Calculus

adds

(globally, piecewise, or locally) constant

PerNumbers

by areas

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Content

Preface

The key to core mathematics is one simple question: “5 4s and 3 2s add to what?”

Adding next-to as areas brigs you directly to integral calculus. And adding on-top after shifting units brings you directly to proportionality that leads on to per-numbers as 2$/5kg when including physical units.

Which again leads to the root of calculus, adding per-numbers as in mixture problems: Adding 2kg at 3$/kg and 4kg at 5$/kg, the unit-numbers 2 and 4 add directly to 6, but the per-numbers 3 and 5 must be multiplied to unit-numbers before adding, thus adding as areas as integral calculus, becoming differential calculus when reversing the question: “2kg at 3$/kg and 4kg at how many $/kg add to 6kg at 5$/kg”, or “5 4s and how many 2s add to 5 6s?”.

Calculus thus, occurs in three versions.

Primary school: adding bundle-numbers by their areas.

Middle school: adding piece-wise constant per-numbers as the area under the per-number graph.

High school: adding locally constant per-numbers as the area under the per-number graph.

A graphing calculator gives the answer directly. Before that, a smart trick gave the answer:

Written as differences, adding area-strips makes all middle terms disappear, and leaves only one difference between the last and the first term. So, differentiation needs to be developed to allow rewriting area-strips as differences, $f * dx = dg$, or $f = dg/dx = g'$.

Likewise, the relation between piece-wise and locally constancy needs to be established by interchanging the epsilon and delta in the formal definition for constancy:

“y is piecewise or locally constant c, if their numeric difference is less than any positive number in a given interval; or can be made so in any given interval.”

The aim of calculus also appears when writing out the number 345 as we say it,

345 = 3tenten + 4ten + 5 = 3 * ten^2 + 4 * ten + 5,

which uncovers the general number formula, $T = a * x$^2 + $b * x$ + c, called a polynomial.

This polynomial shows the formulas for constant change: proportional, linear, exponential, power and accelerated. And it also shows the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration.

Including physical units, we see there can be only four ways to unite numbers. Since in Arabic algebra means to reunite, we may call this beautiful simplicity ‘the algebra square’: addition and multiplication unite changing and constant unit-numbers; and integration and power unite changing and constant per-numbers.

As to the reverse operations: subtraction and division split a total into changing and constant unit-numbers; and differentiation together with factor-counting (logarithm) and factor-finding (root) split a total into changing and constant per-numbers.

Operations **unite**/*split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a*n}$ $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	$\boldsymbol{T = \int a*dn}$ $\frac{\mathrm{d}T}{\mathrm{d}n} = a$	$\boldsymbol{T = a\text{^}n}$ $\sqrt[n]{T} = a \qquad n = \log_a T$

Figure. The ‘algebra-square’ has 4 ways to unite, and 5 to split totals

Big Blunders in Calculus Education

Respecting the philosophical difference between existence and essence will uncover the Five Big Blunders of traditional calculus education:

1. Its name should be an action word as with the other operations. So, the name ‘Add per-numbers’ or ‘add and find per-numbers’ shows the action as well as the object acted upon or finding, where calculus does neither. Of course, ‘integrate’ and ‘differentiate’ are action words, but the objects, functions, are not specified as to units.

2. Integration should come before differentiation, since addition comes before subtraction, and since uniting comes before splitting. And since integral calculus occurs naturally in primary school when adding bundle-numbers with units, and in middle school when adding piecewise constant per-numbers in mixture problems. Also, it will allow and not prevent meeting the beauty of seeing millions of middle terms disappear when adding differences.

3. Integral calculus should be introduced in primary school as adding bundle-numbers with units, and in middle school as adding piecewise constant per-numbers in mixture problems.

4. Limits should be replaced by the three definitions of global, piecewise and locally constancy.

y is *constant c* if their numeric difference is arbitrarily small, i.e., if for all critical distances ε, their numeric difference is less than ε.	$\forall \varepsilon > 0 \ : \ \lvert y - c \rvert < \varepsilon$
y is *piecewise constant* c_A, if an area *A* exists where y is constant c_A:	$\exists A, \ \forall \varepsilon > 0 \ : \ \lvert y - c_A \rvert < \varepsilon$ in A
y is *locally constant* c_o, if for all critical distances ε there is an area *A* where y is constant c_o	$\forall \varepsilon > 0, \ \exists A \ : \ \lvert y - c_o \rvert < \varepsilon$ in A

And the concept should be exemplified by the formulas for π and e.

For n sufficiently big, $\pi = \tan(180/n)*n$, showing the limit for putting a polygon around a circle.

For n sufficiently big, $e = (1+1/n)$^n , showing the limit for compound interest.

5. Calculus should be taught together with its historic importance where Newton’s four corrections of the Pope laid the foundation to a modern industrialized and democratic society. Wanting to sail to India on open sea to avoid the Portuguese fortification of the African coast, the English had to follow the moon, but how does the moon move? The Pope thought he knew until corrected by Newton: “No, the moon does not move among the stars, it falls towards the earth as does the apple. No, the moon does not follow the unpredictable will of a metaphysical Lord, it follows its own gravitational will that is predicted by a formula. No, a will does not uphold order, it changes order. No, algebra does not work on change formulas, so I need to develop a change calculation myself.”

Golden Opportunities in Calculus Education

After the philosophical difference between existence and essence has uncovered the Five Big Blunders of traditional calculus education, it is now possible to avoid these blunders and make calculus accessible to all; in primary school, in middle school and in high school. Carpe Diem.

References.

Tarp, A. (2013). *Deconstructing Calculus.* http://youtu.be/yNrLk2nYfaY.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Content

This booklet contains essays, posters, abstracts and papers on calculus education written in the period from 2000 to 2021.

Essay

With Calculus as Adding Per-Numbers, all will Pass. This essay was written for the magazine of the Danish high school mathematics teachers in 2015, when a discussion began if it was possible to make calculus mandatory in the high school. The teacher union was reluctant, but I expressed a different opinion. However, the essay was not accepted. And it was also rejected as relevant for an in-service course discussing the matter. So, it is a silenced essay.

Posters

The Birth of Per-Numbers, a Poster at the ICME9 Conference in 2000. An American visitor was amazed: “If only you know how many problems you solve with this concept!”

Calculus Grounded in Per-numbers Added Next-to, a Poster at the ICME12 Conference in 2012.

Abstracts sent to conferences and journals, some extended

Calculus Conceptually Changed: From Deified to Reified.

Deconstructed, Calculus Rises in 3 Versions for Primary, Middle and High School.

Rising Again the Dropouts.

To Support Stem, Calculus Must Teach Adding Bundle-Numbers, Per-Numbers and Fractions Also.

From Place Value to Bundle-Bundles: Units, Decimals, Fractions, Negatives, Proportionality, Equations and Calculus in Grade One.

The Power of Per-Numbers.

The Simplicity of Mathematics Designing a Stem-Based Core Curriculum for Refugee Camps.

Mixing Design and Difference Research with Experiential Learning Cycles Allows Creating Classroom Teaching for all Students.

A Fresh-Start Year10 (Pre)Calculus Curriculum.

Papers

One Digit Mathematics. This paper is written together with Saulius Zybartas, and presented at the topic study group 1, new development and trends in mathematics education at pre-school and primary level at the ICME 10. It is published in a journal. Zybartas, S. & Tarp, A. (2005). One Digit Mathematics. *Pedagogika (78/2005)*, Vilnius, Lithuania.

Adding PerNumbers. This paper was presented at the topic study group 2, new development and trends in mathematics education at secondary level at the ICME 10. The paper suggests that to solve the relevance paradox in mathematics education postmodern sceptical Cinderella research could be used to look for new ways to teach mathematics at the secondary school. The paper introduces addition of per-numbers as a more user-friendly approach to the traditional subjects of proportionality, linear and exponential functions and calculus. In was published in a book. Tarp, A. (2004). Adding PerNumbers. In Bock, D. D., Isoda, M., Cruz, J. A. G., Gagatsis, A. & Simmt E. (Eds.) New Developments and Trends in Secondary Mathematics Education. *Proceedings of the Topic Study Group 2* (pp. 69-76). The ICME 10, Copenhagen Denmark.

Per-Number Calculus. The paper was written for the topic study group 12, research and development in the teaching and learning of calculus at the ICME 10. To solve the relevance paradox in mathematics education this paper uses postmodern sceptical fairy tale research to look for new ways to teach calculus in the school. A renaming of ‘calculus’ to ‘adding per-numbers’ allows us to think differently about the reality ‘sleeping’ behind our words, and all of a sudden we see a different calculus taking place both in elementary school, middle school and high school. Being a ‘Cinderella-difference’ by making a difference when tested, this postmodern calculus offers to the classroom an alternative to the thorns of traditional calculus. The paper was accepted for distribution.

FunctionFree PerNumber Calculus. This is a short paper written as my contribution to the Nordic presentation at ICME10. Observing that calculus did not call itself ’calculus’ it is suggested that calculus could also be called something else, thus the non-action word ‘calculus’ could be reworded to the action-word ‘adding per-numbers - taking place from K - 12. Then examples are given on adding per-numbers in primary school, in middle school and in high school. Also, people were invited to the MATHeCADEMY.net stand at the conference to discuss details.

Pastoral Calculus Deconstructed. The paper argues that calculus becomes ‘pastoral calculus’ killing the interest of the student by presenting limit- and function- based calculus as a choice suppressing its natural alternatives. Anti-pastoral sophist research searching for alternatives tochoice presented as nature uncovers the natural alternatives by bringing calculus back to its roots, adding and splitting stacks and per-numbers. The paper was written for the Topic Study Group16: Research and development in the teaching and learning of calculus. The paper war accepted.

Calculus Grounded in Adding Per-numbers. The paper argues that mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? What are the roots of calculus? The paper was written for the Topic Study Group 13: Teaching and learning of calculus. The paper was presented as a poster.

Saving Dropout Ryan With A Ti-82. The paper reports on how to lower the dropout rate in pre-calculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren’t even able to use a TI-30. A compendium called ‘Formula Predict’ replaced the textbook. A formula’s left-hand and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by ‘solve Y1-Y2 = 0’. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects. The paper was written for the Topic Study Group 18: Analysis of uses of technology in the teaching of mathematics. The paper was presented as a full paper.

