.000	7
Total	40	1.000		23.1

Histogram



4. Variance, deviation: IF all the deviations were the same ...

Observations x	Frequency h	Rel. Freq. p	Sum. freq. ∑p	Mean μ = ∑xi*pi	Distance |xi - μ|	Variance v = ∑(xi-μ)^2*pi
0-10	3	3/40=0.075	0.075	5*0.075=0.375	|5-23.1|=18.13	18.13^2*0.075=24.64
10-20	12	0.300	0.375	4.5	8.13	19.80
20-30	18	0.450	0.825	11.25	1.88	1.58
30-50	7	0.175	1.000	7	16.88	49.83
Total	40	1.000		23.1		1 s^2 = 95.86

Deviation s = √95.86 = 9.8

5. Prediction: x = Mean ± 2*deviation =μ ± 2*s = 23.1 ± 19.6 *Confidence-interval* = [3.5 ; 42.7]

6. Using technology
On a GDC the interval midpoints are entered under STAT. Rel. frequency = freq/sum(freq). CumFreq = cumsum(freq).

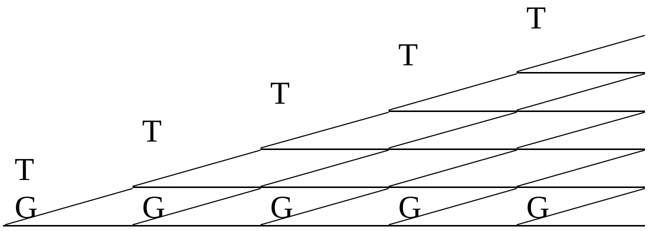
Obs.	Freq	Rel.freq	CumFreq
0	2	.05	.050
1	5	.125	.175
2	9	.225	.400
3	12	.300	.700
4	8	.200	.900
5	4	.100	1.000

The different numbers can be calculated using 1-var statistics:
Mean m = 2.8
Standard deviation, s = 1.3
Confidence-interval = m ± 2*s = [0.2;5.4]
1. quartile = 2
Median = 3
3. quartile = 4

Probability

A game has a 25% winning chance. Repeated 100 times, winning 25 times has the highest probability, which is less than not winning 25 times.	Probability predicts the unpredictable.

Example 1. A game with two results, Gain and Loss, is repeated 5 times. This implies 6 possible outcomes since we can win 0, 1, 2, 3, 4, 5 times. There is only 1 way to win 5 times: GGGGG, there are 5 ways to win 4 times because we can lose the 1^{st} , 2^{nd} , 3^{rd} , 4^{th} and 5^{th} time: TGGGG, GTGGG, GGTGG, GGGTG, GGGGT. How many ways are there to win 3 times? We can count or predict the answer: the number of roads with 3 G out of 5 = nCr (5.3) = 10.
The 6 outcomes are equally possible, but not equally likely. The distribution obtained by repeating a 2-result experiment many times is called a **binomial distribution**. If the gain probability p is 50% = 0.5, then:
The probability of winning just 3 times of 5: p (x = 3) = BinomCdf (5, 0.5, 3, 3) = 0313 = 31.3%.
The probability of winning 2, 3 or 4 times of 5: p (2 ≤ x ≤ 4) = BinomCdf (5, 0.5, 2.4) 0718 = 71.8%
Setting up a statistics of the different outcome, we can calculate the mean and deviation. However, these numbers may be predicted by the formulas m = n*p and s = √ (n*p*(1-p)), where n is the number of repetitions and p is the winning chance.



x	nCr	Cdf
0 G	1	0.031
1 G	5	0.156
2 G	10	0.313
3 G	10	0.313
4 G	5	0.156
5 G	1	0.031

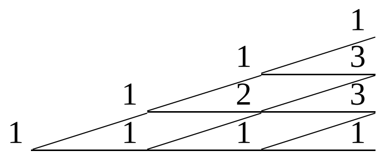
Example 2. Repeating a game with a 2/3 winning chance 30 times will give the mean value m = n*p = 30*2/3 = 20. The deviation is s = √ (n*p*(1-p)) = √ (20*1/3) = 2.6. The confidence interval is 20 ± 2 * 2.6 = [14.8; 25.2].
So there is approximately 95% chance that the next repetition will result in winning between 15 and 25 times.
If we only win 12 times, we must consider rejecting the hypothesis about a 2/3 winning chance.

Normal distribution

With many repetitions, a binomial distribution will approach the normal distribution, which often occurs in nature where there usually will be some variation of, for example, animal height. In such cases, the random variables x may assume decimal values or negative values.
Example 3. Repeating a game with a 60% winning chance 20,000 times will give the mean value m = n*p = 20000*0.60 = 12000, and deviation s = √ (n*p*(1-p)) = √ (12000*0.40) = 69.3.
Binomial distribution: P (0 < x < 12123) = BinomCdf (20000, 0.60, 0,12122) = 0962 = 96.2%
Normal distribution: P (0 < x < 12123) = NormCdf (0, 12123, 12000, 69.3) = 0962 = 96.2%
A normal distribution will have a straight sum-curve on a normal paper.

Exercises

1. Describe example 1 with p = 30%? 2. Describe example 1 with 6 repetitions 3. Describe the example 1 with 8 repetitions. 4. 'when (RAND() < 0.7, 1.0)' is an game with p = 0.7. Combined with itself 5 times is equivalent to	7. In a population height is normal distributed with mean value 132 and spread 24. What is the probability of a maximum of 126? at least 135? between 130 and 140?

performing the experiment in example 1. Perform the game 32 times and make a statistics on the outcome. 5. Describe example 1 with 200 repetitions and different p-values. Compare responses from the binomial and the normal distribution. 6. In a population, weight is normal distributed with mean value 13.2 and deviation 2.4. What is the probability of not exceeding 12.6? at least 13.5? between 13.0 and 14.0?	8. Arrange the nCr numbers systematically in the triangle below to create a ‘Pascal's triangle’. Which features have this triangle? Who was Pascal? 

Polynomials and Calculus

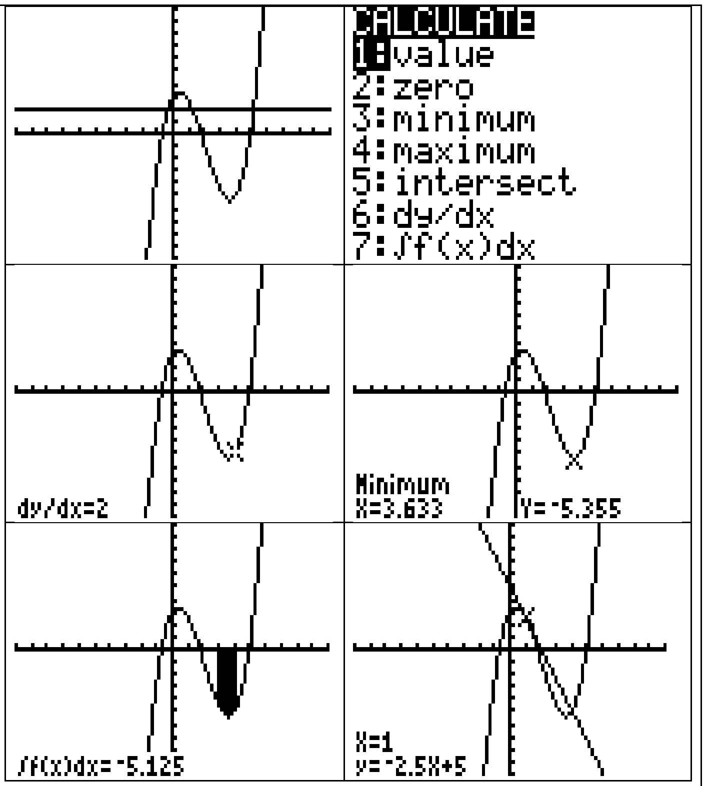
0. degree polynomial tells the (initial) point	y = 5
1. degree polynomial tells the (initial) gradient or steepness	y = 5 + 2*x
2. degree polynomial tells the (initial) bending	y = 5 + 2*x + 0.3*x^2
3. degree polynomial tells the (initial) counter-bending	y = 5 + 2*x + 0.7*x^2 – 0.2*x^3
4. degree polynomial tells the (initial) counter-counter-bending	y = 5 + 2*x + 0.7*x^2 – 0.2*x^3 + 0.3*x^4

Arabic numbers are polynomials: 4352 = 4*10^3 + 3*10^2 + 5*10 + 2. General form: y = 4*x^3 + 3*x^2 + 5*x + 2

Polynomials with bending graphs (degree over 1) have some interesting points:
Turning points, either top-points (maximum) or bottom-points (minimum).
Intersections with the x-axis (zeros), with the y-axis (y-intercept), or with other graphs.

Intersecting other graphs (equations graphically), Intersecting vertical lines (tracing values).
Shifting bending or curvature, where the bending changes its sign.
Tangent-point. A tangent is a straight line practically coinciding with the graph around the contact point, thus showing a scenario: this is how the graph would look like if the steepness stays constant.
If the curve graphs per-numbers, the total is found as the area under the per-number graph, i.e. by integration
If the curve graphs a Total, the per-numbers are found as the steepness of the total graph, i.e. by differentiation
Differentiation twice gives the bending, being positive when bending upwards and negative when bending downwards.
Finding the steepness (gradient, slope) formula is called differentiation. Finding the area under a curve is called integration. Together differentiation and integration are inverse operations called Calculus
Example: y = 0.5x^3-3x^2+2x+3

	Graphics	Formula
Intersecting the y-axis CALC value	y = 3	y1(0)
Intersecting the x-axis CALC zero	x = -0.694 x = 1.748 x = 4.946	Solve(0=Y1)
Intersecting y = 2 CALC intersection	x = -0.329 x = 1.181 x = 5.147	Solve(0=Y1-2)
Top CALC Maximum	x = 0.367 y = 3.355	MATH fMax(Y1,x,0,7)
Bottom CALC Minimum	x = 3.633 y = -5.355	MATH fMin(Y1,x,0,7)
Steepness in x = 4 CALC dy/dx	2	MATH nDeriv(Y1,x,4)
Area from 3 to 4 CALC Sf(x)dx	-5.125	MATH fnInt(Y1,x,3,4)
Tangent in x = 1 DRAW tangent x = 1	y=-2.5x+5	



Exercises

1. Repeat as above with y = 0.7x^3-4x^2+3x+4.
2. Repeat as above with y = -0.4x^3+2x^2-0.5x-3.
3. Produce your own polynomials using randM(4,1).
4. Produce your own polynomials using regression
5. Find the cheapest box without lid containing 1 liter.
6. Find the cheapest pipe without lid containing 1 liter.
7. Find the cheapest box with lid containing 1 liter.
8. Find the cheapest pipe with lid containing 1 liter.
9. Find the cheapest box with double lid containing 1 liter.
10. Find the cheapest pipe with double lid contain. 1 liter.
11. Find the cheapest cone without lid containing 1 liter.
12. Find the cheapest cone with lid containing 1 liter.
13. Find the cheapest cone with double lid contain. 1 liter.
14. y1 is a polynomial of degree 0. If y1 is a Total, what is its per-number? If y1 is a per-number, what is its total?
15. As 14 with polynomials of degree 1.
16. As 14 with polynomials of degree 2.
17. As 14 with polynomials of degree 3.

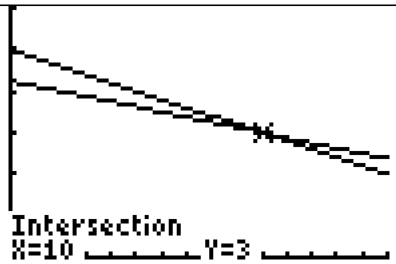
Two equations With Two Unknowns; and Three

Two equations with two unknowns are solved manually, by intersection or by matrices	b \$ + 5kg at a \$/kg = 25 \$ b \$ + 8kg at a \$/kg = 34 \$	x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1&5\\1&8\end{pmatrix}*\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$

2 equations with 2 unknowns: The formula b \$ + 5kg at a \$/kg = 25\$ contains 2 unknowns and cannot be solved, unless we know another example of the same formula as e.g. b \$ + 8kg at a \$/kg = 34 \$.

Written as an equation system	Written as a matrix equation
x + 5*y = 25 x + 8*y = 34	$\begin{pmatrix}1&5\\1&8\end{pmatrix}*\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$

Manually one variable is isolated in the first and inserted in the second equation: x=25-5*y, 25-5*y+8*y=34, y=3 & x=10.

Using graphs, the y's are isolated and inserted into the y- editor. x + 5*y = 25 gives y = (25-x)/5 , x + 8*y = 34 gives y = (34-x)/8 The intersection point is found by 'Calc Intersection' to x = 10 and y = 3. Also we can use Math Solver 0 = Y1 – Y2.		

Matrix-solutions is found by entering the matrices into the matrix-editor as ml and mr (matrix-left & -right):

$\underline{V}=\begin{pmatrix}x\\y\end{pmatrix}=?$	$\underline{\underline{ml}}*\underline{V}=\underline{mr}$
$\underline{\underline{ml}}=\begin{pmatrix}1&5\\1&8\end{pmatrix}$ $\underline{mr}=\begin{pmatrix}25\\34\end{pmatrix}$	$\underline{V}=\underline{\underline{ml}}$^-1 * $\underline{mr}$ $\underline{V}=\begin{pmatrix}1&5\\1&8\end{pmatrix}$ ^-1 * $\begin{pmatrix}25\\34\end{pmatrix}$ $\underline{V}=\begin{pmatrix}10\\3\end{pmatrix}$
Test	$\begin{pmatrix}1&5\\1&8\end{pmatrix}*\begin{pmatrix}10\\3\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$ $\begin{pmatrix}25\\34\end{pmatrix}=\begin{pmatrix}25\\34\end{pmatrix}$

$\underline{V}=\begin{pmatrix}x\\y\\z\end{pmatrix}=?$	$\underline{\underline{ml}}*\underline{V}=\underline{mr}$
$\underline{\underline{ml}}=\begin{pmatrix}2&5&2\\1&0&-1\\4&-3&6\end{pmatrix}$ $\underline{mr}=\begin{pmatrix}18\\-2\\16\end{pmatrix}$	$\underline{V}=\underline{\underline{ml}}$^-1 * $\underline{mr}$ $\underline{V}=\begin{pmatrix}2&5&2\\1&0&-1\\4&-3&6\end{pmatrix}$ ^-1 * $\begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\underline{V}=\begin{pmatrix}1\\2\\3\end{pmatrix}$
Test	$\begin{pmatrix}2&5&2\\1&0&-1\\4&-3&6\end{pmatrix}*\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}18\\-2\\16\end{pmatrix}$ $\begin{pmatrix}18\\-2\\16\end{pmatrix}=\begin{pmatrix}18\\-2\\16\end{pmatrix}$

3 equations with 3 unknowns cannot be solved graphically, but manually and by using matrices:

Written as an equation system	Written as a matrix equation
3*x + 5*y + 2*z = 19 x – z = -2 4*x – 3*y + 6*z = 16	$\begin{pmatrix}3&5&2\\1&0&-1\\4&-3&6\end{pmatrix}*\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}19\\-2\\16\end{pmatrix}$

A matrix-solution is found by entering the matrices into the matrix-editor as ml and mr.

4 equations with 4 unknowns, 5 equations with 5 unknowns etc. Like 3 equations with 3 unknowns.

Equation systems for skill building can be generated by 'randM(3,3)' and 'randM(3,1).

Exercises. Solve the equation systems

1. 4x - 1*y = -9 4x - 4*y = 0 2. 4x + 2*y = 16 5x - 3*y = -2	5. -7*x - 3*y - 7*z = 3 -1*x - 5*y + 1*z = -13 9*y - 5*z = 36 6. 4*x + 3*y + 7*z = 81	8. 2*x + 5*y - 1*z + 9t = 118 1*x + 1*y - 9*z - 5t = -88 - 3*y + 7*z + 5t = -51 -3*x + 5*y + 2*z - 5t = -10

3. 7x + 4*y = -1 -3x + 2*y = 19 4. 2x - 5*y = 16 3x - 4*y = 17	5*x + 3*y + 1*z = 54 2*x + 9*y + 5*z = 57 7. 2*x + 3*y - 1*z = - 6 5*x + 3*y - 4*z = -15 2*x – 2*y + 5*z = 40	9. -6*x - 1*y + 8*z + 8t = 129 -2*x + 2*y - 5*z + 7t = 60 8*x + 6*y + 3*z + 3t = -40 -7*x - 4*y - 8*z - 4t = 12

The Quantitative Literature: The Three Kinds of Models

Geometry and algebra are the classical quantitative literature meaning 'measuring earth' and 'uniting number's in Greek and Arabic respectively. The modern quantitative literature is mainly created by questions coming from the production: How to extract silver and coal from mines? How to navigate at sea? How to predict the orbit of a falling body? How to build machines? How to optimize a production? How to optimize profits? Etc.
Calculations took place to lift silver and water up from the mines, and how to transform crude silver into silver of varying degree of purity. Silver now travelled through Germany to Italy, and from there to the east to be exchanged with pepper and silk, either via the expensive way by land transported by caravans, or via the cheap way over sea transported by Arab merchants. Helped by replacing their own Roman numbers with Arabic numbers allowing multiplication and division to take place, Italian wealth created banks to lend out money for an interest in return. This implies a need for adding interest rates leading to the discovery of power calculations and compound interest: 7 years at 6% yearly = 42% interest rate + 8% compound interest rate = 50% since 106% ^ 7 = 150%. Part of the profit goes to consumption when building magnificent palaces everywhere in Renaissance Italy, and when hiring artists.
Then Portugal took over by decreasing the price on pepper to 1/3 when skipping the Arabic middlemen by sailing silk and pepper home around Africa on their own ships. Spain then tried to find another way to India by sailing west. But in West India there was no spice or silk, but plenty of gold and silver, e.g. in the land of silver, Argentine. The Pope divided the new world between Spain and Portugal. Portugal gets everything east of the 60^{th} longitude, Spain everything west of it. In Spain and Portugal profits went to consumption through the building of churches and monasteries and mansions. The English easily stole Spanish silver returning over the Atlantic, but to avoid Portuguese fortifications of Africa the English had to sail to India on open sea following the moon. But how does the moon move? The church said 'among the stars'. Newton objected: The moon falls towards the earth as does the apple, only the moon has received a push making it bend in the same way as the earth thus being caught in an eternal circular fall to the earth pulled down by a gravitational force, that changes motion thus leading to the creation of change-calculations, calculus.
Once in India, England exchanged silver for cotton to be planted in their North American colonies bought form Spain who had no interest in land without silver. By replacing silk and pepper trade from the Far East with production of cotton in the US, England created a triangular trade on the Atlantic exchanging cotton for weapon, and weapon for labor (slaves) and labor for cotton. The profit was used for investment buying stock and establishing an industrial production. Fighting for colonies led to the second world war, leading to the creation of the computer and operation research.

The three genres: facts, fiction and fiddle

Qualitative and quantitative literature divide into three genres: facts, fiction and fiddle. In qualitative literature, the three qualitative of genres are
Fact: 'SINCE Copenhagen is located on Zealand, THEN Copenhagen is close to water level.'
Fiction: 'IF Copenhagen was located in the Alps, THEN Copenhagen is far from water level.'
Fiddle: 'IF Copenhagen is located first in a sentence, THEN Copenhagen is close to water level.'

Fact

Fact models are 'SINCE-THEN' calculations, quantifying quantities, and calculating the calculable: 'SINCE the price is 4 $/kg, THEN 6 kg costs 6 * 4 = 24$.
'SINCE-THEN' calculations may also be called 'room-calculations: 'SINCE the room has dimensions 3x4x5, THEN the volume is 3*4*5 = 60'
Fact models are re-calculated for testing: T = 3 kg at 4$/kg. = 3*4 $ = 15 $, oops, calculation error, T = 12 $.
Fact models may be wrong if omitting the units. The Mars Climate Orbiter thus failed when mixing cm and inch.

Fiction

Fiction models are 'IF-THEN' calculations, quantifying quantities, and calculating the incalculable: 'IF my daily income is 4$, THEN 6 days will give the income be 6*4 = 24$'. But my income may vary?
'IF-THEN' calculations may also be called 'rate-calculations': 'IF the growth rate is 3% per year, THEN the total growth rate after 5 years be 15.9%, since 103% ^ 5 = 115.9%.' But the growth rate may vary?
Fiction models must be supplemented by parallel scenarios: The daily income is estimated to be between 4$ and 5$, so in 3 days, 'The expected income will be between 3*4$ = 12$ and 3*5$ =15$.

Fiddle

Fiddle models are 'SO-WHAT' calculations, quantifying qualities, and calculate the incalculable: 'If the consequence C= broken bone is set to 2 million $, and if the probability P set to 30%, then will the risk be R = C*P = 2 * 0.3 = 0.6 million $.' But SO-WHAT? Who says that a broken leg costs 2 million $? And who says that a general probability of breaking a leg can be measured?
Another example: 'If the cost of a burial ground is 10 $/day, and the cost of a hospital bed is 10,000 $/day, then it is cheaper to have people lying in the cemetery than at the hospital. But SO-WHAT? Should we increase the speed limit to 200 km/hour to save money?'
'SO-WHAT' calculations may also be called 'risk-calculations': 'IF we can increase the probability of death and reduce the probability of injury, then will the risk of crossing a school road could be reduced.' But SO-WHAT? Should we then dismantle the zebra crossing? Fiddle models must be rejected and referred to a qualitative treatment in the word-language instead of a quantitative one in the number-language.

Letter Calculation, Transposing Formulas

Change the T-formulas to a-formulas, b-formulas and c-formulas, and vice versa.

	T	a	b	c
1	$T = a + b\cdot c$	$a = T - b\cdot c$	$b = \frac{T-a}{c}$	$c = \frac{T-a}{b}$
2	$T = a - b\cdot c$	$a = T + b\cdot c$	$b = \frac{a-T}{c}$	$c = \frac{a-T}{b}$
3	$T = a + \frac{b}{c}$	$a = T - \frac{b}{c}$	$b = (T-a)\cdot c$	$c = \frac{b}{T-a}$
4	$T = a - \frac{b}{c}$	$a = T + \frac{b}{c}$	$b = (a-T)\cdot c$	$c = \frac{b}{a-T}$
5	$T = (a + b)\cdot c$	$a = \frac{T}{c} - b$	$b = \frac{T}{c} - a$	$c = \frac{T}{a+b}$
6	$T = (a - b)\cdot c$	$a = \frac{T}{c} + b$	$b = a - \frac{T}{c}$	$c = \frac{T}{a-b}$
7	$T = \frac{a+b}{c}$	$a = T\cdot c - b$	$b = T\cdot c - a$	$c = \frac{a+b}{T}$
8	$T = \frac{a-b}{c}$	$a = T\cdot c + b$	$b = a - T\cdot c$	$c = \frac{a-b}{T}$
9	$T = \frac{a}{b+c}$	$a = T\cdot (b+c)$	$b = \frac{a}{T} - c$	$c = \frac{a}{T} - b$
10	$T = \frac{a}{b-c}$	$a = T\cdot (b-c)$	$b = \frac{a}{T} + c$	$c = b - \frac{a}{T}$
11	$T = \frac{a}{b} + c$	$a = (T-c)\cdot b$	$b = \frac{a}{T-c}$	$c = T - \frac{a}{b}$
12	$T = \frac{a}{b} - c$	$a = (T+c)\cdot b$	$b = \frac{a}{T+c}$	$c = \frac{a}{b} - T$
13	$T = a\cdot b^{c}$	$a = \frac{T}{b^{c}}$	$b = \sqrt[c]{\frac{T}{a}}$	$c = \frac{\log(\frac{T}{a})}{\log b}$
14	$T = \frac{a}{b^{c}}$	$a = T\cdot b^{c}$	$b = \sqrt[c]{\frac{a}{T}}$	$c = \frac{\log(\frac{a}{T})}{\log b}$
15	$T = (a\cdot b)^{c}$	$a = \frac{\sqrt[c]{T}}{b}$	$b = \frac{\sqrt[c]{T}}{a}$	$c = \frac{\log T}{\log(a\cdot b)}$
16	$T = (\frac{a}{b})^{c}$	$a = \sqrt[c]{T}\cdot b$	$b = \frac{a}{\sqrt[c]{T}}$	$c = \frac{\log T}{\log(\frac{a}{b})}$
17	$T = (a + b)^{c}$	$a = \sqrt[c]{T} - b$	$b = \sqrt[c]{T} - a$	$c = \frac{\log T}{\log(a+b)}$
18	$T = (a - b)^{c}$	$a = \sqrt[c]{T} + b$	$b = a - \sqrt[c]{T}$	$c = \frac{\log T}{\log(a-b)}$
19	$T = a + b^{c}$	$a = T - b^{c}$	$b = \sqrt[c]{T-a}$	$c = \frac{\log(T-a)}{\log b}$
20	$T = a - b^{c}$	$a = T + b^{c}$	$b = \sqrt[c]{a-T}$	$c = \frac{\log(a-T)}{\log b}$

21	$T = a^{(b+c)}$		$a = \sqrt[(b+c)]{T}$	$b = \frac{\log T}{\log a} - c$	$c = \frac{\log T}{\log a} - b$
22	$T = a^{(b-c)}$		$a = \sqrt[(b-c)]{T}$	$b = \frac{\log T}{\log a} + c$	$c = b - \frac{\log T}{\log a}$

Homework

1. In the triangle ABC, C is 90, A = 42, c = 5. Find the rest.
2. In the triangle ABC, C is 90, A = 34, a = 6. Find the rest.
3. In the triangle ABC, C is 90, A = 28, b = 7. Find the rest.
4. In the triangle ABC, C is 90, a = 5, c = 7. Find the rest.
5. In the triangle ABC, C is 90, b = 4, c = 7. Find the rest.
6. In the triangle ABC, C is 90, a = 4, b = 5. Find the rest.
7. In the triangle ABC, A is 32.6, b = 4.6, c = 5.2. Find the rest.
8. In the triangle ABC, A is 34.8, b = 5.6, a = 7.2. Find the rest.
8. In the triangle ABC, A is 42.6, B = 74.6, c = 6.2. Find the rest.
10. In the triangle ABC, A is 34.8, C = 54.6, a = 5.2. Find the rest.

11. (all lin, exp & pow)

x	y
2	10
7	15
9	?
?	30

12

x	y
3	8
7	12
9	?
?	28

13

x	y
1	20
5	30
9	?
?	80

14

x	y
10	80
20	62
30	?
?	30

15

x	y
12	64
18	42
25	?
?	24

16

x	y
3	50
12	28
20	?
?	10

17. In 1993 there was 420 $. In 1998 there was 630 $. In 2005 there was ? $. In ? there was 950 $. Linear and exponential and power change.
18. In 1994 there was 520 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 1250 $. Linear and exponential and power change.
19. In 1992 there was 920 $. In 1996 there was 730 $. In 2005 there was ? $. In ? there was 450 $. Linear and exponential and power change.
20. In 1994 there was 720 $. In 1998 there was 630 $. In 2004 there was ? $. In ? there was 250 $. Linear and exponential and power change.
21. A capital had 753 $. increased with 20% 4 times and became ? $. What is the doubling-time?
22. A capital had 956 $. decreased with 25% 5 times and became ? $. What is the half-time?
23. A capital had 486 $. increased with 30% ? times and became 2345.83 $. What is the doubling-time?
24. A capital had 324 $. decreased with 35% ? times and became 25.88 $. What is the half-time?
25. A capital had 743 $. increased with ?% 4 times and became 2854.32 $. What is the doubling-time?
26. A capital had 896 $. decreased with ?% 5 times and became 45.09 $. What is the half-time?
27. A capital had ? $. increased with 50% 6 times and became 2423.83 $. What is the doubling-time?
28. A capital had ? $. decreased with 55% 7 times and became 2.45 $. What is the half-time?

31. (Polynomial regr.)

x	y
2	10
7	30
9	35
12	?
?	30
?	turn

32

x	y
3	8
7	5
11	12
9	?
?	28
?	turn

33

x	y
1	20
5	30
7	35
9	?
?	10
?	turn

34

x	y
10	60
20	120
30	30
40	70
50	?
?	80
?	turn

35

x	y
12	74
18	22
20	43
25	41
30	?
?	34
?	turn

36

x	y
3	9
12	28
15	8
17	14
20	?
?	10
?	turn

41. (Mean, ogive. & boxplot)

Obs	Freq
0-10	6
10-20	9
20-30	12
30-40	15
40-50	6

42

Obs	Freq
0-10	50
10-20	20
20-30	10
30-40	20
40-50	30

43

Obs	Freq
0-10	16
10-20	29
20-30	52
30-40	25
40-50	16

44

Obs	Freq
0-10	16
10-20	29
20-30	32
30-40	45
40-50	56
50-60	66

45

Obs	Freq
0-10	12
10-20	56
20-30	42
30-40	13
40-50	73
50-60	25

46

Obs	Freq
0-10	23
10-20	45
20-30	25
30-40	12
40-50	86
50-60	23
60-70	45

51. Solve the equation 2+3*(1+x)^4 = 20
52. Solve the equation 4+5*(1+x)^6 = 30
53. Solve the equation 40-3*(1-x)^4 = 20
54. Solve the equation 50-4*(1-x)^5 = 10
55. Transpose the equations T= d-e, $T = d - \frac{e}{f}$, $T = d - \frac{e-f}{g}$
56. Transpose the equations $T = \frac{d}{e}$, $T = \frac{d}{e} - f$, $T = \frac{d-e}{f} - g$

01. Project Forecasting

Problem: How to set up a forecast assuming constant growth?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real world	Problem		Solution

1. The real-world problem

A capital is assumed to grow constantly. From two data sets we would like to establish a forecast predicting the capital at a certain time and when a certain level will be reached.

2. The mathematical problem

We set up a table showing the capital to two different times. x are years, y is 1000 $

x	y = ?
2	10
5	30
8	?
?	60

Growth	Meaning
1. Linear ++ growth: y = a*x + b	x: +1, y: +a (gradient, slope)
2. Exponential +* growth: y = a*b^x = a*(1+r)^x	x: +1, y: + r% (interest rate, b = 1+r)
3. Power ** growth: y = a*x^b	x: +1%, y: + r% (elasticity)

3. Solving the mathematical problem

First we find the y-formulas using regression. We enter the table as lists L1 and L2 und STAT.

’LinReg Y1’ produces a linear model transferred to the y-list as Y1

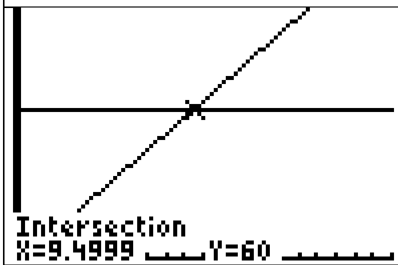
’ExpReg Y1’ produces an exponential model transferred to the y-list as Y1

’PowerReg Y1’ produces a power model transferred to the y-list as Y1

Linear growth

y = ?	y = 6.667*x – 3.333
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 6.667*8 – 3.333 = 50
Test	Trace x = 8 gives y = 50

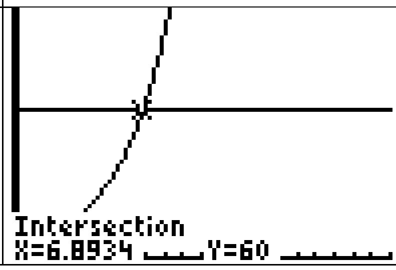
x = ?	y = 6.667*x – 3.333
y = 60	60 = (6.667*x)– 3.333 60 + 3.333 = 6.667*x 63.333/6.667 = x 9.5 = x
Test1	60 = 6.667*9.5– 3.333 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 9.5
Test3	CALC Intersection with y2=60 gives x = 9.5



Exponential growth

y = ?	y = 4.807 * 1.442^x
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.807 * 1.442^8 = 89.9
Test	Trace x = 8 gives y = 89.9

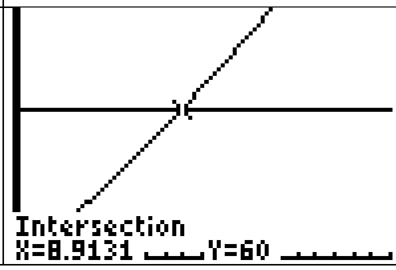
x = ?	y = 4.807 * 1.442^x
y = 60	60 = 4.807 * (1.442^x) 60/4.807 = 1.442^x log(60/4.807)/log1.142 = x 6.89 = x
Test1	60 = 4.807 * 1.442^6.89 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 6.89
Test3	CALC Intersection with y2=60 gives x = 6.89



Power growth

y = ?	y = 4.356*x^1.199
Test Trace	x = 2 and 5 gives y = 10 and 30
x = 8	y = 4.356*8^1.199 = 52.7
Test	Trace x = 8 gives y = 52.7

x = ?	y = 4.356*x^1.199
y = 60	60 = 4.356*(x^1.199) 60/4.356 = x^1.199 1.199√(60/4.356) = x 8.91 = x
Test1	60 = 4.356*8.91^1.199 60 = 60
Test2	MathSolver 0 = Y1-60 Gives x = 8.91
Test3	CALC Intersection with y2=60 gives x = 8.91



4. Solving the real-world problem

We see that forecast can be made by using technology’s regression lines. The forecasts give different answers to the same questions since different forms of growth is assumed.

Linear growth assumes that the gradient is constant.

Exponential growth assumes that the interest rate is constant.

Power growth assumes that the elasticity is constant.

02. Project Population and Food Growth

Problem: When will the population exceed food supply

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

Around the year 1800, the English economist Malthus (1766-1834) predicted a future food crisis, "since the world's population grows exponentially and food supply linearly, the population will one day overtake the food supply with famine to follow" (Malthus ' principle of population). Is Malthus right?

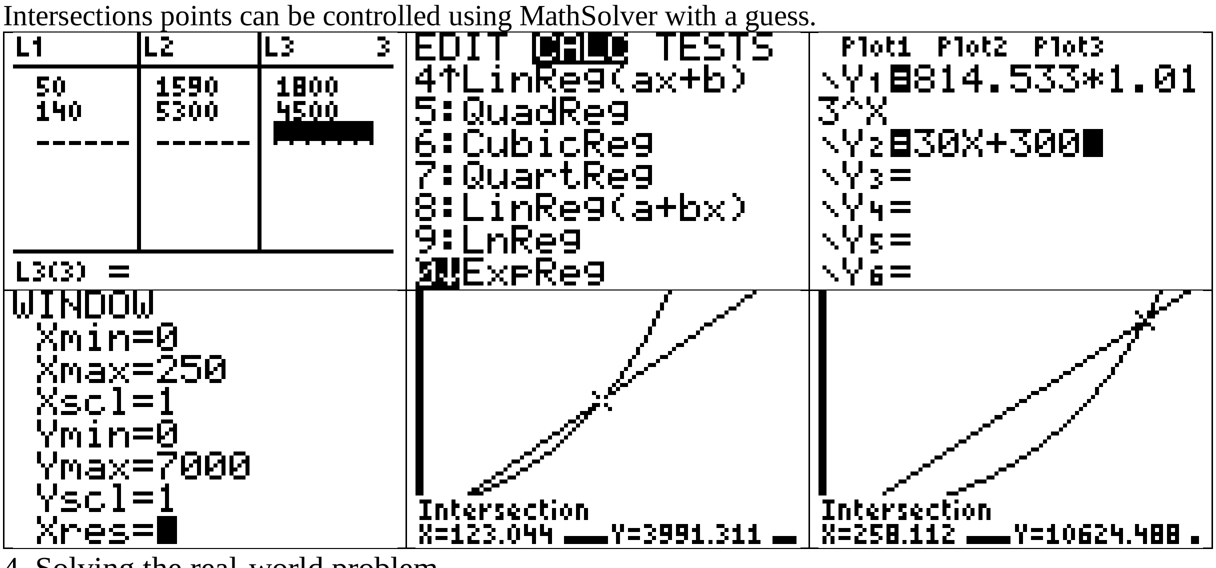
2. The mathematical problem

We set up a table of time x as the number of years after 1850; and the world's population, which is assumed to be 1.59 billion in 1900 and 5.3 billion in 1990; and world food production, which is assumed to be 1,800 billion daily rations in 1900 and 4.5 billion daily rations in 1990. The population is assumed to grow exponentially, and food quantity is assumed to grow linearly. The table scope is assumed to be 0 < x < 250.

x Years after 1850	Y1 World population in mio.	Y2 World food supply in mio. in daily rations
(1900) 50	1590	1800
(1990) 140	5300	4500

3. Solving the mathematical problem

On a TI-82, the x-numbers are on list L1, and the y-numbers are on lists L2 and L3. The formulas for the population are determined by L1, L2, ExpReg Y1. The result is y1 = 815 * 1,013 ^ x. So, when x is 0 in 1850, the population is y = 815; and when x increases by 1, the population increases by 1.3% 1. The formula for the volume of food is determined by LinReg L1, L3, Y2. The result is y1 = 300 + 30 x. So, when x is 0 in the 1850, the food amount is y = 300; and when x increases by 1, the food increases by 30.
Famine occurs where Y1 is greater than Y2. Y1 and Y2 are the intersection found with 'Calc Intersection ' to around x = 30 and 123. So, as to the model there was famine from 1850 to 1880, and again after 1973.
Assume instead that the world population is growing by 1% a year, and the amount of food with 40 per year. Then the formulas become Y3 = 815 * 1.01^ x, and Y4 = 300 + 40 x. They will intersect at approximately x = 16 and x = 258. I.e. in this case, there was famine from 1850 to 1866, and again after 2108.
Intersections points can be controlled using MathSolver with a guess.



4. Solving the real-world problem

Malthus is right in saying that there will be famine if the world's population continues to grow exponentially, and the quantity of food continues to grow linearly, for a curved path will always outpace a straight. If food grows by 30 million daily rations per year, then famine will occur years 1973 if the world's population is growing by 1.3% per year, and in the year 2108 if the world's population is growing by 1% per year and food production grows by 40/year. However, it will be earlier if a part of the food is used for fueling cars instead.

03. Project Saving and Pension

Problem: How much pension will a saving provide?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

A saving comes from sending in a fixed amount each month to a bank. In the end, a saving can be used to drawing out a fixed amount each month. What is the relationship between the monthly saving input and the pension output?

2. The mathematical problem

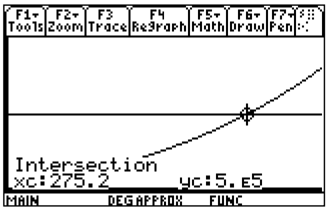
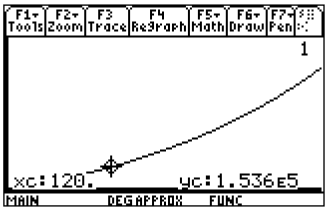
If paying $1000 monthly for 30 years, what will the monthly pension be for 10 years? The interest rate is 0.4% monthly.
By saving two formulas apply, the first applies to a single deposit, the second for many monthly deposits:
1) K = Ko*(1+R), 1+R = (1+r)^n, K/Ko: terminal/initial capital, r: monthly rate, R: total interest, n number of months.
2) K/a = R/r, K: terminal capital, a. monthly deposit, r: monthly rate, R: total interest,

3. Solving the mathematical problem

First we find the total interest rate per year R: 1+R = (1+r)^n = (1+0.004)^12, so R = 1.049 –1 = 0.049 = 4.9% per year.
Then we find the total interest rate for 30 years R: 1+R = (1+r)^n = (1+0.004)^(30*12) = 4.209. So. R = 4.209-1 = 3.209 = 321%.
The simple interest rate is 30*12*0.4% = 144%. So the effect of compound interest is 321% - 144% = 177%.
With a = 1000, r = 0.4% the saving after x deposits will be K = a*R/r = 1000*(1.004^x –1) / 0.004.
We find the saving after 10, 20 and 30 years:

Months	120	240	360
Saving	153632	401675	802147

The total deposit after 30 years is 1000*360 = 360000. The effect of compound interest is 802147 –360000 = 442147.
We observe that the saving is 500000 after 275 deposits.

x = ?	$\frac{1000*(1.004\text{^}x-1)}{0.004} = 500000$	Test 2 	x = 120, K = ? 
	$1.004\text{^}x - 1 = \frac{500000*0.004}{1000}$ $1.004\text{^}x = 2+1$ $x = \frac{\ln(3)}{\ln(1.004)} = 275.2$		
Test1	$\frac{1000*(1.004\text{^}275.2-1)}{0.004} = 500000$ 499994 = 500000		

To use the saving for a 10 years pension we use two accounts.
On the first, the saving grows from 10 years of interest to K = Ko (1 + R) = 802147 * (1 + 0.004) ^ 120 = 1295089.
The second is used for a 'negative saving' with a monthly redraw, a, that makes the two accounts balance after 10 years: K = a*R/r = Ko*(1+R), giving the equation a*(1,004^120 – 1)/0.004 = 1295089 that is solved by a = 8430.
Thus the relationship between output and input is (10*12*8430)/(30*12*1000) = 2.8.
Repeating the calculations with a monthly interest rate of 0.3% and 0.5% gives a different relationship:

Monthly %	Yearly %	Saving	Monthly pension	Relationship between output and input
0.3%	3.7%	646640	6425	(10*12*6425)/(30*12*1000) = 2.1
0.4%	4.9%	802147	8430	(10*12*8430)/(30*12*1000) = 2.8
0.5%	6.2%	1004515	11152	(10*12*11152)/(30*12*1000) = 3.7

4. Solving the real-world problem

When saving, an account grows from three monthly sources: a deposit, and an interest rate of the total deposit as well as of the total interest amount. When terminated, a saving continues to grow, but, used as a pension fund, the monthly input is replaced by a monthly output, the pension. Depositing $1000 monthly for 30 years allows taking out monthly $8430 in 10 years. With a monthly interest rate of respectively 0.3%, 0.4% and 0.5% the output-input ratio is respectively. 2.1, 2.8 and 3.7. However, it should be remembered that 40 years of inflation will reduce this factor.

Proof of the saving-formula: Account 1 contains the amount a/r. Each month we transfer the interest amount, r*a/r = a, to account 2, also receiving the monthly interest of its own amount. Account 2 thus will contain a saving K, but at the same time it contains total interest amount R of account 1, i.e. R*a/r. Therefore K = R*a/r , or K/a = R/r.

04. Project Supply, Demand and Market Price

Problem: How does supply and demand determine the market price?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

We assume we know supply curve and the demand for a given commodity, e.g. apples. That is, we know the market price determines supply and demand. If supply is larger than demand, a decrease in price should increase demand and decrease supply. If supply is less than demand, an increase in price should decrease demand and increase supply. The equilibrium price therefore should occur where supply equals demand.

2. The mathematical problem

We set up a table showing the relationship between price and demand and supply. The table is supposed to be valid for prices between 0 and 10, 0 < x < 10. Regression allows finding the two formulas, becoming an equation when set to be equal, thus finding the point of intersection that determines the equilibrium price.

Linear graphs

Price x	Supply S	Demand D
2	40	80
4	60	50

Bending graphs

Price x	Supply S	Demand D
2	40	80
4	60	50
6	75	30

3. Solving the mathematical problem

Entering a table to the data/matrix-editor of a graphic display calculator allows finding regression formulas.

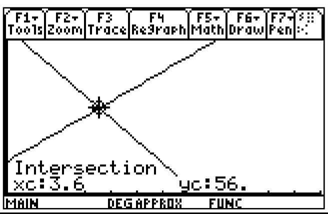
With 2 data-sets we choose LinREg, giving a degree 1 polynomial without bending.

With 3 data-sets we choose QuadREg, giving a degree 2 polynomial with bending.

The intersection point is found graphically, or algebraically by solving two equations with two unknowns.

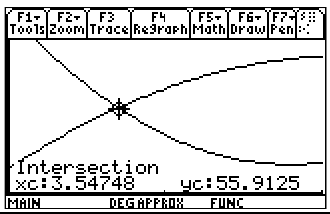
Degree one polynomial

x = ?	Supply = Demand
S = 10x+20 D = -15x+110	10x+20 = -15x+110 10x + 15x = 110-20 25x = 90 x = 90/25 = 3.6
Test1	y1(x) x=3.6 gives y=56 y2(x) x=3.6 gives y=56
Test2	Solve(y1(x)=y2(x),x) gives x=3.6
Test3	Graphical reading gives (x,y)=(3.6,56) (intersection)



Degree two polynomial

x = ?	Supply = Demand
S = -0.625x^2+13.75x+15 D = 1.25x^2-22.5x+120	-0.625x^2+13.75x+15 = 1.25x^2-22.5x+120 -1.875x^2+36.25x-105 = 0
Factorizing Zero rule	-1.875*(x-15.79)*(x-3.55) = 0 x = 15.79 and x = 3.55 15.79 is outside the validity area
Test1	y1(x) x=3.55 gives y=55.91 y2(x) x=3.55 gives y=55.91
Test2	Solve(y1(x)=y2(x),x) gives x=3.55
Test3	Graphical reading gives (x,y)=(3.55,55.91) (intersection)



4. Solving the real-world problem

We see that with linear supply and demand curves, the equilibrium price is $3.6 resulting in an equilibrium level at 56 units. And we see that with bending supply and demand curves, the equilibrium price is \$3.55 resulting in an equilibrium level at 55.9 units. The solution assumes that the tables are unchanged. If changed, the regression formulas will change accordingly, and so will the solutions.

05.Project Collection, Laffer-Curve

Problem: How Which ticket price will optimize a collection income?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

We want to collect a charity fund among the school's 500 students by selling tickets at a fixed price. Which of the following three collection models will provide the highest contribution?

A) No marketing. B) Marketing. C) Marketing and lottery.

2. The mathematical problem

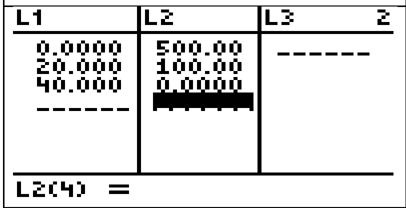
The demand Y1 will depend on the price x. The collected fund then will be Y2 = Y1*x.

3. Solving the mathematical problem

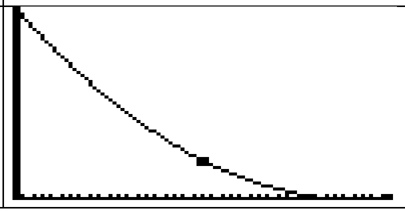
Model A. We assume that all 500 customers will buy a ticket at the price $0, that no one will provide over $40; and that demand is falling rapidly as only 100 customers will give $20.

Demand: 3 data-sets gives a degree2 polynomial. 'QuadReg' gives Y1 = .375*x^2 – 27.5*x + 500

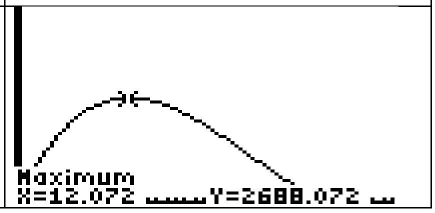
Demand table



Demand curve



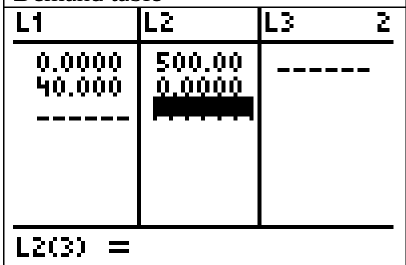
Laffer-income with a maximum



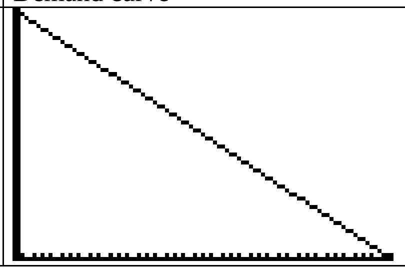
Test: Calc dy/dx ≈ 0 in x = 12.07. dy/dx = gradient, slope

Model B. We assume that with marketing, 500 will buy a ticket at $0, no one will give more than $40, demand is supposed to fall evenly. Demand: 2 data-sets gives a degree1 polynomial. 'LinReg' gives Y1 = -12.5*x + 500

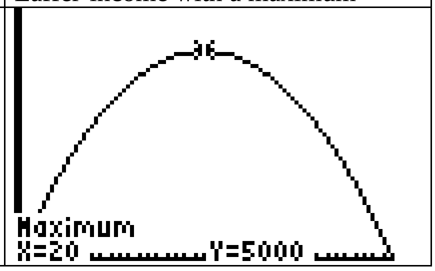
Demand table



Demand curve



Laffer-income with a maximum

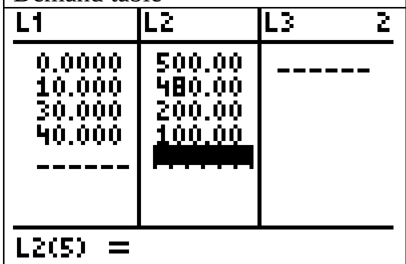


Test: Calc dy/dx ≈ 0 in x = 20. dy/dx = gradient, slope

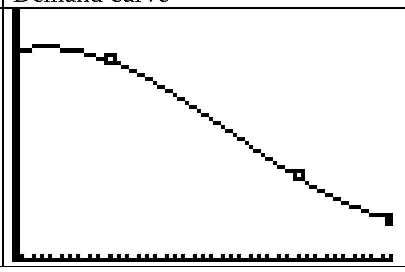
Model C. Here marketing includes a lottery with a Grand Prize $500 and 3 extra prizes of $200. We assume this will result in all 500 customers will buy a ticket at the price $0, 480 customers will give $10, 400 customers will give $20, 200 customers will give $30 and 100 customers will give $40.

Demand: 4 data-sets gives a degree3 polynomial. 'CubicReg' gives Y1 = 0.013*x^3 – 0.933*x^2 + 6*x +500

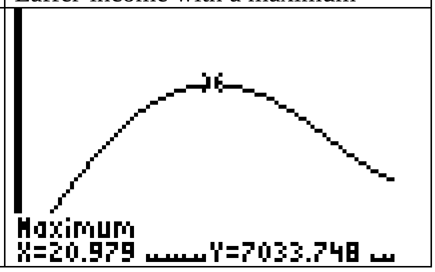
Demand table



Demand curve



Laffer-income with a maximum



Test: Calc dy/dx ≈ 0 in x = 21.28. dy/dx = gradient, slope

4. Solving the real-world problem

Collection without marketing will provide an income of $2688 at a ticket price of $12.
Marketing without lottery will give an income of $5000 at a price at of $20.
Marketing with lottery will provide an income of 7034 – (500 + 3 * 200) = 5934 at a price at $21.
The Laffer-curve is an argument for increasing income tax together with decreased tax percentage.
The demand curve tells then that with growing tax rate, undeclared work will also grow.

06. Project Linear Programming

Problem: How to optimize a product mix?

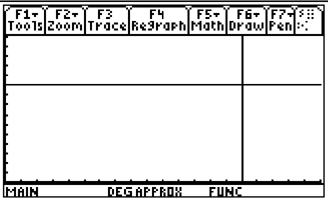
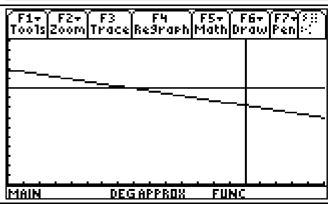
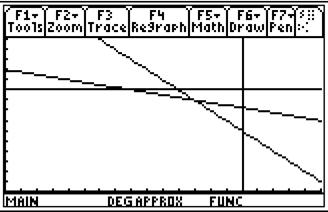
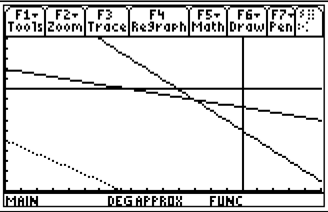
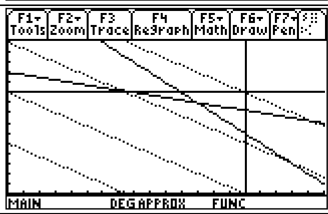
A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

A market booth sells water (max 15 boxes) and beer (max 10 boxes). The cost for water and beer is per box \$25 and \$100 respectively. A maximum of \$1200 DKK can be invested. At most 21 boxes can be sold in opening hours. Income is \$80/120 per box beer/water. How to optimize the income?

2. The mathematical problem

Outside world	Inside equations	Graphics
Boxes of water Boxes of beer	x y	y beer x water
Restriction on goods: The booth has place for a maximum of 15 boxes of water 10 boxes of beer	$0 \le x \le 15$ $0 \le y \le 10$	
Restriction on capital \$1200, given the cost: \$25 per box of water \$100 per box of beer	$25*x + 100*y \le 1200$ $(100*y \le -25*x + 1200$ $y \le -\frac{1}{4}*x + 12)$	
Restriction on labor: At most 21 boxes can be sold during opening hours.	$x + y \le 21$ $(y \le -x + 21)$	
Total income T is \$80/120 per box water/beer.	$T = 80*x + 120*y$ $y = -\frac{2}{3}*x + \frac{T}{120}$ N0: $T = 0$: $y = -2/3*x$ N600: $T = 600$: $y = -2/3*x + 5$	
Solution Buying 12 boxes of water and 9 boxes of beer will maximize the total income to \$2040.	'Solve($-\frac{1}{4}*x+12 = -x+21$,x)' gives x = 12 'y =-x + 21|x=12' gives y = 9 'T = 80*x + 120*y | x=12 and y = 9' gives T = 2040	

3. Solving the mathematical problem

Graphically, the restrictions give a polygon. The level-lines are since the income T only influences the intersection with the y-axis. A parallel translation of the level-lines across the polygon thus will increase or decrease T. So, the optimal value comes where a level-line leaves or is a tangent to the polygon, which will always happen in a corner. We can therefore predict the optimal situation by calculating all corner points (n equations with n unknowns), as well as T's values in these points (simplex method). This method is used when the number of variables is greater than 2.

4. Solving the real-world problem

We see that the maximum income will be \$2040 when12 boxes of water and 9 boxes of beer are sold. Furthermore, we see that the effective restrictions are the opening hours and the invested capital. Linear programming is used to optimize a particular quantity (to maximize profit, to minimize cost, etc.) within a series of restrictions on other quantities.

07. Project Game Theory

Problem: Which strategy will maximize my outcome?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

Two players A and B choose between different strategies. The outcome table shows what B must pay to A. The 2person game is called a ZeroSum game, since what A wins, B loses, and vice versa.

2. The mathematical problems

We look at two different games:

I

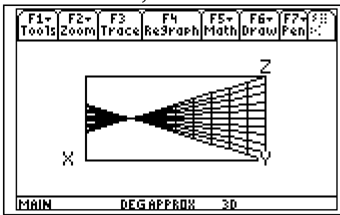
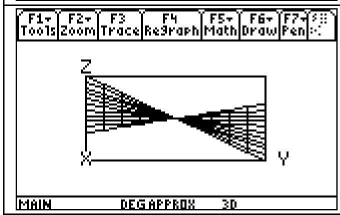
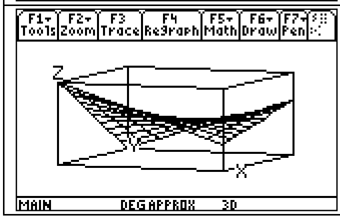
		B: b1	B: b2
A	a1	5	0
	a2	15	10

II

			B: *y* b1	B: *1-y* b2
A	*x*	a1	5	0
	1-x	a2	-5	10

3. Solving the mathematical problem

The two players analyze the game:
A: With a1 is I <u>risk</u> outcome 0, with a2 I risk 10. In order to maximize my minimum-number I should choose a2. And, if I was B, I would choose b2, so I choose a2.
B: With b1 is I <u>risk</u> paying 15, with b2 I risk 10. In order to minimize my maximum-pay I should choose a2. And, if I was A, I would choose a2, so I choose b2
The pair (a2, b2) is called the game **equilibrium** with the game value 10. No player will gain from not choosing the equilibrium strategy: A risks to get 0 instead of 10, so A sticks with the **maximin**-strategy. B risks paying 15 instead of 10, so B sticks with the **minimax**-strategy.

	'Solve (10*x-5 = -10*x+10, x)' gives x= 0.75
	'Solve (-15*y + 10 = 5*y, y)' gives y= 0.5
	Hold down the arrow for 1 second to watch the surface rotate

The two players analyze the game:
A: With a1 is I <u>risk</u> outcome 0, with a2 I risk -5. In order to maximize my minimum-number I should choose a1.
B: With b1 is I <u>risk</u> to pay 5, with b2 I risk 10. In order to minimize my maximum-pay I should choose b1.
But the maximin and minimax strategies a1 and b1 are not in equilibrium, since B gain by choosing b2, making A choose a2, making B choose b1, making A choose a1, etc.
Thus, with no equilibrium point, A should mix a1 and a2 in the ratio x to 1-x, randomly, and likewise B.
If B chooses b1 or b2, the outcomes will be respectively
P1 = 5*x – 5*(1-x) = 10*x – 5.
P2 = 0*x + 10*(1-x) = -10*x +10.
The two outcomes are like with 10*x – 5 = -10*x +10, or 20*x = 15, or x = 15/20 = 3/4 giving P = 10*3/4-5 = 2.5.
So by mixing the strategies a1 and a2 in the ration 3 to 1, player A will secure an average outcome 2.5 no matter what B chooses. So this game has the value 2.5.
Likewise we find that by mixing the strategies b1 and b2 in the ration 1 to 1, player B will secure an average loss 2.5 no matter what A chooses.
With A and B randomly mixing their strategies in the ratios x to 1-x, and y to 1-y respectively, the outcome will be
P = 5*x*y + 0*x*(1-y) – 5*(1-x)*y + 10*(1-x)*(1-y)
P = -10*x – 15*y + 20*x*y + 10
On a graphical display calculator P becomes a surface called a **saddle point**, going down one way, and up the other way.

4. Solving the real-world problem

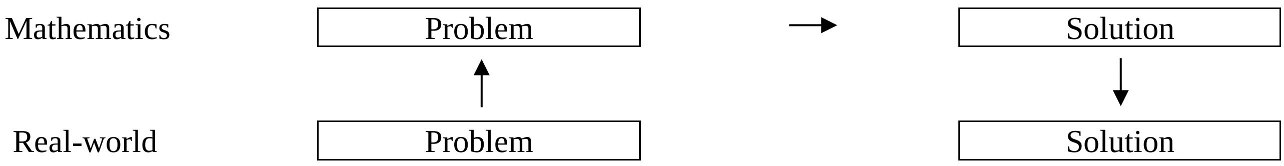
In game I, A and B should choose strategy a2 and b2 respectively resulting B losing 10 per game. In game II, the solution x =. 75, y = .5, P = 2.5 means that with a deck of cards, A should choose a2 when dragging clubs, otherwise a1. And that B should choose b2 when dragging black, otherwise b1. B will have an average loss at 2.5 per game.

We see that a 2person ZeroSum-game always has an equilibrium point in a saddle point going up to one side and down to the other. With the saddle point in a corner, this corner is the solution to the problem. With an inside saddle point, the strategies must be mixed randomly in a ration found by looking at the saddle point from the side.

08. Project Distance to a Far-away Point

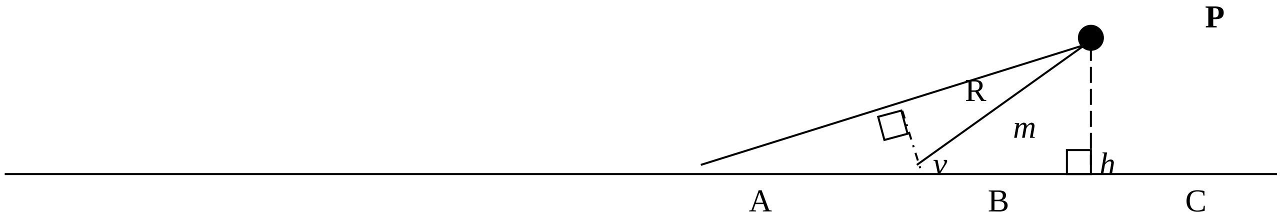
Problem: How to determine the distance to an inaccessible distant point?

A mathematical model



1. The real-world problem

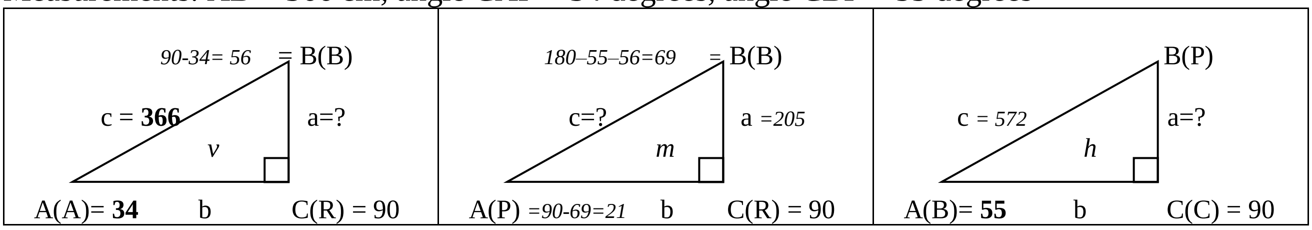
From a given baseline we want to determine the distance to a far-away inaccessible point P.



2. The mathematical problem

From a known baseline AB we measure the angles A and B to the inaccessible point P.
From the three right angled triangles ABR, BRP and BCP we calculate RB, BP as well as the distance PC.

Measurements: AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees



3. Solving the mathematical problem

We set up three formula tables

Triangle ABR

a = ?	$\sin A = \frac{a}{c}$
A = 34 c = 366	$\sin 34 = \frac{a}{366}$ sin34 * 366 = a 205 = a
Test1 ☺	$\sin 34 = \frac{205}{366}$ 0.559 = 0.560
Test2 ☺	Math Solver $0 = \frac{x}{366} - \sin 34$ gives x = 205

Triangle PBR

c = ?	$\sin A = \frac{a}{c}$
A = 21 a = 205	$\sin 21 = \frac{205}{c}$ c * sin21 = 205 $c = \frac{205}{\sin 21}$ c = 572
Test1 ☺	$\sin 21 = \frac{205}{572}$ 0.358 = 0.358
Test2 ☺	Math Solver $0 = \frac{205}{x} - \sin 21$ gives x = 572

Triangle PBC

a = ?	$\sin A = \frac{a}{c}$
A = 55 c = 572	$\sin 55 = \frac{a}{572}$ sin55 * 572 = a 469 = a
Test1 ☺	$\sin 55 = \frac{469}{572}$ 0.819 = 0.820
Test2 ☺	Math Solver $0 = \frac{x}{572} - \sin 55$ gives x = 469

4. Solving the real-world problem

Using trigonometry, we are able to determine the distance to the inaccessible point P to 469 cm.

09. Project Bridge

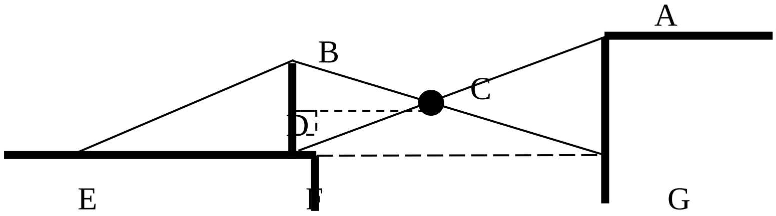
Problem: How to determine the dimensions of a bridge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

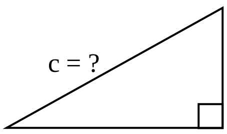
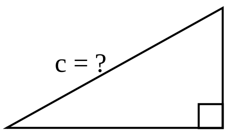
Over a canyon a suspension bridge made of steel is fastened to the cliff and to a vertical upright. We want to determine the length of the 3 beams as well as the welding point. The left fixing angle must be 30 degrees.



2. The mathematical problem

From the right-angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between the lines BG and FA.

Measurements: angle FEB = 30 degrees, FB = 3.5m, FG = 8m + 1m = 9m and AG = 5m.

 B(B) c = ? a = 3.5 A(E) = 30 b C(F) = 90	 B(A) c = ? a = 5 A(F) b = 8+1= 9 C(G) = 90	I a coordinate system with F as zero the following coordinates emerge: F: (0,0) and A: (9,5), as well as B: (0,3.5) and G: (9,0). Using linear regression, we determine the equations for the lines FA and BG.

3. Solving the mathematical problem

We set up formula tables

Triangle EFB

c = ?	$\sin A = \frac{a}{c}$
A = 30 a = 3.5	$\sin 30 = \frac{3.5}{c}$ sin30*c = 3.5 c = 3.5/sin30 c = 7.0
Test1 ☺	$\sin 30 = \frac{3.5}{7}$ 0.5 = 0.5
Test2 ☺	Math Solver $0 = \frac{3.5}{x} - \sin 30$ gives x = 7

Triangle FGA and GFB

c = ?	$a^2 + b^2 = c^2$
a = 5 b = 9	$5^2 + 9^2 = c^2$ $\sqrt{(106)} = c$ 10.30 = c
Test1 & Test2	
c = ?	$a^2 + b^2 = c^2$
a = 3.5 b = 9	$3.5^2 + 9^2 = c^2$ $\sqrt{(93.25)} = c$ 9.66 = c

Lines BG and FA

BG: ?	y = ax + b
Test	y = – 0.389x + 3.5 Found by LinReg L1, L2, Y1 Trace x=0 gives 3.5 Trace x=9 gives 0 StatPlot fits

Likewise we find

FA: ? y = 0.556x

Calc Intersection gives

x = 3.71 and y = 2.06

In the triangle FDC, DC = 3.71 and FD = 2.06

Pythagoras gives:

FC = √(3.71^2 + 2.06^2) = 4.24

In the triangle BDC, DC = 3.71 and

FD = 3.6-2.06 = 1.54

Pythagoras gives:

BC = √(3.71^2 + 1.54^2) = 4.02

4. Solving the real-world problem

Using trigonometry, we have found the lengths of the three steel beams as EB = 7.00 m, FA = 10.30 m and BG = 9.66
The welding point is determined by FC = 4.24 m and BC = 4.02 m.
As an extra control the bridge can be drawn and build by pipe cleaners in the ration 1:100.

10. Project Driving

Problem: How far and how did Peter drive?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

When driving, the velocity 100 km/t is 100*1000/(60*60) = 27.8 m/s. A camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s and 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

2. The mathematical problem

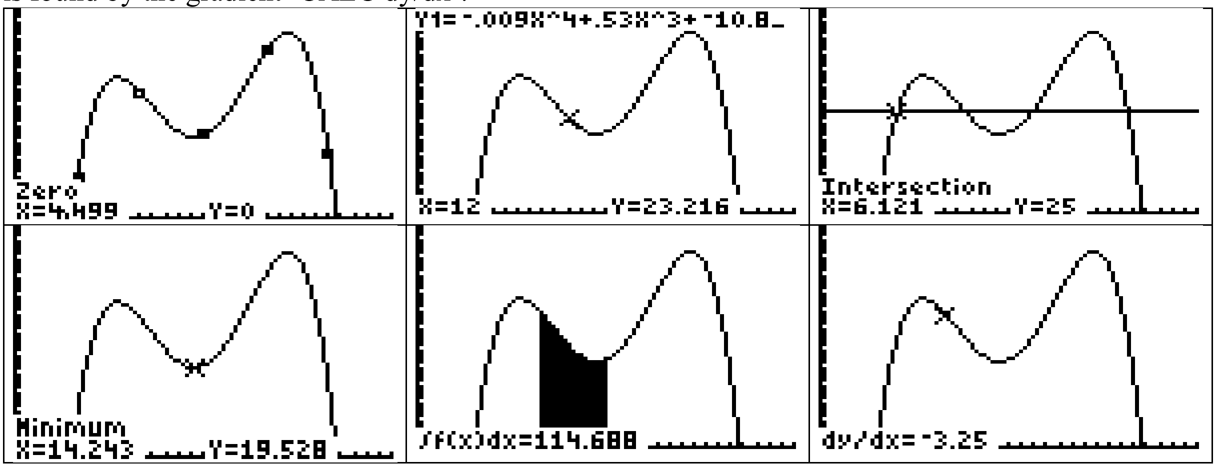
We set up a table showing time x and velocity y. The domain of the table is taken to be 0<x<30.

Time x sec	Velocity y m/s	Accel. dy/dx
5	10	?
10	30	?
15	20	?
20	40	?
25	15	?

3. Solving the mathematical problem

On TI-84 the table is entered as the lists L1 and L2. 5 data sets allow quartic regression (a 4. degree polynomial with a 3-fold parabola) providing the formula y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235 placed as Y1. No the question asked can be answered using formula tables, or using technology, i.e. graphical readings or calculations.

Starting and ending points are found using 'CALC Zero'. Y-numbers are found using 'TRACE'. X-numbers are found using 'CALC Intersection'. Maximum and minimum are found with 'CALC Maximum/Minimum'. The total meter-number is obtained by summing up the m/s*s = ∫Y1dx. Acceleration is found by the gradient 'CALC dy/dx'.



y = ?	y = y1
x=12	y = y1(12) = 3.667
Test	TRACE x = 12 gives y = 23.216

x = ?	y = y1
y = 25	MATH Solver 0 = y1 - 25 gives x =6.12 and …
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC intersection gives x = 6.12, 11.44, 16.86 and 24.47

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx = 0 at x = 5.5

4. Solving the real-world problem

The driving began after 4.50 sec. and ended after 25.62 sec. After 12 sec the velocity was 23.2 m/s. And it was 25m/s after 6.12 sec, 11.44 sec, 16.86 sec and 24.47 sec. Acceleration took place in the time-intervals (4.50; 8.19) and (14.24; 21.74). Deceleration in the intervals (8.19;14.24) and (21.74;25.62). Max-velocity was 44.28 m/s = 159 km/t. after 21.7 sec. In the time-intervals (5;10), (10;15), (15;20) and (20;25) the distance traveled was 142.8 m, 114.7 m, 142.8 m and 189.7 m. The acceleration in the beginning of these time-intervals were 17.75, -3.25, 1.25, 4.25, -21.25 m/s^2. The total distance traveled was 597.4 m.

11. Project Vine Box

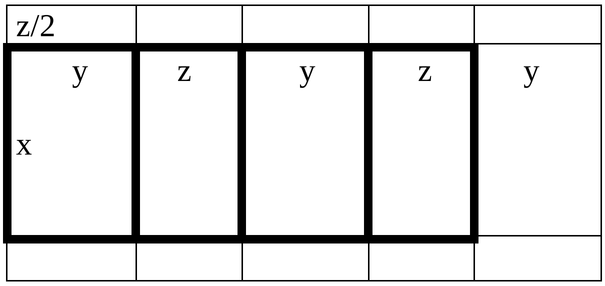
Problem: What are the dimensions of a 3 liters vine bag with the least surface area?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

Vine is sold in bottles or in boxes. A 3 liter bag will be constructed by cutting out a piece of cardboard.



2. The mathematical problem

The cardboard dimensions are called x, y & z all in dm. We express the volume V and the Surface S as formulas:

V = x*y*z = 3,   S = x*(3y+2z) + 2*z/2*(3y+2z)

3. Solving the mathematical problem

We expand the S-formula: S = x*(3y+2z) + 2*z/2*(3y+2z) = 3xy + 2xz + 3yz + 2z^2

We now insert z = 3/(x*y) so that S only depends on two variables x and y:

S = 3xy + 2xz + 3yz + 2z^2 and z = 3/(x*y) gives $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2}$

Scenario A. We assume that y should be half the length of x: y = 0.5*x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 1.5x^2 + \frac{21}{x} + \frac{72}{x^4}$, which gives $\frac{dS}{dx} = 3x - \frac{21}{x^2} - \frac{288}{x^5} = 0$ for x = 2.4

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

x = 2.4 and S = 19.56, so y = 0.5*x = 0.5*2.4 = 1.2, and z = 3/(2.4*1.2) = 1.0

Scenario B. We assume that y should be the same length of x: y = x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 3x^2 + \frac{15}{x} + \frac{18}{x^4}$, which gives $\frac{dS}{dx} = 6x - \frac{15}{x^2} - \frac{72}{x^5} = 0$ for x = 1.7

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

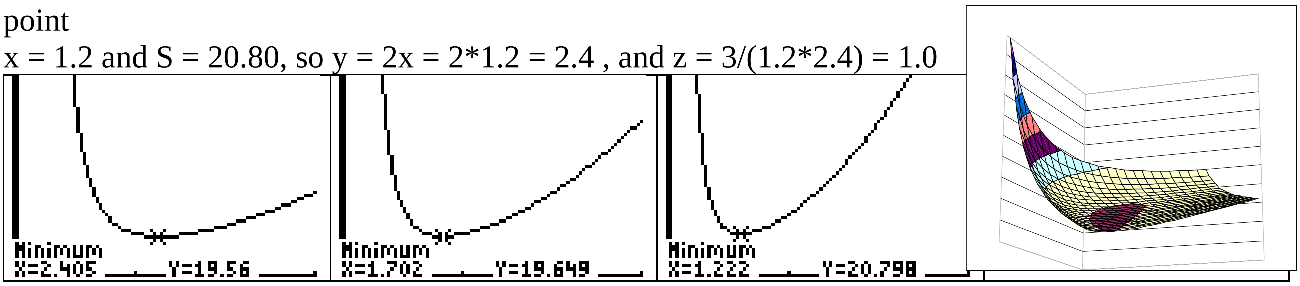
x = 1.7 and S = 19.65, so y = x = 1.7, and z = 3/(1.7*1.7) = 1.0

Scenario C. We assume that y should be double the length of x: y = 2*x. This restriction is inserted:

$S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x^2 y^2} = 6x^2 + \frac{12}{x} + \frac{4.5}{x^4}$, which gives $\frac{dS}{dx} = 12x - \frac{12}{x^2} - \frac{18}{x^5} = 0$ for x = 1.2

Graphing this S-formula in a window with Domain =]0,5] and Range =]0, 100] gives the minimum point

x = 1.2 and S = 20.80, so y = 2x = 2*1.2 = 2.4 , and z = 3/(1.2*2.4) = 1.0



Scenario A Scenario B Scenario C 3 dim. in Excel

4. Solving the real-world problem

We see that the minimum surface area is a little above 19 dm^2. Using an Excel-spreadsheet we can find the optimal solution to be x = 2.1 and y = 1.4 and z = 1.0, giving a minimum surface area at 19.47 dm^3.

(Graphing $S = 3xy + \frac{9}{x} + \frac{6}{y} + \frac{18}{x\hat{}2*y\hat{}2}$ does not give a curve but a surface as shown on the Excel-pict.)

12. Project Golf

Problem: How to hit a golf hole behind a hedge?

A mathematical model

Mathematics	Problem	→	Solution
	↑		↓
Real-world	Problem		Solution

1. The real-world problem

From a position on a 2 meter high flat hill we want to send a golf ball over a 3 meter hedge 2 meter away on the hill to hit a hole situated 12 meters away at level zero.
What is the orbit of the ball? How high is the ball at the distance 10 meters? When does the ball have a height of 6 meters? How high does the ball go? What is the direction of the ball in the beginning, at 10 meters distance and at the impact?

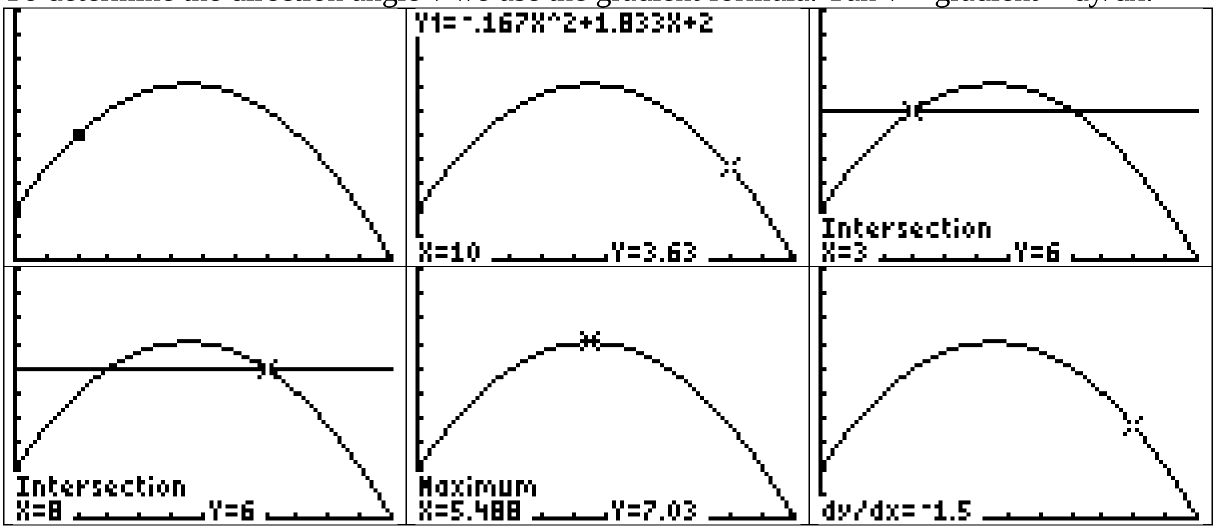
2. The mathematical problem

We set up a table with the length x and the height y having the domain 0<x<12.

Length x	Height y	Direction v
0	2	?
2	5	
12	0	?
10	?	?
?	6	

3. Solving the mathematical problem

We insert the table as lists L1 and L2. Three data-sets allows a 2nd degree polynomial, quadratic regression, which produces the formula y= -0.167x^2 + 1.833x + 2, which is transferred to Y1. Now the questions asked can be answered using formula tables and a calculator for graphing or calculating. The Y-number can be found by CALC Value, the x-number by CALC Intersection, the maximum by CALC Maximum, and the gradient by CALC dy/dx.
To determine the direction angle v we use the gradient formula: Tan v = gradient = dy/dx.



y = ?	y = y1
x=10	y = y1(10) = 3.667
Test	Trace x=10 gives y=3.67

x = ?	y = y1
y = 6	Math solver 0 = y1 - 6 gives x =3 & x = 8
Test1	y1(3) = 6, y1(8)= 6
Test2	CALC Intersection gives x = 3 and 8

ymax=?	y = y1
	Calc maximum gives y = 7.042 at x = 5.5
Test	dy/dx ≈ 0 at x = 5.5

v = ?	tan v = dy/dx
x = 0	tan v = 1.833 v = tan-1(1.833) v = 61.4

v = ?	tan v = dy/dx
x = 12	tan v = -2.167 v=tan-1(-2.167) v =- 65.2

v = ?	tan v = dy/dx
x =10	tan v = -1.5 v = tan-1(-1.5) v = -56.3

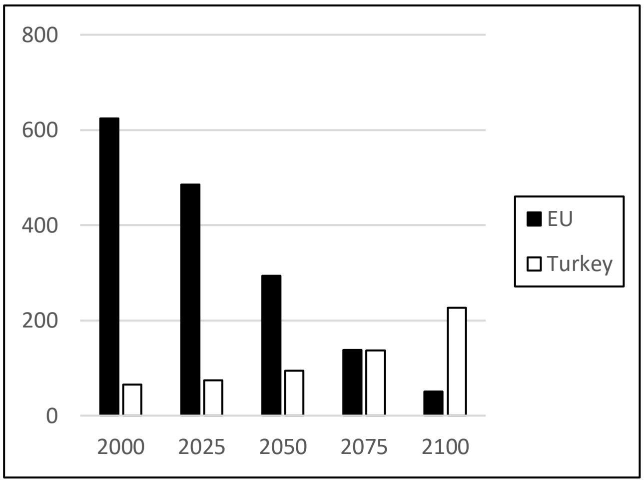
4. Solving the real-world problem

The orbit of the ball is a parabola. The height of the ball at the distance 10 meters is 3.67 meters? At the distances 3 meters and 8 meters the ball has a height of 6 meters. The ball goes to the maximum height 7.04 meters? The direction of the ball in the beginning, at 10 meters distance and at the impact are 61.4 grader, -65.2 grader and -56.3 grader.