PerNumbers Replace Proportionality, Fractions and Calculus. This paper was written for 13th International Conference of The Mathematics Education for the Future Project in Catania, Sicily, September 2015.

Primary, Middle and High School Calculus. At the ICME 13, all of a sudden, we were only allowed to send in one submission. Still, I wrote seven and chose the one focusing on philosophy. My paper was published in Tarp, A. (2016). From Essence to Existence in Mathematics Education. *Philosophy of Mathematics Education Journal No. 31* (November 2016).

Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations. In Växjö January 14-15, 2020, the Swedish Society for Research in Mathematics Education welcome to Madif 12, its twelfth research seminar in connection with the Matematikbiennalen 2020. The theme of the seminar is ‘Sustainable mathematics education in a digitalized world’. I sent in three papers inspired by (01), one on early childhood education (25), and one on middle school (26), and one on precalculus (27) as well as two proposals for a workshop (28, 29). All were rejected.

The Power of Bundle- and Per-Numbers Unleashed in Primary School: Calculus in Grade One – What Else? This paper was written for the 14th International Congress on Mathematical Education in Shanghai, 12th –19th July, 2020. I was allowed to send one submission only, so I chose the Topic Study Group 36 on Research on classroom practice at primary level to send in a paper. I was allowed to give a 10 minutes presentation. However, because of the corona situation, the congress was postponed a year, and I then replaced this paper with another paper.

Aarhus, May 2021, Allan.Tarp@gmail.com

With Calculus as Adding Per-Numbers, all will Pass

The world screams for engineers. And yet, the access road, calculus, is too steep for many. What to do? This article asks: How will calculus look like if it respects the roots of mathematics as a science of the natural fact Many?

Mathematics as a Science on Many

Daily we see many examples of Many: Many people, houses, cars, etc. We deal with Many with two competences, counting and adding. First counting gives numbers with units. Then ratios give per-number as 3$/kg, 4 m/s, and 5m/100m = 5% with like units.

Once counted as totals, these may be united to a grand total, resonating with the Arabic word 'algebra' meaning to re-unite. Written fully a total as T = 354 = 3*10^2 + 5*10 + 4*1 shows the four ways to unite: addition, multiplication, power, and next-to stack addition, also called integration. They are connected to the four kinds of numbers: constant and changing unit-numbers and per-numbers.

Operations predict counting results. Uniting 4$ and 5$ is predicted by addition: T = 4+5. Uniting 4$ 5 times is predicted by multiplication: T = 4*5. Uniting 4% 5 times is predicted by power: 100% + T = 104%^5. Uniting 4kg á 6$/kg with 5kg of 7$/kg is predicted by the area below the per-number curve: $T = 6*4 + 7*5 = \Sigma\, p*\Delta x$, also called integration.

The reverse process, splitting a total into parts, is called reversed calculation or solving equations. We see that the equation x+3 = 15 is solved by moving numbers to the opposite side with the opposite calculation sign. This 'opposite side & sign' also applies to the other operations. The third root is a factor-finder, the third logarithm is a factor-counter, and differentiation is finding per-number formulas.

Uniting/ Splitting into	Changing	Constant
Unit-numbers $, kg, sec	Plus: $T = a + b$ Minus: $T - b = a$	Times: $T = a * b$ Division: $T/b = a$
Per-numbers $/kg, $/100$ = %	Integration: $T = \int f\,dx$ Differentiation: $dT/dx = f$	Power: $T = a \wedge b$ Root: $b\sqrt{T} = a$ Logarithm: $loga(T) = b$

So, algebraically, integration combines multiplication and addition, while, as the reverse operation, differentiation combines subtraction and division.

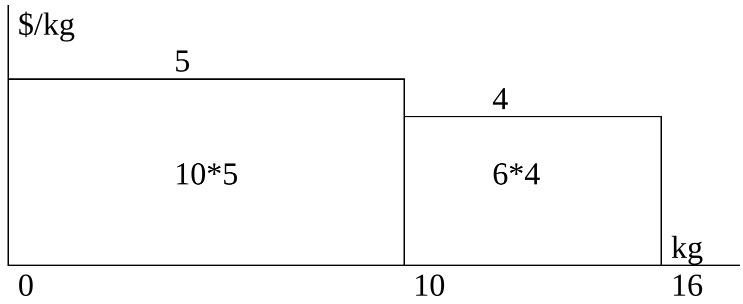
And geometrically, integration is the area below the per-number curve, while differentiation is the slope of the total curve.

Precalculus deals with change with a constant number or percentages, that is, about the second and third way of uniting. Calculus then deals with uniting changing per-numbers.

Averages and fractions as examples of per-numbers: 10 kg at 5 $/kg plus 6 kg at 4 $/kg gives 16 kg at ? $/kg. Her the unit-numbers unite by addition, while the per-numbers unite as the area under the per-number curve.

10 kg at 5 $/kg gives 10*5 = 50 $
6 kg at 4 $/kg gives 6*4 = 24 $

16 kg at ? $/kg gives 50+24 $



The reverse question is: '3 kg at 4 $/kg and 5 kg at how many $/kg gives 8 kg a 7 $/kg?' The per-number p is obtained by subtracting the first total $T1$ from the total T, and then by dividing by 5, i.e., by differentiation: $p = (T-T1)/5 = \Delta T/5$.

Fractions are also per-number: 3 $ per 5 kg = 3$/5kg = 3/5 $/kg. For example, 1 red of 2 apples plus 2 red of 3 apples unite to T = 1/2*2 + 2/3*3 = 3/5*5. Again, addition unites the unit-numbers 2 and 3, where the area under the per-number unites the per-numbers 1/2 and 2/3.

It sounds incredible today, but past textbooks claimed that 1/2 + 2/3 gave 7/6, that is, that the total should be seven red of six apples. Mathematics, which is true in the library but not in the laboratory, may be called 'matematism'. Hopefully, this kind of mathematics is about to disappear.

Three Types of Constancy

With averages, the per-numbers are piecewise constant. But what to do with a changing per-number, such as a free-falling object's meter/second number, also called speed?

Again, the area below the per-number curve gives the answer, as variable numbers are usually locally constant (continuous). Local constancy over a small piece is the third of three kinds of constancy:

A variable number y is globally constant c, if for all positive numbers d, the distance between y and c is less than d.

A variable number y is piecewise constant c, if there is an interval radius r, so that for all positive numbers d, the distance between y and c is less than d within r.

A variable number y is locally constant c, if for all positive numbers d there is an interval radius e, so the distance between y and c is less than d within r.

Because local constancy means constancy over small intervals, the area under the per/number curve is calculated as a sum of many small area strips, which happens quickly with a computer. However, there is a cunning shortcut, because when summarizing single-changes, the middle terms disappear, so that the total change is just the difference between the end number and the start number.

Number T	Single change ΔT	Total change $\Sigma\Delta T$
to		
$t1$	$t1 - to$	$t1 - to$
$t2$	$t2 - t1$	$t2 - to$
$t3$	$t3 - t2$	$t3 - to$

The shortcut for calculating the area under an y-curve is, therefore, to rewrite area strips $y*dx$ as an area-change dA, so that $dA = y*dx$, or $dA/dx = y$.

Suppose we can show that for a small x-change, dx, the area strip $2*x*dx$ can be rewritten to the change $d(x\text{^}2)$. The small area strips below the per-number curve $y = 2x$ then have the size $y*dx = 2*x*dx = d(x\text{^}2)$. Summing the single-changes from $x = 1$ to $x = 4$ then gives the total change of x^2, as the end number minus the start number, i.e., 4^2 – 1^2 = 15. This result is confirmed on a CAS calculator.

If we use an old-fashioned S as a symbol for a sum of small changes, we can now write:

Area = $\int_1^4 2 * x\, dx = \Delta x\text{^}2 \,\Big|_1^4$ = 4^2 - 1^2 = 15

In order to make full use of the shortcut, we need to look more closely at calculating changes.

Change Calculations

We saw above that the number 345 is in fact a sum of power-terms called a poly-nomial. The change of power formulas may be studied by looking at a rectangle with the sides f and g, both of which are assumed to depend on the same variable x.

df	$df * g$	$df * dg$
f	$A = f * g$	$f*dg$
	g	dg

A small change in x, dx, gives a small change in f and g, df and dg, which gives the area $A = f*g$, a small change, dA, which consists of three pieces, $df *g$, $f*dg$, and $df* dg$.

Since $df*dg$ can be made arbitrarily small, dA and $f*dg + f*dg$ will be locally alike.

Forming the per-number with dx, the following formula is available for local change:

$d(f*g)/dx = df/dx*g + f*dg/dx$, or $(f*g)' = f'*g + f*g'$, where $df/dx = f'$.

Or, as ratios: $(f*g)'/(f*g) = f/f' + g/g'$

If $y = x$, then $y' = dy/dx = dx/dx = 1$.

If $y = x^2 = x*x$, then $y' = (x*x)' = x'*x + x*x' = 1*x + x*1 = 2x$

If $y = x^3 = x^2*x$, then $y' = (x^2*x)' = (x^2)'*x + x^2*x' = 2x*x + x^2*1 = 2x^2+x^2 = 3x^2$

Similarly, we see that $(x^4)' = 4x^3$, $(x^5)' = 5x^4$, etc.

Since $(x^2)' = d(x^2)/dx = 2x$, we have thus, shown that $2*x*dx = d(x^2)$.

Slope Calculations

Drawing various examples of the number formula $y = a*x^3 + b*x^2 + c*x + d$, we see that the y-curve begins in d with slope c, which it curves away from, to later (or sooner) to curve back towards. Therefore, we can call d for start-level, c for start-slope, b for start-curvature and a for counter-curvature.

These names correspond to the corresponding level, slope and curvature formulas:

$y = a*x^3 + b*x^2 + c*x + d = d$ for $x = 0$

$y' = 3a*x^2 + 2b*x + c = c$ for $x = 0$

$y'' = 6a*x + 2b = 2b$ for $x = 0$

Since the per-number indicates the slope of the total curve, a per-number sign shift from minus to plus will change curve from decreasing to growing through an intermediate bottom point.

Likewise, there is a top where the per-number has a sign shift from plus to minus. In addition, we see that the top and bottom point are related to negative and positive curvature respectively.

	2		4		x	
/	Local Max	\	Local Min	/	$y = x^3 - 9x^2 + 24x + 5$	
+	0	–	0	+	$y' = 3x^2 - 18x + 24$	$= 0$ for $x = 2$ and $x = 4$
	–	0	+		$y'' = 6x - 18x$	$= 0$ for $x = 3$

If the per-number is constant, the curve is linear. If the per-number is piecewise constant, the curve is piecewise linear. If the per-number is locally constant, the curve is locally linear.

If the local per-number is maintained, the curve will follow a straight line that is locally congruent with the curve around the contact point, called the tangent of the curve.

The tangent equation therefore is: $\Delta y/\Delta x = dy/dx$, where $\Delta y = y – yo$ and $\Delta x = x – xo$.

The Historical Background of Calculus

Calculus arose when England wanted to use stolen Spanish silver to buy pepper and silk in India, and had to sail after the moon on the high seas to avoid Portugal's fortification of the coast of Africa. The church's view of how the moon moves was shunned by Newton's four times NO:

No, the moon does not move between the stars, it falls towards the earth like the apple, but in a fall whose curvature corresponds to that of the earth. Consequently, its fall is eternal.

No, the moon and the apple do not follow the erratic will of a metaphysical Lord, they follow their own physical will, which is predictable as it follows a formula of gravity.

No, a force does not give motion, but a change in motion.

No, algebra cannot be used for change calculations, we need to develop a new operation, calculus.

Model Building with Regression

Number formula $T = 3*x\text{^}2 + 5*x + 4*1$ gives rise to different formula types:

Straight lines of constant growth, $y = a*x + b$, where determining the 2 constants a and b requires a table with 2 rows of matching x and y values.

Curved lines (parabolas) with constant acceleration, $y = a*x\text{^}2 + b*x + c$, where determination of the 3 constants a, b and c requires a table with 3 rows of matching x and y values.

Curvature-changing lines (double parabolas), $y = a*x\text{^}3 + b*x\text{^}2 + c*x + d$, where determining the 4 constants a, b, c and d requires a table of 4 rows of matching x and y values.

In mathematics models, regression can translate table data into formulas that can answer relevant questions. With per-number tables, you will typically ask for the average per-number. And for unit tables, you will typically ask about change conditions and for optimizing top or bottom points.

Conclusion

Calculus often hides its origins as addition of changing per-numbers. This makes calculus difficult, resulting in high failing rates. Conversely, how to add changing per-numbers is so simple to understand that everyone will pass the calculus exam and continue to study the use of per-numbers in economy, engineering or science. Per-number based calculus provides ample time for the repetition of the substance, for making modeling projects, and for additional written tasks both solved with and without a CAS formula calculator. The large failing rate in calculus is therefore not nature, but a conscious or unconscious choice, which is presented as nature. Of course, one can excuse oneself by not being informed of the alternative to the prevailing tradition. This is precisely why difference research has been developed as a method of uncovering hidden differences to choices presented as nature. The per-number difference presents a choice to calculus education. You can no longer excuse yourself by simply following orders. One can realize your nature as a human being through a conscious choice: should I maintain a high failing rate, or should I make the last of the four ways to unite numbers, calculus, accessible to all?

Material

Material on per-number based calculus may be downloaded free of charge from the MATHeCADEMY.net website. http://mathecademy.net/various/us-compendia/

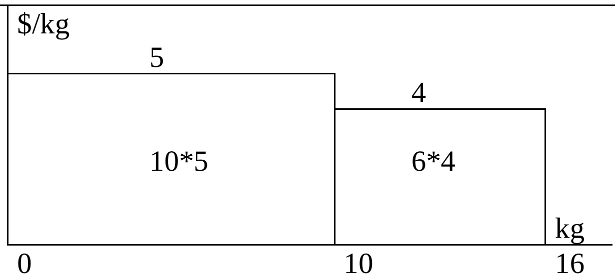
The compendium contains 15 mathematics models for, among other things, distance determination, forecasting, pension, golf, fund-raising, wine cartons, driving, exchange rate fluctuations by takeover attempts, linear programming, game theory, etc.

Also, there is a website with modeling papers, ttp://mathecademy.net/various/mathmodelingpapers/

Finally, there are several MrAlTarp YouTube videos, e.g., Deconstructing Calculus, http://youtu.be/yNrLk2nYfaY.

The Birth of Per-Numbers on a Poster at the ICME9 in 2000

Adding Per-Numbers

Multiplication gives areas: 10 kg @ 5 $/kg = 10*5 = 50 $ 6 kg @ 4 $/kg = 6*4 = 24 $	
16 kg @ *x* $/kg = 16**x* = 74 $ *x* = 74/16 = 4.63 $/kg **Hence 5 $/kg + 4 $/kg = 4.63 $/kg**	**so** **Per-Numbers add as areas:** **Integral Calculus**
2 s @ 3 m/s = 2*3 = 6 m + 6 s @ 6 m/s = 6*6 = 36 m	
8 s @ *x* m/s = 8**x* = 42 m *x* = 42/8 = 5.25 m/s **Hence 3 m/s + 6 m/s = 5.25 m/s**	
5000 $ @ 1/2 = 5000/2*1 = 2500 $ + 3000 $ @ 1/6 = 3000/6*1 = 500 $	
8000 $ @ *x* = 8000**x* = 3000 $ *x* = 3000/8000 = 0.375 = 375/1000 **Hence 1/2 + 1/6 = 375/1000**	
6500 $ @ 3% = 6500/100*3 = 195 $ + 1500 $ @ 6% = 1500/100*6 = 90 $	
8000 $ @ *x* % = 8000/100**x* = 285 $ *x* = 285/80 = 3.6 % **Hence 3 % + 6 % = 3.6 %**	

+ 3% *+ 6%* *+ 9.2%*
5000 -----> 5150 -----> 5459 5000 ----------> 5150
**1.03* **1.06* **1.092*
Hence 3 % + 6 % = 9.2 % = simple + compound interest

Calculus Grounded in Per-numbers Added Next-to

Primary School Calculus is Rooted in Next-To Stack Addition

Mathematics is a natural science dealing with the natural fact Many by bundling and stacking. 1.order counting bundles sticks into icons with five sticks in the 5-icon making 5 1s to 1 5s. 3.order counting bundles in tens needing no icon since 10 means 1 bundle. 2.order counting bundles in icon-bundles. Thus, a total of 7 can be bundled in 3s as T = 2 3s & 1 = 2.1 3s

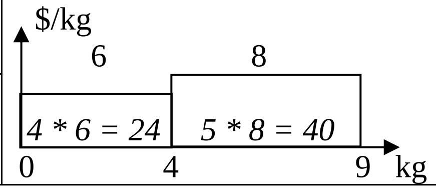
Once counted, totals can be added on-top or next-to. With on-top addition the units must be the same using the recount-formula $T = (T/B)*B$ saying that T/B times Bs can be taken away from T. Thus, 3 7s can be recounted in 5s as T = (3*7/5)*5 = 4.1 5s. Changing units roots proportionality. Next-to addition of 2 3s and 3 4s as areas to 2.4 7s roots integration.

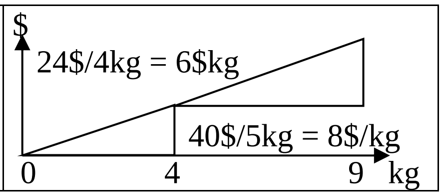
Middle School Calculus is Rooted in Adding Per-Numbers

Recounting creates fractions when, e.g., counting 3 1s in 5s as T = 3 = (3/5)*5 = 0.3 5s, which shows that 3/B = 0.3 for any bundle-size B; and 'per-numbers' when a quantity is recounted first as 2$ then as 5kg thus, containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg.

In primary school addition next-to adds stacks by integrating bundles. Now integration adds per-numbers when adding two recounted quantities, asking, e.g., 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg. This question can be answered by using a table or a graph.

4kg at 6$/kg	= 4 * 6 = 24
5kg at 8$/kg	= 5 * 8 = 40
9kg at ?$/kg	= 9 * ? = 64
	? = 64/9





Using a graph, we see that integration means finding the area under a per-number graph; and opposite that the per-number is found as the gradient on the total-graph.

Hign School Calculus is Adding Locally Constant Per-numbers

Middle school per-numbers are piecewise constant. In high school per-numbers are locally constant, continuous.

Still, the area under its graph adds variable per-numbers f by summing up area strips $f*dx$.

This area can be found by a graphical calculator, or by integration using that the sum of many small changes gives a total change:

$\int dF = \Delta F = F2 - F1$,

using technology or the head to solve the change equation $dF = f*dx$.

Calculus Conceptually Changed: From Deified to Reified

Abstract

Calculus adds areas by writing the strips as differences so addition makes all middle terms disappear and leaving only the first and last terms. It is as simple as that. And, taught as a natural science about Many, ManyMath, calculus occurs in grade one when adding bundle-numbers as 2 3s and 4 5s next-to, and in middle and high school when adding piecewise and locally constant per-numbers. However, taught as self-referring essence-based ‘mathematism’, true inside but seldom outside classrooms, things look completely different.

Here, because of the limit concept, calculus is described as the Mount Everest of high school mathematics, a college course only reachable for the chosen few that in return get an exclusive insight into what is called the divine nature of mathematics.

Outside classrooms, integral and differential calculus are extremely simple. The names show they integrate and differentiate something called per-numbers; a concept that is banned by the simple fact it makes mathematics too easy. For the same reason school also bans the bundle-numbers children develop while adapting to Many. Instead, forcing upon them the orthodoxy of single-numbers adding without units quickly leads to widespread innumeracy and dislike.

However, changing from single numbers without units to bundle-numbers with units allows calculus to enter grade one when asking 2 3s plus 4 5s total how many 8s; and reversely, 2 3s plus how many 5s total 4 8s? And to enter middle and high school when adding per-numbers as, e.g., asking 3kg at 2$/kg plus 5kg at 4$/kg total 8kg at how many $/kg?

So, reified, calculus becomes easy.

Statement of the Theoretical Problem

Created to add per-numbers by their areas, integral calculus normally is the last subject in high school, and only taught to a minority of students. But, since most STEM-formulas express proportionality by means of per-numbers, the question is if integral calculus may be taught earlier and to all students. Difference research (Tarp, 2018) may give an answer.

An Account of the Theoretical Proposal Being Made

Reified, integral calculus occurs in grade one when performing next-to addition of bundle-numbers as, e.g., T = 2 3s + 4 5s = ? 8s, leading on to differential calculus as the reverse question: 2 3s + ? 5s = 3 8s, solved by first removing 2 3s from 3 8s and then counting the rest in 5s, thus, letting subtraction precede division, where integral calculus does the opposite by letting multiplication creating areas precede addition.