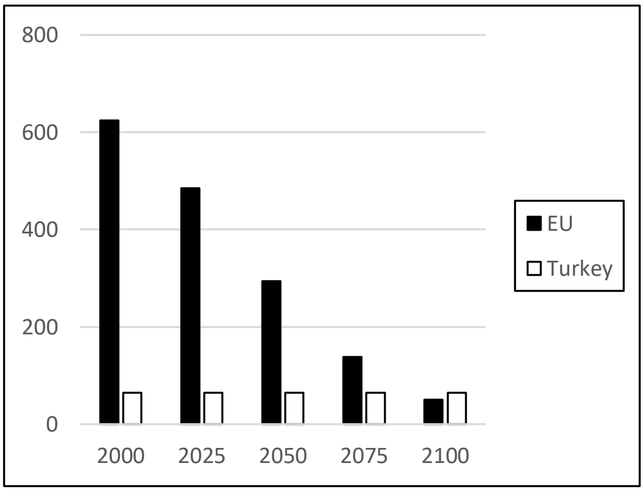
13. Project Population Forecast
Problem:

From year 2000, find a 100year forecast for the populations in the EU and Turkey, building on the assumption that the yearly change percent in the EU is -1% if on average each woman gives birth to 1.5 child.
Please, construct three scenarios: ,
1. In Turkey, the yearly change percent in Turkey is 0,5% if on average each woman gives birth to 2.2 child
2. In Turkey, the yearly change percent in Turkey is 0% if production causes women to have fewer children
3. In Turkey, the yearly change percent in Turkey is 1% if religion causes women to give birth to 4.4 children

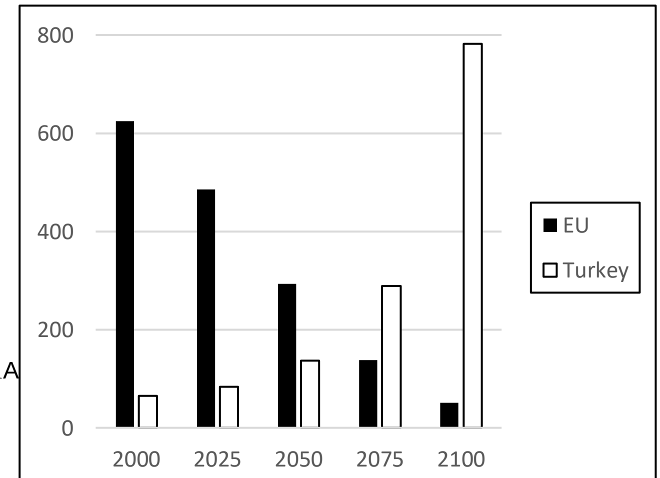
Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	74	94	137	226



Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	65	65	65	65



Year	2000	2025	2050	2075	2100
EU	624	485	294	138	51
Turkey	65	83	137	289	782



Answer: In all three scenarios Turkey will exceed EU in year 2100

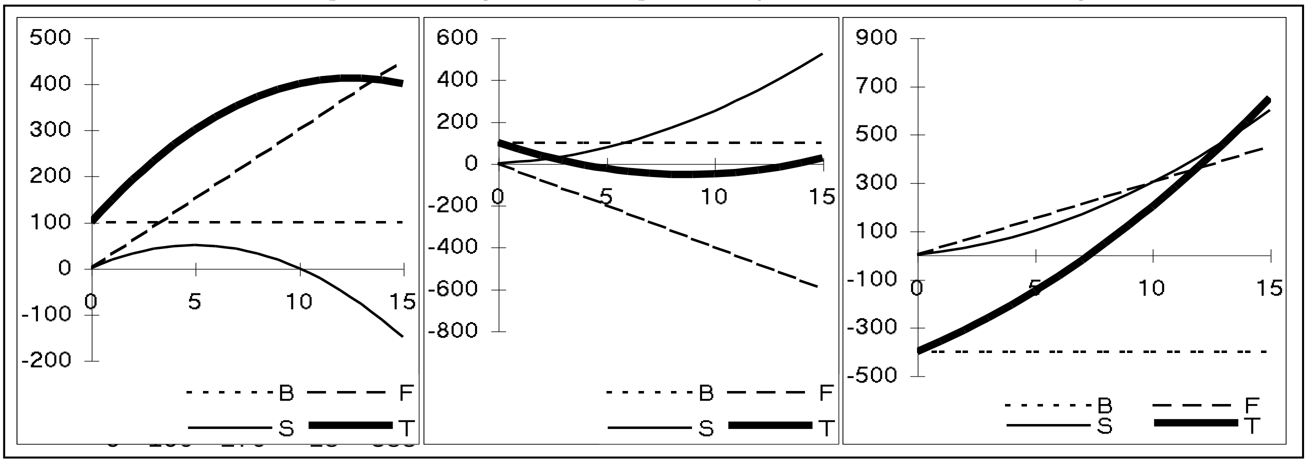
14. Project Family Firm

A family firm builds its total capital T through income from three generations. The grandfather has retired and has left the capital To \$. The father has established a routine which earns b \$/day. The son has just come back from the university, where he has learned a new technology enabling him to slowly raise the daily income d\$/day to s = so + d*n. The total capital after n days, since the family is calculated as a sum of polynomials:

Grandfather B	To	degree 0 polynomial, a constant
Father F	b*n	degree 1 polynomial, a line
Son S	s*n = (so + ½*d*n)* n = so*n + ½*d*n^2	degree 2 polynomial, a bended line
Total T	T = To + (b+so)*n + d*n^2	degree 2 polynomial, a parabola

Gatherers and Spreaders

Some families have both a spreader and a gatherer. The spreader may be the son, the father or the grandfather



Spreader: Son Father Grandfather

Pricing the tea

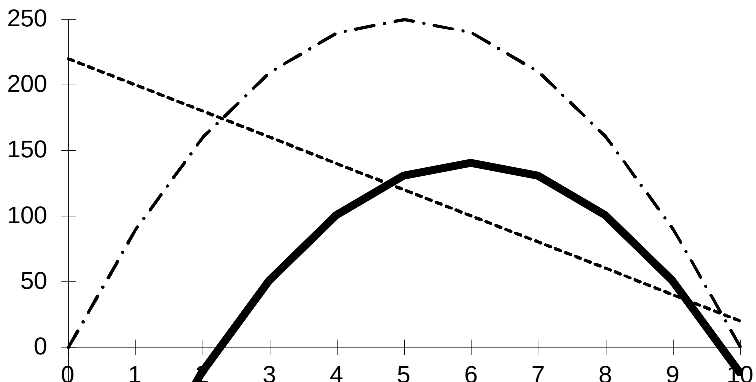
In the family firm they discuss if increasing the unit price by necessity will make the sale decrease.

Grandfather: The sale will decrease with an increasing unit price. I believe in a linear relation y = a + b*x found from the table	Father: The sale will decrease slower at high unit prices. I believe in a degree 2 polynomial y = a + b*x + c*x^2 found from the table	Son: The sale will decrease more at low and high unit prices. I believe in a degree 3 polynomial y = a + b*x + c*x^2+ d*x^3 found from the table
price x / sale y: 0 / 100; 10 / 0	price x / sale y: 0 / 100; 5 / 80; 10 / 0	price x / sale y: 0 / 100; 2 / 60; 8 / 40; 10 / 0

The grandfather scenario

The sale will be y = 100-10*x found by regression. The total income T is T = unit price*sale = x*y = x*(100-10 x) = 100*x-10*x ^ 2, a degree 2 polynomial. The cost C to produce y units consists of a fixed cost co = 20 and a variable unit-cost m = 2. So, C = co + m*y = = 20 + 2*y = 20 + 2*(100–10*x) = 20 + 200–20*x = 220–20*x. The profit P will be when P = T – C = (100*x–10*x^2) – (220–20*x) = -220 + 80x – 10*x^2, i.e. again a degree 2 polynomial.

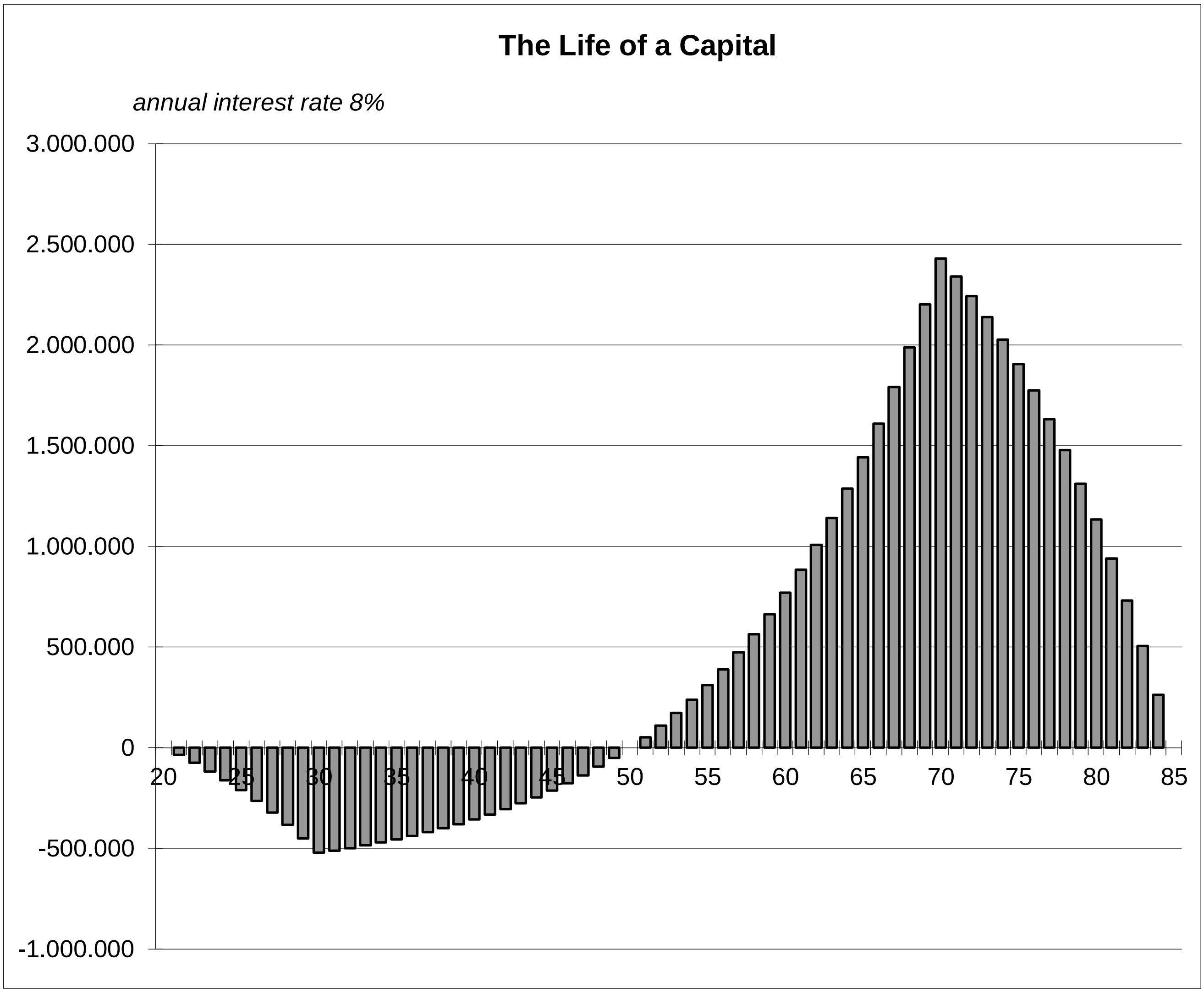
price	sale	income	cost	profit
0	100	0	220	-220
1	90	90	200	-110
2	80	160	180	-20
3	70	210	160	50
4	60	240	140	100
5	50	250	120	130
6	40	240	100	140
7	30	210	80	130
8	20	160	60	100
9	10	90	40	50



Exercises
1. Set up the father's scenario
2. Set up the son's scenario

Project 15. The Life of a Capital

During the lifetime, a person's capital will change. In the following example, life is divided into four periods:
1) 20-30 years: student loans at $36,000 per year for 10 years.
2) 30-50 years: debt settlement, where the student loan is settled by paying $53,000 per year for 20 years.
3) 50-70 years: building a capital by continuing depositing $53,000 per year for 20 years
4) 70-85 years: pension, where assets are settled by paying $284,000 per year for 15 years.
The example assumes an interest rate at 8% per year. Other examples may be set up with different interest rates.



Compound interest

A constant monthly interest rate will doubling-time on T months. Counting in doublings-times, we look at an account A0 receiving 1 unit at time 0. At time 1, the interest of A0 is transferred to account A1. At time 2, the interest of A0 is transferred to account A1, and the interest of A1 is transferred to account A2, etc. What do you observe?

A10											

A9											
A8											
A7											
A6											
A5											
A4					1						
A3				1	1+3 = 4						
A2			1	1+2 = 3	3+3 = 6						
A1		1	1+1 = 2	2+1 = 3	3+1 = 4						
A0	1	1	1	1	1						
Time	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Sum	1	2	4	8	16						

Revision Problems Using TI-84

<table>
<tr><td>1.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>?</td></tr><tr><td>?</td><td>40</td></tr></table></td><td>Answer the question marks in case of a linear model.
Answer the question marks in case of an exponential model.
What is the doubling time?
Answer the question marks in case of a power model.</td></tr>
<tr><td>2.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>18</td></tr><tr><td>15</td><td>?</td></tr><tr><td>?</td><td>10</td></tr></table></td><td>Answer the question marks in case of a quadratic model.
Find maxima or minima.
Find the equation for the tangent line in x=2.
Find the gradient formula.
Find the gradient number in x = 5
Find the area formula
Find the area number from x= 1 to x = 6
Find the intersection points with the line y = 3 + 2x</td></tr>
<tr><td>3.</td><td><table><tr><td>x</td><td>y = ?</td></tr><tr><td>3</td><td>12</td></tr><tr><td>7</td><td>16</td></tr><tr><td>10</td><td>14</td></tr><tr><td>12</td><td>18</td></tr><tr><td>15</td><td>?</td></tr><tr><td>?</td><td>30</td></tr></table></td><td>Answer the question marks in case of a cubic model.
Find maxima and minima.
Find the equation for the tangent line in x=2.
Find the gradient formula.
Find the gradient number in x = 5
Find the area formula
Find the area number from x= 1 to x = 6
Find the intersection points with the line y = 3 + 2x</td></tr>
<tr><td>4.</td><td>3x + 4y = 15 & 5x – 6y = 12</td><td>Solve the simultaneous equations</td></tr>
<tr><td>5.</td><td>Given two points in a coordinate system P(2,4) and Q(6,10)</td><td>Find the midpoint of the line PQ.
Find the equation for the line through P and Q
Find the equation for the normal line to PQ passing through P
Find the angle between PQ and the x-axis.
Find the distance between P and Q
Find the distance from the line PQ to the point S(8,1)
Find the equation for the circle through P and Q and with the midpoint of PQ as center.
Find the intersection point between the circle and the line y = 12-2x</td></tr>
<tr><td>6.</td><td>Let X be a normal random variable with mean m = 100 and standard deviation d = 12</td><td>$P(X<89) = ?$
$P(X>108) = ?$
$P(93<X<109) = ?$</td></tr>
<tr><td>7.</td><td>X counts the numbers of wins in 100 repetitions of a game with 65% winning chance.</td><td>$P(X<70) = ?$ $P(X\leq 60) = ?$
$P(X\geq 58) = ?$ $P(63<X\leq 72) = ?$</td></tr>
<tr><td>8.</td><td>sin(3x) = 0.4, $0\leq x\leq 2\pi$
con(½x) = -0.3, $0\leq x\leq 2\pi$
tan(2x) = 0.7, $0\leq x\leq 2\pi$</td><td>Find the solutions: Remember to adjust the window
Find the solutions:
Find the solutions:</td></tr>
<tr><td>9.</td><td>A = 40, b = 7, C = 90</td><td>Find a, B and c.</td></tr>
<tr><td>10.</td><td>a = 4, c = 7, C = 90</td><td>Find A, B and b.</td></tr>
<tr><td>11.</td><td>A = 40, b = 7, C = 68</td><td>Find a, B and c.</td></tr>
<tr><td>12.</td><td>A = 40, b = 7, c = 6.8</td><td>Find a, B and C.</td></tr>
</table>

13.	A = 40, b = 7, a = 6.2	Find c, B and C.
14.	a = 4, b = 7, c = 6.8	Find A, B and C.
15.	$T = \frac{d}{e-f} + g$	Transpose the T-formula to a d-, e-, f-, and g-formula
16.	The capital 785 increased with 2.7% 5 times and became ?	Find the answer Find the corresponding doubling time.
17.	The capital 785 increased with 2.7% ? times and became 980	Find the answer Find the corresponding doubling time.
18.	The capital 785 increased with ?% 5 times and became 980	Find the answer Find the corresponding doubling time.
19.	-21	As 16-18, but with $ instead of %

Problem 1. Linear model

Equation:	y=ax+b
	y=x+9, found by Stat, Calc, LinReg
Test	y1(3) = 12 ☺

y=?	y=x+9
x=10	y=19 found by y1(10)
Test	y=19 found by CalcValue ☺

x=?	y=x+9
y=40	x=31, found by Math, Solver 0=y1-40
Test	y1(31) = 40 ☺

Exponential model

Equation:	y=a*b^x
	y=9.671*1.075^x, found by Stat, Calc, ExpReg
Test	y1(3) = 12 ☺

y=?	y=9.671*1.075^x
x=10	y=19.853 found by y1(10)
Test	y=19.853 found by CalcValue ☺

x=?	y=9.671*1.075^x
y=40	x=19.740, found by Math, Solver 0=y1-40
Test	y1(19.740) = 40 ☺

Doubling time T = log2/logb = log2/log1.075 = 9.6

Power model

Equation:	y=a*x^b
	y=8.264*x^0.340 found by Stat, Calc, PwrReg
Test	y1(3) = 12 ☺

y=?	y=8.264*x^0.340
x=10	y=18.060 found by y1(10)
Test	y=18.060 found by CalcValue ☺

x=?	y=8.264*x^0.340
y=40	x=104.024 found by Math, Solver 0=y1-40
Test	y1(104.024) = 40 ☺

Problem 2. Quadratic model

Equation:	y=a*x^2+b*x+c
	y=-0.048x^2+1.476x+8 found by Stat, Calc, QuadReg
Test	y1(3) = 12 ☺

y=?	y=-0.048x^2+1.476x+8
x=15	y=19.429 found by y1(15)
Test	y=19.429 found by Graph, Calc, Value ☺

x=?	y=-0.048x^2+1.476x+8
y=10	x=1.420 or 29.580 found by Math, Solver 0=y1-10
Test	y1(1.420) = 10 y1(29.580) = 10 ☺

Maximum:	y=-0.048x^2+1.476x+8
	(x,y) = (15.500,19.140) found by Graph, Calc, Maximum
Test	dy/dx = 0 for x = 15.5 y1(15.5)= 19.14 ☺

Tangent in x=2	y=-0.048x^2+1.476x+8
x=2	y=1.286x + 8.190 found by Graph, Draw, Tangent

Gradient formula	y=-0.048x^2+1.476x+8
	y'= -0.095*x + 1.476, found by TI89
Test	∫y'dx=-0.048x^2+1.476x found by TI89 ☺

Gradient number:	y=-0.048x^2+1.476x+8
x=5	dy/dx = 1 for x=5 found by Graph, Calc, dy/dx
Test	1, found by Math, nDeriv ☺

Area formula:	y=-0.048x^2+1.476x+8
x=2	∫ydx = -0.016*x^3 + 0.738*x^2 + 8.000*x found by TI89
Test	d(∫ydx)/dx = -0.048x^2+1.476x+8 found by TI89 ☺

Area number:	y=-0.048x^2+1.476x+8
	$\int_1^6 ydx$ = 62.421, found by Graph, Calc, ∫f(x)dx
Test	62.421, found by Math, fnInt ☺

Intersection points	y = -0.048x^2+1.476x+8 and y = 3+2x (y1 = y3)
	(x,y) = (-17.130,-31.260) and (x,y) = (6.130, 15. 260), found by Math, Solver 0=y1-y3 and y1(-17.130) = -31.260 etc.

Test	tested by Graph, Calc, Intersect ☺

Problem 3. Cubic model

Equation:	y=a*x^3+b*x^2+c*x+d
	y=0.086x^3-1.952x^2 +13.752x-14, found by Stat, Calc, CubicReg
Test	y1(3) = 12 ☺

y=?	y=0.086x^3-1.952* x^2 +13.752x-14
x=15	y=42.286 found by y(15)
Test	y=42.286 found by Graph, Calc, Value ☺

x=?	y=0.086x^3-1.952x^2 +13.752x-14
y=30	x=13.885 found by Math, Solver 0=y1-30
Test	y1(13.885) = 30 ☺

Maximum Minimum:	y=0.086x^3-1.952x^2+13.752x-14
	Max: (x,y) = (5.552, 16.841) found by Graph, Calc, Maximum Min: (x,y) = (9.634, 13.925) found by Graph, Calc, Minimum
Test	dy/dx = 0 for x = 5.552 y1(5.552)= 16.841 dy/dx = 0 for x = 9.643 y1(9.643)= 13.925 ☺

Tangent in x=2	y=0.086x^3-1.952x^2+13.752x-14
x=2	y = 6.971x – 7.562 found by Graph, Draw, Tangent

Gradient formula	y=0.086x^3-1.952x^2+13.752x-14
	y’= 0.257*x^2 – 3.905*x + 13.752, found by TI89
Test	∫y’dx=0.086x^3-1.952x^2+13.752x found by TI89 ☺

Gradient number:	y=0.086x^3-1.952x^2+13.752x-14
x=5	y’(5) = 0.657 found by Graph, Calc, dy/dx
Test	0.657, 1 found by Math, nDeriv ☺

Area formula:	y=0.086x^3-1.952x^2+13.752x-14
x=2	∫ydx = 0.021*x^4 – 0.651*x^3+6.876*x^2+14*x found by TI89
Test	d(∫ydx)/dx = 0.086x^3-1.952x^2+ 13.752x-14 found by TI89 ☺

Area number:	y=0.086x^3-1.952x^2+13.752x-14
	$\int_1^6 y\,dx$ = 58.496, found by Graph, Calc, ∫f(x)dx
Test	58.496, 62.421 found by Math, fnInt ☺

Intersection points with y=3+2x: (x,y) = (2.129,-7.259) and (x,y) = (6.657, 16.315) and (x,y) = (13.991, 30.981) found by Math, Solver 0=y1-y3 , tested by Graph, Calc, Intersect.

Problem 4

Solutions: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix}$, found by A*B=C, B=A^-1*C, where A = $\begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix}$ and B = $\begin{pmatrix} x \\ y \end{pmatrix}$ and C = $\begin{pmatrix} 15 \\ 12 \end{pmatrix}$

Tested by A*B=C : A*B = $\begin{pmatrix} 3 & 4 \\ 5 & -6 \end{pmatrix} * \begin{pmatrix} 3.632 \\ 1.027 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$ = C ☺

Problem 5

Midpoint:	$(x,y) = \left(\frac{x1+x2}{2}, \frac{y1+y2}{2}\right)$
x1=2 x2=6 y1=4 y2=10	$(x,y) = \left(\frac{2+6}{2}, \frac{4+10}{2}\right)$ (x,y) = (4,7)
Test	Tested geometrically ☺

Gradient PQ:	$a = \frac{y2-y1}{x2-x1}$
x1=2 x2=6 y1=4 y2=10	$a = \frac{10-4}{6-2}$ a= 3/2 a = 1.5
Test	Tested geometrically ☺

Line PQ:	y = y1 + a*(x – x1)
a =1.5 x1=2 y1=4	y = 4 + 1.5*(x – 2) y = 1.5*x + 1
Test	Tested geometrically ☺

Gradient perpend.:	c*a = -1
a=3/2	c = -2/3 found by Math, Solver 0 = c*3/2+1
Test	Tested geometrically ☺

Normal:	y = y1 + a*(x – x1)
a=-2/3 x1=2 y1=4	y = 4 + -2/3*(x – 2) y = -2/3*x + 5.333
Test	Tested geometrically ☺

Distance PQ	$d = \sqrt{(x2-x1)^2 + (y2-y1)^2}$
x1=2 x2=6 y1=4 y2=10	$d = \sqrt{(6-2)^2 + (10-4)^2}$ d = 7.21
Test	Tested geometrically ☺

Distance point-line	$d = \frac{\lvert y1 - a*x1 - b\rvert}{\sqrt{1 + a^2}}$
a=1.5 b=1 x1=8 y1=1	$d = \frac{\lvert 1 - 1.5*8 - 1\rvert}{\sqrt{1 + 1.5^2}}$ d = 6.66
Test	Tested geometrically ☺

Circle equation	$(x-c1)^2 + (y - c2)^2 = r^2$
r=½*7.21 r=3.61 c1=4 c2=7	$(x -4)^2+(y - 7)^2 = 3.61^2$ $(x -4)^2+(y - 7)^2 = 13.03$
Test	Tested geometrically ☺

Intersection	$(x-c1)^2 + (y - c2)^2 = r^2$ and y = 12-2x
r =½*7.21 =3.61 c1 =4 c2 =7	(x,y) = (1.30,9.40) and (4.30,3.40) found by Math, Solver $0 = (x-4)^2 + (12-2x-7)^2 – 3.61^2$
Test	Tested geometrically ☺

Angle: tan(v) = a, a=3/2 ; v = 56.31 found by Math, Solver 0 = tanv - 3/2, v>0 and v<90. Tested geometrically ☺

Problem 6 / **Problem 7**

p(X<115) = 0.894, found by normalCdf(1EE-99,115,100,12)	p(X<70) = 0.827, found by binomCdf(100,0.65,0,69)
p(X<89) = 0.180, found by normalCdf(1EE-99,89,100,12)	p(X≤60) = 0.172, found by binomCdf(100,0.65,0,60)
p(X>108) = 0.253, found by normalCdf(108,1EE99,100,12)	p(X≥58) = 0.941, found by binomCdf(100,0.65,58,100)
p(93<X<109) = 0.494, found by normalCdf(93,109,100,12)	p(63<X≤72) = 0.571, found by binomCdf(100,0.65,64,72)

Problem 8

x=?	sin(3x) = 0.4
	x =0.137, or 0.910, or 2.232 or 3.004 or 4.326 or 5.099 found by Math, Solver 0=y1-0.4
Test	tested by Graph, Calc, Intersect ☺

x=?	cos(½x) = -0.3
	x =3.745 found by Math, Solver 0=y1+0.3
Test	tested by Graph, Calc, Intersect ☺

x=?	tan(2x) = 0.7
	x =0.305, or 1.876, or 3.447 or 5.018 found by Math, Solver 0=y1-0.7
Test	tested by Graph, Calc, Intersect ☺

Problem 9

a = ?	tan A = a/b
A =40 b=7	a = 5.874 found by Math, Solver 0=a/7-tan40
Test	tan40 = 5.874/7 0.839 = 0.839 ☺

c = ?	cos A = b/c
A =40 b=7	c = 9.138 found by Math, Solver 0=7/c-cos40
Test	cos40 = 7/9.138 0.766 = 0.766 ☺

B = ?	A + B = 90
A = 40	B = 50 found by Math, Solver 0=40+B-90
Test	50+40 = 90 90 = 90 ☺

Problem 10

b = ?	a^2 + b^2 = c^2
a =4 c=7	b = 5.745 found by Math, Solver 0=4^2+b^2-7^2
Test	4^2 + 5.745^2 = 7^2 49 = 49 ☺

A = ?	sinA = a/c
a =4 c=7	A =34.85 found by Math, Solver 0=4^2+b^2-7^2
Test	Sin34.85 = 4/7 0.571 = 0.571 ☺

B = ?	A + B = 90
A = 34.85	B = 55.15 found by Math, Solver 0=34.85+B-90
Test	34.85+55.15 = 90 90 = 90 ☺

Problem 11

B = ?	A+B+C=180
A =40 C =68	B = 72 found by Math, Solver 0=40+B+68-180
Test	40+72+68=180 180= 180 ☺

a = ?	a/sinA = b/sinB
A =40 B =72 b = 7	a =4.731 found by Math, Solver 0=a/sin40- 7/sin72
Test	4.731/sin40 = 7/sin72 7.360 = 7.360 ☺

c = ?	c/sinC = b/sinB
C =68 B =72 b = 7	c =6.824 Math, Solver 0=c/sin68 - 7/sin72
Test	6.824/sin68 = 7/sin72 7.360 = 7.360 ☺

Problem 12

a = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
A =40 c =6.8 b = 7	a =4.724 found by Math, Solver $0=a^2-6.8^2-7^2+2*6.8*7*\cos 40$
Test	$4.724^2=6.8^2+7^2–2*6.8*7*\cos 40$ 22.316= 22.316 ☺

B = ?	a/sinA = b/sinB
A =40 b = 7 a=4.724	B = 72.3 found by Math, Solver 0=4.724/sin40- 7/sinB
Test	4.724/sin40 = 7/sin72.3 7.348 = 7.348 ☺

C = ?	A+B+C=180
A =40 B =72.3	C = 67.7 found by Math, Solver 0 = 40+72.3+C-180
Test	40+72.3+67.7=180 180= 180 ☺

Problem 13

B = ?	a/sinA = b/sinB
A =40 a =6.2 b = 7	B = 46.53 or B = 133.47 found by Math, Solver 0 = 6.2/sin40 - 7/sinB
Test	6.2/sin40=7/sin46.53=7/sin133.47 9.645 = 9.645 = 9.645 ☺

C = ?	A+B+C=180
A =40 B =46.53 or B=133.47	C = 93.47or C = 6.53 found by Math, Solver 0 = 40+B+C-180
Test	40+46.53+93.47=180 180= 180 ☺

c = ?	a/sinA = c/sinC
A =40 a=6.2 C =93.47, C=6.53	c = 9.628 or C = 1.097 found by Math, Solver 0 = 6.2/sin40 - c/sinC
Test	6.2/sin40=9.628 /sin93.47=9.628 /sin6.53 9.645 = 9.645 = 9.645 ☺

Problem 14

A = ?	$a^2 = c^2+b^2–2*c*b*\cos A$
a = 4 c =6.8 b = 7	A =33.66 found by Math, Solver $0 = 4^2-6.8^2-7^2+ 2*6.8*7*\cos A$
Test	$4^2 =6.8^2+7^2–2*6.8*7*\cos 33.66$ 16 = 16 ☺

B = ?	$b^2 = a^2+c^2–2*a*c*\cos B$
a = 4 c = 6.8 b = 7	B =75.91 found by Math, Solver $0 = 7^2-4^2-6.8^2+2*6.8*4*\cos B$
Test	$7^2=4^2+6.8^2–2*6.8*4*\cos 75.91$ 49 = 49 ☺

C = ?	A+B+C=180
A =33.66 B =75.91	C = 70.43 found by Math, Solver 0 = 33.66+75.91+C-180
Test	33.66+75.91+70.43=180 180= 180 ☺

Problem 15

d=?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$

e=?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$

f=?	$T = (\frac{d}{(e-f)}) + g$
	(T-g)(e-f) = d

g=?	$T = \frac{d}{e-f} + g$
	$T = (\frac{d}{(e-f)}) + g$

	d = (e-f)*(T-g)
Test	$T=\frac{(e-f)*(T-g)}{e-f}+g = T$

	(T-g)(e-f) = d $e = \frac{d}{T-g} + f$
Test	$T=\frac{d}{\frac{d}{T-g}+f-f}+g = T$

	$e = \frac{d}{T-g} + f$ $e - \frac{d}{T-g} = f$
Test	$T=\frac{d}{e-e-\frac{d}{T-g}}+g = T$

	$T - (\frac{d}{(e-f)}) = g$
Test	$T=\frac{d}{e-f}+T-(\frac{d}{(e-f)}) = T$

Problems 16-18

y = ?	y = a*b^x
a=785	y = 785*1.027^5
b=1.027	y = 896.85
x=5	

T = log(2)/log(1.027) = 26.0

x = ?	y = a*b^x
a=785	x = 8.3
b=1.027	found by Math, Solver
y=980	0=785*1.027^x - 980
Test	980 = 785*1.027^8.3 980 = 980 ☺

T = log(2)/log(1.027) = 26.0

b = ?	y = a*b^x
a=785	b = 1.045 = 1 + 4.5%
y=980	found by Math, Solver
x=5	0=785*b^5 - 980
Test	980 = 785*1.045^5 980 = 980 ☺

T = log(2)/log(1.045) = 15.7

Problems 19-21

y = ?	y = a*x + b
b=785	y = 2.7*5 + 785
a= 2.7	y = 798.5
x=5	

x = ?	y = a*x + b
b=785	x = 72.2
a= 2.7	found by Math, Solver
y=980	0 = 2.7*x+785 - 980
Test	980 = 2.7*72.2 + 785 = 980 ☺

a = ?	y = a*x + b
b=785	a = 39
y=980	found by Math, Solver
x=5	0 = a*5+785 - 980
Test	980 = 39*5 + 785 = 980 ☺

25. SUSTAINABLE ADAPTION TO QUANTITY: FROM NUMBER SENSE TO MANY SENSE

Their biological capacity to adapt to their environment make children develop a number-language based upon two-dimensional block- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language. And math education must accept that its goal is not to mediate the truth regime of setcentric university math, but to develop the child's already existing adaption to Many.

Decreased PISA Performance Despite Increased Research

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden.

However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

> The highest performing education systems across OECD countries are those that combine excellence with equity. A thriving education system will allow every student to attain high level skills and knowledge that depend on their ability and drive, rather than on their social background. Sweden is committed to a school system that promotes the development and learning of all its students, and nurtures within them a desire for lifelong learning. PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. The share of top performers in mathematics roughly halved over the past decade. (OECD 2015, p. 3).

Widespread innumeracy also resides in Denmark, where the use of multi-year office-directed lines with fixed classes from secondary school has lowered the exam passing limit at the end of lower and upper secondary school to about 15% and 20% compared to the North-American limit at 70%, using instead self-chosen half-year blocks to uncover and develop the student's individual talent.

Furthermore, two different forms of mathematics are taught, one accepting and one rejecting the 'New Math' occurring around 1960.

Mathematics and its Education

The Pythagoreans used the word 'mathematics' as a common label for their knowledge about Many by itself and in space and time: arithmetic, geometry, music and astronomy. (Freudenthal, 1973)

Without the two latter, mathematics later became a label for arithmetic, algebra and geometry, which may be called pre-setcentric math, replaced by the present setcentric 'New Math' in 1960 despite it never solved its self-reference problem that became visible when Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not, and vice versa.

In any case, mathematics is a core subject in schools together with reading and writing. However, there is a difference. If we adapt to the outside world by proper actions, it has meaning to learn how to read and how to write since these are action-words. But, we cannot math, we can reckon. Consequently, continental Europe taught reckoning, called ‘Rechnung’ in German, until the arrival of the New Math. And, when opened up, mathematics still contains reckoning in the form of fraction-reckoning, triangle-reckoning, differential-reckoning, probability-reckoning, etc.

Today, Europe only teach set-centric mathematics, whereas the North American republics offer classes in algebra and geometry, both being action words meaning to reunite numbers and to measure earth in Arabic and Greek. But also here precalculus is seen as a very difficult class to teach, discouraging many students from taking calculus classes.

However, in their ‘Learning framework 2030’, OECD (2018) points to the necessity of a solid background for all in literacy and numeracy, which raises the ‘Cinderella question’: with pre-setcentric and setcentric mathematics unable to ‘make the prince dance’, is there a third hidden post-setcentric alternative, that may prove sustainable so it will last?

The nature of education has been studied by different sciences. To discuss how to find a sustainable solution we should begin with biology, specializing in sustainability through adaption.

Biology Looks at Education

As a life science, biology sees life as built from green, grey and black cells.

Green cells form plants able to perform photosynthesis that store the energy form solar photons in carbon hydrate molecules by replacing oxygen with water in carbon dioxide molecules. To survive, plants must access light and water where they are situated since they are unable to move.

Grey cells form animals able to release the energy from plants or other animals by the replacing hydrogen with oxygen when inhaling oxygen and exhaling carbon dioxide through breathing. To survive, animals must move using muscles and limbs, as well as a brain to decide which way to move. Also, according to ethology (Darwin, 2003) they must adapt to the environment.

Black cells exist individually in oxygen depleted areas on the bottom of lakes or in the stomach of animals, surviving by removing oxygen from carbon hydrate molecules, thus being transformed to carbon or carbon hydrogen or oil allowing energy to be used by machines.

The holes in their head allow animals to satisfy their two basic needs for information and food. Animals come in three forms.

Reptiles have one brain allowing it to transform outside information into a choice between alternative actions.

Mammals also have a second brain for feelings binding them to a mate and to the offspring to allow it to gradually adapt to the environment through childhood before having offspring themselves.

Finally, humans also have a third brain to store and share information, made possible by transforming forelegs to arms with hands that can grasp food and things that are named by sounds, thus developing a language for mutual sharing information about what they observe and know about the six core ingredients of their life: I, you1, it, we, you2, and they; or in German: ich, du, es, wir, ihr, sie.

The combination of individual and collective adaption is so effective that to reproduce, humans only need two to three offspring in a lifetime, where other mammals need it per year.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured e.g. by a social institution called a school.

With life existing in space and time, institutional education has to answer two core questions: what things and events in the environment is important to address in education? And will learning take place through a meeting allowing individual representations to be created, or will it need to be mediated through the teaching of socially constructed representations.

To answer this, we now turn to three other sciences: philosophy, psychology and sociology.

Philosophy Looks at Education

Philosophy looks at the relation between outside existence, ontology, and inside representation, epistemology, or, in other words, the ‘it-we’ relation. Within philosophy, precedence is given to outside phenomena by empiricism, to inside rationality by rationalism, and to questioning ruling knowledge claims by skepticism.

A controversy within philosophy began in ancient Greece where the sophists pointed out that to practice democracy, a population must be enlightened, especially about the difference between nature and choice to avoid being patronized by political choice masked as unpolitical nature. In opposition to this, Plato saw choice as an illusion since all physical is but examples of metaphysical forms only visible to philosophers educated at the Plato academy. (Russell, 1945)

Later, the Christian church changed the academies into monasteries, where some changed back into universities after the reformation; and after Newton’s discovery of physical laws controlling nature without being physical or metaphysical patronized. This inspired the Enlightenment Century installing two republics, one in North America taking over empiricism when developing ‘it is right if it works’ pragmatism, symbolic interactionism, grounded theory, and action research; and one in France taking over skepticism after seeing its republic turned over several times because of resistance from its German speaking neighbors.

Today, opposition against rationalism is seen within existentialism, claiming with Sartre (2007) that existence precedes essence. And explicated by Heidegger (1962) arguing that in defining verdict-sentences “subject is predicate”, the subjects should be respected for naming what exist outside, whereas the predicates should be questioned and appealed since they represent an inside choice between alternatives in risk of being masked as nature.

In contrast to this, continental rationalism gives precedence to inside essence developed by rational universities through peer-reviewed research.

Psychology Looks at Education

Psychology looks at cognitive aspects of learning, or, in other words, the ‘it-I’ relation. Here, the philosophical controversy between outside existence and inside essence becomes a controversy between different forms of inside constructivism.

Supporting the philosophical existence stance, Piaget (1971) sees learning as a biological process of adapting inside to the outside environment through outside assimilation and inside accommodation, where assimilation makes the outside conform to inside schemata, whereas accommodation makes inside schemata conform to the outside resistance against assimilation.

Thus, to Piaget, learning takes place in the meeting between outside existence and inside schemata that accommodate through outside operations and inside peer communication. Here, teaching socially constructed schemata should be kept to a minimum to not influence the construction of individual schemata.

Siding with Piaget, Ausubel says that “The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly” (Ausubel, 1968, p. vi).

Supporting the philosophical essence stance, Vygotsky (1986) sees learning as adapting to the socially institutionalized knowledge mediated through good teaching respecting that the knowledge taught must be attachable to what the learners already know in their zone of approximate development. Consequently, high quality must be given to teacher education and textbooks to provide good teaching. And teaching should be structured and well-organized aiming at students being able to reproduce what teachers teach.

Sociology Looks at Education

Sociology looks at the social aspects of human interaction, or, in other words, the 'they-I' relation. Here, the philosophical controversy between outside existence and inside essence carries on as a controversy between different forms of social theory emphasizing individual agency or social structure.

Individual agency is emphasized in the first Enlightenment republic in North America showing strong resistance against institutional answers since they may lead to a goal displacement (Bauman, 1990) becoming an inside goal itself instead of staying as an inside means to an outside goal, thus suppressing the 'sociological imagination' (Mills, 1959) that might keep the answer fluid instead of fossilizing into what Weber (1930) calls an 'disenchanting Iron Cage'.

Structuralism is preeminent in continental Europe seeing established science as being a true inside representation of the outside environment if accepted by the society's knowledge institutions, the universities.

In the second Enlightenment republic in France, institutional skepticism inspired by Heidegger led to French post-structuralism where Derrida, Lyotard, Foucault and Bourdieu warn against hidden patronization in our most basic institutions: words, correctness, diagnosing discourses, curing institutions, and education especially that might make mathematics so difficult it may serve as symbolic violence to establish a new knowledge nobility.

> It seems to me that the real political task in a society such as ours is to criticize the workings of institutions, which appear to be both neutral and independent; to criticize and attack them in such a manner that the political violence which has always exercised itself obscurely through them will be unmasked, so that one can fight against them. (Chomsky & Foucault, 2006: 41)

As to education, Foucault (1995) sees schools as 'pris-pitals' mixing social power techniques from a prison and a hospital. Here students are 'pati-mates' forced to return to the same class, hour after hour, day after day, month after month for several years. Furthermore, self-reference is used to create diagnose 'ignorant' without specifying of what: 'you don't know math, so we must teach you math'. Consequently, humans are placed in a Kafkaesque (2015) situation where they, unable to have their diagnose defined, finally accept it and therefore also accept to be retained and treated to be cured.

In Germany, Habermas (1981) theorizes the possibility of creating a third Enlightenment republic in post-war Germany. Inspired by Weber's warning that rationalization carried too far might lead to an iron cage dis-enchanting the world, Habermas warns against system-worlds tending to colonize the life-world and recommends a power free communication rationality to prevent this from happening where "peers exchange views".

Likewise, Arendt (1963) points out that by definition, institutions lack competition, forcing employees to follow order, which might lead to the banality of evil, where ordinary citizens must act evilly to keep the job.

Agency-based education sees knowledge construction as best taking place in symbolic interaction between peers exchanging views about the sentence subject in order to negotiate a common view. This resonates with Piagetian constructivism, and with philosophical existentialism giving precedence to existence; and with the construction of social knowledge by using Grounded Theory (Glaser and Strauss, 1967). Secondary education should be block-organized to support the student's identity work through self-chosen half-year blocks with teachers teaching only one subject.

Structuralism sees democratically controlled educational institutions as the best way to mediate the university knowledge heritage. This resonates with Vygotskyan constructivism. Secondary education should be line-organized to supply the state with skilled academic and non-academic workers, thus forcing students to choose career line early, and to start all over if changing career.

Meeting Many, Children Bundle to Count and Share

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists: bundles of 2s, and 2 of them. Inside, children thus adapt to outside quantities by using two-dimensional bundle-numbers with units.

Likewise, children use bundle-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 more'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 more'.

Children love placing four cars or dolls in patterns; and they smile when the items form a 4-icon. Likewise, they like to form number-icons with footprints in the sand, with body-parts etc.

Children love counting their fingers in 4s using a rubber band to hold the bundles together. They smile when seeing that the fingers can be counted in 4s as 1Bundle6, 2B2 or 3B less2. Or, if counting in 3s, as 1B7, 2B4, 3B1, or 4Bless2. Some even see that 3 bundles is the same as one bundle of bundles, 3B = 1BB.

Likewise, children love bundle-counting the fingers in e.g. 4s as 0Bundle1, 0B2, 0B3, 0B4 no 1B0, 1B1, 1B2, 1B3, 1B4 no 2B0, 2B1, 2B2.

A special case is counting in pairs or 2s. Here the fingers can be counted as 1B8, 2B6, 3B4, 4B2, 5B0. A different color for the rubber band used for the bundle of bundles will allow the fingers to be counted as 1BundleBundle6, 2BB2, 3BBless2. Some might suggest a new color for the bundles of bundles of bundles, thus counting the fingers as 1BBB2 or 1BBB1B0; or even 1BBB0BB1B0.

And children don't mind writing using 'bundle-writing' with a full sentence containing a subject, a verb and a predicate as in the word-language: T = 8 = 1B5 = 2B2 = 3B-1 3s. Some might even write T = 8 = 3B-1 = 1BB-1 3s.

Also, children smile when they see that, counting in hands, T = 5 = 1B0 5s, thus realizing that ten is written as 10 because ten becomes 1B0 if we count in tens.

Sharing 8 cakes, 2 children take away 2 to have one each; and smile when they see that entering '8/2', a calculator predicts they can have 4 each; thus seeing the division sign as an icon for a broom pushing away 2s. This motivates rooting division by 2 as counting in 2s.

Likewise, when counting 9 cubes in 2s they may stack the 2s on-top as a block of 4 2s, smiling when they see that entering '4x2', a calculator predicts they have a total of 8; thus seeing the multiplication sign as an icon for a lift pushing up 2s.

And again, they smile they see that entering '8 – 4*2', a calculator predicts that 1 is left when pulling away a stack of 4 2s from 8; thus seeing the subtraction sign as an icon for a rope pulling away the 4 2s.

Children thus see that counting involves three processes: pushing away, pushing up and pulling away, that can be performed by a broom, a lift and a rope; and that can be predicted on a calculator by using division, multiplication and subtraction. Some may even accept that the counting prescription 'From the total 8, 8/2 times, 2s can be pushed away' may be shortened to the calculation formula '8 = 8/2x2', later with unspecified numbers becoming a core formula expressing proportionality, the recount-formula 'T = (T/B)*B'.

Exposed to counting, children adapt in a natural way to the three basic operations division, multiplication and subtraction; and typically enjoys using a calculator, or even the recount-formula, to predict the counting result.

A Contemporary Mathematics Curriculum

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g. that 2+3 = 5.

This offers a ‘natural’ curriculum with multidigit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

In some cases, reverse operations create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions and decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about the results of operating on such numbers, e.g. that (a+b) * (a-b) = a^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles and areas. Later, geometry and algebra are coordinated in coordinate geometry.

To be followed by halving a rectangle by its diagonal to create a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

In a calculation, changing the input x will change the output y, making y a function of x, y = f(x), using f for an unspecified calculation. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about the results of operating on calculations, e.g. that (f*g)’/(f*g) = f’/f + g’/g. And with a reverse operation, integration, again creating additional education about the results of operating on calculations, e.g. that ∫6*x^2dx = 2*x^3 + c, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as the qualitative: fact, fiction and fiddle (Tarp, 2001).

The Difference to a Typical Contemporary Mathematics Curriculum

Thus, typically the core of a curriculum is about how to operate on specified and unspecified numbers and calculations.

Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent.

Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles, bundles, and bundles of bundles.

Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting ‘(ten and) 1 left’, ‘(ten and) 2 left’. Seldom children may say ‘bundle’ instead of ten, or to say ‘ten-ten’ or ‘bundle-bundle’ instead of hundred.

Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit.

Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit.

Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed.

Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction.

In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

Mathematics as a Number-Language

Wanting to respect the child's own number-language, Tarp (2018, p. 103) talks about word- and number-languages with word- and number-Sentences:

> Observing the quantitative competence children bring to school, (..) we discover a different 'Many-matics'. Here digits are icons with as many sticks as they represent. Operations are icons also, used when bundle-counting produces two-dimensional block-numbers, ready to be re-counted in the same unit to remove or create overloads to make operations easier; or in a new unit, later called proportionality; or to and from tens rooting multiplication tables and solving equations. Here double-counting in two units creates per-numbers becoming fractions with like units; both being, not numbers, but operators needing numbers to become numbers. Addition here occurs both on-top rooting proportionality, and next-to rooting integral calculus by adding areas; and here trigonometry precedes geometry.

Discussing Number Sense and Number Nonsense

The basic question in grade one mathematics is: shall education be about numbering or about numbers? Shall education guide and support the development of the children's already existing adaption to quantity, or shall education teach numbers? Shall the 'I' keep on adapting to the 'it' directly, or indirectly by having the adaption replaced by what is mediated by the 'they'?

Choosing numbers over numbering, the US National Council of Teachers of Mathematics, NCTM, in their publication 'Principles and Standards for School Mathematics' (2000) says: "Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense (p. 7)."

Likewise choosing numbers over numbering, ICMI study 23 creates a WNA-discourse (Whole Number Arithmetic) asking:

> To what extent is basic number sense inborn and to what extent is it affected by socio-cultural and educational influences? How is the relationship between these precursors/foundations of WNA, on the one hand, and children's whole number arithmetic development?" (Bussi and Sun, 2018, pp 500-501)

Thus, both to the NCTM and in the WHA discourse, the concept 'number sense' is central, although not being that well defined (Griffin, 2004). In the ICMI study there are several references to Sayers and Andrews (2015) that based upon reviewing research in the WHA domain create a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

However, several questions may be raised to this FoNS framework.

In his book, Dantzig (2007) uses the term 'number sense' for a natural property shared by humans and animals. However, from a biological view it is sensing the environment that is fundamental to all grey cells. And as human constructs, numbers are not part of the environment, in contrast to what they number and what is embedded in human language as the singular in plural forms, the physical fact many or 'more-ness'. Using the term 'cardinality' just adds a religious power aspect demanding respect for the Cardinal.

Thus, the term ‘many sense’ is more precise than the term ‘number sense’. Especially since, with its reference to numbers, ‘number sense’ becomes a self-reference that removes meaning from four of the eight categories.

Furthermore, using the word ‘understanding’ makes three categories dubious since there are many different understandings of the word understanding.

What is left is category seven, simple arithmetic competence, which is about adding and subtraction, thus neglecting that division and multiplication come first when counting in bundles.

Thus, it seems difficult to define number-sense without self-reference and without referring to a tradition giving priority to addition and subtraction.

A grounded definition of number-sense or many-sense should come from how numbers emerge in the numbering process counting and recounting a total in bundles, to allow seeing the link between the number and what it numbers by including the ‘missing link’, the bundle and the unit, absent in everyday use: T = 6B7 tens = 67.

Therefore, a short definition could be: Having number-sense or many-sense means including the word ‘bundle’ as a unit for the numbers. That is:

To bridge the outside total with an inside numbering by bundling creating flexible bundle-numbers expressed in a full number-language sentence with an outside subject, a verb and an inside predicate, e.g. T = 2 3s.

To count 5 fingers in fives as 0B1, 0B2, 0B3, 0B4, 0B5 or 1B0; and as 1Bundle less 4, 1B-3, 1B-2, 1B-1, 1B0; and to recount five fingers with ‘flexible bundle-numbers’ with overload, underload or fraction, i.e. as 1B3 2s, 2B1 2s or 3B-1 2s or 2 ½B 2s, and later as 1BB 0B1 2s or 1BB1B-1 2s.

And to recount ten fingers in 3s as 1B7, 2B4, 3B1, 4B-2, 31/3, 1BB0B1, or 1BB1B-2.

And to let 67 = 6B7 = 5B17 = 7B-3 = 6.7 tens = 7.-3 tens.

And 678 = 67B8 = 6BB7B8. (Tarp, 2019)

To see the digits as icons with as many sticks or strokes as they represent if written less sloppy; and with ten needing no icon when used as bundle-size.

To see the operations as icons coming from the counting process, where division iconizes a broom pushing away bundles, where multiplication iconizes a lift pushing up bundles into a block, where subtraction iconizes a rope pulling away the block to find unbundles singles, and where addition iconizes placing blocks next-to or on-top.

To see the counting process predicted by the recount-formula T = T/(B)*B, saying ‘From the total T, T/B times, B-bundles can be pushed way’; and to use a calculator to enter ‘9/4’ giving ‘2’, and ‘9-2*4’ giving ‘1’ to predict that from 9, 4s can be pushed away 2 times, and that pulling away the 2 4s from 9 leaves 1, thus predicting that 9 may be recounted as 2B1 4s.

To see totals as double described both as outside blocks and as inside bundles.

To see 678 as a numbering containing four numbers counting unbundled, bundles, bundles of bundles and specifying the bundle-size.

To see a multiplication task as recounting from icons to tens, facilitated by using flexible block&bundle numbers so that 6*8 = 1B-4 * 1B-2 = 1BB – 4B – 2B + 4*2 = 4B8 = 48, thus realizing that -*- is + since the corner was pulled away twice. And to see that 4*67 may be calculated as 4*6B7 giving 24B28, which may be recounted without an overload as 26B8 or 268.

To see a multiplication equation 4*x = 20 as recounting from tens to icons, solved by the recount-formula.

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**1~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	10**B**10
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

Figure 1. A counting table that includes the bundles in the number names

The WHA discourse defines numbers by internal reference as a set of whole numbers included in the set of integers, included in etc. All created to describe what is called cardinality which is claimed to be linear and represented by a number-line.

The WHA discourse thus presents 678 as one number, or if asked to be more precise, as 6 numbers: 6, 7, 8, ones, tens and hundreds, even if the correct answer is fou4r numbers: 6, 7, 8 and bundles, which typically is ten where it is twenty when the French and the Danes count four twenties instead of eight tens.

Furthermore, 67 is not even a whole number but decimal number that might include a negative number as well:

67 = 6ten7 = 6B7 tens = 7B-3 tens, or 6ten7 = 6.7 tens = 7.-3 tens.

The WNA discourse subscribes to setcentric mathematics. Even if Russell proved that self-reference leads to the nonsense of the classical liar paradox, ‘this sentence is false’, since the set of sets not belonging to itself will belong if and only if it will not.

Russell’s point is that it is OK to talk about elements and sets since that is how a language is organized, but when you talk about sets of sets you talk from a meta-level that should not be mixed with the language level, even if this was precisely what Zermelo and Fraenkel did when trying to save the set theory by disregarding the difference between a set and its elements, thus disregarding the difference between examples and abstractions that is the basis in any language.

Grounded in outside observations, the numbers zero, one and two are rooted in fingers on a hand. Defined inside the WNA discourse, zero is defined as the empty set Ø = {}. With 0 = Ø, 1 is defined as the set containing the set Ø, 1 = {Ø}, but as a set of sets, this places 1 on a different language level where it cannot be added to 0. Then 2 is defined as the set that contains a set, and a set of sets, Ø and 1, 2 = {Ø, {Ø}} thus placing 0, 1 and 2 on three different language levels. Which is nonsense according to Russell.

As to the sociological effect of creating an educational concept ‘number-sense’ we should remember that sociologically, a school is a pris-pital. So, the moment you introduce a new construct you may also introduce a new diagnose: this child lacks number sense, so it must be treated. Especially since it is claimed that children who start with a poorly developed understanding of numbers remain low achievers throughout school (Geary, 2013). And with eight diagnose components, you need eight cures. This might be good news for universities selling teacher education courses, but bad news for the curers, the teachers, now having three times eight additional tasks forced upon them: How to understand the diagnoses, how to find material to use in the cure, and how to evaluate if the cure works.

Introducing diagnoses is an example of what Foucault calls pastoral power:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al, 1982: 213, 215)

In this way Foucault describes the salvation promise of the generalized church: ‘You are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will cure you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and do exactly what we tell you. If not, you only have yourself to than for your decline’.

Introducing diagnoses may also be seen as an example of ‘symbolic violence’ used as an exclusion technique to keep today’s knowledge nobility in power (Bourdieu 1977).

To master Many, humans invented numbers as a means, typically rooted in the hands as the Roman numbers bundling fingers in hands and double hands (Dantzig, 2007). But numbers may lose their outside link and become examples of inside abstractions instead of abstractions from the outside. Likewise, outside quantity may become an example of inside cardinality. In that moment numbers undertake what Bauman calls a goal displacement, where inside derived setcentric numbers become the goal instead with outside quantity as a means thus leaving Many as what Weber calls disenchanted.

The situation with eight components in number sense reminds of the claimed eight ‘mathematical competencies’ (Niss, 2003) also made meaningless by self-reference, but meaningfully reduced to two competences, count and add (Tarp, 2002). Likewise, both situations remind of the eight sacraments in the catholic church, challenged by the two sacraments of the protestant church.

To look for meaningful diagnoses in a sustainable mathematics education adapted to quantity we must ask: What is it in the outside world that the children are not adapted to? Will bringing this inside the classroom allow children to extend their existing adaption?

So, instead of using the eight number sense components as diagnoses, we may use the alternative definition of number sense given above as diagnoses to be cured by guiding questions to outside subjects brought inside to receive common predicates, thus reifying the subject in the number language sentences.

Conclusion and Recommendation

This paper asked if there is a third hidden post-setcentric alternative, that may prove sustainable so it will last? The answer is yes, and maybe, since testing for sustainability has to be carried out on what may be called post-setcentric mathematics respecting instead of colonizing the way children adapt to quantity by using two-dimensional bundle-numbers with units instead of the one-dimensional line-numbers forced upon them by setcentric education. Thus, mathematics education should see itself as a language education allowing children develop their quantitative number-language like their qualitative word-language, both using sentences typically with a subject, a verb and a predicate.

A core question in language education is the following: should education develop further the children’s own language, or should education colonize it by replacing their native language with a foreign language. And should language be taught before, together with or after its grammar?

Word-language education chose to respect the children's native language and to develop it before introducing a grammar. Likewise, with foreign language after the language revolution in the 1970s made language be taught before grammar (Widdowson, 1978, and Halliday, 1973).

Number-language education chose to disrespect the children's native language. Furthermore, its revolution in the 1970s made language be taught after its grammar, that was introduced not through bottom-up reference to examples, but as top-down examples of the abstraction Set.

So, to establish as sustainable tradition that will allow all to learn and practice a number-language, mathematics education must stop using a setcentric grammar-based foreign language to colonize the children's own native language.

The consequences of not decolonizing is seen in the OECD-report on the Swedish school system as well as in the widespread innumeracy documented by various PISA studies. The time has come for a paradigm shift (Kuhn, 1962) in early childhood education in adaption to quantity by developing the children's already existing many-sense.

Therefore, if the goal is a sustainable mathematics education it might be a good idea to respect and develop the natives' own natural number-language; and to say: 'only cure the diagnosed'.

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26. Per-numbers connect Fractions and Proportionality and Calculus and Equations

In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word 'percent' to also talk about other 'per-numbers' as e.g. 'per-five' thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

Mathematics is Hard, or is it

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature.

That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969.

Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Witnessing poor PISA performance, Denmark has lowered the passing limit at the final exam is to around 15% and 20 % in lower and upper secondary school.

Other countries also witness poor PISA performance. And high-ranking countries admit they have a high percentage of low scoring students.

As to finding the cause, Kilpatrick, Swafford, and Findell (2001, p. 36) points out that "what is actually taught in classrooms is strongly influenced by the available textbooks". Personally, working ethnographically in schools in Denmark and abroad, listening to teachers and students confirms the picture that textbooks are followed quite strictly.

So, it seems only natural to look at what is currently being discussed in textbook research e.g. by looking at the Third International Conference on Mathematics Textbook Research and Development, ICMT3, in Germany.

The ICMT3 Conference

The September 2019 ICMT3 conference consisted of 4 keynote addresses, 15 symposium papers, 2 workshops, 40 oral presentations and 13 posters.

The name 'fraction' occurred 212 times in the proceedings, and one of the keynotes addressed the problems students have when asked to find 3/5 of 2/4.

As to fractions, Ripoll and Garcia de Souza writes that "The integer numbers structure and the idea of equivalence are elementary in the mathematical construction of the ordered field of the rational numbers. Hence, the concept of equivalence should not be absent in the Elementary School's classrooms and textbooks." (Rezat et al, 2019, p. 131). Looking at 13 Brazilian textbooks from 4th to 7th grade they conclude that

> The conclusion, with respect to equivalence, was that no (complete) characterization of equivalent fractions is present in the moment the content fractions is carried on in the 6th grade Brazilian textbooks, like "Two given fractions a/b and c/d are equivalent if and only if ad = bc." In most cases only a partial equivalence criterion is presented, like "Two fractions are equivalent if one can transform one into the other by multiplying (or dividing) the numerator and the denominator by the same natural number."

The authors thus take it that fractions should obey the New Math ‘set-centrism’ (Derrida, 1991) by saying: in a set-product of integers, a fraction is an equivalence class created by the equivalence relation statin that a/b ~c/d if a*d = b*c; and thus neglect the pre-setcentric version mentioned above where a fraction keeps its value by being expanded or shortened; as well as the post-setcentric version seeing a fractions as an example of a per-number, described later in this paper.

Confirming in the afterwards discussion that fractions are introduced by the part-whole model, an argument was made that if a fraction is defined as a part of a whole then a fraction must always be a fraction of something; thus being an operator needing a number to become a number, and not a number in itself.

Of course, in a 30 minutes presentation there is little time to discuss the nature of fractions thoroughly, so this question needs to be addressed in more details.

Also addressing middle school problems, Watanabe writes that “Ratio, rate and proportional relationships are arguably the most important topics in middle grades mathematics curriculum before algebra. However, many teachers find these topics challenging to teach while students find them difficult to learn.” (p. 353)

And, talking about proportionality, Memis and Yanik writes that “Proportional reasoning is an important skill that requires a long process of development and is a cornerstone at middle school level. One of the reasons why students cannot demonstrate this skill at the desired level is the learning opportunities provided by textbooks.“ (p. 245)

Textbooks must follow curricula, and middle school problems were also mentioned at the International Commission on Mathematical Instruction Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, in Japan November 2018. Here in his plenary talk, McCallum after noting that “a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships” challenged the audience by asking “What is the difference between 5/3 and 5÷3?” (ICMI, 2018, p. 4).

So, this paper will focus on these challenges by asking: “Is there a hidden different way to see and teach core middle school concepts as fractions, quotients, ratios, rates and proportionality?” A question that might be answered answer by Difference-research (Tarp, 2018) using sociological imagination (Mills, 1959) to search for differences making a difference by asking two questions: ‘Can this be different – and will the difference make a difference?’

Different Ways of Seeing Fractions

In a typical curriculum using a ‘part-whole’ approach, fractions are introduced after division has been taught as sharing a whole in equal parts: 8/4 is 8 split in 4 parts or 8 split by 4.

Representing the whole geometrically as a bar or a circle, dividing in 4 parts creates 4 pieces each called ¼ of the total. Assigning numbers to the whole allows finding ¼ of e.g. 8 by the division 8/2. Then the fraction ¾ means taking ¼ three times, so that taking ¾ of 8 involves two calculations, first 8/4 as 2, then 3*2 as 6, so that ¾ of 8 is 8/4*3, later reformulated to one calculation, 8*¾, multiplying the integer 8 with the rational number ¾.

However, in the ‘part-whole’ approach a fraction is a fraction of something, thus introducing a fraction as an operator needing a number to become a number.

This becomes problematic when the fraction later is claimed to be a point on a number line, i.e. a number in its own right, a rational number, defined by set-centrism as an equivalence class in a set-product as described above.

Furthermore, set-centrism is problematic in itself by making mathematics a self-referring ‘Meta-matics’, defined from above as examples from abstractions instead of from below as abstractions from examples.

And, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox ‘this sentence is false’ being false if true and true if false: If M = $\{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

To avoid self-reference Russell introduced a type theory allowing reference only to lower degree types. Consequently, fractions could not be numbers since they refer to numbers in their setcentric definition.

Neglecting the Russell paradox by defining fractions as rational numbers leads to additional educational questions: When are two fractions equal? How to shorten or expand a fraction? What is a fraction of a fraction? Which of two fractions is the bigger? How to add fractions? Etc.

Fraction later leads on to percentages, the special fractions having 100 as the denominator; which leads to the three percentage questions coming from the part-whole formula defining a fraction, fraction = part/whole.

Seeing fractions as, not numbers, but operators still raises the first three questions whereas the two latter are meaningless since the answer depends on what whole they are taken of as seen by ‘the fraction paradox’ where the textbook insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes, and never 7 cokes of 6 bottles.

Adding numbers without units may be called ‘mathe-matism’, true inside but seldom outside the classroom. And strangely enough the two latter questions are only asked with fractions and seldom with percentages.

Ratios and Rates

When introduced, ratios are often connected to fractions by saying that splitting a total in the ratio 2:3 means splitting it in 2/5 and 3/5.

Where fractions and ratios typically are introduced without units, rates include units when talking e.g. about speed as the ratio between the meter-number and the second-number, speed = 2m/3s.

Per-numbers Occur when Double-counting a Total in two Units

The question “What is 2/3 of 12?” is typically rephrased as “What is 2 of 3 taken from 12?” Seldom it is rephrased as “What is 2 per 3 of 12?”. Even if the word ‘per’ occurs in many connections, meter per second, per hundred, etc.

When we rephrase “taking 30% of 400” as “taking 30 per 100 of 400”, why don’t we rephrase “taking 3/5 of 400” as “taking 3 per 5 of 400” ?

In short, why don’t we rephrase 3/5 both as ‘3 of 5’ and as ‘3 per 5’?

In his conference paper, Tarp (p. 332) introduces per-numbers and recounting:

> An additional learning opportunity is to write and use the ‘recount-formula’ T = (T/B)*B, saying “From T,T/B times B can be taken away”, to predict counting and recounting examples. (..) Another learning opportunity is to observe how double-counting in two physical units creates ‘per-numbers’ as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Of course, you might argue that we cannot write ‘6$ = 9kg’ since the units are not the same. But then again, we write ‘2 meter = 200 centimeter’ even if the units are different, and we are allowed to do so since the bridge between the two units is the per-number 1m/100cm. Likewise, we should be allowed to write ‘6$ = 9kg’ since the bridge between the two units for now is the per-number 2$/3kg.

The difference is that the per-number between meter and centimeter is globally valid, whereas the per-number between kilogram and dollar is only locally valid. Still, it has validity as long as you are talking about the same outside total.

The interesting thing is that by including units, per-numbers connects fractions and proportionality. And that by including units, the recount-formula gives an introduction to fractions saying that 1/3 is ‘1 counted in 3s’: 1 = (1/3)*3 = 1/3 3s.

Fractions as Per-numbers

With per-numbers coming from double-counting the same total in two units, we see that when double-counting in the same unit, the unit cancels out and we get a ratio between two numbers without units, a fraction as e.g. 3$/8$ = 3/8.

Reversely, inside fractions without units may be ‘de-modeled’ outside by adding new units, e.g. ‘good’ and ‘total’ transforming 3/8 to 3g/8t. This allows per-numbers and recounting to be used when solving the three fraction questions:

“What is 3/4 of 60?”, and “20 is what of 60?”, and “20 is 2/3 of what?”

Asking “What is ¾ of 60” means asking “What is 3 per 4 of 60”, or de-modeled with units, “What is 3g per 4t of 60t”,

Of course, 60t is not 4t, but 60 can be recounted in 4s by the recount-formula, 60t = (60/4)*4t = (60/4)*3g = 45g, giving the inside answer “¾ of 60 is 45”.

Asking “20 is which fraction of 60” means asking “What fraction is 20 per 60”, or with units, “Which per-number is 20g per 60t”, giving the answer directly as 20g/60t or 20/60 g/t. Here we might look for a common unit in 20 and 60 to cancel out, e.g. 20, giving 20/ 60 = 1 20s/3 20s = 1/3. This allows transforming the outside answer “20 per 60 is 1 per 3” to the inside answer “20 is 1/3 of 60”.

Asking “20 is 2/3 of what” means asking “20 is 2 per 3 of what”, or with units, “20g is 2g per 3t of which total”. Of course, 20g is not 2g, but 20 can be recounted in 2s by the recount-formula, 20g = (20/2)*2g = (20/2)*3t = 30t. This allows transforming the outside answer “20 is 2 per 3 of 30” to “20 is 2/3 of 30.”

Expanding and Shortening Fractions

With fractions as per-numbers coming from double counting in the same unit that has cancelled out, we are always free to add a common unit to both numbers.

Using numbers as units will expand the fraction:

2/3 = 2 7s/3 7s = 2*7/3*7 = 14/21

Reversely, if both numbers contain a common unit, this will cancel out:

14/21= 2*7/3*7 =2 7s/ 3 7s = 2/3

Taking Fractions of Fractions, the Per-number Way

One of the keynotes pointed out that to understand that 6/20 is the answer to the question “What is 3/5 of 2/4?” we must draw a rectangle with 4 columns if which 2 are yellow, and with 5 rows of which 3 are blue. Then 6 double-colored squares out of a total of 20 squares gives an understanding that 3/5 of 2/4 is 6/20, which also comes from multiplying the numerators and the denominators.

Seeing fractions as per-numbers the question “What is 3/5 of 2/4?” translates into “What is 3 per 5 of 2 per 4”. Knowing that using per-numbers to bridge two units involves recounting them in the per-number which again involves division, we might begin with a number that is easily recounted in 4s and 5s, e.g. 4*5 = 20, and reformulate the question to “3 per 5 of 2 per 4 is what per 20?”.

To find 2 per 4 of 20 means finding 2g per 4t of 20t, so we recount 20 in 4s:

20t = (20/4)*4t = (20/4)*2g = 10g, so 2 per 4 of 20 is 10.

To find 3 per 5 of 10 means finding 3g per 5t of 10t, so we recount 10 in 5s:

10t = (10/5)*5t = (10/5)*3g = 6g, so 3 per 5 of 10 is 6

Thus, we can conclude that 3 per 5 of 2 per 4 is the same as 6 per 20, or, with fractions, that 3/5 of 2/4 is 6/20, again coming from multiplying the numerators and the denominators.

Of course, we could discuss, which method gives a better understanding, but we might never reach an answer, given the many different understandings of the word ‘understanding’

Direct and Inverse Proportionality

Using a coordinate system with decimal numbers comes natural if bundle-writing totals in tens so e.g. T = 26 becomes T = 2.6 tens. This allows fixing a 3x5 box in the corner with the base and the height on the x- and y-axes. The recount-formula T = (T/B)*B then shows a total T as a box with base x = B and height y = T/B.

To keep the total unchanged, increasing the base will decrease the height (and vice versa) making the upper right corner create a curve called a hyperbola with the formula height = T/base, or y = T/x, showing inverse proportionality.

In a 3x5 box, the raise of the diagonal is the per-number 3/5. Expanding or shortening the per-number by adding or removing extra units will make the diagonal longer or shorter without changing direction. This will make the upper right corner move along a line with the formula 3/5 = height/base = y/x, or y = 3/5*x, showing direct proportionality.

Adding Fractions, the Per-number Way

Adding per-numbers occurs in mixture problems asking e.g. “What is 2kg at 3$/kg plus 4kg at 5$/kg?”. We see that the unit-numbers 2 and 4 add directly, whereas the per-numbers cannot add before multiplication changes them to unit-numbers. However, multiplication creates the areas 2*3 and 4*5, which gives the answer: 2kg at 3$/kg + 4kg at 5$/kg gives (2+4)kg at (2*3+4*5)/(2+4)$/kg.

So we see that per-numbers add by the areas under the per-number graph in a coordinate system with the kg-numbers and the per-numbers on the axes.

But adding area under a graph is what integral calculus is all about. Only here, the per-number graph is piecewise constant, where the velocity graph in a free fall, is not piecewise, but locally constant, which means that the total area comes from adding up very many small area-strips.

This may be done by observing that the total area always changes with the last area-strip thus creating a change equation ΔA = p*Δx, which motivates differential calculus to answer questions as dA/dx = p, thus finding the area formula that differentiated gives the give per-number formula p, e.g. d/dx (x^2) = 2*x.

Interchanging epsilon and delta to change piecewise constancy to local may be postponed to high school, that would benefit considerably by a middle school introduction of integral calculus as adding locally constant per-numbers by the area under the per-number graph, using differential calculus to find the area in a quicker way than asking a computer to add numerous small area-strips.

Solving Proportionality Equations by Recounting

Reformulating the recount-formula from T = (T/B)*B to T = c*B shows that with an unknown number u it may turn into an equation as 8 = u*2 asking how to recount 8 in 2s, which of course is found by the recount-formula, u*2 = 8 = (8/2)*2, thus providing the equation u*2 = 8 with the solution u = 8/2 obtained by isolating the unknown by moving a number to the opposite side with the opposite sign.

This resonates with the formal definition of division saying that 8/2 is the number u that multiplied by 2 gives 8: if u*2 = 8 then u = 8/2.

Set-centrism of course prefers applying and legitimizing all concepts from abstract algebra's group theory (commutativity, associativity, neutral element and inverse element) to perform a series of reformulations of the original equation: 2*u = 8, so (2*u)*½ = 8*½, so (u*2)*½ = 4, so u*(2*½) = 4, so u*1 = 4, so u = 4.

Seven Ways to Solve the two Proportionality Questions

The need to change units has made the two proportionality questions the most frequently asked questions in the outside world, thus calling for multiple solutions.

With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g. Q1: "7 meters need how many seconds?", and Q2: "How many meters is covered in 12 seconds?"

● Europe used 'Regula-de-tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s. Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.

● Find the unit rate: Q1: Since 2meter needs 5second, 1meter needs 5/2second, so 7meter needs 7*(5/2) second = 17.5second. Q2: Since 5second give 2meter, 1second gives 2/5meter, so 12second give 12*(2/5) meter = 4.8meter.

● Equating the rates. The velocity rate is constantly 2meter/5second. So we can set up an equation equating the rates. Q1: 2/5 = 7/x, where cross-multiplication gives 2*x = 7*5, which gives x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, where cross-multiplication gives 5*x = 12*2, which gives x = (12*2)/5 = 4.8.

● Recount in the per-number. Double-counting produces the per-number 2m/5s used to recount the total T. Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s; Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.

● Recount the units. Using the recount-formula on the units, we get m = (m/s)*s, and s = (s/m)*m, again using the per-numbers 2m/5s or 5s/2m coming from double-counting the total T. Q1: T = s = (s/m)*m = (5/2)*7 = 17.5; Q2: T = m = (m/s)*s = (2/5)*12 = 4.8.

● Multiply with the per-number. Using the fact that T = 2m, and T = 5s, division gives T/T = 2m/5s = 1, and T/T = 5s/2m = 1. Q1: T = 7m = 7m*1 = 7m*5s/2m = 17.5s. Q2: T = 12s = 12s*1 = 12s*2m/5s = 4.8m.

● Modeling a linear function f(x) = c*x, with f(2) = 5, f(7) = ?, and f(x) =12.

A Case: Peter, about to Peter Out of Teaching

As a new middle school teacher, Peter is looking forward to introducing fractions to his first-year class coming directly from primary school where the four basic operations have been taught so that Peter can build upon division when introducing fractions in the traditional way. However, Peter is shocked when seeing many students with low division performance, and some even showing dislike when division is mentioned. So, Peter soon is faced with a class divided in two, a part that follows his introduction of fractions, and a part that transfers their low performance or dislike from divisions to fractions.

The following year seeing his new class behaving in the same way, Peter is about to give up teaching when a colleague introduces him to a different approach where division is used for bundle-counting instead of sharing called 'Recounting fingers with flexible bundle-numbers'. The colleague also recommends some YouTube videos to watch and some material to download from the MATHeCADEMY.net to try it yourself.

Inspired by this, Peter designs a micro-curriculum for his class aiming at introducing the class to bundle-counting leading to the recount-formula leading to double-counting in two units leading to per-numbers having fractions as the special case with like units.

“Welcome class, this week we will not talk about fractions!” “?? Well, thank you Mr. teacher, then what will we do?” “We will count our five fingers.” “Ah, Mr. teacher we did that in preschool!” “Correct, in preschool we counted our fingers in ones, now we will bundle-count them in 2s and 3s and 4s using bundle-writing. In this way we will see that a total can be counted in three different ways: overload, standard and underload. Look here:

Outside we have IIIII = ǁ III = ǁ ǁ I = ǁ ǁ ǁ

Inside we write: T = 5 = 1B3 2s = 2B1 2s = 3B-1 2s

We will call this to recount 5 with flexible bundle-numbers. Now count the five fingers in 3s and 4s in the same way. Later, we will count all ten fingers.”

The following class, Peter began by rehearsing.

“Welcome class. Yesterday we saw that an outside total can be recounted in different units, and that the result inside can be bundle-written in three ways, with overload, standard and underload. Today we will begin by recounting twenty in hands, in six-packs and in weeks. Why twenty? Because counting in twenties was used by the Vikings who also gave us the words eleven and twelve, meaning one-left and two-left in Viking language.”

Later, Peter introduced the recount-formula:

“Here we have 6 cubes that we will count in 2s. We do that by pushing away 2-bundles, and write the result as T = 6 = 3B 2s. We see that the inside division stroke looks like an outside broom pushing away the bundles. And asking the calculator, 6/2, and we get the answer 3 predicting it can be done 3 times. We can illustrate this prediction with a recount formula ‘T = (T/B)xB’ saying that ‘from the total T, T/B times, B can be pushed away’. So, from now on, 6/2 means 6 recounted in 2s; and 3x2 means 3 bundles of 2s. And since it is counted in tens, 42 is seen as 4B2 or 3B12 or 5B-8 using flexible bundle-numbers.

Now let us read 42/3 as 4bundle2 tens recounted in 3s; and let us use flexible bundle-numbers to rewrite 4B2 with an overload as 3B12. Then we have T = 42 /3 = 4B2 /3 = 3B12 / 3 = 1B4 = 14. We notice that squeezing a box from base 10 to base 3 will increase the height, here from 4.2 to 14.

And, by the way, flexible bundle-numbers also come in handy when multiplying: Here 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 bundles and 56 unbundled singles, which can be recounted to remove the overload:

T = 7 x 4B8 = 28B56 = 33B6 = 336.”

The third day Peter repeated the lesson with 7 cubes counted in 3s to show that where the unbundled single was placed would decide if the total should be written using a decimal number when placed next-to as separate box of ones, T = 2B1 3s = 2.1 3s. Placed on-top means missing 2 to form a bundle, thus written as T = 3B-2 3s = 3.-2 3s. Or it means recounting 1 in 3s as 1 = (1/3)x3 = 1/3 3s, a fraction.

Later, Peter introduced per-numbers and fractions as described above, which allowed Peter to work with fractions and ratios and proportionality at the same time; and later to introduce calculus as adding fractions and per-numbers by areas.

Observing the increase of performance and the disappearance of dislike, the headmaster suggested to the headmaster of the nearby primary school that Peter be used as a facilitator for in-service teacher training. This would allow primary school children to meet fractions and negative numbers and proportionality when recounting and double-counting a total in a new bundle-unit.

Discussion and Recommendation

This paper asked “Is there a hidden different way to see and teach core middle school concepts as fractions, quotients ratios, rates and proportionality?“ The answer is yes: per-numbers includes them all as examples, as well as integral calculus and equations.

So introducing per-numbers through double-counting the same total in two units makes a difference by allowing fractions, quotients, rates and ratios to be seen and taught as examples of per-numbers, and by allowing integral calculus to be introduced in middle school, and by allowing a more natural way to solve multiplication equations, and by allowing STEM examples in the classroom since most STEM formulas are proportional formulas.

Furthermore, introducing recounting with flexible bundle-numbers allows math dislike to be cured by taking the hardness out of division, seen traditionally as the basis for fractions but becoming a tumbling stone instead if not learned well.

Consequently, it is recommended that primary school accepts and develops the double-numbers children bring to school. And that middle school introduces students to recounting in flexible bundle-numbers from the start to provide a strong division foundation for fractions that becomes connected with quotients, ratios, rates, proportionality, equations and calculus if introduced as per-numbers coming from double-counting in two units that may be the same.

So yes, mathematics is hard, not by nature, but by a choice replacing it with a mixture of top-down meta-matics and mathe-matism seldom true outside the class.

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27. SUSTAINABLE ADAPTION TO DOUBLE-QUANTITY: FROM PRE-CALCULUS TO PER-NUMBER CALCULATIONS

Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

A need for curricula for all students

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed e.g. by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

Traditionally, a school system is divided into a primary school for children and a secondary school for adolescents, typically divided into a compulsory lower part, and an elective upper part having precalculus as its only compulsory math course. So, looking for a change we ask: how can precalculus be sustainably changed?

A Traditional Precalculus Curriculum

Typically, basic math is seen as dealing with numbers and shapes; and with operations transforming numbers into new numbers through calculations or functions. Later, calculus introduces two additional operations now transforming functions into new functions through differentiation and integration as described e.g. in the ICME-13 Topical Survey aiming to "give a view of some of the main evolutions of the research in the field of learning and teaching Calculus, with a particular focus on established research topics associated to limit, derivative and integral." (Bressoud et al, 2016)

Consequently, precalculus focuses on introducing the different functions: polynomials, exponential functions, power functions, logarithmic functions, trigonometric functions, as well as the algebra of functions with sum, difference, product, quotient, inverse and composite functions.

Woodward (2010) is an example of a traditional precalculus course. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

A Different Precalculus Curriculum

Inspired by difference research (Tarp, 2018) we can ask: Can this be different; is it so by nature or by choice?

In their 'Principles and Standards for School Mathematics' (2000), the US National Council of Teachers of Mathematics, NCTM, identifies five standards: number and operations, algebra, geometry, measurement and data analysis and probability, saying that "Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century (p. 2)." In the chapter 'Number and operations', the Council writes: "Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number (p. 7)."

Their biological capacity to adapt to their environment make children develop a number-language allowing them to describe quantity with two-dimensional block- and bundle-numbers. Education could profit from this to teach children primary school calculus that adds blocks (Tarp, 2018). Instead, it imposes upon children one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must be learned before it can be applied to operate on the functions introduced at the precalculus level.

However, listening to the Ausubel (1968) advice "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly (p. vi).", we might want to return to the two-dimensional block-numbers that children brought to school.

So, let us face a number as 456 as what it really is, not a one-dimensional linear sequence of three digits obeying a place-value principle, but three two-dimensional blocks numbering unbundled singles, bundles, bundles-of-bundles, etc., as expressed in the number-formula, formally called a polynomial:

T = 456 = 4*B^2 + 5*B + 6*1, with ten as the international bundle-size, B.

This number-formula contains the four different ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant 'double-numbers' or 'per-numbers'. We might call this beautiful simplicity 'the algebra square' inspired by the Arabic meaning of the word algebra, to re-unite.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $log_a(T) = b$

Figure 01. The 'algebra-square' has 4 ways to unite, and 5 to split totals

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers, creating the operations integration and its reverse operation, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power, and its two reverse operations, root and logarithm.

Precalculus, building on or rebuilding?

In their publication, the NCTM writes "High school mathematics builds on the skills and understandings developed in the lower grades (p. 19)."

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they may have from the lower grades?

Furthermore, what kind of mathematics has been taught? Was it 'grounded mathematics' abstracted 'bottom-up' from its outside roots as reflected by the original meaning of 'geometry' and 'algebra' meaning 'earth-measuring' in Greek and 're-uniting' in Arabic? Or was it 'ungrounded mathematics' or 'meta-matics' exemplified 'top-down' from inside abstractions, and becoming 'meta-matism' if mixed with 'mathe-matism' (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept 'function', Euler saw it as a bottom-up name abstracted from 'standby calculations' containing specified and unspecified numbers. Later meta-matics defined a function as an inside-inside top-down example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may also be seen as an outside-inside bottom-up number-language sentence containing a subject, a verb and a predicate allowing a value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group operation applying the associative and commutative law as well as the neutral element and inverse elements, thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2:

2*u =6, so (2*u)*½ =6*½, so (u*2)*½ =3, so u*(2*½) =3, so u*1 =3, so u =3.

However, 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula 'T = (T/B)*B' (Tarp, 2018), present all over mathematics as a proportionality formula, thus solved in one step:

2*u = 6 = (6/2)*2, giving u = 6/2 = 3.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Bauman (1990) as the core of sociology.

Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as a number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar.

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction for an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

A Grounded Outside-Inside Fresh-start Precalculus from Scratch

Let students see that both the word-language and the number-language provide ’inside’ descriptions of ‘outside’ things and actions by using full sentences with a subject, a verb, and an object or predicate, where a number-language sentence is called a formula connecting an outside total with an inside number or calculation, shortening ‘the total is 2 3s’ to ‘T = 2*3’;

Let students see how an outside degree of Many at first is iconized by an inside digit with as many strokes as it represents, five strokes in the 5-icon etc. Later the icons are reused when counting by bundling, which creates icons for the bundling operations as well. Here division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation. Writing ‘y = f(x)’ means that the y-number is found by specifying the x-number and the f-calculation. Thus, with x = 3, and with f(x) = 2+x, we get y = 2+3 = 5.

Let students see how calculations predict: how 2+3 predicts what happens when counting on 3 times from 2; how 2*5 predicts what happens when adding 2$ 5 times; how 1.02^5 predicts what happens when adding 2% 5 times; and how adding the areas 2*3 + 4*5 predicts adding the ‘per-numbers’ when asking ‘2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?’

Solving Equations by Moving to Opposite Side with Opposite Sign

Let students see the subtraction ‘u = 5-3’ as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers ‘to opposite side with opposite calculation sign’; a rule that applies also to the other reversed operations:

- the division u = 5/3 is the number u that multiplied with 3 gives 5, thus solving the equation u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, thus solving the equation u^3 = 5, and making root a ‘factor-finder’
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, thus solving the equation 3^u = 5, and making logarithm a ‘factor-counter’.

Let students see multiple calculations reduce to a single calculation by unhiding ‘hidden brackets’ where 2+3*4 = 2+(3*4) since, with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s.

This prevents solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5. Allowing to unhide the hidden bracket we get:

2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4

This solution is verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy a ‘Hymn to Equations’: “Equations are the best we know, they’re solved by isolation. But first the bracket must be placed, around multipli-cation. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don’t want to stop!”

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as e.g. T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc. ; as well as formulas as e.g. T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let students place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, three terms disappear if adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = b^2/4 – c.

Moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c, so u + b/2 = ±$\sqrt{b^\wedge 2/4 - c}$, so u = -b/2 ±$\sqrt{b^\wedge 2/4 - c}$

Recounting Grounds Proportionality

Let students see how recounting in another unit may be predicted by a recount-formula ‘T = (T/B)*B’ saying “From the total T, T/B times, B may be pushed away” (Tarp, 2018). In primary school this formula recounts 6 in 2s as 6 = (6/2)*2 = 3*. In secondary school the task is formulated as an equation u*2 = 6 solved by recounting 6 in 2s as u*2 = 6 = (6/2)*2 giving u = 6/2, thus again moving 2 ‘to opposite side with opposite calculation sign’.

Thus an inside equation u*b = c can be ‘demodeled’ to the outside question ‘recount c from ten to bs’, and solved inside by the recount-formula: u*b = c = (c/b)*b giving u = c/b.

Let students see how recounting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. And see how filling a circle with right triangles from the inside allows phi to be found from a formula: circumference/diameter = □ ≈ n*tan(180/n) for n large.

Double-counting Grounds Per-numbers and Fractions

Let students see how double-counting in two units create ‘double-numbers’ or ‘per-numbers’ as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number:

- Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg.
- Asking ‘9kg = ?$’ we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Let students see how double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 40%.

To find 40% of 20$ means finding 40$ per 100$, so we re-count 20 in 100s:

T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$:

T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

Let students see how double-counting physical units create per-numbers all over STEM (Science, Technology, Engineering and mathematics):

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter;
- meter = (meter/second) * second = velocity * second;
- joule = (joule/second) * second = watt * second

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, T = 456 = 4*B^2 + 5*B + 6*1, containing the formulas for constant change:

T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change. So where constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics to ‘post-dict’ what we can’t ‘pre-dict’; and using confidence for predicting intervals.

Combining linear and exponential change by n times depositing a$ to an interest percent rate r, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest percent rate R is predicted by the formula 1+R = (1+r)^n. This saving may be used to neutralize a debt Do, that in the same period changes to D = Do*(1+R).

This formula and its proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Uniting Constant Per-Numbers as Factors

Adding 7% to 300$ means 'adding' the change-factor 107% to 300$, changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, the formula for change with a constant change-factor, also called exponential change.

Reversing the question, this formula entails two equations. Asking 600 = 300*a^5, we look for an unknown change-factor. So here the job is 'factor-finding' which leads to defining the fifth root of 2, i.e. 5√2, found by moving the exponent 5 to opposite side with opposite calculation sign, root.

Asking instead 600 = 300*1.2^n, we now look for an unknown time period. So here the job is 'factor-counting' which leads to defining the 1.2 logarithm of 2, i.e. log1.2(2), found by moving the base 1.2 to opposite side with opposite calculation sign, logarithm.

Calculus Deals with Uniting Changing Per-Numbers as Areas

In mixture problems we ask e.g. '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?' Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before added, thus adding by areas. So here multiplication precedes addition.

Asking inversely '2kg at 3$/kg + 4kg at how many $/kg gives 6kg at 5 $/kg?', we first subtract the areas 6*5 – 2*3 before dividing by 4, a combination called differentiation, ΔT/4, thus meaningfully postponed to after integration.

Statistics Deals with Unpredictable Change

Once mastery of constant change is established, it is possible to look at time series in statistical tables asking e.g. "How has the unemployment changed over a ten-year period?" Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b+a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n.

The average numbers allow calculating all totals in the period, assuming the numbers are predictable. However, they are not, so instead of predicting the number with a formula, we might 'post-dict' the numbers using statistics dealing with unpredictable numbers. This, in turn, offers a likely prediction interval by describing the unpredictable random change with nonfictional numbers, median and quartiles, or with fictional numbers, mean and standard deviation.

Calculus as adding per-numbers by their areas may also be introduced through cross-tables showing real-world phenomena as unemployment changing in time and space, e.g. from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add. First, they must multiply back to unit-numbers to find the total percent. However, multiplying creates areas, so per-numbers add by areas, which is what calculus is about.

Modeling in Precalculus Exemplifies Quantitative Literature

Furthermore, graphing calculators allows authentic modeling to be included in a precalculus curriculum thus giving a natural introduction to the following calculus curriculum, as well as introducing 'quantitative literature' having the same genres as qualitative literature: fact, fiction and fiddle (Tarp, 2001).

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator thus using a natural science proof, valid until finding counterexamples.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A' = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A' = x' = 1; and that if A = ½*x^2, then y = A' = (½*x^2)' = x; and that if A = 1/3*x^3, then y = A' = (1/3*x^3)' = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to 3, instead of adding small strips we just calculate the change in the area over this distance, later called the fundamental theorem of calculus.

A Literature Based Compendium

An example of an ideal precalculus curriculum is described in 'Saving Dropout Ryan With a Ti-82' (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' (Tarp, 2009) replaced the textbook. A formula's left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

An Example of a Fresh-start Precalculus Curriculum

This example was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

An Example of an Exam Question

Authentic material was used both at the written and oral exam. The first had specific, the second had open questions as the following asking ‘What does the table tell?

Agriculture: Number of agricultural farms allocated over agricultural area

	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977
Farms in total	**161142**	**154 694**	**148 512**	**144 070**	**143093**	**141 137**	**137712**	**13424S**	**130 7S3**	**127117**
0,0- 4,9 ha	25 285	23 493	21 533	21623	22123	21872	21093	19915	18 852	17 833
5.0- 9.9-	34 644	32129	30 235	28 404	27693	26 926	26109	25072	24066	23152
10,0-19.9-	48 997	46482	43 971	41910	40850	39501	38261	36 702	35 301	34 343
20.0-29.9-	25670	25 402	25181	24 472	24 195	23 759	23 506	23134	22737	22376
30,0-49.9-	18 505	18 779	18 923	18 705	18 968	18 330	19 095	19 304	10 305	19 408
50,0-99.9-	6 552	6 852	7 076	7 275	7 549	7956	7 847	8247	8 556	8723
100.0 ha and over	1489	1 557	1611	1681	1 715	1791	1801	1871	1934	1882

Figure 02. A table found in a statistical survey used at an oral exam.

Discussion and Conclusion

Asking “how can precalculus be sustainably changed?” an inside answer would be: “By its nature, precalculus must prepare the ground for calculus by making all function types available to operate on. How can this be different?”

An outside answer could be to see precalculus, not as a goal but as a means, an extension to the number-language allowing us to talk about how to unite and split into changing and constant per-numbers. This could motivate renaming precalculus to per-numbers calculations.

In this way, precalculus becomes sustainable by dealing with adding, finding and counting change-factors using power, roots and logarithm. Furthermore, by including adding piecewise constant per-numbers by their areas, precalculus gives a natural introduction to calculus by letting integral calculus precede and motivate differential calculus since an area changes with the last strip, thus connecting the unit number, the area, with the per-number, the height.

Finally, graphing calculators allows authentic modeling to take place so that precalculus may be learned through its use, and through its outside literature.

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28. A Lyotardian Dissension to the Early Childhood Consensus on Numbers and Operations

Can Sociological Imagination Improve Mathematics Education?

Decreasing Swedish PISA results made OECD (2015) write the report ‘Improving Schools in Sweden’ describing its school system as “in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)”

As a social institution, mathematics education might improve by inspiration form sociological imagination, seen by Mills (1959) and Bauman (1990) as the core of sociology; and also emphasized in Lyotard’s report on knowledge in a postmodern digitalized condition (1984):

> “We no longer have recourse to grand narratives (..) But as we have seen, the little narrative remains the quintessential form of imaginative invention most particularly in science. In addition, the principle of consensus as a criterion of validation seems to be inadequate. (..) consensus is a component of the system, which manipulates it in order to maintain and improve its performance. It is the object of administrative procedures (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. The problem is therefore to determine whether it is possible to have a form of legitimation based solely on paralogy. Paralogy must be distinguished from innovation: the latter is under the command of the system, or at least used by it to improve its efficiency; the former is a move (the importance of which is often not recognized until later) played in the pragmatics of knowledge. (..) It is necessary to posit the existence of a power that destabilizes the capacity for explanation, manifested in the promulgation of new norms for understanding (p. 60-61).”

As a number-language, mathematics would create a paradigm shift (Kuhn, 1962) if copying the communicative turn in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by connecting to its outside world before its inside grammar,

In the workshop we focus on early childhood mathematics education as described in the ICME study 23 (Sun et al, 2015); and with a dissension by Tarp (2018).

Consensus and Dissension on Early Childhood Numbers & Operations

Question 01: There seems to be a consensus saying ‘Of course numbers must be learned before being applied in numbering. And as one-dimensional, numbers are names for points along a number line obeying a place value principle when containing more digits’. Thus, a dissension may ask: ‘From the age around four, children seem to distinguish between four ones and two twos thus developing double-numbers with units when adapting to outside quantity. So, why not develop the double-numbers with units children bring to school?’

Question 02: There seems to be a consensus saying ‘Of course addition must be learned before subtraction, multiplication and division since they are all defined from addition’. Thus, a dissension may ask: “Counting an outside total in bundle-counted by a broom pushing away the bundles, iconized as division, to be stacked by a lift iconized as multiplication, to be pulled away by a rope iconized as subtraction, thus finding unbundled singles that placed next-to or on-top the block roots decimals, fractions and negative numbers. This creates a ‘recount-formula’ T = (T/B)xB saying ‘From T, T/B times, B is pushed away’, present all over mathematics and science. Once counted, blocks may be added, but on-top needing units to be changed by recounting, or next-to as areas as in integral calculus? This ambiguity leaves addition not that well defined. So, why not accept the opposite order of the operations as the natural?’

Question 03: There seems to be a consensus saying ‘Of course functions are postponed to secondary school since their algebra builds upon the algebra of letter expressions.’ Thus, a dissension may ask: ‘The word- and the number-language both offer an inside description of an outside object or action by using sentences with a subject, a verb and a predicate, abbreviating 'the total is 2 3s' to 'T = 2x3'. So, why not use functions as number-language sentences from the start?

Time Table for the Workshop

A 20minutes introduction will focus on the core question: As to the goal of mathematics education, is it to master inside mathematics as the means to later master outside Many; or is it to master outside Many by choosing among its three inside versions; the present setcentric Skemp-based 'meta-matics' defining concepts as examples of abstractions instead of as abstractions from examples, the pre setcentric Skinner-based 'mathe-matism' true inside but seldom outside classrooms by adding numbers and fractions without units; and the post setcentric Lyotard-based 'many-math', accepting the number-language children develop when adapting to Many before school.

A 30minutes group discussion on the three questions below is followed by 20 minutes in exchange-groups, and a 20minutes plenum for summing up.

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29. Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis?

Does Mathematics Education Research have an Irrelevance Paradox?

The Swedish Centre for Mathematics Education is meant to mediate research findings and facilitate their implementation. Still, decreasing Swedish PISA results made OECD (2015) write the report ‘Improving Schools in Sweden’ describing its school system as ‘in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life’ (p. 3).

Increasing research together with decreasing student performance points to an ‘irrelevance paradox’ in mathematics education research, possibly caused by peer reviewing failing to assure research quality. The so-called ‘replication crisis’ suggests that this might indeed be the case. First noticed in medical science, the crisis may also occur in schools seen by Foucault (1995) as ‘pris-pitals’, i.e. prison-like hospitals using education to cure humans from the diagnose ‘uneducated’.

Consequently, there is a need for a workshop discussing this hypothesis, as well as ways to make peer reviewed conferences produce more quality. We may ask: When mathematics itself has abandoned peer review, why shouldn’t also mathematics education?

The Replication Crisis in Science

In the article “How Science goes Wrong”, The Economist writes:

> A rule of thumb among biotechnology venture-capitalists is that half of published research cannot be replicated. Even that may be optimistic. Last year researchers at one biotech firm, Amgen, found they could reproduce just six of 53 "landmark" studies in cancer research. (..) The most enlightened journals are already becoming less averse to humdrum papers. (..) But these trends need to go much further. Journals should allocate space for "uninteresting" work, and grant-givers should set aside money to pay for it. Peer review should be tightened - or perhaps dispensed with altogether, in favour of post-publication evaluation in the form of appended comments. That system has worked well in recent years in physics and mathematics (The Economist, 19 Oct. 2013).

The replication crisis thus comes from the ‘metascience’ observation that many research studies are difficult or impossible to replicate or reproduce. It applies to different fields, e.g. psychology where Pashler and Wagenmakers (2012) writes:

> Is there currently a crisis of confidence in psychological science reflecting an unprecedented level of doubt among practitioners about the reliability of research findings in the field? It would certainly appear that there is (p. 528).

The authors refer among others to Ioannidis (2005) who writes:

> Scientists in a given field may be prejudiced purely because of their belief in a scientific theory or commitment to their own findings. Many otherwise seemingly independent, university-based studies may be conducted for no other reason than to give physicians and researchers qualifications for promotion or tenure. (..) Prestigious investigators may suppress via the peer review process the appearance and dissemination of findings that refute their findings, thus condemning their field to perpetuate false dogma (p. 0698).

As to the peer review process, LeBel (2015) writes:

> In recent years, there has been a growing concern regarding the replicability of findings in psychology (..) I propose a new replication norm that aims to further boost the dependability of findings in psychology (p. 1).

Addressing case series studies, Horton (1996) writes:

> The importance of the case series in surgical research is beyond doubt. Therefore, it seems reasonable to ask whether we can trust this study method to yield a valid result. According to conventional epidemiological wisdom, the answer is no (p. 984).

The quality of research was also questioned by Lyotard (1984) distinguishing between consensus and dissension:

> Consensus is a component of the system, which manipulates it (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. (..) Returning to the description of scientific pragmatics, it is now dissension that must be emphasized (p. 60-61).

Time Table for the Workshop

A 20minute introduction to the replication crisis and to conflicting theories within sociology, psychology and philosophy also includes examples on peer-reviews from MADIF 10, CERME 11, ICMT 3, and a journal (Tarp, 2018); and a proposal for a 'Salon des Refusés' created in France in 1863 to display rejected paintings later inspiring important innovation.

Then a 30minutes group discussion will use a short reader with excerpts of the authors cited above to discuss questions as: What kind of dissension risks being silenced by a peer review consensus? Will master-level papers applying existing theory oust research-level papers questioning or expanding it? Also, the groups are invited to exchange experiences on peer reviews; and to exchange opinions on how to increase the quality of the peer review process.

The next 20minutes, the groups split up to join the other groups to exchange views. Finally, a 20minutes plenum will sum up and formulate recommendations as to how to add quality to the coming MADIF sessions.

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30. A BUNDLE COUNTING TABLE

A guide to bundle-counting in pre-school.

Bundle-counting clarifies that we count by bundling, typically in tens

Example 01. Counting Mikado Sticks

The Mikado sticks are positioned next to each other to the right. Counting is done by taking one stick at a time to the left and assembling them in a bundle with an elastic band when we reach ten.

When counting, we say: "0 Bundle 1, 0 bundle 2,. .. "

"Why 0 bundle?" "Because we don't have a bundle yet, before we'll reach ten."

"..., 0 bundle 8, 0 bundle 9, 0 bundle ten, well no, 1 bundle 0".

Example 02. Counting matches

The box says 39, which we read as '3 bundles 9'. We bundle-count as with Mikado sticks.

Extra-option

Some children may find it fun later to count ' 1 bundle less 2, 1 bundle less 1, 1 bundle and 0, 1 bundle and 1 ' as a new way to count ' 0 bundle 8, 0 bundle 9, 1 bundle 0, 1 bundle 1 '. Later again, some children may find It fun to say ' 1 bundle-bundle 0 ' instead of ' ten bundles 0 ' or ' hundred '.

Example 03. Counting ten fingers or ten matches

The ten fingers (or ten matches) bundle are counted in 4s and in 3s while saying "The total is..." and possibly writing "T =..."

Ten counted in 4s	Ten counted in 3s
T = I I I I I I I I I I = ten 1s	T = III I I I I I I I = 1B7 3s
T = IIIIIIIIII = 1 tens = 1B0 tens	T = III III I I I I = 2B4 3s
T = IIII I I I I I I = 1B6 4s	T = III III III I = 3B1 4s
T = IIII IIII I I = 2B2 4s	T = III III III III = 4B-2 3s
T = IIII IIII IIII = 3B-2 4s	T = III III III I = 1BB 0B 1 3s

Table for counting ten tens, or 1 bundle bundles, or 1 hundred:

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~

31. PROPOSALS FOR THE 2020 SWEDISH MATH BIENNALE

Start-math for children and migrants: bundle-count and recount before adding

Assembling 4 fingers 2 and 2, a 3-year-old will protest: "It is not 4, but two 2s". The child counts in bundle- and block-numbers just like we: 456 = 4 Bundle Bundles + 5 bundles + 6 unbundled. And recounts 3 4s to 5s, which leads to proportionality. And recounts 42 to 7s, which leads to equations. And adds 2 3s and 4 5s to 3Bundle2 8s that leads to calculus. The child is directed directly to core mathematics if allowed to keep its 2D bundle- and block-numbers, and to count and recount before adding.

Start-matte for børn og migranter: Bundt-tæl og om-tæl før addition

Samles 4 fingre 2 og 2, vil en 3årig protestere: ”Det er ikke 4, men to 2ere”. Barnet tæller i bundt- og bloktal ligesom vi: 456 = 4 bundtbundter + 5 bundter + 6 ubundtede. Og om-tæller 3 4ere til 5ere, der fører til proportionalitet. Og om-tæller 42 til 7ere, der fører til ekvationer. Og adderer 2 3ere og 4 5ere til 3Bundt2 8ere, der fører til calculus. Barnet føres direkte til kernematematikken hvis det må beholde sine 2D bundt- og bloktal, samt må tælle og om-tælle før addition.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

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Hvordan ville matematikken se ud, hvis den grundfæstes i sin udvendige rod, den fysiske faktum Mange?

For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Bundt-tæl og om-tæl før addition.

Cifre samler mange streger til ét ikon: Fem streger i 5tallet, osv. indtil ti = 1bundt0, 10.

Med en kop til bundter kan en total T på 7 pinde bundt-tælles i ikon-bundter, fx T = 7 = 2 3ere & 1 = 2B1 3ere. Herefter kan totalen om-tælles i samme enhed og skabe overlæs og underlæs: T = 7 = 2B1 3ere = 1B4 3ere = 3B-2 3ere.

En total kan også om-tælles i en ny enhed (proportionalitet), fx 2 4ere = ? 5ere, forudsagt af en regner som 2*4/5 = 1, og 2*4-1*5 = 3, altså T = 2 4ere = 1B3 5ere.

Vi tæller ved at bundte og stakke forudsagt med regnearter, som også er ikoner: Ved op-tælling af en total T i B-bundter, T/B, viser division den kost, der fra T fejer Bere væk. Multiplikation er den kran der løfter bundter op i en stak, og subtraktion er den snor, der trækker stakken væk for at finde de ubundtede. Resultatet kan derfor forudsiges af en 'omtællings-formel' T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere'.

Om-tælling fra ikon-bundter til 10ere fører til multiplikationstabellen: T = 3 4ere = 3*4 = 12 = 1ti2 = 1B2 10ere.

Tilbage-tælling fra 10ere til ikon-bundter bliver til ligninger, som løses ved at bruge om-tællingsformlen: 'Hvor mange 5ere giver 40' fører til ligningen: x*5 = 40, der løses ved at om-tælle 40 til 5ere: 40 = (40/5)*5, så x = 40/5. Så en ligning løses ved at flytte til modsat side med modsat regnetegn. For flere detaljer, se det web-baserede lærerakademi MATHeCADEMY.net og MrAlTarp YouTube videoer.

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Counting before adding will strengthen the number sense by children and migrants

We master many with a number-language sentences, formulas, e.g. T = 4 5ere = 4*5. Which shows that we enumerate totals T by bundling and stacking. So, 4*5 is 4 5s that can be recounted to another unit, e.g. 7s.

Math issues are prevented by bundle-numbers that can be trained as counting ‘6,..., 10’ also as ‘bundle less 4, B-3, B-2, B-1, Bundle’. And ‘10,..., 15’ as ‘Bundle, 1left, 2left, 3left, 4left, 5left’ to show that ‘eleven’ and ‘twelve’ originate from Viking counting.

Tælling før addition styrker talsansen hos børn og migranter

Vi mestrer Mange med et tal-sprog med talsprogs-sætninger, formler, fx T = 4 5ere = 4*5. Som viser, at vi ital-sætter totaler T ved at bundte og stakke. Så 4*5 er altså 4 5ere, der kan om-tælles til en anden enhed, fx 7ere.

Matte-problemer forebygges med bundt-tal. Og kan indøves ved at ’6, …, 10’ også tælles som ’bundt pånær 4, B-3, B-2, B-1. B’. Og ’10, …, 15’ som ’bundt, 1levnet, 2levnet, 3levnet, 4levnet, 5levnet’ for at vise, at ’eleven’ og ’twelve’ stammer fra vikingetiden.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Tælling før addition styrker talsansen

Vi mestrer Mange med et tal-sprog med talsprogs-sætninger, formler, fx T = 4 5ere = 4*5. Som viser, at vi ital-sætter totaler T ved at bundte og stakke. Så 4*5 er altså 4 5ere, der kan om-tælles til en anden enhed, fx 7ere. Eller tiere, som er den internationale bundt-størrelse.

At se tal som bundt-formler gør matte let og forebygger matte-problemer og dyskalkuli. Og bør derfor indøves via forskellige tælleremser, så ’5, 6, 7, 8, 9, 10’ også tælles som ’5, bundt pånær 4, B-3, B-2, B-1. B’, og som ’½budt, ½bubndt&1, ½bB&2, ½B&3, ½B&4, Bundt. Og ’10, 11, 12, 13, 14, 15’ tælles som ’bundt, 1bundt&1, 1B&2, 1B&3,1B&4, 1B&5’, og som ’bundt, 1levnet, 2levnet, 3levnet, 4levnet, 5levnet’ for at vise, at ’elleven’ og ’twelve’ stammer fra vikingetiden.

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Med en kop til bundter kan en total T på 7’bundt-tælles’ i ikon-bundter, fx T = 7 = 2B1 3ere. Herefter kan totalen om-tælles i samme enhed og skabe overlæs og underlæs: T = 7 = 2B1 3ere = 1B4 3ere = 3B-2 3ere. Tilsvarende med totaler optalt i tiere, T = 68 = 6B8 = 5B18 = 7B-2 tiere.

Før addition opøves talsansen med multiplikationstabellen, som reduceres til en 5x5-tabel ved at omskrive tal over 5, fx 6 = ½bundt&1 = bundt-4. Først fordobling, fx T = 2*7 = 2*(½bundt&2) = bundt&4 = 14, eller T = 2*7 = 2*(bundt-3) = 20-6 = 14. Herefter med bundt-tælling, fx T = 2*38 = 2*3B8 = 6B16 = 7B6 = 76. Så halvering, fx ½*38 = ½* 3B8 = ½*4B-2 = 2B-1 = 19.

At gange med 5 er at gange med halve bundter, 5*7 = ½bundt*7 = ½70 = ½ af 6B10 = 3B5 = 35.

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Division dislike cured with 5 sticks and 1 cup and bundle-writing

A class dislikes division, e.g. 336/7. The solution is to see 336/7, not as 336 divided between 7, but as 336 counted in 7s; and to use bundle-writing 336 = 33B6 = 28B56, since a total can be recounted in three ways: normal and with overload or underload. Now, with T = 336 = 33B6 = 28B56, we have T/7 = 4B8 = 48.

Recounting may be trained with bundle-counting 5 sticks in 2s.

Normal: T = 5 = 2B1 2s. With overload: T = 5 = 1B3 2s. With underload: T = 5 = 3B-1 2s.

Likewise with: T = 74 = 7B4 = 6B14 = 8B-6.

Ulyst til division kureret med 5 pinde og 1 kop og bundt-skrivning

En klasse har problemer med division, fx 336/7. Løsningen er at opfatte 336/7, ikke som 336 delt mellem 7, men som 336 optalt i 7ere; samt benytte bundt-skrivning 336 = 33B6 = 28B56, idet totaler kan omtælles på tre måder: normal og med overlæs eller underlæs. Så med T = 336 = 33B6 = 28B56, er T/7 = 4B8 = 48.

Omtælling indøves med 5 pinde, som bundt-tælles i 2ere.

Normal: T = 5 = 2B1 2ere. Med overlæs: T = 5 = 1B3 2ere. Med underlæs: T = 5 = 3B-1 2ere.

Ligeledes med: T = 74 = 7B4 = 6B14 = 8B-6.

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Fokus: Ulyst til division kureret

En klasse har problemer med division, fx 336/7. Løsningen er at opfatte 336/7, ikke som 336 delt mellem 7, men som 336 optalt i 7ere; samt benytte bundt-skrivning 336 = 33B6, hvor koppen opdeler totalen i bundtede inden for koppen og u-bundtede udenfor.

Samt ved øvelser i at 'bundt-tælle' totaler på tre måder: normal og med overlæs eller underlæs.

Først med 5 pinde, som bundt-tælles i 2ere med en kop til bundterne.

Normal: T = IIIII = II II I = 2B1 2ere. Med overlæs: T = IIIII = II III = 1B3 2ere. Med underlæs: T = IIIII = II II I I̶I = 3B-1 2ere.

På samme måde hvis vi optæller i 10ere: T = 74 = 7B4 = 6B14 = 8B-6.

Så med en total på 336 (dvs. 33.6 tiere) er der 33 bundter indenfor koppen og 6 ubundtede udenfor. Men vi fortrækker 28 indenfor, så 5 bundter flytter udenfor som 50, dvs. nu med 56 udenfor, som divideret med 7 giver 4 indenfor og 8 udenfor:

T = 336 = 33B6 = 28B56, og T/7 = 4B8 = 48.

Bundt-skrivning kan bruges ved alle regne-operationer.

T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

T = 7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

T = 7* 48 = 7* 5B-2 = 35B-14 = 33B6 = 336

T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

T = 338 /7 = 33B8 /7 = 28B58 /7 = 4B8 + 2/7= 48 2/7

Bundt-skrivning kan også bruges ved multiplikationstabellen:

T = 4*8 = 4*1B-2 = 4B-8 = 32 og 7*8 = 7*1B-2 = 7B-14 = 6B-4 = 5B6 = 56

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Fractions and percentages as per-number

Factions dislike disappear if viewing a fraction as a per-number obtained from double- counting in the same unit, 3/5 = 3$ per 5$; or as percent 2% = 2/100 = 2$ per 100$.

Recounting and double-counting uses the ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away.’

To find 2/3 of 12 means finding 2$ per 3$ of 12$. Here 12 recounts in 3s as 12$ = (12/3)*3$, giving (12/3)*2kr = 8$. So 2/3 of 12 is 8.

Brøker og procenter som per-tal

Problemer med brøker forsvinder ved at se en brøk som et per-tal, der fremkommer ved en dobbelt-tælling i samme enhed, 2/3 = 2kr per 3kr, eller som procent 2% = 2/100 = 2kr per 100kr.

Ved om-tælling og dobbelt-tælling bruges tælle-formlen T = (T/B)*B, der siger: ’Fra T kan vi T/B gange fjerne Bere’.

Herved findes 2/3 af 12 som 2kr per 3kr af 12 kr. Altså ved at om-tælle i 12 i 3ere som 12kr = (12/3) * 3kr, der giver (12/3) * 2kr = 8kr. Så 2/3 af 12 er 8.

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Fokus: Brøker og procenter som per-tal.

En klasse har problemer med brøker. Dels med at finde en brøkdel af en total, dels med at forlænge og forkorte, hvor mange adderer og subtraherer i stedet for at multiplicere og dividere.

Løsningen er at se en brøk som et per-tal, der fremkommer ved en dobbelt-tælling i samme enhed, 2/3 = 2kr per 3kr, eller som procent 2% = 2/100 = 2kr per 100kr.

Ved investering forventes et afkast, der kan være højere eller lavere, fx 7kr pr 5kr eller 3kr per 5kr.

Ved om-tælling og dobbelt-tælling bruges tælle-formlen T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere'.

Herved findes 2/3 af 12 som 2kr per 3kr af 12 kr. Altså ved at om-tælle i 12 i 3ere som 12kr = (12/3) * 3kr, der giver (12/3) * 2kr = 8kr. Så 2/3 af 12 er 8.

Opgaven' Hvor mange procent er 3 per 5?' løses ved at om-tælle 100 i 5ere or erstatte 5kr med 3kr: T = 100kr = (100/5) * 5kr, der giver (100/5) * 3kr = 60kr. Så 3/5 er det samme som 60 pr. 100, eller 3/5 = 60%.

At forlænge eller forkorte brøker sker ved at indsætte eller fjerne den samme enhed ovenfor og nedenfor brøklinjen: T = 2/3 = 2 4ere / 3 4ere = (2*4)/(3*4) = 8/12; og T = 8/12 = 4 2ere / 6 2ere = 4/6.

Faktisk kan og bør brøker og decimaltal introduceres i første klasse i forbindelse med optælling i ikoner under ti. 7 optalt i 3ere giver en stak på 2 3ere samt 1. Anbringes denne ved siden af i sin egen stak, fås et decimaltal, T = 7 = 2.1 3ere. Anbringes den ovenpå optalt som 3ere, fås en brøk: T = 7 = 2 1/3 3ere.

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Fractions and per-numbers added as integration

Factions dislike disappear if viewing a fraction as a per-number obtained from double- counting in the same unit, 3/5 = 3$ per 5$; and by respecting that also fractions are added with units: 1/2 of 2 + 2/3 of 3 gives 3/5 of 5. And not 7/6, as the school says.

Per-numbers also ad with units: 2kg at 3$/kg + 4kg at 5$/kg gives 6kg at (2*3$ + 4*5$)/6kg. Thus, here 3$/kg+5$/kg = 4.33$/kg, where the per-number add as the area under the piecewise constant per-number graph, called integration) Similarly, with fractions.

Brøker og per-tal adderet som integration

Problemer med at addere brøker forsvinder ved at se en brøk som et per-tal fremkommet fra dobbelt-tælling i samme enhed, 3/5 = 3kr per 5kr. Samt ved at respektere, at brøker adderes med enheder: ½ af 2 + 2/3 af 3 giver 3/5 af 5. og ikke 7/6, som skolen siger.

Også per-tal adderes med enheder: 2kg á 3kr/kg + 4kg á 5kr/kg giver 6kg á (2*3kr + 4*5kr)/6kg. Da 3kr/kg + 5kr/kg = 4.33kr/kg, adderes per-tal som arealet under den stykkevis konstante per-tals kurve (integration). Tilsvarende med brøker.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Brøker og per-tal adderet som integration.

En klasse har problemer med at addere brøker. Mange adderer tæller og nævner hver for sig.

Løsningen er at se en brøk som et per-tal fremkommet fra dobbelt-tælling i samme enhed, 3/5 = 3kr per 5kr, eller som procent 3% = 3/100 = 3kr per 100kr.

Samt at begynde med at addere brøker med enheder, som fx ½ af 2 + 2/3 af 3, der netop giver 1+2 af 2+3, altså 3/5 af 5. Her er altså ½+2/3 = 3/5, som fås ved at addere tæller og nævner hver for sig.

Tilsvarende adderes per-tal med enheder: 2kg á 3kr/kg + 4kg á 5kr/kg. Her adderes styktallene 2kg og 4kg direkte til 6kg, medens pertallene skal opganges til styktal før de kan adderes: 3*2kr + 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg. Så her er 3kr/kg + 5kr/kg = 4.33kr/kg, kaldet det vægtede gennemsnit.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tals kurve. Tilsvarende med brøker.

At addere to brøker a/b og c/d uden enheder er i princippet meningsløst, men kan gives mening ved at brøkerne tages af den samme total, b*d. Man får da additionen:

a/b af b*d + c/d af b*d, hvilket giver en total på a*d + c*b af b*d. Altså er a/b + c/d = (a*d + c*b)/b*d.

At addere brøker og pertal med enheder giver en god introduktion til calculus. Som vist er multiplikation før addition det samme som integration. Og omvendt integration er det samme som differentiation: Opgaven 2kg á 3kr/kg + 4kg á ?kr/kg = 6 kg á 5kr/kg fører til 6 + 4*x = 30 eller T1 + 4*x = T2, som løses med subtraktion før division, altså differentiation: x = (T2-T1)/4 = ΔT/4.

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Proportionality as double-counting with per-numbers

Proportionality dislike disappears by renaming it to ‘unit-shift by double-counting’, which leads to ‘per-numbers’ such as e.g. 2$ per 3kg or 2$/3kg or 2/3 $/kg. Recounting uses the ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away.’

Thus, a total of 16$ is recounted kg as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg can be recounted in $ as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

Proportionalitet som dobbelt-tælling med per-tal

Problemer med proportionalitet forsvinder ved at omdøbe proportionalitet til ’enheds-skift ved dobbelt-tælling’, som fører til ’per-tal’ som fx 2kr pr 3kg eller 2kr / 3 kg eller 2/3 kr/kg. Til om-tælling bruges ’omtællings-formlen’ T = (T/B)*B, der siger: ’Fra T kan vi T/B gange fjerne Bere’.

Herved kan 16kr om-tælles i 2ere som T = 16kr = (16/2) * 2kr = (16/2) * 3kg = 24 kg. Ligeledes kan de 12kg om-tælles i 3ere som T = 12kg = (12/3) * 3kg = (12/3) * 2kr = 8kr.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Proportionalitet som dobbelt-tælling med per-tal

En klasse har problemer med proportionalitet. Prisen er 2kr / 3kg. Alle finder kr-tallet for 12kg, men kun få finder kg-tallet for 16kr. Løsningen er at omdøbe proportionalitet til 'enheds-skift' ved ' dobbelt-tælling', som fører til 'per-tal' som fx 2kr pr 3kg eller 2kr / 3 kg eller 2/3 kr/kg. Enhederne forbindes så ved at om-tælle det kendte antal i per-tallet.

Op-tælling og om-tælling bruger begge 'tælle-formlen' T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere'.

Herved kan 16kr om-tælles i 2ere som T = 16kr = (16/2) * 2kr = (16/2) * 3kg = 24 kg. Ligeledes kan de 12kg om-tælles i 3ere som T = 12kg = (12/3) * 3kg = (12/3) * 2kr = 8kr. Vil denne forskel gøre en forskel? I teorien, ja, da proportionalitet forbindes med optælling, en basal fysisk aktivitet.

Faktisk findes proportionalitet i første klasse ved at op-tælle totaler i ikon-bundter forskellig fra standardbundtet ti og ved bagefter at om-tælle i en ny enhed. Dette fører direkte til tælleformlen, der har samme form som y = k*x.

Således kan en total på 8 op-tælles i 2ere som T = (8/2)*2 = 4*2 = 4 2ere.

Og en total på 3 4ere kan om-tælles til 5ere som T = (3*4/5)*5 = 2*5 + 2.

Og per-tal fører direkte videre til brøktal, som fremkommer ved dobbelt-tælling i samme enhed, fx 2kr per 3kr = 2kr/3kr = 2/3 = 2 per 3.

2/3 af 15 svarer til at få 2kr per 3kr af 15kr fundet ved at om-tælle 15 i 3ere og deraf tage 2: T = 15kr = (15/3)*3kr giver (15/3)*2kr = 10 kr. Så 2/3 af 15 er 10.

Tilsvarende findes 20% af 15 ved at om-tælle 15 i 100ere: T = 15 = (15/100)*100 giver (15/100)*20 = 3.

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Equations solved by moving across, inverse reckoning or recounting

Equations such as 2+3u = 14 and 25–u = 14 and 40/u = 5 are easily solved by the rule for reverse operations: ‘Move to opposite side with opposite calculation sign’.

The equation u+3 = 8 asks for a number u that added to 3 gives 8, which by definition is u = 8-3; thus +3 moves to the opposite side with the opposite calculation sign. Similarly with u*2 = 8 solved by u = 8/2; and with u^3 = 12 solved by u = 3√12, where the root is a ‘factor-finder’; and with 3^u = 12 solved by u = log3(12), where the logarithm is a ‘factor counter’.

Ekvationer løst ved overflytning, tilbageregning eller omtælling

Ekvationer som 2 + 3u = 14 og 25 – u = 14 og 40/u = 5' løses let ved reglen for omvendte operationer: ’Flyt til modsat side med modsat regnetegn'.

I u+3 = 8 søges det tal u, der adderet med 3 giver 8, hvilket pr. definition er u = 8-3; så +3 flytter til modsat side med modsat regnetegn. Tilsvarende med u*2 = 8, som løses af u = 8/2; og med u^ 3 = 12, som løses af u = 3√12, hvor roden er en faktor-finder; og med 3^u = 12, som løses af u = log3(12), hvor logaritmen er en faktor-tæller.

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Fokus: Ekvationer løst ved overflytning.

En klasse har problemer med ekvationerne 2 + 3u = 14 og 25 – u = 14 og 40/u = 5, hvor ekvationen er sammensat eller den ubekendte har omvendt regnetegn. Løsningen er at bruge definitionerne af de omvendte operationer til at fastlægge den grundlæggende løsningsregel: 'Flyt til modsat side med modsat regnetegn'.

I u+3 = 8 søges det tal u, der adderet med 3 giver 8, hvilket pr. definition er u = 8-3; så +3 flytter til modsat side med modsat regnetegn. Tilsvarende med u*2 = 8, som løses af u = 8/2; og med u^3 = 12, som løses af u = 3√12, hvor roden er en faktor-finder; og med 3^u = 12, som løses af u = log3(12), hvor logaritmen er en faktor-tæller.

Ekvationen 2 + 3*u = 14 kan ses som en dobbelt beregning, der reduceres til en enkelt af en parentes omkring den stærkere operation, 2 + (3*u). Flyttes 2 til modsat side med modsat regnetegn fås 3*u = 14-2. Så flyttes 3 til modsat side, hvor en parentes sættes om det, der først skal beregnes: u = (14-2)/3 = 12/3 = 4.

Ekvationen kan også løses ved frem-og-tilbage-gang: Frem ganges med 3 og adderes med 2. Tilbage subtraheres 2 og divideres med 3, så u = (14-2)/3 = 4.

I ekvationen 25 – u = 14 har u modsat regnetegn, og flytter derfor til modsat side for at få et normalt regnetegn. Herefter flyttes 14 til modsat side med modsat regnetegn: 25 = 14 + u; 25 – 14 = u; 11 = u.

Tilsvarende med 40/u = 5 som giver 40 = 5*u; 40/5 = u; 8 = u.

Har klassen lært bundt-tælling og om-tælling vil en dobbelt om-tælling gives 40 = (40/u)*u = 5*u, og 40 = (40/5)*5, så u = 40/5.

For flere detaljer, se det web-baserede lærerakademi MATHeCADEMY.net og MrAlTarp YouTube videoer.

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Calculus: Adding and splitting into locally constant per-numbers

Calculus is made easy by beginning with integral calculus for adding piecewise or locally constant per-numbers by their areas. Adding ‘2kg á 3$/kg + 4kg á 5$/kg’, the unit numbers 2 and 4 add directly to 6, while the per-numbers 3 and 5 must be multiplied to unit-numbers before they can add: 3*2 + 5*4 = 26. Thus, the answer is 6 kg á 26/6 $/kg. However, multiplication creates areas, so per-numbers add by the area under the piecewise constant per-number graph.

Calculus: Addition af og opdeling i lokalt konstante per-tal

Calculus lettes ved at begynde med integralregning til addition af stykkevis eller lokalt konstant per-tal ved deres arealer.

I additionen ’2kg á 3kr/kg + 4kg á 5kr/kg’ adderes styktallene 2kg og 4kg direkte til 6kg, medens pertallene skal opganges til styktal før de kan adderes: 3*2kr + 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tal graf.

Baggrund: Faldende PISA-resultater

Forskningen i matematikuddannelse er vokset siden dens første internationale kongres ICME1 i 1969. Ligeledes har finansiering, se fx ’National Center for Matematik’. På trods af ekstra forskning og finansiering og til trods for at være blevet advaret mod den mulige irrelevans af en voksende forskningsindustri (Tarp, 2004) har faldende svenske PISA-resultater forårsaget OECD til at skrive rapporten "Improving Schools in Sweden" (2015), der beskriver den svenske skolens som "havende brug for akut ændring", da "mere end en ud af fire studerende ikke engang opnår basisniveauet 2 i matematik, hvor eleverne begynder at demonstrere kompetencer for aktivt at deltage i livet (s. 3)."

Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

Fantasi som kernen i sociologi er beskrevet af Mills (1959). Bauman (1990) er enig ved at sige, at sociologisk tænkning "genindfører fleksibilitet til en verden, der er fastfrosset i rutiner ved at vise en alternativ verden, som den kunne være forskellig fra hvad den er nu (s. 16). "

Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

Et sådant eksempel er at sige, at formålet med matematikuddannelse er at lære matematik, da en sådan målsætning er åbenbart meningsløs ved sin selvreferencemåde.

Forbindelsen mellem et mål og dets midler er også til stede i den eksistentialistiske filosofi, der er beskrevet af Sartre (2007) som at fastholde, at "Eksistens går forud for essens (s. 20)". På samme måde påpeger Arendt (1963) at det at praktisere et middel blindt uden at reflektere over sit mål kan føre til at praktisere "ondskabets banalitet".

Inspireret af de gamle græske sofister, der ønsker at undgå at blive patroniseret af valg præsenteret som natur, søger ’differensforskning’ efter skjulte forskelle, der gør en forskel (Tarp, 2017). For at undgå en målforskydning i matematikuddannelsen spørger differensforskning: Hvordan ville matematikken se ud, hvis den grundfæstes i sin udvendige rod, den fysiske faktum Mange?

For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Calculus: Addition af og opdeling i lokalt konstante per-tal

En klasse har problemer med calculus. Løsningen er at udskyde differentialregning indtil efter integralregning er lært som et middel til at addere stykkevis eller lokalt konstant per-tal ved deres arealer.

I additionen '2kg á 3kr/kg + 4kg á 5kr/kg' adderes styktallene 2kg og 4kg direkte til 6kg, medens pertallene skal opganges til styktal før de kan adderes: 3*2kr + 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tal graf, altså som få arealstrimler, $S = \Sigma p^*\Delta x$.

Et ikke-konstant per-tal kan anses som lokalt konstant (kontinuert), hvilket betyder addition af uoverskueligt mange arealstrimler, $S = \int p^*dx$. Der dog kan lettes ved at omskrive strimlerne som tilvækster, $p^*dx = dy$ eller $dy/dx = p$. For ved opsummering af tilvækster vil alle midterled forsvinde og blot efterlade differensen mellem y-slut og y-start.

Dette giver en ægte motivation for differentialregning: Kan strimlen 2x*dx skrives som tilvæksten d(x^2), bliver summen ∫2x*dx differensen mellem x^2-slut og x^2-start.

Tilvækst- formler kan observeres i et rektangel, hvor ændringerne Δb og Δh i basen b og højden h giver den samlede ændring af arealet ΔT som summen af en lodret strimmel, Δb*h, og en horisontal strimmel, b*Δh, og et hjørne, Δb*Δh, der kan negligeres ved små ændringer.

Dvs. d(b*h) = db * h + b * dh eller optalt i Tere: dT/T = db/b + dh/h, eller med T' = dT/dx, T'/ T = b'/ b + h'/ h.

Derfor er (x^2)'/ x^2 = x'/x + x'/x = 2*x'/x, hvilket giver (x ^ 2)' = 2*x, da x' = dx/dx = 1.

Så differentialregning er først og fremmest nyttig til hurtig opsummering af mange tal. Sidenhen også til at beskrive grafers vækstforhold med henblik på optimering.

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Calculus in primary school, in middle school and in high school

Counted and recounted, blocks may be added, but on-top or next-to? Next-to addition of 2 3s and 4 5s as 8s means integrating their areas, called integral calculus where multiplication precedes addition. Asked oppositely '2 3s +? 5s gives 3 8s', the answer is obtained by letting subtraction precede division, called differential calculus. So, with block numbers, calculus occurs already in primary school.

In middle school calculus occurs when adding per-numbers in blending tasks as '2kg á 3$/kg + 4kg á 5$/kg = ?'

Calculus ved skolestart, i mellemskolen og i highskolen

Efter optælling følger addition af blokke, men skal de adderes ovenpå eller sidelæns?

Skal 2 3ere og 4 5ere adderes sidelæns som 8ere, sker det via deres areal, altså ved integration, hvor multiplikation kommer før addition.

Spørges modsat '2 3ere + ? 5ere giver 3 8ere', fås svaret ved at lade subtraktion komme før division, dvs. ved differentiation.

Så med bloktal, vil calculus forekomme allerede ved skolestarten.

I mellemskolen findes calculus i blandingsregning '2kg á 3kr/kg + 4kg á 5kr/kg = ?'

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces og hvordan man kan forbedre skolerne i Sverige og andre steder.

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

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For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

Fokus: Calculus ved skolestart, i mellemskolen og i highskolen.

Matematik betyder viden på græsk, der valgte ordet som fællesbetegnelse for vidensområderne aritmetik og geometri og musik og astronomi, som de så som studiet af mange for sig selv, i rum, i tid og i tid og rum.

Med musik og astronomi ude, er matematik i dag blot en fællesbetegnelse for algebra og geometri, begge med rødder i mange, som det fremgår af deres betydning på arabisk og græsk: at genforene tal og at måle jord.

Når vi møder mange, spørger vi 'hvor mange totalt?' Svaret får vi ved at tælle, før vi regner. Tælling sker ved at bundte og stakke, forudsagt af tælleformlen T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere', fx T = 3 4ere = (3*4)/5*5 = 2 5ere & 2.

Efter optælling følger addition af stakke, men skal de adderes ovenpå eller sidelæns?

Skal stakkene 2 3ere og 4 5ere adderes sidelæns som 8ere, sker det via deres areal, altså ved integration, hvor multiplikation kommer før addition.

Spørges modsat '2 3ere + ? 5ere giver 3 8ere', fås svaret ved at lade subtraktion komme før division, dvs. ved differentiation.

Så ved at optælle totaler i stak-tal, vil calculus forekomme allerede ved skolestarten.

I mellemskolen optræder calculus ved blandingsregning:

I additionen '2kg á 3kr/kg + 4kg á 5kr/kg' adderes styktallene 2kg og 4kg direkte til 6kg, medens per-tallene skal opganges til styk-tal før de kan adderes: 3*2kr + 5*4kr = 26kr. Så svaret er 6 kg á 26/6 kr/kg.

At addere gangestykker betyder geometrisk at addere arealer, hvilket kaldes integration. Så per-tal adderes som arealet under den stykkevis konstante per-tal graf, altså addition af arealstrimler, $S = \Sigma p*\Delta x$, eller $S = \int p*dx$ i highskolen, hvor per-tallene er lokalt konstante (kontinuerte), og igen først skal adderes før de kan subtraheres ved differentiation.

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STEM-based core-math makes migrants pre-engineers

Our word- and number-language assign words and numbers to the world with sentences and formulas that contain a subject, a verb, and a predicate. With a number-language, young migrants can access core-math directly: Recounting in a new unit is predicted by a ‘recount-formula’ T = (T/B)*B, saying ‘From the total T, T/B times, Bs can be pushed away’, e.g., T = 3 4s = (3*4)/5*5 = 2 5s & 2. After recounting and recounting, blocks may add next-to as areas (integration) or on-top if the units are made equal by recounting (proportionality).

STEM-baseret kerne-matte gør migranter til præ-ingeniører

Vore tale- og tal-sprog itale- og ital-sætter verden med sætninger og formler, som indeholder et subjekt, et verbum og et prædikat. Med tal-sprog får unge migranter adgang til med kerne-matte: Omtælling i ny enhed forudsiges af en ’tælle-formel’ T = (T/B)*B, der siger: ’Fra T kan vi T/B gange fjerne Bere’, fx T = 3 4ere = (3*4)/5*5 = 2 5ere & 2. Efter op- og omtælling kan blokke adderes sidelæns som arealer (integration) eller ovenpå hvis enhederne gøres ens ved omtælling (proportionalitet).

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere, at "Den ideelle rationalitet indeholder en iboende fare for en anden afvigelse fra dets formål - faren for såkaldt målforskydning (s. 84)."

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Fokus: STEM-baseret kerne-matte.

Vi mestrer omverdenen med tale-sprog og tal-sprog, der itale-sætter og ital-sætter ting med sætninger og formler, som indeholder et subjekt, et verbum og et prædikat: 'Bordet er gult' og 'Totalen er 3 4ere'. Til et sprog hører et metasprog, en grammatik og en matematik, som bør læres efter sproget.

Unge migranter får direkte adgang til tal-sproget med kerne-matte: A) Cifre er ikoner med det antal streger, det repræsenterer. B) Regnearter er ikoner for bundt-tælling: division fjerner bundter, multiplikation stakker bundter, subtraktion fjerner stakken for at finde u-bundtede, addition forener stakke ovenpå eller sidelæns. C) Bundt-tælling og bundtskrivning viser bundter inden for koppen og u-bundtede udenfor, fx T = 4B5 = 4.5 tiere = 45. D) Totaler skal op-tælles og om-tælles og dobbelt-tælles før de adderes. E) Om-talt i samme enhed kan en total skrives på 3 måder: normal, med overlæs eller underlæs, fx T = 46 = 4B6 = 3B16 = 5B-4. F) Om-tælling i ny enhed (proportionalitet) forudsiges af en 'tælle-formel' T = (T/B)*B, der siger: 'Fra T kan vi T/B gange fjerne Bere', fx T = 3 4ere = (3*4)/5*5 = 2 5ere & 2. G) Om-tælling fra tiere til ikoner giver ekvationer, fx x*5 = 40 = (40/5)*5 med løsning x = 40/5.

Dobbelttælling giver per-tal og proportionalitet med om-tælling i pertallet: med 2kr per 3 kg er 12 kg = (12/3)*3kg = (12/3)*2kr = 8kr. H) Efter optælling kommer addition, ovenpå og sidelæns, der fører til proportionalitet og integration. I) Omvendt addition fører til ligninger og differentiation. J) Per-tal fører til brøker, der som operatorer begge skal opganges for at blive tal, og derfor adderes som arealer, altså ved integration. K) Calculus er addition af og opdeling i lokalt konstante per-tal. L) Undervejs inddrages centrale STEM-områder under temaet 'vand i bevægelse'.

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Math blocks for the block-organized secondary school

In the EU, secondary school is a line-organized with compulsory classes forcing the teenagers to follow the year-group despite boys being two years behind in maturity. For economic reasons, low achievers are forced to stay in the class which cannot be repeated.

In the United States, secondary school supports the teenager's identity work by welcoming them with esteem: 'Inside, you carry a talent that together, we will now uncover and develop through daily homework in self-selected half-year blocks of a practical or theoretical nature together with teachers who have one subject only.'

Matte-blokke til den blok-opdelte sekundærskole

I EU er sekundærskolen linjeopdelt med 'tvangsklasser', hvor de unge følger årgangen til skade for drenge, der er to år bagud i modenhed. Af økonomiske grunde fastholdes langsomme unge, som får lave karakterer og lavt selvværd. I USA støtter den identitetsarbejdet ved at byde den unge velkommen med agtelse: 'Du bærer et talent, som vi i fællesskab nu afdækker og udvikler gennem daglige lektier i selvvalgte halvårsblokke af praktisk eller teoretisk art sammen med lærere, der kun har ét fag.'

Baggrund: Faldende PISA-resultater på trods af øget forskning.

Forskningen i matematikuddannelse er vokset siden dens første internationale kongres ICME1 i 1969. Ligeledes har finansiering, se fx 'National Center for Matematik'. På trods af ekstra forskning og finansiering og til trods for at være blevet advaret mod den mulige irrelevans af en voksende forskningsindustri (Tarp, 2004) har faldende svenske PISA-resultater forårsaget OECD til at skrive rapporten "Improving Schools in Sweden" (2015), der beskriver den svenske skolens som "havende brug for akut ændring", da "mere end en ud af fire studerende ikke engang opnår basisniveauet 2 i matematik, hvor eleverne begynder at demonstrere kompetencer for aktivt at deltage i livet (s. 3)."

Da matematikuddannelse er en social institution, kan social teori muligvis give et fingerpeg om den manglende forskningssucces.

Fantasi som kernen i sociologi er beskrevet af Mills (1959). Bauman (1990) er enig ved at sige, at sociologisk tænkning "genindfører fleksibilitet til en verden, der er fastfrosset i rutiner ved at vise en alternativ verden, som den kunne være forskellig fra hvad den er nu (s. 16). "

Med hensyn til institutioner, hvoraf matematikuddannelsen er et eksempel, taler Bauman om rationel handling "hvor målet er tydeligt angivet, og aktørerne koncentrerer deres tanker og bestræbelser på at vælge sådanne midler til målet som viser sig at være mest effektive og økonomiske (s. 79) ". Han påpeger endvidere faren for en målforskydning (s. 84).

Et sådant eksempel er at sige, at formålet med matematikuddannelse er at lære matematik, da en sådan målsætning er åbenbart meningsløs ved sin selvreferencemåde.

Forbindelsen mellem et mål og dets midler er også til stede i den eksistentialistiske filosofi, der er beskrevet af Sartre (2007) som at fastholde, at "Eksistens går forud for essens (s. 20)". På samme måde påpeger Arendt (1963) at det at praktisere et middel blindt uden at reflektere over sit mål kan føre til at praktisere "ondskabets banalitet".

Inspireret af de gamle græske sofister, der ønsker at undgå at blive patroniseret af valg præsenteret som natur, søger 'differensforskning' efter skjulte forskelle, der gør en forskel (Tarp, 2017). For at undgå en målforskydning i matematikuddannelsen spørger differensforskning: Hvordan ville matematikken se ud, hvis den grundfæstes i sin udvendige rod, den fysiske faktum Mange?

For at finde et svare bruges Grounded Theory (Glaser & Strauss, 1967), til at løfte Piaget videns-tilegnelse (Piaget, 1970) fra et personligt til et socialt niveau for at give fænomenet Mange mulighed for at åbne sig for os og skabe egne kategorier og egenskaber.

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Så alle forlader halvårsblokken med ros. Og med lyst til som 18-årige at prøve kræfter med de tertiære jobrettede veje, hvor netop deres personlige talent kan udfoldes. Og som også er opdelt i halvårsblokke, så man hurtigt kan supplere eller udbygge sin grad med nye blokke ved jobskifte eller arbejdsløshed.

På en blokopdelt skole kan matte tilbydes i forskellige blokke med teoretisk eller praktisk udgangspunkt, så alle kan få kompetencer til at påbegynde en tertiær STEM-uddannelse (Science, Technology, Engineering, Math). Der kan også tilbydes blokke, som gentager primærskolens matte (tal, regnearter, addition, subtraktion, multiplikation, division, brøker, osv.). Og som kan kurere matte ulyst og give en introduktion til kerneområderne proportionalitet, ligninger og calculus ved at bruge en anderledes tilgang (mange-matik) med bloktal, bundt-tælling, om-tælling og dobbelt-tælling med per-tal før addition.

En blokopdelt skole er særlig effektiv for unge migranter som ønsker hurtigt at opnå STEM-kompetence for at bidrage til genopbygning af deres oprindelige hjemland.

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The teacher as a difference researcher

Difference research finds hidden differences that make a difference. It is used to solve problems in class. Or by teacher-researchers with shared time between academic work at a university and intervention research in a class. Or by full-time researchers working together with teachers: The teacher observes problems, the researcher identifies differences. A mutual micro-curriculum is created, tested by the teacher and reported by the researcher in a pretest-posttest study.

Læreren som differens-forsker

Differensforskning finder skjulte forskelle, der gør en forskel. Den bruges til at løse problemer i klassen, eller af lærer-forskere, der deler tid mellem akademisk arbejde på et universitet og interventionsforskning i en klasse. Eller af fuldtids forskere, der samarbejder med lærerne om fælles brug: læreren observerer problemer, forskeren identificerer forskelle. Sammen udarbejdes et mikro-curriculum, der testes af læreren og rapporteres af forskeren efter en pretest-posttest undersøgelse.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Læreren som differens-forsker.

Når traditioner giver problemer, kan differensforskning afdække skjulte forskelle der gør en forskel. Eksempelvis siger traditionen, at en funktion er et eksempel på en mængderelation, hvor førstekomponentidentitet medfører andenkomponent-identitet, hvilke du unge hører som 'bublibub er et eksempel som bablibab', som ingen finder meningsfyldt. En forskel er at bruge Eulers oprindelige definition, som alle unge godtager problemløst: 'En funktion er et fællesnavn for regnestykker med både kendte og ukendte tal.'

Differensforskning kan bruges af lærere til at løse problemer i klassen, eller af lærer-forskere, der deler deres tid mellem akademisk arbejde på et universitet og interventionsforskning i en klasse. Eller af fuldtids forskere, der samarbejder med lærerne om fælles brug af differensforskning: læreren observerer problemer, forskeren identificerer forskelle. Sammen udarbejdes et mikro-curriculum, der testes af læreren og rapporteres af forskeren efter en pretest-posttest undersøgelse. En typisk differensforsker begynder som en almindelig lærer, der reflekterer over, om alternativer kan løse observerede læringsproblemer.

En differens-forsker kombinerer elementer fra aktionslæring og aktionsforskning og interventionsforskning og designforskning. Først identificeres en forskel, så designes et mikro-curriculum, der testes for at se, hvad der gør en forskel. Forløbet rapporteres internt og drøftes med kolleger. Efter at have gentaget denne cyklus af design, undervisning og intern rapportering, kommer den eksterne rapportering til magasiner eller konferencer eller tidsskrifter af forskellen og hvilken forskel den gør i en pretest-posttest undersøgelse.

Forskning bør skabe viden til at forklare naturen og forbedre sociale forhold. Men som institution risikerer den det, sociologen Bauman kalder en målforskydning, så forskningen bliver selvrefererende i stedet for at finde forskelle.

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MATHeCADEMY.net: Math as a Natural Science about Many, a Booth Exhibit

MATHeCADEMY.net provides online courses for teachers who want to teach mathematics as ‘Many-Math’, i.e. as a natural science about the physical fact many; as well as wanting to see mathematics as a number-language in family with the word-language, both using full sentences with a subject, a verb and a predicate, and where two competencies are sufficient: counting and adding in space and time. Many-Math respects the child’s own number-language with flexible 2D bundle-numbers like T = 2*3 = 2 3s; and counting before addition.

MATHeCADEMY.net: Læreruddannelse i matte som naturvidenskaben om Mange, ideudstilling

MATHeCADEMY.net giver online-kurser til lærere, der ønsker at undervise i matematik som ’mange-matik’, dvs. som en naturvidenskab om det fysiske faktum Mange; samt ønsker at se matematik som et tal-sprog i familie med tale-sproget, hvor begge bruger fulde sætninger med subjekt, verbum og prædikat, og hvor to kompetencer er tilstrækkeligt: at tælle og regne i rum og tid. Mange-matik respekterer barnets eget tal-sprog, med fleksible 2D bundt-tal som T = 2*3 = 2 3ere, og tælling før addition.

Baggrund: Faldende PISA-resultater på trods af øget forskning.

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Fokus: Læreruddannelse i matte som naturvidenskaben om Mange.

Peter er matematiklærer i en klasse af hvor mange har opgivet division og brøker, så Peter er ved at opgive at være lærer, da han hører om '1kop & 5pinde' metoden til at helbrede matematik-modvilje (Tarp, 2018), og da han ser 'CupCount and ReCount before you add’ (www.youtube.com/watch?v=IE5nk2YEQIAxx).

Her bundt-tælles 5 pinde i 2ere med en kop til bundterne. Han ser, at en total kan optælles i samme enhed på tre måder: overlæs, standard og underlæs:

T = 5 = I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = IIII II I = 3B-1 2s

Så optalt i bundter giver indvendigt et antal bundter og et udvendigt et antal singler; og flyttes en pind ud eller ind skaber det overlæs eller underlæs.

Ved multiplikation kan 7 x 48 bundt-skrives som 7 x 4B8, hvilket giver 28 indvendige og 56 udvendige, der er et overlæs, der kan omtælles: T = 7 x 4B8 = 28B56 = 33B6 = 336.

Og når du deler, kan 336/7 bundt-skrives som 33B6 / 7, der omtælles til 28 indeni og 56 udenfor i henhold til multiplikationstabellen. Så 33B6 / 7 = 28B56 / 7 = 4B8 = 48.

For at prøve det selv, downloader Peter 'CupCount & ReCount Booklet'. Han giver en kopi til sine kolleger, og de beslutter at arrangere et gratis 1-dags Skype-seminar.

Om morgenen ser de PowerPoint-præsentationen 'Curing Math Dislike' og diskuterer seks spørgsmål: først problemerne med moderne matematik, MetaMatism; så potentialerne i postmoderne matematik, ManyMath; så forskellen mellem de to; så et forslag til en ManyMath læseplan på grundskolen og mellem- og gymnasiet; så teoretiske aspekter; og endelig hvor man kan lære om ManyMath.

Her er MetaMatisme en blanding af MateMatism, sand inde i et klasseværelse, men sjældent udenfor hvor '2 + 3 = 5' modsiges af 2 uger + 3dage = 17dage; og MetaMatik, der præsenterer begreber TopDown som et eksempel på en abstraktion i stedet for BottomUp som en abstraktion fra mange eksempler: ”En funktion er et eksempel på et mængdeprodukt” i stedet for ”En funktion er en tal-sprogs sætning”

Om eftermiddagen arbejder gruppen med en udvidet version af CupCount & ReCount Booklet, hvor Peter hjælper de nyankomne. På seminaret er der to Skype-sessioner med en ekstern instruktør, en middag og en om eftermiddagen.

Ved at bringe ManyMath til sit klasseværelse, ser Peter, at mange vanskeligheder forsvinder, så han tager en 1 års fjernundervisning på MATHeCADEMY.net for at undervise i matematik som ManyMath, en naturvidenskab om mange. Peter og 7 andre afprøver PYRAMIDeDUCATION, hvor de er organiseret i 2 hold af 4 lærere, der vælger 3 par og 2 instruktører på tur. En ekstern træner hjælper instruktørerne med at instruere resten af deres hold. Hvert par arbejder sammen for at løse tæl&regn problemer og rutinemæssige problemer; og at udføre en pædagogisk opgave, der skal rapporteres i et essay rigt på observationer af eksempler på kognition, både genkognition og ny kognition, dvs. både assimilering og akomodering. I et par korrigerer hver lærer den andens rutineopgave. Hvert par er opponeknt på essayet i et andet par. Hver lærer betaler ved at coaching en ny gruppe på 8 lærere.

På akademiet kaldes 2x4 sektionerne CATS til grundskole og gymnasium inspireret af, at vi mestrer mange ved at tælle og regne i tid og rum.

På akademiet læres børnenes matematik gennem guidede sætningsfrie møder med sætningens subjekt, hvilket udvikler tavs viden og kompetencer samt individuelle sætninger, der kommer fra abstraktioner og valideringer i laboratoriet, dvs. gennem automatisk ’grib for at begribe’-læring.

De unges matematik læres gennem pædagogiske sætningsbelastede fortællinger abstraheret fra og valideret i laboratoriet, det vil sige gennem automatisk "sladderlæring": Tak for at fortælle mig noget nyt om noget, jeg allerede vidste.

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Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. Journal of Mathematics Education, 11(1), 103-117.

Introducing Allan Tarp

As a curriculum architect at the MATHeCADEMY.net, Allan Tarp uses difference research to find differences that make a difference, such as icon-numbers, bundle-counting, recounting and double-counting and per-numbers, see MrAlTarp YouTube videos, and the article ‘Mastering Many’ in Journal of Mathematics Education, 11 (1), 103-117. As an instructor at MATHeCADEMY.net, he teaches teachers to teach mathe-matics as ‘Many-Math’, a natural science about many, with two competencies only, count and add in time and space.

Som curriculum arkitekt ved MATHeCADEMY.net, bruger Allan Tarp ’Differens-forskning' til at finde forskelle, der gør en forskel, som fx ikontal, bundt-tælling, om-tælling og dobbelt-tælling og per-tal, se MrAlTarp YouTube videoer, samt artiklen ‘Mastering Many’ i Journal of Mathematics Education, 11(1), 103-117. Som instruktør på MATHeCADEMY.net lærer han lærere at undervise i mate-matik som ’mange-matik’, en naturvidenskab om Mange, med blot to kompetencer, tæl og regn i tid og rum

A Case: Peter, stuck in division and fractions

Being a mathematics teacher in a class of ordinary students and repeaters flunking division and fractions, Peter is about to give up teaching when he learns about the ‘1cup & 5sticks’ method to cure mathematics dislike by watching ‘CupCount and ReCount before you Add’ (https://www.youtube.com/watch?v=IE5nk2YEQIAxx).

Here 5 sticks are CupCounted in 2s using a cup for bundles. He sees that a total can be recounted in the same unit in 3 different forms: overload, standard and underload:

T = 5 = I I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = II II II I = 3B-1 2s

So counted in bundles, a total has an inside number of bundles and an outside number of singles; and moving a stick out or in creates an over-load or an under-load.

When multiplying, 7 x 48 is bundle-written as 7 x 4B8 resulting in 28 inside and 56 outside as an overload that can be recounted: T = 7 x 4B8 = 28B56 = 33B6 = 336.

And when dividing, 336/7 is bundle-written as 33B6 /7 recounted to 28 inside and 56 outside according to the multiplication table. So 33B6 /7 = 28B56 /7 = 4B8 = 48.

To try it himself, Peter downloads the ‘CupCount & ReCount Booklet’. He gives a copy to his colleagues and they decide to arrange a free 1day Skype seminar.

In the morning they watch the PowerPoint presentation ‘Curing Math Dislike’, and discuss six issues: first the problems of modern mathematics, MetaMatism; next the potentials of postmodern mathematics, ManyMath; then the difference between the two; then a proposal for a ManyMath curriculum in primary and middle and high school; then theoretical aspects; and finally where to learn about ManyMath.

Here MetaMatism is a mixture of MatheMatism, true inside a classroom but rarely outside where ‘2+3 = 5’ is contradicted by 2weeks+3days = 17days; and MetaMatics, presenting a concept TopDown as an example of an abstraction instead of BottomUp as an abstraction from many examples: A function IS an example of a set-product.

In the afternoon the group works with an extended version of the CupCount & ReCount Booklet where Peter assists newcomers. At the seminar there are two Skype sessions with an external instructor, one at noon and one in the afternoon.

Bringing ManyMath to his classroom, Peter sees that many difficulties disappear, so he takes a 1year distance learning education at the MATHeCADEMY.net teaching teachers to teach MatheMatics as ManyMath, a natural science about Many. Peter and 7 others experience PYRAMIDeDUCATION where they are organised in 2 teams of 4 teachers choosing 3 pairs and 2 instructors by turn. An external coach assists the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. In a pair each teacher corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays by coaching a new group of 8 teachers.

At the academy, the 2x4 sections are called CATS for primary and secondary school inspired by the fact that to deal with Many, we Count & Add in Time & Space.

At the academy, primary school mathematics is learned through educational sentence-free meetings with the sentence subject developing tacit competences and individual sentences coming from abstractions and validations in the laboratory, i.e. through automatic ‘grasp-to-grasp’ learning.

Secondary school mathematics is learned through educational sentence-loaded tales abstracted from and validated in the laboratory, i.e. through automatic ‘gossip-learning’: Thank you for telling me something new about something I already knew.

Et tilfælde: Peter, kørt fast i division og fraktioner

Peter er matematiklærer i en klasse af hvor mange har opgivet division og brøker, så Peter er ved at opgive at være lærer, da han hører om '1kop & 5pinde' metoden til at helbrede matematik-modvilje, og da han ser 'CupCount and ReCount before you add (https://www.youtube.com/watch?v=IE5nk2YEQIAxx).

Her bundt-tælles 5 pinde i 2ere med en kop til bundterne. Han ser, at en total kan optælles i samme enhed på tre måder: overlæs, standard og underlæs:

T = 5 = I I I I = II I I I = 1B3 2s = II II I = 2B1 2s = IIII II I = 3B-1 2s

Så optalt i bundter giver indvendigt et antal bundter og et udvendigt et antal singler; og flyttes en pind ud eller ind skaber det overlæs eller underlæs.

Ved multiplikation kan 7 x 48 bundt-skrives som 7 x 4B8, hvilket giver 28 indvendige og 56 udvendige, der er et overlæs, der kan omtælles: T = 7 x 4B8 = 28B56 = 33B6 = 336.

Og når du deler, kan 336/7 bundt-skrives som 33B6 / 7, der omtælles til 28 indeni og 56 udenfor i henhold til multiplikationstabellen. Så 33B6 / 7 = 28B56 / 7 = 4B8 = 48.

For at prøve det selv, downloader Peter 'CupCount & ReCount Booklet'. Han giver en kopi til sine kolleger, og de beslutter at arrangere et gratis 1-dags Skype-seminar.

Om morgenen ser de PowerPoint-præsentationen 'Curing Math Dislike' og diskuterer seks spørgsmål: først problemerne med moderne matematik, MetaMatisme; så potentialerne i postmoderne matematik, ManyMath; så forskellen mellem de to; så et forslag til en ManyMath læseplan på grundskolen og mellem- og gymnasiet; så teoretiske aspekter; og endelig hvor man kan lære om ManyMath.

Her er MetaMatisme en blanding af MateMatisme, sand inde i et klasseværelse, men sjældent udenfor hvor '2 + 3 = 5' modsiges af 2 uger + 3dage = 17dage; og MetaMatik, der præsenterer begreber TopDown som et eksempel på en abstraktion i stedet for BottomUp som en abstraktion fra mange eksempler: ”En funktion er et eksempel på et mængdeprodukt” i stedet for ”En funktion er en tal-sprogs sætning”

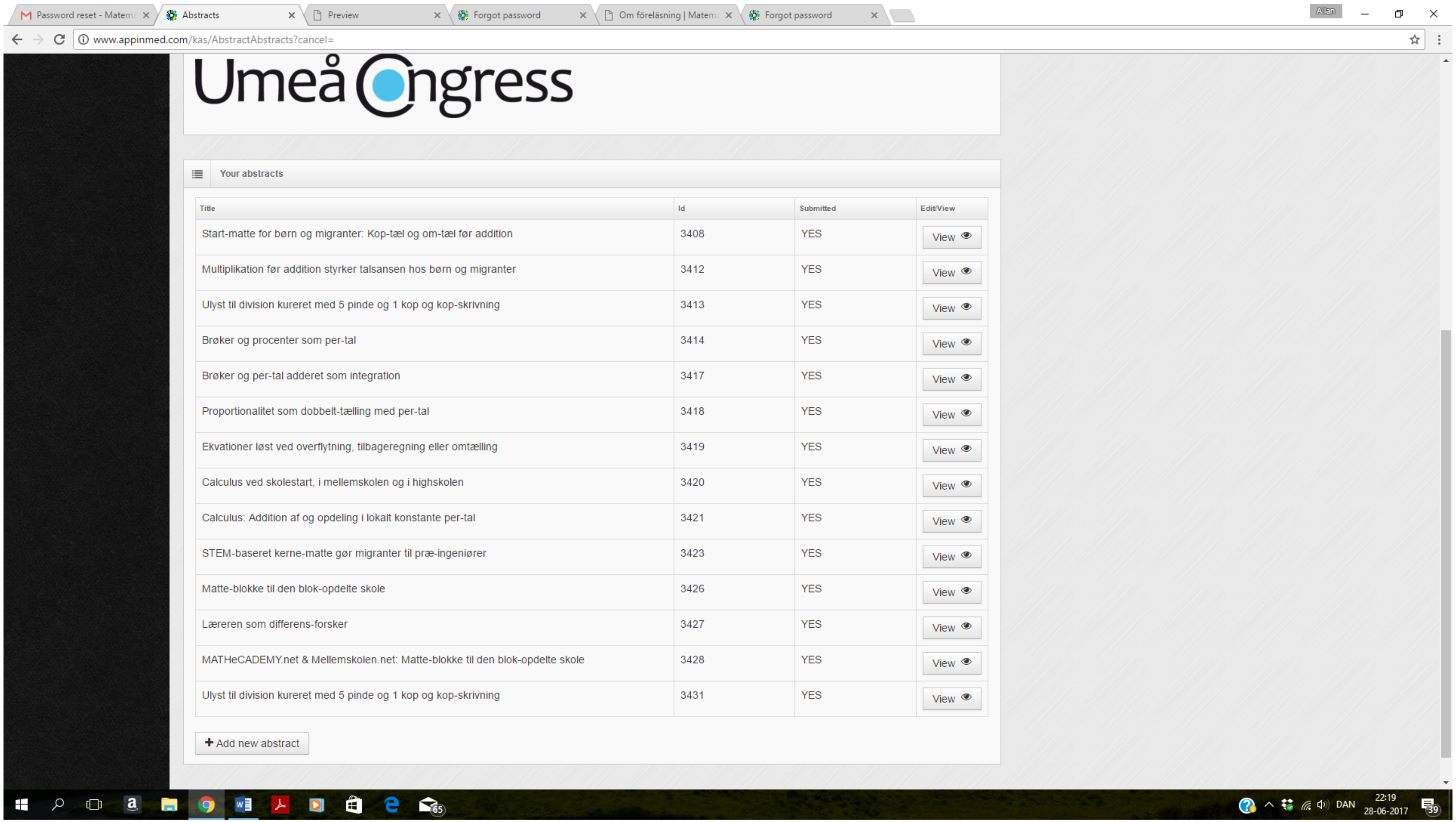
Om eftermiddagen arbejder gruppen med en udvidet version af CupCount & ReCount Booklet, hvor Peter hjælper de nyankomne. På seminaret er der to Skype-sessioner med en ekstern instruktør, en middag og en om eftermiddagen.

Ved at bringe ManyMath til sit klasseværelse, ser Peter, at mange vanskeligheder forsvinder, så han tager en 1 års fjernundervisning på MATHeCADEMY.net for at undervise i matematik som ManyMath, en naturvidenskab om mange. Peter og 7 andre afprøver PYRAMIDeDUCATION, hvor de er organiseret i 2 hold af 4 lærere, der vælger 3 par og 2 instruktører på tur. En ekstern træner hjælper instruktørerne med at instruere resten af deres hold. Hvert par arbejder sammen for at løse tæl&regn problemer og rutinemæssige problemer; og at udføre en pædagogisk opgave, der skal rapporteres i et essay rigt på observationer af eksempler på kognition, både genkognition og ny kognition, dvs. både assimilering og akkomodering. I et par korrigerer hver lærer den andens rutineopgave. Hvert par er opponent på essayet i et andet par. Hver lærer betaler ved at coaching en ny gruppe på 8 lærere.

På akademiet kaldes 2x4 sektionerne CATS til grundskole og gymnasium inspireret af, at vi mestrer mange ved at tælle og regne i tid og rum.

På akademiet læres børnenes matematik gennem guidede sætningsfrie møder med sætningens subjekt, hvilket udvikler tavs viden og kompetencer samt individuelle sætninger, der kommer fra abstraktioner og valideringer i laboratoriet, dvs. gennem automatisk 'grib for at begribe'-læring.

De unges matematik læres gennem pædagogiske sætningsbelastede fortællinger abstraheret fra og valideret i laboratoriet, det vil sige gennem automatisk "sladderlæring": Tak for at fortælle mig noget nyt om noget, jeg allerede vidste.



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Title	Id	Submitted	Edit/View
Start-matte for børn og migranter: Kop-tæl og om-tæl før addition	3408	YES	View
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Ulyst til division kureret med 5 pinde og 1 kop og kop-skrivning	3413	YES	View
Brøker og procenter som per-tal	3414	YES	View
Brøker og per-tal adderet som integration	3417	YES	View
Proportionalitet som dobbelt-tælling med per-tal	3418	YES	View
Ekvationer løst ved overflytning, tilbageregning eller omtælling	3419	YES	View
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Calculus: Addition af og opdeling i lokalt konstante per-tal	3421	YES	View
STEM-baseret kerne-matte gør migranter til præ-ingeniører	3423	YES	View
Matte-blokke til den blok-opdelte skole	3426	YES	View
Læreren som differens-forsker	3427	YES	View
MATHeCADEMY.net & Mellemskolen.net: Matte-blokke til den blok-opdelte skole	3428	YES	View
Ulyst til division kureret med 5 pinde og 1 kop og kop-skrivning	3431	YES	View

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32. DE-MODELING NUMBERS, OPERATIONS AND EQUATIONS: FROM INSIDE-INSIDE TO OUTSIDE-INSIDE UNDERSTANDING

ABSTRACT

Adapting to the outside fact Many, children internalize social number-names, but how do they externalize them when communicating about outside numerosity? Mastering Many, children use bundle-numbers with units; and flexibly use fractions and decimals and negative numbers to account for the unbundled singles. This suggests designing a curriculum that by replacing abstract-based with concrete-based psychology mediates understanding through de-modeling core mathematics, thus allowing children to expand the number-language they bring to school.

Keywords: number; operation; equation; numeracy; proportionality; early childhood

1. Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish Centre for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA results caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life (p. 3)". In Germany the corresponding number is one of five students, according to a plenary address at the Educating Educators conference in Freiburg in October 2019.

This raises some questions: Is mathematics so hard that one out of four or five students cannot master even basic numeracy? Is it mathematics we teach? Do we use the proper psychological learning theories? Can we design a different mathematics curriculum where most students become successful learners? In short: could this be different?

2. Materials/ Subjects and Methods

To get an answer we use difference research (Tarp, 2018) to create a design research cycle (Bakker, 2018) consisting of reflection, design, and implementation.

2.1. Reflections on Different forms of Mathematics

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric (Russell, 1945).

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. So, as a label, mathematics has no existence itself, only its content has, algebra and geometry; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-math' rooted in Many was replaced by the 'New Mathematics'.

Here the invention of the concept Set created a 'setcentric' (Derrida, 1991), 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant self-reference defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. To avoid self-reference, Russell developed a type theory defining concepts from examples at the abstraction level below. This implies that fractions cannot be numbers, but operators needing numbers to become numbers.

Wanting fractions to be rational numbers, the setcentric mathematics neglected Russell's paradox and insisted that to be well defined, a concept must be derived from the mother-concept set above.

In this way, the concept Set changed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism', true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2 x 3 = 6' stating that 2 3s can always be re-counted as 6 1s (Tarp, 2018).

Rejecting setcentrism as making mathematics to abstract, many Anglo-Saxon countries went back to basics and now teach what they call 'school mathematics' although this still is mathe-matism adding numbers and fractions without units.

So, today we have three different forms of mathematics: a pre setcentric mathe-matism saying that a function is a calculation with specified and unspecified numbers; a present setcentric meta-matism saying that a function is a subset of a set-product where first-component identity implies second-component identity; and a post setcentric many-math (Tarp, 2018) saying that a function is a number-language sentence with a subject, a verb and a predicate as in the word-language; and that a communicative turn learning language through communication instead of through grammar is needed in the number-language also (Widdowson, 1978).

In its pre and present setcentric versions, a mathematics curriculum typically begins with digits together with addition, later to be followed by subtraction as reversed addition, multiplication as repeated addition, and division as reversed multiplication - sometimes as repeated subtraction also. Then follows fractions, percentages and decimals as rational numbers. Then comes negative numbers, to be followed by expressions with unspecified letter numbers, and by solving equations.

Present setcentric meta-matics defines numbers by inside abstract self-reference as examples of sets. Zero is defined as the empty set. One is defined as the set containing the empty set as its only element. The next numbers then are generated by a follower principle.

With natural numbers defined, integers are defined as equivalence classes in the set- product of natural numbers created by the equivalence relation saying that (a,b) is equivalent to $(c.d)$ if cross addition holds, $a+d = b+c$. This makes (-2,0) equivalent to (0,2) thus geometrically forming straight lines with gradient 1 in a coordinate system.

With integers defined, rational numbers are defined as equivalence classes the set- product of integers created by the equivalence relation saying that (a,b) is equivalent to $(c.d)$ if cross multiplication holds, $a \times d = b \times c$, thus making (2,3) equivalent to (8,12) thus geometrically forming straight lines with various gradients in a coordinate system.

Equations are examples of open statements that may be transformed to a solution by using abstract algebra's group theory to neutralize numbers by their inverse numbers.

In geometry, halfplanes define lines that are parallel if a subset relation exists among their halfplanes. And an angle is the intersection set of two halfplanes.

Post setcentric many-math is grounded in the observation that when asked "How old next time?", a 3year-old will answer "4", but will object to 4 fingers held together 2 by 2: "That is not 4; that is 2 2s." So, when adapting to the outside fact Many children count in bundles, and use double-numbers to describe both the numbers of bundles and the bundle-unit. And it turns out that double-numbers contain the core of mathematics since recounting to change units implies proportionality and equations; and when adding double-numbers, on-top addition leads to proportionality making the units like, and next-to addition means adding areas, which leads to integral calculus.

2.2. Reflections on Different forms of Psychology

As institutionalized learning, education is meant to help human brains adapt to the outside world by accommodating schemas failing to assimilate it (Piaget, 1970); or to mediate institutionalized schemas that may colonize the brain (Habermas, 1981).

Adaption is theorized by psychology, often seen as the science of behavior and mind, thus being a sub-discipline of life science, where biology sees life as communities of green and grey cells, plants and animals. Plants stay and get the energy directly from the sun. Animals move to get the energy from plants or other animals, thus needing holes in the head for food and information, making the brain transform stimuli to behavior responses.

Besides the reptile and mammal brains for routines and feelings, humans also have a third human brain for balancing and for storing and sharing information, made possible by transforming forelegs to arms with hands that can grasp (and share) food and things that accompanied by sounds develop a language about the six core outside components: I, you, he-she-it, we, you, and they; or in German: ich, du, er-sie-es, wir, ihr, sie.

Receiving information may be called learning; and transmitting information may be called teaching. Together, learning and teaching may be called education, that may be unstructured or structured e.g. by a social institution called education.

Educational psychology first focused on behavior by studying stimulus-response pairings, called classical conditioning where Pavlov showed how dogs would salivate when hearing a sound previously linked to food. Later Skinner (1953) developed this into an operant conditioning by adding the concepts of reinforcement and punishment as stimuli following a student response coming from building routines through repetition.

But, does correct responses imply understanding? So, the educational psychology called constructivism focuses on what happens in the mind when constructing inside meaning to outside stimuli. Here especially Piaget (1970), Vygotsky (1986), and Bruner (1977) have contributed in creating teaching methods and practices.

Piaget found four different development stages for children: the sensorimotor stage below 2 years old, the preoperational state from 2 to 7 years old, the concrete operational stage from 7 to 10 years old, and formal operational stage from 11 years old and up.

In philosophy, existentialism sees existence as preceding essence (Sartre, 2007). Where Piaget sees learning taking place through adaption to outside existence, Vygotsky focuses on adaption to inside institutionalized essence, i.e. through enculturation allowing learners to expand their 'Zone of Proximal Development' (ZPD) under guidance of a more knowledgeable other. "What a child can do today with assistance, she will be able to do by herself tomorrow" (azquotes.com).