In middle school adding per-numbers by areas occurs in mixture problems: 3kg at 2$/kg + 5kg at 4$/kg = 8kg at ? $/kg, again with differential calculus coming from the reverse question: 3kg at 2$/kg + 5kg at ? $/kg = 8kg at 4$/kg. Here the per-number graph is piecewise constant c, i.e., there exists a delta-interval so that for all positive epsilons, the distance between y and c is less than epsilon. With like units, per-numbers become fractions thus, also added by their areas, and never without units.

In high school adding per-numbers occurs when the distance travelled with a varying per-number P is found as the area under the per-number graph now being locally constant, formalized by interchanging epsilon and delta. Here the area A under the per-number graph P is found by slicing the area thinly. If writing the area strips as differences, addition will make all middle terms disappear and leave only the first and last terms. Alternatively, the last strip represents the change of the area, dA, and may be written as $dA = P*dx$, thus, motivating developing differential calculus to find a formula A with the property that $A' = dA/dx = P$.

Here, looking at the shadow of an $f*g$ book directly gives the formula $(f*g)'/(f*g) = f'/f + g'/g$, exemplified, e.g., by $(x^2)' = 2x$.

Review of the Relevant Literature

Mathematics education sees its goal as mastering university mathematics, seen as a pure self-supporting science theorizing one-dimensional number sets organised by different operations. And later introducing differentiation and integration as operations on functions seen as set-relations or subsets of set-products where first-component identity implies second-component identity.

As an alternative Kuhnian paradigm, ManyMath sees mathematics as a natural science about the outside fact Many as shown by geometry and algebra meaning earth-measuring and reuniting in Greek and Arabic; and developing children's already existing mastery of Many with two-dimensional bundle-numbers that when added next-to by their areas presents integral calculus before differential, which makes irrelevant the existing calculus literature doing the opposite and neglecting bundle-numbers.

Instead, theoretical guidance comes from seeing mathematics education as an institutionalized goal-directed treatment of human brains, thus, being theorized by sociology, philosophy, and psychology. Here however, internal controversies necessitate choices to be made. In sociology, this project chooses agency over structure by using Bauman, Habermas and Foucault; in philosophy it chooses empiricism over rationalism by using existentialism; and in psychology it chooses nature over culture by choosing Piaget over Vygotsky.

Clarifying the Novel Contribution of this Particular Project

ManyMath education has the goal to outside master Many, where traditional mathematics education has the goal to inside master mathematics so other subjects later may apply it outside.

Accepting and developing the bundle-numbers children create when adapting to Many, ManyMath teach, not numbers, but numbering, using functions from grade one as number-language sentences that, as in the word-language, contain a subject, a verb, and a predicate. The tradition insists on teaching single-numbers, and postpones functions to high school.

In grade one, on-top addition of bundle-numbers leads to proportionality making the units the same, and next-to addition leads to integral calculus by adding areas, and to differential calculus when reversed. In middle school both reappear when adding or subtracting piecewise constant per-numbers by their areas in mixture problems, to be followed in high school by locally constant per-numbers instead, again with integration preceding differentiation. The tradition only teaches on-top addition of single numbers, and teaches fractions as mathematism being added without units. Mixture problems are not illustrated geometrically to show the relationship with calculus, and the concepts of piecewise and local constancy are absent. Finally, differential calculus is taught before integral calculus, and local constancy and linearity is called continuous and differentiable.

Empirical Research that Could Test the Validity of the Theoretical Proposal

Being very costly to change expensive textbooks and long-term teacher education makes testing the validity of Reifying Calculus difficult inside a traditional education, except for where it is stuck, e.g., adding fractions without units. But it may be tested outside: in preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

Implication of this Work for Further Theory Development

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows Reifying Calculus to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental learning curricula as, e.g., in Vietnam.

Implications for Practice

Institutionalized education systematizes adaption by teaching children about their outside world, and teenagers about their inside talents and potentials. Thus, to develop the students' mastery of Many, outside abstracted ManyMath must replace todays inside derived university mathematics by presenting calculus at all three school levels.

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Deconstructed, Calculus Rises in 3 Versions for Primary, Middle and High School

Extended Abstract

"Calculus is hard - and therefore only for the few, and only late in high school".

This tradition has established itself worldwide (Rasmussen et al, 2014; Bressoud et al, 2016). Which motivates the core sociological question "Is calculus hard by nature, or by choice?" (Bauman, 1990).

Postmodern deconstruction may provide an answer by using 'Difference Research' (Tarp, 2018) searching for hidden differences that may make a difference.

Thus, to deconstruct calculus education means asking the philosophical questions "Which ontological fact lies behind the epistemological construct calculus? How does this fact present itself phenomenologically to the user? And what psychological consequences does this have for making calculus a tool for all students?"

Algebraically, Hindu-Arabic numbers as 456 = 4*B^2 + 5*B + 6*1 are sums of monomials, i.e., a sum of areas geometrically. They show that there are four ways to unite numbers: number-addition, multiplication, power, and area-addition; and that only monomials add, thus, making simple numbers operators, that may be multiplied, but needing a unit-factor to add.

In this way, area-addition roots primary school calculus if allowing children to add what they bring to school, bundle-numbers or area-numbers as 2 3s. Here, adding two areas when asking "2 3s + 4 5s = ? 8s" roots integral calculus, that reversed roots differential calculus when asking "2 3s + ? 5s total 3 8s", answered by removing the 2*3 area before counting the rest in 5s, i.e., by letting subtraction precede division:

(3*8 – 2*3)/5 = ΔArea/5.

Later, in middle school, double-counting a quantity in two different units leads to per-numbers becoming fractions with like units: 2$/5kg, and 2$/5$ = 2/5 = 40/100 = 40%.

As with simple numbers, also per-numbers and fractions are operators to be multiplied to areas before adding, now as the area under a per-number graph; and again, rooting differential calculus when reversed.

Examples: "2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg"; and "1/2 of 30$ + 4/5 of 60$ = ? of 90$".

An a by b area shows the geometrical meaning of per-numbers. Multiplied, $a*b$ gives the area. Divided, their per-number b/a gives the diagonal's turning angle, tan$A$ = b/a, as well as its steepness, $\Delta y/\Delta x = b/a$.

Another example of adding per-numbers is speed: "2sec at 3m/sec + 4sec at 5m/sec = 6sec at ? m/sec".

Observing that a ball falls with accelerated speed makes high school physics formulate speed-questions as "5sec at 0m/sec increasing gradually to 50m/sec totals how many meters?"

Here, the speed per-number, s, is not piecewise, but locally constant. So again, the answer is the area under the per-number s-graph, now split into many small slices to add up. But how to add many small numbers? Very simple, if the numbers may be written as differences making all middle terms cancel out and leaving only the difference between the terminal and initial numbers.

So, finding the area under a locally constant s-graph roots differentiation as a difference-equation: Find a formula F so that the slice $s*\Delta x$ can be written as a difference ΔF: $s*\Delta x = \Delta F$, or $s = \Delta F/\Delta x$? Alternatively, we may say that the area always changes with the last slice, ΔArea = $s*\Delta x$, or s =

$\Delta A/\Delta x$. So, also high school differentiation lets subtraction precede division as in primary and middle school calculus.

Again, the area gives the answer. In an x-by-x area, a small change in the sides, dx, will make the area change with two slices $x*dx$, and a very small corner dx^2, so d(x^2) ≈ 2$x*dx$, or d/dx(x^2) = 2x. Likewise, in an x by x^2 area, d/dx(x^3) = 3x^2.

Once created to find the area under a per-number graph, differential calculus may also be applied for optimization purposes since the per-number gives the steepness of the diagonal in the dy-by-dx area.

We can formalize the difference between piecewise and locally constancy by observing that a variable y is globally constant c if their difference is less than any positive number epsilon.

Likewise, y is piecewise constant c if an interval delta exists so that I y-c I < epsilon in delta.

Finally, y is locally constant c if for any epsilon an interval delta exists so that I y-c I < epsilon in delta. So, to go from piecewise to locally constant, we just interchange epsilon and delta.

Another traditional learning obstacle is that calculus builds on pre-calculus, seen by many teachers as the most difficult course to teach since here all lacking student knowledge surface.

However, here a difference making a difference is the ‘Algebra Square’ showing the four ways to reunite and split-into constant and changing unit-numbers and per-numbers: Addition/subtraction unites/split-into changing unit-numbers; and multiplication/division unites/split-into constant unit-numbers. Whereas power/root&log unites/split-into constant per-numbers; and integration/differentiation unites/split-into changing per-numbers (Tarp, 2012).

Thus, where the focus of calculus is to unite and split-into changing per-numbers, the focus of pre-calculus is to unite and split-into constant per-numbers.

Or, rephrased from a change-perspective we may say that where pre-calculus is about constant change, calculus is about changing predictable change, and statistics is about changing unpre- but post-dictable change.

Reversing the exponential formula for constant change-percentage, $y = b*a$^x, equations as and 5 = 2^x and 5 = x^3 root a logarithm as a factor-counter, and a root as a factor-finder.

Consequently, calculus is not hard by nature, but by choosing to neglect its three outside ontological roots: reuniting and splitting-into bundle numbers and piecewise or locally constant per-numbers. And to neglect that only numbers with units, i.e., monomials, add meaningfully. A fact that is not accepted until calculus.

These roots present themselves in a natural way to the learners having therefore little problems learning to do calculus if allowed to meet existence before essence by choosing a Piagetian over a Vygotskian approach.

Thus, in education, primary and middle school calculus as well as presenting integral before differential calculus in high school remains to be tested and thoroughly researched, e.g., experientially (Kolb, 1983) using design research (Bakker, 2018).

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Rising Again the Dropouts

Children come to school with a number-language developed through communicating about Many using two-dimensional bundle-numbers with units as 2 3s. However, the school ignores this and forces upon them one-dimensional line-numbers without units making many drop out.

A math dropout feels the full burden of the classroom as a 'pris-pital' (Foucault, 1995), i.e., a mixture of a prison forcing you to return again and again, and a hospital curing you for a self-referring diagnose 'as math-ignorant you must learn math.' Just entering makes your stomach turn because of negative associations. The teacher's look shows the negative expectations you also feel inside. You go to your place in the back where you are neglected or bullied. You are down with no chance to rise again. Then a miracle happens. A virus comes and forces all out of the classroom to home education, where all of a sudden you get a second chance to again do what you once mastered so fully, to communicate about Many (Widdowson, 1978). Your school even offers rise-again curricula (Tarp, 2021) saying that the goal of mathematics education is to master, not inside mathematics, but its outside root Many.