Likewise pointing to the importance of good teaching, Bruner developed the concept of instructional scaffolding providing a ladder leading from the ZPD up to a school subject. This should be structured as its university version to help the teacher structure the subject in a way that would give the meaning that the students need for understanding.

Holding that no children master logical thinking before 11 years, and therefore needing to be taught using concrete objects and examples, Piaget instead warned against too much teaching by saying: "Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover for himself will remain with him visible for the rest of his life" (azquotes.com).

2.3. Merging Mathematics and Psychology

Behaviorism is the educational psychology of pre setcentric mathematics. Present setcentric mathematics instead uses Vygotskian constructivism offering scaffolding from the learners ZPD to the institutionalized setcentric university mathematics as defined by e.g. Freudenthal (1973). However, by its self-referring setcentrism, concepts are no longer defined from examples and counterexamples, but as examples themselves of the more abstract set concept. So now not both rules, procedures and concepts should be understood. Freudenthal therefore recommended a special conference be created called PME, Psychology of Mathematics Education, focusing on how to understand mathematics as described by Skemp (1971) saying "The first part of the book will be concerned with this most basic problem: what *is* understanding, and by what means can we help to bring it about? (p. 14)”. Skemp then uses 123 pages to give an understanding of understanding, even if the inherent self-reference should make one skeptical towards such an endeavor.

Heidegger more directly points to four options when defining something by an is-statement: ‘is for example’ points down to examples and counter-examples, ‘is an example of’ points up to an abstraction, ‘is like’ points over to a metaphor, and ‘is.’ describes existence as something to experience without predicates.

Skemp understands numbers as equivalence cardinality classes in the set of sets being equivalent if connected by bijections. Consequently, children should begin drawing arrows between sets to see if they have the same cardinality that then can be named.

However, this approach met resistance in the classroom as illustrated by in this story:

Teacher: “Here is a set of hats and a set of heads. Is one bigger than the other?” Student: “There are more heads”. Teacher: “Why?” Student: “There are six heads and only five hats.” Teacher: “Can you please draw arrows from the hats to the heads!” Student: “No, then one person will not get a hat, and that is unfair.”

In his book ‘Why Johnny Can’t Add’, Morris Kline describes other examples on classroom resistance to the New Math, finally rejected by North America, choosing to go ‘Back to Basics’ even if this meant going back to mathe-matism.

Educational psychology thus has various schools. As an alternative, we might use the observation that children’s initial language consists of words that are exemplified in the outside world, thus using personal names instead of a pronominal as I and you, and protesting when grandma is named ‘Ann’.

Observing that brains easily takes in concepts naming outside examples allows formulating a research question: Can core mathematics as numbers, operations and equations be exemplified, de-modeled, or reified by concrete outside generating examples?

2.4. De-modelling Digits

Looking at a modern watch in front of an old building, we realize that Roman numbers and modern Hindu-Arabic numbers are different ways of describing Many.

The Romans used four icons to describe four, or they used one stroke to the left of the letter V iconizing a full hand. Modern numbers use one icon only when rearranging the four sticks or strokes into one 4-icon, which then serves as a unit when counting a total in fours as e.g. T = 3 4s.

We might even say that all digits from zero to nine are icons with as many sticks or strokes as they represent if written less sloppy, where the zero-digit iconizes a magnifying glass finding nothing.

The Romans bundled in 5, 10, 50, 100, 500 and 1000. Modern numbers bundle in tens only, which is written as 10 meaning 1 Bundle and none. However, in education we may want to symbolize ten with the letter B for ‘Bundle’.

2.4.1. Designing and Implementing a micro-curriculum

Based upon the above reflections we now design and implement a micro-curriculum having as its goal to de-model and reify digits as icons. As means we ask the learners to rearrange four sticks in different connecting forms, then five sticks, then six sticks. This is followed by rearranging also other things in icons including themselves, and by walking the icons, etc. Then the learners build routine by exercising writing all digits as icons. As end product, the learner should be able to rearrange a collection of things in an icon and write down a report using a full number-language sentence with a subject, a verb and a predicate, e.g. “T = 5”; and writing T = B, B1, B2 etc. for ten, eleven, twelve etc.

2.5. *Reflections on how to De-model Bundle-counting Sequences*

From early childhood children memorize the inside sequence of number names ‘one, two,…, ten, eleven, twelve, three-ten, four-ten’ etc. Later they learn the symbols corresponding to the different number-names. In some languages they are lucky to word ‘eleven, twelve, thirteen’ as one-ten, two-ten, three-ten’ etc. In English, number rationality begins with three-ten, making whole populations wonder what eleven and twelve means.

History shows that as most basic English words also these are ‘Anglish’ coming from the Danes settling in England long before the Romans arrived. Thus, with Danish you hear that eleven and twelve means ‘one-left’ and ‘two-left’ coming from Viking counting: ‘eight, nine, ten, 1-left, 2-left, 3-ten’; and ‘1-twotens’ where English shift to ‘twenty-1’.

Likewise, many children and adults wonder why ten has no icon since it has its own name as the rest of the digits. Only few realize that when counting by bundling in tens, ten becomes 1 bundle, or 1B0, or 10 if leaving out the bundle when writing it; even if ten is included when saying it, as e.g. in 63 = sixty-three = 6ten3 = 6B3.

So, it may be an idea to practice different counting sequences that include the name ‘bundle’ so that ‘ten, eleven and twelve’ become ‘1 bundle none, 1 bundle 1, 1 bundle 2’. And it may also be an idea to also count in fives as did the Romans and several East Asian cultures as shown by Chinese and Japanese abacuses. So, we design a lesson about counting fingers first in 5s, then in tens, and later in 4s, 3s and 2s or pairs.

2.5.1. Designing and Implementing a micro-curriculum

One hand counted in 5s using B for Bundle: First 1, 2, 3, 4, 5 or B or 1B1; then 0B1, 0B2, 0B3, 0B4, 0B5 or B or 1B0; then 1Bundle less 4, 1B-3, 1B-2,1B-1,1B.

Two hands counted in 5s: First 1, 2, 3, 4, 5 or B or 1B0, 1B1, …, 1B4, 1B5 or 2B or 2B0; then 0B1, 0B2, 0B3, 0B4, 0B5 or B or 1B0, etc.; then 1Bundle less 4, 1B-3, 1B-2,1B-1,1B0, 2B-4, …, 2B-1, 2B or 2B0.

Two hands counted in tens: First 1, 2, 3, 4, 5 or half Bundle, 6, 7, 8, 9, ten or full Bundle or 1B0; then 0B1, 0B2,…, 0B9, 0B10 or B or 1B0; then 1B-9, 1B-8, …, 1B-1, 1B.

Two hands counted in 4s is similar to counting in 5s.

Two hands counted in 3s provides the end result T = ten = 3B1 3s. But 3 bundles, 3B, is also 1 bundle of bundles, making 9 = 1BB 3s. So we can also write:

T = ten = 3B1 3s = 1BB1 3s, or T = 1BB0B1 3s, or T = 101 3s.

Two hands counted in 2s provides the end result T = ten = 5B0 2s. But, 2 bundles, 2B, is also 1 bundle of bundles, making 4 = 1 BB 2s; and 2 bundles of bundles, 2BB, is also 1 bundle of bundles of bundles, making 8 = 1 BBB 2s. So we can also write:

T = ten = 5B0 2s = 1BBB 1B 2s = 1BBB 0BB 1B 0 2s = 1010 2s.

This can be illustrated with Lego bricks having different colors where a green 1x2 brick is B, a blue 2x2 brick is BB and a red 4x2 brick is BBB.

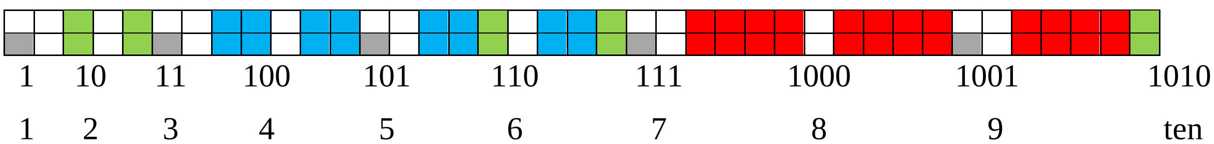


Figure 1. *Ten fingers counted by bundling in 2s and shown by Lego bricks.*

2.6. Reflections on how to De-model Operations

Counting a total of eight ones in 2s, we push away 2s using e.g. a playing card that may be iconized as a sloping stroke named division. So, the outside action ‘from 8, push away 2s’ may inside be iconized as ‘8/2’ that gives an inside prediction of what will happen outside: 8/2 times we can perform the action ‘from 8 push away 2’, or T = 8 = 8/2 2s = 4 2s.

Once pushed away, the bundles of 2s may be lifted into a stack of 4 2s. An outside lifting process may be iconized inside by a wooden scissor lifting up things when compressed, and named multiplication. So, the outside action ‘4 times lifting 2s into a stack’ may inside be iconized as ‘T = 4 x 2 = 4 2s’. And, the reverse outside process ‘de-stack 4 2s into ones’ may inside be predicted by a multiplication T = 4 2s = 4 x 2 = 8.

The total outside process ‘from 8 push away 2s to be stacked as 8/2 2s’ then may be iconized inside as ‘8 = (8/2) x 2’, or ‘*T = (T/B)* x *B*’ if we use *T* for the total 8, and *B* for the bundle-unit. By changing units, this ‘bundle-count’ or ‘re-count to change unit’ formula is perhaps the most fundamental formula in mathematics and science, also called the proportionality or linearity formula.

2.6.1. Designing and Implementing a micro-curriculum

Outside action	Inside prediction
From 8, 8/2 times 2s can be pushed away.	8/2 = 4
So, 8 can be recounted in 2s as 8/2 2s	8 = (8/2) x 2
And, *T* can be recounted in Bs as *T/B* Bs	*T = (T/B)* x *B*

Figure 2. *Counting 8 in 2s is an example of the recount formula T = (T/B) x B*

Asking “How many 2s will give 8” may be reformulated as an equation ‘*u* x 2 = 8’ using a letter *u* for the unknown number; and solved by recounting 8 in 2s. So, an equation is solved by moving a number to the opposite side with the opposite calculation sign. Here, solving equations is just another name for recounting in icon-units.

Outside action	Inside prediction
How many 2s will give 8?	*u* x 2 = 8 = (8/2) x 2 so
To answer, we recount 8 in 2s	*u* = 8/2

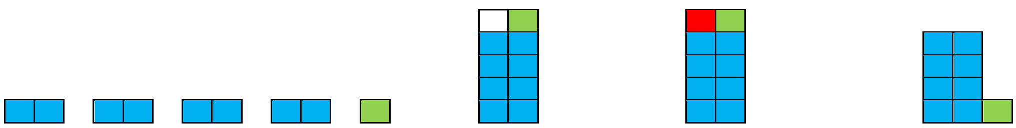
Figure 3. *An equation solved: to the opposite side with the opposite calculation sign*

Having pushed bundles away to stack, some unbundled may be left. So, from the total, we pull away the stack with a rope iconized as a horizontal stroke called subtraction.

Outside action	Inside prediction
From 9, 9/2 times 2s can be pushed away.	9/2 = 4.some
From 9, pull away 4 2s leaves 1	9 – 4 x 2 = 1
Prediction: T = 9 = 4B1 2s	Prediction: 9 = 4 x 2 + 1

Figure 4. *To predict unbundled, we pull away the stack from the original total*

Counting a total of 9 in 2s thus is predicted by the division 9/2 = 4.some. To predict leftovers, we pull away the stack of 4 2s from the total 9, predicted by saying ‘9-4x2’ giving the expected answer 1. So, a total of 9 may be counted in 2s as T = 9 = 4B1 2s, or as T = 4 x 2+1. Here a cross called addition iconizes the two ways to place the unbundled: next-to the stack iconized by a dot named a decimal point, T = 9 = 4.1 2s; or on-top of the stack counting in bundles as 1 = (1/2) x 2 giving T = 9 = 4B½ 2s, or counting what is missing in having a full bundle, T = 9 = 5B-1 2s, thus reifying fractions and negatives.



T = 9 = 4B 2s + 1 = 4B½ 2s = 5B-1 2s = 4B1 2s = 4.1 2s

Figure 5. *Unbundled singles may be placed on-top or next-to the stack of bundles*

Likewise, when counting in tens:

T = 48 = 4B tens + 8 = 4B8/10 tens = 5B-2 tens = 4B8 tens = 4.8 tens

Changing bundles to unbundled or vice versa gives ‘flexible bundle-numbers’ with or without an overload, or with an underload: T = 48 = 4B8 tens = 3B18 tens = 5B-2 tens.

2.7. Reflections on how to Recount into Tens

Once counted, a total may be recounted in the same unit, in a different unit, from tens into icons, or into tens. The first three cases are described in the chapter above.

Recounting from icons into tens, the recount formula cannot be used since there is no ten button or bundle button on the calculator. However, multiplication gives the result directly, only without units and with the decimal point moved one place.

Question: T = 3 8s = ? tens; answer: T = 3 8s = 3 x 8 = 24 = 2.4 tens

Recounting into tens includes multiplying two one-digit numbers, called setting up a multiplication table: a small table for the numbers 1-5, and a large for the numbers till 10.

2.7.1. Designing and Implementing a micro-curriculum

Turning over a stack will change e.g. 2 3s to 3 2s without changing the total. So, in multiplication, the order does not matter, the units may be commuted.

The small table follows directly from using fingers. It is obvious in the case of 2.

In the case of 3,

4x3 = 2x2x3 = 2 6s = 12 seeing a hand as a pawn with six extremities leaving it; and

5x3 = 3x5 = 3 5s = 3 hands = 1B5 = 15.

In the case of 4,

4x4 = 2x2x4 = 2 8s = 2 B-2s = 2B-4 = 1B6 = 16; and

5x4 = 4x5 = 4 hands = 2B0 = 20.

Finally, in the case of 5, 5x5 = 5 5s = 5 hands = 2B5 = 25.

In the large table we recount the numbers from 6 to 10 in bundles as B-4, B-3, etc.; and use a bead pegboard square with two rubber bands to show the actual stack as e.g.

6 7s = 6x7 = (B-4) x (B-3) = BB – 4B –3B + 4 3s removed twice = 3B12 = 4B2 = 42.

This roots the algebraic formula *(a – b)* x *(c – d)* = *a* x *c* – *a* x *d* – *b* x *c* + *b* x *d*.

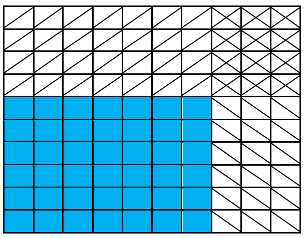


Figure 6. *A pegboard square shows that 6x7 = (B-4) x (B-3) = 10B – 4B – 3B + 4 x 3*

Recounting into tens also includes multiplying multi-digit numbers as e.g. 27 x 36 = 27 36s = ? tens. We may use a square or write the result in lines. It makes sense that changing the unit base from 36 to 10 will increase the height of the stack from 27 to 97.2.

Question: T = 27 x 36 = 27 36s = ? tens. Answer: T = 2B7 x 3B6 = (2B+7) x (3B+6) = 6BB+12B+21B+42 = 6BB + 33B + 4B2 = 6BB + 37B + 2 = 9BB7B2 = 972 = 97.2 tens.

Vice versa, recounting from tens includes division as the opposite of multiplication. Asking 16.8 tens is how many 7s thus gives the division 168/7. We may use the square bottom-up, or write the result in lines using flexible bundle-numbers. Here, it makes sense that changing the base from 10 to 7 will increase the height of the stack from 16.8 to 24.

Question: ? 7s = 16.8 tens.

Answer: u x 7 = 168; u = 168 /7 = 16B8 /7 = 14 B28 /7 = 2B 4 = 24. So 24 7s = 16.8 tens.

T = 27 x 36 = ? tens			
	2B	7	
	6BB	21B	3B
	12B	42	6
6BB	33B	4B2	
6BB	37B	2	
9BB	7B	2	
9	7	2	

? x 7 = 168, or 168/7 = ?; answer: 2B4 = 24			
		7	
		14B	? = 2B
		28	? = 4
	14B	28	
	16B	8	
1BB	6B	8	
1	6	8	

Figure 7. *A 2D schema used top-down for multiplication and bottom-up for division*

2.8. Reflections on how to Model Double-counting with Per-numbers and Fractions

On a Lego brick we can double-count the dots and the rows, e.g. giving 2 rows per 8 dots on a 2x4 brick, thus producing the 'per-number' 2r/8d, or 2/8 r/d.

Double-counting a basket filled with 3red per 5 apples, the like units makes the per-numbers a fraction, both being operators needing numbers to become numbers.

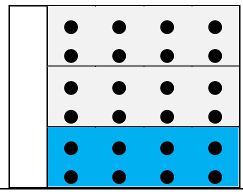
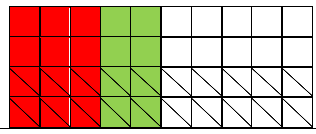
Asking "6 rows gives how many dots?", we recount 6 rows in the per-number as: T = 6r = (6/2) x 2r = (6/2) x 8d = 24d Likewise, when asking "how many rows gives 56 dots?": T = 56d = (56/8) x 8d = (56/8) x 2r = 14r	
To find the number of red apples among 20 apples, we set up a per-number equation u/20 = 3/5, or recount 20 in 5s: u/20 = 3/5; so u = 3/5 x 20 = 12. 20 a = (20/5) x 5a giving (20/5) x 3r = 12 red apples	

Figure 8. *Double-counting in two units creates per-numbers bridging the units*

Likewise with percent. 5% of 40 asks '5 per 100 is what per 40', giving the equation u/40 = 5/100; solved by moving 40 to opposite side with opposite sign: u = 5/100 x 40 = 2.

The unit-changing recount formula may also be used on units as e.g.: \$ = (\$/kg) x kg. Thus with the per-number 2\$/3kg, we may ask '?\$ = 12kg' and '10\$ = ?kg'. Recounting then gives: \$ = (\$/kg) x kg = 2/3 x 12 = 8; and kg = (kg/\$)x\$ = 3/2 x 10 = 15.

2.9. Reflections on how to De-model Trigonometry

In an a x b rectangle halved by its diagonal c, double-counting the sides creates the per-numbers sin $A = a/c$, cos $A = b/c$, and tan $A = a/b$. Filling a circle from the inside, we find a formula for the per-number perimeter per radius: $\pi = n$ x tan(180/n) for n large.

2.10. Reflections on how to Add Next-to and On-top, and how to Add Per-numbers

Once counted or recounted, blocks may be added next-to or on-top. Here adding 2 3s and 4 5s next-to as 8s means adding by areas, called integral calculus. Whereas adding 2 3s and 4 5s on-top means making the units like, using the recount formula to change units.

Adding 2kg at 3$/kg with 4kg at 5$/kg, the unit-numbers add directly as 2+4 = 6, whereas the per-numbers add by their areas (2x3 + 4x5)$ / 6kg, which is integral calculus where multiplication precedes addition. Vice versa, asking what to add to 4kg at 5$/kg to have 6 kg at 4$/kg, we subtract the initial block 4x5 from 6x4 before counting in 2s, which is differential calculus where subtraction precedes division.

3. Results and Discussion

This study asked: Can core mathematics as numbers, operations and equations be exemplified, de-modeled, or reified by concrete outside generating examples?

The answer is yes, if we de-model digits as icons with as many sticks as they represent if written less sloppy; if we use the flexible bundle-numbers children develop when adapting to Many; if we de-model operations as means for bundle-counting 8 as 8/2 2s, leading directly to the recount formula $T = (T/B) \times B$, used to change units, and to solve the question 'How many 2s in 8?' by recounting 8 in 2s: $u \times 2 = 8 = 8/2 \times 2$, so $u = 8/2$.

The operations are de-modeled as bundle-counting where division pushes away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to find unbundled singles to be placed next-to or on-top of the stack as decimal numbers, negative numbers or fractions; and later added with other stacks next-to as integral calculus, or on-top after making the units the same by using the proportionality of the recount formula.

Exemplifying assigns concrete meaning to abstract concepts, so de-modeling and reifying needs no psychological learning theories about how meaning is constructed.

As expected, concrete meaning makes mathematics easy to learn, as confirmed when tested in pilot projects in preschool, special education, and in adult and migrant education.

Of course, the effect of using flexible bundle-numbers and recounting operations should be studied in detail in other cases also to open up a completely new research paradigm (Kuhn, 1962), that may make obsolete all single-number material and research on mathematics in its grammar-based form before undergoing a communicative turn.

4. Conclusion

As to the questions asked in the introduction, the answers are: Yes, mathematics is hard if taught as pre setcentric mathe-matism true inside but seldom outside classrooms; and if taught as present setcentric meta-matism that by defining its concepts by abstract self-reference forces learners to construct a meaning themselves. And no, mathematics is not hard if taught as post setcentric many-math, allowing learners to further develop what they bring to school, a quantitative competence created by adaption to outside quantity.

When writing a mathematics curriculum, we must ask: Is its goal to master inside mathematics first as the only means to master outside Many later? Because then Vygotskian constructivism is needed to assign meaning to what is made meaningless by abstract self-reference or by tradition. Or is its goal to master outside Many directly, or via other means if the mathematics tradition is hindering 1 out of 4 or 5 students in acquiring basic numeracy? Because then Piagetian constructivism is better suited to allow students to accommodate existing schemas created by natural adaption to outside numerosity.

As to research, maybe the time has come to reread the Marx inscription in the entrance hall at the Humboldt mother-university in Berlin: "The philosophers have hitherto only interpreted the world in various ways. The point, however, is to change it."

Changing from abstract single- to concrete double-numbers, mathematics education will meet The Universal Declaration of Human Rights Article 26, saying "Everyone has the right to education. (..) Education shall be directed to the full development of the human personality"; as well as Article 4, saying "No one shall be held in slavery or servitude; slavery (..) shall be prohibited in all their forms.” This also applies to the slavery of an abstract-referring mathematics wanting to colonize children’s own number-language.

Likewise, mathematics education should respect the UN Global Goals for Sustainable Development, where goal 4 about quality education writes in target 4.6 on universal literacy and numeracy: “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy.”

So why teach children different, if they already know?

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34. Visit to Ho Chi Minh City University of Education December 7-13 2019

In December 2019 I paid a visit to the Ho Chi Minh City University of Education. Via Amsterdam and Kuala Lumpur, I arrived at Ho Chi Minh City Friday December 6 in the evening and was welcomed by Le Thai Bao Thien Trung, PhD Associate Professor, Didactic of Mathematics Team, Department of Mathematics, Ho Chi Minh City University of Education who took me to my hotel, a ten minutes' walk from the University.

Saturday, I gave a keynote address at the conference 'Psychology and Mathematics Education' at the University of Education. Sunday, I gave a talk on modeling to a group of master students. Monday, I gave a talk to a class of senior students on a poster presentation from the 'Educating the Educators' conference in Freiburg, Germany, in October, based upon my paper (http://mathecademy.net/educate-educators-2019/) and handed out the notes 'What is Math - and Why Learn it?' and 'Bundle Counting Table'. Tuesday, I gave a talk to the staff on research in mathematics education and networks to join and design research as a methodology to use when researching the implementation of the new activity-based curriculum inspired by Kolb's experimental learning theory.

Saturday December 7: keynote address to the conference

At the conference on Psychology and Mathematics education I gave a keynote address called 'De-modelling Numbers, Operations, and Equations' (http://mathecademy.net/demodel-mathematics/) where I introduced the concept of 'Adaptive Mathematics: Kids' own BundleNumbers with Units, Bundle- & Per-Numbers in Primary & Secondary School', and where I raised the M&M-question: The Goal of Math Education is that to Master outside Many, or Master inside Mathematics (as a means to later, possibly, Master outside Many).

I began by answering the question 'Is mathematics well defined?' by pointing to its three different versions: MetaMatics defining concepts as examples of abstractions instead of as abstractions form examples, MatheMatism true inside but seldom outside classrooms as when adding without units, and ManyMath, a natural science about the outside fact Many.

Then I described the observation that when adapting to Many, Children create Flexible BundleNumbers with units as e.g. 2 2s or 3 4s, which I used to point out that Bundle-Numbers can Shift unit and create a Recount-Formula 8 = (8/2)*2, or T = (T/B)*B recounting a total T in B-bundles by T/B times pushing away Bs; a formula that allows solving equations as u*2 = 8 by recounting 8 in 2s, thus moving 2 to opposite side with opposite sign; and that leads direct to core STEM formulas all expressing proportionality.

Next, I talked about green cells uniting to plants, and grey cells uniting to animals as reptiles, mammals or humans having one, two or three brains for routines, feeling and language. I ended by pointing to Darwin saying: "To SURVIVE, you must ADAPT to the outside world", adding the question "and so must math?"

Thus, as to math education, humans use the reptile brain for routines and rote learning; they use the mammal brain for developing positive or negative feelings towards mathematics; and they use the human brain for creating formulas that allows predicting the behavior of quantities.

As to how a brain adapts to the outside world, I mentioned stimulus and response allowing the brain to develop a word- and a number-language both using full sentences with a subject, a verb and a predicate.

Then I described three kinds of how brains adapt through learning:

Inside-outside Skinner-learning using the reptile brain for rote learning, and using the mammal brain for wishing reward and for fearing punishment. Although widespread in time and space, it is constantly met with the objection: Skinner-learning might lead to skills, but understanding is typically lacking.

Then I described two kinds of learning focusing on understanding, one adapting to inside culture, the other adapting to outside nature.

Inside-Inside Vygotsky-learning focuses on enculturation to the institutionalized university mathematics that by choosing abstract top-down understanding uses the reptile brain for learning definitions. Here Bruner says that school subjects must mirror university subjects to structure a good teaching, providing a scaffolding as the ladder down to the learner's ZPD, Zone of Proximal Development, as described by Vygotsky. Likewise, Skemp says that to understand mathematics, first you must understand understanding; numbers thus must be seen as cardinality, defined as equivalence classes in the set of sets. So, children first must draw arrows between sets before learning number-names. Finally, Vygotsky says that "What a child can do today with assistance, she will be able to do by herself tomorrow". So, learning comes from teaching by a more knowledgeable other. Therefore, good teacher education and good professional development matters. Peer brains must be taught by a major brain even if this might infer colonization as to the nature of numbers when 'preaching' that 0 = Ø, 1 = {Ø}, 2 = {Ø,{Ø}}, etc; and that functions are defined as subset of a set-product {(x,y)} where first-component identity implies second-component identity: if x1 = x2, then y1 = y2.

Outside-inside Piaget-learning focuses on bottom-up understanding through meeting concrete outside examples creating inside schemata meant to assimilate the outside examples, or to accommodate if not validated by meeting resistance. Adaption thus creates flexible schemata about what is outside: I, you, he, she, it, we, you, and they.

Piaget makes the point that "Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover for himself will remain with him visible for the rest of his life". In research, Grounded Theory means creating collective flexible schemata. So, when adapting to Many, learners should use grounded theory to answer the guiding learning questions listed in the curriculum. And teaching should be minimized to supplying concrete material and extra guiding questions, and to be opponents on the learners' findings negotiated through peer-brain communication and learning.

These three ways of adapting brains lead to asking: Can children discover/invent mathematics themselves to obtain a concrete exemplified understanding? To look for an answer we now look for outside roots for inside mathematics by re-rooting or de-modeling digits, operations, equations, functions, and fractions.

First, we de-model digits as icons with as many sticks as they represent: 4 sticks in the 4-icon, etc.

Then, we de-model division, multiplication, subtraction, and addition as icons also: From 9 push away 4s we write 9/4 iconized by a broom, called division. 2 times lifting the 4s to a stack we write 2x4 iconized by a lift called multiplication. From 9 pull away 2 4s' to find un-bundled we write 9 – 2x4 iconized by a rope, called subtraction. Uniting two stacks or blocks A and B next-to or on-top we write A+B iconized by the two directions, called addition.

Now, we bundle-count a Total of 9 in 2s: From 9, 9/2 times we push away 2; then from 9, we pull away 4 2s, leaving 1. Thus, the prediction by the recount-formula is T = 9 = 4B1 2s.

The unbundled can be placed in three ways: next-to the stack iconized by a dot named a decimal point, 4.1 2s; or on-top of the stack counted in bundles as 1 = (1/2) x 2 giving 4½B 2s; or counting what is missing in a full bundle, 5B-1 2s. This de-models decimals, fractions & negatives.

Now, bundle-counting ten fingers in 3s counting singles, bundles, and bundle-bundles gives the counting-sequence: 0B1, 0B2, 0B3 no 1B0, 1B1, 1B2, 1B3 no 2B0, 2B1, 2B2, 2B3 no 3B0, 3B1 or ten.

But 3 Bundles, is 1 Bundle-of-Bundles. So, T = 9 = 1BB 3s or T = ten = 3B1 3s = 1BB1 3s = 1BB0B1 3s = 1BB1B-2 3s = 101 3s

With stacks, bundle-counting ten fingers in 3s create three ways: over-load, normal, under-load.

Likewise, bundle-counting fingers in 2s makes 8 a bundle-bundle-bundle, and ten 1BBB0B1B0, or 1010.

Double-counting in two units creates per-numbers & proportionality. Double-counting in kg & $, we get a per-number 4kg per 5$ = 4kg/5$ = 4/5 kg/$.

With 4kg bridged to 5$, we recount in the per-number. Or we recount the units directly. Or we equate the per-numbers. Or we use the before 1900 'Rule of 3' (regula-de-tri) alternating the units, and, from behind, first multiply, then divide.

Typical question one: 12kg = ?$;

Answers: 12kg = (12/4) x 4kg = (12/4) x 5$ = 15$; or $ = ($/kg) x kg = 5/4 x 12 = 15; or *u*/12 = 5/4, so *u* = 5/4 x 12 = 15; or 'If 4kg is 5$, then 12kg is ?$'; regula-de-tri answer: 12x5/4 = 15

Typical question two: 20$ = ?kg

Answers: 20$ = (20/5) x 5$ = (20/5) x 4kg = 16kg; or kg = (kg/$) x $ = 4/5 x 20 = 16; or u/20 = 4/5, so u = 4/5 x 20 = 16; or 'If 5$ is 4kg, then 20$ is ?kg'; regula-de-tri answer: 20x4/5 = 16

With like units, per-numbers become fractions, both operators needing numbers to become numbers.

In a box filled with 3 red per 5 apples, double-counting reds and apples gives the fraction 3/5 reds/apples.

Question: ? red in 20 apples. Answer: Recount 20 in 5s (the per-number): T = 20 a = (20/5) x 5a gives (20/5) x 3r = 12 red apples.

Or, we equal the per-numbers: *u*/20 = 3/5; so *u* = 3/5 x 20 = 12 found by moving 20 to opposite side with opposite sign

Double-counting the sides in a block. Geometry means to measure earth in Greek. The earth can be divided in triangles; that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula. And connected with the angles by per-number formulas double-counting the sides pairwise.

a = (a/c) x c = sinA x c; tanA = a/b = $\Delta y/\Delta x$ = gradient

Circle: circumference/diameter = π = n*tan(180/n) for n large = n*sin(180/n) for n large

Counted and re-counted, Totals may be added, but how: next-to or on-top

Thus next-to addition of 4 5s to 2 3s to 3B2 8s means adding areas, and adding areas is integration. And on-top addition of 4 5s to 2 3s to 5B1 5s means making the units like, and changing units is proportionality.

Likewise, per-numbers add as areas, i.e. as integration: Asking "2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg?", the unit-numbers add on-top, but the per-numbers must be multiplied to unit-numbers, thus adding as areas under the per-number graph. Here, multiplication before addition

Reversely, subtracting per-numbers is called differential calculus: Asking "2kg at 3$/kg + 4kg at what = 6kg at 5$/kg?", first we remove the initial 2x3 block and recount the rest in 4s. So here subtraction giving a change, Δ, comes before division, as expected when being the opposite of integration.

Flexible bundle-numbers ease operations since over-load and under-load come in handy:

T = 65 + 27 = 6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92
T = 65 - 27 = 6**B**5 - 2**B**7 = 4**B**-2 = 3**B**8 = 38
T = 7 x 48 = 7 x 4**B**8 = 28**B** 56 = 33**B**6 = 336
T = 336 /7 = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48
T = 336 /7 = 33**B**6 /7 = 35**B**-14 /7 = 5**B**-2 = 48

Adding or subtracting unspecified numbers, we look for a common unit. So, finding T = $4ab^2$ + 6abc we remember that factors are the units, and we use a factor-filter to find the common unit 2*a*b, thus giving

T = $4ab^2$ + 6abc = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b) + 3*c*(2*a*b) = (2b+3c)*2ab = 2b+3c 2abs

Conclusion: Asking, can children discover/invent mathematics themselves to obtain a concrete exemplified understanding, the answer is YES, if we

- de-model digits as icons with as many sticks as they represent
- use the flexible bundle-numbers children develop when adapting to Many
- de-model operations as means for bundle-counting 8 as 8/2 2s, leading directly to the recount-formula T = (T/B) x B, used to change units, and to
- solve equations as ‘How many 2s in 8?’ by recounting 8 in 2s
- use double-counting to construct per-numbers, fractions and trigonometry
- add both next-to and on-top, making calculus be addition of per-numbers

Discussion: What is the Difference?

		Traditional math	**Adaptive math**
Digits	4	Symbol	Icon with four strokes
Numbers	456	One number	Three numberings, 4**BB**5**B**6
Division	8/2	8 split in 2	8 counted in **2s**
Multiplication	6 x 7	42	6 **7s** or 4**B**2 **tens**
Addition	2+3	2+3 = 5	2 **4s** + 3 **5s** = 2**B**3 **9s** 2 **4s** + 3 **5s** = 4**B**1 **5s**
Equations	3**u* = 12	Neutralize (3**u*)*1/3 = 12*1/3 (*u**3)*1/3 = 4 *u**(3*1/3) = 4 *u**1 = 4 *u* = 4	Opposite side & sign *u**3 = 12 = (12/3)*3 *u* = 12/3 = 4
Fractions	2/3	Numbers 1/2 + 2/3 **IS** 7/6	Per-numbers, i.e. operators, needing numbers to become numbers: 1/2 of 2 + 2/3 of 3 **IS** 3/5 of 5

Four Ways to Unite and Split a Total

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2 + 4***B** + 5 (a polynomial) shows the 4 ways to add: +, *, ^, next-to block-addition (integration). <u>Addition</u> and <u>multiplication</u> add changing and constant unit-numbers. <u>Integration</u> and <u>power</u> add changing and constant per-numbers. We might call this beautiful simplicity the ‘Algebra Square’ since in Arabic, algebra means to reunite.

The 4 uniting operations each has a reverse splitting operation:

Addition has subtraction (–), and multiplication has division (/). Power has factor-finding (root, √) and factor-counting (logarithm, log). Integration has per-number finding (differentiation dT/dn = T').

Reversing operations is solving equations, done by moving to **opposite side** with **opposite sign**.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$\mathbf{T = a + n}$ $T - n = a$	$\mathbf{T = a*n}$ $T/n = a$
Per-numbers m/s, $/kg, $/100$ = %	$\mathbf{T = \int a*dn}$ $dT/dn = a$	$\mathbf{T = a\hat{}n}$ $n\sqrt{T} = a \quad log_a T = n$

The 'algebra-square' shows the four ways to unite or split numbers.

Recommendation: Learners should be researchers, Extending their already existing adaption to many

- To survive, also math must adapt to the outside world. So, it should adopt the double-numbers children develop before school; and accept fractions as per-numbers, both operators needing numbers to become numbers.
- Hence to survive math must learn from children, not the other way around.
- Designing a micro- or macro-curriculum we should always ask: What is it out there that the learners need to adapt to?
- When adapting, learners should use grounded theory to answer the guiding learning questions listed in the curriculum.
- Teaching should be minimized to supplying concrete material and extra guiding questions, and to be opponents on the learners' findings.

Question Guided Teacher Education may be found at the MATHeCADEMY.net offering free teacher training in many-math, a natural science about Many, using the **CATS** approach, **C**ount & **A**dd in **T**ime & **S**pace; and using PYRAMID**e**DUCATION where a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

- Each pair works together to solve **C**ount&**A**dd problems.
- The coach assists the instructors when instructing their team and when correcting the **C**ount&**A**dd assignments.
- Each teacher pays by coaching a new group of 8 teachers.

Theoretical Background

Tarp, A. (2018). Mastering Many by counting, recounting and double-counting before adding on-top and next-to. *Journal of Math Education, March 2018, 11*(1), 103-117.

Monday, December 8: meeting a group of master students

My talk focused on modelling, which plays a core role in the Vietnamese 2018-curriculum. In the morning I discussed my booklet Mathematics, Modelling and Models (http://mathecademy.net/math-modeling-models/) with professor Trung Le.

I began by repeating core points from my lecture the day before, that humans adapt to the outside world with three brains where the third brain develops two languages, a word-language and a number-language assigning words and numbers to outside things and actions, both using sentences with a subject, a verb and a predicate.

Then I repeated that psychology works with three kinds of learning: inside-outside Skinner-learning and inside-inside Vygostky-learning and outside-inside Piaget-learning. This corresponds to three kinds of mathematics that exists, MetaMatics defining concepts as examples of abstractions instead of as abstractions form examples, MatheMatism true inside but seldom outside classrooms as when adding without units, and ManyMath, a natural science about the outside fact Many.

I recalled that in ancient Greece, the Pythagoreans used the word 'mathematics' meaning 'what we know' in Greek as a common label for their four areas of knowledge: Music, astronomy, geometry and arithmetic studying many in time, in time and space, in space and by itself.

Then I suggested that since the yesterday lecture focused on primary school, today we could talk about upper secondary school and focus on the pre-calculus level and discuss how modelling could be used at this level. Precalculus is the level where the international ICMI 24 curriculum conference the year before in Japan discussed if all students should have the same curriculum with different degrees of details or whether a totally different curriculum focusing on applications should be offered to students not wanting to continue in a STEM-direction.

Instead, I proposed a different approach: Start from scratch and give a new outside-inside understanding or definition of the operations addition, multiplication and power occurring when writing out fully Arabic numbers as 345 = 3*B^2 + 4*B + 5*1.

This allows the inverse operations subtraction and division and root and logarithm to be understood as solutions to equations solved by moving numbers to opposite side with opposite sign.

With 15-3 as the number x that added to 3 gives 15, the equation x+3 = 15 thus is solved by x = 15-3.

With 15/3 as the number x that multiplied with 3 gives 15, the equation x*3 = 15 is solved by x = 15/3.

With 3√15 as the base x that powered to 3 gives 15, the equation x^3 = 15 is solved by x = 3√15.

With log3(243) as the exponent x that with base 3 gives 15, the equation 3^x = 15 is solved by x = log3(243).

This provides a grounded understanding of root as a 'factor-finder', and logarithm as a 'factor-counter'.

This allows modeling right away the two core forecasting questions: Given start and end values in a time-series, what will a future value be; and when will a certain level be reached?

Here the two standard models assume a constant yearly change-number or change-percent.

Adding 5$/year to an initial number 200$ will after 7 years give a total T = 200 + 5*7; or with unspecified numbers, T = b + a*n where the constants b and a may be found by linear regression using an IT-tool.

Adding 5%/year to an initial number 200$ will after 7 years give a total T = 200 *105%^7 since 5% will change 100% to 105% as enlarging-factor; or with unspecified numbers, T = b*a^n with a = 1+ r where the constants b and a may be found by exponential regression using an IT-tool.

Now the two question about a future value or time may be found by solving equations as T = 200 + 5*7 and 300 = 200 + 5*x in the case of a constant yearly change-number; and T = 200*1.05^7 and 300 = 200*1.05^x in the case of a constant yearly change-percent.

The equations may now be solved manually by moving to opposite side with opposite sign, and by using technology to apply the solver-function algebraically, or to identify intersection points geometrically.

Furthermore, the basic number formula T = 345 = 3*B^2 + 4*B + 5*1 contains the core formulas for constant change: proportionality and linear change, T = a*x and T = b+a*x; exponential and power change T = b*a^x and T = b*x^a; as well as accelerated change T = a*x^2 + b*x + c, graphically shown as a bending line, that might also have a cubic term, k*x^3, for counter-bending.

As the next example of modelling I chose the model called 'Project Collection, LafferCurve'. Here the Real-world problem is that we want to collect a charity fund among the school's 500 students by selling tickets at a fixed price. We therefore ask which of the following three collection models will provide the highest contribution: no marketing, marketing without or with a lottery modelled by three different demand-formulas all assuming that The demand would be 500 and 0 at the prices 0$ and 40$, but differing in assuming that the demand would fall quickly or slowly without marketing or with a lottery. This allows three different demand tables to be created with regression giving three different demand formulas showing how the demand Y1 would change with changing unit price x, thus giving a collected fund Y2 = Y1*x that should be maximized.

As the last example I talked about modeling a table containing per-numbers taken from my paper 'Saving Dropout Ryan by a TI-82' included in the booklet 'Math, modeling and models'. Here the Real-world problem is that while driving, a camera shows that at each 5th second Peter's velocity was 10m/s, 30m/s, 20m/s, 40m/s and 15m/s. When did his driving begin and end? What was the velocity after 12 seconds? When was the velocity 25m/s? What was his maximum velocity? When was Peter accelerating? When was he decelerating? What was the acceleration in the beginning of the 5 second intervals? How many meters did Peter drive in the 5 second intervals? What was the total distance traveled by Peter?

Here the table contains 5 data sets that allows quartic regression, i.e. a 4. degree 4 polynomial with a 3-fold parabola changing curvature 3 times, to provide the formula y = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235. Now the question asked can be answered using formula tables, or using technology, i.e. graphical readings or solver calculations.

Here a need for adding per/numbers arise. Consequently, an example is introduced to draw inspiration from saying: 2 kg at 3$/kg plus 4kg at 5$/kg. Here the unit-numbers 2 and 4 add directly to 6 since the units are like. However, the per-numbers 3 and 5 must be multiplied to $-numbers before being added, but the moment you multiply you create areas, so per-numbers add by their area under the pe-number graph showing how the per-number change with the kg-number. Consequently, per-numbers add by the formula T = Σp*Δx in the case of piecewise constancy, or t = ∫p*dx in the case of local constancy. Likewise, the change of y may always be recounted in the change of x as Δy = (Δy/Δx)*Δx = p*Δx in the case of linearity, or dy = (dy/dx)*dx = y'*dx in the case of local linearity.

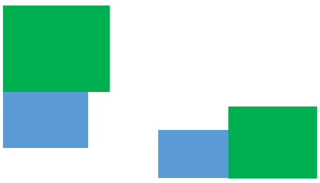
Then as an example of different understandings of mathematical concepts I described the content of my phd-work: At the Danish second chance high school, many students didn't pass the precalculus exam making the ministry consider canceling mathematics as a mandatory subject at the 2005 reform. Then I showed that what was taught was, not mathematics, but meta-matics; and that if they changed this to mathematics, everybody would pass. I showed that the curriculum was built on top-down Vygotsky learning saying that before being applied, student must be taught about linear and exponential functions as examples of functions, defined top-down by saying that a function is an example of a set-relation assigning to each element in one set one and only one element in anther set. Which the students heard as “bublibub is an example of bablibab”, something the was meaningless and therefore uninteresting unless you wanted to learn it by heart to pass the exam.

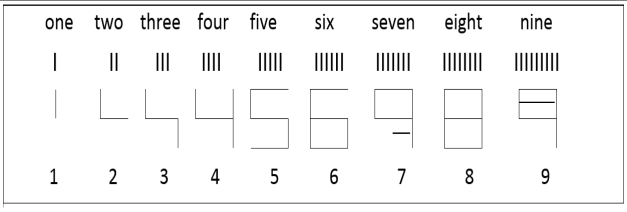
By looking at the history of the function concept I saw that Euler defined a function as a name for a calculation contain both specified and unspecified numbers. This allows defining a function by exemplifying bottom up as e.g. y = 2+x but not y = 2+3; which again allowed linear and exponential functions to be defined by outside examples, e.g. saving at home where an initial amount at b$ grows by adding a$ per month, or in a bank where an initial amount at b$ grows by adding r% per month, which let the students ask if this could be called change by adding and change by multiplying as well. Allowing the students own phrasings as parallel labels to the official labels turned out to be so successful that I recommended that the function concept be replaced by variables at the pre-calculus level despite fierce teacher resistance. The Ministry followed my recommendation and kept per-calculus as a mandatory subject.

Later, using regression and a graphical display calculator to transform tables to formulas, the learning process became so quick that a great part of physics could be included in the curriculum as reported in the paper ‘Saving Dropout Ryan with a TI-82’ (http://mathecademy.net/math-modeling-models/).

I ended by congratulating the students for becoming teachers in a country with perhaps the best curriculum in the world since it emphasizes modeling. A student teacher asked what to do in big classrooms with 40 students. I recommended dividing the class in groups of 5s working as private math consultants modeling the problems in the modeling compendium above thus practicing peer-brain learning.

Tuesday, December 9: A Senior Class Visited to a Poster Exhibition in the Staff Room

Wrong Numbers ~~LineNumbers~~ ~~with place values~~ ☹ IconNumbers BundleNumbers PerNumbers ☺ *Respect & Develop* ***Kids' own Flexible BundleNumbers*** ~~T is 48~~ No: T is **4*B*8 = 3*B*18 = 5*B*-2**	Looking at a calculator we observe the core of mathematics as digits, operations, and equations. However, we teach numbers wrong by presenting them as combinations of digits using a place value system since writing out fully T = 345 = 3*B^2 + 4*B + 5*1 shows that 345 is three numberings of singles, bundles and bundles of bundles, i.e. a bundle-number. Likewise, we silence that digits are icons with as many sticks as they represent; and that fractions are per-numbers, both operators needing numbers to become numbers, as do digits. A 3year old child describing four fingers held together two by two as two twos shows that when adapting to Many, children develop bundle-numbers that are flexible by accepting that five fingers may be counted both as 1B3, 2B1 and 3B-1 2s, i.e. with an overload, normal or with an under-load. Thus, school should teach flexible bundle-numbers instead of line-numbers.
Wrong Operations ~~8/2 is 8 split by 2~~ NO: 8/2 is 8 counted in **2s** ~~5x8 is 40~~ NO: 5x8 is 5 **8s** ~~9-8 = 1~~ NO: 9 – 4x2 = 1, so 9 = 4**B**1 **2s** 2 **3s** + 4 **5s** = ???  OnTop or NextTo Wrong Math = **Dislike**	As to operators, the tradition teaches addition as repeated adding 1, multiplication as repeated addition; and subtraction and division as reversed addition and multiplication using a carry-principle. This means that 5x8 is taught as 40 and that 8/2 is taught as 8 split in 2. However, counting 8 in 2s means pushing away 2s 8/2 times, making division an icon for a broom pushing away bundles to be stacked making multiplication an icon for lifting bundles, so that 5x8 is 5 8s and only 40 if recounted in tens later. Counting 9 in 2s, we pull away the stack 4 2s to look for unbundled singles, which makes subtraction an icon for a rope pulling stacks away. As to addition, it is not well defined since the two totals 2 3s and 4 5s may be added both on-top and next-to.
Numbers are Icons 5 sticks in the 5-icon etc.	To describe the world, we use a word-language combining letters to words, and words to sentences with a subject, a verb and a predicate; and a number-language, combining sticks to icons with as many sticks



	WORD language	NUMBER language
language, grammar	*'is' is a verb*	*'x' is an operation*
Language	This is a table	T = 3x4

as they represent if written less sloppy, four sticks in the 4-icon etc., and combining digits to numbers, and combining numbers and operators to formulas or sentences, also containing a subject, a verb and a predicate.

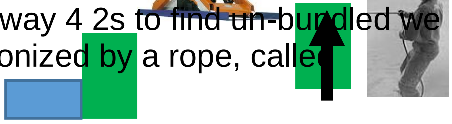
Both languages have a meta-language, a grammar describing the language that describes the world. The word-language teaches the language before the meta-language, the number-language does the opposite thus needing a communicative turn as the one that took place in foreign language education in the 1970s.

Operations are Icons

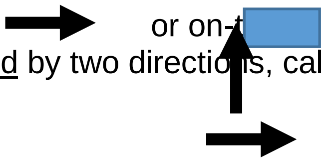
From 9 PUSH away 2s we write 9/2 iconized by a broom, called *division*.

4 times LIFTING 2s to a stack we write 4x2 iconized by a lift called *multiplication*.

From 9 PULL away 4 2s to find un-bundled we write 9 – 4x2 iconized by a rope, called *subtraction*.



UNITING next-to or on-top we write A+C iconized by two directions, called *addition.*



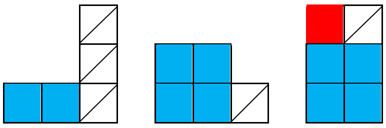
Counting 9 in 2s means pushing away 2s making division an icon for a broom pushing away bundles to be stacked making multiplication an icon for lifting bundles, so that 5x8 is 5 8s and only 40 if recounted in tens later.

MATHeCADEMY.net : Math as MANYmath - a Natural Science MANY

Counting 9 in 2s, we pull away the stack 4 2s to look for unbundled singles, which makes subtraction an icon for a rope pulling stacks away.

As to addition, it is not well defined since the two totals 2 3s and 4 5s may be added both on-top and next-to.

Flexible Bundle-Numbers



Overload *Normal* *Underload*

| | | | | = ╫ | | | = ╫ ╫ | = ╫ ╫ ╫

5 = 1**B**3 = 2**B**1 = 3**B**-1 2s

5 = 1.3 = 2.1 = 3.-1 2s

= 2 ½ 2s

48 = 4**B**8 = 3**B**18 = 5**B**-2

T = 65 + 27 = ? = 6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92

T = 65 – 27 = ? = 6**B**5 – 2**B**7 = 4**B**-2 = 3**B**8 = 38

T = 7* 48 = ? = 7* 4**B**8 = 28**B**56 = 33**B**6 = 336

T = 336 /7 = ? = 33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

We count 5 in 2s by pushing away bundles of 2s.

Pushing away 1 bundle leaves 3 unbundled as an overload that might be placed next to the stack of bundles as a stack of unbundled singles described as 1**B**3 2s or 1.3 2s thus rooting decimal numbers.

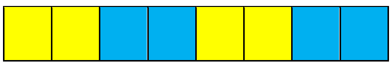
Pushing away 2 bundles leaves 1 unbundled that might be placed next to the stack as 2**B**1 2s or 2.1 2s again using a decimal number.

Or, the unbundled might be placed on-top of the stack counted as bundles as 2 ½ **B** 2s thus rooting fractions; or rooting negative numbers if counting what is missing for an additional bundle, 3**B**-1 2s or 3.-1 2s.

Allowing recounting using overloads or underloads eases calculations. Thus re-counting 336 to 28**B**56 makes division by 7 easier.

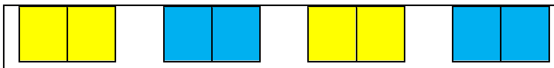
The Recount-formula

recounts a total T in B-bundles, e.g. 8 in 2s



_______ _______ _______ _______

Recounting 8 in 2s means pushing away 2s 8/2 times giving the equation 8 = (8/2)*2, becoming a **recount-formula** using T for the total and B for the bundle: **T = (T/B)*B** saying "from T, T/B times, push B away".



8 = (8/2)*2 = 4*2

T = (T/B)*B

From T, T/B times, push B away

The Recount-formula solves equations:

u*2 = 8 = (8/2)*2

u = 8/2 (opposite side & sign)

u + 2 = 8	u*2 = 8	u^8 = 2	2^u = 8
u = 8 – 2	u = 8/2	u = $\sqrt[8]{2}$	u = $\log_2(8)$

The Recount-formula is used in STEM-formulas

m = (m/sec)*sec = speed*sec

$ = ($/hour)*hour = rate*hour

The recount formula solves multiplication equations as u*2 = 8, since recounting 8 in 2s gives u*2 = 8 = (8/2)*2 thus giving the solution u = 8/2 obtained by moving to the opposite side with opposite sign. Thus 8/2 is the number that multiplied with 2 gives 8.

Likewise, the equations u+2 = 8, u^8 = 2 and 2^u = 8 define 8-2, 8√2 and log2(8), making root a 'factor-finder', and logarithm a 'factor-counter'; and again, solving the equations by the opposite side & sign.

STEM-formulas typically involve recounting in another unit thus also using the recount-formula.

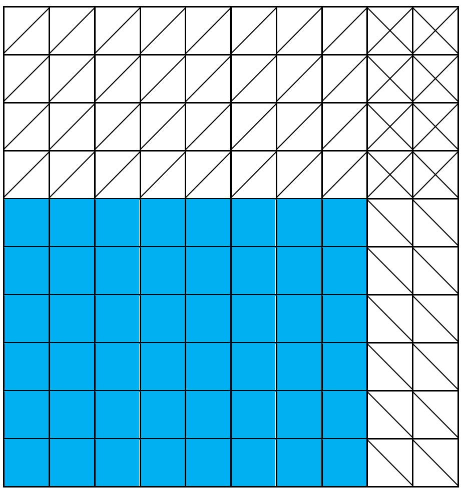
Recounting from icons to ten-numbers

Tables: recount from icons to tens

6 8s = ? tens

longer base - shorter height:

Using underloads:



T = **6 8s** = 6*8

= (***B***-4)*(***B***-2)

= ***BB*** - 4***B*** - 2***B*** - - 8

= 10***B*** - 6***B*** + 8

= 4***B***8 = 4.8 **tens** = 48

Turning over a stack will change e.g. 2 3s to 3 2s without changing the total. So, in multiplication, the order does not matter, the units may be commuted.

The small table follows directly from using fingers. It is obvious in the case of 2.

In the case of 3, 4x3 = 2x2x3 = 2 6s = 12 seeing a hand as a pawn with six extremities leaving it; and 5x3 = 3x5 = 3 5s = 3 hands = 1B5 = 15.

In the case of 4, 4x4 = 2x2x4 = 2 8s = 2 B-2s = 2B-4 = 1B6 = 16; and 5x4 = 4x5 = 4 hands = 2B0 = 20.

Finally, in the case of 5, 5x5 = 5 5s = 5 hands = 2B5 = 25.

In the large table we recount the numbers from 6 to 10 in bundles as B-4, B-3, etc.; and use a bead pegboard square with two rubber bands to show the actual stack as e.g.

6 8s = 6*8 = (B-4)*(B-2) = BB – 4B –2B + 4 2s removed twice = 4B8 = 42.

So, increasing the base from 8 to 10 means decreasing the height from 6 to 4.2.

This roots the algebraic formula

(a – b)*(c – d) = a*c – a*d – b*c + b*d.

Per-numbers

DoubleCounting in kg & $ gives

a **Per-number 2$/3kg**

<u>8$ = ?kg</u>

8$ = (8/2) x 2$

= (8/2) x 3kg = 12kg

<u>9kg = ?$</u>

9kg = (9/3) x 3kg

= (9/3) x 2$ = 6$

Like units make per-numbers fractions: **2$/3$ = 2/3**

STEM-formulas typically contain per-numbers coming from double-counting:

m = (m/sec) * sec = speed * sec

kg = (kg/m^3) * m^3 = density * m^3

Joule = (Joule/sec) * sec = Watt * sec

Double-counting in two physical units, we observe that this creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg.

To bridge units, we recount in the per-number:

Asking '6$ = ?kg' we recount 6 in 2s:

T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and

T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Double-counting in the same unit creates fractions and percentages as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%.

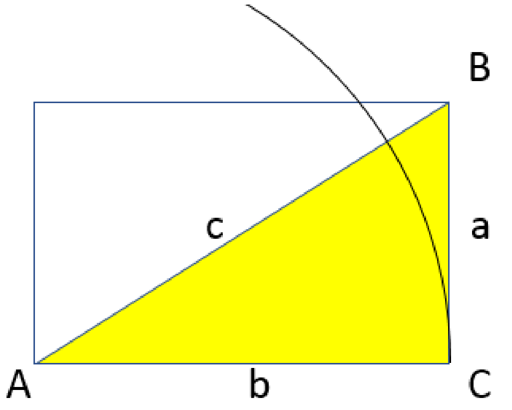
Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%.

We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number.

Recounting Sides in a Box: Trigonometry

Recount sides in a box halved by its diagonal



T = (T/B)*B

a = (a/c)*c = **sinA***c

a = (a/b)*b = **tanA***b, or

Δy = (Δy/Δx)*Δx = **tanA** * Δx = gradient * Δx

π = n*sin(180/n) for n large

c*c = a*a + b*b

Geometry means to measure earth in Greek. The earth can be divided into triangles that can be divided in right triangles; that can be seen as a block halved by its diagonal thus having three sides: the base b, the height a and the diagonal c connected by the Pythagoras formula.

The sides connect with the angles by formulas recounting one side in the other side or in the diagonal:

a = (a/c)*c = sinA * c; or sinA = a/c

b = (b/c)*c = cosA * c; or cosA = b/c

a = (a/b)*b = tanA * b; or tanA = a/b

In a circle, the circumference recounted in diameters is called phi.

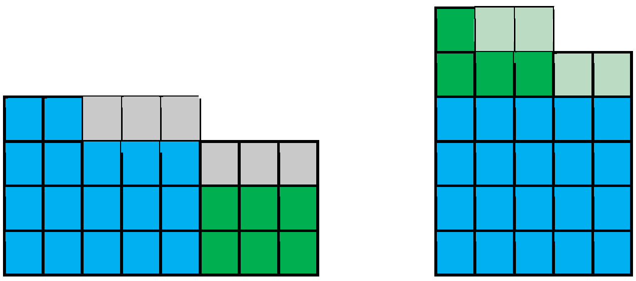
Filling the circle from the inside by right triangles allows phi to be found from a formula:

π ≈ n*sin(180/n) for n large.

Addition is not Well Defined

Counted & Recounted, Totals may be Added

BUT: NextTo	or OnTop
4 **5s** + 2 **3s** = 3**B**2 **8s**	4 **5s** + 2 **3s** = 5**B**1 **5s**
The areas are integrated *Adding areas = Integration*	The units changed to the same *Change unit = Proportionality*



Once counted and recounted, totals may be added, but should they be added next-to or on-top? So, addition is not well defined.

Next-to addition of 4 5s and 2 3s as 8s means adding areas, which is also called integral calculus.

And the reverse question as e.g. 4 5s and ? 3s total 5 8s leads to differential calculus since first we pull away the 4 5s before we recount the rest in 3s by division:

? = (5*8 – 4*5)/3 = ΔT/3.

On-top addition of 4 5s and 2 3s means recounting one or both so that the units are changed to the same unit.

And changing units is proportionality.

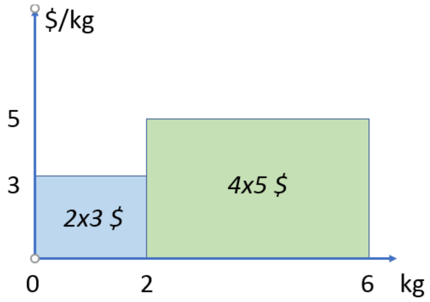
Adding fractions and per-numbers: Calculus

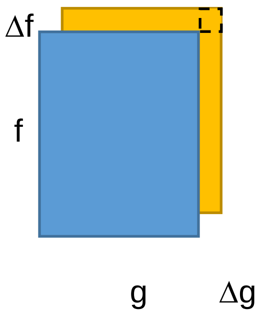
2 kg at 3 $/kg

+ 4 kg at 5 $/kg

(2+4) kg at ? $/kg

Unit-numbers add on-top. Per-numbers add next-to as areas under the per-number graph:





Before adding totals, the units must be made the same. With unit-numbers, recounting will do the job. Per-numbers must be multiplied to unit numbers first. But multiplication creates areas. So, per-numbers add by the area under the per-number graph, which is called integral calculus.

With piecewise constant per-numbers this is a quick job. But with locally constant (or continuous) per-numbers this means adding extremely many area-strips.

However, the area always changes with the last strip, so the area can be found by the formula DA = p*Dx, becoming dA = p*dx or d/dx(A) = p or A’ = p in the case of locally constancy.

This motivates the development of differential calculus studying what is the derivative d/dx of known formulas.

Pushing slightly the top of two playing cards, we see that

Δ(f*g) ≈ Δf*g + f*Δg leading to

(f*g)’ = f’*g + f*g’, leading to

(x^2)’ = 2*x and (x^n)’ = n*(x^n-1).

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- Teaches Teachers to Teach MatheMatics as **Many**Math, a natural science about **Many**
- Cures **Math Dislike** when counting fingers in flexible bundle-numbers
- YouTube videos
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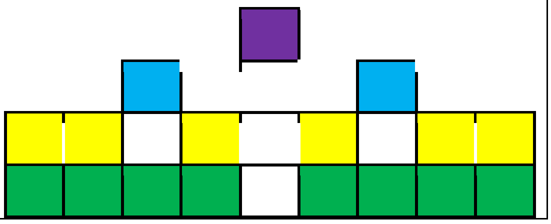
IconNumbers • ReCounting 7 in **5s** & **3s** & **2s**

MATHeCADEMY.net offers free Question Guided Teacher Education by Teaching Teachers to Teach MatheMatics as ManyMath, a Natural Science about MANY.

To learn Math, Count & Add MANY, using the CATS method: Count & Add in Time & Space. Primary: C1 & A1 & T1 & S1. Secondary: C2 & A2 & T2 & S2

In PYRAMIDeDUCATION a group of 8 teachers are organized in 2 teams of 4 choosing 2 instructors and 3 pairs by turn.

Each pair works together to solve Count & Add problems. The coach assists the instruc-tors when instructing their team and when correcting the Count&Add assignments.

	Each teacher pays by coaching a new group of 8 teachers. 

4 Ways to Unite & Split

Operations unite/ *split into*	changing	constant
Unit-numbers *m, s, \$, kg*	**T = a + n** *T – a = n*	**T = a * n** *T/n = a*
Per-numbers *m/s, \$/kg, m/(100m) = %*	**T = ∫ a dn** *dT/dn = a*	**T = a^n** $\log_a T = n$, $\sqrt[n]{T} = a$

We call this beautiful simplicity the '**Algebra Square**' since in Arabic, algebra means to reunite.

A number-formula T = 345 = 3**BB**4**B**5 = 3***B**^2+4***B**+5 (a polynomial) shows the four ways to add: +, *, ^, next-to block-addition (integration.)

<u>Add</u> & <u>multiply</u> add changing and constant unit-numbers. <u>Integrate</u> & <u>power</u> add changing and constant per-numbers.

The 4 uniting operations have a reverse splitting operation:

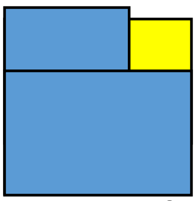
Add has <u>subtract</u> (–), and multiply has <u>divide</u> (/).

Power has factor-find (<u>root</u>, √) and factor-count (<u>logarithm</u>, log).

Integrate has per-number find (<u>differentiate</u> dT/dn = T').

Reversing operations solve equations by moving to **opposite side** with **opposite sign**.

Quadratic Equations
with 2 playing Cards

Solve the quadratic equation	**u^2 + 6u + 8 = 0**
3  8	(u+3)^2 = u^2 + 6u + 8 + 1
u	(u+3)^2 = 0 + 1
u 3	u+3 = ± 1
	u = -3 ± 1
	Solution: **u = -4, u = -2**

In the quadratic equation x ^ 2 + 6*x + 8 = 0 there are two unknown x's so it needs to be rewritten, so there is only one x.

Two playing cards has the width k and the height x + k. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that

(x+k)^2 = x^2 + 2*k*x + k^2, or, have the unknown x only once on the right side:

(x+k)^2 – k^2 = x^2 + 2*k*x, or 'x plus k squared, minus k squared gives x squared + double-k x.

We now rewrite the equation x^2 + 6x + 8 = 0 first to (x^2 + 2*3*x) + 8 = 0, then to (x+3)^2 – 3^2 + 8 = 0, and then to (x+3)^2 – 1 = 0, solved by three times moving to the opposite side.

Solving Equations

ManyMath: Recount

2 x u = 6 = (6/2) x 2	Solved by recounting 6 in 2s
u = 6/2 **= 3**	Test: 2 x 3 = 6 OK

MatheMatics: Neutralize with Abstract Algebra

2 x u = 6	Multiply has 1 as neutral element, and 2 has ½ as inverse element
(2 x u) x ½ = 6x½	Multiply 2's inverse element to both number-names

(u x 2) x ½ = 3	Apply the commutative law to ux2, 3 is the short number-name for 6x½
u x (2 x ½) = 3	Apply the associative law
u x 1 = 3	Apply the definition of an inverse element
u = 3	Apply definition of a neutral element. *With arrows, a test is not needed*

WHAT IS MATH - AND WHY LEARN IT?

"What is math - and why learn it?" Two questions you want me to answer, my dear nephew.

0. What does the word mathematics mean?

In Greek, 'mathematics' means 'knowledge'. The Pythagoreans used it as a common label for their four knowledge areas: Stars, music, forms and numbers. Later stars and music left, so today it only includes the study of forms, in Greek called geometry meaning earth-measuring; and the study of numbers, in Arabic called algebra, meaning to reunite. With a coordinate-system coordinating the two, algebra is now the important part giving us a number-language, which together with our word-language allows us to assign numbers and words to things and actions by using sentences with a subject, a verb and a predicate or object:

"The table is green" and "The total is 3 4s" or "T = 3*4". Our number-language thus describes Many by numbers and operations.

1. Numbers and operations are icons picturing how we transform Many into symbols

The first ten degrees of Many we unite: five sticks into one 5-icon, etc. The icons become units when counting Many by uniting unbundles singles, bundles, bundles of bundles. Operations are icons also:

Counting 8 in **2s** can be predicted by <u>division</u>, iconized by a broom pushing away **2s**: 8/2 = 4, so 8 = 4 **2s**.

Stacking the **2s** into a block can be predicted by <u>multiplication</u>, iconized by a lift pushing up the **2s**: 8 = 4x2.

Looking for unbundled can be predicted by <u>subtraction</u>, iconized by a rope pulling away the 4 **2s**: 8 – 4x2.

Uniting bundles and singles is predicted by <u>addition</u>, iconized by a cross, +, placing blocks next-to or on-top.

Recounting a total T in B-bundles is predicted by a 'recount-formula':	**T = (T/B)*B**	
saying 'From T, T/B times, B can be pushed away'.	9/2	4.some
Recounting 9 in **2s**, the calculator predicts the result	9 – 4x2	1
9 = 4B1= 4.1 **2s** = 4 **2s** + 1		

Now, let us write out the total 345 as we say it when bundling in ones, tens, and ten-tens, or hundreds, we get T = 3*B^2 + 4*B + 5*1.

This shows that uniting takes place with four operations: number-addition unite unlike numbers, multiplication unite like numbers, power unite like factors, and block-addition (integration) unite unlike areas. So, one number is really many numberings united by calculations.

Thus, mathematics may also be called calculation on specified and unspecified numbers and formulas.

2. Placeholders

A letter like x is a placeholder for an unspecified number. A letter like f is a placeholder for an unspecified calculation formula. Writing 'y = f(x)' means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Writing y = f(2) is meaningless, since 2 is not an unspecified number. The first letters of the alphabet are used for unspecified numbers that do not vary.

3. Calculation formula predict

The addition calculation T = 5+3 predicts the result without having to count on. So, instead of adding 5 and 3 by 3 times counting on from 5, we can predict the result by the calculation 5+3 = 8.

Likewise, with the other calculations:

- The multiplication calculation 5*3 predicts the result of 3 times adding 5 to itself.
- The power calculation 5^3 predicts the result of 3 times multiplying 5 with itself.

4. Reverse calculations may also be predicted

'5 + 3 = ?' is an example of a forward calculation. '5 + ? = 8' is an example of a reversed calculation, often written as 5 + x = 8, called an equation that asks: which is the number that added to 5 gives 8?

An equation may be solved by guessing, or by inventing a reverse operation called subtraction, x = 8 – 5; so, by definition, 8-5 is the number x that added to 5 gives 8. The calculator says that 8-5 is 3.

We now test to see if this is the solution by calculating separately the left and right side of the equation. The left side gives 5 + x = 5 + 3 = 8. The right side is already calculated as 8.

When the left side is equal to the right side, the test shows that x = 3 is indeed a solution to the equation.

Likewise, with the other examples of reverse calculations:

- $\frac{8}{5}$ is the number x, that multiplied with 5 gives 8. So, it solves the equation 5*x = 8.
- $\sqrt[5]{8}$ is the number x, that multiplied with itself 5 times gives 8. So, it solves the equation x^5 = 8.
- $\log_5(8)$ is the number x of times to multiply 5 with itself to give 8. So, it solves the equation 5^x = 8.

Thus, where the root is a factor-finder, the logarithm is a factor-counter.

Together we see that an equation is solved by 'moving to <u>opposite side</u> with <u>opposite sign</u>'

5 + x = 8	5*x = 8	x^5 = 8	5^x = 8
x = 8 – 5	x = $\frac{8}{5}$	x = $\sqrt[5]{8}$	x = $\log_5(8)$

5. Double-counting creates per-numbers and fractions

Double-counting in two units creates per-numbers as e.g. 3$ per 4kg or 3$/4kg or ¾ $/kg.

To bridge the units, we just recount the per-number: 15$ = (15/3)*3$ = (15/3)*4kg = 20kg.

With the same unit, a per-number becomes a fractions or percent: 3$/4$ = ¾, 3$/100$ = 3%.

Again, the per-number bridges: To find ¾ of 20, we recount 20 in 4s. 20 = (20/4)*4 gives (20/4)*3 = 15.

6. Change formulas

The unspecified number-formula T = a*x^2 + c*x + d contains basic change-formulas:

- T = c*x; proportionality, linearity
- T = c*x + d; linear formula, change by adding, constant change-number, degree1 polynomial
- T = a*x^2 + c*x + d; parabola-formula, change by acceleration, constant changing change-number, degree2 polynomial
- T = a*b^x; exponential formula, change by multiplying, constant change-percent
- T = a*x^b; power formula, percent-percent change, constant elasticity

7. Use

- Asking '3kg at 5$ per kg gives what?', the answer can be predicted by T = 3*5 = 15$.
- Asking '10 years at 5% per year gives what?', the answer can be predicted by the formula T = 105%^10 – 100% = 62.9% = 50% in plain interest plus 12.9% in compound interest.

- Asking ‘If an x-change of 1% gives a y-change of 3%, what will an x-change of 7% give?’, the answer can be predicted by the approximate formula T = 1.07^3 – 100% = 22.5% = 21% plus 1.5% extra elasticity.
- Asking ’Will 2kg at 3$/kg plus 4kg at 5$/kg total (2+4)kg at (3+5)$/kg?’, the answer is ‘yes and no’.

The unit-numbers 2 and 4 can be added directly, whereas the per-numbers 3 and 5 must first be multiplied to unit-numbers 2*3 and 4*5 before they can be added as areas.

Thus, geometrically per-numbers add by the area below the per-number curve, also called by integral calculus.

A piecewise (or local) constant p-curve means adding many area strips, each seen as the change of the area, p*Δx = ΔA, which allows the area to be found from the equation A = Δp/Δx, or A = dp/dx in case of local constancy, called a differential equation since changes are found as differences. We therefore invent d/dx-calculation also called differential calculus.

Geometrically, dy/dx is the local slope of a locally linear y-curve. It can be used to calculate a curve's geometric top or bottom points where the curve and its tangent are horizontal with a zero slope.

8. Conclusion.

So, my dear Nephew, Mathematics is a foreign word for calculation, called algebra in Arabic. It allows us to unite and split totals into constant and changing unit- and per-numbers. *Love, your uncle Allan.*

Algebra **unites**/*splits into*	**Changing**	**Constant**
Unit-numbers (meter, second, dollar)	**T = a + b** $T - b = a$	**T = a*b** $\frac{T}{b} = a$
Per-numbers (m/sec, m/100m = %)	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Bundle Counting Table

Bundle-counting clarifies that we count by bundling, typically in tens

Example 01. Counting Mikado Sticks
The Mikado sticks are positioned next to each other to the right. Counting is done by taking one stick at a time to the left and assembling them in a bundle with an elastic band when we reach ten.

When counting, we say: "0 Bundle 1, 0 bundle 2,. .. "

"Why 0 bundle?" "Because we don't have a bundle yet, before we'll reach ten."

"..., 0 bundle 8, 0 bundle 9, 0 bundle ten, well no, 1 bundle 0".

Example 02. Counting matches
The box says 39, which we read as '3 bundles 9'. We bundle-count as with Mikado sticks.

Extra-option
Some children may find it fun later to count ' 1 bundle less 2, 1 bundle less 1, 1 bundle and 0, 1 bundle and 1 ' as a new way to count ' 0 bundle 8, 0 bundle 9, 1 bundle 0, 1 bundle 1 '. Later again, some children may find It fun to say ' 1 bundle-bundle 0 ' instead of ' ten bundles 0 ' or ' hundred '.

Example 03. Counting ten fingers or ten matches
The ten fingers (or ten matches) bundle are counted in 4s and in 3s while saying "The total is..." and possibly writing "T =..."

Ten counted in 4s	Ten counted in 3s
T = I I I I I I I I I I = ten 1s	T = III I I I I I I I = 1B7 3s
T = IIIIIIIIII = 1 tens = 1B0 tens	T = III III I I I I = 2B4 3s
T = IIII I I I I I I = 1B6 4s	T = III III III I = 3B1 4s
T = IIII IIII I I = 2B2 4s	T = III III III III = 4B-2 3s
T = IIII IIII IIII = 3B-2 4s	T = III III III I = 1BB 0B 1 3s

Table for counting ten tens, or 1 bundle bundles, or 1 hundred:

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~

Wednesday, December 10: talk to the staff

What is research and how to do research? Broadly speaking, research is a search for knowledge, or a search for knowing about something and for knowing about how to do something. So research about mathematics education research is search for knowing about mathematics, about teaching and about learning.

Once you have conducted a research project you would like to publish it. You might send it to a journal, but typically it is very difficult to have your paper accepted here and it often takes a long time. Furthermore, research accepted in journals does not necessarily benefit practice as witnessed by the peer-review crises and replication-crisis described later.

So it might be a bettr idea to send your contribution to specific network conferences to try to have it accepted for presentation and included in the proceedings.

An overview over existing networks can be find in the invitation to and proceedings of the 4year global conferences called ICME, International Congress on Mathematical Education, held for the first time in 1969 and next time in 2020 in Shanghai arranged by the ICMI, International Commission on Mathematics Instruction that also arranges local 2 or 3year conferences as e.g. EARCOME, the ICMI-East Asia Regional Conferences in Mathematics Education, held last time in 2018 in Taiwan, and next time in 2021 in Seoul.

There are also yearly conferences as e.g. CTRAS, Classroom Teaching Research for All Students Conference, held yearly in China, or before the ICME conference each 4 year. This is an including conference that allows networks to be created, and possibly the 2021 conference could take place in Vietnam since that may attract more participants attracted by not having to apply for and pay for a visa to China. Also, it is connected to a welcoming Journal called 'Journal of Mathematics Education', see e.g. my contribution 'Tarp, A, 2018, "Mastering Many", Journal of Mathematics Education, vol 11(1), pp. 103-117'. Its focus on mathematics for all students may be of extra interest to the Vietnamese curriculum described below.

As to deadlines, the ICME has expired, but the CTRAS 2020 has a deadline in February, and the EARCOME 9 2021 in Seoul the deadline is probably the summer 2020. (http://www.ksicmi.org/about/about_01.php).

The coming years Vietnam will implement a new curriculum (issued with Circular No. 32/2018/TT-BGDDT Dec. 26, 2018 of Minister of Education and Training). The curriculum put emphasis on experiments, modelling, STEM and mathematics for everyone:

> Mathematics has more and more applications in life, knowledge and basic mathematical skills helped people solve problems in real life in a systematic and accurate way, contributing to the promotion of social development. Mathematics at schools contributes to the formation and development of key qualities, general competence and mathematical competence for students; develop knowledge, key skills and create opportunities for students to experience and apply mathematics to their practices; establish a connection between mathematical ideas, between mathematics and practice, between mathematics with subjects and other educational activities, in particular with science, natural sciences, physics, chemistry, biology, technology, information to implement STEM education. Mathematics content is often logical, abstract and essential. Therefore, in order to understand and learn mathematics, the mathematics program in the school should ensure the balance between "learning" knowledge and "applying" knowledge into specific problem solving. In the course of learning and applying mathematics, students always have the opportunity to use modern technology, teaching equipment, especially electronic computers and handheld computers to support the performances, explore knowledge, solve mathematical problems. (..) Mathematics helps students to have a relatively general view of mathematics, understand the role and applications of mathematics in practice, the professions related to mathematics so that students have a career orientation, as well as having the ability to explore issues that are related to mathematics throughout life. In addition to core education content, during each academic year, students (especially those with a natural science and technology orientation) are choosing to learn some learning topics. These topics are aimed at enhancing mathematics knowledge,

> and carefully applying mathematical knowledge to practical practices that meet students ' interests, needs, and career orientation.(..) The program is the spirit of "Mathematics for everyone", but everyone can learn math in a way that suits personal interests and abilities. The mathematics program attaches importance to the application, cohesion with practices or subjects, other educational activities, especially with subjects aimed at the implementation of STEM education, tied to the modern trend of economic development, science, social life, and global-level issues (such as climate change, sustainable development, financial education,,...). This is also reflected in the practical activities and experiences in mathematics education in various forms such as: implementing mathematical topics and learning projects, especially in practical applications and projects of mathematics.

The new Vietnamese curriculum is based upon Kolb's experimental learning cycle that applies both to the learning of students and teachers. The website 'https://www.simplypsychology.org/learning-kolb.html' writes that the cycle has four phases: 1. Concrete Experience - a new experience or situation is encountered, or a reinterpretation of existing experience; 2. Reflective Observation of the New Experience - of particular importance are any inconsistencies between experience and understanding; 3. Abstract Conceptualization reflection gives rise to a new idea, or a modification of an existing abstract concept (the person has learned from their experience); 4. Active Experimentation - the learner applies their idea(s) to the world around them to see what happens.

Being heavily inspired by Piaget, the four phases may be condensed to two core phases of adaption, assimilation and accommodation, where outside experiences create inside schemata assimilating the outside world, or being accommodated in case of outside resistance.

A Kolb-based curriculum allows teachers to become learners also by doing research in their own classroom: Work out plan A. Observe strong and weak points when implementing plan A. Use this to modify plan A to plan B. Observe strong and weak points when implementing plan B. Use this to modify plan C. Observe strong and weak points when implementing plan C. Etc.

Here Design Research (see e.g. A. Bakker's Design Research in Education Design Research in Education, Routledge 2018) presents itself as a natural method to use in master level and phd level research. To meet the genre-claim of research, the data gathered must be reliable, and the conclusion must be tested for validity. And in design research reliability comes when making systematic observations through notes, interviews, questionnaires etc. And test for validity here means holding on to the strong parts of the actual micro curriculum and changing the weak parts.

At a master level it suffices to work out and test plan A and B. A phd level should also include plan C as well as a detailed collection of data using mixed methods with written leaner reactions leading to questionnaires allowing everybody to react to exemplary statements on an agreement scale 1-5 with 3 as neutral, followed by focus group interviews, and individual learning trajectories.

A plan reflects and writes down goals and means: What are the goals as to skills, understandings, and modeling? What are the means chosen, and not chosen?

Typically, plan A is determined by the choice of the textbook and the teaching tradition. Here difference research (see Tarp Math ed & research 2017. Retrieved from //mathecademy.net/2017-math-articles/) is a method to identify the choices made in a traditional plan A as well as alternatives that may be included and tested in plan B.

As an example of difference research including also STEM in the pre-calculus curriculum I pointed to my paper "Saving Dropout Ryan with a TI-82" in my booklet 'Math, modelling and models'.

There are many examples on alternatives micro-curricula. In primary school: flexible bundle-numbers, next-to and on-top addition. In middle school: per-numbers integrating rates and ratios and fractions, trigonometry before plane geometry, integral calculus used to add per-numbers and fractions. In high school: Regression used to model tables with polynomials. Integral calculus before differential calculus. Other examples on alternatives micro-curricula may be found in my trilogy: Math Ed & Research 2017-2019 found on MATHeCADEMY.net.

The talk also included the present crisis in research replication and peer review. Thus in the article "How Science goes Wrong", The Economist writes:

> A rule of thumb among biotechnology venture-capitalists is that half of published research cannot be replicated. Even that may be optimistic. Last year researchers at one biotech firm, Amgen, found they could reproduce just six of 53 "landmark" studies in cancer research. (..) The most enlightened journals are already becoming less averse to humdrum papers. (..) But these trends need to go much further. Journals should allocate space for "uninteresting" work, and grant-givers should set aside money to pay for it. Peer review should be tightened - or perhaps dispensed with altogether, in favour of post-publication evaluation in the form of appended comments. That system has worked well in recent years in physics and mathematics (The Economist, 19 Oct. 2013).

The replication crisis thus comes from the 'metascience' observation that many research studies are difficult or impossible to replicate or reproduce. It applies to different fields, e.g. psychology where Pashler and Wagenmakers (2012) writes:

> Is there currently a crisis of confidence in psychological science reflecting an unprecedented level of doubt among practitioners about the reliability of research findings in the field? It would certainly appear that there is (p. 528).

The authors refer among others to Ioannidis (2005) who writes:

> Scientists in a given field may be prejudiced purely because of their belief in a scientific theory or commitment to their own findings. Many otherwise seemingly independent, university-based studies may be conducted for no other reason than to give physicians and researchers qualifications for promotion or tenure. (..) Prestigious investigators may suppress via the peer review process the appearance and dissemination of findings that refute their findings, thus condemning their field to perpetuate false dogma (p. 0698).

As to the peer review process, LeBel (2015) writes:

> In recent years, there has been a growing concern regarding the replicability of findings in psychology (..) I propose a new replication norm that aims to further boost the dependability of findings in psychology (p. 1).

Addressing case series studies, Horton (1996) writes:

> The importance of the case series in surgical research is beyond doubt. Therefore, it seems reasonable to ask whether we can trust this study method to yield a valid result. According to conventional epidemiological wisdom, the answer is no (p. 984).

The quality of research was also questioned by Lyotard (1984) distinguishing between consensus and dissension:

> Consensus is a component of the system, which manipulates it (..) its only validity is as an instrument to be used toward achieving the real goal, which is what legitimates the system - power. (..) Returning to the description of scientific pragmatics, it is now dissension that must be emphasized (p. 60-61).

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DNM

35. Review 01 ICMT3

The paper contains 6 pages in the specified format. It is reviewed as an empirical study. For each category, the proposal is evaluated as -2: below the standard -1: slightly below the standard 0: meets the standard 1: good 2: excellent

1) Rationale, aim/goals, research questions (0)

There seems to be three research questions for the six pages paper, thus raising the question: Is there room for addressing three research questions in a six pages paper?

The first two is on page 1, chapter 2 section 2: “In our study, we ask how artefacts can be designed for inclusive education, and which inclusive practice can be reconstructed in lessons, when teachers use of the designed artefacts.”

The third is on page 3, chapter 2 sections 2: “RQ1: How are plenaries in inclusive classes characterised?”

2) Theoretical framework and related literature (-1)

It would be nice with definitions, discussions and literature on the core concepts: artefacts, on inclusive education and on practices.

On page 2, the paper emphasizes the concept of “differential sensitivity”, which does not seem to be a theoretical concept but one found in a textbook and used to allow dividing a lesson into two parts, a common part, and a part differentiating slow and quick learners with adaptive and in-depth activities respectively.

Part of the literature is not related to the research question, if any, but has more the character of name dropping.

3) Methodology / statement of authors position and argumentation (-1)

It would be nice to see how the artefact the ‘twenty ten frame’ was used by the three categories of students.

It would be nice to discuss or design alternative artefacts or alternative applications for subtraction tasks.

It would be nice to discuss the truth regimes adempted when choosing subtraction tasks as more relevant than giving meaning to the artefact ‘minus’ occurring both when subtracting as a process and when describing a lack in a total as e.g. T = 8 = Bundle less 2 = B-2 where B is the actual bundle-size ten used when counting.

It would be nice to discuss what kind of mathematics is served by serving the subtraction regime: a North American pre-setcentric version, a present set-centric version or a post-setcentric version.

The paper contains very few data if any on plenaries, mentioned in research question three.

4) Results (-2)

In relation to the first research question you would expect a detailed description and a discussion of different artefacts designed for inclusive education, together with additional adaptive and in-depth questions and tasks. As to the second research question you would expect a description of a wide variety of practices as well as arguments for them being inclusive.

None of this is given here. In the table 1, classroom situation starter plenary, the horizontal and vertical headings are not explained in details, consequently it is difficult to estimate the relevance of these data in connection with the research questions.

The discussion is not related to the research question(s). Furthermore, it introduces new concepts as ‘complexity reduction’ and ‘holistic nature’ that should have been introduced and discussed before choosing a methodology. Finally, the discussion ends with postulating several expectations without any grounding and with little relevance to the research question.

5) Clarity (0)

The chapter headings are: INTRODUCTION, DESIGN PRINCIPLES FOR INCLUSIVE MATHEMATICS EDUCATION, DESIGN AND METHODS, ANALYSIS OF AN EPISODE: INCLUSIVE PRACTICES, DISCUSSION

Thus, there is no chapter called ‘conclusion’ or ‘findings’. The word ‘design’ seems to be used in two different ways, as a design of material, and as a design for the study where the term ‘study-plan’ might prevent confusion. The method described in the method chapter seems to be one chosen from a more general research project, and not for this study specifically. Thus, the chapters seem to lack coherence and relevance to the research question.

Proof-reading by a native speaker would improve the semantical and grammatical quality.

Question: Can textbooks be artefacts? The ‘twenty ten frame’ is an outside artefact with independent existence in the world, but is an inside description of it an artefact? According to Heidegger an Existentialism, in a verdict sentence ‘A is B’, the outside fact A exists, but the inside predicate B is an essence that is chosen and could be different. So including inside descriptions and outside existence as being both artefacts seems to water out the root of the word artefact, to distinguish between ‘nature-fact’ and human-fact’, between a stone and a stone-ax. One of the main points about textbook is that they work best if accompanied by concrete artefacts that exist outside the text in the real world.

A clear recommendation, ACCEPT for presentation without further modification; TO BE CORRECTED as detailed below; REJECT

My recommendation will be “To be corrected”

Suggestions for improvement in a free text field for the authors

The first part of the focus and the curiosity of the paper is very important: Materials for inclusive mathematics education, likewise is asking “how artefacts can be designed for inclusive education”.

Since there seems to be little time to collect data for a quality empirical paper, why not change it to a theoretical essay using sociological imagination and ‘difference research’ to include also narratives (tales about change in time and space), so the research question would be something like: “how can artefacts and narratives be integrated within inclusive education”

This would allow testing on a small scale a new innovative approach possibly with some in-depth descriptions of examples on “making losers users”

An example. The textbook example shows the subtraction 10-3 illustrated by an artefact ‘a twenty ten frame’. Integrating a narrative means re-embedding and bridging the inside calculation in an outside problem by using a full number-language sentence with a subject and a verb and a predicate as in the word-language: the total is 7, abbreviated to T = 7.

This allows formulating several outside narratives:

01. “Yesterday there was ten, what happened?”; expecting the answer, “3 has been taken away” leading to the bridging or modeling narrative “T = 7 = 10 – 3” enacted by adding additional 3 to be immediately taken away. “See, we get 7 by taking 3 away from ten”

02. “Tomorrow there will be ten. Can we tell about this by writing differently what we have today?”; expecting the answer “today we have the ten less 3” leading to the bridging or modeling narrative “T = / = 10 less 3” or “T = 10 – 3” enacted by adding additional 3 in the ten-line below to be added first thing tomorrow. “See, we only lack three. So tomorrow we get the ten by then adding the three to the seven we have today”

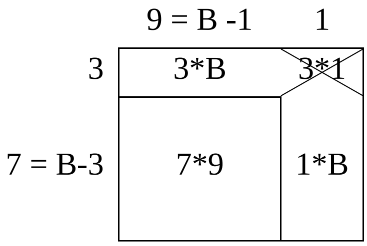
03. “How can a calculator predict what is needed tomorrow?”; expecting the answer “We enter 10-7”

04. “How can we tell a robot what the problem is?”; expecting the answer “Tell me what I need in order to enlarge 7 to ten?” abbreviated to “7 + what = 10” or “7 + ? = 10” or “7 + u = 10” called an unsolved question or equation to be answered or solved by the subtraction u = 10-7, thus isolating the unknow by “moving the other number, to the other side, with the other sign”, a method that works for all future equations with other operations.

Renaissance used a vertical and a horizontal stroke to separate negative numbers and divisors. The horizontal stroke still exists, but the vertical has disappeared except within book-keeping. Using full number-language sentences contains more narrative information about where the result came from so what has importance is not only the end result but also from where it came and why. By bridging the outside existence with the inside essence full number-language sentences also serve as modeling.

In this way we don’t serve a special inside ‘truth regime’ called ‘teaching subtraction’. Instead we help the learners to bridge meaning to the two outside ways the inside horizontal stroke occurs: as ‘taking-away’ in time, or as ‘lack’ in space, thus also introducing negative numbers together with subtraction, i.e. in a natural way. Thus being prepared to allow learning multiplication tables using ‘flexible bundle-numbers” where B stands for the bundle B used when counting totals, in this case tens.

Example 7*9 = 7 9s = (B-3)*(B-1) = BB, less 3B, less 1B, lessless3 = (ten-3-1)B +3 = 6B3 = 63, thus experiencing that ‘lessless’ or negative times negative gives positive as shown by illustrating the product by a ten by ten square where 7*9 is 7 9s. Here the 7 9s come from taking away the horizontal block 3B and the vertical block 1B, only we must add the corner block 3 since it has been taken away twice.



References:

Tarp, A. (2017). *Difference-Research Powering PISA Performance*. http://mathecademy.net/difference-research/

Tarp, A. (2018a). Mastering Many by Counting, Recounting and Double-counting before Adding On-top and Next-to. *Journal of Mathematics Education*, *11*(1), 103-117.

Tarp, A. (2018b). A Heidegger view on how to improve mathematics education. *Philosophy of Mathematics Education Journal, 33*.

Comment to the second version of the paper

The authors have now revised their papers or poster proposals according to reviewer queries and suggestions. Every author, whose paper or poster proposal required corrections was asked to revise the paper and upload a) a revised version of the paper, b) a revised version with tracked changes and c) a letter to the reviewers with a comment on required changes to the latest revision.

Now, it is time to decide if the revisions have improved the papers or poster proposals in a way that they can be accepted for publication in the conference proceedings. Therefore, we kindly ask you to go to the reviewing area and check if the authors have improved their papers or poster proposals sufficiently according to your suggestions. In the reviewing area you can access the revisions of the same papers that were assigned to you for reviewing.

As to the paper, the second version includes some minor changes as sentences left out or reformulated, and the language is now more fluent; but the original structure is unchanged.

I have received no “letter to the reviewers with a comment on required changes to the latest revision” in order to discuss the relevance of my original comments.

When describing the results, I gave the grade -2 mainly because “The discussion is not related to the research question(s). “ as you would expect of a research paper.

Likewise, we have not have an opportunity to discuss my suggestion as to how the paper could be improved: “Since there seems to be little time to collect data for a quality empirical paper, why not change it to a theoretical essay using sociological imagination and ‘difference research’ to include also narratives (tales about change in time and space), so the research question would be something like: “how can artefacts and narratives be integrated within inclusive education”. This would allow testing on a small scale a new innovative approach possibly with some in-depth descriptions of examples on “making losers users”.”

I am certain that, for the next conference, it will be possible to write an extraordinary interesting paper on materials for inclusive mathematics education; and, if wanted I would like to contribute with offering a cooperation on some of the ideas mentioned in the review.

However, in its present state and missing the dialogue part in its second version, I fail to see that the paper is to be accepted for publication in the conference proceedings.

36. Review 02 ICMT3

The paper contains 6 pages, almost in the specified format. For each category, the proposal is evaluated as -2: below the standard -1: slightly below the standard 0: meets the standard 1: good 2: excellent.

1) Rationale, aim/goals, research questions (-1)

There seems to be no explicit research question. Implicitly, however, the research question might be: ”But you can start to give them feeling and knowledge about the dimensions of this activity [how to write good books] and to let them practise their first steps to become an author.” (page 1)

2) Theoretical framework and related literature (-2)

There are no references in the end even if there is space for it within the 6 pages, especially if the five-six examples of a double page break is removed which will give an additional half page. Some references to authors are given in the text without specifying the work referred to (Example of a lecture, page 3, Reader, page 5).

Several parts have no reference: The importance section, page 1-2; The art of writing section, page 2; Theoretical Component, page 2-3.

3) Methodology / statement of authors position and argumentation (-1)

It would be nice to have a discussion about how to set up a research design and how to choose a research methodology to answer the research question.

Likewise, it would be nice to have some reflections on the fact that the project seems to be an example of export from a highly developed country to less developed countries thus calling for discussions based upon post-colonial theory: Can we be sure that the course will not be another example of a new-colonizing project?

Especially, Germans with direct access to the rich German social theories, are somewhat obliged to include Habermas’ theory about the system-world colonizing the life-world into this colonization discussion.

It would be nice to discuss or design alternative courses and to argue why they have not been chosen.

It would be nice to discuss the truth regimes adopted, e.g. when choosing subtraction tasks as more relevant than giving meaning to the artefact ‘minus’ occurring both when subtracting as a process and when describing a lack in a total as e.g. T = 8 = Bundle less 2 = B-2 where B is the actual bundle-size ten used when counting.

It would be nice to discuss what kind of mathematics is served by serving the subtraction regime: a North American pre-setcentric version, a present set-centric version or a post-setcentric version.

The paper contains very few data if any. Thus it would be nice to have the following statement substantiated: “Experiences (for example in Kosovo) show me that teaching without rich textbooks does not work properly.” (page 2)

Likewise, it would be nice to know what supports the following claim: “Textbooks generate and grant • Steadiness, • Consistency, • Perpetualness, • Continuity, • Sustainability for students and for teachers.“ How are these concepts defined? From what theory are they derived or taken? And what is meant by claiming “There is no Teaching 4.0 or Learning 4.0 or higher”; and how can this claim be supported?

In short, it would be nice to know the reasons for what choices have been made, and between which alternatives.

4) Results (-1)

As results is given a description of the contents of the course with an example of a lecture, again without discussing which alternatives have not been chosen and why, thus in Denmark a discussion has taken place between Mogens Niss' 8 competecies and Allan Tarp's 2 competences, counting and adding.

A short description of three sections called Practical component, Examples of assignments, and the Reader is also enclosed. Here the Reader section perhaps should inform about the alternatives a textbook writer must choose between by informing about the controversies in the grand theories surrounding mathematics and its education: in philosophy between structuralists and existentialists, where German can benefit from their direct access to Heidegger and Nietzsche; in psychology between Vygotsky and Piaget disagreeing about the need for extensive teaching; and within sociology where the agent-structure debate discusses if systems should adapt to actors or the other way around.

Finally, a chapter is included called "THE ORGANISATION AND THE PRAXIS OF THE WORKSHOP".

As to a discussion related to the research question, very little data and discussion is included on the aim "to let them practise their first steps to become an author" . It seems as if the course is more eager to supply the participants with a certain (subject-ideological?) background instead of allowing them to write themselves exemplary texts on core primary and secondary arithmetic, algebra and geometry and allowing their voices to be heard as to how they experienced this writing experience. Instead only a short evaluation to the course is included, which coheres only little to the research question.

5) Clarity (0)

The chapter headings are: INTRODUCTION, THE DESIGN AND FRAMEWORK OF THE WORKSHOP, and The Reader (not following the format given). Inside these chapters there are several sections. Thus in the Reader chapter there are two additional chapters: THE ORGANISATION AND THE PRAXIS OF THE WORKSHOP, SOME BUT IMPORTANT RESULTS AND EXPERIENCES, FURTHER STUDIES AND PROJECTS.

Thus, there is no chapter named after the four review criteria, nor is there a chapter with references even if there is half a page for it if complying with the format criteria. Thus, the chapters seem to lack coherence and relevance to the research question.

Proof-reading by a native speaker would improve the semantical and grammatical quality.

A clear recommendation, ACCEPT for presentation without further modification; TO BE CORRECTED as detailed below; REJECT

My recommendation will be "To be corrected"

Suggestions for improvement in a free text field for the authors

The aim of the report is very essential, to design and realize a workshop in writing math textbooks.

However, the moment we want to export a national workshop to a foreign culture a lot of consideration have to be made, first by informing us about culture encounters, especially in the case of a possible postcolonial situation as here, then to reflect if the participants should be informed about the choices they have to made before writing, or if is better to choose for them a specific knowledge regime.

So, the paper would gain immensely in relevance and quality if the research question could focus on this bey asking e.g. "which considerations should be made when exporting workshops in writing math textbooks to less developed countries?"

And again, here we that are living outside Germany could benefit enormously from scholars with direct access to the great variety of leading German speaking theorists. Also including the Glocksee project with its sociological imagination and exemplary education.

So maybe the paper should be rewritten as a theoretical essay, perhaps based upon Marx Feuerbach these 11 about changing instead of interpreting. After all, with close to 50 years of math education research still leaving many problems unsolved, maybe traditional setcentric mathematics is not the best thing to export if the less developed countries if they should have a chance to meet the OECD 2030 learning framework and supply all citizens with basic numeracy.

Thus, a core question in future workshops could be: how can we write textbooks "making losers users"

An example on the use of 'difference research' in textbook writing. In the textbook 'Das Zahlenbuch' an example shows the subtraction 10-3 illustrated by an artefact 'a twenty ten frame'. Integrating a narrative means re-embedding and bridging the inside calculation in an outside problem by using a full number-language sentence with a subject and a verb and a predicate as in the word-language: the total is 7, abbreviated to T = 7.

This allows formulating several outside narratives:

01. "Yesterday there was ten, what happened?"; expecting the answer, "3 has been taken away" leading to the bridging or modeling narrative "T = 7 = 10 – 3" enacted by adding additional 3 to be immediately taken away. "See, we get 7 by taking 3 away from ten"

02. "Tomorrow there will be ten. Can we tell about this by writing differently what we have today?"; expecting the answer "today we have the ten less 3" leading to the bridging or modeling narrative "T = / = 10 less 3" or "T = 10 – 3" enacted by adding additional 3 in the ten-line below to be added first thing tomorrow. "See, we only lack three. So tomorrow we get the ten by then adding the three to the seven we have today"

03. "How can a calculator predict what is needed tomorrow?"; expecting the answer "We enter 10-7"

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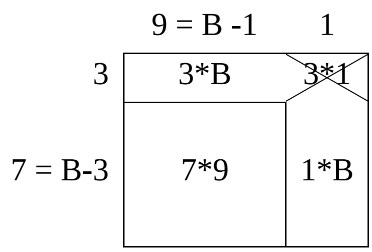
Renaissance used a vertical and a horizontal stroke to separate negative numbers and divisors. The horizontal stroke still exists, but the vertical has disappeared except within book-keeping. Using full number-language sentences contains more narrative information about where the result came from so what has importance is not only the end result but also from where it came and why. By bridging the outside existence with the inside essence full number-language sentences also serve as modeling.

In this way we don't serve a special inside 'truth regime' called 'teaching subtraction'. Instead we help the learners to bridge meaning to the two outside ways the inside horizontal stroke occurs: as 'taking-away' in time, or as 'lack' in space, thus also introducing negative numbers together with subtraction, i.e. in a natural way. Thus being prepared to allow learning multiplication tables using 'flexible bundle-numbers" where B stands for the bundle B used when counting totals, in this case tens.

Example 7*9 = 7 9s = (B-3)*(B-1) = BB, less 3B, less 1B, lessless3 = (ten-3-1)B +3 = 6B3 = 63, thus experiencing that 'lessless' or negative times negative gives positive as shown by illustrating the product by a ten by ten square where 7*9 is 7 9s. Here the 7 9s come from taking away the horizontal block 3B and the vertical block 1B, only we must add the corner block 3 since it has been taken away twice.



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Tarp, A. (2018b). A Heidegger view on how to improve mathematics education. *Philosophy of Mathematics Education Journal*, *33*.

Tarp, A. (2019). *A new curriculum - but for which of the 3x2 kinds of mathematics education*. http://mathecademy.net/the-3x2-kinds-of-math-education/

I received two reviews of my paper “Report on a Workshop ‘Writing Maths Textbooks’”. I appreciate these reviews and I am grateful to get these comments**. I will think about them and change my paper trying to follow their proposals and ideas as far as I am able to do that and to understand them.** **But I like to comment those reviews. When I did two reviews myself I discovered that the “Review Guidelines” are not extremely useful in their standardized and normalised perspective on research to judge on every kind of paper:** 1) Rationale, aim/goals, research questions 2) Theoretical framework and related literature 3) Methodology / statement of authors position and argumentation 4) Results 5) Clarity Even so I was quite sceptical about these review categories I thought it is not up to me to criticize them. But being reviewed myself I think it is worth to discuss them	Thank you for wanting to discuss the criteria of the research genre. I hope we agree on the importance of constructing labels to differentiate between texts reporting on physical things and actions, and texts portraying physical things and actions as examples of meta-physical constructs, and texts proposing meta-physical constructs for physical things and actions. In short that differentiates between journalism, bachelor or master level essays, and research. Also within research genre a distinction should be made between top-down and bottom-up research, the former working inside-outside or top-down by testing hypotheses about operationalized theoretical constructs, and the latter working outside-inside or bottom-up by creating new categories from observations. Thus in the former, the theoretical framework and related literature comes before the search begins, whereas in the latter it comes when discussing if the finding represents new knowledge. Furthermore, since research is a search for new knowledge, such a search should be guided by a question and a method to guide the search for an answer of some form The former then uses the order: Focus, existing theory, expanding methodologically, results. The latter using the order Focus, observations generating categories to be refined by additional observations, comparing with existing categories, result.

In maths education you find research where something is analysed maybe in a theoretical or an empirical way. But there is also mathematics education as a ‘design science’ (Erich Christian Wittman, Educational Studies in Mathematics December 1995, Volume 29, Issue 4, pp 355–374).	Thank you very much for referring to the Wittman paper MATHEMATICS EDUCATION AS A 'DESIGN SCIENCE' where he points to the danger that “However, there is the risk that by adopting standards, methods and research contexts from other well-established disciplines, the applied nature of mathematics education may be undermined. In order to preserve the specific status and the relative autonomy of mathematics education, the suggestion to conceive of mathematics education as a 'design science' is made.” In the chapter ‘THE 'CORE' AND THE 'RELATED AREAS' OF MATHEMATICS EDUCATION’ Wittman begins by phrasing Goethe: “The sciences should influence the outside world only by an enlightened practice; basically they all are esoteric and can become exoteric only by improving some practice. Any other participation leads to nowhere.” This warning can be used to warn against being too rigorous when applying grand theory to mathematics education; but it can also be a warning against being too rigorous when applying contemporary university theory to mathematics education having as the goal to improve the practise of mastering Many, the outside root of mathematics disciplines as geometry and algebra, both rooted in the outside world as shown by their Greek and Arabic meaning: to measure earth and to reunite quantities. Wittman apparently choses the former understanding by saying “Generally speaking, it is the task of mathematics education to investigate and to develop the teaching of mathematics at all levels (..) Scientific knowledge about the teaching of mathematics (..). This is a shame because what Goethe meant was that such a mathematics becomes the esoteric science he warns against, hoping that instead mathematics would see itself as becoming “exoteric only by improving some practice”, in this case the practice of mastering the outside physical fact Many.
I tried to report about a workshop I developed on ‘Writing Maths Textbooks’. Now to those review categories:	To make a report part of a research genre it could ask questions as e.g. “Which options exist when designing a workshop on writing Maths textbooks, and which reflections made me chose some options to others that I rejected.”

The aim was to develop and deliver such a workshop. I think this is research even so I did not come up with a bundle of research questions. I think the simple triple jump ‘research questions -> methods -> results’ is more a persiflage on research than research.	As mentioned above developing a workshop becomes research if options and reflexions are included. Likewise, to report on delivering a workshop may become research if also reporting on how the participants reacted to the number of options to choose between letting their own voice be heard and possibly creating categories from these reactions.
While I used in my workshop a lot of literature, I did not find any example or model in the literature how design such a workshop. There is no related literature in narrower sense and no theoretical framework beside my own one	Also, within literature it would be interesting to report on options and reflexions behind the choices made. Especially since there are conflicting views within both philosophy and psychology and sociology. Which raises the question if these controversies should be mediated to the participants or not. And especially if the basic question should be raised, if mathematics education has mathematics as its goal or as a means to the outside goal of mastering many thus informing about the sociological warning against a goal displacement in mathematics education (Bauman, Weber).
I know it is nowadays quite common to use ‘methods’. But I am quite sceptical if this is the only warrant for real research or a warrant for research at all. Often these methods are just a thinking corset. Developing my workshop I could try to make the (my) underlining philosophy more explicit. But I did not work with any kind of method – beside thinking.	
The result of my efforts is my design of the workshop. It is not a bundle of answers to a bundle of questions.	
If I could not explain my design it is of course my fault. But I am totally unable to explicate my approach to such a workshop and all my decision on six pages.	
Finally I like to comment the critic: “Proof-reading by a native speaker would improve the semantical and grammatical quality” (Allan Tarp). You are right! But wait a minute: By globalisation of educational research we all should speak now English. If you are not a native speaker you are suddenly disabled. I am born and brought up in Germany. As far as thinking is language based I am thinking German. And a lot of concepts like for example “BIldung” and “Erziehung” and “Stoff” and “Stoffdidaktik” as we use them in the German tradition and history of philosophy and humanities are based and established in our German language. So it is nearly impossibly to translate them into English. My imperfect English should always	Thank you for pointing to the question about a common language. A hundred years ago we would be writing and discussing in German because of the great work and fundamental influence of German work in mathematics. As a Dane I had to learn both English and German as foreign languages. English was not so difficult because it is basically ‘Anglish’ i.e. a western Danish dialect around the Harbor region on the Danish west coast from where the settlers sailed, and some French on top because of the Norman invasion by Normans, also Danes but bilingual and thus importing French parallel words to show superiority.

remind you that I think German even so I try to formulate my thoughts in English.	As I see it, when using a foreign language I should respect its meaning and its spelling and its grammar, which a native speaker can help me correcting without knowing anything about the content. But besides I should inform about the existence of special concepts from my own language and gladly accept learning new concepts from another language. Especially, I eagerly listen to and try to read Germans talking about the core concept of mathematics education, called Bildung at the continent and having a history from Herder and Hegel and Humboldt as a sister to marxism and nationalism. Likewise I envy Germans for having direct access to the great thinkers as Nietzsche, Husserl, Heidegger, Weber and Habermas just to mention a few. It would be a great gift to mathematic education if Germans could bring light from these thinkers into it.

37. Comments to ICMT3 Reviewers

Hereby, we forward the reviewers' comments to you. The reviewers indicate weaknesses and make suggestions for improvement of your paper. You are kindly requested to react to their objections and consider their suggestions in the revision of your submission. Please upload

a) a revised version of your submission with tracked changes,

b) a comment to the reviewers,

c) a final version of the paper

before 15 May 2019.

Please note that the organizers of the conference take no liability for the comments of the reviewers. If there are any questions regarding the reviews you might contact the reviewers directly through the reviewing area.

Comments to reviewers by Allan Tarp, April 29, 2019

Below quotes from reviewer comments are inside "", and my comments are inside <>.

Part 01 is a self-review written to clarify the comments. Part 02 is comments to reviewer R2. Part 03 is comments to reviewer R3

<Thank you very much for the two reviews they have been very inspirational in different ways when rewriting the paper.>

Part 01. Self-review of the paper 'Developing the Child's own Mastery of Many'.

1) Rationale, aim/goals, research questions

On the background of the fact demonstrated by OECD studies that 'Research thus still leaves many issues unsolved after half a century (p. 1)', and of the similarity of the word-language and the number-language, the paper asks the question 'IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE? (p. 1)'

2) Theoretical framework and related literature

A theoretical framework is chosen within the grand theories within which mathematics education resides. In philosophy the framework is existentialism giving precedence to existence over essence; and within sociology it is the warning against a goal-displacement where an institutionalized means establishes itself as a goal instead.

As related literature is chosen the 23rd ICMI Study 'Building the Foundation: Whole Numbers in the Primary Grades' to describe and compare with the present tradition, as well as my own article from the Journal of Mathematics Education 11(1).

3) Methodology / statement of authors position and argumentation

As methodology is chosen observations examining and discussing the mastery of Many that children bring to school, thus uncovering a different number type: two-dimensional bundle numbers with units that implies a need for recounting in another unit both icons and tens; as well as a different order of operations that postpones addition until after counting and recounting have taken place.

4) Results

Examining how bundle-numbers can be developed and systematized, a twofold curriculum is designed allowing the children and the teacher to co-examine how research-like guiding questions leads to developing the children's already existing mastery of Many. This leads to the conclusion that 'the number-language children bring to school contain core mathematics as proportionality and calculus, which allows designing a curriculum for all students without splitting it up into tracks (p.6)'.

5) Clarity

The paper respects the genre-criteria of the research genre wanting to generate new knowledge by asking a question, by choosing a method to find an answer, by comparing the answer with already existing knowledge, and by discussing the answer as to its character and as to consequences as to further research.

Part 02. Comments to reviewer R2:

1) Rationale, aim/goals, research questions

“The aim of the paper is to characterise typical questions for a new kind of textbook curriculum, which aims to enable children to develop mathematical knowledge by counting, bundling and adding activities. The paper contains 24 listed questions and accompanying learning opportunities (13 for counting and 11 for adding).”

<I would say that the aim of the paper is to answer the research question ‘IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE?’>

2) Theoretical framework and related literature

“The theoretical framework seems to consist in the differentiation between number-language and world-language, but it is not pointed out comprehensibly how these languages can be characterised. Related literature is missing, too. “

< I would say that the theoretical framework is chosen within the grand theories within which mathematics education resides. And that as the related literature is chosen the 23rd ICMI Study ‘Building the Foundation: Whole Numbers in the Primary Grades’ as well as my own article from the Journal of Mathematics Education 11(1). >

“Additionally, some links are made to philosophical theories (Satre & Marx). However, the theories are neither linked to mathematics education nor are their meanings explained in the context of the paper. Mainly names and phrases are dropped without comprehensible connection to the text.”

< Thank you for this remark. Consequently, I have added to the conclusion the point that mathematics education always resides in a greater philosophical and sociological context and therefore should be able to answer to core philosophical and sociological questions as to whether existence precedes essence or whether a goal displacement has taken place in mathematics education. That such questions are rarely if ever addressed or answered traditionally does not mean that they are not important. On the contrary, an activity calling itself scientific research should always have a dialogue with its overarching grand theories.>

“There is a of lack of theories and empirical findings dealing with counting, grouping or the place value system.”

<I think that some can be found in the reference in the Journal of Mathematics Education 11(1), 103-117. Furthermore, the page format of six pages make you restrict yourself to answer the research question stated.>

3) Methodology/statement of authors position and argumentation

“The author advocates for a change in curriculum, but the suggested questions/tasks cannot be identified as a curriculum. Some questions seem to be new and contain interesting ideas, but the connection between them is insufficient. They are labelled as “textbook for a question guided curriculum”, but information about the grade, competencies or learning principles is missing. Questions/tasks for a “recounting” and “adding curriculum” are listed over almost three pages. However, it is neither clear how the questions are related to each other nor how they are connected to the “language theories”. The educational goal does not become clear to me.”

<Thank you for commenting on the lack of clarity. This made me rewrite the final chapter. As to the aim of the textbook I write on page 3: ‘This calls for a different kind of textbooks that instead of

mediating institutionalized knowledge allow the students to develop their existing knowledge trough guiding outside research-like questions.’ This means that the textbook is given the form of a series of ‘research-like questions’ to ‘allow the students to develop their existing knowledge’, which again means that we are talking about first year students. As to their internal connection, in many cases the following question occurs when working with the pervious question, thus counting sequences leads to asking what bundle to use which leads to what to do with the unbundled which leads to asking how to recount in another unit etc.>

“The bundling activities use non-decimal grouping as well as decimal grouping. It could be a good idea to deal with non-decimal grouping in school but an explanation or a discussion about the aim of non-decimal grouping is missing.”

<To me the important thing is to observe that children use grouping in their number-language so numbers become two-dimensional bundle-numbers, and then to respect this as a fact so that the development is guided by their own numbers instead of by an external abstract theory. Of course, in a longer paper with a different question it could be interesting to compare the two.>

“The “addition curriculum” is introduced by a short explanation of the relation between the operations and followed by a list of questions towards addition, subtraction, multiplication and division. It is not clear if the lists of questions contain the total of school activities. Furthermore, we do not get any information about how the ideas can be transformed into tasks and in which grades they should be realised. “

<Again, detail may be found in the reference Journal of Mathematics Education 11(1), 103-117. Here the important thing is to observe, that with two-dimensional bundle-numbers instead of one-dimensional line-numbers, adding has two forms, next-to addition and on-top addition leading directly to integral calculus and proportionality in grade one. This is a new and original result as reported in details in the journal reference, which leads to a need of designing alternatives textbooks and learning materials that accepts this and also accepts that addition is postponed to after counting and recounting has been practiced, and that organizes mathematics education as a coworking situation between the teacher and the students when investigating mastery of Many guided by research-like questions.>

4) Results/implications for research in the respective area

“The author does not make suggestions how the curriculum ideas could be evaluated or how the questions/tasks could be tested or developed to concrete textbooks task following the idea of design research.”

< Thank you for this remark. Consequently, I have changed and rewritten the conclusion by adding ‘Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees.’>

5) Clarity

“The paper lacks clarity in it´s argumentation, as mentioned above. The limit of pages is exceeded.”

<Thank you for being so detailed and structured in your comments about a lacking clarity. As to the limit of pages, the version I send in and the new version as well contains 6 pages as asked for, so since there is no specification as to the nature of exceedance, I fail to see how this should be the case.>

Part 03. Comments to reviewer R3:

1) Rationale, aim/goals, research questions

“The article is interesting and thought provoking and should be published as a discussion paper presenting different theses and assumptions. But I would not call it a research paper. It is maybe a paper to push or induce research.”

<As shown in the self-review, I would say that the paper is written in the research genre by stating and trying to answer the research question ‘IS ONE CURRICULUM AND TEXTBOOK FOR ALL STUDENTS POSSIBLE?’>

2) Theoretical framework and related literature

“The quotes come from a quite vast extent: PISA, Sartre and Marx “

< As shown in the self-review, I would say that the theoretical framework is chosen within the grand theories within which mathematics education resides. And that as the related literature is chosen the 23rd ICMI Study ‘Building the Foundation: Whole Numbers in the Primary Grades’ as well as my own article from the Journal of Mathematics Education 11(1). >

“The concepts of mathematics (‘goal displacement’) and number-language (?) are use in a wrong way. Probably the authors means ‘numeracy’ as other papers call it.”

<Thank you for this remark. Consequently, I have added on page 2 ‘Numeracy as ‘the ability to understand and work with numbers’ (Oxford Dictionary) thus has an outside interpretation by the child’s own mastery of Many that contrasts the inside interpretations seeing numeracy as applying institutionalized mathematics. ‘

“But mathematics is neither numeracy nor a number language.”

<The nature of mathematics has been left undecided after centuries of competing and conflicting views. Of course, a present dogmatic exist as it has always done and will always do in the future probably. But to think that the present dogmatic is the final is to lack a sense of historic humility. As to the goal of mathematics education, it cannot be to learn mathematics because of the meaningless self-reference in such a statement. Mathematics is an inside discourse of an outside source, the physical fact Many, as expressed by the ancient Greeks choosing the label mathematics for their four areas of knowledge about Many by itself, in space, in time and in time and space, also called arithmetic, geometry, music and astronomy. So, of course, the goal of mathematics education is to develop mastery of Many, or more precisely to develop the mastery of Many the children already possess when coming to school, anything else would be an attempt to brainwash and may be the reason for fifty years of less successful research in mathematics education.>

3) Methodology/statement of authors position and argumentation

<No comments is found as to this criterium>

4) Results/implications for research in the respective area

“I do not understand Q1 to Q24 on the pages 3 to 6. Are they examples for the text of a primary textbook? I would not give them to children in this way.”

< Thank you for this remark. Consequently, I have changed and rewritten the first part of page 3 to ‘Typically, a mediating curriculum sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a developing curriculum sees mathematics as an exoteric means to develop the children’s existing ability to master Many by numbering outside totals and blocks with inside two-dimensional bundle-numbers. This calls for different textbooks from grade 1 that don’t mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).’

“The questions in your CONCLUSION “Is the educational goal ...? What to choose ...? Must textbooks ...?” Could be answered in various ways. Your approach is certainly not the only one or only correct one.”

< Thank you for this remark. Consequently, I have rewritten the conclusion.>

5) Clarity

“And you should clarify conceptually the area of:

School-mathematics, primary maths, mathematics, application of mathematics, daily life of children and their use of numbers.”

< Thank you for this remark. I can only say that Instead of a top-down conceptualization of these concepts I have chosen a bottom-up approach and methodology that is grounded in the outside educational need to master Many as well as in the mastery of Many that children already possess.>

38. Educating Educators Reviews

Submission ID: 34 Abstract title: Addition-free Math Make Migrants and Refugees STEM Educators Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Declined Reviews: Submission 34 | Review 1 Numerical criteria Content: 1 Significance: 2 Originality: 3 Relevance: 2 Style: 1 Comment I suggest that this is reviewed by a Mathematics Education person because there were some things that I could not follow and this could be because I'm from science education. However I do not feel that the proposal was clear enough. My first reaction is that it is not a good proposal. Having said that, if this is going to be presented as a poster, the presenter can be asked to clarify by participants on a one to one basis according to their needs and backgrounds. The proposal states that it is research-based but no research is reported in the description. Recommend for acceptance or rejection Not suitable Recommend as oral or poster Poster	Submission 34 | Review 2 Numerical criteria Content: 2 Significance: 1 Originality: 4 Relevance: 1 Style: 2 Comment Recommend for acceptance or rejection Not suitable Recommend as oral or poster Poster Reviewers' comments: The proposal states that it is research-based but no research is described in the proposal. It is also not clear how refugees will benefit as a result of the addition-free recounting math. You also need to clarify terms such as Many, for participants who are not familiar. No details are included about what will be presented in the actual poster. Although the approach of ManyMath as described in this proposal is original and interesting it does not fit the conference theme of scaling up PD. Neither the role of the teacher nor PD is addressed in the proposal. Chair's comment: Sorry, your contribution does not fit the theme of the conference (scaling up) professional development of science and mathematics teachers. We suggest you to try to submit your contribution at another conference. Success!
Submission ID: 36 Abstract title: Recounting Before Adding makes Teachers Course Leaders and Facilitators Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Accepted Reviews: Submission 36 | Review 1 Numerical criteria Content: 2 Significance: 2 Originality: 3 Relevance: 2 Style: 2 Comment The idea of the inquiry-based material repository Mathecademy is interesting. It's a	Submission 36 | Review 2 Numerical criteria Content: 3 Significance: 3 Originality: 4 Relevance: 3 Style: 3 Comment interesting idea of the self sustaining learning community Recommend for acceptance or rejection Suitable Recommend as oral or poster Oral Reviewers' comments: We think it is better to make a poster of this proposal, we are not convinced this proposal fits the format of a workshop, but are positive

pity you only have one reference (only by yourself), we know (worldwide) more teachers/researchers try to broaden the educational possibilities of blended learning (combinations of real meetings and online support). Recommend for acceptance or rejection Suitable Recommend as oral or poster Poster	to have this contribution at the Ete. Please further specify the target group and what they gain from this presentation for their daily practice. Format of the presentation is not specified is this a PowerPoint? or web-based? Chair's comment: Please update your proposal for a poster presentation and take account of the comments from the reviewers.
Submission ID: 37 Abstract title: Self-explanatory Learning Material to Improve your Mastery of Many Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Accepted Reviews: Submission 37 | Review 1 Numerical criteria Content: 1 Significance: 2 Originality: 3 Relevance: 3 Style: 2 Comment Topic 2, material dimension, research-based workshop. Reevant materials, but workshop not clearly described in proposal. Recommend for acceptance or rejection Suitable Recommend as oral or poster Oral	Submission 37 | Review 2 Numerical criteria Content: 1 Significance: 2 Originality: 3 Relevance: 1 Style: 3 Comment Not really a workshop described in the proposal, that is why I have to reject it. Recommend for acceptance or rejection Not suitable Recommend as oral or poster Poster Reviewers' comments: In the current form it does not describe the role of teachers or PD in the workshop. Creative and interesting approach does not fit the conference theme. The way this proposal is set up I think it will be better suited as a poster rather than a workshop. Relevant materials to present at a workshop, well linked to the conference theme and the materials dimension. But, the proposal do not give any theoretical grounding, or any structure for the workshop. On the other hand it gives lots of examples, but these are more or less presented in a too complicated way. Do you mean that you have 8 topics to present at the workshop? What about paragraph 10? What do you try to say with this? I assume that the examples are more understandable in practice (hands-on), then written here. I suggest that you add the meta level as an introduction to the workshop (incl reasons behind the examples, and add references if this is meant to be research-based), and some experiences with use of the materials at the end. Chair's comment: Please take account of the comments from the reviewers to update your proposal.
Submission ID: 42	Submission 42 | Review 2

Abstract title: Can Grounded Math and Education and Research Become Relevant to Learners Submitted by: Allan Tarp (Allan.Tarp@gmail.com) Status: Accepted Reviews: Submission 42 | Review 1 Numerical criteria Content: 3 Significance: 3 Originality: 4 Relevance: 3 Style: 3 Comment The paper of the discussion group addresses a very interesting (philosophical) topic. The paper contains also many good ideas, but also a few questionable statements. I suggest the authors to 1) link the DG more clearly to the overall conference theme (scaling up) 2) reflect whether (well) defining education, teacher and learner is the main challenge (I doubt that we will never reach a world-wide accepted solution) 3) avoid stating that 'Lehrer' means using the same word as for learning (Lehrer=Teacher, lehren=teaching; Schüler=learner/student, lernen=learning); by the way: use "Unterricht" not "Unterrichtung" 4) avoid stating that constructism means only working with peers or with manupulatives (this is a misuse; Glasersfeld et al. mean that also lecturing by a teacher leads to students' constructions; teachers, peers etc. can only send sonics, there is no direct transmission of knowledge (however, some teachers and teacher educators think so). Constructivism is understood by Glasersfeld etc. as an epistemological stance, not as a didactical principle) Recommend for acceptance or rejection Suitable Recommend as oral or poster Oral	Numerical criteria Content: 2 Significance: 2 Originality: 3 Relevance: 2 Style: 3 Comment Seems to be a discussion group proposal. Not clear what Grounded math would be. Recommend for acceptance or rejection Not suitable Recommend as oral or poster Poster Reviewers' comments: Chair's comment: Please update your proposal for an oral presentation and take account of the comments from the reviewers.

Math Ed & Research 2020

No Power Point Presentations

The same Mathematics Curriculum for Different Students
Addition-free STEM-based Math for Migrants
Math Dislike Cured with Inside-Outside Deconstruction
Developing the Child's Own Mastery of Many
What is Math - and Why Learn it?
Flexible BundleNumbers

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Preface

● The texts 01 and 02 concern the ICMI Study 24, School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities, held in Tsukuba, Japan, 26-30 November 2018. My paper, ‘A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many’, is included in the 2018 Articles. The discussion document theme B, “Analysing school mathematics curriculum reforms for coherence and relevance” had five sub-questions, and I was asked to contribute to writing a chapter addressing key-question B2, “How are mathematics content and pedagogical approaches in reforms determined for different groups of students (for e.g. in different curriculum levels or tracks) and by whom? How do curriculum reforms establish new structures in content, stakeholders (e.g. students and teachers), and school organisations; and what are their effects?”, in short called mathematics for all.

The first outline was scheduled to February 15, 2019. Having almost finished the outline, on February 12 we received a mail saying “The deadline for the chapter outlines has been extended until at least the end of February (from 15 February). I would like to open discussion between us on several matters.” On February 20 I sent in my response (see 02).

I never got any reaction so on March 30 I sent a mail to the organizing committee saying “To stimulate our work, would it be an idea to send out a monthly or quarterly newsletter reporting on the progress and challenges being made and met?” No response came so I began writing a proposal for a contribution (se 01). Then on May 10 I got a mail saying “There will not be a separate chapter on key question 2”, to which I responded “I think that the question ‘math for all’ as focused on in the question B2 is so important that it deserves an answer. When mentioned at the conference that no paper addressed this I objected since my paper is addressing the question if it with a different way of organizing math education will be able to include all. Furthermore, I have written a first draft that I send on February 12. Moreover, I have collected a substantial amount of material to include, just waiting for an answer to my mail on February 20 and March 28. So I am going to write a chapter with the focus originally decided upon since I think the research question deserves an answer as mentioned above. I will send it to you as planned before at the end of June. You might then decide not to include it, it will be your choice, then I will publish it elsewhere since the question is very important and since the material, I have collected shows that it is indeed possible to have mathematics for all in different ways.”

I then sent in my proposal before July 1 but heard nothing then or after the time limit for a reaction on August 15. So apparently my contribution will be the chapter that was commissioned and rejected without even being read.

01. The same Mathematics curriculum for different students.

The paper has the following chapters: 01. A need for curricula for all students 02. Addressing the need 03. Coherence and relevance 04. Parallel tracks to the main curriculum, examples 05. Pre-calculus, typically the last mandatory curriculum 06. Precalculus in the Danish parallel high school, a case study 07. A refugee camp curriculum 08. Do we really need parallel curricula 09. Conclusion.

02. Comments to a discussion paper

● At the Genoa University April 8-11, Paolo Boero held an international workshop called ‘Habermas’ elaboration on rationality and mathematics education’ over the Habermas text ‘Some further clarifications of the concept of communicative rationality’. I was allowed to give a short presentation.

03. A Mathematics Teacher Using Communicative Rationality Towards Children

● At Beijing Normal University June 28-30 2019, the 2019 Classroom Teaching Research for All Students Conference (CTRAS) took place with the conference theme ‘innovative practices and research initiatives in STEM integration that supports all students’ mathematics learning (..) The conference will provide participants from around the world with the opportunity to share: 1) best practices of STEM integration; 2) the most contemporary STEM research initiatives; 3) innovative curriculum of STEM integration; and 4) professional development approaches for STEM educators.' I contributed with a paper with a power point presentation, a proposal for a short presentation, a poster and a workshop, 04-08

● At Paderborn University September 16-19 the third international conference on mathematics textbooks research and development, ICMT3, will take place. Invitations were sent out to contribute with oral communications, workshops, posters, papers and symposia. I sent in one oral presentation (09), one paper (14), one workshop (11) and three posters (10, 12, 13). All were rejected except for the paper that was accepted for a ten minutes oral presentation.

09. Developing the Child's Own Mastery of Many, oral presentation

10. Math Dislike Cured with Inside-Outside Deconstruction, poster

11. Learning from The Child's Own Mathematics, workshop

12. Five Alternative Ways to Teach Proportionality, poster

13. New Textbooks, but for Which of the 3x2 Kinds of Mathematics Education, poster

14. Developing the Child's Own Mastery of Many, paper. The abstract says: Sociological imagination sees continuing educational problems as possibly caused by a goal displacement making mathematics see itself as the goal instead of its outside root, mastery of Many. Typically, the number-language is taught inside-inside as examples of its meta-language. However, as the word-language, it can also be taught inside-outside, thus bridging it to the outside world it describes. So, textbooks should not reject, but further guide the mastery of Many that children bring to school.

The chapters are called 'is one curriculum and textbook for all students possible, meeting many, children bundle to count and share, textbooks for a question guided counting curriculum, textbook for a question guided adding curriculum, discussion and future research.'

15. The PowerPointPresentation is called 'The Child's Own Mastery of Many, Count & ReCount & DoubleCount, before Adding NextTo & OnTop' and contains 43 slides.

● At Freiburg Pädagogische Hochschule October 7-8 the third Educating the Educators International Conference on approaches to scaling-up professional development in maths and science education will take place. Invitations were sent out to contribute with oral presentation sessions in the three dimensions (personal, material and structural) to report on projects, approaches and research, workshop sessions actively involving all participants, discussion group sessions also actively involving all participants, poster sessions and materials market, allowing participants to exhibit interesting professional development materials (including classroom materials) and learn about other materials.

The conference focused on three topics wanting to 'serve as a lever and platform for international exchange about concepts and experiences. The aim is to present and discuss different approaches which ensure a high quality of the education of educators:

* Personal dimension: Which roles, contents and activities have to be considered in the professional development courses for PD course leaders and facilitators in professional learning?

* Material dimension: Which role can materials play in professional development for maths and science teachers (classroom materials, face-to-face PD materials and e-learning PD materials)?

* Structural dimension: How can projects or initiatives for scaling up professional development look like and how can they be evaluated?

I sent in four proposals. One was rejected (16, sent as a poster for topic 3), two were accepted as posters (17 sent as a presentation for topic 1, 18 sent as a workshop for topic 2), one was accepted for presentation (19 sent as a discussion group 3). The proposal for a material market (21) was accepted.

16. Addition-Free Math Make Migrants and Refugees Stem Educators

17. Recounting Before Adding Makes Teachers Course Leaders and Facilitators

18. Self-explanatory Learning Material to Improve your Mastery of Many

19. Can Grounded Math and Education and Research Become Relevant to Learners

20. The PowerPointPresentation is called 'Can Grounded Mathematics & Education & Research become Relevant to Learners?' and contains 54 slides.

21. Recounting in Icon-Units and in Tens Before Adding Totals Next-To and On-Top, together with the posters presented at the stand.

● The following note is handed out to students and to teacher to have a basic discussion of the need and form of mathematics education.

22. What is Math - and Why Learn it?

● This material is meant for high school to illustrate how algebra and geometry should be always together and never apart.

23. Mathematics with Playing Cards

● This math compendium is meant for a high school pre-calculus course to illustrate the point made in (01) that it is possible to start all over from the bottom in a pre-calculus course, and also to give an introduction to calculus presenting integral calculus before differential calculus. The compendium also includes several projects modeling real world problems.

24. Mathematics Predicts, PreCalculus

● In Växjö January 14-15, 2020, the Swedish Society for Research in Mathematics Education welcome to Madif 12, its twelfth research seminar in connection with the Matematikbiennalen 2020. The theme of the seminar is 'Sustainable mathematics education in a digitalized world'. I sent in three papers inspired by (01), one on early childhood education (25), and one on middle school (26), and one on precalculus (27) as well as two proposals for a workshop (28, 29). All were rejected.

25. Sustainable Adaption to Quantity: From Number Sense to Many Sense

The abstract says: Their biological capacity to adapt to their environment make children develop a number-language based upon two-dimensional block- and bundle-numbers, later to be colonized by one-dimensional place-value numbers with operations derived from a self-referring setcentric grammar, forced upon them by institutional education. The result is widespread innumeracy making OECD write the report 'Improving Schools in Sweden'. To create a sustainable quantitative competence, the setcentric one-dimensional number-language must be replaced by allowing children develop their own native two-dimensional language. And math education must accept that its goal is not to mediate the truth regime of setcentric university math, but to develop the child's already existing adaption to Many.

The chapters are called: Decreased PISA Performance Despite Increased Research Mathematics and its Education, Biology Looks at Education, Philosophy Looks at Education, Psychology Looks at Education, Sociology Looks at Education, Meeting Many, Children Bundle to Count and Share, A Contemporary Mathematics Curriculum, The Difference to a Typical Contemporary Mathematics Curriculum, Mathematics as a Number-Language, Discussing Number Sense and Number Nonsense, Conclusion and Recommendation.

26. Per-numbers connect Fractions and Proportionality and Calculus and Equations Sustainable Adaption to Quantity: From Number Sense to Many Sense

The abstract says: In middle school, fractions and proportionality are core subjects creating troubles to many students, thus raising the question: can fractions and proportionality be seen and taught differently? Searching for differences making a difference, difference-research suggests widening the word ‘percent’ to also talk about other ‘per-numbers’ as e.g. ‘per-five’ thus using the bundle-size five as a unit. Combined with a formula for recounting units, per-numbers will connect fractions, quotients, ratios, rates and proportionality as well as and calculus when adding per-numbers by their areas, and equations when recounting in e.g. fives.

The chapters are called: Mathematics is Hard, or is it, The ICMT3 Conference, Different Ways of Seeing Fractions, Ratios and Rates, Per-numbers Occur when Double-counting a Total in two Units, Fractions as Per-numbers, Expanding and Shortening Fractions, Taking Fractions of Fractions, the Per-number Way, Direct and Inverse Proportionality, Adding Fractions, the Per-number Way, Solving Proportionality Equations by Recounting , Seven Ways to Solve the two Proportionality Questions, A Case: Peter, about to Peter Out of Teaching, Discussion and Recommendation

27. Sustainable Adaption to Double-Quantity: From Pre-Calculus to Per-Number Calculations

The abstract says: Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

The chapters are called: A need for curricula for all students, A Traditional Precalculus Curriculum, A Different Precalculus Curriculum, Precalculus, building on or rebuilding?, Using Sociological Imagination to Create a Paradigm Shift, A Grounded Outside-Inside Fresh-start Precalculus from Scratch, Solving Equations by Moving to Opposite Side with Opposite Sign, Recounting Grounds Proportionality, Double-counting Grounds Per-numbers and Fractions, The Change Formulas, Precalculus Deals with Uniting Constant Per-Numbers as Factors, Calculus Deals with Uniting Changing Per-Numbers as Areas, Statistics Deals with Unpredictable Change, Modeling in Precalculus Exemplifies Quantitative Literature, A Literature Based Compendium, An Example of a Fresh/start Precalculus Curriculum, An Example of an Exam Question, Discussion and Conclusion.

28. A Lyotardian dissension to the early childhood consensus on numbers and operations. The chapter are called: Can sociological imagination improve mathematics education? Consensus and Dissension on Early Childhood Numbers & Operations. Time Table for the Workshop.

29. Salon des Refusés, a Way to Assure Quality in the Peer Review Caused Replication Crisis? The chapter are called: Does Mathematics Education Research have an Irrelevance Paradox? The Replication Crisis in Science. Time Table for the Workshop.

30. Bundle Counting Table. A guide to bundle-counting in pre-school. Written for the stand at the Matematikbiennale.

31. Proposals for the 2020 Swedish Math Biennale. All were rejected.

● At the Ho Chi Minh City University of Education on December 7, a conference was held called ‘Psychology and Mathematics education’. I was invited to give the plenary talk Saturday, which I named after the paper I send in (32), together with a Power Point Presentation (33). Sunday, I gave a talk on modeling to a group of master students. Monday, I gave a talk to a class of senior students on a poster presentation from the ‘Educating the Educators’ conference in Freiburg, Germany, in October, and handed out the notes ‘What is Math - and Why Learn it?’ and ‘Bundle Counting

Table’. Tuesday, I gave a talk to the staff on research in mathematics education and networks to join and design research as a methodology to use when researching the implementation of the new activity-based curriculum inspired by Kolb’s experimental learning theory.

32. De-Modeling Numbers, Operations and Equations: From Inside-Inside to Outside-Inside Understanding

The abstract says: Adapting to the outside fact Many, children internalize social number-names, but how do they externalize them when communicating about outside numerosity? Mastering Many, children use bundle-numbers with units; and flexibly use fractions and decimals and negative numbers to account for the unbundled singles. This suggests designing a curriculum that by replacing abstract-based with concrete-based psychology mediates understanding through de-modeling core mathematics, thus allowing children to expand the number-language they bring to school.

The chapters are: 1. Introduction, 2. Materials/ Subjects and Methods, 2.1. Reflections on Different forms of Mathematics, 2.2. Reflections on Different forms of Psychology, 2.3. Merging Mathematics and Psychology, 2.4. De-modelling Digits, 2.4.1. Designing and Implementing a micro-curriculum, 2.5. Reflections on how to De-model Bundle-counting Sequences, 2.5.1. Designing and Implementing a micro-curriculum, 2.6. Reflections on how to De-model Operations, 2.7. Reflections on how to Recount into Tens, 2.7.1. Designing and Implementing a micro-curriculum, 2.8. Reflections on how to Model Double-counting with Per-numbers and Fractions, 2.9. Reflections on how to De-model Trigonometry, 3. Results and Discussion, 4. Conclusion.

33. De-Model Numbers, Operations and Equations, PPP.

34. Visit to Ho Chi Minh City University of Education December 7-13 2019.

● The ICMT3 and Educating Educators conferences used peer-reviews, and in the first you were allowed to comment on the reviews

35. Review 01 ICMT3

36. Review 02 ICMT3

37. Comments to ICMT3 Reviewers

38. Educating Educators Reviews

Aarhus, December 2019, Allan Tarp

01. Conceptual Change and Compulsion in Primary Mathematics Education

Statement of the Theoretical Problem

The OECD 2015-report 'Improving Schools in Sweden' describes how "one out of four students" do not "demonstrate competencies to actively participate in life (p. 3)". So, we ask: Is it so by nature, or could it be different? Difference research (Tarp, 2018) may give an answer.

An Account of the Theoretical Proposal Being Made

Adapting to outside quantity, children develop two-dimensional double-numbers as 2 3s. Using sticks, children see digits as icons with as many sticks as they represent if written less sloppy. And, recounting 8 in 2s, children see bundle-counting as pushing away bundles, iconised as a broom called division; stacked as 4x2, iconized as a lift called multiplication; to be pulled away, iconized as a rope called subtraction in order to add the unbundled singles next-to or on-top, iconized as a cross showing the two directions.

Writing 8 = (8/2)x2, unspecified numbers gives a recount-formula T = (T/B)xB, saying "from a total T, T/B times, bundles of Bs may be taken away and stacked." By changing units, this formula expresses proportionality, thus becoming the typical STEM-formula.

Added next-to or on-top, unbundled become decimals, fractions or negatives thus introducing 'flexible bundle-numbers': T = 7 = 2B1 3s = 2.1 3s = 2 1/3 3s = 3.-2 3s = 1.4 3s.

Rephrasing "recount 8 in 2s" to "how many 2s are there in 8?" creates the equation ux2 = 8 solved be recounting 8 in 2s, ux2 = 8 = (8/2)x2, where the solution u = 8/2 comes from moving numbers to opposite side with opposite sign.

Recounting between icons and tens leads to tables and equations when asking T = 6 7s = ? tens, and T = 78 = 7.8 tens = ? 9s.

Double-counting in different physical units creates per-numbers as 3$/4kg that bridge the units when recounting in the per-number; and become fractions when recounted in the same unit; and add by areas as integral calculus since multiplying before adding transform per-numbers into areas.

Recounting the sides in a block halved by its diagonal creates trigonometry.

Review of the Relevant Literature

Mathematics education sees its goal as mastering university mathematics, seen as a pure self-supporting science theorizing one-dimensional number sets organised by different operations: addition, multiplication and power; all with inverse operations: subtraction, division, root and logarithm; and together creating additional number sets when introducing letters for unspecified numbers.

As an alternative Kuhnian paradigm, ManyMath sees mathematics as a natural science about the outside fact Many as shown by geometry and algebra meaning earth-measuring and reuniting in Greek and Arabic; and developing children's already existing mastery of Many with two-dimensional double-numbers, which makes the existing literature on one-dimensional single-numbers little relevant. Instead theoretical guidance comes from seeing mathematics education as an institutionalized goal-directed treatment of human brains, thus being theorized by sociology, philosophy, and psychology. Here however, internal controversies necessitate choices to be made. In sociology, this project chooses agency over structure by using Bauman, Habermas and Foucault; in philosophy it chooses empiricism over rationalism by using existentialism; and in psychology it chooses nature over culture by choosing Piaget over Vygotsky.

Clarifying the Novel Contribution of this Particular Project

ManyMath education has the goal to outside master Many, where traditional mathematics education has the goal to inside master mathematics so other subjects later may apply it outside.

Accepting and developing the double-numbers children create when adapting to Many, ManyMath teach, not numbers, but numbering, using functions from grade one as number-language sentences that, as in the word-language, contains a subject, a verb, and a predicate; and where numbers always carry units.

By counting and recounting before adding, double-numbers with units let children meet the core of mathematics in grade one including equations and calculus, allowing the following years to study footnotes and expand what is already known.

Finally, the concept per-number is a new concept that unite e.g. proportionality, fractions and calculus. The recount-formula shows that per-numbers and fractions are not numbers, but operators needing numbers to become numbers, so adding without units creates 'mathematism', true inside but seldom outside classrooms where 2weeks+3days is 17days.

Empirical Research that Could Test the Validity of the Theoretical Proposal

Being very costly to change expensive textbooks and long-term teacher education makes testing the validity of ManyMath difficult in a traditional education, except for where it is stuck, e.g. division and fractions. But ManyMath may be tested outside the main track: in preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

Implication of this Work for Further Theory Development

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows Mastering Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental learning curricula as e.g. in Vietnam.

Implications for Practice

Brains help animals adapt to the outside world by making proper behavioural choices to satisfy their basic needs as expressed by the holes in their heads: food for the stomach, and information for the brain. Additional brain capacity allows humans to share information by developing a word- and a number-language describing outside qualities and quantities, thus needing only few children in a lifetime to reproduce if systematizing adaption through institutionalized education teaching children about their outside world, and teenagers about their inside talents and potentials, using and developing the students' native languages, and refraining from colonizing students' brains by forcing foreign languages upon them. Thus, to develop the students' mastery of Many, outside abstracted ManyMath must replace today's inside derived university mathematics.

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Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

02. Calculus Conceptually Changed: From Deified to Reified

Statement of the Theoretical Problem

Created to add per-numbers by their areas, integral calculus normally is the last subject in high school, and only taught to a minority of students. But, since most STEM-formulas express proportionality by means of per-numbers, the question is if integral calculus may be taught earlier and to all students. Difference research (Tarp, 2018) may give an answer.

An Account of the Theoretical Proposal Being Made

Reified, integral calculus occurs in grade one when performing next-to addition of bundle-numbers as e.g. T = 2 3s + 4 5s = ? 8s, leading on to differential calculus as the reverse question: 2 3s + ? 5s = 3 8s, solved by first removing 2 3s from 3 8s and then counting the rest in 5s, thus letting subtraction precede division, where integral calculus does the opposite by letting multiplication creating areas precede addition.

In middle school adding per-numbers by areas occurs in mixture problems: 3kg at 2$/kg + 5kg at 4$/kg = 8kg at ? $/kg, again with differential calculus coming from the reverse question: 3kg at 2$/kg + 5kg at ? $/kg = 8kg at 4$/kg. Here the per-number graph is piecewise constant c, i.e. there exists a delta-interval so that for all positive epsilons, the distance between y and c is less than epsilon. With like units, per-numbers become fractions thus also added by their areas, and never without units.

In high school adding per-numbers occurs when the distance travelled with a varying per-number P is found as the area under the per-number graph now being locally constant, formalized by interchanging epsilon and delta. Here the area A under the per-number graph P is found by slicing the area thinly. If writing the area strips as differences, addition will make all middle terms disappear and leave only the first and last terms. Alternatively, the last strip represents the change of the area, dA, and may be written as dA = P*dx, thus motivating developing differential calculus to find a formula A with the property that A’ = dA/dx = P. Here, looking at the shadow of an f*g book directly gives the formula (f*g)’/(f*g) = f’/f + g’/g, exemplified e.g. by (x^2)’ = 2x.

Review of the Relevant Literature

Mathematics education sees its goal as mastering university mathematics, seen as a pure self-supporting science theorizing one-dimensional number sets organised by different operations. And later introducing differentiation and integration as operations on functions seen as set-relations or subsets of set-products where first-component identity implies second-component identity.

As an alternative Kuhnian paradigm, ManyMath sees mathematics as a natural science about the outside fact Many as shown by geometry and algebra meaning earth-measuring and reuniting in Greek and Arabic; and developing children’s already existing mastery of Many with two-dimensional double-numbers that when added next-to by their areas presents integral calculus before differential, which makes irrelevant the existing calculus literature doing the opposite and neglecting double-numbers.

Instead theoretical guidance comes from seeing mathematics education as an institutionalized goal-directed treatment of human brains, thus being theorized by sociology, philosophy, and psychology. Here however, internal controversies necessitate choices to be made. In sociology, this project chooses agency over structure by using Bauman, Habermas and Foucault; in philosophy it chooses empiricism over rationalism by using existentialism; and in psychology it chooses nature over culture by choosing Piaget over Vygotsky.

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ManyMath education has the goal to outside master Many, where traditional mathematics education has the goal to inside master mathematics so other subjects later may apply it outside.

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In grade one, on-top addition of double-numbers leads to proportionality making the units the same, and next-to addition leads to integral calculus by adding areas, and to differential calculus when reversed. In middle school both reappear when adding or subtracting piecewise constant per-numbers by their areas in mixture problems, to be followed in high school by locally constant per-numbers instead, again with integration preceding differentiation. The tradition only teaches on-top addition of single numbers, and teaches fractions as mathematism being added without units. Mixture problems are not illustrated geometrically to show the relationship with calculus, and the concepts of piecewise and local constancy are absent. Finally, differential calculus is taught before integral calculus, and local constancy and linearity is called continuous and differentiable.

Empirical Research that Could Test the Validity of the Theoretical Proposal

Being very costly to change expensive textbooks and long-term teacher education makes testing the validity of Reifying Calculus difficult inside a traditional education, except for where it is stuck, e.g. adding fractions without units. But it may be tested outside: in preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

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Implications for Practice

Institutionalized education systematizes adaption by teaching children about their outside world, and teenagers about their inside talents and potentials. Thus, to develop the students' mastery of Many, outside abstracted ManyMath must replace today's inside derived university mathematics by presenting calculus at all three school levels.

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Extended summary, synopsis

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03. The Power of Bundle- & Per-Numbers Unleashed in Primary School: Calculus in Grade One – What Else?

In middle school, fraction, percentage, ratio, rate, and proportion create problems to many students. So, why not teach it in primary school instead where they all may be examples of per-numbers coming from double-counting a total in two units. And bundle-numbers with units is what children develop when adapting to Many before school. Here children love counting, recounting, and double-counting before adding totals on-top or next-to as in calculus, also occurring when adding per-numbers. Why not accept, and learn from the mastery of Many that children possess until mathematics takes it away?

MATHEMATICS IS HARD, OR IS IT?

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature. That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969. Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries, and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Also among the countries with poor PISA performance, Denmark has lowered the passing limit at the final exam to around 15% and 20 % in lower and upper secondary school. And, at conferences as e.g. The Third International Conference on Mathematics Textbook Research and Development, ICMT3 2019, high-ranking countries admit they have a high percentage of low scoring students. Likewise at conferences, discussing in the breaks what is the goal of mathematics education, the answer is almost always 'to learn mathematics'. When asked to define mathematics, some point to schoolbooks, others to universities; but all agree that learning it is important to master its outside applications.

So, we may ask, is the goal of mathematics education to master outside Many, or to first master inside mathematics as a means to later master outside Many. Here, institutionalizing mathematics as THE only inside means leading to the final outside goal may risk creating a goal displacement transforming the means to the goal instead (Bauman, 1990) leading on to the banality of evil (Arendt, 1963) by just following the orders of the tradition with little concern about its effect as to reaching the outside goal. To avoid this, this paper will answer the question about the hardness by working backwards, not from mathematics to Many, but from Many to mathematics. So here the focus is not to study why students have difficulties mastering inside mathematics, but to observe and investigate the mastery of outside Many that children bring to school before being forced to learn about inside mathematics instead.

RESEARCH METHOD

Difference research searching for differences has uncovered hidden differences (Tarp, 2018c). To see if the differences make a difference, phenomenology (Tarp, 2018a), experiential learning (Kolb, 1984), and design research (Bakker, 2018) may create cycles of observations, reflections, and designs of micro curricula to be tested in order to create a new cycle for testing the next generation of curricula.

OBSERVATIONS AND REFLECTIONS 01

Asked "How old next time?" a three-year-old will say four showing four fingers, but will react to seeing the fingers held together two by two: "That is not four. That is two twos!" The child thus describes what exists, bundles of 2s, and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)x2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving a formula with unspecified numbers '$T = (T/B)xB$' saying "from T, T/B times, B may be pushed away and stacked". Consequently, search questions about 'bundle-numbers' and 'recounting' may be given to small groups of four preschool children to get ideas about how to design a generation-1 curriculum.

GUIDING QUESTIONS

The following guiding questions were used: "There seems to be five strokes in the symbol five. How about the other symbols?", "How many bundles of 2s are there in ten?", "How to count if including the bundle?", "How to count if using a cup for the bundles?", "Can bundles also be bundled, e.g. if counting ten in 3s?", "What happens if we bundle to little or too much?", "How to recount icon-numbers in tens?", "How to manually recount 8 in 2s, and recount 7 in 2s?", "What to do if a bundle is not full?", "How to bundle-count seconds, minutes, hours, and days?", "How to double-count lengths in centimeters and inches?", "A dice decided my share in a lottery ticket, how to share a gain?", "Which numbers can be folded in other numbers than 1s?", "Asking how many 2s in 8 may be written as $u*2 = 8$, how can this equation be solved?", "How to recount from tens to icons?", "How to add 2 3s and 4 5s next-to?", "How to add 2 3s and 4 5s on-top?", " 2 3s and some 5s gave 3 8s, how many?", "How to add totals bundle-counted in tens?", "How to subtract totals bundle-counted in tens?", "How to add per-numbers?", "How to enlarge or diminish bundle-bundle squares?", "What happens when recounted stacks are placed on a squared paper?", "What happens when turning or stacking stacks?"

OBSERVATIONS AND REFLECTIONS 02

Data and ideas allowed designing Micro Curricula (MC) with guiding questions and answers (Q, A).

MC 01: Digits as Icons

With strokes, sticks, dolls, and cars we observe that four 1s can be bundled into 1 fours that can be rearranged into a 4-icon if written less sloppy. So, for each 4 1s there is 1 4s, or there is 1 4s per 4 1s. In this way, all digits may be iconized, and used as units for bundle-counting (Tarp, 2018b).

MC 02: Bundle-counting Ten Fingers

A total of ten ones occurring as ten fingers, sticks or cubes may be counted in ones, in bundles, or with 'underloads' counting what must be borrowed to have a full bundle. Count ten in 5s, 4s, 3s, and 2s.

In 5s with bundles: 0*B*1, …, 0*B*4, 0*B*5 no 1*B*0, 1*B*1, …, 1*B*4, 1*B*5 no 2*B*0.

In 5s with bundles and underloads: 1*B*-4, 1*B*-3, …, 1*B*0, 2*B*-4, … , 2*B*0.

MC 03: Counting Sequences Using Tens and Hundreds

In oral counting-sequences the bundle is present as tens, hundreds, thousands, ten thousand (wan in Chinese) etc. By instead using bundles, bundles of bundles etc. it is possible to let power appear as the number of times, bundles have been bundled thus preparing the ground for later writing out a multi-digit number fully as a polynomial, $T = 345 = 3BB4B5 = 3*B\text{^}2 + 4*B + 5*1$.

Count 10, 20, 30, …, 90, 100 etc. Then 1*B*, 2*B*, …, 9*B*, ten*B* no 1*BB*.

Count 100, 200, 300, …, 900, ten-hundred no thousand. Then 1*BB*, 2*BB*, …, 9*BB*, ten*BB* no 1*BBB*.

Count 100, 110, 120, 130, …, 190, 200 etc. Then 1*BB*0*B*, 1*BB*1*B*, …, 1*BB*9*B*, 1*BB*ten*B* no 2*BB*0*B*.

A dice shows 3 then 4. Name it in five ways: thirty-four, three-ten-four, three-bundle-four, four-bundle-less6, and forty less 6. Travel on a chess board while saying 1*B*1, 2*B*1, 3*B*1, 3*B*2, …; 3*B*4.

MC 04: Cup-counting and Bundle-bundles

When counting a total, a bundle may be changed to a single thing representing the bundle to go to a cup for bundles, later adding an extra cup for bundles of bundles. Writing down the result, bundles and unbundled may be separated by a bundle-letter, a bracket indicating the cups, or a decimal point.

Q. T = two hands, how many 3s?

A. With 1 3s per 3 1s we count 3 bundles and 1 unbundled, and write T = 3B1 3s = 3]1 3s = 3.1 3s showing 3 bundles inside the cup, and 1 unbundled outside. However, 3 bundles are 1 bundle-of-bundles, 1BB, so with bundle-bundles we write T = 1BB0B1 3s = 1]0]1 3s = 10.1 3s with an additional cup for the bundle-bundles.

Q. T = two hands, how many 2s?

A. With 1 2s per 2 1s we count 5 bundles, T = 5B0 2s = 5]0 2s = 5.0 2s. But, 2 bundles is 1 bundle-of-bundles, 1BB, so with bundle-bundles we write T = 2BB1B0 2s = 2]1]0 2s = 21.0 2s. However, 2 bundles-of-bundles is 1 bundle-of-bundles-of-bundles, 1BBB, so with bundle-bundle-bundles we write T = 1BBB0BB1B0 2s= 1]0]1]0 2s = 101.0 2s with an extra cup for the bundle-bundle-bundles.

MC 05: Recounting in the Same Unit Creates Underloads and Overloads

Recounting 8 1s in 2s gives T = 4B0 2s. We may create an underload by borrowing 2 to get 5 2s. Then T = 5B-2 2s = 5]-2 2s = 5.-2 2s. Or, we may create an overload by leaving some bundles unbundled. Then T = 3B2 2s = 2B4 2s = 1B6 2s. Later, such 'flexible bundle-numbers' will ease calculations.

MC 06: Recounting in Tens

With ten fingers, we typically use ten as the counting unit thus becoming 1B0 needing no icon.

Q. T = 3 4s, how many tens? Use sticks first, then cubes.

A. With 1 tens per ten 1s we count 1 bundle and 2, and write T = 3 4s = 1B2 tens = 1]2 tens = 1.2 tens, or T = 2B-8 tens = 2.-8 tens using flexible bundle-numbers. Using cubes or a pegboard we see that increasing the base from 4s to tens means decreasing the height of the stack. On a calculator we see that 3x4 = 12 = 1.2 tens, using a cross called multiplication as an icon for a lift stacking bundles. Only the calculator leaves out the unit and the decimal point. Often a star * replaces the cross x.

Q. T = 6 7s, how many tens?

A. With 1 tens per ten 1s we count 4 bundles and 2, and write T = 6 7s = 4B2 tens = 4]2 tens = 4.2 tens. Using flexible bundle-numbers we write T = 6 7s = 5B-8 tens = 5]-8 tens = 5.-8 tens = 3B12 tens. Using cubes or a pegboard we see that increasing the base from 7s to tens means decreasing the height of the stack. On a calculator we see that 6*7 = 42 = 4.2 tens.

Q. T = 6 7s, how many tens if using flexible bundle-numbers on a pegboard?

A. T = 6 7s = 6*7 = (B-4) * (B-3) = BB-3B-4B+4*3 = 10B-3B-4B+1B2 = 4B2 since the 4 3s must be added after being subtracted twice.

MC 07: Recounting Iconizes Operations and Creates a Recount-formula for Prediction

A cross called multiplication is an icon for a lift stacking bundles. Likewise, an uphill stroke called division is an icon for a broom pushing away bundles. Recounting 8 1s in 2s by pushing away 2-bundles may then be written as a 'recount-formula' 8 = (8/2)*2 = 8/2 2s, or T = (T/B)*B = T/B Bs, saying "From T, T/B times, we push away B to be stacked". Division followed by multiplication is called changing units or proportionality. Likewise, we may use a horizontal line called subtraction as an icon for a rope pulling away the stack to look for unbundled singles.

These operations allow a calculator predict recounting 7 1s in 2s. First entering '7/2' gives the answer '3.some' predicting that pushing away 2s from 7 can be done 3 times leaving some unbundled singles that are found by pulling away the stack of 3 2s from 7. Here, entering '7-3*2' gives the result '1', thus predicting that 7 recounts in 2s as 7 = 3B1 2s = 3]1 2s = 3.1 2s.

Recounting 8 1s in 3s gives a stack of 2 3s and 2 unbundled. The singles may be placed next-to the stack as a stack of unbundled 1s, written as T = 8 = 2.2 3s. Or they may be placed on-top of the stack counted in bundles as 2 = (2/3)*3, written as T = 8 = 2 2/3 3s thus introducing fractions. Or, as T = 8 = 3.-1 3s if counting what must be borrowed to have another bundle.

Q. T = 9, 8, 7; use the recount-formula to predict how many 2s, 3s, 4s, 5s before testing with cubes.

MC 08: Recounting in Time

Counting in time, a bundle of 7days is called a week, so 60days may be recounted as T = 60days = (60/7)*7days = 8B4 7days = 8weeks 4days. A bundle of 60 seconds is called a minute, and a bundle of 60 minutes is called an hour, so 1 hour is 1 bundle-of-bundles of seconds. A bundle of 12hours is called a half-day, and a bundle of 12months is called a year.

MC 09: Double-counting in Space Creates Per-Numbers or Rates

Counting in space has seen many units. Today centimeter and inches are common. 'Double-counting' a length in inches and centimeters approximately gives a 'per-number' or rate 2in/5cm shown with cubes forming an L. Out walking we may go 3 meters each 5 seconds, giving the per-number 3m/5sec. The two units may be bridged by recounting in the per-number, or by physically combining Ls.

Q. T = 12in = ?cm; and T = 20cm = ?in

A1. T = 12in = (12/2)*2in = (12/2)*5cm = 30cm; and A2. T = 20cm = (20/5)*5cm = (20/5)*2in = 8in

MC 10: Per-numbers Become Fractions

Double-counting in the same unit makes a per-number a fraction. Recounting 8 in 3s leaves 2 that on-top of the stack become part of a whole, and a fraction when counted in 3s: T = 2 = (2/3)*3 = 2/3 3s.

Q. Having 2 per 3 means having what per 12?

A. We recount 12 in 3s to find the number of 2s: T = 12 = (12/3)*3 giving (12/3) 2s = (12/3)*2 = 8. So, having 2/3 means having 8/12. Here we enlarge both numbers in the fraction by 12/3 = 4.

Q. Having 2 per 3 means having 12 per what?

A. We recount 12 in 2s to find the number of 3s: T = 12 = (12/2)*2 giving (12/2) 3s = (12/2)*3 = 18. So, having 2/3 means having 12/18. Here we enlarge both numbers in the fraction by 12/2 = 6.

MC 11: Per-numbers Become Ratios

Recounting a dozen in 5s gives 2 full bundles, and one bundle with 2 present, and 3 absent: T = 12 = 2B2 5s = 3B-3 5s. We say that the ratio between the present and the absent is 2:3 meaning that with 5 places there will be 2 present and 3 absent, so the present and the absent constitute 2/5 and 3/5 of a bundle. Likewise, if recounting 11 in 5s, the ratio between the present and the absent will be 1:4, since the present constitutes 1/5 of a bundle, and the absents constitute 4/5 of a bundle. So, splitting a total between two persons A and B in the ration 2:4 means that A gets 2, and B gets 4 per 6 parts, so that A gets the fraction 2/6, and B gets the fraction 4/6 of the total.

MC 12: Prime Units and Foldable Units

Bundle-counting in 2s has 4 as a bundle-bundle. 1s cannot be a unit since 1 bundle-bundle stays as 1. 2 and 3 are prime units that can be folded in 1s only. 4 is a foldable unit hiding a prime unit since 1 4s = 2 2s. Equal number can be folded in 2s, odd numbers cannot. Nine is an odd number that is foldable in 3s, 9 1s = 3 3s. Find prime units and foldable units up to two dozen.

MC 13: Recounting Changes Units and Solves Equations

Rephrasing the question "Recount 8 1s in 2s" to "How many 2s are there in 8?" creates the equation 'u*2 = 8' that evidently is solved by recounting 8 in 2s since the job is the same:

If u*2 = 8, then u*2 = 8 = (8/2)*2, so u = 8/2 = 4.

The solution $u = 8/2$ to $u*2 = 8$ thus comes from moving a number to the opposite side with the opposite calculation sign. The solution is verified by inserting it in the equation: $u*2 = 4*2 = 8$, OK.

Recounting from tens to icons gives equations: “42 is how many 7s” becomes $u*7 = 42 = (42/7)*7$.

MC 14: Next-to Addition of Bundle-Numbers Involves Integration

Once recounted into stacks, totals may be united next-to or on-top, iconized by a cross called addition.

To add bundle-numbers as 2 3s and 4 5s next-to means adding the areas 2*3 and 4*5, called integral calculus where multiplication is followed by addition.

Q. Next-to addition of 2 3s and 4 5s gives how many 8s?

A1. T = 2 3s + 4 5s = (2*3+4*5)/8 8s = 3.2 8s; or A2. T = 2 3s + 4 5s = 26 = (26/8) 8s = 3.2 8s

MC 15: On-top Addition of Bundle-Numbers Involves Proportionality

To add bundle-numbers as 2 3s and 4 5s on-top, the units must be made the same by recounting.

Q. On-top addition of 2 3s and 4 5s gives how many 3s and how many 5s?

A1. T = 2 3s = (2*3/5)*5 = 1.1 5s, so 2 3s and 4 5s gives 5.1 5s

A2. T = 2 3s + 4 5s = (2*3+4*5)/5 5s = 5.1 5s; or T = 2 3s + 4 5s = 26 = (26/5) 5s = 5.1 5s

MC 16: Reversed Addition of Bundle-Numbers Involves Differentiation

Reversed addition may be performed by a reverse operation, or by solving an equation.

Q. Next-to addition of 2 3s and how many 5s gives 3 8s?

A1: Removing the 2*3 stack from the 3*8 stack, and recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called differentiation.

A2: The equation 2 3s + u*5 = 3 8s is solved by moving to opposite side with opposite calculation sign

u*5 = 3 8s – 2 3s = 3*8 – 2*3, so u = (3*8 – 2*3)/5 = 18/5 = 3 3/5, giving 3.3 5s.

MC 17: Adding and Subtracting Tens

Bundle-counting typically counts in tens, but leaves out the unit and the decimal point separating bundles and unbundled: T = 4B6 tens = 4.6 tens = 46. Except for e-notation with a decimal point after the first digit followed by an e with the number of times, bundles have been bundled: T = 468 = 4.68e2.

Calculations often leads to overloads or underloads that disappear when re-bundling:

Addition: 456 + 269 = 4BB5B6 + 2BB6B9 = 6BB11B15 = 7BB12B5 = 7BB2B5 = 725.

Subtraction: 456 – 269 = 4BB5B6 – 2BB6B9 = 2BB-1B-3 = 2BB-2B7 = 1BB8B7 = 187

Multiplication: 2 * 456 = 2 * 4BB5B6 = 8BB10B12 = 8BB11B2 = 9B1B2 = 912

Division: 154 /2 = 15B4 /2 = 14B12 /2 = 7B6 = 76

MC 18: Next-to Addition & Subtraction of Per-Numbers and Fractions is Calculus

Throwing a dice 8 times, the outcome 1 and 6 places 4 cubes on a chess board, and the rest 2 cubes. When ordered it may be 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s called the average: If alike, the per-numbers would be 2.6 cubes per square. Thus per-numbers add by areas, i.e. by integration. Reversing the question to 2 5s + ? 3s total 3 8s then leads to differentiation: 2 5s + ? 3s = 3 8s gives the equation

$2*5 + u*3 = 3*8$, so $u*3 = 3*8 – 2*5$, so $u = (3*8 – 2*5)/3 = 4\ 2/3$, or $u = (T2\text{-}T1)/3 = \Delta T/3$

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC 19: Having Fun with Bundle-Bundle Squares

On a pegboard we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas:

$n*n + 2*n + 1 = (n+1)*(n+1)$; and $n*n – 2*n + 1 = (n-1)*(n-1)$; and $(n–1)*(n+1) = n*n – 1$.

Two $s*s$ bundle-bundles form two squares that halved by their diagonal d gives four half-squares called right triangles. Rearranged, they form a diagonal-square $d*d$. Consequently, $d*d = 2*s*s$

Four $c*b$ playing cards with diagonal d are placed after each other to form a $(b+c)*(b+c)$ bundle-bundle square. Below to the left is a $c*c$ square, and to the right a $b*b$ square. On-top are 2 playing cards. Inside there is a $d*d$ square and 4 half-cards. Since 4 half-cards is the same as 2 cards, we have the formula $c*c + b*b = d*d$ making it easy the add squares, you just square the diagonal.

MC 20: Having Fun with Halving Stacks by its Diagonal to Create Trigonometry

Halving a stack by its diagonal creates two right triangles. Traveling around the triangle we turn three times before ending up in the same direction. Turning 360 degrees implies that the inside angles total 180 degree, and that a right angle is 90 degrees. Measuring a 5up_per_10out angle to 27 degrees we see that tan(27) is 0.5 approximately. So, the tan-number comes from recounting the height in the base.

MC 21: Having Fun with a Squared Paper

A dozen may be 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed on a squared paper with the lower left corners coinciding, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. Traveling by saying “3up_per_1out, 2up_per_1out, …, 3down_per_1out” allows the end points to follow a parabola. With a per-number 2G/3R, a dozen R can be changed to 2G+9R, 4G+6R, 6G+3R, and 8G. Plotted on a square paper with R horizontally and G vertically will give a line sloping down with the per-number.

MC 22: Having Fun with Turning and Combining Stacks

Turned over, a 3*5 stack becomes a 5*3 stack with the same total, so multiplication-numbers may commute (the commutative law). Adding 2 7s on-top of 4 7s totals (2+4) 7s, 2*7+4*7 = (2+4)*7 (the distributive law). Stacking stacks gives boxes. Thus 2 3s may be stacked 4 times to the box T = 4*(2*3) that turned over becomes a 3*(2*4) box. So, 2 may freely associate with 3 or 4 (the associative law).