In MC01, micro-curriculum 1, at your folded right hand you point to 4, to 5, and to 6 when including the arm. You then rearrange a total of four sticks as one 4 icon, reported inside on paper by writing *T* = I I I I = 4. Likewise with 5 and 6. Thus, observing that digits are icons with their number of sticks if written less sloppy.

In MC02, inside you write *T* = 56 = 5ten6 = 5Bundle 6 = 5*B* 6 = 5x*B* + 6 = 5x*B* + 6x1. Thus, observing that 56 is not one number, but two numberings of bundles, and of unbundled singles, to be shown outside in different ways, and as two boxes of bundles and singles.

In MC03, inside you write *T* = 456 = 4hundred 5ten 6 = 4tenten 5ten 6 = 4*BB* 5*B* 6 = 4x*B*^2 + 5x*B* + 6x1. Thus, observing that 456 is three numberings of bundles-of-bundles, of bundles, and of unbundled singles, to be shown outside in different ways, and as three boxes.

In MC04, inside you draw the 'algebra-square' (Tarp, 2018) showing the 4 ways to unite as shown in the formula above. Here addition and multiplication unite changing and constant unit numbers, and box-addition (integration) and power unite changing and constant per-numbers. Now you see the simplicity of math.

In MC05, outside you re-count eight cubes in 2s by 4 times pushing away 2s with a card, inside iconized by an uphill stroke called division, and predicted by a calculator as '8/2 = 4'. Then you stack the bundles, inside iconized by a lift called multiplication, and predicted by a calculator as '4x2 = 8'. Then inside you report by writing, first 8 = (8/2)x2, then *T* = (*T*/*B*)x*B* with unspecified numbers, called a 'recount-formula' solving equations: *u*x2 = 8 is solved by recounting 8 in 2s, *u*x2 = 8 = (8/2)x2, so *u* = 8/2. And also changing units.

In MC06, outside you re-count cubes in dollars, given that there is 3$ per 5cubes called a per-number 3/5 $/cube. You then answer the two questions by recounting in the per-number:

Q01: "12$ = ? cubes." *T* = 12$ = (12/3) x 3$ = (12/3) x 5 cubes = 20 cubes

Q02: "? $ = 15 cubes." *T* = 15 cubes = (15/5) x 5 cubes = (15/5) x 3$ = 9$

In MC07, outside you use the diagonal to halve a box with base *b*, and height *h*, which inside allows you to set up trigonometry formula when mutually recounting the sides: *h* = (*h*/*b*) x *b* = tan*A* x *b*, etc.

In MC08, outside you observe that two totals 2 3s and 4 5s may add both on-top after recounting provides like units, and next-to by areas as 8s, also called integration; becoming differentiation when asking the reverse question '2 3s and how many 5s total 3 8s?', thus, subtracting the 2 3s before dividing by 5.

In MC09, outside in the bill ‘2kg at 3$/kg and 4kg at 5$/kg’, the unit-numbers add directly, but the per-numbers must be multiplied to unit-numbers before adding. So, per-numbers add by areas, called integral calculus, that again may be reversed to differential calculus finding per-numbers.

Now, you are far ahead in mathematics, and will return to the classroom as a star teaching how to master Many.

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To Support Stem, Calculus Must Teach Adding Bundle-Numbers, Per-Numbers and Fractions Also

Created to add locally constant per-numbers by their areas, integral calculus normally is the last subject in high school, and only taught to a minority of students. But, since most STEM-formulas express proportionality by means of per-numbers, the question is if integral calculus may be taught earlier. Difference research searching for hidden differences finds that the answer is yes.

Integral calculus occurs in grade one when performing next-to addition of bundle-numbers as, e.g., *T* = 2 3s + 4 5s = ? 8s, leading on to differential calculus as the reverse question: 2 3s + ? 5s = 3 8s, solved by first removing 2 3s from 3 8s and then counting the rest in 5s, thus, letting subtraction precede division, where integral calculus does the opposite by letting multiplication creating areas precede addition.

In middle school adding per-numbers by areas occurs in mixture problems: 2kg at 3$/kg + 4kg at 5$/kg = 6kg at ? $/kg, again with differential calculus coming from the reverse question: 2kg at 3$/kg + 4kg at ? $/kg = 6kg at 5 $/kg. Here the per-number graph is piecewise constant c, i.e., there exists a delta-interval so that for all positive epsilon, the distance between y and *c* is less than epsilon. With like units, per-numbers become fractions thus, also added by their areas, and never without units.

In high school adding per-numbers occurs when the meters traveled with varying m/s speed *P* is found as the area under the per-number graph now being locally constant, formalized by interchanging epsilon and delta. Here the area *A* under the per-number graph *P*, is found by slicing the area thinly so that its change may be written as d*A* = *P***dx* in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus, motivating developing differential calculus to find *A*' = d*A*/*dx* = *P*.

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From Place Value to Bundle-Bundles: Units, Decimals, Fractions, Negatives, Proportionality, Equations and Calculus in Grade One

Traditionally, a multi-digit number as 2345 is presented top-down as an example of a place value notation counting ones, tens, hundreds, thousands, etc.; and seldom as four numberings of unbundled, bundles, bundle-bundles, bundle-bundle-bundles, etc., to provide a bottom-up understanding abstracted from concrete examples, which would introduce exponents in primary school as the number of bundle-repetitions. Counting ten fingers in 3s thus, introduces bundle-bundles:

T = ten = 3*B*1 3s = 1*BB*1 3s.

Stacking bundles, the unbundles singles may be placed as a stack next-to leading to decimals, e.g., *T* = 7 = 2.1 3s; or on-top of the stack counted as bundles thus, leading to fractions, *T* = 7 = 2 1/3 3s; or to negative numbers counting what is needed for another bundle, *T* = 7 = 3.-2 3s.

Bundles and negative numbers may also be included in the counting sequence:

0*B*1, 0*B*2, …, 0*B*7, 1*B*-2; 1*B*-1, 1*B*0, 1*B*1, …, 9*B*7, 1*BB*-2, 1*BB*-1, 1*BB*.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s may be written as a recount-formula:

8 = (8/2)*2, or *T = (T/B)*B*,

used to solve the equation *u**2 = 8 by recounting 8 in 2s to give the solution *u* = 8/2; thus, solving most STEM-equations, typically expressing proportionality.

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

At the end of grade one, recounting between digits and tens leads to tables and equations when asking, e.g., *T* = 4 6s = ? tens, and *T* = 42 = ? 7s.

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The Power of Per-Numbers

Uniting unit-numbers as 4$ and 5$, or per-numbers as 6$/kg and 7$/kg or 6% and 7%, we observe that addition and multiplication unite changing and constant unit-numbers into a total, and integration and power unite changing and constant per-numbers. Reversely, subtraction and division split a total into changing and constant unit-numbers, and integration and power split a total into changing and constant per-numbers.

Recounting 8 in 2s as 8 = (8/2)*2 creates a recount-formula $T = (T/B)*B$, saying 'From T, T/B times, T may be pushed away'; and used to change units when asking, e.g., 2 6s = ? 3s, giving the prediction T = (2*6/3)*3 = 4*3 = 4 3s.

Recounting 8 in 2s also provides the solution u = 8/2 to the equations as u*2 = 8 = (8/2)*2; thus, solving most STEM-equations, since the recount-formula occurs all over. In proportionality, $y = c*x$; in coordinate geometry as line gradients, $\Delta y = \Delta y/\Delta x = c*\Delta x$; in calculus as the derivative, $dy = (dy/dx)*dx = y'*dx$. In science as meter = (meter/second)*second = speed*second, etc. In economics as price formulas: $ = ($/kg)*kg = price*kg, $ = ($/day)*day = price*day, etc.

With physical units, recounting gives per-numbers bridging the units. Thus, 4$ per 5kg or 4/5 $/kg gives T = 15kg = (15/5)*5kg = (15/5)*4$ = 3$; and T = 16$ = (16/4)*4$ = (16/4)*5kg = 20kg. With like units, per-numbers become fractions.

Trigonometry occurs as per-numbers when mutually recounting sides in a rectangle halved by its diagonal, $a = (a/c)*c = \sin A*c$, etc.

Modeling mixtures as 2kg at 3$/kg + etc, unit-numbers add directly, but per-numbers P add by the area A under the per-number graph, found by slicing it thinly so that the change may be written as dA = $P*dx$ in order to use that when differences add, all middle terms disappear leaving just the endpoint difference, thus, motivating developing differential calculus to find the per-number A' = dA/dx = P.

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The Simplicity of Mathematics Designing a Stem-Based Core Curriculum for Refugee Camps

Numbers as 2345 evade the place value notation if seen as numbering unbundled, bundles, bundle-bundles, bundle-bundle-bundles. Here exponents occur as the number of bundling-repetitions, e.g., when counting ten fingers as *T* = ten = 3*B*1 3s = 1*BB*1 3s.

Stacking bundles in blocks, the unbundled singles may be placed as a stack next-to leading to decimals, e.g., *T* = 7 = 2.1 3s; or on-top of the stack counted as bundles leading to fractions, *T* = 7 = 2 1/3 3s, or to negative numbers counting what is needed for another bundle, *T* = 7 = 3.-2 3s.

Bundles and negative numbers may be included in the counting sequence: 0*B*1, 0*B*2, …, 0*B*7, 1*B*-2; 1*B*-1, 1*B*0, 1*B*1, …, 9*B*7, 1*BB*-2, 1*BB*-1, 1*BB*.

Counting makes operations icons: division is a broom pushing away bundles, multiplication is a lift stacking bundles, subtraction is a rope pulling away stacks to find unbundled, and addition is the two ways to unite stacks, on-top and next-to.

Recounting 8 in 2s gives a recount-formula, 8 = (8/2)*2 or *T* = *(T/B)*B*, that changes unit from 1s to 2s (proportionality), that gives the equation u*2 = 8 the solution u = 8/2 (moving to opposite side with opposite sign), and that shows that per-numbers as 8/2 must be multiplied to areas before being added (integral calculus).

Once counted, stacks may add on-top after recounting changes the units to the same, or next-to by adding areas as in integral calculus. And reverse addition leads to differential calculus by pulling away the initial stack before pushing away bundles.

Recounting between digits and tens leads to tables and equations when asking, e.g., *T* = 4 6s = ? tens, and *T* = 42 = ? 7s. Recounting in different units gives per-numbers bridging the units, becoming fractions with like units, and adding by areas. Mutually recounting sides in a block halved by its diagonal gives trigonometry.

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Mixing Design and Difference Research with Experiential Learning Cycles Allows Creating Classroom Teaching for all Students

International tests show that not all students benefit from mathematics education. This poor-performance-problem raises a Cinderella question: is there a hidden difference that can make the Prince dance? If so, design research can create Kolb's experiential learning cycles to adapt a given micro-curriculum so that all students may benefit.

In primary school, difference research searching for hidden differences has identified several alternatives: Digits are icons. Numbers are double-numbers with bundles as units, e.g., T = 2 3s. Flexible bundle-numbers have over- and underloads, e.g.,

$T = 53 = 5B3 = 4B13 = 6B\text{-}7$ tens, and ease operations as, e.g.,

$329/7 = 32B9/7 = 28B49/7 = 4B7 = 47$, or $23*8 = 2B3*8 = 16B24 = 18B4 = 184$.

Operations are icons also where division is a broom pushing away bundles, multiplication a lift stacking bundles, subtraction a rope pulling away stacks to find unbundled, and addition the two ways to unite stacks, on-top and next-to.

Changing units may be predicted by a recount-formula $T = (T/B)*B$ coming from recounting 8 in 2s as 8 = (8/2)*2, or, and used to solve the equation $u*2 = 8$ by recounting 8 in 2s to give the solution $u = 8/2$; thus, solving most STEM-equations, typically predicting proportionality: meter = (meter/sec)*sec = speed*sec.

In middle school, double-counting leads to per-numbers becoming fractions with like units, and adding by their areas as integral calculus. In algebra, factors are units placed outside a bracket. Trigonometry occurs when mutually double-counting sides in a rectangle halved by its diagonal.

In high-school, redefining inverse operations allows equations to be solved by moving to opposite side with opposite sign. And adding per-numbers by areas allows introducing integral calculus before differential calculus.

Designing and redesigning micro-curricula as experiential learning cycles allows teachers perform design research in their own classroom, to be reported as master projects first, and later perhaps as PhD projects including more details.

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A Fresh-Start Year10 (Pre)Calculus Curriculum

Often precalculus suffers from lacking student knowledge. Three options exist: make mathematics non-mandatory, choose an application-based curriculum; or, to rebuild student self-confidence, design a fresh-start curriculum that also includes the core of calculus by presenting integral calculus first.

Writing a number out fully as a polynomial, e.g., T = 345 = 3*B^2 + 4*B + 5 shows the four ways to unite numbers, resonating with the Arabic meaning of the word algebra, to reunite: addition and multiplication unite changing and constant unit-numbers into totals; and next-to-block-addition (integration) and power unite changing and constant per-numbers, all having reverse operations that split totals into parts.

Addition, multiplication, and power are defined as counting-on, repeated addition and repeated multiplication. As reverse operations, x = 7-3 is defined as the number that added to 3 gives 7, thus, solving the equation x+3 = 7 by moving to opposite side with opposite sign. Likewise, x = 7/3 solves x*3 = 7, the factor-finder (root) x = 3√7 solves x^3 = 7, and the factor-counter (logarithm) x = log3(7) solves 3^x = 7, again moving to opposite side with opposite sign.

Hidden brackets allow reducing a double calculation to a single:

2+3*x = 14 becomes 2+(3*x) = 14, solved by x = (14-2)/3.

Next transposing letter-equations as $T = a+b*c$^d really boost self-pride.

Future behavior of 2set unit-number tables is predicted by linear, exponential, or power models assuming constant change-number, change-percent, or elasticity.

1-4set per-number speed tables are modeled with lines, parabolas and double-parabolas, allowing technology to calculate the distance covered, thus, introducing integral calculus, that also occurs when adding per-numbers in mixture-problems, and when adding percent in cross tables generated by statistical questionnaires.

Trigonometry comes from mutual double-counting sides in a rectangle halved by its diagonal, and is used to model distances to far away points, bridges, roads on hillsides, motion down an incline, and jumps from a swing.

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Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

One Digit Mathematics

*To solve the relevance paradox in mathematics education this paper suggests that LIB-mathematics as 2+3 = 5, valid only in the library and not in the laboratory, should be replaced by LAB-mathematics as 2*3 = 6, also valid in the laboratory. Replacing modern authorized routines with postmodern authentic routines turns elementary mathematics upside down by bringing the authority back to the Many-laboratory where mathematics may be learned from one-digit numbers only.*

Lib-Mathematics and Lab-Mathematics

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998) and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371). From the first day at school mathematics teaches that 2+3 = 5 and 2*3 = 6. That 2*3 = 6 is easily validated in the laboratory by re-counting 2 3s as 6 1s: *** *** -> * * * * * *.

2+3 = 5 is true in the library, but not in the laboratory where countless counter-examples exist: 2 weeks+3 days = 17 days, 2m+3cm = 203 cm etc. This '2&3-paradox' makes it possible to distinguish between LIB-mathematics from the library and LAB-mathematics from the laboratory where only the latter can be validated in the laboratory. And we may assume that the relevance paradox disappears when LIB-mathematics is replaced with LAB-mathematics. This hypothesis can be tested with a proper sceptical method.

Skepticism Towards Words

Modern natural science has established research as a number-based 'LAB-LIB research' where the LIB-statements of the library are induced from and validated by reliable LAB-data from the laboratory as illustrated by, e.g., Brahe, Kepler and Newton, where Brahe by studying the motion of the planets provided LAB-data, from which Kepler induced LIB-equations that later were deduced from Newton's LIB-theory about gravity. So, research is based upon numbers, not upon words.

The scepticism towards words is validated by a simple 'number&word-observation': Placed between a ruler and a dictionary a thing can point to a number but not to a word, so a thing can falsify a number-statement in the laboratory but not a word-statement in the library; thus, numbers carry research, while words carry interpretations, which presented as research become seduction - to be met by sceptical counter-research replacing LIB-authority with LAB-authenticity (Tarp 2003).

From Authorized Routines to Authentic Routines

Postmodernism means scepticism, especially towards authorized routines creating problems to modern society (Bauman 1989: 21). To replace authorized routines with authentic routines the authority must be moved from the library back to the laboratory to allow 12 educational meetings with the root of mathematics, Many, teaching us through educational questions and activities.

1 Repetition Becomes Many

The first educational question is: How can we represent temporal repetition in space? One answer is iconisation: Put a finger to the throat and add a match or a stroke for each beat of the heart:

. . . . -> I I I I I I I I

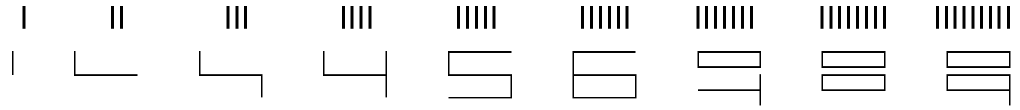
2 Many Becomes Bundles

The next educational question is: How can we organize Many? One answer is bundling: line up the total and divide it into bundles:

I I I I I I I I I -> II II II II I or I I I I I I I I I -> IIIII IIII or I I I I I I I I I -> III III III or …

3 Bundles Become Icons

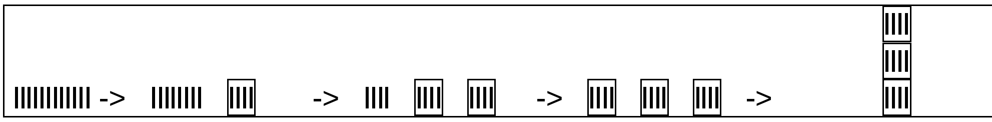
The next educational question is: How can we represent the different degrees of Many? Again ionization is an answer: the strokes of the different degrees of Many are rearranged as icons, realizing that there would be four strokes in the number-icon 4 etc., if written in a less sloppy way.



1 2 3 4 5 6 7 8 9

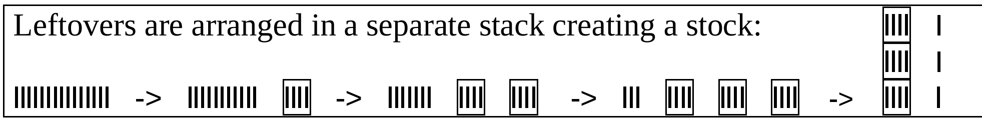
4 Many is Counted as a Stack

The next educational question is: How can we account for the different degrees of Many? One answer is counting by bundling and stacking: First the total is lined up, then it is bundled and equal bundles are stacked and finally the height is counted as, e.g., *T* = 3 4s = 3*4.



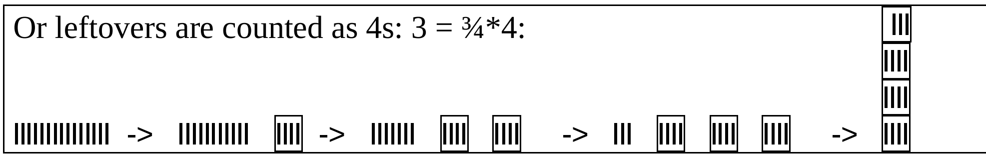
T = 3 4s = 3*4
(a stack)

Leftovers are arranged in a separate stack creating a stock:



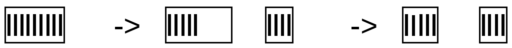
T = 3*4 + 3*1
(a stock)

Or leftovers are counted as 4s: 3 = ¾*4:



T = 3*4 + ¾ *4
= 3 ¾ *4

We count in 4s by taking away 4s. The process ‘from *T* take away 4’ may be iconized as ‘*T*-4’ and worded as ‘T minus 4’. The 4 taken away does not disappear, they are just put aside so the original total T is divided into two totals, one containing T-4 and the other containing 4:

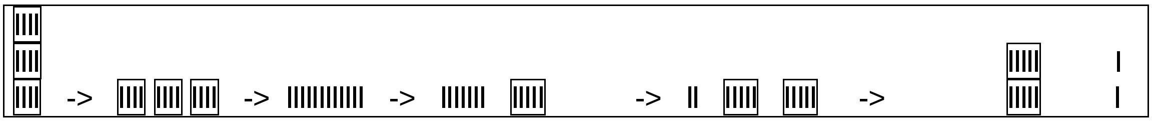


9 = (9-4) + 4 = 5 + 4 as predicted by the ‘restack-equation’ *T* = (T-B)+B.