DISCUSSION AND RECOMMENDATION

This paper asks: what mastery of Many does the child develop before school? The question comes from observing that mathematics education still seems to be hard after 50 years of research; and from wondering if it is hard by nature or by choice, and if it is needed to achieve its goal, mastery of Many.

To find an answer, phenomenology, experiential, and design research is used to create a cycle of observations, reflections, and testing of micro curricula designed from observing the reflections of preschoolers to guiding questions on mastering Many. The first observation is that children use two-dimensional bundle-numbers with units instead of the one-dimensional single numbers without units that is taught in school together with a place value system. Reflecting on this we see that units make counting, recounting, and double-counting core activities leading to proportionality by combining division and multiplication, thus reversing the order of operations: first division pulls away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to identify unbundled singles that becomes decimal, fractional or negative numbers. And that recounting between icons and tens leads to equations when asking e.g. 'how many 5s are 3 tens?' And that units make addition ambiguous: shall totals add on-top after proportionality has made the units like, or shall they add next-to as an example of integral calculus adding areas, and leading to differentiation when reversed? Finally, we see that flexible bundle-numbers ease traditional calculations on ten-based numbers.

Testing the micro curricula will now show if mathematics is hard by nature or by choice. Of course, investments in traditional textbooks and teacher education, all teaching single numbers without units, will deport testing to the outskirts of education, to pre-school or post-school; or to special, adult, migrant, or refugee education; or to classes stuck in e.g. division, fractions, precalculus, etc. All that is needed is asking students to count fingers in bundles. Recounting 8 in 2s thus directly gives the proportionality recount-formula $8 = (8/2)*2$ or $T = (T/B)*B$ used in STEM, and to solve equations. Likewise, direct and reversed next-to addition leads directly to calculus. Furthermore, testing micro curricula will allow teachers to practice action learning and action research in their own classroom.

Phenomenologically, it is important to respect and develop the way Many presents itself to children thus providing them with the quantitative competence of a number-language. Teaching numbering instead of numbers thus creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education (Widdowson, 1978). The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not lean from children instead?

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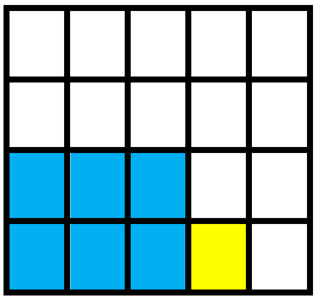
LEARN
CORE MATH
THROUGH YOUR KID'S
TILE-MATH



T = 2***B***1 ***3S***

RECOUNTING BUNDLE-NUMBERS

EARLY TRIGONOMETRY

CALCULUS

STEM

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March 2020

04. Learn Core Mathematics Through Your Kid's Tile-Math

Asked 'How old next time?', a 3-year-old says 'Four' showing four fingers; but objects when seeing them held together two by two: 'That is not four, that is two twos!' A child thus sees what exists in the world, bundles of 2s, and 2 of them. So, adapting to Many, children develop bundle-numbers with units as 2 2s having 1 1s as the unit, i.e. a tile, also occurring as bundle-of-bundles, e.g. 3 3s, 5 5s or ten tens.

Recounting 8 in 2s as 8 = (8/2)x2 gives a recount-formula $T = (T/B)\text{x}B$ saying 'From the total T, T/B times, B can be pushed away' occurring all over mathematics and science. It solves equations: ux2 = 8 = (8/2)x2, so u = 8/2. And it changes units when adding on-top, or when adding next-to as areas as in calculus, also occurring when adding per-numbers or fractions coming from double-counting in two units. Finally, double-counting sides in a tile halved by its diagonal leads to trigonometry.

The following papers present close to 50 micro-curricula in **Mastering Many** inspired by the bundle-numbers children bring to school.

Learn Core Mathematics Through Your Kid's Tile-Math: Recounting Bundle-Numbers and Early Trigonometry

This first paper is written for the conference 'The Research on Outdoor STEM Education in the digiTal Age (ROSETA) Conference' planned to take place between 16th and 19th June 2020 at Instituto Superior de Engenharia do Porto in Portugal.

The Power of Bundle- & Per-Numbers Unleashed in Primary School: Calculus in Grade One – What Else?

This second paper is written for the International Congress for Mathematical Education, ICME 14, planned to be held in Shanghai from July 12th to 19th, 2020, but postponed one year.

Allan Tarp, Aarhus, Denmark, March 2020

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Learn Core Mathematics Through Your Kid's Tile-Math: Recounting Bundle-Numbers And Early Trigonometry

Fifty years of mathematics education research has failed to create a mathematics for all. This raises the Cinderella question: are there hidden unnoticed alternatives that may make the prince dance? There are. Education may be different, and also math may be different from today's 'meta-matism'. Adapting to Many, children develop bundle-numbers with units as 2 3s having 1 1s as the unit, i.e. a tile also occurring as bundle-of-bundles, e.g. 3 3s. Recounting a total in bundles gives a recount-formula to solve equations, to change units when adding on-top or when adding areas next-to as in calculus, also occurring when adding per-numbers or fractions coming from double-counting in two units. Double-counting sides in a tile halved by its diagonal leads to trigonometry. Asking 'What kind of mathematics may grow from tiles?' this paper uncovers 25 micro-curricula for outdoor STEM education.

POOR PISA PERFORMANCE, A PERMANENT PANDEMIA?

When evaluating the effect of mathematics education, poor PISA performance occurs all over the world, despite 50 years of increasing research and funding. Thus, decreasing Swedish PISA result made OECD (2015) write the report 'Improving Schools in Sweden' describing its school system as "in need of urgent change (..) with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life" (p. 3).

Thus, it seems that it is now demonstrated beyond any doubt that mathematics is indeed difficult to learn so research should continue to understand why this is the case. But then again, can we be sure that what is called mathematics education is not something else.

Education can be different as seen when comparing continental Europe using multi-year lines to prepare teenagers for offices in the public or private sector, with North America using self-chosen half-year blocks to support the identity work of their teenagers.

History shows that also mathematics can be different. The Pythagoreans used it as a label for their knowledge about Many by itself, in space, in time and in space and time, also called arithmetic, geometry, music and astronomy. Where North America still uses specific names, Europe teaches 'metamatism' combining set-derived 'meta-matics', defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples, with 'mathematism' true inside but seldom outside by adding without units (Tarp, 2018).

Finally, math education may suffer from a goal displacement (Bauman, 1990) by changing mathematics from being an inside means to master the outside goal, Many, to being a goal in itself, thus unable to develop the mastery of Many, children bring to school through adaption to Many. Seeing mastery of Many as the end goal, phenomenology has found that Many presents itself to children as bundle-numbers, e.g. 2 3s. (Tarp, 2018). This creates a basis for cycles of experiential learning (Kolb, 1984) and design research to design and test micro-curricula (MC) leading to new cycles. So, respecting children's own two-dimensional bundle-numbers we now ask: What kind of mathematics may grow from tiles?

MC01. Counting by Bundling and Iconizing

Chopsticks placed on tiles show that digits are icons with as many sticks as they represent if written less sloppy. Counting fingers in 3s, we may include the word bundle in the counting sequence by saying '0*B*1, 0*B*2, 0*B*3 no 1*B*0; 1*B*1, 1*B*2, 1*B*3 no 2*B*0; 2*B*1, 2*B*2, 2*B*3 no 3*B*0, 3*B*1'. Counting and stacking tiles and using a folding ruler or a rope to show a bundle, we see that 3 bundles is 1 bundle-of-bundles or a 3x3 square of tiles, so we should instead say '2*B*3 no 3*B*0 no 1*BB*0*B*0'.

To include time in bundle-counting in 3s, we place a cube on each of ten neighboring tiles e.g. on a chess board. A cube moves to the next tile, and both move on to the next tile where they unite to 1 bundle that moves to the tile above, from where it moves to the next tile. This is repeated until tile 9 where 3 bundles unite to 1 bundle-of-bundles that move to the tile above, from where it moves to the last tile, thus showing that ten recounts as 1*BB*0*B*1 3s. Now the same is repeated with bundle-counting in 4s, then in 2s.

MC02. Formulas Predict

Eight persons are bundle-counted in 2s by asking 2s to go to neighboring tiles 4 times. Observing the total 8 splits into 4 2s, we write T = 8 = 4*2. Using an uphill stroke to iconize a broom pushing away bundles, the action 'from 8, push away 2s' may be entered on a calculator as '8/2', thus predicting 4 before carrying out the action. This allows rewriting 8 = 4*2 to 8 = (8/2)*2, or $T = (T/B)*B$ using unspecified numbers, saying 'From T, T/B times, B can be pulled away'. This 'recount-formula' predicts changing units. And, rephrasing recounting to "how many 2s in 8?" allows formulating recounting as an equation $u*2 = 8$ solved by $u = 8/2$, i.e. by moving a number to opposite side with opposite calculation sign.

MC03. Unbundled Become Decimals, Fractions or Negative Numbers

Can we predict the result of rearranging in 4s persons placed on tiles as 2 7s? Entering 2*7/4, a calculator says '3.some'. Using a horizonal stroke to iconize a rope pulling away stacks, entering 2*7-3*4 gives the answer '2' thus predicting that 2 7s recount as 3 4s and 2. The unbundled 2 may be placed next-to the stack reported as a decimal number, T = 3*B*2 4s = 3.2 4s, or on-top counted as bundles, 2 = (2/4)*4 = 2/4 4s, reported as a fraction, T = 3 2/4 *B* 4s; or, if counting what is needed for an extra bundle, reported by a negative number, T = 4 *B*-2 4s = 4.-2 4s. The prediction is then tested with persons or cubes placed on tiles.

MC04. Double-counting Creates Per-numbers

Traveling through a row of tiles is rewarded with 3 cubes per 2 tiles thus creating the 'per-number' 3/2 cubes/tiles. Travelling 12 tiles thus gives 3 cubes 6 times, which can be precited by recounting in the per-number: T = 12 tiles = (12/2)*2 tiles = (12/2)*3 cubes = 18 cubes. Alternatively, we can equate the per-numbers in an equation $u/12 = 3/2$ solved by moving to opposite side with opposite sign, $u = 3/2*12 = 18$. Per-numbers are all over mathematics and science, e.g. meter = (meter/second)*second = speed*second.

Double-counting in the same unit creates fractions: Marking 2 tiles with a dot for each 3 tiles traveled thus creates a per-number 2 tiles/3 tiles = 2/3. Having travelled 12 tiles, we mark 2 dots 4 times. Again, this can be precited by recounting in the per-number: T = 12 tiles = (12/3)*3 tiles marking dots on (12/3)*2 tiles, i.e. 8 tiles with dots.

MC05: Bundle-Numbers Add Next-to or On-top, Directly or Reversed

Once counted as stacks, totals may unite next-to or on-top, iconized by a cross showing the two directions. Adding 2 3s and 4 5s next-to as 8s means adding the areas 2*3 and 4*5, called integral calculus where addition follows multiplication. Adding them on-top, first recounting must change the units to the same. This is called proportionality. Reversed addition asks e.g. '2 3s and how many 5s give 3 8s?'. Here, first the 2*3 stack is pulled away from the 3*8 stack, then recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called reversed integration or differentiation.

MC06: Next-to Addition & Subtraction of Per-numbers and Fractions is Calculus

Throwing a dice 8 times, the outcomes 1 and 6 place 4 cubes on a chess board, and the rest place 2 cubes. When ordered, we may have 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 = 8 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s. This average says that all per-numbers would be 2.6 if alike.

Per-numbers thus add by areas, i.e. by integration. Reversing the question to '2 5s + how many 3s total 3 8s' leads to the equation $2*5 + u*3 = 3*8$ solved by differentiation:

$2*5 + u*3 = 3*8$, so $u*3 = 3*8 – 2*5$, so $u = (3*8 – 2*5)/3 = 4\ 2/3$, or $u = (T2\text{-}T1)/3 = \Delta T/3$

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC07. Double-Counting Sides in a Rectangle Halved by its Diagonal

Two neighboring tiles form a rectangle, that halved by its diagonal creates a right triangle with base *b*, height *h* and diagonal *d*. Recounting pair of sides produces the trigonometry formulas: *h* = (*h*/*d*)**d* = sin*A***d; b* = (*b*/*d*)**d* = cos*A***d*, and *h* = (*h*/*b*)**b* = tan*A***b* that allows a (*b*,*h*) = (+3,+2) angle to be predicted by tan-1(2/3) to give 33.7 degrees. This again allows predicting the diagonal: *h* = sin*A***d*, or 2 = sin33.7**d*, or *d* = 2/sin33.7 = 3.60.

MC08. Meeting Pythagoras

Four tiles with base *b* and diagonal *d* form a squared tile. Here 4 diagonals form a square containing 4 half-tiles, i.e. 2 tiles. Consequently *d***d* = 2**b***b*, or *d*^2 = *b*^2 + *b*^2.

A tile-pair has base *b*, height *h*, and diagonal *d*. Turned 90 degrees a copy is placed on-top. Repeated three times, this creates a square with the side *b*+*h*. Inside we find a diagonal square and four half tile-pairs; as well as a *b***b* square and a *h***h* square and two tile-pairs. But 4 half tile-pairs is 2 full tile-pairs, so we see that *d***d* = *b***b* + *h***h*, or *d*^2 = *b*^2 + *h*^2, making it easy the add squares, you just square the diagonal.

The normal from the right angle divides the diagonal in *p* and *q*. Seeing *b* as *d**cos*A*, and *p* as *b**cos *A*, we get *b***b* = (*d**cos*A*)**b* = *d**(cos*A***b*) = *d***p*. So, the extension of the normal divides the diagonal square in two parts equal to the squares of the neighboring rectangle side. Since only the angle *A* is involved this applies to all triangles with angles not above 90 degrees, thus leading to the extended Pythagoras: *a*^2 = *b*^2 + *c*^2 – 2**b***c**cos*A*, etc.

MC09. The Height of an Accessible Flagpole

The point *P* forms a right triangle with a flagpole of height *h*. From a point *Q* in the distance *s* from *P*, the vertical distance to the diagonal is *k*. The angle *P* may now be found in two ways, tan*P* = *h*/*b* = *k*/*s*, allowing the unknown height *h* to be found by moving to opposite side with opposite sign, *h* = *k*/*s***b*. Solved in a tile-system (coordinate-system), the diagonal is a line passing through three points with the coordinates (0,0), (*s*,*k*) and (*b*,*h*) providing the slope *c* = tan*P* = *k*/*s* and the equation *y* = *k*/*s***x* that with *x* = *b* gives *y* = *k*/*s***b*.

MC10. The Height of an Inaccessible Flagpole

Three points *A*, *B* and *C* with distances *AB* = *r* and *BC* = *s* are placed on a line towards the foot of an inaccessible flagpole with height *h*. With the flagpole, *A* and *B* form two right triangles with diagonals *d1* and *d*2. The vertical distance is *p* from *d1* to *B*, and *q* from *d*2 to *C*. The distance from *C* to the foot of the flagpole pole is *c*. Using tan*A* = *p*/*r* and tan*B* = *q*/*s*, the triangles give two formulas for *h*: *h* = (*r*+*s*+*c*)**p*/*r* = (*s*+*c*)**q*/*s*. Solved for *c*, this gives a formula that inserted in the *h*-formula gives *h* = *p***q***r*/(*p***r* – *p***s*).

MC11. How High the Moon?

A vertical stick with height *h* helps finding the position of the moon or sun. If the shadow has the length *s*, the angle to the sun is predicted by tan*A* = *s*/*h*. A compass helps finding a direction line segment north with the same length as the shadow. The segment between the two has the length *a*. The angle *A* then may be predicted by the formula sin(*A*/2) = ½*a*/*s*.

MC12. The Slope of a Tile

A folding ruler allows creating a right triangle with the bottom line following a sloped tile. A lead line placed in the distance *d* along the ruler from the bottom line will mark on it a distance *b*. The slope of the tile is the same as the top angle *A*, thus predicted by tan*A* = *b*/*d*.

MC13. How Many Turns on a Steep Tile?

On a 30-degree squared tile, a 10-degree road is constructed. How many turns will there be? We let *A* and *B* label the ground corners of the hillside. *C* labels the point where a road from *A* meets the edge for the first time, and *D* is vertically below *C* on ground level. We want to find the distance *BC* = *u*. First, in the triangle *BCD*, the angle *B* is 30 degrees, and *BD* = *u**cos(30). With Pythagoras we get *u*^2 = *CD*^2 + *BD*^2 = *CD*^2 + *u*^2*cos(30)^2, or *CD*^2 = *u*^2(1-cos(30)^2) = *u*^2*sin(30)^2. Next, in the triangle *ACD*, the angle *A* is 10 degrees, and *AD* = *AC**cos(10). With Pythagoras we get *AC*^2 = *CD*^2 + *AD*^2 = *CD*^2 + *AC*^2*cos(10)^2, or *CD*^2 = *AC*^2(1-cos(10)^2) = *AC*^2*sin(10)^2. Finally, in the triangle *ACB*, *AB* = 1 and *BC* = *u*, so with Pythagoras we get *AC*^2 = 1^2 + *u*^2, or *AC* = √(1+*u*^2).

Consequently, *u*^2*sin(30)^2 = *AC*^2*sin(10)^2, or *u* = *AC**sin(10)/sin(30) = *AC***r*, or *u* = √(1+*u*^2)**r*, or *u*^2 = (1+*u*^2)**r*^2, or *u*^2*(1-*r*^2) = *r*^2, or *u*^2 = *r*^2/(1-*r*^2) = 0.137, giving the distance *BC* = *u* = √0.137 = 0.37. So, two turns: 3.70 cm and 7.40 cm up the tile.

MC14. Rectangles as Extended Squares

A rectangle with base *b* and height *h* = *c***b* may be called a '*c* extended square'. It consists of a lower square, *b***b*, and an upper rectangle (*h*-*b*)**b*, becoming a square also if *c* is 2. The two diagonals *d1* and *d*2 are raised the angles *A1* and *A*2 that may be predicted by tan*A1* = *h*/*b* = *c*, and tan*A*2 = (*h*-*b*)/*b* = *h*/*b*-1 = *c*-1. The diagonal *d1* is predicted by Pythagoras: *d1*^2 = *b*^2 + *h*^2 = *b*^2 + *c*^2**b*^2 = *b*^2*(1+*c*^2), or *d1* = *b**√(1+*c*^2). The diagonal *d*2 is predicted by *d*2 = *b**√(1+(1-*c*)^2). Finally, the normal *n* to the diagonal *d1* is predicted by *n***d1* = *h***b* = *c***b*^2, or *n** *b**√(1+c^2) = *c***b*^2, or *n* = *b***c*/√(1+c^2) = *b** 1/√(1+1/*c*^2).

So, combining a tile with half of its neighbor will provide a rectangle as a 1.5 extended square where the diagonal angels and lengths may be predicted by proper formulas: tan*A1* = 1.5 giving *A1* = 56.3, tan*A1* = 0.5 giving *A1* = 26.6. Likewise with the two long diagonals: *d1* = *b**√(1+*c*^2) = *b**√(1+1.5^2)= *b**1.80; and *d*2 = *b**√(1+(1-*c*)^2)= *b**√(1+1.5^2)= *b** 1.12. Finally, the normal: *n* = *b**1/√(1+1/*c*^2) = *b**1/√(1+1/1.5^2) = *b**0.832 = *b**sin*A1*.

MC15. The Golden Factor Pervades Art

On a tile, a circle with center in the midpoint of an edge and passing through the opposite corners marks two points that extend the tile to one side or the other with the golden factor ½(1+√5) ≈ 1.62, i.e. 62%. This extends the original edge with the same factor as will extend the new edge to a length equivalent to adding an extra tile.

Likewise, with a radius half the edge the half-diagonal from the midpoint to the corner will mark a length that will divide the edge in two parts connected by the golden factor.

MC16. Meeting Pi on the Pavement

Two neighboring tiles are circumscribed by a semicircle, again circumscribed by two tiles. On the right tile, the diagonal creates two triangles enveloping a quarter of the semicircle, i.e. 180/4 degrees or pi/4. Consequently, 4*sin(180/4) < pi < 4*tan (180/4). In other words, pi = *n**sin(180/*n*) = *n**tan(180/*n*) = 3.14… for *n* sufficiently large.

MC17. Meeting Algebra

Half of a *b***b* tile will extend a tile upwards to a *h***b* playing card. Removing from a *h***h* square two playing cards, and adding the bottom right tile that has been removed twice will leave the square (*h* – *b*)^2 = *h*^2– 2**a***b* + *b*^2. And, removing from a *h***b* playing card the bottom *b***b* tile will leave the top (*h* – *b*)**b* = *h***b* – *b***b*.

Four playing cards are arranged to form a (*h*+*b*)*(*h*+*b*) square. Inside we find a *h***h* square, a *b***b* square and two playing cards, so, (*h*+*b*)*(*h*+*b*) = *h***h* + *b***b* + 2**h***b*.

Pulling away a $b*b$ tile from the $h*h$ square leaves a $(h–b)*h$ and a $(h–b)*b$ rectangles that add up to a $(h–b)*(h+b)$ rectangle. Consequently, $(h+b)*(h–b) = h^2 – b^2$.

To solve the quadratic equation $x^2+6x+8 = 0$ we use four tiles forming a square, labeling the first side x and the next 6/2. The $(x+6/2)$ square now contains two $6/2*x$ rectangles and two squares, x^2 and $(6/2)^2$ split in two parts, 8 below and $(6/2)^2-8$ above if possible, all disappearing except for last part. So $(x+6/2)^2 = (6/2)^2– 8 = 1$ giving $x = -6/2 \pm 1 = -2$ and -4. Looking instead at $x^2+bx+c = 0$ gives the solution $x = -b/2 \pm \sqrt{((b/2)^2 –c)}$.

MC18. Predicting Change

Two $b*b$ tiles form a $h*b$ playing card that is extended with a tape on-top and to the left to show that a change in h and b, Δh and Δb, will give a change in the area, $\Delta(b*h) = \Delta b*h + b*\Delta h$, or with per-numbers, $\Delta(b*h)/(b*h) = \Delta b/b + \Delta h/h$. Thus, with products, the change-percentages almost just add: Changing a kg-number with 3% and a $/kg-number with 5% will make the $-number change with approximately 3% + 5% = 8%. This rule applies to changes less than 10% with decreasing precision. Here we neglect the upper right tape-corner, which is allowed for sufficiently small changes, giving $(b*h)'/(b*h) = b'/b + h'/h$. So, with $y = x^n$ we get that $dy/y = n*dx/x$, or $dy/dx = n*y/x = n*x^n/x = n*x^{(n-1)}$.

MC19. Following Change Formulas e.g. when Playing Golf

Tiles form a coordinate system to move in. Person A starts a (+1,+1) trip in (0,3). Person B starts a (+1,-2) trip in (0,9). Predict where they meet. Person A starts a (+1,+s) trip in (0,0) where s decreases with 1 from +4 to -4. Person B starts a (+1,+s) trip in (10,0) where s increases with 1 from -4 to +4. Person A starts a (+1,+s) trip in (0,½) where s is doubled from ½ the first 4 steps, then halved the next 6 steps. Person B starts a (+1,+s) trip in 0,0 where s decreases with 1 from +3 to -3, then increases with 1 from -3 to +3. Person A and B start a (+1,+s) trip in (0,0) and (2,0) wanting to end closest to a golf hole in (10,0).

MC20. The Saving Formula

A saving combines a deposit amount a with an interest percent r, illustrated by two tiles, K1 and K2. K2 receives a one-time deposit a/r, and each period its interest amount $a/r*r = a$ is transferred to K1 after K1 has received its own interest amount. After n periods, K1 will contain a saving A growing from a deposit amount a and an interest percent r. But, at the same time, K1 will contain the total interest percent R of the initial amount a/r in K2, so $A = a/r*R$, or $A/R = a/r$, where $1+R = (1+r)^n$. Using the doubling-time as the period, a 1$ deposit will after 5 doubling periods save 31$ = 5$ deposit + 26$ compound interest.

MC21: Having Fun with a Tile System and with Bundle-bundle Squares

A dozen recounts as 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed in a tile system, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. With a per-number 2G/3R, a dozen R may be changed to 2G+9R, 4G+6R, 6G+3R, and 8G, thus traveling along a line sloping down with the per-number. With Bundle-Bundle squares we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas: $n*n + 2*n + 1 = (n+1)*(n+1)$; and $n*n – 2*n + 1 = (n-1)*(n-1)$; and $(n–1)*(n+1) = n*n – 1$.

MC22. Pascal’s Triangle

A triangle of tiles consists of 1 tile in column 1, 2 in column 2, etc. until column 5. Traveling the triangle with a (+1,+1) win-step or a (+1,+0) loose-step we observe how many roads lead to each tile. Could it be predicted? Start over, but now let a coin decide the next step. Mark the tile with a short stroke. How many times did you win? Could it be predicted?

MC23. Game Theory

Two players A and B choose column and row at 2x2 tiles carrying the numbers 1, 2, 3, 4 in the top and bottom row, indicating what A pays back to B after having received a fix fee from B. Showing paper or stone means choosing the first or second strategy. Thus, if A chooses stone and B paper, A will pay back 2 to B. Which fee makes the game fair? Use cubes to show that the fee is 2.5 if 4 and 2 change places. In the first game, 3 is a stable ‘saddle point’ going up if A changes, and down if B changes, which they don’t want. In the second game both players will be tempted to change, so both will mix strategies, but how?

MC24. Geometry with Handles

Graph theory and topology is geometry where neither distances nor angles matters, but only the relative positions between the points. A classic problem is the supply problem shown with two separated rows with 3 tiles each: How can three houses A, B and C be supplied with electricity, gas and water with no crossing wires? Hint: Connect A and B with gas and water. Conclusion: the task cannot be solved unless we add a bridge whereby the plan changes its topology to a torus which is a plane with a handle.

MC25. The Electric Circuit

To work properly, a device demands energy coming from a supplier, thus creating a circuit of carriers with the demander as a sink and the supplier as a source. If the demand is 16 energy-units per second, and the supplier provides 8 units per carrier, a flow of 2 carriers per second is needed, enabled by a device resistance at 4, calculated as the ratio between the supply and the flow. In technical terms: If a device needs 16 Joule per second (Watt) it needs 2 electrical carriers (Coulombs) per second (Ampere) as current from a battery delivering 8 Joule per Coulomb (Volt). To realize this, the device needs the resistance 4 Ohm.

A circuit thus follows two formulas: Demand = Supply * Flow, or Watt = Volt * Ampere, or $P = U * I$; and Supply = Resistance * Flow, or Volt = Ohm * Ampere, or $U = R * I$.

Built into the device, the resistance cannot change, but the voltage can. Supplying a (4ohm, 16watt) device from a 4volt instead of an 8volt battery will give the current 4volt/4ohm or 1 ampere, supplying 4volt*1ampere = 4watt instead of 16watt, i.e. only a quarter of what is needed.

Supplying a (4ohm, 16watt) device from a 16volt instead of an 8volt battery will give the current 16volt/4ohm or 4ampere, supplying 16volt*4ampere = 64watt instead of 16watt, i.e. 4 times what is needed.

With a 12volt battery, a 3ohm device will produce the current 12volt/3ohm or 4ampere supplying 12volt*4ampere = 48watt. With a 1ohm it will be 12ampere and 144watt.

Using a 12volt battery to supply both a (3ohm, 48watt) and a (1ohm, 144watt) device, the total resistance 4ohm will produce the current 12volt/4ohm or 3ampere which on the first device will use 3ohm*3A = 9volt supplying 9volt*3A = 27W ≈ 60% of what is needed; and on the next device will use 1ohm*3A = 3volt supplying 3volt*3A = 9W ≈ 6% of what is needed. Thus, the bigger consumer receives the smaller part. Consequently, multiple devices are connected, not serial as here, but parallel increasing the current to 16A.

Three tiles serve at simulating how 2 cups supply a device labeled (4ohm, 16watt) with energy from an 8volt battery. LEGO-bricks serve as energy-units, and a slow metronome tells when a ‘second’ has passed. From the battery, 8ers are placed in 2 cups that move to the devise to deliver 2*8 units at the time signal, and then move back empty to refill.

In the case of a 4volt battery, 1 cup carries a 4er and delivers 1*4 watt, only 25% of what is needed. In the case of 1ohm, 8cups deliver 8*4 = 32watt.

With a 12volt battery supplying first a (3ohm, 48watt) and then a (1ohm, 144watt) device, 3 cups each supply 9 to the first device needing 48, and 3 to the second device needing 144 before returning to refill.

Discussion and Recommendation

This paper asked 'What kind of mathematics may grow from tiles?' The background was the phenomenological observation, that Many presents itself to children as bundle-numbers with units as e.g. 2 3s, thus having squared tiles as 1 1s or bundle-bundles as the unit, which allows geometry and algebra to go hand in hand from grade 1.

With units, a core question is how to change it, traditionally leading to proportionality in its classical regula-de-tri form multiplying before dividing, or to its modern form doing the opposite by first finding the per-unit-number. And thus postponed until after teaching all four operations from addition to division.

Recounting turns this order around. First division pushes away bundles to be stacked by multiplication to be pulled away by subtraction in order to find unbundled singles that become decimals, fractions or negative numbers depending on where they are placed. Recounting produces a formula $T = (T/B)*B$ present all over mathematics and science, and showing there things: How to change units, how to solve equations by moving to opposite side with opposite sign, and how per-numbers must be multiplied before being added.

Double-counting in two units produces per-numbers, becoming fractions with like units. As geometrical representations of bundle-numbers, squares and rectangles lead directly to double-counting the sides in rectangles halved by their diagonals, thus allowing trigonometry and tile or coordinate geometry to precede traditional plane geometry.

Addition comes last in two forms, on-top needing proportionality to change units, and next-to adding areas as integral calculus, also occurring when adding per-numbers and fractions with units to avoid mathematism. And reversed addition leads directly to differentiation.

Quadratic expressions, equations and functions also relate to tiles in a natural way, as does differential equations through change formulas directing trips through a tile system

So, it turns out that the core of mathematics springs from tiles once we accept the two-dimensional bundle-numbers children develop while adapting to Many before school. Education needs not teaching 'metamatism' as the only means leading to the end goal, Mastery of Many. Tiles will show the way directly in a concrete way including also outdoor and STEM education; and will perhaps be able to answer the Cinderella question with a yes, there is a hidden unnoticed alternative that makes the prince dance. Tile-mathematics may offer a new Kuhnian paradigm that will finally create a mathematics for all. This paper has taken a first step in an experiential learning cycle by designing more than a score of micro-curricula to be tested inside and outside classrooms in the hope that after several cycles of redesigning they will become scores making math dislike evaporate.

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THE POWER OF BUNDLE- & PER-NUMBERS UNLEASHED IN PRIMARY SCHOOL: CALCULUS IN GRADE ONE – WHAT ELSE?

In middle school, fraction, percentage, ratio, rate, and proportion create problems to many students. So, why not teach it in primary school instead where they all may be examples of per-numbers coming from double-counting a total in two units. And bundle-numbers with units is what children develop when adapting to Many before school. Here children love counting, recounting, and double-counting before adding totals on-top or next-to as in calculus, also occurring when adding per-numbers. Why not accept, and learn from the mastery of Many that children possess until mathematics takes it away?

MATHEMATICS IS HARD, OR IS IT?

“Is mathematics hard by nature or by choice?” is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature. That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented e.g. at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969. Likewise, increased funding used e.g. for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries, and significantly below the OECD average at 494. This caused OECD (2015) to write the report ‘Improving Schools in Sweden’ describing the Swedish school system as being ‘in need of urgent change’.

Also among the countries with poor PISA performance, Denmark has lowered the passing limit at the final exam to around 15% and 20 % in lower and upper secondary school. And, at conferences as e.g. The Third International Conference on Mathematics Textbook Research and Development, ICMT3 2019, high-ranking countries admit they have a high percentage of low scoring students. Likewise at conferences, discussing in the breaks what is the goal of mathematics education, the answer is almost always ‘to learn mathematics’. When asked to define mathematics, some point to schoolbooks, others to universities; but all agree that learning it is important to master its outside applications.

So, we may ask, is the goal of mathematics education to master outside Many, or to first master inside mathematics as a means to later master outside Many. Here, institutionalizing mathematics as THE only inside means leading to the final outside goal may risk creating a goal displacement transforming the means to the goal instead (Bauman, 1990) leading on to the banality of evil (Arendt, 1963) by just following the orders of the tradition with little concern about its effect as to reaching the outside goal. To avoid this, this paper will answer the question about the hardness by working backwards, not from mathematics to Many, but from Many to mathematics. So here the focus is not to study why students have difficulties mastering inside mathematics, but to observe and investigate the mastery of outside Many that children bring to school before being forced to learn about inside mathematics instead.

RESEARCH METHOD

Difference research searching for differences has uncovered hidden differences (Tarp, 2018c). To see if the differences make a difference, phenomenology (Tarp, 2018a), experiential learning (Kolb, 1984), and design research (Bakker, 2018) may create cycles of observations, reflections, and designs of micro curricula to be tested in order to create a new cycle for testing the next generation of curricula.

OBSERVATIONS AND REFLECTIONS 01

Asked “How old next time?” a three-year-old will say four showing four fingers, but will react to seeing the fingers held together two by two: “That is not four. That is two twos!” The child thus describes what exists, bundles of 2s, and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)x2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving a formula with unspecified numbers '$T = (T/B)\text{x}B$' saying "from T, T/B times, B may be pushed away and stacked". Consequently, search questions about 'bundle-numbers' and 'recounting' may be given to small groups of four preschool children to get ideas about how to design a generation-1 curriculum.

GUIDING QUESTIONS

The following guiding questions were used: "There seems to be five strokes in the symbol five. How about the other symbols?", "How many bundles of 2s are there in ten?", "How to count if including the bundle?", "How to count if using a cup for the bundles?", "Can bundles also be bundled, e.g. if counting ten in 3s?", "What happens if we bundle to little or too much?", "How to recount icon-numbers in tens?", "How to manually recount 8 in 2s, and recount 7 in 2s?", "What to do if a bundle is not full?", "How to bundle-count seconds, minutes, hours, and days?", "How to double-count lengths in centimeters and inches?", "A dice decided my share in a lottery ticket, how to share a gain?", "Which numbers can be folded in other numbers than 1s?", "Asking how many 2s in 8 may be written as $u*2 = 8$, how can this equation be solved?", "How to recount from tens to icons?", "How to add 2 3s and 4 5s next-to?", "How to add 2 3s and 4 5s on-top?", " 2 3s and some 5s gave 3 8s, how many?", "How to add totals bundle-counted in tens?", "How to subtract totals bundle-counted in tens?", "How to add per-numbers?", "How to enlarge or diminish bundle-bundle squares?", "What happens when recounted stacks are placed on a squared paper?", "What happens when turning or stacking stacks?"

Observations and reflections 02

Data and ideas allowed designing Micro Curricula (MC) with guiding questions and answers (Q, A).

MC31: Digits as Icons

With strokes, sticks, dolls, and cars we observe that four 1s can be bundled into 1 fours that can be rearranged into a 4-icon if written less sloppy. So, for each 4 1s there is 1 4s, or there is 1 4s per 4 1s. In this way, all digits may be iconized, and used as units for bundle-counting (Tarp, 2018b).

MC32: Bundle-counting Ten Fingers

A total of ten ones occurring as ten fingers, sticks or cubes may be counted in ones, in bundles, or with 'underloads' counting what must be borrowed to have a full bundle. Count ten in 5s, 4s, 3s, and 2s.

In 5s with bundles: 0*B*1, …, 0*B*4, 0*B*5 no 1*B*0, 1*B*1, …, 1*B*4, 1*B*5 no 2*B*0.

In 5s with bundles and underloads: 1*B*-4, 1*B*-3, …, 1*B*0, 2*B*-4, … , 2*B*0.

MC33: Counting Sequences Using Tens and Hundreds

In oral counting-sequences the bundle is present as tens, hundreds, thousands, ten thousand (wan in Chinese) etc. By instead using bundles, bundles of bundles etc. it is possible to let power appear as the number of times, bundles have been bundled thus preparing the ground for later writing out a multi-digit number fully as a polynomial, $T = 345 = 3BB4B5 = 3*B\text{^}2 + 4*B + 5*1$.

Count 10, 20, 30, …, 90, 100 etc. Then 1*B*, 2*B*, …, 9*B*, ten*B* no 1*BB*.

Count 100, 200, 300, …, 900, ten-hundred no thousand. Then 1*BB*, 2*BB*, …, 9*BB*, ten*BB* no 1*BBB*.

Count 100, 110, 120, 130, …, 190, 200 etc. Then 1*BB*0*B*, 1*BB*1*B*, …, 1*BB*9*B*, 1*BB*ten*B* no 2*BB*0*B*.

A dice shows 3 then 4. Name it in five ways: thirty-four, three-ten-four, three-bundle-four, four-bundle-less6, and forty less 6. Travel on a chess board while saying 1*B*1, 2*B*1, 3*B*1, 3*B*2, …; 3*B*4.

MC34: Cup-counting and Bundle-bundles

When counting a total, a bundle may be changed to a single thing representing the bundle to go to a cup for bundles, later adding an extra cup for bundles of bundles. Writing down the result, bundles and unbundled may be separated by a bundle-letter, a bracket indicating the cups, or a decimal point.

Q. *T* = two hands, how many 3s?

A. With 1 3s per 3 1s we count 3 bundles and 1 unbundled, and write *T* = 3*B*1 3s = 3]1 3s = 3.1 3s showing 3 bundles inside the cup, and 1 unbundled outside. However, 3 bundles are 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 1*BB*0*B*1 3s = 1]0]1 3s = 10.1 3s with an additional cup for the bundle-bundles.

Q. *T* = two hands, how many 2s?

A. With 1 2s per 2 1s we count 5 bundles, *T* = 5*B*0 2s = 5]0 2s = 5.0 2s. But, 2 bundles is 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 2*BB*1*B*0 2s = 2]1]0 2s = 21.0 2s. However, 2 bundles-of-bundles is 1 bundle-of-bundles-of-bundles, 1*BBB*, so with bundle-bundle-bundles we write *T* = 1*BBB*0*BB*1*B*0 2s= 1]0]1]0 2s = 101.0 2s with an extra cup for the bundle-bundle-bundles.

MC35: Recounting in the Same Unit Creates Underloads and Overloads

Recounting 8 1s in 2s gives *T* = 4*B*0 2s. We may create an underload by borrowing 2 to get 5 2s. Then *T* = 5*B*-2 2s = 5]-2 2s = 5.-2 2s. Or, we may create an overload by leaving some bundles unbundled. Then *T* = 3*B*2 2s = 2*B*4 2s = 1*B*6 2s. Later, such 'flexible bundle-numbers' will ease calculations.

MC36: Recounting in Tens

With ten fingers, we typically use ten as the counting unit thus becoming 1*B*0 needing no icon.

Q. *T* = 3 4s, how many tens? Use sticks first, then cubes.

A. With 1 tens per ten 1s we count 1 bundle and 2, and write *T* = 3 4s = 1*B*2 tens = 1]2 tens = 1.2 tens, or *T* = 2*B*-8 tens = 2.-8 tens using flexible bundle-numbers. Using cubes or a pegboard we see that increasing the base from 4s to tens means decreasing the height of the stack. On a calculator we see that 3x4 = 12 = 1.2 tens, using a cross called multiplication as an icon for a lift stacking bundles. Only the calculator leaves out the unit and the decimal point. Often a star * replaces the cross x.

Q. *T* = 6 7s, how many tens?

A. With 1 tens per ten 1s we count 4 bundles and 2, and write *T* = 6 7s = 4*B*2 tens = 4]2 tens = 4.2 tens. Using flexible bundle-numbers we write *T* = 6 7s = 5*B*-8 tens = 5]-8 tens = 5.-8 tens = 3*B*12 tens. Using cubes or a pegboard we see that increasing the base from 7s to tens means decreasing the height of the stack. On a calculator we see that 6*7 = 42 = 4.2 tens.

Q. *T* = 6 7s, how many tens if using flexible bundle-numbers on a pegboard?

A. *T* = 6 7s = 6*7 = (*B*-4) * (*B*-3) = *BB*-3*B*-4*B*+4*3 = 10*B*-3*B*-4*B*+1*B*2 = 4*B*2 since the 4 3s must be added after being subtracted twice.

MC37: Recounting Iconizes Operations and Creates a Recount-formula for Prediction

A cross called multiplication is an icon for a lift stacking bundles. Likewise, an uphill stroke called division is an icon for a broom pushing away bundles. Recounting 8 1s in 2s by pushing away 2-bundles may then be written as a 'recount-formula' 8 = (8/2)*2 = 8/2 2s, or *T* = (*T*/*B*)**B* = *T*/*B* *B*s, saying "From *T*, *T*/*B* times, we push away *B* to be stacked". Division followed by multiplication is called changing units or proportionality. Likewise, we may use a horizontal line called subtraction as an icon for a rope pulling away the stack to look for unbundled singles.

These operations allow a calculator predict recounting 7 1s in 2s. First entering '7/2' gives the answer '3.some' predicting that pushing away 2s from 7 can be done 3 times leaving some unbundled singles that are found by pulling away the stack of 3 2s from 7. Here, entering '7-3*2' gives the result '1', thus predicting that 7 recounts in 2s as 7 = 3*B*1 2s = 3]1 2s = 3.1 2s.

Recounting 8 1s in 3s gives a stack of 2 3s and 2 unbundled. The singles may be placed next-to the stack as a stack of unbundled 1s, written as *T* = 8 = 2.2 3s. Or they may be placed on-top of the stack counted in bundles as 2 = (2/3)*3, written as *T* = 8 = 2 2/3 3s thus introducing fractions. Or, as *T* = 8 = 3.-1 3s if counting what must be borrowed to have another bundle.

Q. *T* = 9, 8, 7; use the recount-formula to predict how many 2s, 3s, 4s, 5s before testing with cubes.

MC38: Recounting in Time

Counting in time, a bundle of 7days is called a week, so 60days may be recounted as *T* = 60days = (60/7)*7days = 8*B*4 7days = 8weeks 4days. A bundle of 60 seconds is called a minute, and a bundle of 60 minutes is called an hour, so 1 hour is 1 bundle-of-bundles of seconds. A bundle of 12hours is called a half-day, and a bundle of 12months is called a year.

MC39: Double-counting in Space Creates Per-Numbers or Rates

Counting in space has seen many units. Today centimeter and inches are common. 'Double-counting' a length in inches and centimeters approximately gives a 'per-number' or rate 2in/5cm shown with cubes forming an L. Out walking we may go 3 meters each 5 seconds, giving the per-number 3m/5sec. The two units may be bridged by recounting in the per-number, or by physically combining Ls.

Q. *T* = 12in = ?cm; and *T* = 20cm = ?in

A1. *T* = 12in = (12/2)*2in = (12/2)*5cm = 30cm; and A2. *T* = 20cm = (20/5)*5cm = (20/5)*2in = 8in

MC40: Per-numbers Become Fractions

Double-counting in the same unit makes a per-number a fraction. Recounting 8 in 3s leaves 2 that on-top of the stack become part of a whole, and a fraction when counted in 3s: *T* = 2 = (2/3)*3 = 2/3 3s.

Q. Having 2 per 3 means having what per 12?

A. We recount 12 in 3s to find the number of 2s: *T* = 12 = (12/3)*3 giving (12/3) 2s = (12/3)*2 = 8. So, having 2/3 means having 8/12. Here we enlarge both numbers in the fraction by 12/3 = 4.

Q. Having 2 per 3 means having 12 per what?

A. We recount 12 in 2s to find the number of 3s: *T* = 12 = (12/2)*2 giving (12/2) 3s = (12/2)*3 = 18. So, having 2/3 means having 12/18. Here we enlarge both numbers in the fraction by 12/2 = 6.

MC41: Per-numbers Become Ratios

Recounting a dozen in 5s gives 2 full bundles, and one bundle with 2 present, and 3 absent: *T* = 12 = 2*B*2 5s = 3*B*-3 5s. We say that the ratio between the present and the absent is 2:3 meaning that with 5 places there will be 2 present and 3 absent, so the present and the absent constitute 2/5 and 3/5 of a bundle. Likewise, if recounting 11 in 5s, the ratio between the present and the absent will be 1:4, since the present constitutes 1/5 of a bundle, and the absents constitute 4/5 of a bundle. So, splitting a total between two persons A and B in the ration 2:4 means that A gets 2, and B gets 4 per 6 parts, so that A gets the fraction 2/6, and B gets the fraction 4/6 of the total.

MC42: Prime Units and Foldable Units

Bundle-counting in 2s has 4 as a bundle-bundle. 1s cannot be a unit since 1 bundle-bundle stays as 1. 2 and 3 are prime units that can be folded in 1s only. 4 is a foldable unit hiding a prime unit since 1 4s = 2 2s. Equal number can be folded in 2s, odd numbers cannot. Nine is an odd number that is foldable in 3s, 9 1s = 3 3s. Find prime units and foldable units up to two dozen.

MC43: Recounting Changes Units and Solves Equations

Rephrasing the question "Recount 8 1s in 2s" to "How many 2s are there in 8?" creates the equation 'u*2 = 8' that evidently is solved by recounting 8 in 2s since the job is the same:

If u*2 = 8, then u*2 = 8 = (8/2)*2, so u = 8/2 = 4.

The solution u = 8/2 to u*2 = 8 thus comes from moving a number to the opposite side with the opposite calculation sign. The solution is verified by inserting it in the equation: u*2 = 4*2 = 8, OK.

Recounting from tens to icons gives equations: "42 is how many 7s" becomes u*7 = 42 = (42/7)*7.

MC44: Next-to Addition of Bundle-Numbers Involves Integration

Once recounted into stacks, totals may be united next-to or on-top, iconized by a cross called addition.

To add bundle-numbers as 2 3s and 4 5s next-to means adding the areas 2*3 and 4*5, called integral calculus where multiplication is followed by addition.

Q. Next-to addition of 2 3s and 4 5s gives how many 8s?

A1. T = 2 3s + 4 5s = (2*3+4*5)/8 8s = 3.2 8s; or A2. T = 2 3s + 4 5s = 26 = (26/8) 8s = 3.2 8s

MC45: On-top Addition of Bundle-Numbers Involves Proportionality

To add bundle-numbers as 2 3s and 4 5s on-top, the units must be made the same by recounting.

Q. On-top addition of 2 3s and 4 5s gives how many 3s and how many 5s?

A1. T = 2 3s = (2*3/5)*5 = 1.1 5s, so 2 3s and 4 5s gives 5.1 5s

A2. T = 2 3s + 4 5s = (2*3+4*5)/5 5s = 5.1 5s; or T = 2 3s + 4 5s = 26 = (26/5) 5s = 5.1 5s

MC46: Reversed Addition of Bundle-Numbers Involves Differentiation

Reversed addition may be performed by a reverse operation, or by solving an equation.

Q. Next-to addition of 2 3s and how many 5s gives 3 8s?

A1: Removing the 2*3 stack from the 3*8 stack, and recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called differentiation.

A2: The equation 2 3s + u*5 = 3 8s is solved by moving to opposite side with opposite calculation sign

u*5 = 3 8s – 2 3s = 3*8 – 2*3, so u = (3*8 – 2*3)/5 = 18/5 = 3 3/5, giving 3.3 5s.

MC47: Adding and Subtracting Tens

Bundle-counting typically counts in tens, but leaves out the unit and the decimal point separating bundles and unbundled: T = 4B6 tens = 4.6 tens = 46. Except for e-notation with a decimal point after the first digit followed by an e with the number of times, bundles have been bundled: T = 468 = 4.68e2.

Calculations often leads to overloads or underloads that disappear when re-bundling:

Addition: 456 + 269 = 4BB5B6 + 2BB6B9 = 6BB11B15 = 7BB12B5 = 7BB2B5 = 725.

Subtraction: 456 – 269 = 4BB5B6 – 2BB6B9 = 2BB-1B-3 = 2BB-2B7 = 1BB8B7 = 187

Multiplication: 2 * 456 = 2 * 4BB5B6 = 8BB10B12 = 8BB11B2 = 9B1B2 = 912

Division: 154 /2 = 15B4 /2 = 14B12 /2 = 7B6 = 76

MC48: Next-to Addition & Subtraction of Per-Numbers and Fractions is Calculus

Throwing a dice 8 times, the outcome 1 and 6 places 4 cubes on a chess board, and the rest 2 cubes. When ordered it may be 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s called the average: If alike, the per-numbers would be 2.6 cubes per square. Thus per-numbers add by areas, i.e. by integration. Reversing the question to 2 5s + ? 3s total 3 8s then leads to differentiation: 2 5s + ? 3s = 3 8s gives the equation

2*5 + *u**3 = 3*8, so *u**3 = 3*8 – 2*5, so *u* = (3*8 – 2*5)/3 = 4 2/3, or *u* = (*T*2-*T*1)/3 = Δ*T*/3

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC49: Having Fun with Bundle-Bundle Squares

On a pegboard we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas:

*n*n* + 2**n* + 1 = *(n+1)*(n+1);* and *n*n* – 2**n* + 1 = *(n*-1*)*(n*-1*);* and *(n*–1*)*(n*+1*)* = *n*n* – 1.

Two *s*s* bundle-bundles form two squares that halved by their diagonal *d* gives four half-squares called right triangles. Rearranged, they form a diagonal-square *d*d*. Consequently, *d*d* = 2**s*s*

Four *c*b* playing cards with diagonal *d* are placed after each other to form a *(b+c)*(b+c)* bundle-bundle square. Below to the left is a *c*c* square, and to the right a *b*b* square. On-top are 2 playing cards. Inside there is a *d*d* square and 4 half-cards. Since 4 half-cards is the same as 2 cards, we have the formula *c*c* + *b*b* = *d*d* making it easy the add squares, you just square the diagonal.

MC50: Having Fun with Halving Stacks by its Diagonal to Create Trigonometry

Halving a stack by its diagonal creates two right triangles. Traveling around the triangle we turn three times before ending up in the same direction. Turning 360 degrees implies that the inside angles total 180 degree, and that a right angle is 90 degrees. Measuring a 5up_per_10out angle to 27 degrees we see that tan(27) is 0.5 approximately. So, the tan-number comes from recounting the height in the base.

MC51: Having Fun with a Squared Paper

A dozen may be 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed on a squared paper with the lower left corners coinciding, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. Traveling by saying “3up_per_1out, 2up_per_1out, …, 3down_per_1out” allows the end points to follow a parabola. With a per-number 2G/3R, a dozen R can be changed to 2G+9R, 4G+6R, 6G+3R, and 8G. Plotted on a square paper with R horizontally and G vertically will give a line sloping down with the per-number.

MC52: Having Fun with Turning and Combining Stacks

Turned over, a 3*5 stack becomes a 5*3 stack with the same total, so multiplication-numbers may commute (the commutative law). Adding 2 7s on-top of 4 7s totals (2+4) 7s, 2*7+4*7 = (2+4)*7 (the distributive law). Stacking stacks gives boxes. Thus 2 3s may be stacked 4 times to the box *T* = 4*(2*3) that turned over becomes a 3*(2*4) box. So, 2 may freely associate with 3 or 4 (the associative law).

DISCUSSION AND RECOMMENDATION

This paper asks: what mastery of Many does the child develop before school? The question comes from observing that mathematics education still seems to be hard after 50 years of research; and from wondering if it is hard by nature or by choice, and if it is needed to achieve its goal, mastery of Many.

To find an answer, phenomenology, experiential, and design research is used to create a cycle of observations, reflections, and testing of micro curricula designed from observing the reflections of preschoolers to guiding questions on mastering Many. The first observation is that children use two-dimensional bundle-numbers with units instead of the one-dimensional single numbers without units that is taught in school together with a place value system. Reflecting on this we see that units make counting, recounting, and double-counting core activities leading to proportionality by combining division and multiplication, thus reversing the order of operations: first division pulls away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to identify unbundled singles that becomes decimal, fractional or negative numbers. And that recounting between icons and tens leads to equations when asking e.g. ‘how many 5s are 3 tens?’ And that units make addition ambiguous: shall totals add on-top after proportionality has made the units like, or shall they add next-to as an example of integral calculus adding areas, and leading to differentiation when reversed? Finally, we see that flexible bundle-numbers ease traditional calculations on ten-based numbers.

Testing the micro curricula will now show if mathematics is hard by nature or by choice. Of course, investments in traditional textbooks and teacher education, all teaching single numbers without units, will deport testing to the outskirts of education, to pre-school or post-school; or to special, adult, migrant, or refugee education; or to classes stuck in e.g. division, fractions, precalculus, etc. All that is needed is asking students to count fingers in bundles. Recounting 8 in 2s thus directly gives the proportionality recount-formula 8 = (8/2)*2 or $T = (T/B)*B$ used in STEM, and to solve equations. Likewise, direct and reversed next-to addition leads directly to calculus. Furthermore, testing micro curricula will allow teachers to practice action learning and action research in their own classroom.

Phenomenologically, it is important to respect and develop the way Many presents itself to children thus providing them with the quantitative competence of a number-language. Teaching numbering instead of numbers thus creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education (Widdowson, 1978). The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not lean from children instead?

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A BUNDLE COUNTING TABLE

A guide to bundle-counting in pre-school.

Bundle-counting clarifies that we count by bundling, typically in tens.

Example 01. Counting Mikado Sticks

The Mikado sticks are positioned next to each other to the right. Counting is done by taking one stick at a time to the left and assembling them in a bundle with an elastic band when we reach ten.

When counting, we say: "0 Bundle 1, 0 bundle 2,. .. "

"Why 0 bundle?" "Because we don't have a bundle yet, before we'll reach ten."

"..., 0 bundle 8, 0 bundle 9, 0 bundle ten, well no, 1 bundle 0".

Example 02. Counting matches

The box says 39, which we read as '3 bundles 9'. We bundle-count as with Mikado sticks.

Extra-option

Some children may find it fun later to count ' 1 bundle less 2, 1 bundle less 1, 1 bundle and 0, 1 bundle and 1 ' as a new way to count ' 0 bundle 8, 0 bundle 9, 1 bundle 0, 1 bundle 1 '. Later again, some children may find It fun to say ' 1 bundle-bundle 0 ' instead of ' ten bundles 0 ' or ' hundred '.

Example 03. Counting ten fingers or ten matches

The ten fingers (or ten matches) bundle are counted in 4s and in 3s while saying "The total is..." and possibly writing "T =..."

Ten counted in 4s	Ten counted in 3s
T = I I I I I I I I I I = ten 1s	T = III I I I I I I I = 1B7 3s
T = IIIIIIIIII = 1 tens = 1B0 tens	T = III III I I I I = 2B4 3s
T = IIII I I I I I I = 1B6 4s	T = III III III I = 3B1 4s
T = IIII IIII I I = 2B2 4s	T = III III III III = 4B-2 3s
T = IIII IIII IIII = 3B-2 4s	T = III III III I = 1BB 0B 1 3s

Table for counting ten tens, or 1 bundle bundles, or 1 hundred:

1BB0	1BB1	1BB2	1BB3	1BB4	1BB5	1BB6	1BB7	1BB8	1BB9	~~1BB10~~
~~10B0~~	~~10B1~~	~~10B2~~	~~10B3~~	~~10B4~~	~~10B5~~	~~10B6~~	~~10B7~~	~~10B8~~	~~10B9~~	~~10B10~~
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	~~9B10~~
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	~~8B10~~
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	~~7B10~~
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	~~6B10~~
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	~~5B10~~
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	~~4B10~~
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	~~3B10~~
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	~~2B10~~
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	~~1B10~~
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	~~0B10~~

05. LEARN CORE MATHEMATICS THROUGH YOUR KID'S TILE-MATH: RECOUNTING BUNDLE-NUMBERS AND EARLY TRIGONOMETRY

Fifty years of mathematics education research has failed to create a mathematics for all. This raises the Cinderella question: are there hidden unnoticed alternatives that may make the prince dance? There are. Education may be different, and also math may be different from today's 'meta-matism'. Adapting to Many, children develop bundle-numbers with units as 2 3s having 1 1s as the unit, i.e. a tile also occurring as bundle-of-bundles, e.g. 3 3s. Asking 'What kind of mathematics may grow from tiles?' this paper uncovers 22 micro-curricula for outdoor STEM education.

Key words: mathematics education, STEM, proportionality, trigonometry, *equation, fraction, calculus.*

POOR PISA PERFORMANCE, A PERMANENT PANDEMIA?

When evaluating the effect of mathematics education, poor PISA performance occurs all over the world, despite 50 years of increasing research and funding. Seeing mastery of Many as the end goal of math education, phenomenology has found that Many presents itself to children as bundle-numbers, e.g. 2 3s. (Tarp, 2018). This creates a basis for cycles of experiential learning (Kolb, 1984) and design research to design and test micro-curricula (MC) leading to new cycles. So, respecting children's own two-dimensional bundle-numbers we now ask: What kind of mathematics may grow from tiles?

MC01. Counting by Bundling and Iconizing

Chopsticks placed on tiles show that digits are icons with as many sticks as they represent if written less sloppy. Counting fingers in 3s, we may include the word bundle in the counting sequence by saying '0*B*1, 0*B*2, 0*B*3 no 1*B*0; 1*B*1, 1*B*2, 1*B*3 no 2*B*0; 2*B*1, 2*B*2, 2*B*3 no 3*B*0, 3*B*1'. Counting and stacking tiles and using a folding ruler or a rope to show a bundle, we see that 3 bundles is 1 bundle-of-bundles or a 3x3 square of tiles, so we should instead say '2*B*3 no 3*B*0 no 1*BB*0*B*0'.

To include time in bundle-counting in 3s, we place a cube on each of ten neighboring tiles e.g. on a chess board. A cube moves to the next tile, and both move on to the next tile where they unite to 1 bundle that moves to the tile above, from where it moves to the next tile. This is repeated until tile 9 where 3 bundles unite to 1 bundle-of-bundles that move to the tile above, from where it moves to the last tile, thus showing that ten recounts as 1*BB*0*B*1 3s. Now the same is repeated with bundle-counting in 4s, then in 2s.

MC02. Formulas Predict

Eight persons are bundle-counted in 2s by asking 2s to go to neighboring tiles 4 times. Observing the total 8 splits into 4 2s, we write T = 8 = 4*2. Using an uphill stroke to iconize a broom pushing away bundles, the action 'from 8, push away 2s' may be entered on a calculator as '8/2', thus predicting 4 before carrying out the action. This allows rewriting 8 = 4*2 to 8 = (8/2)*2, or $T = (T/B)*B$ using unspecified numbers, saying 'From *T*, *T*/*B* times, *B* can be pulled away'. This 'recount-formula' predicts changing units. And, rephrasing recounting to "how many 2s in 8?" allows formulating recounting as an equation $u*2 = 8$ solved by $u = 8/2$, i.e. by moving a number to opposite side with opposite calculation sign.

MC03. Unbundled Become Decimals, Fractions or Negative Numbers

Can we predict the result of rearranging in 4s persons placed on tiles as 2 7s? Entering 2*7/4, a calculator says '3.some'. Using a horizonal stroke to iconize a rope pulling away stacks, entering 2*7-3*4 gives the answer '2' thus predicting that 2 7s recount as 3 4s and 2. The unbundled 2 may be placed next-to the stack reported as a decimal number, T = 3*B*2 4s = 3.2 4s, or on-top counted as bundles, 2 = (2/4)*4 = 2/4 4s, reported as a fraction, T = 3 2/4 *B* 4s; or, if counting what is needed for an extra bundle, reported by a negative number, T = 4 *B*-2 4s = 4.-2 4s. The prediction is then tested with persons or cubes placed on tiles.

MC04. Double-counting Creates Per-numbers

Traveling through a row of tiles is rewarded with 3 cubes per 2 tiles thus creating the 'per-number' 3/2 cubes/tiles. Travelling 12 tiles thus gives 3 cubes 6 times, which can be precited by recounting in the per-number: T = 12 tiles = (12/2)*2 tiles = (12/2)*3 cubes = 18 cubes. Alternatively, we can equate the per-numbers in an equation u/12 = 3/2 solved by moving to opposite side with opposite sign, u = 3/2*12 = 18. Per-numbers are all over mathematics and science, e.g. meter = (meter/second)*second = speed*second.

Double-counting in the same unit creates fractions: Marking 2 tiles with a dot for each 3 tiles traveled thus creates a per-number 2 tiles/3 tiles = 2/3. Having travelled 12 tiles, we mark 2 dots 4 times. Again, this can be precited by recounting in the per-number: T = 12 tiles = (12/3)*3 tiles marking dots on (12/3)*2 tiles, i.e. 8 tiles with dots.

MC05: Bundle-Numbers Add Next-to or On-top, Directly or Reversed

Once counted as stacks, totals may unite next-to or on-top, iconized by a cross showing the two directions. Adding 2 3s and 4 5s next-to as 8s means adding the areas 2*3 and 4*5, called integral calculus where addition follows multiplication. Adding them on-top, first recounting must change the units to the same. This is called proportionality.

Reversed addition asks e.g. '2 3s and how many 5s give 3 8s?'. Here, first the 2*3 stack is pulled away from the 3*8 stack, then recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called reversed integration or differentiation.

MC06: Next-to Addition & Subtraction of Per-numbers and Fractions is Calculus

Throwing a dice 8 times, the outcomes 1 and 6 place 4 cubes on a chess board, and the rest place 2 cubes. When ordered, we may have 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 = 8 squares, but the per-numbers add as stack-numbers, i.e. as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s. This average says that all per-numbers would be 2.6 if alike.

Per-numbers thus add by areas, i.e. by integration. Reversing the question to '2 5s + how many 3s total 3 8s' leads to the equation 2*5 + u*3 = 3*8 solved by differentiation:

2*5 + u*3 = 3*8, so u*3 = 3*8 – 2*5, so u = (3*8 – 2*5)/3 = 4 2/3, or u = ($T2$-$T1$)/3 = ΔT/3

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus creating areas added by integration.

MC07. Double-Counting Sides in a Rectangle Halved by its Diagonal

Two neighboring tiles form a rectangle, that halved by its diagonal creates a right triangle with base b, height h and diagonal d. Recounting pair of sides produces the trigonometry formulas: h = (h/d)*d = sin$A$*d; b = (b/d)*d = cos$A$*d, and h = (h/b)*b = tan$A$*b that allows a (b,h) = (+3,+2) angle to be predicted by tan-1(2/3) to give 33.7 degrees. This again allows predicting the diagonal: h = sin$A$*d, or 2 = sin33.7*d, or d = 2/sin33.7 = 3.60.

MC08. Meeting Pythagoras

Four tiles with base b and diagonal d form a squared tile. Here 4 diagonals form a square containing 4 half-tiles, i.e. 2 tiles. Consequently d*d = 2*b*b, or d^2 = b^2 + b^2.

A tile-pair has base b, height h, and diagonal d. Turned 90 degrees a copy is placed on-top. Repeated three times, this creates a square with the side b+h. Inside we find a diagonal square and four half tile-pairs; as well as a b*b square and a h*h square and two tile-pairs. But 4 half tile-pairs is 2 full tile-pairs, so we see that d*d = b*b + h*h, or d^2 = b^2 + h^2, making it easy the add squares, you just square the diagonal.

The normal from the right angle divides the diagonal in p and q. Seeing b as d*cos$A$, and p as b*cos A, we get b*b = (d*cos$A$)*b = d*(cos$A$*b) = d*p. So, the extension of the normal divides the diagonal square in two parts equal to the squares of the neighboring rectangle side. Since only the angle A is involved this applies to all triangles with angles not above 90 degrees, thus leading to the extended Pythagoras: a^2 = b^2 + c^2 – 2*b*c*cos$A$, etc.

MC09. The Height of an Accessible Flagpole

The point P forms a right triangle with a flagpole of height h. From a point Q in the distance s from P, the vertical distance to the diagonal is k. The angle P may now be found in two ways, tan$P$ = h/b = k/s, allowing the unknown height h to be found by moving to opposite side with opposite sign, h = k/s*b. Solved in a tile-system (coordinate-system), the diagonal is a line passing through three points with the coordinates (0,0), (s,k) and (b,h) providing the slope c = tan$P$ = k/s and the equation y = k/s*x that with x = b gives y = k/s*b.

MC10. The Height of an Inaccessible Flagpole

Three points A, B and C with distances AB = r and BC = s are placed on a line towards the foot of an inaccessible flagpole with height h. With the flagpole, A and B form two right triangles with diagonals $d1$ and d2. The vertical distance is p from $d1$ to B, and q from d2 to C. The distance from C to the foot of the flagpole pole is c. Using tan$A$ = p/r and tan$B$ = q/s, the triangles give two formulas for h: h = (r+s+c)*p/r = (s+c)*q/s. Solved for c, this gives a formula that inserted in the h-formula gives h = p*q*r/(p*r – p*s).

MC11. How High the Moon?

A vertical stick with height h helps finding the position of the moon or sun. If the shadow has the length s, the angle to the sun is predicted by tan$A$ = s/h. A compass helps finding a direction line segment north with the same length as the shadow. The segment between the two has the length a. The angle A then may be predicted by the formula sin(A/2) = ½a/s.

MC12. How Many Turns on a Sloped Tile?

A folding ruler allows creating a right triangle with the bottom line following a sloped tile. A lead line placed in the distance d along the ruler from the bottom line will mark on it a distance b. The slope of the tile is the same as the top angle A, thus predicted by tan$A$ = b/d.

On a 30-degree squared tile, a 10-degree road is constructed. How many turns will there be? We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u. First, in the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2. Next, in the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2. Finally, in the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r, or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37. So, two turns: 3.70 cm and 7.40 cm up the tile.

MC13. Rectangles as Extended Squares

A rectangle with base b and height h = c*b may be called a 'c extended square'. It consists of a lower square, b*b, and an upper rectangle (h-b)*b, becoming a square also if c is 2. The two diagonals $d1$ and d2 are raised the angles $A1$ and A2 that may be predicted by tan$A1$ = h/b = c, and tan$A$2 = (h-b)/b = h/b-1 = c-1. The diagonal $d1$ is predicted by Pythagoras: $d1$^2 = b^2 + h^2 = b^2 + $c^{2*}b$^2 = b^2*(1+c^2), or $d1$ = b*√(1+c^2). The diagonal d2 is predicted by d2 = b*√(1+(1-c)^2). Finally, the normal n to the diagonal $d1$ is predicted by $n*d1 = h*b = c*b^2$, or $n* b*\sqrt{(1+c^2)} = c*b^2$, or $n = b*c/\sqrt{(1+c^2)} = b* 1/\sqrt{(1+1/c^2)}$.

So, combining a tile with half of its neighbor will provide a rectangle as a 1.5 extended square where the diagonal angels and lengths may be predicted by proper formulas: $\tan A1 = 1.5$ giving $A1 = 56.3$, $\tan A1 = 0.5$ giving $A1 = 26.6$. Likewise with the two long diagonals: $d1 = b*\sqrt{(1+c^2)} = b*\sqrt{(1+1.5^2)}= b*1.80$; and $d2 = b*\sqrt{(1+(1-c)^2)}= b*\sqrt{(1+1.5^2)}= b* 1.12$. Finally, the normal: $n = b*1/\sqrt{(1+1/c^2)} = b*1/\sqrt{(1+1/1.5^2)} = b*0.832 = b*\sin A1$.

MC14. Meeting Pi on the Pavement

Two neighboring tiles are circumscribed by a semicircle, again circumscribed by two tiles. On the right tile, the diagonal creates two triangles enveloping a quarter of the semicircle, i.e. 180/4 degrees or pi/4. Consequently, $4*\sin(180/4) < \pi < 4*\tan(180/4)$. In other words, $\pi = n*\sin(180/n) = n*\tan(180/n) = 3.14\ldots$ for n sufficiently large.

MC15. Meeting Algebra

Half of a $b*b$ tile will extend a tile upwards to a $h*b$ playing card. Removing from a $h*h$ square two playing cards, and adding the bottom tile that has been removed twice will leave the square $(h - b)^2 = h^2- 2*h*b + b^2$. And, removing from a $h*b$ playing card the bottom $b*b$ tile will leave the top $(h - b)*b = h*b - b*b$.

Four playing cards are arranged to form a $(h+b)*(h+b)$ square. Inside we find a $h*h$ square, a $b*b$ square and two playing cards, so, $(h+b)*(h+b) = (h+b)^2 = h*h + b*b + 2*h*b$.

Pulling away a $b*b$ tile from the $h*h$ square leaves a $(h-b)*h$ and a $(h-b)*b$ rectangles that add up to a $(h-b)*(h+b)$ rectangle. Consequently, $(h+b)*(h-b) = h^2 - b^2$.

To solve the quadratic equation $x^2+6x+8 = 0$ we use four tiles forming a square, labeling the first side x and the next 6/2. The $(x+6/2)$ square now contains two $6/2*x$ rectangles and two squares, x^2 and $(6/2)^2$ split in two parts, 8 below and $(6/2)^2-8$ above if possible, all disappearing except for last part. So $(x+6/2)^2 = (6/2)^2- 8 = 1$ giving $x = -6/2 \pm 1 = -2$ and -4. Looking instead at $x^2+bx+c = 0$ gives the solution $x = -b/2 \pm \sqrt{((b/2)^2 -c)}$.

MC16. Predicting Change

Two $b*b$ tiles form a $h*b$ playing card that is extended with a tape on-top and to the left to show that a change in h and b, Δh and Δb, will give a change in the area, $\Delta(b*h) = \Delta b*h + b*\Delta h$, or with per-numbers, $\Delta(b*h)/(b*h) = \Delta b/b + \Delta h/h$. Thus, with products, the change-percentages almost just add: Changing a kg-number with 3% and a \$/kg-number with 5% will make the \$-number change with approximately 3% + 5% = 8%. This rule applies to changes less than 10% with decreasing precision. Here we neglect the upper right tape-corner, which is allowed for sufficiently small changes, giving $(b*h)'/(b*h) = b'/b + h'/h$. So, with $y = x^n$ we get that $dy/y = n*dx/x$, or $dy/dx = n*y/x = n*x^n/x = n*x^{(n-1)}$.

MC17. Following Change Formulas e.g. when Playing Golf

Tiles form a coordinate system to move in. Person A starts a (+1,+1) trip in (0,3). Person B starts a (+1,-2) trip in (0,9). Predict where they meet. Person A starts a (+1,+s) trip in (0,0) where s decreases with 1 from +4 to -4. Person B starts a (+1,+s) trip in (10,0) where s increases with 1 from -4 to +4. Person A starts a (+1,+s) trip in (0,½) where s is doubled from ½ the first 4 steps, then halved the next 6 steps. Person B starts a (+1,+s) trip in 0,0 where s decreases with 1 from +3 to -3, then increases with 1 from -3 to +3. Person A and B start a (+1,+s) trip in (0,0) and (2,0) wanting to end closest to a golf hole in (10,0).

MC18. The Saving Formula

A saving combines a deposit amount a with an interest percent r, illustrated by two tiles, K1 and K2. K2 receives a one-time deposit a/r, and each period its interest amount $a/r*r = a$ is transferred to K1 after K1 has received its own interest amount. After n periods, K1 will contain a saving A growing from a deposit amount a and an interest percent r. But, at the same time, K1 will contain the total interest percent R of the initial amount a/r in K2, so $A = a/r*R$, or $A/R = a/r$, where $1+R = (1+r)^n$. Using the doubling-time as the period, a 1$ deposit will after 5 doubling periods save 31$ = 5$ deposit + 26$ compound interest.

MC19: Having Fun with a Tile System and with Bundle-bundle Squares

A dozen recounts as 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed in a tile system, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. With a per-number 2G/3R, a dozen R may be changed to 2G+9R, 4G+6R, 6G+3R, and 8G, thus traveling along a line sloping down with the per-number. With Bundle-Bundle squares we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas: $n*n + 2*n + 1 = (n+1)*(n+1)$; and $n*n – 2*n + 1 = (n-1)*(n-1)$; and $(n–1)*(n+1) = n*n – 1$. A triangle of tiles consists of 1 tile in column 1, 2 in column 2, etc. until column 5. Traveling the triangle with a (+1,+1) win-step or a (+1,+0) loose-step we observe how many roads lead to each tile. Could it be predicted? Start over, but now let a coin decide the next step. Mark the tile with a short stroke. How many times did you win? Could it be predicted?

MC20. Game Theory

Two players A and B choose column and row at 2x2 tiles carrying the numbers 1, 2, 3, 4 in the top and bottom row, indicating what A pays back to B after having received a fix fee from B. Showing paper or stone means choosing the first or second strategy. Thus, if A chooses stone and B paper, A will pay back 2 to B. Which fee makes the game fair? Use cubes to show that the fee is 2.5 if 4 and 2 change places. In the first game, 3 is a stable ‘saddle point’ going up if A changes, and down if B changes, which they don’t want. In the second game both players will be tempted to change, so both will mix strategies, but how?

MC21. Geometry with Handles

Graph theory and topology is geometry where neither distances nor angles matters, but only the relative positions between the points. A classic problem is the supply problem shown with two separated rows with 3 tiles each: How can three houses A, B and C be supplied with electricity, gas and water with no crossing wires? Hint: Connect A and B with gas and water. Conclusion: the task cannot be solved unless we add a bridge whereby the plan changes its topology to a torus which is a plane with a handle.

MC22. The Electric Circuit

Three tiles serve at simulating how 2 cups supply a device labeled (4ohm, 16watt) with energy from an 8volt battery. LEGO-bricks serve as energy-units, and a slow metronome tells when a ‘second’ has passed. From the battery, 8ers are placed in 2 cups that move to the devise to deliver 2*8 units at the time signal, and then move back empty to refill.

In the case of a 4volt battery, 1 cup carries a 4er and delivers 1*4 watt, only 25% of what is needed. In the case of 1ohm, 8cups deliver 8*4 = 32watt. With a 12volt battery supplying first a (3ohm, 48watt) and then a (1ohm, 144watt) device, 3 cups each supply 9 to the first device needing 48, and 3 to the second device needing 144 before returning to refill.

Conclusion

It turns out that many different micro-curricula springs form tiles once you accept the two-dimensional bundle-numbers children develop while adapting to Many before school. Bundle-numbers have units, leading to a recount-formula used to change units, and to solve STEM per-number equations coming from double-counting in different units, thus allowing trigonometry and coordinate geometry to precede plane geometry; and calculus to add per-numbers. Tile-math may be a new Kuhnian paradigm that will finally create a mathematics for all. Are outdoor tile-based micro-curricula scores that makes math dislike evaporate?

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06. Proposals for Papers To the CTRAS 2020 Conference, Invitation to Co-authorship

STEM prevents a goal-displacement that makes mathematics a goal instead of a means 1
Stop teaching wrong numbers and operations, start guiding children's mastery of many 2
To support STEM, trigonometry and coordinate geometry should precede plane geometry 3
A fresh-start year10 (pre)calculus curriculum 4
To support STEM, calculus must teach adding bundle-numbers, per-numbers and fractions also ... 5
Conflicting grand theories create 2x3x2 different mathematics educations 6
Replacing STEAM with STE3M will include also economics and English 7
The power of per-numbers 8
Mixing design and difference research with experiential learning cycles allows creating classroom teaching for all students 9
From place value to bundle-bundles: units, decimals, fractions, negatives, proportionality, equations and calculus in grade one 10
Sociological imagination designs micro-curricula for experiential learning cycles 11
Concrete STEM subjects allow mathematics learning by modeling and peer-brain teaching 12
The simplicity of mathematics designing a stem-based core curriculum for refugee camps 13
Calculation models, fact or fiction 14

STEM PREVENTS A GOAL-DISPLACEMENT THAT MAKES MATHEMATICS A GOAL INSTEAD OF A MEANS

Asking what is the purpose of mathematics education, US and UK mathematics educators say "to learn school mathematics". Others say "to learn set-based mathematics as defined by university mathematics." Focusing on competences leads to saying "to learn mathematical competences" or "to master mathematics". Seldom, if ever, is heard that the goal is "to master many" or "to develop the number-language that children bring to school."

Sociological imagination (Bauman, 1990) may prevent a goal displacement where a means becomes a goal instead. Historically, the Pythagoreans chose the word 'mathematics' meaning 'knowledge' in Greek as a common name for their knowledge about Many in space and time and by itself: astronomy, music, geometry and arithmetic. And today in North America, mathematics is still a common name for geometry and algebra, showing their outside goals in their original meanings, earth-measuring in Greek, and reuniting in Arabic. Integration and differentiation also name their tasks directly, to integrate small changes, and to differentiate a total change in small changes.

To avoid a goal displacement, mathematics must de-model (Tarp, 2019) its core ingredients: digits, operations, equations, fractions, functions etc. to allow primary school develop the flexible bundle-numbers children bring to school by teaching, not numbers to add, but numbering totals by counting, recounting and double-counting, where recounting 8 fingers in 2s as 8 = (8/2)*2 leads directly to the recount-formula $T = (T/B)*B$ with per-numbers that solve equations, that occur in most STEM-formulas typically predicting proportionality, and that become fractions when double-counting in the same unit.

Liberated from its goal displacement, mathematics education may have its own communicative turn as in the 1970s (Widdowson, 1978) such that from now on both the word- and the number-language are taught and learned through their use and not through their grammar, thus allowing all students to model outside quantities as to levels, change and distribution.

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STOP TEACHING WRONG NUMBERS AND OPERATIONS, START GUIDING CHILDREN'S MASTERY OF MANY

Learning means adapting the inside brain to outside nature and culture. Vygotsky prioritizes culture and wiser-brain teaching, Piaget nature and peer-brain learning.

Adapting to Many, children answer the question 'How many?' with bundle-numbers as T = 2 3s containing two digits: 3 is a quantity-number in space, also called a cardinal-number taking on positive integer values; 2 is a counting-number in time taking on also decimal, fractional, and negative values as T = 7 = 2.1 3s = 2 1/3 3s = 3.-2 3s.

Quantity-numbers may add, and so may counting-numbers, but not in between. So, digits must be categorized before adding. Digits are not numbers but operators, needing a multiplier to become a number, T = 2 3s = 2*3, as seen when writing numbers fully as polynomials, as e.g. $T = 345 = 3*B^2 + 4*B + 5*1$

So, teaching digits as numbers is teaching wrong numbers. And bundle-numbers need not to be taught since children bring them to school, that should guide them to develop their number-language by learning that

digits are icons with as many strokes as they represent.

operations are icons also, rooted in the counting process: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

recounting 8 in 2s gives a recount-formula: 8 = (8/2)*2, or $T = (T/B)*B$, used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus solving most STEM-equations, typically predicting proportionality.

Later recounting between digit-units and tens leads to tables, and to equations when asking e.g. T = 4 6s = ? tens, and T = 42 = 4.2 tens = ? 7s.

So, childhood education should guide children develop the quantitative competence they bring to school using Kolb's experiential learning cycles.

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TO SUPPORT STEM, TRIGONOMETRY AND COORDINATE GEOMETRY SHOULD PRECEDE PLANE GEOMETRY

Halved by its diagonal c, a rectangle becomes a right triangle ABC with base b and height h. Using the recount-formula $T = (T/B)*B$ coming from recounting 8 in 2s as 8 = (8/2)*2, mutual recounting gives trigonometry: $h = (h/c)*c = \sin A*c$, $b = (b/c)*c = \cos A*c$, $h = (h/b)*b = \tan A*b$.

Splitting the diagonal in $c1$ and $c2$ by the triangle-height produces two triangles where $\cos A = c1/b = b/c$, making $b\text{^}2 = c*c1$, and $\cos B = c2/h = h/c$, making $h\text{^}2 = c*c2$, thus giving the Pythagoras rule $h\text{^}2 + b\text{^}2 = c\text{^}2$.

Finding √70 means squeezing 7 tens until becoming a square $(8+t)$^2 situated between 8^2 and 9^2. And having four parts as shown by two playing cards placed like an L: 8^2, and $8*t$ twice, and t^2. Neglecting t^2, we get the equation $8*t$ = (70-8^2)/2 = 3 = (3/8)*8, solved by recounting 8 in 3s, giving t = 3/8 = 0.375, so 8.375^2 = 70,14 = 70 approximately.

In a coordinate system, a circle with center in the origin and radius r gets the equation $x\text{^}2 + y\text{^}2 = r\text{^}2$, else $(\Delta x)\text{^}2 + (\Delta y)\text{^}2 = r\text{^}2$. In a horizontal right triangle, moving along the diagonal will change x and y with Δx and Δy. Recounting Δy in Δx gives $\Delta y = (\Delta y/\Delta x)*\Delta x = m*\Delta x = \tan A*\Delta x$ that allows drawing lines from tables.

Intersection points between lines are predicted by a linear equation solved by technology or by moving to opposite side with opposite sign.

Intersection points between lines and circles or parabolas are predicted by a quadratic equation $x\text{^}2 + b*x + c = 0$, solved by two L-placed playing cards showing that $(x+t)$^2 = $x\text{^}2 + 2*x*t + c + (t\text{^}2 – c)$ where the first three terms disappear with $t = b/2$.

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A FRESH-START YEAR10 (PRE)CALCULUS CURRICULUM

Often precalculus suffers from lacking student knowledge. Three options exist: make mathematics non-mandatory, choose an application-based curriculum; or, to rebuild student self-confidence, design a fresh-start curriculum that also includes the core of calculus by presenting integral calculus first.

Writing a number out fully as a polynomial, e.g. $T = 345 = 3*B$^2 + 4*B + 5 shows the four ways to unite numbers, resonating with the Arabic meaning of the word algebra, to reunite: addition and multiplication unite changing and constant unit-numbers into totals; and next-to-block-addition (integration) and power unite changing and constant per-numbers, all having reverse operations that split totals into parts.

Addition, multiplication, and power are defined as counting-on, repeated addition and repeated multiplication. As reverse operations, x = 7-3 is defined as the number that added to 3 gives 7, thus solving the equation x+3 = 7 by moving to opposite side with opposite sign. Likewise, x = 7/3 solves x*3 = 7, the factor-finder (root) x = 3√7 solves x^3 = 7, and the factor-counter (logarithm) x = log3(7) solves 3^x = 7, again moving to opposite side with opposite sign.

Hidden brackets allow reducing a double calculation to a single: 2+3*x = 14 becomes 2+(3*x) = 14, solved by x = (14-2)/3. Next transposing letter-equations as T = a+b*c^d really boost self-pride.

Future behavior of 2set unit-number tables is predicted by linear, exponential, or power models assuming constant change-number, change-percent, or elasticity.

1-4set per-number speed tables are modeled with lines, parabolas and double-parabolas, allowing technology to calculate the distance covered, thus introducing integral calculus, that also occurs when adding per-numbers in mixture-problems, and when adding percent in cross tables generated by statistical questionnaires.

Trigonometry comes from mutual double-counting sides in a rectangle halved by its diagonal, and is used to model distances to far away points, bridges, roads on hillsides, motion down an incline, and jumps from a swing.