The repeated process ‘from *T* take away 4s’ may be iconized as ‘*T*/4’ and worded as ‘*T* counted in 4s’. So, the ‘recount-equation’ *T* = (*T*/4)*4 predicts the result of recounting the total *T* in 4-bundles: *T* =(T/4)*4 =3*4+3*1 =3 ¾*4. *T*/4 is called a per-number, *T* a stack-number or a total, and 4 a unit.

5 Stacks are Recounted

The next educational question is: How can we change the bundle-size in a stack (*T* = 3 4s = ? 5s). One answer is de-stacking, de-bundling, re-bundling and re-stacking: First the stack is de-stacked into separate bundles, then the bundles are de-bundled into a total, then the total is bundled and equal bundles are stacked and finally the heights are counted.



3*4 = 4 +4 +4 = *T* = (*T*-5) +5 = (*T*/5)*5 = 2*5 + 2*1

Again the result can be predicted by the recount-equation *T* = (*T*/5)*5 = (3*4/5)*5 = 2*5 + 2*1 and displayed on a calculator able to do integer division as, e.g., the Texas Instruments’ Math Explorer.

6 Stacks are Symbolized

The next educational question is: How can we symbolize a stack? One answer is to use symbols as C and S to symbolize different bundle sizes, e.g., C = 4 and S = 5. Then the recount-question ‘*T* = 3*4 = ?*5’ is reformulated to ‘*T* = 3*C = ?*S ‘.

C	
C	S I
C -> C C C -> IIIIIIIIIIII -> IIIIIII S -> II S S ->	S I

3*C = C+C+C = T = (T-S)+ S = (T/S)*S = 2*S + 2*1

Also, the recounting can be done as in trade by skipping the 1s: Thus, if 1*C = 2*S we have

CCC	-> SS SS SS	-> SSSSSS
T = 3*C	= 3*2*S	= 6*S

SSSSSS	-> SS SS SS	-> CCC	
T = 6*S	= 3*2*S	= 3*C	predicted by 6*S = ((6/2)*2)*S = 3*2*S = 3*C

SSSSSSS	-> SS SS SS S	-> CCC S	
T = 7*S	= 3*2*S + S	= 3*C + 1*S	predicted by 7*S = ((7/2)*2)*S = 3*2*S+1*S = 3*C+1*S

With units a stack can by counted in Cs to practise tables. Thus, C C C C can be counted as ‘2 4 6 8’ or ‘ 3 6 9 twelve’. Here ‘twelve’ regains its original meaning ‘two left’ making it possible to count from ten to ‘two-ten’ as: ten and: one left, two left, three left, …, nine left, two ten etc.

7 Stocks are Symbolized

The next educational question is: How can we symbolise stocks? One answer is to use short writing by leaving out the stack symbols: *T* = 2*C+3*1 -> 2]3] = 23. Here we need a symbol ‘0’ for ‘none’ to distinguish *T* = 2*C = 20 from *T* = 2*1 = 2. Thus, stocks slowly become many-digit numbers.

With this short writing we can still practise recounting stocks by reformulating the recount-question ‘*T* = 3*C + 1 = ?*S’ to ‘*T* = 31 = ?*S’, e.g., with C = 4 and S = 5:

		T = 3*C + 1
C	I	= 31
C	S I	*T* = 2*S + 3*1
C I -> C C C I -> IIIIIIIIIIIII -> IIIIIIII S -> III S S ->	S I	= 23

3 1 = *T* = (T/S)*S = 2 3

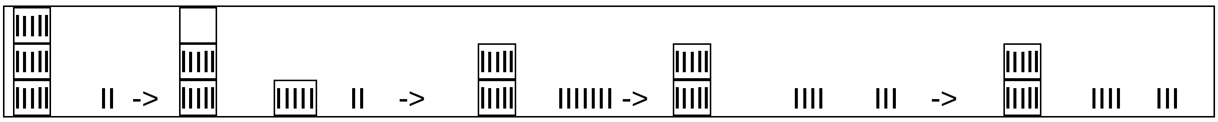
When the icons stop, we can use symbols. Thus, the Romans used the symbol X for the number ten.

The recount-question ‘*T* = 3*8 = ?*X’ recounts 3 8s as 2 tens and 4 1s: *T* = 3*8 = 2*X+4*1 = 24. This leads to traditional multiplication. However, 3*8 is 3 8s; only if recounted in tens it is 24.

The recount-question ‘*T* = 3*X = ?*8’ recounts 3 tens as 3 8s and 6 1s: *T*= 3*X= 30/8*8= 3*8+6*1. This connects traditional division with recounting, and not with sharing. Thus, sharing is applying and not creating division.

8 Stacks are Sold

The next educational question is: How can we sell from an empty stack? From the stock *T* = 3 5s + 2 1s we want to sell 3 1s, but we only have 2 1s in stock. The answer is to restack 1 5s to 5 1s:



3*5 + 2 = (3*5-1*5) + 1*5 + 2 = 2*5 + 7 = 2*5 + (7-3) + 3 = 2*5 + 4 + 3

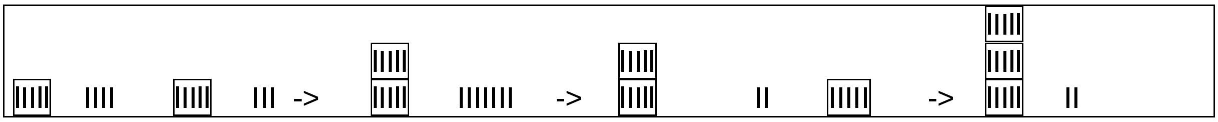
Working with symbols we use ‘internal trade’:

T = 32 -3 = 3]2] -3 = 3-1]5+2] -3 = 27 -3 = 24.

In the case of tens, we have *T* = 32 -3 = 3]2] -3 = 3-1]10+2] -3 = 2]12] -3 = 29

9 Stacks are Bought

The next educational question is: How can different stocks be added? To the stock *T* = 1 5s + 4 1s we add the stock T' = 1 5s + 3 1s. Adding the 1s we are able to recount 7 1s to 1 5s + 2 1s as predicted by the restack-equation: *T* = 7 = (7-5) + 5 = 2 + 1*5



1*5 + 4 + 1*5 + 3 = 2*5 + 7 = 2*5 + (7-5) + 5 = 3*5 + 2

Working with symbols we have *T* = 14 + 13 = 1]4] + 1]3] = 2]7] = 2+1]-5+7] = 3]2] = 32.

In the case of tens we have *T* = 14 + 17 = 1]4] + 1]7] = 2]11] = 2+1]-10+11] = 3]1] = 31

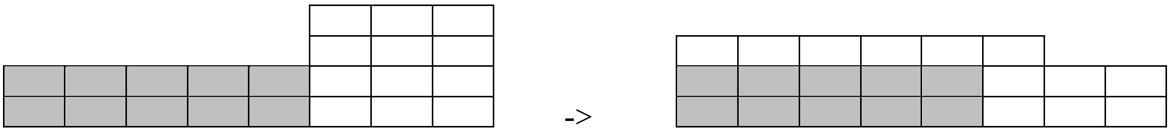
10 One Digit Calculus

The next educational question is: How can stocks be added differently? The stacks 2 5s and 4 3s can be 'added in time' as 3s or as 5s, or 'added in space' as 8s, which is called integration or calculus.

Added as 3s: *T* = 2 5s + 4 3s = 2*5 + 4*3 = (2*5)/3*3 + 4*3 = 3 1/3 * 3 + 4*3 = 7 1/3 * 3

Added as 5s: *T* = 2 5s + 4 3s = 2*5 + 4*3 = 2*5 + (4*3/5)*5 = 2*5 + 2 2/5 * 5 = 4 2/5 * 5

Added as 8s: *T* = 2 5s + 4 3s = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8



T = 2*5 + 4*3 = 2 6/8 *8 = (2*5+4*3)/8*8

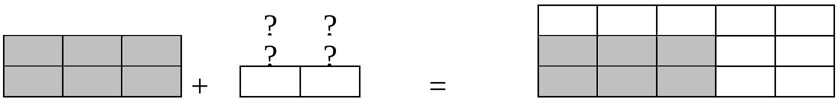
Thus, integration adds the per-numbers 2 and 4 as heights in stacks: 2 + 4 = 2 6/8.

So, 2 + 4 can give many different results, unless the units are the same:

T = 2*3 + 4*3 = 6*3 if added in time; and

T = 2*3 + 4*3 = (2*3 + 4*3)/6*6 = 3*6 if added in space.

The addition process can be reversed by asking 2 3s + ? 2s = 3 5s:

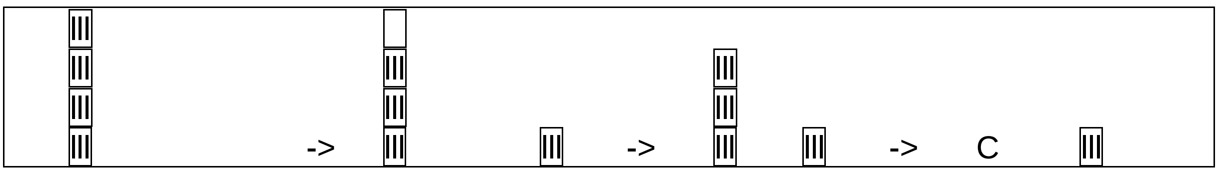


The answer can be obtained by removing the 2 3s from the 3 5s and then recounting the remaining 9 in 2s as (9/2)*2 = 4 1/2 * 2. Thus, ? = 4 ½.

This process is later called differentiation.

11 Stacks are Overloaded

The next educational question is: What do we do about overloads? If a stack is higher than its unit, the stack is overloaded. The overload then can be restacked to a new stack leaving a full stack C.



T = 4*3 = overload = (4*3 – 1*3) + 1*3 = 3*3 + 1*3 = 1*C + 1*3

12 One Digit Equations

The last educational question is: How can we reverse addition?