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TO SUPPORT STEM, CALCULUS MUST TEACH ADDING BUNDLE-NUMBERS, PER-NUMBERS AND FRACTIONS ALSO

Created to add locally constant per-numbers by their areas, integral calculus normally is the last subject in high school, and only taught to a minority of students. But, since most STEM-formulas express proportionality by means of per-numbers, the question is if integral calculus may be taught earlier. Difference research searching for hidden differences finds that the answer is yes.

Integral calculus occurs in grade one when performing next-to addition of bundle-numbers as e.g. T = 2 3s + 4 5s = ? 8s, leading on to differential calculus as the reverse question: 2 3s + ? 5s = 3 8s, solved by first removing 2 3s from 3 8s and then counting the rest in 5s, thus letting subtraction precede division, where integral calculus does the opposite by letting multiplication creating areas precede addition.

In middle school adding per-numbers by areas occurs in mixture problems: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg, again with differential calculus coming from the reverse question: 2kg at 3$/kg + 4kg at ? $/kg = 6kg at 5 $/kg. Here the per-number graph is piecewise constant c, i.e. there exists a delta-interval so that for all positive epsilon, the distance between y and c is less than epsilon. With like units, per-numbers become fractions thus also added by their areas, and never without units.

In high school adding per-numbers occurs when the meters traveled with varying m/s speed P is found as the area under the per-number graph now being locally constant, formalized by interchanging epsilon and delta. Here the area A under the per-number graph P, is found by slicing the area thinly so that its change may be written as $dA = P*dx$ in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus motivating developing differential calculus to find $A' = dA/dx = P$.

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CONFLICTING GRAND THEORIES CREATE 2X3X2 DIFFERENT MATHEMATICS EDUCATIONS

As part of institutionalized education, mathematics falls under the focus of the three grand theories, philosophy, sociology and psychology, discussing different kinds of mathematics, of education and of learning; and recommending appropriate means to institutional goals. However, is the goal to master mathematics first, as a means to later master many; or to master many directly if mastering mathematics proves difficult?

As to learning, psychology sees coping coming from brains adapting to outside nature and culture, but which is more important? Vygotsky points to culture, mediated by a more knowledgeable wiser-brain, a teacher. Piaget points to nature, automatically creating inside schemata that accommodate if meeting outside resistance from nature or from peer-brain communication.

As to mathematics, philosophy has three conflicting views: Pre-modern mathematics is inspired by the Pythagoreans seeing mathematics as knowledge about Many in space and time and by itself as expressed in astronomy, music, geometry and arithmetic; and as part of the three basic Rs: reading, writing and 'rithmetic called reckoning in Germanic countries. Modern mathematics needs no outside examples for its concepts. Alternatively, postmodern scepticism sees mathematics as a number-language abstracting inside concepts from outside examples, and parallel to the word-language.

As to institutions, sociology recommends imagination to prevent a goal displacement making a means a goal instead. As to education, two conflicting views exits. One sees the student as a servant of the state forcing its population to choose between different multiyear tracks from upper secondary school, and forcing students back to start if changing track. One sees the state as a servant of the student by helping students to uncover and develop their personal talent in self-chosen half-year blocks after puberty. So, two different learning forms, three different mathematics forms, and two different education forms create 2x3x2 different ways of conducting mathematics education.

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REPLACING STEAM WITH STE3M WILL INCLUDE ALSO ECONOMICS AND ENGLISH

STEM integrates mathematics with its roots in science, technology and engineering, all using formulas from algebra and trigonometry to pre-dict the behavior of physical quantities. Statistics post-dicts unpredictable quantities by setting up probabilities for future behavior, using factual or fictitious numbers as median and fractals or average and deviation. Including economics and English in STEM opens the door to statistics also. Art may be an appetizer, but not a main course since geometry and algebra should be always together and never apart to play a core role in STEM.

Macroeconomics describes households and factories exchanging salary for goods on a market in a cycle having sinks and sources: savings and investments controlled by banks and stock markets; tax and public spending on investment, salary and transferals controlled by governments; and import and export controlled by foreign markets experiencing inflation and devaluation. Proportionality and linear formulas may be used as first and second order models for this economic cycle, using regression to set up formulas and spreadsheet for simulations using different parameters.

Microeconomics describes equilibriums in the individual cycles. On a market, shops buy and sell goods with a budget for fixed and variable cost, and with a profit depending on the volume sold and the unit-prices, all leading to linear equations. In the case of two goods, optimizing leads to linear programming. Competition with another shop leads to linear Game Theory. Market supply and demand determines the equilibrium price. Market surveys leads to statistics, as does insurance. In the households, spending comes from balancing income and transferals with saving and tax. In a bank, income come from simple and compound interest, from installment plans as well as risk taking. At a stock market, courses fluctuate. Governments must consider quadratic Laffer-curves describing a negative return of a tax-raise. To avoid units, factories use variations of Cobb-Douglas power elasticity production functions for modeling.

In English sentences may be analyzed on a word level as to the frequency of subjects, verbs, direct and indirect objects, predicates, and unspecified words.

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THE POWER OF PER-NUMBERS

Uniting unit-numbers as 4$ and 5$, or per-numbers as 6$/kg and 7$/kg or 6% and 7%, we observe that addition and multiplication unite changing and constant unit-numbers into a total, and integration and power unite changing and constant per-numbers. Reversely, subtraction and division split a total into changing and constant unit-numbers, and integration and power split a total into changing and constant per-numbers.

Recounting 8 in 2s as 8 = (8/2)*2 creates a recount-formula $T = (T/B)*B$, saying 'From T, T/B times, T may be pushed away'; and used to change units when asking e.g. 2 6s = ? 3s, giving the prediction T = (2*6/3)*3 = 4*3 = 4 3s.

Recounting 8 in 2s also provides the solution $u = 8/2$ to the equations as $u*2 = 8 = (8/2)*2$; thus solving most STEM-equations, since the recount-formula occurs all over. In proportionality, $y = c*x$; in coordinate geometry as line gradients, $\Delta y = \Delta y/\Delta x = c*\Delta x$; in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In science as meter = (meter/second)*second = speed*second, etc. In economics as price formulas: \$ = (\$/kg)*kg = price*kg, \$ = (\$/day)*day = price*day, etc.

With physical units, recounting gives per-numbers bridging the units. Thus 4\$ per 5kg or 4/5 \$/kg gives T = 15kg = (15/5)*5kg = (15/5)*4\$ = 3\$; and T = 16\$ = (16/4)*4\$ = (16/4)*5kg = 20kg. With like units, per-numbers become fractions.

Trigonometry occurs as per-numbers when mutually recounting sides in a rectangle halved by its diagonal, $a = (a/c)*c = \sin A*c$, etc.

Modeling mixtures as 2kg at 3\$/kg + etc, unit-numbers add directly, but per-numbers P add by the area A under the per-number graph, found by slicing it thinly so that the change may be written as dA = P*dx in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus motivating developing differential calculus to find the per-number $A' = dA/dx = P$.

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MIXING DESIGN AND DIFFERENCE RESEARCH WITH EXPERIENTIAL LEARNING CYCLES ALLOWS CREATING CLASSROOM TEACHING FOR ALL STUDENTS

International tests show that not all students benefit from mathematics education. This poor-performance-problem raises a Cinderella question: is there a hidden difference that can make the Prince dance? If so, design research can create Kolb's experiential learning cycles to adapt a given micro-curriculum so that all students may benefit.

In primary school, difference research searching for hidden differences has identified several alternatives: Digits are icons. Numbers are double-numbers with bundles as units, e.g. T = 2 3s. Flexible bundle-numbers have over- and underloads, e.g. T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 tens, and ease operations as e.g. 329/ 7 = 32*B*9/ 7 = 28*B*49 /7 = 4*B*7 = 47, or 23*8 = 2*B*3*8 = 16*B*24 = 18*B*4 = 184.

Operations are icons also where division is a broom pushing away bundles, multiplication a lift stacking bundles, subtraction a rope pulling away stacks to find unbundled, and addition the two ways to unite stacks, on-top and next-to.

Changing units may be predicted by a recount-formula $T = (T/B)*B$ coming from recounting 8 in 2s as 8 = (8/2)*2, or, and used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus solving most STEM-equations, typically predicting proportionality: meter = (meter/sec)*sec = speed*sec.

In middle school, double-counting leads to per-numbers becoming fractions with like units, and adding by their areas as integral calculus. In algebra, factors are units placed outside a bracket. Trigonometry occurs when mutually double-counting sides in a rectangle halved by its diagonal.

In high-school, redefining inverse operations allows equations to be solved by moving to opposite side with opposite sign. And adding per-numbers by areas allows introducing integral calculus before differential calculus.

Designing and redesigning micro-curricula as experiential learning cycles allows teachers perform design research in their own classroom, to be reported as master projects first, and later perhaps as PhD projects including more details.

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FROM PLACE VALUE TO BUNDLE-BUNDLES: UNITS, DECIMALS, FRACTIONS, NEGATIVES, PROPORTIONALITY, EQUATIONS AND CALCULUS IN GRADE ONE

Traditionally, a multi-digit number as 2345 is presented top-down as an example of a place value notation counting ones, tens, hundreds, thousands, etc.; and seldom as four numberings of unbundled, bundles, bundle-bundles, bundle-bundle-bundles, etc., to provide a bottom-up understanding abstracted from concrete examples, which would introduce exponents in primary school as the number of bundle-repetitions. Counting ten fingers in 3s thus introduces bundle-bundles: T = ten = 3*B*1 3s = 1*BB*1 3s.

Stacking bundles, the unbundles singles may be placed as a stack next-to leading to decimals, e.g. T = 7 = 2.1 3s; or on-top of the stack counted as bundles thus leading to fractions, T = 7 = 2 1/3 3s; or to negative numbers counting what is needed for another bundle, T = 7 = 3.-2 3s.

Bundles and negative numbers may also be included in the counting sequence: 0*B*1, 0*B*2, …, 0*B*7, 1*B*-2; 1*B*-1, 1*B*0, 1*B*1, …, 9*B*7, 1*BB*-2, 1*BB*-1, 1*BB*.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s may be written as a recount-formula: 8 = (8/2)*2, or $T = (T/B)*B$, used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus solving most STEM-equations, typically expressing proportionality.

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

At the end of grade one, recounting between digits and tens leads to tables and equations when asking e.g. T = 4 6s = ? tens, and T = 42 = ? 7s.

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SOCIOLOGICAL IMAGINATION DESIGNS MICRO-CURRICULA FOR EXPERIENTIAL LEARNING CYCLES

Forced by peer review to focus on existing research, many education research articles fail to be validated in the classroom by observing if its educational goal is reached. However, the peer review crisis creates a need for a different research meeting its proper genre demands: reliable data and valid findings to a research question.

To help student brains adapt to the outside world, mathematics education must decide if its goal is to master inside mathematics as a means to later master outside quantity, thus risking what sociology calls a goal displacement (Bauman, 1990) where a means becomes a goal instead; or to master quantity directly if first mastering contemporary university mathematics becomes too difficult to many students.

Many curriculum reforms include competences. But again, we must ask: is the goal to obtain inside mathematical competence, or outside quantitative modeling competence?

A learning-by-doing curriculum calls for experiential learning cycles as described by Kolb's learning theory (Kolb, 1983) being adapted e.g. in the new Vietnamese curriculum; and containing cyclic phases. First micro-curriculum A is taught and validated if meeting its expected goals, next systematic observations gather reliable data as to which goals are met, and which are not, then reflections modifies the micro-curriculum into version B. Then plan B is taught, etc.

Combined with design research (Bakker, 2018), experiential learning cycles allows teachers to become action learners or action researchers in their own classroom reporting their work in master or PhD papers. To meet the genre demands of research, the data gathered must be reliable, and the findings must be tested for validity. In design research, reliability comes by making systematic observations through notes, interviews, questionnaires etc. And testing validity here means holding on to the strong parts of the actual micro-curriculum and changing the weak parts.

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CONCRETE STEM SUBJECTS ALLOW MATHEMATICS LEARNING BY MODELING AND PEER-BRAIN TEACHING

Traditionally, mathematics is considered one of the core subjects in education because of the many 'applications of mathematics'. This phrasing leads directly to the view that "of course mathematics must be learned first before it can be applied by others". Consequently, mathematics teaches the operation order addition, subtraction, multiplication and division with cardinal numbers, later expanded to integers, rational and real numbers, again followed by expressions including also unspecified numbers.

Talking instead about outside roots leads to the opposite view that "of course, mathematics must be learned through its outside roots, also constituting its basic applications". This 'de-modeling' view resonates with the fact that historically, the Pythagoreans chose the name mathematics, meaning knowledge in Greeks, as a common label for their four areas of knowledge about Many in time and pace, in time, in space and by itself: astronomy, music, geometry and arithmetic. Later the Arabs added algebra with polynomial numbers created by systematic bundling. Here the outside roots are evident through the original meanings of geometry and algebra: earth-measuring and reuniting.

So, mathematics grew and may still grow from counting, recounting and double-counting bundles, and from applying science, describing forces pumping motion in and out of matter when having the same or opposite directions.

Working in groups with science applications allows students to learn through peer-brain teaching instead of through wiser-brain teaching. As to matter, tasks could be to find its mass, its center, its density, and the heat transfer under collision between visible macro-matter and invisible micro-matter, applied in steam power, or when placing ice-cubes in water.

As to motion, tasks could be to describe traveling with constant or changing speed horizontally or on an incline, vertical motion, projectile orbits; and circular motion, swings or see-saws on a market place. As well as how to use electrons to store or transport motion and information.

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THE SIMPLICITY OF MATHEMATICS DESIGNING A STEM- BASED CORE CURRICULUM FOR REFUGEE CAMPS

Numbers as 2345 evade the place value notation if seen as numbering unbundled, bundles, bundle-bundles, bundle-bundle-bundles. Here exponents occur as the number of bundling-repetitions, e.g. when counting ten fingers as T = ten = 3B1 3s = 1BB1 3s.

Stacking bundles in blocks, the unbundled singles may be placed as a stack next-to leading to decimals, e.g. T = 7 = 2.1 3s; or on-top of the stack counted as bundles leading to fractions, T = 7 = 2 1/3 3s, or to negative numbers counting what is needed for another bundle, T = 7 = 3.-2 3s.

Bundles and negative numbers may be included in the counting sequence: 0B1, 0B2, …, 0B7, 1B-2; 1B-1, 1B0, 1B1, …, 9B7, 1BB-2, 1BB-1, 1BB.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s gives a recount-formula, 8 = (8/2)*2 or T = (T/B)*B, that changes unit from 1s to 2s (proportionality), that gives the equation u*2 = 8 the solution u = 8/2 (moving to opposite side with opposite sign), and that shows that per-numbers as 8/2 must be multiplied to areas before being added (integral calculus).

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

Recounting between digits and tens leads to tables and equations when asking e.g. T = 4 6s = ? tens, and T = 42 = ? 7s. Recounting in different units gives per-numbers bridging the units, becoming fractions with like units, and adding by areas. Mutually recounting sides in a block halved by its diagonal gives trigonometry.

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CALCULATION MODELS, FACT OR FICTION

As qualitative literature, also quantitative literature has the genres fact and fiction when modeling real world situations.

Fact is ‘since-then’ calculations using numbers and formulas to quantify and to predict predictable quantities as e.g. ‘since the base is 4 and the height is 5, then the area of the rectangle is T = 4*5 =

20’. Fact models can be trusted once the numbers and the formulas and the calculation have been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter.

Fiction is ‘if-then’ calculations using numbers and formulas to quantify and to predict unpredictable quantities as e.g. ‘if the unit-price is 4 and we buy 5, then the total cost is T = 4*5 = 20’. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made.

This paper looks at three infection models, the standard logistic model and two alternatives, one formulated as a differential equation, one as a difference equation.

The models all assume that the population change is proportional to the population itself thus giving a doubling factor that is assumed to decrease with the number of infected as the first model assumes, or with time as the two others assume.

However, where the standard model cannot be validated since infection data are difficult or impossible to achieve, reliable data from the number of hospital beds points to the other fiction models. Here, the scientific principle of simplicity, known as Occam's razor, points to the difference equation, easy to set up in a spreadsheet.

It may be simple but it provides important information: its high degree of elasticity recommends a gradual reduction of the two central factors, group size and meeting time, instead of a complete shutdown.

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Remedial Math MicroCurricula – When Stuck in a Traditional Curriculum

Its many applications make mathematics useful; and of course, it must be learned before applied. Or, can it be learned through its original roots? Observing the mastery of Many children bring to school we discover, as an alternative to the present set-based mathe-matics, a Many-based 'Many-matics'. Asking 'How many in total?' we count and recount totals in the same or in a different unit, as well as to and from tens; also, we double-count in two units to create per-numbers, becoming fractions with like units. To predict, we use a recount-formula occurring all over mathematics. Once counted, totals can be added next-to or on-top rooting calculus and proportionality. From this 'Count-before-Adding' curriculum, Many-matics offers remedial micro-curricula for classes stuck in a traditional curriculum.

Background

A city council is challenged by the economical and organizational costs of many teachers seeking different jobs after only a few years of service, especially in the case for math teachers. Seeing a valuable resource lost, the council tries persuading the teachers to stay and give mathematics a second chance. However, most teachers recline arguing that the teaching material present at the school and at conferences and seminars are very much alike.

Consequently, the city council brought their concern to a university professor asking if not other ways to teach primary math was available. The professor witnessed the author's off-Broadway cafeteria poster at the ICME 2016 showing how to cure math dislike with flexible bundle numbers and cup-writing, accepting the three ways children count 5 in 2s, using overload and underload besides the traditional way (Tarp, 2017a)

T = I I I I I = II I I I = 1B3 = II II I = 2B1 = II II II = 3B-1 2s.

Flexible bundle-numbers and cup-writing provides alternatives to tradition calculations:

65 + 27 = 6]5 + 2]7 = 8]12 = 9]2 = 92

65 – 27 = 6]5 – 2]7 = 4]-2 = 3]8 = 38

7x 48 = 7x 4]8 = 28]56 = 33]6 = 336

336 /7 = 33]6 /7 = 28]56 /7 = 4]8 = 4

Inspired by a paper on mastering many (Tarp, 2018b), the professor asked the author to design a list of remedial math micro-curricula meant for teachers stuck in traditional curricula allowing them to see unnoticed alternatives. Furthermore, the professor wanted the suggestion to have the form of a scientific paper, so that it may be published in a journal. This paper then is the reaction to the demand.

Introduction

Research in mathematics education has grown since the first International Congress on Mathematics Education, ICME, in 1969. Likewise, funding has increased as seen e.g. by the creation of a Swedish center for Mathematics Education. Yet, despite increased research and funding, decreasing Swedish PISA result caused OECD (2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change'. Since mathematics education is a social institution, social theory may give a clue to the lacking success and how to improve schools in Sweden and elsewhere.

Social Theory Looking at Mathematics Education

Imagination as the core of sociology is described by Mills (1959). Bauman (1990) agrees by saying that sociological thinking 'renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now' (p. 16).

Mathematics education is an example of 'rational action (..) in which the end is clearly spelled out, and the actors concentrate their thoughts and efforts on selecting such means to the end as promise to be most effective and economical (p. 79)'. However

The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right. (p. 84)

One such goal displacement is saying that the goal of mathematics education is to learn mathematics since, by its self-reference, such a goal statement is meaningless. So, if mathematics isn't the goal of mathematics education, what is? And, how well defined is mathematics after all?

Mathematics, Before and Now

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many by itself, Many in space, Many in time and Many in time and space. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here the invention of the concept SET created a Set-based 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts. However, 'well-defined' meant defining by self-reference, i.e. defining top-down as examples of abstractions instead of bottom-up as abstractions from examples. And by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false' being false if true and true if false:

If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, SET changed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meet counter-examples as e.g. 2weeks + 3days is 17 days; in contrast to '2x3 = 6' stating that 2 3s can always be re-counted as 6 1s.

Difference Research Looking at Mathematics Education

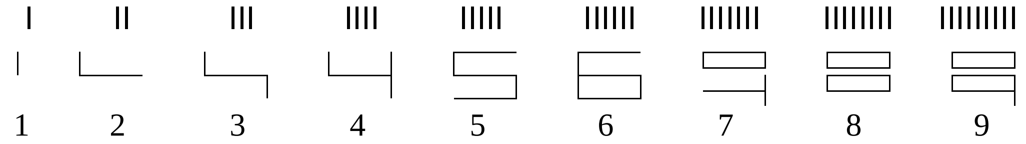
Inspired by the ancient Greek sophists (Russell, 1945), wanting to avoid being patronized by choices presented as nature, 'Difference-research' (Tarp, 2017b) is searching for hidden differences making a difference. So, to avoid a goal displacement in mathematics education, difference-research asks: How will mathematics look like if grounded in its outside root, Many?

To answer we allow Many to open itself for us, so that, as curriculum architects, sociological imagination may allow us to construct a list of remedial micro-curricula for classes stuck in a traditional mathematics curriculum. So, we now return to the original Greek meaning of mathematics as knowledge about Many by itself and in time and space; and use Grounded Theory (Glaser & Strauss, 1967), lifting Piagetian knowledge acquisition (Piaget, 1969) from a personal to a social level, to allow Many create its own categories and properties.

Meeting Many Creates a 'Count-before-Adding' Curriculum

Meeting Many, we ask 'How many in Total?' To answer, we total by counting and adding to create number-language sentences, T = 2 3s, containing a subject and a verb and a predicate as in a word-language sentence (Tarp, 2018b).

Rearranging many 1s in 1 icon with as many strokes as it represents (four strokes in the 4-con, five in the 5-icon, etc.) creates icons to be used as units when counting:

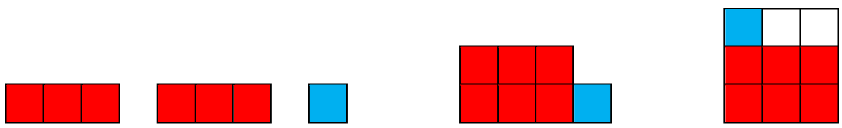


We count in bundles to be stacked as bundle-numbers or block-numbers, which can be re-counted and double-counted and processed by on-top and next-to addition, direct or reversed.

To count a total T, we take away bundles B (thus rooting and iconizing division as a broom wiping away the bundles) to be stacked (thus rooting and iconizing multiplication as a lift stacking the bundles into a block) to be moved away to look for unbundled singles (thus rooting and iconizing subtraction as a trace left when dragging the block away). A calculator predicts the result by a re-count formula T = (T/B)*B saying that ‘from T, T/B times, B can be taken away’: 7/3 gives 2.some, and 7 – 2x3 gives 1, so T = 7 = 2B1 3s.

7 / 3	2.some
7 – 2 x 3	1

Placing the singles next-to or on-top of the stack counted as 3s, roots decimals and fractions to describe the singles: T = 7 = 2.1 3s = 2 1/3 3s



T = 7 = 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s

A total counted in icons can be re-counted in tens, which roots multiplication tables; or a total counted in tens can be re-counted in icons, T = 42 = ? 7s, which roots equations.

Double-counting in physical units roots proportionality by per-numbers as 3$/4kg bridging the units. Per-numbers become fractions if the units are the same. Since per-numbers and fractions are not numbers but operators needing a number to become a number, they add by their areas, thus rooting integral calculus.

Once counted, totals can be added on-top after being re-counted in the same unit, thus rooting proportionality; or next-to as areas, thus rooting integral calculus. And both on-top and next-to addition can be reversed, thus rooting equations and differential calculus:

2 3s + ? 4s = 5 7s gives differentiation as: ? = (5*7 – 2*3)/4 = ΔT/4

In a rectangle halved by a diagonal, mutual re-counting of the sides creates the per-numbers sine, cosine and tangent. Traveling in a coordinate system, distances add directly when parallel; and by their squares when perpendicular. Re-counting the y-change in the x-change creates change formulas, algebraically predicting geometrical intersection points, thus observing the ‘geometry & algebra, always together, never apart’ principle.

Predictable change roots pre-calculus (if constant) and calculus (if variable). Unpredictable change roots statistics to ‘post-dict’ numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for numbers we else cannot pre-dict.

A typical mathematics curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling.

Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1B14 tens is not allowed. Adding without units always precede bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, let us use difference research and imagination to uncover or invent remedial micro-curricula for classes stuck in the tradition.

Remedial micro-curricula for classes stuck in the tradition

01. A preschool or year 1 class is stuck with the traditional introduction of one-dimensional line-numbers and addition without counting. Here a difference is to use two-dimensional block-numbers and bundle-counting, recounting in the same and in a different unit, and calculator prediction before next-to and on-top addition using LEGO-bricks and a ten-by-ten abacus. Teaching counting before adding and next-to addition before on-top addition allows learning core mathematics as proportionality and integral calculus in early childhood.

02. A class is stuck in addition. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create or remove an over- or an underload. Thus T = 65 + 27 = 6B5 + 2B7 = 8B12 = 8+1B12-10 = 9B2 = 92.

03. A class is stuck in subtraction. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 65–27 = 6B5 – 2B7 = 4B-2 = 4-1B-2+10 = 3B8 = 38.

04. A class is stuck in multiplication. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to recount in the same unit to create/remove an over/under-load. Thus T = 7* 48 = 7* 4B8 = 28B56 = 28+5B56-50 = 33B6 = 336.

05. A class is stuck in multiplication tables. Here a difference is to see multiplication as a geometrical stack that recounted in tens will increase its width and therefore decrease its height to keep the total unchanged. Thus T = 3*7 means that the total is 3 7s that may or may not be recounted in tens as T = 2.1 tens = 21 if leaving out the unit and misplacing the decimal point.

Another difference is to reduce the full ten-by-ten table to a small 2-by-2 table containing doubling, since 4 is doubling twice, 5 is half of ten, 6 is 5&1 or 10 less 4, 7 is 5&2 or 10 less 3 etc. Thus T = 2*7 = 2 7s = 2*(5&2) = 10&4 = 14, or 2*(10-3) = 20 – 6 = 14; and T = 3*7 = 3 7s = 3*(5&2) = 15&6 = 21, or 3*(10-3) = 30 – 9 = 21; T = 6*9 = (5+1) * (10-1) = 50 – 5 +10 – 1 = 54, or (10-4)*(10-1) = 100 – 10 – 40 + 4 = 54. These results generalize to a*(b – c) = a*b – a*c and vice versa; and (a – d)*(b – c) = a*b – a*c – b*d + d*c.

06. A class is stuck in short division. Here a difference is to Here a difference is to talk about 8/2 as '8 counted in 2s' instead of as '8 divided between 2'; and to rewrite the number as '10 or 5 times less something' and use the results from the small 3-by-3 multiplication table. Thus T = 28 /7 = (35-7) /7 = (5-1) = 4; and T = 57 /7 = (70-14+1)/7 = 10-2+1/7 = 8 1/7. This result generalizes to (b – c)/a = b/a – c/a, and vice versa.

07. A class is stuck in long division. Here a difference is to rename numbers using bundle names, e.g. sixty-five as 6ten5, together with bundle-writing, and to introduce recounting in the same unit to create/remove an over/under-load. Thus T = 336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48.

08. A class is stuck in ratios and fractions greater than one. Here a difference is stock market simulations using dices to show the value of a stock can be both 2 per 3 and 3 per 2; and to show that a gain must be split in the ratio 2 per 5 if you owe two parts of the stock.

09. A class is stuck in fractions. Here a difference is to see a fraction as a per-number and to recount the total in the size of the denominator. Thus 2/3 of 12 is seen as 2 per 3 of 12 that can be recounted

in 3s as 12 = (12/3)*3 = 4*3 meaning that we get 2 4 times, i.e. 8 of the 12. The same technique may be used for shortening or enlarging fractions by inserting or removing the same unit above and below the fraction line: T = 2/3 = 2 4s/ 3 4s = (2*4)/(3*4) = 8/12; and T = 8/12 = 4 2s/ 6 2s = 4/6

10. A class is stuck in adding fractions. Here a difference is to stop adding fractions since this is an example of 'mathe-matism' true inside but seldom outside classrooms. Thus 1 red of 2 apples plus 2 red of 3 apples total 3 red of 5 apples and not 7 red of 6 apples as mathe-matism teaches. The fact is that all numbers have units, fractions also. By itself a fraction is an operator needing a number to become a number. The difference is to teach double-counting leading to per-numbers, that are added by their areas when letting algebra and geometry go hand in hand. In this way, the fraction 2/3 becomes just another name for the per-number 2 per 3; and adding fractions as the area under a piecewise constant per-number graph becomes 'middle school integration' later to be generalized to high school integration finding the area under a locally constant per-number graph.

11. A class is stuck in algebraic fractions. Here a difference is to observe that factorizing an expression means finding a common unit to move outside the bracket: T = (a*c + b*c) = (a+b)*c = (a+b) cs.

12. A class stuck in proportionality can find the $-number for 12kg at a price of 2$/3kg but cannot find the kg-number for 16$. Here a difference is to see the price as a per-number 2$ per 3kg bridging the units by recounting the actual number in the corresponding number in the per-number. Thus 16$ recounts in 2s as T = 16$ = (16/2)*2$ = (16/2)*3kg = 24 kg. Likewise, 12kg recounts in 3s as T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$.

13. A class is stuck in equations as 2+3*u = 14 and 25 – u = 14 and 40/u = 5, i.e. that are composite or with a reverse sign in front of the unknown. Here a difference is to use the basic definitions of reverse operations to establish the basic rule for solving equations 'move to the opposite side with the opposite sign': In the equation u+3 = 8 we seek a number u that added to 3 gives 8, which per definition is u = 8 – 3. Likewise with u*2 = 8 and u = 8/2; and with u^3 = 12 and u = 3√12; and with 3^u = 12 and u = log3(12). Another difference is to see 2+3*u as a double calculation that can be reduced to a single calculation by bracketing the stronger operation so that 2+3*u becomes 2+(3*u). Now 2 moves to the opposite side with the opposite sign since the u-bracket doesn't have a reverse sign. This gives 3*u = 14 – 2. Since u doesn't have a reverse sign, 3 moves to the other side where a bracket tells that this must be calculated first: u = (14-2)/3 = 12/3 = 4. A test confirms that u = 4: 2+3*u = 2+3*4 = 2+(3*4) = 2 + 12 = 14. With 25 – u = 14, u moves to the other side to have its reverse sign changed so that now 14 can be moved: 25 = 14 + u; 25 – 14 = u; 11 = u. Likewise with 40/u = 5: 40 = 5*u; 40/5 = u; 8 = u. Pure letter-formulas build routine as e.g. 'transform the formula T = a/(b-c) so that all letters become subjects.' A hymn can be created: "Equations are the best we know / they're solved by isolation. / But first the bracket must be placed / around multiplication. / We change the sign and take away / and only x itself will stay. / We just keep on moving, we never give up / so feed us equations, we don't want to stop."

14. A class is stuck in classical geometry. Here a difference is to replace it by the original meaning of geometry, to measure earth, which is done by dividing the earth into triangles, that divide into right triangles, seen as half of a rectangle with width w and height h and diagonal d. The Pythagorean theorem, w^2 + h^2 = d^2, comes from placing four playing cards after each other with a quarter turn counter-clockwise; now the areas w^2 and h^2 is the full area less two cards, which is the same as the area d^2 being the full area less 4 half cards. In a 3 by 4 rectangle, the diagonal angles are renamed a 3per4 angle and a 4per3 angle. The degree-size can be found by the tan-bottom on a calculator. Here algebra and geometry go hand in hand with algebra predicting what happens when a triangle is constructed. To demonstrate the power of prediction, algebra and geometry should always go hand in hand by introducing classical geometry together with algebra coordinated in Cartesian coordinate geometry.

15. A class is stuck in stochastics. Here a difference is to introduce the three different forms of change: constant change, predictable change, and unpredictable or stochastic change. Unable to 'pre-dict' a number, instead statistics can 'post-dict' its previous behavior. This allows predicting an interval that

will contain about 95% of future numbers; and that is found as the mean plus/minus twice the deviation, both fictitious numbers telling what the level- and spread-numbers would have been had they all been constant. As factual descriptors, the 3 quartiles give the maximal number of the lowest 25%, 50% and 75% of the numbers respectively. The stochastic behavior of n repetitions of a game with winning probability p is illustrated by the Pascal triangle showing that although winning n*p times has the highest probability, the probability of not winning n*p times is even higher.

16. A class is stuck in the quadradic equation x^2 + b*x + c = 0. Here a difference is to let algebra and geometry go hand in hand and place two m-by-x playing cards on top of each other with the bottom left corner at the same place and the top card turned a quarter clockwise. With k = m-x, this creates 4 areas combining to (x + k)^2 = x^2 + 2*k*x + k^2. With k = b/2 this becomes (x + b/2)^2 = x^2 + b*x + (b/2)^2 + c – c = (b/2)^2 – c since x^2 + b*x + c = 0. Consequently the solution is x = -b/2 ±√((b/2)^2 – c).

17. A class is stuck in functions having problems with its abstract definition as a set-relation where first component identity implies second component identity. Here a difference is to see a function f(x) as a placeholder for an unspecified formula f containing an unspecified number x, i.e. a standby-calculation awaiting the specification of x; and to stop writing f(2) since 2 is not an unspecified number.

18. A class is stuck in elementary functions as linear, quadratic and exponential functions. Here a difference is to use the basic formula for a three-digit number, T = a*x^2 + b*x + c, where x is the bundle size, typically ten. Besides being a quadratic formula, this general number formula contains several special cases: proportionality T = b*x, linearity (affinity, strictly speaking) T = b*x+c, and exponential and power functions, T = a*k^x and T = a*x^k. It turns out they all describe constant change: proportionality and linear functions describe change by a constant number, a quadratic function describes change by a constant changing number, an exponential function describes change with a constant percentage, and a power function describes change with a constant elasticity.

19. A class is stuck in roots and logarithms. With the 5th root of 20 defined as the solution to the equation x^5 = 20, a difference is to rename a root as a factor-finder finding the factor that 5 times gives 20. With the base3-log of 20 defined as the solution to the equation 3^x = 20, a difference is to rename logarithm as a factor-counter counting the numbers of 3-factors that give 20.

20. A class is stuck in differential calculus. Here a difference is to postpone it because as the reverse operation to integration this should be taught first. In Arabic, algebra means to reunite, and written out fully, T = 345 = 3*B^2 + 4*B + 5*1 with B = ten, we see the four ways to unite: Addition and multiplication unite variable and constant unit numbers, and integration and power unite variable and constant per-numbers. And teaching addition and multiplication and power before their reverse operations means teaching uniting before splitting, so also integration should be taught before its reverse operation, differentiation.

21. A class is stuck in the epsilon-delta definition of continuity and differentiability. Here a difference is to rename them ‘local constancy’ and ‘local linearity’. As to the three forms constancy, y is globally constant c if for all positive numbers epsilon, the difference between y and c is less than epsilon. And y is piecewise constant c if an interval-width delta exists such that for all positive numbers epsilon, the difference between y and c is less than epsilon in this interval. Finally, y is locally constant c if for all positive numbers epsilon, an interval-width delta exists such that the difference between y and c is less than epsilon in this interval. Likewise, the change ratio □y/□x can be globally, piecewise or locally constant, in which case it is written as dy/dx.

22. A class of special need students is stuck in traditional mathematics for low achieving, low attaining or low performing students diagnosed with some degree of dyscalculia. Here a difference is to accept the two-dimensional block-numbers children bring to school as the basis for developing the children’s own number-language. First the students use a folding ruler to see that digits are not symbols but icons containing as many sticks as they represent. Then they use a calculator to predict

the result of recounting a total in the same unit to create or remove under- or overloads; and also to predict the result of recounting to and from a different unit that can be an icon or ten; and of adding both on-top and next-to thus learning proportionality and integration way before their classmates, so they can return to class as experts.

23. A class of migrants knows neither letters nor digits. Her a difference is to integrate the word- and the number-language in a language house with two levels, a language describing the world and a meta-language describing the language. Then the same curriculum is used as for special need students. Free from learning New Math's meta-matics and mathe-matism seeing fractions as numbers that can be added without units, young migrants can learn core mathematics in one year and then become STEM teachers or technical engineers in a three-year course.

24. A class of primary school teachers expected to teach both the word- and the number-language is stuck because of a traumatic prehistory with mathematics. Here a difference is to excuse that what was called mathematics was instead 'meta-matism', a mixture of meta-matics presenting concepts from above as examples of abstractions instead of from below as abstractions from examples as they arose historically; and mathe-matism, true inside but seldom outside a classroom as adding without units. Instead, as a grammar of the number language, mathematics should be postponed since teaching grammar before language creates traumas. So, the job in early childhood education is to integrate the word- and the number-language with their 2x2 basic questions: 'What is this? What does it do?'; and 'How many in total? How many if we change the unit?'

25. In an in-service education class, a group of teachers are stuck in how to make mathematics more relevant to students and how to include special need students. The abundance of material just seems to be more of the same, so the group is looking for a completely different way to introduce and work with mathematics. Here a difference is to go to the MATHeCADEMY.net, teaching teachers to teach MatheMatics as ManyMatics, a natural science about Many, and watch some of its YouTube videos. Then to try out the 'FREE 1day SKYPE Teacher Seminar: Cure Math Dislike' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows before the coffee break.

To learn more, the group can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e. both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Inspirational purposes have led to the creation of several DrAlTarp YouKu.com, SoKu.com videos, and MrAlTarp YouTube videos: Deconstructing Fractions, Deconstructing Calculus, Deconstructing PreCalculus Mathematics, Missing Links in Primary Mathematics, Missing Links in Secondary Mathematics, Postmodern Mathematics, PreSchool Math.

Conclusion

For centuries, mathematics was in close contact with its roots, the physical fact Many. Then New Math came along claiming that it could be taught and researched as a self-referring meta-matics with no need for outside roots. So, one alternative presents itself directly for future studies creating a paradigm shift (Kuhn, 1962): to return to the original meaning of mathematics as many-matics grounded as a natural science about the physical fact Many; and to question existing theory by using curriculum architecture to invent or discover hidden differences, and by using intervention research to see if the difference makes a difference.

In short, to be successful, mathematics education research must stop explaining the misery coming from teaching meta-matism. Instead, mathematics must respect its origin as a mere name for algebra and geometry, both grounded in Many. And research should search for differences and test if they make a difference. Then learning the word-language and the number-language together may not be that difficult, so that all leave school literate and numerate and use the two languages to discuss how to treat nature and its human population in a civilized way.

Inspired by Heidegger, an existentialist would say: In a is-sentence, trust the subject since it exists, but doubt the predicate, it is a verdict that might be gossip. So, maybe we should stop teaching essence and instead start letting learners meet and experience existence.

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Deconstructed, Calculus Rises in Three Versions for Primary, Middle and High School

Extended Abstract

"Calculus is hard - and therefore only for the few, and only late in high school".

This tradition has established itself worldwide (Rasmussen et al, 2014; Bressoud et al, 2016). Which motivates the core sociological question "Is calculus hard by nature, or by choice?" (Bauman, 1990).

Postmodern deconstruction may provide an answer by using 'Difference Research' (Tarp, 2018) searching for hidden differences that may make a difference.

Thus, to deconstruct calculus education means asking the philosophical questions "Which ontological fact lies behind the epistemological construct calculus? How does this fact present itself phenomenologically to the user? And what psychological consequences does this have for making calculus a tool for all students?"

Algebraically, Hindu-Arabic numbers as 456 = 4**B*^2 + 5**B* + 6*1 are sums of monomials, i.e. a sum of areas geometrically. They show that there are four ways to unite numbers: number-addition, multiplication, power, and area-addition; and that only monomials add, thus making simple numbers operators, that may be multiplied, but needing a unit-factor to add.

In this way, area-addition roots primary school calculus if allowing children to add what they bring to school, bundle-numbers or area-numbers as 2 3s. Here, adding two areas when asking "2 3s + 4 5s = ? 8s" roots integral calculus, that reversed roots differential calculus when asking "2 3s + ? 5s total 3 8s", answered by removing the 2*3 area before counting the rest in 5s, i.e. by letting subtraction precede division:

(3*8 – 2*3)/5 = ΔArea/5.

Later, in middle school, double-counting a quantity in two different units leads to per-numbers becoming fractions with like units: 2$/5kg, and 2$/5$ = 2/5 = 40/100 = 40%.

As with simple numbers, also per-numbers and fractions are operators to be multiplied to areas before adding, now as the area under a per-number graph; and again, rooting differential calculus when reversed.

Examples: "2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg"; and "1/2 of 30$ + 4/5 of 60$ = ? of 90$".

An *a* by *b* area shows the geometrical meaning of per-numbers. Multiplied, *a*b* gives the area. Divided, their per-number *b/a* gives the diagonal's turning angle, tan*A* = *b/a*, as well as its steepness, Δ*y*/Δ*x* = *b/a*.

Another example of adding per-numbers is speed: "2sec at 3m/sec + 4sec at 5m/sec = 6sec at ? m/sec".

Observing that a ball falls with accelerated speed makes high school physics formulate speed-questions as "5sec at 0m/sec increasing gradually to 50m/sec totals how many meters?"

Here, the speed per-number, *s*, is not piecewise, but locally constant. So again, the answer is the area under the per-number *s*-graph, now split into many small slices to add up. But how to add many small numbers? Very simple, if the numbers may be written as differences making all middle terms cancel out and leaving only the difference between the terminal and initial numbers.

So, finding the area under a locally constant *s*-graph roots differentiation as a difference-equation: Find a formula *F* so that the slice *s**Δ*x* can be written as a difference Δ*F*: *s**Δ*x* = Δ*F*, or *s* = Δ*F*/Δ*x*? Alternatively, we may say that the area always changes with the last slice, ΔArea = *s**Δ*x*, or *s* = Δ*A*/Δ*x*. So, also high school differentiation lets subtraction precede division as in primary and middle school calculus.

Again, the area gives the answer. In an x by x area, a small change in the sides, dx, will make the area change with two slices x*dx, and a very small corner dx^2, so d(x^2) ≈ 2x*dx, or d/dx(x^2) = 2x. Likewise, in an x by x^2 area, d/dx(x^3) = 3x^2.

Once created to find the area under a per-number graph, differential calculus may also be applied for optimization purposes since the per-number gives the steepness of the diagonal in the dy by dx area.

We can formalize the difference between piecewise and locally constancy by observing that a variable y is globally constant c if their difference is less than any positive number epsilon.

Likewise, y is piecewise constant c if an interval delta exists so that | y-c | < epsilon in delta.

Finally, y is locally constant c if for any epsilon an interval delta exists so that | y-c | < epsilon in delta. So, to go from piecewise to locally constant, we just interchange epsilon and delta.

Another traditional learning obstacle is that calculus builds on pre-calculus, seen by many teachers as the most difficult course to teach since here all lacking student knowledge surface.

However, here a difference making a difference is the ‘Algebra Square’ showing the four ways to reunite and split-into constant and changing unit-numbers and per-numbers: Addition/subtraction unites/split-into changing unit-numbers; and multiplication/division unites/split-into constant unit-numbers. Whereas power/root&log unites/split-into constant per-numbers; and integration/differentiation unites/split-into changing per-numbers (Tarp, 2012).

Thus, where the focus of calculus is to unite and split-into changing per-numbers, the focus of pre-calculus is to unite and split-into constant per-numbers.

Or, rephrased from a change-perspective we may say that where pre-calculus is about constant change, calculus is about changing predictable change, and statistics is about changing unpre- but post-dictable change.

Reversing the exponential formula for constant change-percentage, $y = b*a$^x, equations as and 5 = 2^x and 5 = x^3 root a logarithm as a factor-counter, and a root as a factor-finder.

Consequently, calculus is not hard by nature, but by choosing to neglect its three outside ontological roots: reuniting and splitting-into bundle numbers and piecewise or locally constant per-numbers. And to neglect that only numbers with units, i.e. monomials, add meaningfully. A fact that is not accepted until calculus.

These roots present themselves in a natural way to the learners having therefore little problems learning to do calculus if allowed to meet existence before essence by choosing a Piagetian over a Vygotskian approach.

Thus, in education, primary and middle school calculus as well as presenting integral before differential calculus in high school remains to be tested and thoroughly researched e.g. experientially (Kolb, 1983) using design research (Bakker, 2018).

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Towards Truth, Beauty and Goodness in Mathematics Education

Purpose. What May be Different in Math Education

Pausing for a moment at the entry to the third decennium of the 21. century we may ask, what can and what cannot be different in mathematics education, leading on to the question: how will a civilized mathematics education for the rest of the century look like? To get an answer, we may rephrase the question from behind: how to make mathematics education comply with the three classical virtues, truth and beauty and goodness.

To use a suitable methodology, difference-research (Tarp, 2018) recommends looking for hidden differences, e.g. from the viewpoint of the three grand theories, philosophy and sociology and psychology, typically absent in research except for the latter. Here philosophy and psychology may inform the discussion on truth and beauty, and sociology may address the question on goodness.

What we find is a third way to mastering Many respecting the quantitative competence and mastery of Many, children develop before coming to school. This alternative to the classical and set-based version may fulfill the dream of mathematics education for all students, once developed and tested.

Research Design. Grand Theory Looks at Mathematics Education

Within philosophy, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that 'existence precedes essence', exemplified by the Heidegger-warning: In a sentence, trust the subject, it exists; doubt the predicate, it is essence coming from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber (1930) was the first to theorize the increasing goal-oriented rationalization that de-enchant the world and create an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking "renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now" (p. 16). But he also formulates a warning (p. 84): "The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right". Which may lead to 'the banality of evil' (Arendt, 1963).

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologycally, how we perceive and verbalize it. University mathematics holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB + 4*B + 5*1, resonating with what children bring to school, e.g. T = 2 5s.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? University mathematics holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian psychology.

Sociologically, a Weberian viewpoint would ask if SET is a totalitarian rationalization of Many gone too far leaving Many de-enchanted and the learners in an iron cage. A Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, university mathematics has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word 'mathematics' must go from goal descriptions. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a 'pris-pital', a Foucaultian viewpoint, would ask, first which structure to choose, European line-organization forcing a return to the same cell after each hour, day and month for several years; or the North American block-organization changing cell each hour, and changing the daily schedule twice a year? Next, as prisoners of a 'the goal of math education is to learn math' discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g. by examining and developing the existing mastery children bring to school?

Findings. Meeting Many, Children use Bundle-Numbers to Count with Units

How children master Many can be seen at preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists, and with units: bundles of 2s, and 2 of them.

Children also use bundle-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 extra'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 4 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 4s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as T = 7 = 1B5 = 2B3 = 3B1 = 4B1; or even as 1BB3 or 1BB1B1. Also, children love to count in 3s, 4s, and in hands.

Children thus show an unnoticed quantitative competence before they come to school. So why not develop instead of rejecting the core mastery of Many that children bring to school?

Conclusion

We asked: how to make mathematics education comply with the three classical virtues, truth and beauty and goodness. Using sociological imagination (Mills, 1959) on pre-school observation we can suggest that the following differences may be worthwhile testing.

It is truth and beauty:

To reserve names for what exists. And since the end-goal of mathematics education is to master many, many should be present in each statement as the total T, thus using full sentences with a subject, a verb and a predicate in both the word-language and the number-language.

To transform a collection of sticks into an icon for a digit with five sticks in the 5- icon etc.

To use the word bundling when using a counting sequence to answer the question 'How many?' to find the degree of many in a given total: 0B1, 0B2, …, 0B9, 1B0, 1B1, etc.

To use other bundle-sizes than ten to meet bundle-of bundles e.g. when counting ten fingers in 3s as 0B1, 0B2, 0B3 or 1B0, 1B1, …2B2, 2B3 or 3B0 or 1BB0B0, 1BB0B1. Thus, allowing ten and hundred and thousand to be called bundle and bundle of bundles and bundle-of-bundles-of-bundles.

To show that the unbundled may be accounted for in three different ways using a decimal, a fraction or a negative number as in T = 7 = 2.1 3s = 2 1/3 3s = 3.-2 3s.

To see a total as a flexible rectangular tile with a height that increases if squeezed, and decreases if expanded to hold the same total. And to see a total transformed into a square thus showing approximate values of a square root.

To see the operation 'push-away' named division iconized as a broom brushing away bundles; the operation 'stack bundles' named multiplication iconized as a lift; the operation 'pull-away' a stack named subtraction iconized as a rope; the operation 'uniting stacks' named on-top or next-to addition iconized as a cross showing the 2 directions; and the operation of bundling bundles iconized as a hat.

To see the process of (re)counting a total of 8 in 2s written as T = 8 = (8/2)*2 that, with letters for unspecified numbers, becomes a recount formula, T = (T/B)*B, also called a proportionality formula present all over mathematics and science, and able to let a calculator predict the result of a recounting before performing it.

To see integral calculus coming from next-to addition of two stacks asking e.g. 2 3s + 4 5s = ? 8s, thus letting multiplication precede addition; and to see differential calculus coming from reversing next-to addition of two stacks thus letting subtraction precede division. And to include units when adding.

To see that double-counting creates per-numbers as 3$/4kg allowing problems to be solved by recounting in the per-number: 20kg = ?$; 20kg = (20/4)*4kg = (20/4)*3$ = 15$.

To see the algebra square showing the four ways to 2x2 numbers, constant and changing unit- and per-numbers, together with the four ways to unite them, addition and multiplications and power and integration, as well as the five ways to split them, subtraction and division and root and logarithm and differentiation. And thus, to see the problem of backward or reversed calculation formulated as an equation with the answer predicted by the reverse operations.

To use a tile divided with horizontal and vertical rubber bands to multiply multi-digit numbers as well as algebraic expressions.

To see the quadratic equation solved by two identical tiles turned a quatre round.

To see the Pythagorean theorem coming from rearranging 4 identical tiles.

To see the multiplication formula of differential calculus appearing as the shadows of a tile.

To see the different change formulas as special examples of the number formula T = a*x^2+b*x+c.

To use two tiles to show the most beautiful formula in mathematics: when saving, the ratio between the final and single $-and %-input are identical.

To treat integral before differential calculus, using that if adding differences, middle terms disappear.

It is goodness:

To accept and develop the flexible-bundle numbers children bring to school.

To allow totals to be present simultaneously in time and in space as bundle-numbers and tiles.

To postpone addition to after division, multiplication and subtraction has changed a pile into a total.

To allow calculus in primary school as direct and reversed next-to addition of two bundle-numbers; and in middle school as adding per-numbers asking 2kg at 3$/kg + 4kg at 5$/kg total 6kg at ?$/kg.

To treat fractions as, not numbers, but operators needing numbers to become numbers, i.e. as per-numbers with like units.

To parallel technical words with rooted words: function/formula, root/factor-finder, logarithm/factor-counter, divided by/counted in, continuous/locally constant, differentiable/locally linear, equation/reversed calculation, fraction/per-number, etc.

Originality and Recommendation

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting ‘(ten and) 1 left’, ‘(ten and) 2 left’. Never children are asked to use full number-language sentences, T = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as

2.-3 tens and 24 as 1B14 tens is not allowed. Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many should be an overall goal of a curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

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A Refugee Camp Math Curriculum

The name 'refugee camp curriculum' is a metaphor for a situation where mathematics is taught from the beginning and with simple manipulatives. Thus, it is also a proposal for a curriculum for early childhood education, for adult education, for educating immigrants and for learning mathematics outside institutionalized education. It considers mathematics a number-language parallel to our word-language, both describing the outside world in full sentences, typically containing a subject and a verb and a predicate. The task of the number-language is to describe the natural fact Many in space and time, first by counting and recounting and double-counting to transform outside examples of Many to inside sentences about the total; then by adding to unite (or split) inside totals in different ways depending on their units and on them being constant or changing. This allows designing a curriculum for all students inspired by Tarp (2018) that focuses on proportionality, solving equations and calculus from the beginning, since proportionality occurs when recounting in a different unit, equations occur when recounting from tens to icons, and calculus occurs when adding block-numbers next-to and when adding per-numbers coming from double-counting in two units.

Talking about 'refugee camp mathematics' thus allows locating a setting where children do not have access to normal education, thus raising the question 'What kind and how much mathematics can children learn outside normal education especially when residing outside normal housing conditions and without access to traditional learning materials?'. This motivates another question 'How much mathematics can be learned as 'finger-math' using the examples of Many coming from the body as fingers, arms, toes and legs?'

Focus 01. Digits as Icons with as Many Outside Sticks and Inside Strokes as They Present

Focus 02-04. Counting Ten Fingers, Sticks, Cubes in Various Ways

Focus 05-07. Counting a Dozen Finger-parts, Sticks, Cubes in Various Ways

Focus 08. Counting Numbers with Underloads and Overloads.

Focus 09. Operations as Icons Showing Pushing, Lifting and Pulling

Focus 10. The Inside Recount-Formula T = (T/B)xB Predicts Outside Bundlecounting Results

Focus 11. Discovering Decimals, Fractions and Negative Numbers.

Etc. until Focus 30

Conclusion. A curriculum for a refugee camp assumes that the learners have only the knowledge they acquire outside traditional education. The same is the case for street children living outside traditional homes; and for nomadic people always moving around.

However, a refugee camp curriculum might also be applied in a traditional school setting allowing the children to keep on to the two-dimensional bundle numbers they bring to school allowing them to learn core mathematics as proportionality, equations, functions and calculus in the first grade, thus not needing parallel curricula later on.

So, needing parallel curricula after grade 9 is not there by nature, but by choice. It is the result of disrespecting the mastery of many children bring to school and force them to adopts numbers as names along a number line, and force them to add numbers that are given to them without allowing them to find them themselves by counting, recounting and double-counting.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

DEVELOPING THE CHILD'S OWN MASTERY OF MANY

Abstract

In an isolated covid-19 household we may ask: How much mathematics is a child able to learn by itself? Or even: How much mathematics can we learn from a child isolated form traditional schooling?

As a matter of fact, amazingly much I learned first from our three Korean foster girls, and now from my two grandchildren. Changing from teaching to learning mode you observe that meeting many, children bundle and count with units. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly if held together 2 by 2: 'That is not four, that is two twos', thus insisting that the outside existing bundles should inside be predicated by a 'bundle-number' including the unit. When asked "How many 3s when uniting 2 3s and 3 4s they may say '5 3s and 3'; and when asked "How many 4s?" they may say '5 4s less 2'; and, integrating them next-to each other, they typically say '2 7s and 4'. Children have fun 'bundle-counting' their fingers in 3s in various ways: as 1Bundle7 3s, 'bundle-written' as T=1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Children thus master numbering before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. And they easily accept that 8 recounted in 2s means 8/2 times pushing away 2, giving 8 = (8/2)x2 leading directly to the core mathematical formula of proportionality, T = (T/B)xB, present all over STEM. Furthermore, they typically propose a horizontal stroke as an icon for a rope pulling away the stack to look for unbundles, to be placed next-to the stack as a decimal, or on-top counted in bundles, or reporting whet is needed for an additional bundle, e.g. T = 7 = 2 3s & 1 = 2B1 3s = 2.1 3s = 2 1/3 3s = 3.-2 3s.

And when asked to unite the two blocks 2 3s and 3 4s, they do so both on-top after recounting the blocks in the same unit, and next-to as areas thus performing integral calculus. Corona-isolation thus shows that core university mathematics as linearity and calculus is present at the child before school begin. So why not develop instead of rejecting the mastery of Many that children bring to school, counting before adding?

The paper designs two question-guided curricula, one for counting, and one for adding.

MEETING MANY, CHILDREN BUNDLE AND COUNT WITH UNITS

How children master Many I observed from my three Korean foster girls. Asked "How old next time?", a 3year old would say "Four" and show 4 fingers; but would react strongly if held together 2 by 2: 'That is not four, that is two twos', thus insisting that the outside existing bundles should inside be predicated by a 'bundle-number' including the unit. When asked "How many 3s when uniting 2 3s and 3 4s they would say '5 3s and 3'; and when asked "How many 4s?" they would say '5 4s less 2'; and, integrating them next-to each other, they typically said '2 7s and 4'.

Children have fun 'bundle-counting' their fingers in 3s in various ways: as 1Bundle7 3s, 'bundle-written' as T=1B7 using a full sentence with the outside total T as the subject, a verb, and an inside predicate, that could also be 2B4, 3B1 or 4B less2.

Children thus master numbering before school; only they see 8/2 as 8 counted in 2s, and 3x5 as a stack of 3 5s in no need to be restacked as tens. So why not develop instead of rejecting the mastery of Many that children bring to school, counting before adding?

MATERIALS FOR QUESTIONGUIDED COUNTCURRICULUM

Typically, a 'mediating curriculum' sees mathematics as its esoteric goal and teaches about numbers as inside names along a one-dimensional number line, respecting a place value system, to be added, subtracted, multiplied and divided before applied to the outside world. In contrast, a 'developing curriculum' sees mathematics as an exoteric means to develop the children's existing ability to master Many by numbering outside totals and stacks with inside two-dimensional bundle-numbers. This calls for different materials from grade 1 that don't mediate institutionalized knowledge but let students and the teacher co-develop knowledge by guiding outside research-like questions (Qs).

The design is inspired by Tarp (2018, 2020) holding that only two competences are needed to master Many, counting and adding. The corresponding pre-service and in-service teacher education may be found at the MATHeCADEMY.net.

QC01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the 'learning opportunity (L.O.) is to change many ones to one icon with as many sticks or strokes as it represents if written less sloppy. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

QC02, counting sequences: "How to count fingers?" Here the L.O. is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H".

QC03, icon-counting: "How to count fingers by bundling?" Here the L.O. is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported by a number-language sentence with subject, verb and predicate: T = 5 = 1Bundle3 2s = 2B1 2s = 3B-1 2s = 1BB1 2s, called an 'inside bundle-number' describing the 'outside stack-number'. Turning over a two- or three-dimensional stack or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

QC04, calculator-prediction: "How can a calculator predict a counting result?" Here the L.O. is to see the division sign as an icon for a broom pushing away bundles: 7/2 means 'from 7, push away bundles of 2s'. The calculator says '3.some', thus predicting it can be done 3 times. Now the multiplication sign iconizes a lift stacking the bundles. Finally, the subtraction sign iconizes a rope pulling away the stack to look for unbundled singles. By showing '7-3*2 = 1' the calculator indirectly predicts that a total of 7 can be recounted as 3B1 2s. An additional L.O. is to write 8 = (8/2)*2 as a 'recount-formula' T = (T/B)*B, saying "From T, T/B times B can be pushed away", to predict counting and recounting.

QC05, recounting in another unit: "How to change a unit?" Here the L.O. is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s. Counting 3 in 5s gives fractions: T = 3 = (3/5)*5. Another L.O. is to observe how double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

QC06, unbundled becomes decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the L.O. is to see that the unbundled occur in three ways: Next-to the stack as a stack of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles; or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside stack 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

QC07, prime or foldable units: "Which stacks can be folded?" Here the L.O. is to examine the symmetry of a stack. The stack T = 2 4s = 2*4 has 4 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a 'prime unit' and 4 a 'foldable unit', 4 = 2 2s. A number is called even or symmetrical if it can be folded in 2s, else the number is called odd.

QC08, finding units: "What are possible units in T = 12?". Here the L.O. is that units come from factoring in prime units, 12 = 2*6 and 6 = 2*3, so 12 = 2*2*3.

QC09, recounting from tens to icons: “How to change unit from tens to icons?” Here the L.O. is that asking ‘T = 2.4 tens = 24 = ? 8s’ can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign.

QC10, recounting from icons to tens: “How to change unit from icons to tens?” Here the L.O. is that without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking ‘T = 3 7s = ? tens’. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional L.O. uses ‘less-numbers’, geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3*(ten less 3) = 3*ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. So, ‘less less 1’ means adding 1.

QC11, recounting stack-sides. “How to recount sides in a stack halved by its diagonal?” Here, in a stack with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.

QC12. On squared paper a point has an out-number x and and up-number y, A(x,y). The per-number $\Delta y/\Delta x$ allows moving on a line.

With A(2,5) and B(4,6), the line per-number is $\Delta y/\Delta x$ = (6-5)/(4-2) = ½. Changing position to C(8,y) gives $\Delta y = (\Delta y/\Delta x)*\Delta x$ = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

MATERIALS FOR QUESTION GUIDED ADD CURRICULUM

Counting ten fingers in 3s gives T = 1BundleBundle1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula T = a*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity ‘the algebra square’.

QA01, next-to addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?” Here the L.O. is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

QA02, reversed next-to addition: “If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?” Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

QA03, on-top addition: “With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?” Here the L.O. is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

QA04, reversed on-top addition: “If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?” Here the L.O. is that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

QA05, adding tens: “With T1 = 23 and T2 = 48, what is T1+T2 when added as tens?” Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

QA06, subtracting tens: “If T1 = 23 and T2 add to T = 71, what is T2?” Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

QA07, multiplying tens: “What is 7 43s recounted in tens?” Here the L.O. is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

QA08, dividing tens: “What is 348 recounted in 6s?” Here the L.O. is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349/6 = 34B9/6 = 30B49/6 = (30B48 +1) /6 = 58 + 1/6.

QA09, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the L.O. is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining local constancy by epsilon and delta.

QA10, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the L.O. is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

QA11, solving the quadratic equation x^2 + 6x + 8 = 0. Two playing cards have the width 3 and the height x + 3. One is rotated a quarter turn and placed on top of the other so their lower left corners are congruent. We now see that (x+3)^2 = x^2 + 2*3*x + 3^2, or, (x+3)^2 = x^2 + 6*x + 8+1, or (x+3)^2 = 1 since x^2 + 6x + 8 = 0. So x = -3 ± 1 = -4 and -2.

QA12, finding common units: “Only add with like units, so how add T = 4ab^2 + 6abc?”. Here units come from factoring:

T = 2*2*a*b*b + 2*3*a*b*c = (2b+3c) *2ab.

DISCUSSION

Meeting Many makes children bring flexible bundle-numbers to school with core math as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social ‘colonization’ of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child’s existing mastery of Many.

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SELF-EXPLANATORY LEARNING MATERIAL TO IMPROVE LEARNERS MASTERY OF MANY

In an isolated covid-19 household we may ask: How to create simple material supporting the children in improving their mastery of Many?

This workshop is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but will protest when held together two by two by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. Meeting Many, children develop a number-language with full sentences including a subject and a verb and a predicate as in the word-language, as well as 2-dimensional bundle-numbers with units, neglected by the school's 1-dimensional line-names making some children count-over by saying 'twenty-ten'. So, the goal of the workshop is to inquire into the mastery of Many children bring to school to see what kind of mathematics occur if allowing the children to develop their already existing quantitative competence under proper guidance.

CHILDREN SHOW MASTERY OF MANY BEFORE SCHOOL

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This workshop-design is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but, held together two by two, protests by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. Meeting Many, children develop a number-language with full sentences including a subject and a verb and a predicate as in the word-language, as well as 2-dimensional bundle-numbers with units, neglected by the school's 1-dimensional line-names making some children count-over by saying 'twenty-ten'. So, the goal of the workshop is to inquire into the mastery of Many children bring to school to see what kind of mathematics occur if allowing the children to develop their already existing quantitative competence under proper guidance (Tarp 2018, 2020). In the workshop you will need 12 sticks, 12 cubes, a pegboard with rubber bands, squared paper and a pencil.

BRIDGE OUTSIDE EXISTEN-CE TO INSIDE ESSENCE

E01. Many become icons.

Pushing sticks away, transform many OUTSIDE ones into one INSIDE many-icon with as many sticks or strokes as it represents. Repeat with cubes.

Sticks and a folding ruler show that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent: five sticks in the 5-icon, etc. A looking glass finding nothing when looking for more will iconize zero. If used as bundle-size, ten has no icon but is reported as 1 bundle or 1B or 1B0.

E02. Flexible bundle-counting roots negative numbers.

Count ten fingers in 5s writing 6 in three different ways. Then count in 4s, 3s and 2s.

Including bundles, ten fingers may be bundle-counted in fives as '0Bundle1, 0B2, 0B3, 0B4, 1B0, 1B1' that also may be counted as 0B6 or 2B less 4 or 2B-4 with an overload or an underload, thus writing 6 in three ways as a 'flexible bundle-number':

T = 6 = 0B6 = 1B1 = 2B-4

Using flexible bundle-numbers, ten fingers may also be counted in 5s as '1B-4, 1B-3, etc.'

Counting on from ten, we meet 'Viking-counting' where eleven is 'one-left', twelve is 'two-left', and thirteen is 'three-ten', while 'three-twenty' becomes twenty-three.

Counting in scores (twenties) from forty, the Danish Viking-descendants still count: half-three-scores, three-scores, half-four-scores, four-scores, and half-five-scores for ninety. Unable to understand the half-notion, the French instead counts over when expressing 87 as '4 scores and 1 ten and 7.'

E03. Flexible bundle-counting roots polynomials.

Bundle-counting ten fingers in 3s, nine becomes 3B or 1BB, 1bundle-bundle, called hundred when using ten-bundles. And ten becomes 3B1 3s or 1BB0B1 3s, leading on to the general number-formula or polynomial T = ten = 1*B^2 + 0*B + 1*1 3s, showing the four ways to unite numbers (the Arabic meaning of Algebra): on-top addition, multiplication, power and next-to stack-addition called integration, all with reverse splitting operations: subtraction, division, factor-finding (root), factor-counting (logarithm), and differentiation.

Counting ten fingers in 2s, eight becomes 1BBB, called thousand when using ten-bundles.

Bundle-counting thus allows meeting power before the other operations which allows replacing the place-value-system with bundle-writing: T = 345 = 3BB4B5.

E04. Recounting as flexible bundle numbers will ease traditional calculations.

T = 53 = 5B3 = 4B13 = 6B-7 tens

65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

E05. Counting creates icons.

Count 7 cubes in 2s by 3 times pushing away 2s with a phone iconized as a division sign, predicted by a calculator as '7/2 = 3.some'. Stack the bundles by a lift iconized as a cross, 3x2. To look for unbundled, pull away the stack with a rope iconized as a subtraction sign, predicted by a calculator as '7-3x2 = 1'. So T = 7 = 3 2s & 1. So operations predict.

E06. Placing the unbundled roots decimals, negative numbers and fractions.

Counting 7 cubes in 3s gives 2 3s & 1 as predicted: T = 7 = (7/3) = 2.some; 7-2x3= 1.

Placing the unbundled next-to the stack roots decimals and negative numbers:

T = 7 = 2.1 3s = 3.-2 3s

Placing the unbundled instead on-top of the stack counted in bundles roots fractions:

T = 7 = 2 1/3 3s

Counting in tens, T = 68 = 6.8 tens = 7.-2 tens = 6 8/10 tens.

E07. OUTSIDE bundle-counting with icons as units is predicted INSIDE by a recount-formula T = (T/B)*B.

Recounting 8 in 2s by 8/2 times pushing away 2s is predicted on a calculator as T = 8 = (8/2)*2, which with unspecified numbers becomes a recount-formula T = (T/B)*B, (from T, T/B times, push away Bs) using a full number-language sentence with a subject, a verb and a predicate.

This recount ‘proportionality’ formula occurs all over mathematics and science: when relating proportional quantities as y = c*x; in trigonometry as sine and cosine and tangent, a = (a/c)*c = sinA*c; in coordinate geometry as line gradients, □y = (Dy/□x)*□x = c*□x; in calculus as the derivative, dy = (dy/dx)*dx = y’*dx; speed in science: □s = (□s/□t)*□t = v*□t.

E08. Recount from tens to icons

OUTSIDE, to answer ‘40 = ? 5s’ we use a pegboard or a squared paper to transform the stack 4.0 tens to 8.0 5s. So decreasing the base will increase the height. INSIDE, we formulate an equation to be solved by recounting 40 in 5s:

u*5 = 40 = (40/5)*5, so

u = 40/5 giving 40 = 8*5 = 8 5s

Notice that recounting gives the equation solution rule ‘move to opposite side with opposite calculation sign’.

E09. Recount from icons to tens

OUTSIDE, to answer ‘4 7s = ? tens’ we use a pegboard or a squared paper to transform the stack 4 7s to 2.8 tens. So, increasing the base will decrease the height.

INSIDE: oops, with no calculator ten-button we can’t use the recount-formula? Oh, we just multiply, thus creating multiplication tables.

Using flexible bundle-numbers on a pegboard or a squared paper we see that

T = 4 7s = 4*7 = (B-6)*(B-3) = 10B-6B-3B - - 6 3s = 1B + 18 = 28, making - - to +.

E10. Double-counting in two physical units.

Double-counting in two physical units gives a ‘per-number’ as 2m per 3sec or 2m/3sec or 2/3 m/sec. To answer the question ‘T = 6m = ?sec’ we just recount 6 in the per-number:

T = 6m = (6/2)*2m = (6/2)*3sec = 9sec.

Double-counting in the same unit, per-numbers become fractions: 2m per 3m = 2 per 3 = 2/3, and 2 per 100 = 2/100 = 2%.

To answer the question ’20 per hundred is ? per 400’ we just recount 400 in 100s: T = 400 = (400/100)*100 giving (400/100)*20 = 80.

To answer the question ’20 per 400 is what per hundred?’ we just recount 100 in 400s: T = 100 = (100/400)*400 giving (100/400)*20 = 5.

STEM formulas contain per-numbers:

meter = (meter/sec)*sec = velocity*sec,

kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter;

force = (force/square-meter)*square-meter = pressure*square-meter;

energy = (energy/sec)*sec = Watt*sec;

energy = (energy/kg)*kg = heat * kg.

Lego-bricks: number = (number/meter)*meter = density*meter.

E11. Mutual double-counting the sides in a stack with base b and height a, halved by its diagonal c, creates per-numbers:

a = (a/c)*c = sinA*c

b = (b/c)*c = cosA*c

a = (a/b)*b = tanA*b

$\pi \approx$ n*sin(180/n)

Draw a vertical tangent to a circle with radius 5. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/5 with tangent of the angle on a calculator.

E12. On squared paper a point has an out-number x and an up-number y, A(x,y). The per-number Δy/Δx allows moving on a line.

With A(2,5) and B(4,6), the line per-number is Δy/Δx = (6-5)/(4-2) = ½. Changing position to C(8,y) gives Δy = (Δy/Δx)* Δx = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

E13. Next-to addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s? Here we see that next-to addition OUTSIDE means adding by areas where multiplication precedes addition. INSIDE, the recount-formula predicts the result. Next-to addition is called integral calculus.

E14. Reversed next-to addition: If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

E15. On-top addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s? Here we see that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

E16. Reversed on-top addition: If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

E17. Adding tens: If T1 = 23 and T2 = 48, what is T1+T2 when added as tens? Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

E18. Subtracting tens: If T1 = 23 and T2 add to T = 71, what is T2? Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

E19. Multiplying tens: What is 7 43s recounted in tens? Here we see that also multiplication may create overloads:

T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301

T = 27*43 = 2B7*4B3 = 8BB+6B+28B+21 = 8BB34B21 = 8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

E20. Dividing tens: What is 348 recounted in 6s? Here we see that recounting a total with overload often eases division:

T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58;

T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

E21. Adding per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at what? Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

E22. Subtracting per-numbers: 2kg of 3$/kg + 4kg of what = 6kg of 5$/kg? Here we see that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a 'locally constant' situation, subtracting per-numbers is called differential calculus.

Grand theory holds conflicting conceptions on concepts

Within philosophy, Platonism and Existentialism discuss if concepts are examples of abstractions or abstractions from examples. Within psychology, Vygotsky and Piaget discuss if concepts are constructions mediated socially or experienced individually. Within sociology, the agent-structure debate is about establishing inclusion by accepting the agent's own concepts or establishing exclusion by insisting on teaching and learning institutionalized concepts.

DISCUSSION

The physical fact Many makes children bring flexible bundle-numbers to school containing core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many.

References

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WORKSHOP EXERCISES IN FLEXIBLE BUNDLE-NUMBERS

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<u>E01.</u> Pushing sticks away, transform many OUTSIDE ones into one INSIDE many-icon with as many strokes as it represents. Repeat with cubes transforming 3 1s to 1 3s.

<u>E02</u>. Bundle-count ten fingers in 5s writing 6 in three different ways. Then count in 4s, 3s and 2s: Using 'flexible BundleNumbers', T = 6 = 0**B**6 = 1**B**1 = 2**B**-4 **5s** (overload, standard, underload). And 0**B**1 = 1**B**-4, 0**B**2 = 1**B**-3, … **5s**

<u>E03</u>. Bundle-count ten fingers in 3s using bundle-bundles. Then in 2s. T = ten = 1**BB**0**B**1 = 101 **3s**.

Write traditional numbers as flexible BundleNumbers: T = 53 = 5**B**3 = 4**B**13 = 6**B**-7 tens

<u>E04</u>.

Flexible BundleNumbers ease Operations

65 + 27 = ? =	6**B**5 + 2**B**7 = 8**B**12 = 9**B**2 = 92
65 – 27 = ? =	6**B**5 – 2**B**7 = 4**B**-2 = 3**B**8 = 38
7* 48 = ? =	7* 4**B**8 = 28**B**56 = 33**B**6 = 336
336 /7 = ? =	33**B**6 /7 = 28**B**56 /7 = 4**B**8 = 48

<u>E05</u>. With cubes, transform the three OUTSIDE parts of a counting process, PUSH & LIFT & PULL, into three INSIDE operation-icons: division / & multiplication x & subtraction -.

Five counted in **2s**: I I I I I (push away **2s**) II II I (lift to stack) $\begin{matrix}II\\II\end{matrix}$ I (pull to find unbundles ones) $\begin{matrix}II\\II\end{matrix}$ I.

<u>E06</u>. Counting 7 cubes in 3s gives 2 3s & 1 as predicted: T = 7 = (7/3) = 2.some; 7-2x3= 1.

Placing the unbundled next-to the stack roots decimals and negative numbers:	T = 7 = 2.1 **3s** = 3.-2 **3s**
Placing the unbundled instead on-top of the stack counted in bundles roots fractions:	T = 7 = 2 1/3 **3s**

Recount traditional numbers: T = 68 = 6.8 tens = 7.-2 tens = 6 8/10 tens

<u>E07</u>. OUTSIDE bundle-counting with icons as units is predicted INSIDE by a **recount-formula T = (T/B)*B** (from T, T/B times, push away Bs) coming from recounting 8 in 2s by 8/2 times pushing away 2s as predicted on a calculator as T = 8 = (8/2)*2, thus using a full number-language sentence with a subject, a verb and a predicate.

OUTSIDE: T = I I I I I; T counted in **2s**: ~~II~~ ~~II~~ I; T – 2x2: = ~~II~~ ~~II~~ I ; INSIDE:

$5/2$	$2.\text{some}$
$5-2\text{x}2$	1

<u>E08</u>. Recount from tens to icons (decreasing the base will increase the height)

OUTSIDE, to answer the question '40 = ? **5s**', on squared paper transform the stack 4.0 **tens** to **5s**.

INSIDE, formulate an equation to be solved by recounting 40 in **5s**:

u*5 = 40 = (40/5)*5, so u = 40/5.

Notice that recounting gives the solution rule 'move to opposite side with opposite calculation sign'.

<u>E09</u>. Recount from icons to tens (increasing the base will decrease the height)

OUTSIDE, to answer '3 **7s** = ? **tens**', on squared paper or a pegboard change the stack 3 **7s** to **tens**.

INSIDE: oops, with no ten-button on a calculator we can't use the recount-formula? Oh, we just multiply! Use flexible bundle-numbers on a pegboard or a squared paper we see that

T = 4 7s = 4*7 = (B-6)*(B-3) = 10B-6B-3B - - 6 3s = 1B + 18 = 28, making - - to +.

<u>E10</u>. DoubleCounting in two physical units

DoubleCounting in two physical units gives a 'per-number' as e.g. 2m per 3sec, or 2m/3sec.

To answer the question 'T = 6m = ?sec', we just recount 6 in the per-number: T = 6m = (6/2)*2m = (6/2)*3sec = 9sec. Answer the question 'T = 12sec = ? m'.

Find formulas with per-numbers in science and mathematics.

E11. Mutual double-counting the sides in an axb stack halved by its diagonal c creates trigonometry: a = (a/b)*b = tanA*b, etc

Draw a vertical tangent to a circle with radius r. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/radius with tangent of the angle on a calculator.

A 12x12 square ABCD has AB on the ground and is inclined 20 degrees. From B, a straight road is to be constructed intersecting the borderline AD in the point E, inclined 5 degrees. Find the length DE. (Hint: Show that if DE = 2, then the incline of the road is 3.2 degrees).

E12. On squared paper a point has an out-number x and an up-number y, A(x,y). The per-number $\Delta y/\Delta x$ allows moving on a line.

With A(2,5) and B(4,6), the line per-number is $\Delta y/\Delta x = (6-5)/(4-2) = ½$. Changing position to C(8,y) gives $\Delta y = (\Delta y/\Delta x)*\ \Delta x = ½*(8-2) = 3$, and y = 5+3 = 8, giving C(8,8).

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E14. Reversed next-to addition: If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?

E15. On-top addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?

E16. Reversed on-top addition: If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?

E17. E19. Multiplying tens: What is 27 43s recounted in tens? T = 27*43 = 2B7*4B3 = 8BB+6B+28B+21 = 8BB34B21 = 8BB36B1 = 11BB6B1 = 1161

E18. Adding per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at what?

E19. Subtracting per-numbers: 2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?

E20. Solving STEM proportionality heating problems with recounting

With a heater giving 20 J in 30 sec, what does 40 sec give, and how many seconds is needed for 50J?

With 40 Joules melting 5kg, what will 60 Joules melt and what will 7 kg need?

With 3 degrees needs 50 Joules, what does 7 degrees need; and what does 70 Joules give?

With 4 deg. in 20kg needing 50 Joules, what does 9 deg. in 30 kg need? What does 70 Joules give in 40 kg?

1**BB**0	1**BB**1	1**BB**2	1**BB**3	1**BB**4	1**BB**5	1**BB**6	1**BB**7	1**BB**8	1**BB**9	~~1**BB**10~~
~~10**B**0~~	~~10**B**1~~	~~10**B**2~~	~~10**B**3~~	~~10**B**4~~	~~10**B**5~~	~~10**B**6~~	~~10**B**7~~	~~10**B**8~~	~~10**B**9~~	~~10**B**10~~
9**B**0	9**B**1	9**B**2	9**B**3	9**B**4	9**B**5	9**B**6	9**B**7	9**B**8	9**B**9	~~9**B**10~~
8**B**0	8**B**1	8**B**2	8**B**3	8**B**4	8**B**5	8**B**6	8**B**7	8**B**8	8**B**9	~~8**B**10~~
7**B**0	7**B**1	7**B**2	7**B**3	7**B**4	7**B**5	7**B**6	7**B**7	7**B**8	7**B**9	~~7**B**10~~
6**B**0	6**B**1	6**B**2	6**B**3	6**B**4	6**B**5	6**B**6	6**B**7	6**B**8	6**B**9	~~6**B**10~~
5**B**0	5**B**1	5**B**2	5**B**3	5**B**4	5**B**5	5**B**6	5**B**7	5**B**8	5**B**9	~~5**B**10~~
4**B**0	4**B**1	4**B**2	4**B**3	4**B**4	4**B**5	4**B**6	4**B**7	4**B**8	4**B**9	~~4**B**10~~
3**B**0	3**B**1	3**B**2	3**B**3	3**B**4	3**B**5	3**B**6	3**B**7	3**B**8	3**B**9	~~3**B**10~~
2**B**0	2**B**1	2**B**2	2**B**3	2**B**4	2**B**5	2**B**6	2**B**7	2**B**8	2**B**9	~~2**B**10~~
1**B**0	1**B**1	1**B**2	1**B**3	1**B**4	1**B**5	1**B**6	1**B**7	1**B**8	1**B**9	~~1**B**10~~
0**B**0	0**B**1	0**B**2	0**B**3	0**B**4	0**B**5	0**B**6	0**B**7	0**B**8	0**B**9	~~0**B**10~~

Flexible Bundle-Numbers Respect & Develop Kids' Own Math

Abstract

In an isolated covid-19 household we may ask: How can accepting children's own flexible bundle-numbers help us improve their mastery of Many?

This poster is based on the observation that when asked 'How old next time?', a 3year old will say 4 showing 4 fingers; but will protest when held together two by two by saying 'That is not 4. That is 2 2s', thus rejecting the predication 'four' by insisting on describing what exists, bundles of 2s and 2 of them. The poster gives an overview of what may be done with children's 'flexible bundle-numbers'.

01. Many exists in space as a total. Instead viewing the total in time by counting 1s allows transforming many 1s into 1 many-icon with as many strokes or sticks as it represents, stopping at ten where also counting bundles begin, so that ten = 1B0. Including units, the counting sequence then becomes 0B1, 0B2, …, 0B9 or 1B-1, 1B or 1B0, 1B1, etc. Counting in 3s, ten fingers become a total of T = 3B1 = 1BB0B1 3s. Including units makes the place value system superfluous.

02. Counting in bundles instead involves four tasks: First division iconized as a broom pushes away bundles, then multiplication iconized as a lift stacks the bundles, then subtraction iconized as a rope pulls away the stack to find unbundled, finally addition iconized as a two-way cross unite the two on-top or next-to. This natural order is the opposite of the tradition.

03. Five fingers can be flexibly bundle-counted in 2s with over- or underload: T = 5 = 1B3 = 2B1 = 3B-1 2s thus rooting negative numbers. The unbundled can be placed next-to the stack, or on-top counted in bundles, thus rooting decimals and fractions. T = 5 = 2B1 = 2.1 = 2 ½ 2s.

04. Flexible bundle-numbers ease operations. Example: T = 336 = 33B6 = 28B56 = 35B-14, so 336/7 = 4B8 = 5B-2 = 48.

05. Recounting 8 in 2s giving 8 = (8/2)x2 generalizes into a proportionality 'recount formula', T = (T/B)*B, predicting that 'from T, T/B times, Bs can be pushed away'; and occurring all over STEM. And allowing a calculator predict the result of changing units.

06. Recounting from tens to icons by asking '? 7s = 35' is called an equation u*7 = 35. It is easily solved by recounting 35 in 7s: u*7 = 35 = (35/7)*7. So u = 35/7, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking '2 7s = ? tens' is eased by using underloads: T = 2*7 = 2*(B-3) = 2B-6 = 20-6 = 14; and 6*8 = (B-4)*(B-2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48.

07. Double-counting a quantity in units gives a 'per-number' as e.g. 2$ per 3kg, or 2$/3kg. To change units we simply recount in the per-number: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

08. Double-counting an axb block halved by its diagonal c produces trigonometric per-numbers: a = (a/b)*b = tan A*b, etc.

09. Next-to addition geometrically means adding areas where multiplication precedes addition. Next-to addition is also called integral calculus, or differential if reversed. On-top addition involves recounting to get like units, or solving equations if reversed.

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BRIDGE OUTSIDE EXISTEN-CE TO INSIDE ESSENCE

E01. Many become icons.

Pushing sticks away, transform many OUTSIDE ones into one INSIDE many-icon with as many sticks or strokes as it represents. Repeat with cubes.

Sticks and a folding ruler show that digits are, not symbols as the alphabet, but sloppy writings of icons having in them as many sticks as they represent: five sticks in the 5-icon, etc. A looking glass finding nothing when looking for more will iconize zero. If used as bundle-size, ten has no icon but is reported as 1 bundle or 1B or 1B0.

E02. Flexible bundle-counting roots negative numbers.

Count ten fingers in 5s writing 6 in three different ways. Then count in 4s, 3s and 2s.

Including bundles, ten fingers may be bundle-counted in fives as '0Bundle1, 0B2, 0B3, 0B4, 1B0, 1B1' that also may be counted as 0B6 or 2B less 4 or 2B-4 with an overload or an underload, thus writing 6 in three ways as a 'flexible bundle-number':

T = 6 = 0B6 = 1B1 = 2B-4

Using flexible bundle-numbers, ten fingers may also be counted in 5s as '1B-4, 1B-3, etc.'

Counting on from ten, we meet 'Viking-counting' where eleven is 'one-left', twelve is 'two-left', and thirteen is 'three-ten', while 'three-twenty' becomes twenty-three.

Counting in scores (twenties) from forty, the Danish Viking-descendants still count: half-three-scores, three-scores, half-four-scores, four-scores, and half-five-scores for ninety. Unable to understand the half-notion, the French instead counts over when expressing 87 as '4 scores and 1 ten and 7.'

E03. Flexible bundle-counting roots polynomials.

Bundle-counting ten fingers in 3s, nine becomes 3B or 1BB, 1bundle-bundle, called hundred when using ten-bundles. And ten becomes 3B1 3s or 1BB0B1 3s, leading on to the general number-formula or polynomial T = ten = 1*B^2 + 0*B + 1*1 3s, showing the four ways to unite numbers (the Arabic meaning of Algebra): on-top addition, multiplication, power and next-to stack-addition called integration, all with reverse splitting operations: subtraction, division, factor-finding (root), factor-counting (logarithm), and differentiation.

Counting ten fingers in 2s, eight becomes 1BBB, called thousand when using ten-bundles.

Bundle-counting thus allows meeting power before the other operations which allows replacing the place-value-system with bundle-writing: T = 345 = 3BB4B5.

E04. Recounting as flexible bundle numbers will ease traditional calculations.

T = 53 = 5B3 = 4B13 = 6B-7 tens

65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92

65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38

7* 48 = 7* 4B8 = 28B56 = 33B6 = 336

336 /7 = 33B6 /7 = 28B56 /7 = 4B8 = 48

E05. Counting creates icons.

Count 7 cubes in 2s by 3 times pushing away 2s with a phone iconized as a division sign, predicted by a calculator as ‘7/2 = 3.some’. Stack the bundles by a lift iconized as a cross, 3x2. To look for unbundled, pull away the stack with a rope iconized as a subtraction sign, predicted by a calculator as ‘7-3x2 = 1’. So T = 7 = 3 2s & 1. So operations predict.

E06. Placing the unbundled roots decimals, negative numbers and fractions.

Counting 7 cubes in 3s gives 2 3s & 1 as predicted: T = 7 = (7/3) = 2.some; 7-2x3= 1.

Placing the unbundled next-to the stack roots decimals and negative numbers:

T = 7 = 2.1 3s = 3.-2 3s

Placing the unbundled instead on-top of the stack counted in bundles roots fractions:

T = 7 = 2 1/3 3s

Counting in tens, T = 68 = 6.8 tens = 7.-2 tens = 6 8/10 tens.

E07. OUTSIDE bundle-counting with icons as units is predicted INSIDE by a recount-formula T = (T/B)*B.

Recounting 8 in 2s by 8/2 times pushing away 2s is predicted on a calculator as T = 8 = (8/2)*2, which with unspecified numbers becomes a recount-formula T = (T/B)*B, (from T, T/B times, push away Bs) using a full number-language sentence with a subject, a verb and a predicate.

This recount ‘proportionality’ formula occurs all over mathematics and science: when relating proportional quantities as y = c*x; in trigonometry as sine and cosine and tangent, a = (a/c)*c = sinA*c; in coordinate geometry as line gradients, □y = (Dy/□x)*□x = c*□x; in calculus as the derivative, dy = (dy/dx)*dx = y’*dx; speed in science: □s = (□s/□t)*□t = v*□t.

E08. Recount from tens to icons

OUTSIDE, to answer ‘40 = ? 5s’ we use a pegboard or a squared paper to transform the stack 4.0 tens to 8.0 5s. So decreasing the base will increase the height. INSIDE, we formulate an equation to be solved by recounting 40 in 5s:

u*5 = 40 = (40/5)*5, so

u = 40/5 giving 40 = 8*5 = 8 5s

Notice that recounting gives the equation solution rule ‘move to opposite side with opposite calculation sign’.

E09. Recount from icons to tens

OUTSIDE, to answer ‘4 7s = ? tens’ we use a pegboard or a squared paper to transform the stack 4 7s to 2.8 tens. So, increasing the base will decrease the height.

INSIDE: oops, with no calculator ten-button we can’t use the recount-formula? Oh, we just multiply, thus creating multiplication tables.

Using flexible bundle-numbers on a pegboard or a squared paper we see that

T = 4 7s = 4*7 = (B-6)*(B-3) = 10B-6B-3B - - 6 3s = 1B + 18 = 28, making - - to +.

E10. Double-counting in two physical units.

Double-counting in two physical units gives a ‘per-number’ as 2m per 3sec or 2m/3sec or 2/3 m/sec. To answer the question ‘T = 6m = ?sec’ we just recount 6 in the per-number:

T = 6m = (6/2)*2m = (6/2)*3sec = 9sec.

Double-counting in the same unit, per-numbers become fractions: 2m per 3m = 2 per 3 = 2/3, and 2 per 100 = 2/100 = 2%.

To answer the question ’20 per hundred is ? per 400’ we just recount 400 in 100s: T = 400 = (400/100)*100 giving (400/100)*20 = 80.

To answer the question ’20 per 400 is what per hundred?’ we just recount 100 in 400s: T = 100 = (100/400)*400 giving (100/400)*20 = 5.

STEM formulas contain per-numbers:

meter = (meter/sec)*sec = velocity*sec,

kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter;

force = (force/square-meter)*square-meter = pressure*square-meter;

energy = (energy/sec)*sec = Watt*sec;

energy = (energy/kg)*kg = heat * kg.

Lego-bricks: number = (number/meter)*meter = density*meter.

E11. Mutual double-counting the sides in a stack with base b and height a, halved by its diagonal c, creates per-numbers:

a = (a/c)*c = sinA*c

b = (b/c)*c = cosA*c

a = (a/b)*b = tanA*b

$\pi \approx$ n*sin(180/n)

Draw a vertical tangent to a circle with radius 5. With a protractor, mark the intersection points on the tangent for angles from 10 to 80. Compare the per-number intersection/5 with tangent of the angle on a calculator.

E12. On squared paper a point has an out-number x and an up-number y, A(x,y). The per-number Δy/Δx allows moving on a line.

With A(2,5) and B(4,6), the line per-number is Δy/Δx = (6-5)/(4-2) = ½. Changing position to C(8,y) gives Δy = (Δy/Δx)* Δx = ½*(8-2) = 3, and y = 5+3 = 8, giving C(8,8).

E13. Next-to addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s? Here we see that next-to addition OUTSIDE means adding by areas where multiplication precedes addition. INSIDE, the recount-formula predicts the result. Next-to addition is called integral calculus.

E14. Reversed next-to addition: If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus.

E15. On-top addition: If T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s? Here we see that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit.

E16. Reversed on-top addition: If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2? Here we see that when finding the answer by removing the initial stack and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting.

E17. Adding tens: If T1 = 23 and T2 = 48, what is T1+T2 when added as tens? Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

E18. Subtracting tens: If T1 = 23 and T2 add to T = 71, what is T2? Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

E19. Multiplying tens: What is 7 43s recounted in tens? Here we see that also multiplication may create overloads:

T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301

T = 27*43 = 2B7*4B3 = 8BB+6B+28B+21 = 8BB34B21 = 8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 stack.

E20. Dividing tens: What is 348 recounted in 6s? Here we see that recounting a total with overload often eases division:

T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58;

T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

E21. Adding per-numbers: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at what? Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

E22. Subtracting per-numbers: 2kg of 3$/kg + 4kg of what = 6kg of 5$/kg? Here we see that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a 'locally constant' situation, subtracting per-numbers is called differential calculus.

Grand theory holds conflicting conceptions on concepts

Within philosophy, Platonism and Existentialism discuss if concepts are examples of abstractions or abstractions from examples. Within psychology, Vygotsky and Piaget discuss if concepts are constructions mediated socially or experienced individually. Within sociology, the agent-structure debate is about establishing inclusion by accepting the agent's own concepts or establishing exclusion by insisting on teaching and learning institutionalized concepts.

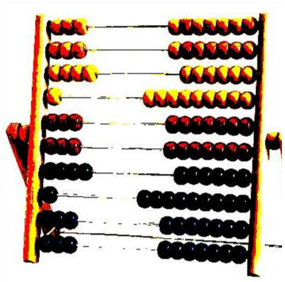
DISCUSSION

The physical fact Many makes children bring flexible bundle-numbers to school containing core mathematics as proportionality, calculus, solving equations, and modeling by number-language sentences with a subject, a verb and a predicate. Of course, a curriculum with counting before adding is contrary to the present tradition, and calls for huge funding for new textbooks and for extensive in-service training. However, it can be researched outside the tradition in special education, and when educating migrants and refugees. Likewise, applying grand theory in mathematics education is uncommon, but with education as a social 'colonization' of human brains, sociological warnings should be observed. Quality education, the fourth of the United Nations Sustainable Development Goals, thus should develop the child's existing mastery of Many.

References

Tarp, A. (2018). Mastering many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education 11*(1), 103-117.

Tarp, A. (2020). De-modeling numbers, opera-tions and equations: from inside-inside to outside-inside understanding. *Ho Chi Minh City University of Education Journal of Science 17*(3), 453-466.



Count before you Add

MatheMatics as ManyMath

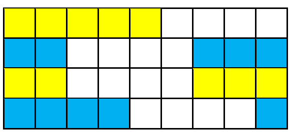
a Natural Science about MANY

MATHeCADEMY.net

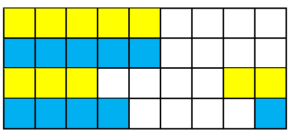
Cure Math Dislike with Kid's own **BundleNumbers** with **Units:** 2 **3s**

Count in ***Icons*** in ***Bundles***	T = I I I I = 4 = 4 T = 7 = IIII I I I I = IIII IIII I = IIII IIII IIII = 1B4 3s or 2B1 3s or 3B-2 3s
ReCount in *same* ***Unit*** in *new* ***Unit***	**ReBundle** to create Overload or Underload T = 7 = I I I I I I I = 1B4 3s = 2B1 3s = 3B-2 3s T = 7 = 2B1 3s = 1B3 4s = 1B2 5s = 3B1 2s = 1BB1B1 2s = 11B1 2s
ReCounting Predicted by a ***Recount-Formula***	Push ● Lift ● Pull ● Unite: **/ X – +** 8 = ? **2s.** 8 count in **2s** 8/2 times, stack as 8/2 **2s.** **So** 8 = (8/2)x2 **T = (T/B) x B = T/B** Q: 2 **4S** = ? **5S** 2x4/5 1.some 2x4 – 1x5 3 A: 2 **4s** = 1B3 **5s**
ReCount in ***Tens*** from ***Tens***	3 7s = ? tens Answer: 3x7 = 21 = 2B1 tens ? 7s = 3 tens Answer: (30/7) x 7 = 4B2 7s
DoubleCount in *PerNumbers* in *PerFive, 3/5* in *PerHundred, %*	With 4$ per 5kg or 4/5 $/kg, T = 20kg = (20/5) x 5kg = (20/5) x 4$ = 16$ 3/**5** = 3$/**5**$ of 200$ = ?$. 200$ = (200/5) x **5**$ gives (200/5) x 3$ = 120$ 70**%** = 70$/100$ of 300$ = ?$. 300$ = (300/100) x 100$ = gives (300/100) x 70$ = 210$
Add *NextTo* *OnTop*	T = 2 3s + 4 5s = 3B2 8s ... *gration* T = 2 3s + 4 5s = 1B1 5s + 4 5s = 5B1 5s P... *rtionality*
Multiply, Divide *BundleWriting*	7 x 63 = 7 x 6B3 = 42B21 = 44B1 = 441 245 /7 = 24B5 /7 = 21B35 /7 = 3B5 = 35

Abacus in 2 modes: T = 7 = 2B1 **3s**



Geometry-mode



Algebra-mode

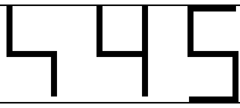
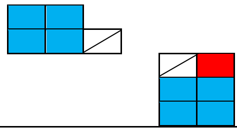
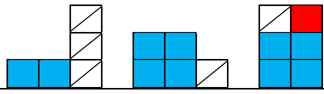
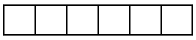
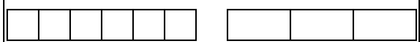
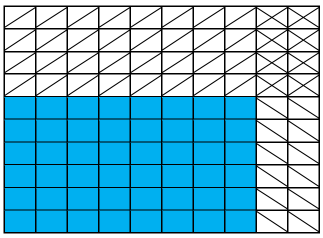
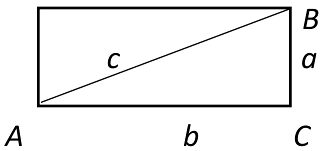
MrAlTarp
YouTube Videos:
Allan.Tarp@MATHeCADEMY.net

MATHeCADEMY.net	*Teaching Teachers to Teach MatheMatics as* *ManyMath* PYRAMIDeDUCATION **CATS**: **C**ount & **A**dd in **T**ime & **S**pace

Flexible BundleNumbers

Develops when **Kids** adapt to Many

Outside & Inside Math

Digits as ICONS III IIII IIIII		3 4 5
Operations as ICONS	Push ● Lift ● Pull ● Unite	/ X – +
Count Fingers in 5s using BundleCounting & BundleNumbers	I●●●● II●●● III●● IIII● IIIII IIIII I●●●●	T = 0*B*1 = 1*B*-4 **5s** T = 0*B*2 = 1*B*-3 **5s** T = 0*B*3 = 1*B*-2 **5s** T = 0*B*4 = 1*B*-1 **5s** T = 1*B*0 = 1*B*0 **5s** T = 1*B*1 = 2*B*-4 **5s**
Unbundled creates Decimals & Fractions & Negative Numbers IIIII ➔ II II I		T = 5 = 2*B*1 **2s** = 2.1 **2s** T = 2 1/2 **3s** T = 3*B*-1 2s = 3.-1 **2s** T = 1*BB* 0*B*1 (T = *p*x^2 + q*x + r*)
ReCount in Same Unit creates Flexible Numbers ▌▌▌▌▌III ➔ 53	5: IIII I I I IIII II I IIII II II 	T = 1*B*3 **2s** *Overload* T = 2*B*1 **2s** *Standard* T = 3*B*-1 **2s** *Underload* T = 53 = 5*B*3 = 4*B*13 = 6*B*-7 **tens**
Flexible BundleNumbers ease Operations	65 + 27 = ? = 65 – 27 = ? = 7 x 48 = ? = 336 /7 = ? =	6*B*5 + 2*B*7 = 8*B*12 = 9*B*2 = 92 6*B*5 – 2*B*7 = 4*B*-2 = 3*B*8 = 38 7 x 4*B*8 = 28*B*56 = 33*B*6 = 336 33*B*6 /7 = 28*B*56 /7 = 4*B*8 = 48
ReCount in New Unit  6 = ? **2s** *ReCount-Formula:*	 6 = (6/2) x 2 T = (T/*B*) x *B*	T = 5 = (5/2) x 2 = ? = 2*B*1 **2s** $\boxed{\begin{array}{ll} 5/2 & 2.\text{some} \\ 5 - 2*2 & 1 \end{array}}$
ReCount: Tens to Icons ▌▌▌IIIII = ? **7s**	3*B*5 **tens** = *u**7	*u**7 = 35 = (35/7)*7 so *u* = 35/7
ReCount: Icons to Tens 6 **8s** = ? **tens** 		T = 6 **8s** = 6 x 8 = (*B*-4) x (*B*-2) = *BB* - 4*B* - 2*B* - - 8 = 10*B* - 6*B* + 8 = 4*B*8 = 4.8 **tens** = 48
DoubleCount gives PerNumbers	2$ per 3kg = 2$/3kg	T = 6$ = (6/2) x 2$ = (6/2) x 3kg = 9kg
Like Units: Fractions 5% of 40	5$/100$ of 40$	T = 40$ = (40/100) x 100$ gives (40/100) x 5$ = 2$
DoubleCount a Block halved by its Diagonal		*a = (a/c)*c* = sin *A*c* *a = (a/b)*b* = tan *A*b* π = *n**tan(180/*n*) *for n large* *c*c = a*a + b*b*

Flexible Bundle-Numbers Respect & Develop Kids Own Math

01. Meeting Many inspires transforming five ones into one five-icon containing five strokes or sticks. Likewise, with the other digits from one to nine, also containing as many strokes or sticks as they represent if written less sloppy. Icon-building may be illustrated with a folding ruler. Transforming five ones to one fives allows using five as a unit when counting a total T by bundling and stacking, to be reported in a full number-language sentence with a subject, a verb and a predicate, e.g. $T = 2$ 5s.

02. Icons thus inspires ‘bundle-counting’ and ‘bundle-writing’ where a total T of 5 1s is recounted in 2s as $T = 1B3$ 2s = $2B1$ 2s = $3B$-1 2s, i.e. with or without an overload, or with an underload rooting negative numbers. The unbundled 1 can be placed next to the bundles separated by a decimal point, or on-top of the bundles counted in bundles, thus rooting fractions, $T = 5 = 2B1$ 2s = 2.1 2s = 2 ½ 2s. Recounting in the same unit to create or remove over- or underloads eases operations. Example: $T = 336 = 33B6 = 28B56 = 35B\text{-}14$, so $336/7 = 4B8 = 5B\text{-}2 = 48$.

03. Bundle-counting makes operations icons also. First a division-broom pushes away the bundles, then a multiplication-lift creates a stack, to be pulled away by a subtraction-rope to look for unbundles singles separated by the stack by an addition-cross. A calculator uses a ‘recount formula‘, $T = (T/B)*B$, to predict that ‘from T, T/B times, Bs can be taken away’. This recount formula occurs all over mathematics and science: when relating proportional quantities as $y = c*x$; in trigonometry as sine and cosine and tangent, e.g. $a = (a/c)*c = \sin A*c;$ in coordinate geometry as line gradients, $\Delta y = (\Delta y/\Delta x)*\Delta x = c*\Delta x$; and in calculus as the derivative, $\mathrm{d}y = (\mathrm{d}y/\mathrm{d}x)*\mathrm{d}x = y'*\mathrm{d}x$.

04. Recounting in a different unit is called proportionality. Asking ‘3 4s = ? 5s’, sticks say $2B2$ 5s. Entering ‘3*4/5’ we ask a calculator ‘from 3 4s we take away 5s’. The answer ‘2.some’ predicts that the singles come by taking away 2 5s, thus asking ‘3*4 – 2*5’. The answer ‘2’ predicts that 3 4s can be recounted in 5s as $2B2$ 5s or 2.2 5s.

05. Recounting from tens to icons by asking ‘35 = ? 7s’ is called an equation $u*7 = 35$. It is easily solved by recounting 35 in 7s: $u*7 = 35 = (35/7)*7$. So $u = 35/7$, showing that equations are solved by moving to opposite side with opposite calculation sign.

06. Recounting to tens by asking ‘2 7s = ? tens’ is eased by using underloads: $T = 2*7 = 2*(B\text{-}3) = 20\text{-}6 = 14$; and $6*8 = (B\text{-}4)*(B\text{-}2) = BB - 4B - 2B -- 8 = 100 – 60 + 8 = 48$.

07. Double-counting a quantity in units gives a ‘per-number’ as e.g. 2$ per 3kg, or 2$/3kg. To answer the question ‘T = 6$ = ?kg’, we recount 6 in 2s since the per-number is 2$/3kg: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. Double-counting in the same unit creates fractions and percent: 2$/3$ = 2/3, and 2$/100$ = 2/100 = 2%.

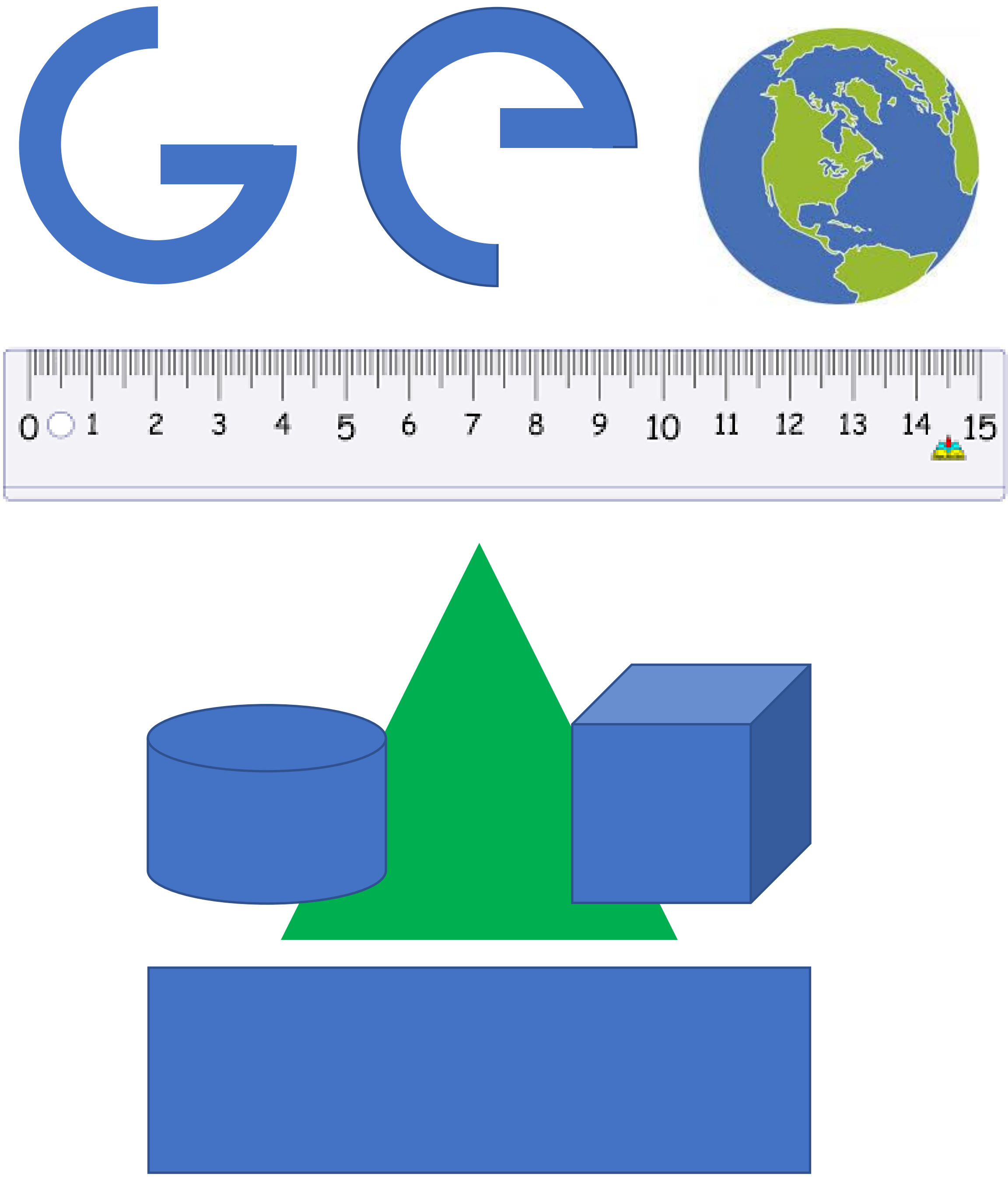
08. Next-to addition geometrically means adding by areas, so multiplication precedes addition. Next-to addition is also called integral calculus, or differential if reversed.

09. On-top addition means using the recount-formula to get like units. Changing units is also called proportionality, or solving equations if reversed.

References

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Tarp, A. (2020). De-modeling numbers, operations and equations: from inside-inside to outside-inside understanding. *Ho Chi Minh City University of Education Journal of Science 17*(3), 453-466.



GeoMetry

Earth Measurement

from below

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GEOMETRY - FROM BELOW

‘Geometry from below’ means geometry as tales about a social practice, in this case about ‘earth-measurement’, to which the Greek word ’geo-metry’ can be directly translated.

The earth is what we live on and what we live on. We divide the earth between us by drawing dividing lines. This divides the soil into areas limited by lines, multi-edges, polygons.

If these boundaries disappear, it is important to be able to re-establish them, and this restoration of lines and corners requires that these can be measured.

In ancient Egypt, the Nile thus crossed its banks once a year and brought manure to the fields. After retiring, the divisions had to be re-established.

Geometry from below can be understood as the opposite of geometry from above, deducing geometry from metaphysical truths, axioms.

The following material is not a traditional textbook, but rather an activity guide with suggestions for a range of activities that the reader can perform and report.

So, the idea is that the reader builds his own textbook.

However, some proposals for definitions and rules are included.

Rules can be proved either through evidence or conviction. It is recommended to work with the last based on the task: “Try to convince someone else of the validity of the rule.” Likewise, the idea is that the reader performs his own illustrations, which is why very few illustrations are included in the material.

The following exercises should be performed both on paper, on the floor and on earth, as well as, if possible, on computer programs.

We work out a report of questions, techniques, inventions, and discoveries we meet along the way.

Jutland, Denmark, Allan Tarp

DK version January 2000, US version May 2020

http://mathecademy.net/geometry-from-below/

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GE00 LAND SHARING

In this chapter we will look at a number of traditional land-sharing problems. No solutions are provided, but the problems can hopefully serve as motivation for the following chapters.

Basic problem: How to share an area?

Exercise 1. Two people stand in random places on a restricted floor surface and are tasked with dividing it between them. represent groups of different sizes. What are the different sharing principles? Note: Random location in a room may be possible. achieved with a cube: I stand with my back and right shoulder against the wall in a corner. Then I roll a dice 4-6 times. The cube tells me how many steps I need to walk alternately forward and to the left. Same with the other participants.

Exercise 2. Perform a sharing based on the sharing principle: No matter to everyone.

Exercise 3. Perform a sharing based on the sharing principle: Just far to the limit.

Exercise 4. Perform a division based on the sharing principle: The distances to the border must be 1:2, as there are twice as many people living in the other area.

Exercise 5. Repeat exercise 1-4, but now with three people.

Exercise 6. Repeat exercise 1-4, but now with more than three people.

GE01 POLYGONS

Basic problem: How to talk about and divide areas.

Exercise 1. Set up a triangle. Measure the triangle in order to re-establish it. Delete the triangle. Re-establish the triangle on the basis of the objectives, on the one hand at any point and partly in the same place. Insert names (definitions) based on this exercise.

Exercise 2. Construct a square with skewed angles. Measure the square in order to re-establish it. Delete the square. Re-establish the square on the basis of the dimensions, on the one hand, in any place and partly in the same place.

Exercise 3. Set up a triangle. Divide it into two right triangles. Measure the right triangles in order to restore them. Re-establish the triangles on the basis of the objectives, on the one hand, at any point and partly in the same place.

Exercise 4. Set two parallel lines. Measure the distance of the lines.

Exercise 5. What is A4 paper? And what are A1, A2, A3, A5 etc?

Exercise 6. Length can be measured in many ways. Today, meters are usually used. Other targets were used in the past. Which? Is the meter system used all over the world? What is the link between meters and other targets?

Exercise 7. Can angles be measured in ways other than degrees? Why is a right angle 90 degrees and not 100 degrees?

Exercise 8. Find rules for the defined concepts and convince someone else that these rules are true.

DEFINITION 1. One point is

DEFINITION 2. A straight line is

DEFINITION 3. A polygon is

DEFINITION 4. The circumference of a polygon is

DEFINITION 5. An angle is

DEFINITION 6. A polygon's parts are

DEFINITION 7. A diagonal is

DEFINITION 8. A rectangle is

DEFINITION 9. A right triangle is

DEFINITION 10. The distance between a point and a line is

DEFINITION 11. Parallel lines are

DEFINITION 12. A normal is

DEFINITION 13. A convex polygon is

Rule 1. A polygon can be divided into

Rule 2. A triangle can be divided into

GE02 TRIANGLES

Basic problem: How are triangles designated and calculated?

Exercise 1. Construct a triangle and enter names for the different components of a triangle. How many targets can we only achieve in order to uniquely construct a triangle? How many different triangle types are there?

Exercise 2. A triangle can be divided by different lines. Name some of these and discuss what sharing principles the different lines might embodies.

Exercise 3. Find rules for the named lines in practice 2 and convince someone else that these rules are true.

Exercise 4. Construct an equilateral triangle. Do special rules apply to straight-legged triangles?

Exercise 5. Construct an equilateral triangle. Do special rules apply to equilateral triangles?

Exercise 6. Cut out a triangle and hang it in a corner. Draw the load line, how is it? Switch to the other corners. Where is the center of gravity of the triangle?

Exercise 7. The foot point of a dividing line in a triangle is the intersection of the line with a side. Do special rules apply to foot points for heights, medians, angular halves and bisectors?

Exercise 8. A triangle can be wrapped in a rectangle in different ways. Which rectangle has the smallest circumference?

Exercise 9. Construct a triangle ABC. Move A to A* without changing the circumference. What will be the new points?

DEFINITION 1. A triangle or a three-corner is

DEFINITION 2. The three angles of a triangle are

DEFINITION 3. The three sides of a triangle are

DEFINITION 4. An acute triangle is

DEFINITION 5. An obtuse triangle is

DEFINITION 6. In triangle ABC the height ha is

DEFINITION 7. In triangle ABC the median ma is

DEFINITION 8. In triangle ABC, the angular bisector bA is

DEFINITION 9. In triangle ABC the bisector na is

DEFINITION 10. An SSA triangle is a triangle in which two sides and one angle are known. Similarly, an SAA and an SSS triangle are defined.

Rule 1. In a triangle is the angle sum.

Rule 2. In a triangle, the heights intersect

Rule 3. In a triangle, the angular half-lines intersect

Rule 4. In a triangle, the medians intersect

Rule 5. In a triangle, the bisectors intersect

Rule 6. In a triangle, the intersection of the heights has the following property:

Rule 7. In a triangle, the intersection of the angular halves has the following property:

Rule 8. In a triangle, the intersection of the medians has the following property:

Rule 9. In a triangle, the intersection of the bisectors has the following property:

GE03 RIGHT TRIANGLES

Basic problem: How is a right triangle designated and calculated?

Exercise 1. Construct a rectangle and divide it into two right triangles using a diagonal. Introduce names for the different components of the right triangle.

Exercise 2. Construct a rectangle and divide it into two right triangles using a diagonal. Measure the angles. Measure the diagonal and side lengths, partly in cm, and partly by diagonal lengths (as a percentage of the diagonal). Compare to the calculator's sin and cos button.

Exercise 3. Construct a rectangle and divide it into two right triangles using a diagonal. Measure the angles. Measure the diagonal and side lengths, partly in cm, partly in floor lengths (as a percentage of the floor, i.e. the horizontal side). Compare to the calculator's tan button.

Exercise 4. Set on millimeter paper a quartz circle with a radius of 10 cm, or set on the floor a quartz circle with a radius of 1 meter. Sign a series of right triangles ABC with A in the center of the circle, B on the circular arc and C on the 0-degree line. A shall pass degrees 10, 20, 30 up to 80. Table for each A value the length of BC and AC as well as the BC/AC ratio. Compare to the calculator's sin, cos and tan button.

Exercise 5. Construct a right triangle from abandoned dimensions. What's the minimum number of goals we can settle for? How many different types of right triangles are there? we can measure up to the unknown goals, but can we also count for the unknown goals?

Exercise 6. Construct a random triangle and divide it into two right triangles. Measure the parts in the right triangles and calculate them afterwards. Finally, enter the dimensions of the original triangle's parts, and check when measuring.

Exercise 7. Construct a known triangle and divide it into two right triangles. Measure the parts in the right triangles and calculate them afterwards.

Exercise 8. Set off a known SAA triangle. Measure and calculate the other parts.

Exercise 9. Set off a known SSA triangle. Measure and calculate the other parts.

Exercise 10. Set off a known rectangle. Measure and calculate the length and angles of the diagonal.

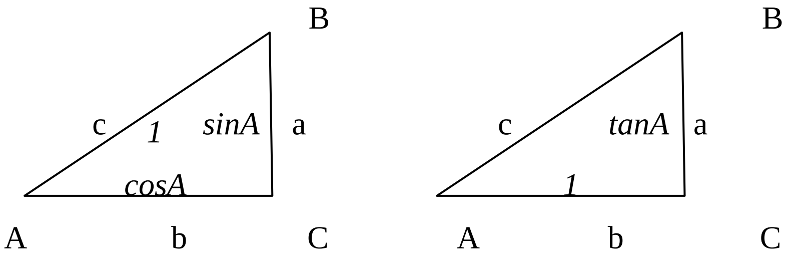
Exercise 11. Construct using the different known angle sizes of the seaweed, e.g. 27°, 42° and 133°.

Exercise 12. Construct a rectangle and construct new rectangles with the same diagonal. How are they?

Exercise 13. Use a PC spreadsheet to set up the different types of triangular calculation.

GREEK DEFINITIONS: In a right triangle, the two short sides are called catheter, and the long hypotenuse.

ARABIC DEFINITIONS: Triangle ABC is right with C as the right angle. b is horizontal and a vertical. The floor side b is called the cosine side as seen from A. The wall side a is called the sine side or the tangent side as seen from the A. Sine and cosine sides are indicated as a percentage of the sloping wall c, or with c as a unit of measurement ("recounting" in c's: a = a/c*c = sinA*c). The tangent side is indicated as a percentage of the floor b.



sinA = a/c

cosA = b/c

tanA = a/b

The calculation problem

In a right triangle there are three unknowns. Calculation therefore requires three equations.

The Greeks knew only two, one angle- and one area equation (Pythagoras):

A+B = 90   a^2 + b^2 = c^2

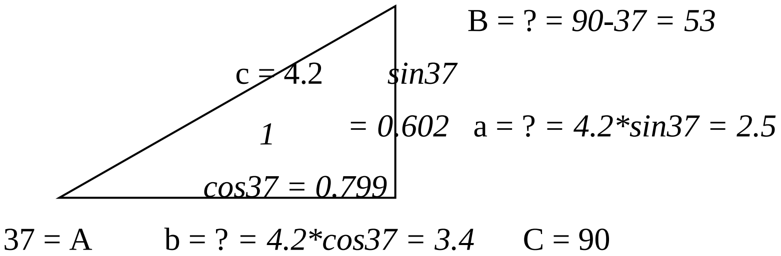
The Arabs knew three, one angle and two side equations:

A+B = 90   a = c*sinA   b = c*cosA

The following calculations use the two sides of the triangle: the outer side with the actual numbers and the inside with the percentages. Alternatively, we could use equation schemes, see the appendix.

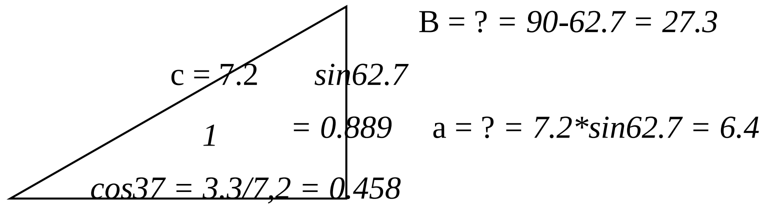
Example 1. *Given SAA, Find SSA.*

In triangle ABC is C = 90°, A = 37° and c = 4.2. Find a = ?, b = ? and B = ?



Example 2. *Given SSA, Find SAA.*

In triangle ABC, C = 90°, b = 3.3, and c = 7.2. Find a = ?, A = ? and B = ?



INVcos 0.458 = 62.7° = A = ? b = 3.3 C = 90

GE04 DIFFERENT APPLICATIONS

Problem 1: How is the height of something high, for example, a flagpole?

Problem 2: How to move a corner thing?

Problem 3: How are shortcuts calculated?

Problem 4: How does one thing move fastest from a point in one area to a point in another area when we move at different speeds in the two areas?

Problem 5: How are the different openings calculated at a door that is ajar?

Problem 6: How to build a path up a steep mountainside?

Problem 7: How steep can a ladder be set so as not to break glass or ice?

Problem 8: How are astronomical distances calculated?

Problem 9: How should a triangular bridge over a river be sized?

Problem 10: During a swing, when do we have to jump off to get the furthest away?

Exercise 1. Determine the height of a tall thing (a flagpole) in two different ways: the lightweight, where we can get all the way to the thing, and the hard one where we can't. Determine in the same way the width of a wide thing (a house wall or a "river"). If possible, check the calculations by measurement.

Exercise 2. Set off once with the width 2 meters, which forms a crack of 90 degrees. A stick should be turned around the corner without lifting. What is the maximum length of the stick? A box with a width of 1 meter must be turned around the corner without lifting. What is the maximum length of the box? If necessary, use the other objectives.

Exercise 3. Once with the width 3 meters forms a crack of 90 degrees and continues with a width of 4 meters. A stick should be turned around the corner without lifting. What is the maximum length of the stick? A box with a width of 2 meters must be turned around the corner without lifting. What is the maximum length of the box?

Exercise 4. A treasure can be found by first traveling 4 meters to the east, then 2. 5 meters to the south, and finally 3. 1 meter to the west. Find a shortcut to the treasure.

Exercise 5. A treasure can be found by first traveling 4 meters to the east in 32 degrees northerly direction, then 2. 5 meters to the south in 68 degrees westerly direction, and finally 3. 1 meter to the west in a 14-degree northerly direction. Find a shortcut to the treasure.

Exercise 6. A boat rows over a 50-meter-wide river at a speed of 20 meters/minute. The current in the river is 10 meters/minute. What angle should you praise for landing right across the face on the opposite side? How long does the trip take?

Exercise 7. Construct a straight line and two points on either side of the line. Go from one point to the other with steps equal to 1 shoe length. 1 second corresponds to 1 shoe length in one area and 2 shoe lengths in the other. Find the fastest route. Could the fastest route be figured out? If necessary, use a PC spreadsheet to calculate different routes. A rule applies to the approach and refraction angle at the point on the boundary line which the fastest route passes. Which?

Exercise 8. Open a door ajar. There are now three openings, one perpendicular to the wall, one parallel to the wall and one parallel to the door. How big are these openings? How much will 10 degrees extra increase these openings by?

Exercise 9. Tip a plate 30 degrees and sign up a way up that can rise no more than 20 degrees (a hairpin bend). Repeat the exercise with other degree numbers. How much does the gravitational pull of a car increase when the rise of the road increases 10 degrees?

Exercise 10. Place a heavy book on a scale. Tilt it to different positions. What happens to the weight? Could this result be calculated? How much is the pressure against a vertical hand supporting it?

Exercise 11. Construct and load a triangular bridge and control the pressure against the surface by placing the bridge on two scales. Can these numbers be calculated?

Exercise 12. Distances in space cannot be measured, but must be calculated. How is the radius of the earth calculated? How far is it to the moon? What is the radius of the moon? How far is the sun? What is the radius of the sun?

Exercise 13 (difficult). A person sits in a swing that is suspended in ropes that are 3 meters long. Pull out the swing so that it is in height 1 meter above the bottom point. What angle does this correspond to?

If the swing is released, the speed v can be calculated from the formula: v^2 = 19.6*h, where h is the distance from the maximum height in the outer position. The speed consists of a horizontal and a vertical part.

Where do we have to jump off to get the furthest away?

(After the jump, the horizontal part of the speed will be unchanged, while the vertical part will grow downwards by 9.8 m/s every second.)

DEFINITION 1. One incident angle is

DEFINITION 2. A reflective angle is

Rule 1. The fastest route between two points in two different areas with speed v1 and v2 will meet the wrestling law at the crack point between the two areas:

sin (incident angle)/sin (reflective angle) = v1/v2.

GE05 AREA, SURFACE AREA, COVERAGE

Basic problem: How is the size of an area designated and calculated?

Exercise 1. A piece of squared A4 paper covers a certain area. How can the extent, area, coverage, extent of this surface be measured?

Exercise 2. A piece of squared A4 paper is divided by a transverse diagonal. What is the area of a right triangle?

Exercise 3. A piece of squared A4 paper is divided by lines into three triangles. What is the area of an ordinary triangle? Does this rule also apply to blunt-angled triangles?

Exercise 4. Construct a triangle and determine its area.

Exercise 5. Construct a triangle and divide it into 2 parts with the same area. In 3 parts. In 4 parts.

Exercise 6. Same as 5, but now the divider must be a normal to the baseline.

Exercise 7. A height can be difficult to measure accurately. Can a triangle area be calculated solely from sides and angles?

Exercise 8. An angle can be difficult to measure accurately. Can the area of a triangle be calculated solely from sides?

Exercise 9. Find some other surface areas that are used elsewhere and at other times.

Exercise 10. Use a PC spreadsheet to set up the different types of area calculation.

DEFINITION 1. An area unit is a square (a tile) of 1x1. 1m x 1m = 1m^2

Rule 1. A rectangle with side lengths a and b has an area of A =

Rule 2. A right triangle with catheter a and b has an area of A =

Rule 3. A triangle with a height and a surface g has an area of A =

Rule 4. Triangle ABC area can be calculated using the sine formula: A = 1/2*a*b*sinC.

Rule 5. The area of Triangle ABC can be calculated using the Herons formula:

A^2 = s*(s–a)*(s–b)*(s–c) where s = 1/2*(a+b+c) = half the circumference.

GE06 SHAPE CHANGE 1: SCALING

Basic problem: How is magnification or shrinking of a triangle is termed and calculated?

Exercise 1. When we walk away from a triangle, it seems to resize, but not shape. Construct a triangle ABC (it may be right). Set triangle AB*C* where the side lengths are half the size. What's the angles? What applies to the sides? What is the area?

Exercise 2. Construct a triangle ABC and enlarge it by 20% to 120% (scaling percentage and factor) to triangle AB*C*. Draw through B and C lines parallel to b and c (parallel transversals). These lines cut a in B** and C**. How big are the triangles ABC, AB*C*, BB*B** and CC*C** relative to each other?

Exercise 3. Construct a triangle ABC. Reseal it, but now in the 1:2 ratio. What is the scaling percentage and factor?

Exercise 4. Construct a triangle on a floor. Draw the triangle of paper in the 1:10 ratio.

Exercise 5. Construct a triangle on paper. Draw the triangle of a floor in the 1:10 ratio.

Definition 1. In a triangle, all side lengths are made k times as large. The number k is then called scaling factor, and k-1 is called scaling percent.

Rule 1. Scaling a triangle preserves the size of the angles and the direction of the sides.

Rule 2. If two triangles are similar, one is a scaling of the other.

Rule 3. In the triangle ABC and AB*C* we know that AB* = k*AB (AB* "re-counted" in AB's: AB* = (AB*/AB)*AB = k*AB) and AC* = k*AC.

Triangle AB*C* will then be a triangle of ABC. BB* = (k-1)*AB (provided k>1).

GE07 CHANGE OF FORM 2: LAND CONSERVATION

Basic problem: How can a shape change shape without changing the area (property exchange)?

Exercise 1. Construct a triangle ABC. Construct triangles A*BC with the same area. How will the A* items lie?

Exercise 2. Construct a right triangle ABC. Set triangles A*B*C with the same area. How will points A* and B* lie?

Exercise 3. Set off a rectangle ABCD. Construct rectangles AB*C*D* with the same area. How will the b*, c* and d* points lie?

Exercise 4. Construct two rectangles with the same area, increasing the longest side by 30%.

Exercise 5. Construct two right triangles with the same area, increasing the height by 25%.

Exercise 6. Construct two triangles with the same area, reducing the height by 40%.

Exercise 7. Set off a rectangle ABCD. Construct a square AB*C*D* with the same area.

Exercise 8. Construct a right triangle ABC. Construct the area a^2, as well as a rectangle with the area a^2 if one side length is c. Set the area b^2, as well as a rectangle with the area b^2, if a side length is c. What is the total area of these two rectangles?

Definition 1. A rectangle with the sides b and c must be changed to a rectangle with the sides a and s to preserve the area: a*s = b*c or a/b = c/s.

Definition 2. A rectangle with the sides b and c must be changed to a square with the side s to preserve the area: s^2 = b*c or a/s = s/b. s is then called a medium proportional to a and b, or a geometric average of a and b.

Definition 3. In triangle ABC, the BC side is enlarged by the factor k to B*C. The line through B parallel to B*A cuts AC into A*. A*C is then called a k-incision for AC and BC.

Rule 1. A fourth proportional s to a, b and d is a d/a cut to a and b.

Rule 2. A triangle retains area by

Rule 3. A right triangle retains area at

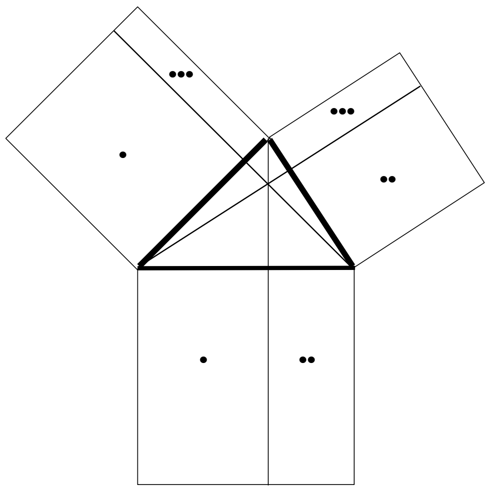
Rule 4. A rectangle retains area at

Rule 5. An mean proportional s to a and b can be constructed as the height of the semicircle arc above the a+b at the divider point.

GE08 OUTER SIDE SQUARES OF THE TRIANGLE, PYTHAGORAS

Basic problem: How can we make sure a triangle is right?

What applies to the outer side squares of a triangle?



Exercise 1. Construct a pointed triangle ABC with outer side squares. Set the three heights. The heights will divide the outer side squares into 3x2 parts. What is the case with these parts?

Exercise 2. Construct a right triangle ABC with outer side squares. Set the height. The height will divide one of the outer side squares into two parts. What is the case with these parts?

Exercise 3. Construct a blunt-angled triangle ABC with outer side squares. Set the three heights. The heights will divide the outer side squares into 3x2 parts. What is the case with these parts?

Exercise 4. Draw a triangle with heights and outer squares on cardboard. Cut out and road as a means of conviction.

Exercise 5. Use the land conservation technique as a means of conviction.

Exercise 6. Place a rectangle on a floor and use the diagonal to control the angles.

Exercise 7. Find some full numbers a, b, and c that meet the requirement a^2 + b^2 = c^2.

Exercise 8. Make of a string a closed loop with knots to check if an angle is right.

Exercise 9. Use a PC worksheet to set up diagonal calculation in a rectangle.

Definition 1. Triangle ABC's outer side squares are the squares a^2, b^2 and c^2.

Rule 1. In a triangle, the heights divide the outer squares of the triangle into one's "resettlement areas".

Rule 2. In triangle ABC is c^2 = a^2 + b^2 - 2*a*b*cosC

Rule 3. In the right triangle ABC (C right) is c^2 = a^2 + b^2 (Pythagoras' theorem)

Rule 4. In triangle ABC, C is right if c^2 = a^2 + b^2

GE09 NON-RIGHT TRIANGLE

Basic problem: How are the unknown parts calculated in a non-right triangle?

Exercise 1. Set off a known SSS triangle. Measure and calculate the three unknown angles.

Exercise 2. Set off a known SSA triangle. Measure and calculate the three unknown parts.

Exercise 3. Set off a known SAA triangle. Measure and calculate the three unknown parts.

Comment. An AAA triangle is only a AA triangle, as the last angle cannot be selected freely, but is determined by the other two. Thus, there is no information on a side, whereby the triangle becomes an SAA triangle.

Rule 1. In triangle ABC, the cosine relationships apply:

a^2 = b^2 + c^2 - 2*b*c*cos A

b^2 = a^2 + c^2 - 2*a*c*cos B

c^2 = a^2 + b^2 - 2*a*b*cos C,

Rule 2. In triangle ABC, the sinus relations apply:

a/sinA = b/sinB = c/sinC (watch out for the angles! why?)

GE10 OUTER BASE LENGTHS AND BASE ANGLES OF THE TRIANGLE

Basic problem: How can a triangle be measured from an outer base length? How can a triangle be re-established and calculated on the basis of external base dimensions?

Exercise 1. Construct a triangle ABC. Set off outside the triangle (possibly on the other side of the "river") an outer base line with known length KL. How can point A be determined from KL?

Exercise 2. Remove triangle ABC and restore it based on the outer base dimensions.

Exercise 3. How can triangle ABC's outer base dimensions (lengths or angles) form the basis for calculations of the triangle's parts, heights and area? Check by measurement.

Exercise 4. Use a PC spreadsheet to set up the calculation of the triangle's parts, heights, and area based on the outer base dimensions.

Definition 1. The base lengths of point A in relation to a base line KL are the lengths of the AK and AL line parts.

Definition 2. The base angles of point A in relation to a base line KL are the angles AKL and ALK.

Rule 1. The base angles can be calculated from the base lengths as follows:

AKL =

ALK =

Rule 2. The base lengths can be calculated from the base angles as follows:

h = height from K =

AK =

AL =

Rule 3. A right triangle ABC is considered from an outer baseline KL with length k.

Base lengths of the corners are referred to as:

KA = a1, KB = b1, KC = c1, LA = a2, LB = b2, LC = c2.

Base angles of the corners are referred to as:

AKL = A1, ALK = A2, BKL = B1, BLK = B2, CKL = C1, CLK = C2

The sides of the triangle can be calculated as follows:

a =

b =

c =

GE11 SQUARES

Basic problem: How are squares are designated and calculated?

Exercise 1. Construct different types of squares and name them.

Exercise 2. Find rules for a square's angles.

Exercise 3. Find rules for a square's diagonals.

Exercise 4. Find rules for a square area.

Exercise 5. Set a square and divide into two equal parts. Divide a square into three equal parts.

Exercise 6. Clip a square and find its center of gravity. Could we figure and make it to the center of gravity?

Definition 1. A trapeze is a square where a set of facing sides are parallel. A parallelogram is a square where both sets of facing sides are parallel. A rhombus is a parallelogram where all sides are equally long. A rectangle is a square where all angles are right. A square is a rectangle where all sides are equally long.

Rule 1. In a square, the angular sum

Rule 2. A trapeze with height h and parallel sides a and b has an area of A =

Rule 3. A parallelogram with height h and surface g has an area of A =

GE12 SPATIAL ANGULAR SHAPES, SURFACE AND VOLUME

Basic problem: How do describe spatial figures?

Exercise 1. Find different types of spatial shapes from everyday life and name them.

Exercise 2. Choose a box from everyday life. Measure the sides and diagonals. Calculate surface and volume, diagonal lengths and diagonal angles. If possible, check the volume by filling the box with water or sand, which can be poured into a cylinder glass that can measure volume.

Exercise 3. Construct a box according to given dimensions, partly of paper, partly of cardboard and partly on computer. If possible, check by calculating and measuring diagonal lengths.

Exercise 4. Choose an slanted cut-off box from everyday life, for example, a box that is cut off. Measure sides and angles. Calculate surface and volume. Measure and calculate diagonal lengths and diagonal angles.

Exercise 5. Set up a paper and floor tent. Measure or calculate side and diagonal lengths, angles, surface and volume.

Exercise 6. Construct a pyramid of paper and on the floor. Measure or calculate side and diagonal lengths, angles, surface and volume.

Exercise 7. A closed box with square bottoms must hold 1 liter. Describe different options in table or formula. Find the corresponding surface. Which box has the least surface. If necessary, use a PC spreadsheet.

Exercise 8. Same as exercise 7, but now the box is open at one end.

Exercise 9. Same as exercise 7, but now the box is open at both ends.

Exercise 10. A closed box with a square bottom shall have an outer surface of 1 m^2. Describe different options in table or formula. Find the corresponding volume. Which box has the largest volume. If necessary, use a PC spreadsheet.

Exercise 11. Same as exercise 10, but now the box is open at one end.

Exercise 12. Same as exercise 10, but now the box is open at both ends.

Exercise 13. Same as exercise 7-9, but now with a prism with an equilateral triangle as the base surface.

Exercise 14. Same as exercise 10-12, but now the prism has an equilateral triangle as the base surface.

Exercise 15. Same as exercise 7-9, but now with a pyramid with square surface instead of a box.

Exercise 16. Same as exercise 10-12, but now with a pyramid with square surface instead of a box.

Exercise 17. Same as exercise 7-9, but now with a pyramid with an equilateral triangular surface instead of a box.

Exercise 18. Same as exercise 10-12, but now with a pyramid with equilateral triangular surface instead of a box.

Exercise 19. Same as exercise 7-9, 13, 15 and 17, but now the material for the end surface is twice as expensive as aside. Minimum cost is requested.

Exercise 20. Where is the center of gravity of a box? How many degrees can the box be tilted before it topples over? Try and rain out.

Exercise 21. What is a polyhedron and what is a regular polyhedron?

Definition 1. A plane is

Definition 2. The angle between two planes is the angle between the normal of the plans.

Definition 3. A polyhedron or a multi-face is

Definition 4. A regular polyhedron is

Definition 5. A prism is

Definition 6. A fair prism is

Definition 7. A box is

Definition 8. A cube or cube is

Definition 9. A pyramid is

Definition 10. The volume or volume unit is a cube of 1x1x1. 1 liter = 1dm^3.

Rule 1. A box with side lengths a, b and c has the volume V =

Rule 2. A box of surface G and height h has the volume V =

Rule 3. A pyramid with a base G and height h has the volume V =

Rule 4. A box with side lengths a, b and c has the surface O =

Rule 5. A box with side lengths a, b and c has diagonal length D =

Rule 6. A roll with a side length a has diagonal length D =

Rule 7. The number of regular polyhedrons is

Rule 8. There are many rules for regular polyhedrons:

GE13 Round shapes: Circles, cylinders, spheres, etc.

Basic problem: How are round plane and spatial shapes calculated and calculated?

Exercise 1. Find a series of circular things from everyday life. How can we find the center? Measure the circumference and diameter with a sewing thread. How many times can the diameter be around the circle?

Exercise 2. Draw a circle on paper. Cut it into narrow "Pizza pieces." If these are placed alternately in contrast, a rectangle-like figure will appear. What is the relationship between the two sides? What can the rectangle tell us about a circle's circumference and area.

Exercise 3. What is the circumference and area of a pie section and a circle section?

Exercise 4. What is the surface and volume of a sphere? Road the garbage of a mandarin and way then 1 cm^2 of the garbage. Way an apple and way behind a cube of 1 cm^3 carved from the apple.

Exercise 5. What is the surface and volume of a sphere? Way a plastic sphere. Cut a piece of 1 cm^2 and way it.

Exercise 6. What is the surface of a bullet? Take a mandarin or an orange. Make an equator cut and two on each other perpendicular polar cuts. This divides the surface into eight equal round triangles, each containing an equilateral triangle. Show that the approximate area of these triangles is A = 8*1.41*(1.5*r)^2/2 = 4* 3.17*r^2.

Exercise 7. What is the surface and volume of a sphere? Take a mandarin or an orange. Place three circular cuts: at the equator, at 30° and at 60° north latitude. The parts can be flattened like a circle and the chops can be flattened like circular rings. What is the total area of the two circular rings and the top? Arrow the mandarin in boats and place them alternately opposite. This produces something resembling part of a box. Assess the total volume of the boats.

Exercise 8. What is the surface of a cylinder? Make a cylinder out of an A4 paper. What is radius and height?

Exercise 9. What is the volume of a cylinder? Make a small cylinder of paper. What is radius and height? Fill the cylinder with sand and then pour the sand into a measuring glass.

Exercise 10. What is the surface and volume of a leg, a head, one, human body?

Exercise 11. Cut a slice out of a circle and fold it to form a fussy house (a cone). What is the surface and volume of a cone? Fill the cone with sand and then pour the sand into a measuring glass.

Exercise 12. Compare the volume between a cylinder, a hemisphere and a cone of the same height.

Exercise 13. Cut a triangular and square pyramid out of a piece of paper. What is the surface and volume of a pyramid? Fill the pyramid with sand and then pour the sand into a measuring glass.

Exercise 14. A closed cylinder must hold 1 liter. Describe different options in table or formula. Find the corresponding surface. Which cylinder has the least surface. If necessary, use a PC spreadsheet.

Exercise 15. Same as exercise 14, but now the cylinder is open at one end.

Exercise 16. Same as exercise 14, but now the cylinder is open at both ends.

Exercise 17. A closed cylinder shall have an outer surface of 1 m^2. Describe different options in table or formula. Find the corresponding volume. Which cylinder has the most volume. If necessary, use a PC spreadsheet.

Exercise 18. Same as exercise 17, but now the cylinder is open at one end.

Exercise 19. Same as exercise 17, but now the cylinder is open at both ends.

Exercise 20. Same as exercise 14-15, but now with a cone instead of a cylinder.

Exercise 21. Same as exercise 17-18, but now with a cone instead of a cylinder.

Exercise 22. Same as exercise 14-16 and 20, but now the material for the end surface is twice as expensive as aside. Minimum cost is requested.

Exercise 23 (difficult). A tube with diameter 1 m is filled by three equal tubes with which diameter?

Exercise 24. Place a bicycle lamp on the rim of a bicycle wheel. What is called the curve that the lamp will form if the bike ride is seen from the side (best seen in the dark)? This curve can also be produced on paper by letting a circle move along the edge of the paper. From the front, a circular motion will be a swing up and down. Is it reasonable for physics to describe the fluctuation u by a harmonic oscillation as u = R*sin(wt)? What does it say? R, w and t too?

Exercise 25. How can we make gear for a bike? When is a bike easy to tread? When will a pedal turn move the bike the most?

Exercise 26. Construct a triangle and place it in a circle. How is the center of the circle? Can we calculate the radius in this circle (the circumscribed circle of the triangle)?

Exercise 27. Construct a triangle and place a circle in it. How is the center of the circle? Can we calculate the radius in this circle (the inscribed circle of the triangle)?

Exercise 28. Under what conditions can a square be wrapped in a circle that touches all four corners? How is the center of the circle? Can we calculate the radius in this circle (the square's rewritten circle)?

Exercise 29. Under what conditions can a square wrap a circle that touches all four sides? How is the center of the circle? Can we calculate the radius in this circle (the square's inscribed circle)?

Exercise 30. Find some space targets that are used in other locations and at other times.

Exercise 31 (difficult). What can be understood by the outer piping circles of a triangle and how are they constructed? What can be said about radius and center?

Definition 1. A circle with radius r and center C is

Definition 2. A pie slice is

Definition 3. A circle section is

Definition 4. A sphere with radius r and center C is

Definition 5. A cylinder is

Definition 6. A cone is

Rule 1. In a circle, the circumference-diameter ratio is always $\pi = 3.1416$

Rule 2. A circle with radius r has the circumference O =

Rule 3. A circle with radius r has the area A =

Rule 4. From the circular arc, the diameter will always be seen at a right angle.

Rule 5. A sphere with radius r has the surface O =

Rule 6. A sphere with radius r has the volume V =

Rule 7. A cylinder with a radius r and height h has the volume V =

Rule 8. A cone with the base surface radius r and height h has the volume V =

Rule 9. A cylinder with a radius r and height h has the diagonal D =

GE14 GEOMETRY ON A SPHERE SURFACE

Basic problem: How to designate and calculate points, distances, angles, triangles, areas, etc. on a sphere surface?

Exercise 1. Look at a globe. How are points on the surface described (London, Paris, Rome, etc.)?

Exercise 2. What is the distance between two points and how is it (London-New York, Paris-Tokyo, etc.)?

Exercise 3. When are three points on a straight line (Oslo, Prague, Cairo)?

Exercise 4. Madrid-Paris-Bangkok makes up a triangle. What are the angles? What is the area?

Exercise 5. Can we count on the answers to the questions in exercise 2-4?

Definition 1. A grand circle goes through two diametrically opposite points. A pole circle or meridian circle passes through the two poles. An equator circle stands perpendicular to a pole circle midway between the poles.

Definition 2. On a sphere surface with center C and poles N and S, the equator circle and a standard pole circle are loaded. The intersections of the circles are called A and A*. The coordinates of a point P are determined as follows: through the point, a pole circle is placed that cuts the equator circle into P*. The width of the point is angle P*CP with one of the indications north or south. The length of the point is angle P*CA with one of the indications west or east.

Definition 3. The distance between two points P and Q means the smallest of the arc lengths of the grand circle through P and Q.

Definition 4. Three points A, B and C determine three large circle arcs AB, AC and BC. If these large circle arches do not coincide, they will delineate a spherical triangle ABC as well as another 7 spherical triangles.

Definition 5. The spherical exces e for a spherical triangle ABC is degree surplus above 180 degrees: e = A+B+C-180.

Rule 1. For a right spherical triangle ABC applies:

sinA = sina/sinc,

cosA = tanb/tanc,

tanA = tana/sinb

Rule 2. For a crooked spherical triangle ABC applies:

sina/sinA = sinb/sinB = sinc/sinC,

cosa = cosb*cosc + sinb*sinc*cosA

Rule 3. On a sphere with radius r, a spherical triangle ABC has the area e*4πr^2/720

GE15 MOVING: TURNING, MIRRORING AND PARALLEL DISPLACEMENT

Basic problem: How to talk turning, mirroring and parallel displacement of a triangle?

Exercise 1. Set up a triangle. Make a copy of the triangle. With the copy, perform each of the operations turning, mirroring and parallel displacement. Then set up definitions for these operations.

Exercise 2. Set up a triangle. Make a copy of the triangle and place it randomly a distance from the original triangle. Describe how the two triangles can be brought to cover each other.

Exercise 3. Put a triangle on the floor. Shift the triangle 3 meters to the south and 2 meters to the north and turn it 40 degrees.

Definition 1. If triangle ABC is turned to triangle AB*C* the angle of rotation will be angle BAB*.

Definition 2. If triangle ABC is turned about AB to triangle ABC*, ABC* is said to be a reflection of ABC on the mirror axis AB.

Definition 3. A triangle ABC is moved to triangle A*B*C*. If the sides do not change direction, a parallel displacement is indicated, the size of which is indicated by the length AA* and if the direction is specified, for example, if the direction is indicated. at angle A*AC.

Rule 1. Any movement can be performed on one or more of the operations of turning, mirroring and parallel displacement.

GE16 BLUEPRINTS

Basic problem: How is a spatial shape transferred to a plane drawing that can be used to construct the shape?

Exercise 1. Build a character into LEGO bricks. Draw the shape as a blueprint with both front, side, top vision (FST drawing).

Exercise 2. Draw the shape from exercise 1 as a blueprint in skewed vision on ISO(metric) paper. From where should the shape be seen to be correctly enrolled?

Exercise 3. Build a shape out of bricks, some of which are round (hemispheres, cylinders, cones, etc.) and some triangular. Draw the shape as FST drawing. Also draw the figure on ISO paper.

Exercise 4. Make an FST drawing of a LEGO brick shape. Build the shape from the blueprint. Draw the shape on ISO paper.

Exercise 5. Make an FST drawing of a shape with round bricks. Build the shape from the blueprint. Draw the shape on ISO paper.

Exercise 6. Make a drawing on ISO paper of a LEGO brick shape. Build the shape from the drawing. Draw the shape as FST drawing.

Exercise 7. Make a drawing on ISO paper of a shape with round bricks. Build the shape from the drawing. Draw the shape as FST drawing.

Exercise 8. we can get work in a shipyard if we can read an ISO drawing of piping. Take a clip, copper wire or similar and bend it at 90-degree angles as a model of a piping. Sign on ISO paper and let others try to reconstruct the model based on the drawing.

Exercise 9. A kitchen is set on the wall of a room with cm-dimensions 250 x 250 x 250 (height x width x depth). At the bottom is a 25 x 250 x 50 socket. Upstairs on the left is a 150 x 50 x 50 cabinet, and on the right four 50 x 50 x 50 cabinets. Above these is with a clearance of 50 cm suspended four 50 x 50 x 25 wall cabinets. Make a blueprint of the kitchen. Make a drawing of the kitchen on ISO paper. (If necessary, use the more realistic numbers 30, 60, 180 and 300 cm.)

Exercise 10. Make an FST drawing of a house. Build the house in cardboard from the blueprint. Draw the house on ISO paper.

Exercise 11. As 10, but now the opposite.

DEFINITION 1. Place a cube so that it rests on the 4-side. The cube can be seen in three different ways: the 1-Side then indicates the front vision of the cube, the 2-side side vision and the 3-side top-vision of the cube.

DEFINITION 2. From an (isometric) skewed view of a spatial figure, all three visions are seen simultaneously. A shape is viewed from a point of view that is below the figure on the left. A skewed vision can be recorded on ISO(metric) paper, where one's distances are drawn the same. we are therefore disregarding the impact of perspective.

GE17 SHADOWS AND PROJECTION

Basic problem: How to calculate shadow shapes

Exercise 1. Put a book on a table under a lamp. Turn the book upwards in 10 degrees. Measure each time the angle of rotation and shadow length. Can the shadow length be calculated?

Exercise 2. Make a triangle of a thumb stick. Place a pocket lamp and triangle on a table so that the shade falls on a wall. Move the flashlight in a 20 cm spurt and measure each time the distance to the triangle and the parts of the shadow triangle. Can the shading triangle parts be calculated?

Exercise 3. Make a triangle of a thumb stick. Place a pocket lamp and triangle on a table so that the shade falls on a vertical plate at the end of the table. Turn the plate in a 10-degree spur. Measure each time the angle of rotation and the parts of the shadow triangle. Can the shading triangle parts be calculated?

GE18 PERSPECTIVE DRAWING

Basic problem: How does a spatial shape transfer to a plane drawing that shows what's being seen?

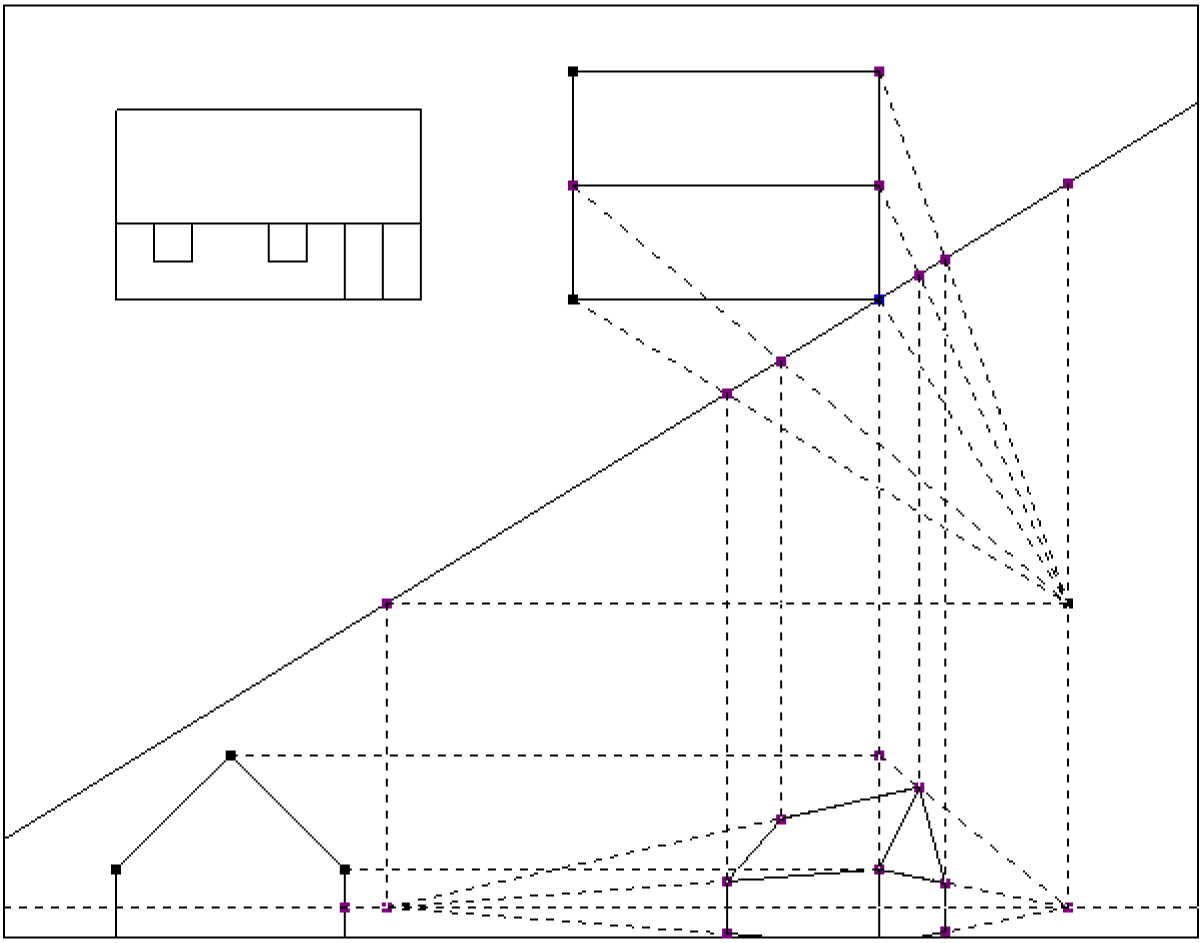
Exercise 1. Look along a surface on which there are parallel lines, such as lines on a brick wall. Move the eye in a vertical direction first up, then down. Which lines are horizontal and which fall and which increase? Move the eye in a horizontal direction first towards, then away from the surface. How does the steepness of the surface's lines change? What are the terms "horizontal line" and "disappearance point of parallel lines"?

Exercise 2. View a box-shaped thing through a window and copy the boxes' lines onto a piece of transparent plastic laid on the window. Mark the position of the eye on the plastic. Then copy the drawing onto a sheet of paper laid on top of the plastic. Sign horizontal line and vanishing points.

Exercise 3. The same exercise as exercise 2, this time, just used Albrecht Dürer's approach: Consider a box-shaped thing through a grid with horizontal and vertical lines, and copy the box lines onto a piece of A4 paper with the same number of horizontal and vertical lines as the grid. Mark the location of the eye. Sign horizontal line and vanishing points.

DEFINITION 1. Two points P and Q cast the shadows P' and Q' on a line l.

Front view Top view



Side view Perspective

Exercise 4. A house has meters 3 x 12 x 9 (height x width x depth). On top of the house there is a normal saddle roof that is 4 m high in the middle. Make an FTS drawing of the house. Use Top vision and side view to make a perspective drawing of the house seen in height 1 m from a position located 3 meters to the right and 6 meters in front of the house.

Exercise 5. Draw a tile floor seen from an eye level of 1.5 m when the horizontal distance to the tile floor is 1 m.

Exercise 6. Draw the shapes from the working drawing chapter as a perspective drawing.

Exercise 7. Draw the kitchen from the chapter on the drawing as a perspective drawing from a position that is 1 meter inside and 1 meter up in the kitchen and 1.25 meters from the wall of the closet.

Exercise 8. Draw a perspective drawing of a grid of 10 equally tall vertical bars with the same distance between them. Leave the first rod 10 cm high and let the horizontal line pass through the center of the rod. Set up two series of measurements from the drawing: one that measures the length of the rods and one that measures the distance from the vanishing point to the rods. What can be said about the two series of figures? Can the grid be perceived as a fractal?

Exercise 9. Draw a box from the supermarket in perspective (e.g. washing powder).

Exercise 10. Draw a house in perspective.

Exercise 11. Explore how we can draw in perspective on a PC.

NOTE. A perspective drawing takes into account the fact that a fixed length view decreases with increasing distance. A spatial figure is said to be drawn in perspective if all the points of the shapes are construct on a drawing plane perpendicular to the direction of vision using a point projection from the pupil of the eye (or a photographic film).

DEFINITION 1. A drawing plane is a plane perpendicular to the eye's direction of vision. A horizontal line at eye level is called the horizon line. The point on the horizon next to the eye is called the eye point. All points in the drawing plan are assumed to have the same distance from the eye.

DEFINITION 2. Parallel lines perpendicular to the direction of vision are drawn parallel. Otherwise, they are drawn so that they run together in the vanishing point of the lines on the horizon line.

COMMENT. Parallel planes: Subjects in the drawing plan are drawn from distance. Subjects in plans parallel to the drawing plan are drawn from distance, albeit on a different scale, determined by comparing top vision and side vision.

Rule 1. If a tile is drawn in perspective, its center point can be found as the intersection of the diagonals.

GE19 PARABOLAS AND PARABOLOIDS

Basic problem: Why are signals being transmitted in all directions being collected in the same direction?

Exercise 1. Set a line and a point outside the line. Select all points that are the same distance to the point and line. This shape is called a parable. The normal to the line through the point is called the axis of the parable.

Exercise 2. Construct a parable. Send a light ray (any laser light) in parallel to the axis. Examine the ray corridor after reflection on the side of the parable. Exercise can be performed by drawing or using light rays reflected in mirrors set up as keys in the reflection points.

Exercise 3. Try to prove that axis parallel rays pass the focal point of the parabola.

Exercise 4. Cut a parable in wood and use this to make a dish in clay or else. Place tin foil in the dish and try to use the dish as the sender and recipient of light.

Exercise 5. Borrow a car light from an auto-scrapper. Is it a dish?

Exercise 6. For a level clip in a cone, the following shapes can appear: Circle, ellipse, parable, and hyperbola (the so-called cone sections). How should the clips be laid in the different cases?

DEFINITION 1. A parable consists of all the points that have the same distance to a straight line (guideline) and a point (focal point). The normal to guide the guideline through the focal point is called the axis of the parable. A Paraboloid is produced by rotating a parable 360 degrees on its axis.

DEFINITION 2. An ellipsis consists of all the points whose distances to two given points (focal points) constitute a constant sum.

DEFINITION 3. A hyperbola consists of all the points whose distances to two given points (focal points) represent a constant differential.

Rule 1. At a parable, axis parallel rays will gather at the focal point, and rays emitted from the focal point will move axis parallels.

Rule 2. In the case of an ellipse, rays emitted from the focal point

Rule 1. At a hyperbola, rays emitted from the focal point

GE20 LIGHT BEND IN WATER AND IN GLASSES

Basic problem: How to calculate the bending of light rays when passing through water or glass?

Exercise 1. Fill a box-shaped glass with water. Allow a ray of light (any laser ray) to fall slanted onto the side of the glass. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the law of wrestling from physics.

Exercise 2. Let a light ray (possibly laser ray) fall slanted onto the side of a solid transparent glass box. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the law of wrestling from physics.

Exercise 3. Let a ray of light (possibly laser ray) fall slanted onto the side of a triangular transparent glass prism. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the law of wrestling from physics.

Exercise 4. Fill a cylinder-shaped glass with water. Allow a ray of light (any laser ray) to fall slanted onto the side of the glass. Mark with pins the ray passage through the glass and out on the other side. Measure the different angles and compare with the angle of a rainbow.

Exercise 5. Allow a ray of light (any laser ray) to fall perpendicular to a lens. Mark with pins the ray passage through the glass and out on the other side. Perform the test five times when the ray hits the lens in the middle, halfway and all the way out to both sides. Measure the different angles and compare with the law of wrestling from physics.

DEFINITION 1. A ray of light hits a wall. The angle between the normal and incoming ray of the wall is called the angle of approach. The angle between the normal and outgoing ray of the wall is called the outage angle. The angle between the normal of the wall and the broken ray is called the refraction angle.

Rule 1. On reflection, the incident ray angle = reflected ray angle.

Rule 2. In the case of refraction, sin(incident ray angle)/sin(reflected ray angle) = refractive index.

GE21 REPEATING FIGURES, FRACTALS

Basic problem: What aggregated figure comes from by figure repetition?

Exercise 1. Construct a square. Construct a new square whose side length is the diagonal of the previous square. Repeat this process many times.

Exercise 2. Construct a square. Construct a new square whose diagonal is the side length of the previous square. Repeat this process many times.

Exercise 3. Like 1 and 2, but now with a rectangle instead.

Exercise 4. Construct a right triangle ABC. Set a new right triangle one-angled with the first that has the short catheter as hypotenuse. Repeat this process.

Exercise 5. Construct a right triangle ABC. Set a new right triangle one-angled with the first that has the long catheter as hypotenuse. Repeat this process.

Exercise 6. Construct a right triangle ABC. Construct a new right triangle one-angled with the first that has the hypotenuse as the long catheter. Repeat this process.

Exercise 7. Construct a right triangle ABC. Construct a new right triangle one-angled with the first that has the hypotenuse as the short catheter. Repeat this process.

Exercise 8. Divide a right triangle using the height of the hypotenuse. What is the scaling factor? Repeat this process. Reverse the process.

Exercise 9. A horizontal line divides into two lines, each forming an angle of 30 degrees with the original line and half the length. Repeat this process many times.

Exercise 10. Make a number row starting with 0 and 1, with the next indent being the sum of the previous two. What are the properties of these Fibonacci numbers?

Exercise 11. Then divide a line AB with the point C/ AC = AC/BC (elevation, golden cut). What is the AC/CB scaling factor?

Exercise 12. Construct a square of the line AB. Let P be the golden cut in AB. Repeat the construction, but now from AP.

Exercise 13. Make a spiral with ray lengths $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, etc.

Exercise 14. Halves a square with side length 1 dm. Halves one part.

Exercise 15. Find some fractals yourself, e.g. in the program GeomeTricks.

Exercise 16. Find some fractals in nature.

DEFINITION 1. C is the golden cut in the line AB if AB/AC = AC/BC, i.e. if AC is mean proportional between AB and BC.

GE22 GEOMETRY OF AN OATMEAL BOX

Basic problem: Which geometry is hiding in an oatmeal box?

Suggestions for questions:

1. What is the surface and volume of an oatmeal box?

2. What is the smallest assault containing the given volume?

3. What is the largest volume that can be contained in the given surface?

4. What is the length and angles of the box's four diagonals?

5. Where is the center of gravity of the box? How much can the box be tilted without tipping over? (Full, half-filled)

6. Make a cylinder, a sphere, a cone, a pyramid, a tetrahedron with the same volume. What is the surface?

7. Make a cylinder, a sphere, a cone, a pyramid, a tetrahedron with the same surface. What's the volume?

8. Tilt the box so it looks like a steep mountain wall. Install a "hairpins" way up the box. What is the gravitational acceleration in this position?

9. Tilt the box to a specific position. How much is the load on the substrate reduced?

10. Draw the box, partly as blueprints, partly in perspective.

11. Draw the side of the box in scaled-down versions so that the wide side becomes diagonal on the wide side of the new box. What is the scaling factor? Repeat this process.

12. What is the smallest corner around the box?

13. If we can travel twice as fast on the wide side as on the narrow side, what is the fastest journey between two given points on the two sides?

14. Place 2-3 boxes on the dining table as islands in a large sea, and divide the sea between the islands.

GE23 NEWSPAPER GEOMETRY

Basic problem: Which geometry is hiding on a newspaper side?

Suggestions for questions:

1. What are the goals on a newspaper side? What are the relationships between these objectives? What is understood by Al, A2, A3, A4, A5, A6, etc. Paper?

2. Measure and calculate the diagonal (at least two ways) and its angles.

3. The diagonal divides the side into two triangles. Measure and calculate their height and elevation angles.

4. Calculate the area of the triangles in three different ways.

5. The height divides the triangle into two new triangles. How are these compared to the original? What are the scaling factors?

6. These new triangles can again be divided into new triangles. Repeat this split a few times and enjoy the resulting fractals.

7. As 2-6, but now with the newspaper folded once.

8. Calculate the right edge of the newspaper side from the lower left corner A when neither of the two center lines may be crossed.

9. A cracked corridor can be illustrated by three newspaper sides arranged appropriately in relation to each other. What is the length of the longest rod (bending ruler) that can travel around the corner? At what angle does the rod bump against both walls. Show that the angle can be calculated by the equation tanv^3 = a/b, where a and b are the two walking widths.

10. Draw the cheapest route between the two opposite corners when it is twice as expensive on one half. Show that sini/sinb = 1/2 where in and b is the angle of approach and refraction at the center line.

11. Fold a newspaper and open it 30 degrees. Sign a way up the newspaper that rises 20 degrees.

12. In two different ways, a newspaper side can be rolled around and form a cylinder. The surface is the same, is the volume also the same? What is the largest volume that can be formed by a newspaper side with the same area?

13. Fold the newspaper side and repeat the previous exercise. Will the volume be halved?

14. Fold the short side towards the long. Fold the remaining part to form a triangle. There will now be three triangles. What are their scaling factors?

GE24 Supplements: Equation Schemas

For calculations based on formulas, it is possible to use an equation schema that says:

What to figure out	Which equation to use
Which numbers are used	How the equation is reshaped

Example. In triangle ABC is C = 90°, A = 37° and c = 4.2. Find a and b and B.

a = ?	sinA = a/c
A = 37	c*sinA = a
c = 4.2	4.2*sin37 = a
	2.53 = a

b = ?	cosA = b/c
A = 37	c*cosA = a
c = 4.2	4.2*cos37 = a
	3.35 = a

B = ?	A+B = 90
A = 37	B = 90-A
	B = 90-37
	B = 53

Routine tasks

Tasks **Answers**

	A	B	C	A	B	C		A	B	C	A	B	C
1			3.917	33.3		90		2.151	3.274			56.7	
2			6.519	42.4		90		4.396	4.814			47.6	
3	2.534			23.8		90			5.745	6.279		66.2	
4	3.772			21.4		90			9.625	10.338		68.6	
5	2.707		4.628			90			3.754		35.8	54.2	
6	3.883		6.265			90			4.917		38.3	51.7	
7	7.502	3.611				90				8.326	64.3	25.7	
8	8.588	8.093				90				11.801	46.7	43.3	

Non-right triangles **Answers**

	A	B	C	A	B	C		A	B	C	A	B	C
9	1.075			57.2		72.4			0.985	1.219		50.4	
10	2.212			42.5		59.6			3.201	2.824		77.9	
11		4.104		66.9		72.6		5.812		6.029		40.5	
12		4.915		21.5		86.4		1.893		5.155		72.1	
13			2.165	35.1		68.6		1.337	2.259			76.3	
14			3.041	38.4		76.2		1.945	2.847			65.4	
15	1.748	3.562				68.6				3.346	29.1	82.3	
16	1.433	4.346				88.4				4.538	18.4	73.2	
17		3.078	4.892	60.8				4.326				38.4	80.8
18		3.938	3.706	59.3				3.787				63.4	57.3
19	4.298		5.027		42.9				3.477		57.3		79.8
20	4.861		4.439		81.8				6.097		52.1		46.1