One answer is reversed calculations, also called solving equations. The recount- and the restack-equation show that equations are solved when moving a number to the other side of the equation sign reversing its calculation sign:

Recounting:	T	$= (T/4) * 4$		Restacking:	T	$= (T\text{-}4) + 4$	
Equation	T	$=$	$x \quad * 4$	Equation	T	$=$	$x \quad + 4$
Solution	$T/4$	$=$	x	Solution	$T\text{-}4$	$=$	x

The equation $2*x+1 = 9$ is solved by combining restacking and recounting, not by neutralizing.

$2*x+1$	$= 9 = (9\text{-}1) +1 = 8 + 1$	restacking 9 into 8 + 1	or by moving:	$2*x + 1$	$= 9$
$2*x$	$= 8 = (8/2)*2 = 4*2$	recounting 8 into 4*2		$2*x$	$= 9 - 1 = 8$
x	$= 4$	the solution		x	$= 8/2 = 4$

Educated by Many

Through these 12 educational meetings with Many we are introduced to a radical different mathematics, a postmodern LAB-mathematics turning the modern LIB-mathematics upside down:

- To LIB-mathematics number-symbols are cultural artefacts to be learned by heart. To LAB-mathematics number-symbols are natural icons representing the number of strokes they contain.

- LIB-mathematics has many number-types: Whole numbers, integers, rational and irrational numbers etc. LAB-mathematics only has two number-types, stack-numbers and per-numbers.

- LIB-mathematics has addition as its fundamental operation to be followed by subtraction and multiplication, and finally the hard part, division. LAB-mathematics has division as its fundamental operation to be followed by multiplication, subtraction, and finally the hard part, addition.

- LIB-mathematics introduces many-digit numbers before mathematics. LAB-mathematics introduces mathematics before many-digit numbers.

- LIB-mathematics introduces many-digit numbers as the abstract idea of a positions system. LAB-mathematics introduces many-digit numbers as a physical fact, a stock, i.e., a many-stack.

- LIB-mathematics introduces addition of many-digit numbers as the abstract idea of carrying. LAB-mathematics introduces addition of many-digit numbers as an internal trade inside a stock.

- LIB-mathematics postpones calculus to tertiary education. LAB-mathematics includes calculus in both primary and secondary education.

- LIB-mathematics postpones algebra and equations to secondary education. LAB-mathematics includes algebra and equations in primary education.

Summing up, LIB-mathematics is defined as examples of abstractions, derived from the concept set, part of a metaphysical axiom system; thus, becoming a purely abstract mental activity, that cannot always by validated in the laboratory, as shown by the 2&3-paradox. LAB-mathematics is defined as abstractions from examples, derived from Many, part of a physical reality; thus, becoming a natural science induced from and validated in a Many-laboratory.

Tested in a pre-calculus classroom LAB-mathematics solved the relevance paradox (Tarp 2003).

Tested in the classroom of teacher education student teachers overwhelmingly voted for including postmodern LAB-mathematics in teacher education (Zybartas et al 2001). This positive response has led to the development of teaching and learning material in postmodern 'LAB-mathematics from below' for free teacher PYRAMIDeDUCATION at www.MATHeCADEMY.net (Tarp 2003).

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Adding PerNumbers

To solve the relevance paradox in mathematics education this paper uses postmodern skeptical Cinderella research to look for new ways to teach mathematics at the secondary school. The paper introduces addition of per-numbers as a more user-friendly approach to the traditional subjects of proportionality, linear and exponential functions and calculus.

1 The Focus: Unnoticed Ways to Teach Mathematics

Mathematics education has a relevance paradox 'formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994). At a talk in the late 1970's Freudenthal described the didactical expert as the reflective practitioner. This inspired the author to use the mathematics classroom as a laboratory to develop different ways to teach secondary mathematics at the pre-calculus and the calculus level. Later as a research student the author used the inspiration from postmodern thinking to develop a new action focused methodology called skeptical Cinderella research. This postmodern methodology has proved to be a Cinderella-difference making a difference by solving the relevance paradox in the classrooms of the authors and cooperating teachers. So, now the time has come to make its results more widely known.

2 The Theoretical Framework: Institutional Skepticism

The theoretical framework of the study is institutional skepticism, as it appeared in the Enlightenment and was implemented in its two democracies, the American in the form of pragmatism and symbolic interactionism, and the French in the form of post-structuralism and post-modernism.

3 Literature: Symbolic Interactionism & Postmodernism

The source of inspiration is American symbolic interactionism (Blumer etc.) and its methodology 'Grounded Theory' (Glaser & Strauss, etc.) as well as French poststructuralism (Derrida, Lyotard, Foucault) and postmodern thinking (Bauman). These sources have led to a definition saying 'postmodernism means institutional skepticism towards and deconstruction of the pastoral power of non-democratic traditions'.

4 Methodology: Skeptical Cinderella-Research

The methodology 'skeptical Cinderella research' is a postmodern counter-seduction research based upon a post-structuralist 'pencil-paradox': Placed between a ruler and a dictionary a thing can point to numbers, but not to words - thus, a thing can falsify a number-statement about its length, but not a word-statement about its name;, i.e., a thing can defend itself against a number-accusation by making a statement of difference in a laboratory; but is forced to pay deference to any word-accusation from the library. (Tarp 2001, 2003).

A number is an ill written icon showing the degree of Many (there are 4 strokes in the number sign 4, etc.); a word is a sound made by a person and recognized in some groups and not in others. Words can be questioned and put to a vote in a courtroom, numbers cannot. Numbers can carry valid conclusions based upon reliable data, i.e., research. Words can carry only interpretations, that if presented as research become seduction; words carry no truth, but hide differences to be uncovered by counter-seduction as, e.g., skeptical research, having quality if it can produce a 'Cinderella-difference', i.e., a difference that makes a difference.

Thus, the skeptical part of the methodology comes from the ancient Greek sophists always distinguishing between choice and necessity, between political and natural correctness. This part allows the researcher to see the difference between what could be different and what could not. Once an area with a hidden difference has been identified, the Cinderella part is used to identify hidden alternatives either by discovering forgotten or unnoticed alternatives at different times and places, inspired by the genealogy of Foucault (Dreyfus et al 1982); or by inventing alternatives using sociological imagination, inspired by Mills (Mills 1959).

Counter-seduction research is always skeptical towards word-based research in the library, especially if it has not been generated from and validated in the laboratory, thus, unable to meet the 'LibLab' criteria of the research genre saying that a Lib-acceptance implies a Lab-validation of findings induced from Lab-observations or deduced from a Lib-hypothesis.

Thus, reference to the existing research literature is not possible within a counter-seduction methodology. The aim of skeptical Cinderella research is not to extend the existing seduction of the library. The aim is to search for hidden Cinderella-alternatives in the learning-laboratory, i.e., in the classroom, by

1) finding a non-democratic tradition hiding its alternatives,

2) finding hidden alternatives through discovery and imagination,

3) testing the alternative in the laboratory to see if it is a Cinderella-difference making a difference, and

4) publish the alternative so it can be an option in the classroom.

5 The Classroom Tradition: Fractions and Their Applications

Having developed the mathematics of natural numbers and decimal numbers and their applications in the primary school the time has come to introduce two new types of numbers in the secondary school. The rational numbers are treated in the pre-calculus classes, and the real numbers are introduced in the calculus classes.

Both of these numbers base their definition on the concept of 'sets of sets'. Thus, the set Q of rational numbers is defined as a set of equivalence sets in a product set of two sets of sets of equivalence sets in a product set of two sets of sets of equivalence sets in a product set of two sets of Peano-numbers; such that the pair (a,b) is equivalent to the pair (c,d) if a*d = b*c, which makes, e.g., (2,4) and (3,6) represent then same rational number 1/2 (see, e.g., Griffith et al 1970).

The set-concept however is controversial. Numbers differentiate between degrees of Many; and sets differentiate between degrees of infinity.

Kronecker objected that infinity is not a quantity, but a quality; so potential infinity exists, but actual infinity does not. Instead, Kronecker meant that mathematics should be built upon the natural numbers, our gift from God. The intuitionists later pursued this view.

Russell objected to sets of sets through his paradox: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$. Instead, Russell made self-reference illegal by introducing his type-theory saying that a given type can only be described by a higher type. Computer science later pursued this view.

Historically fractions are left-overs from the medieval times, when strokes were used to separate the two processes of using numbers as input and output: '3 | 4' meant that 3 is an input-number and that 4 is an output-number as used in book-keeping: 20 + (3 | 4) = 20 + 3 – 4 = 19. And 3/4 meant that 3 is a repeated input-number and that 4 is a repeated output-number: 15*3/4 = (15/4)*3.

Modern mathematics instead talks about minus and reciprocal signs of numbers: '- 4' (take away 4) and '/4' (take away 4s).

Another problem is illustrated by the typical 'welcome ceremony' in the Danish upper secondary school:

The teacher:	The students:
Welcome! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No, 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
Inside this classroom 1/2 + 2/3 = 7/6!	

Apparently, we have a fraction-paradox:

Inside the classroom	20/100 + 10/100 = = = 20% + 10% =	30/100 = 30%
Outside the classroom, e.g., in the laboratory	20% + 10% = or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

Of course, there are examples where 20% + 10% = 30% as in 20% of 300 + 10% of 300 = 30% of 300. But as the fraction-paradox shows, that there are also many counter-examples, so there is no general rule saying that 20% + 10% = 30%

If research is valid conclusions based on reliable data, then addition of fractions is not research since it cannot be validated outside the classroom. So, we can distinguish between ‘mathematics’, which is a science that can be validated in the laboratory, and ‘mathematism’, which is a doctrine, an ideology, which cannot be validated in the laboratory.

This gives a possible solution to the relevance paradox: What we call ‘mathematics’ is really ‘mathematism’ teaching ‘killer-mathematics’ only existing inside classrooms, where it kills the relevance of mathematics. Presenting the students to mathematics instead of mathematism may validate this solution.

6 A Classroom Alternative: The History of Per-Numbers

Spices and silk from the East have always been popular in the West; but being a higher culture, the East only accepted silver and gold as means of exchange. So, history has witnessed a steady flow of silver from the West to the East creating wealth and culture along the road.

Thus, silver mines outside Athens in Greece financed the antique culture. Later the silver mines in Spain financed the Roman Empire. After the Arabic conquest of Spain, the dark Middle Ages came to Europe until silver was found again, this time in Germany. Italy became the middleman between Germany and the East and this position created the wealth that financed the Renaissance. The next trade centers were Portugal and Spain using the seaway around Africa to the East, and using America for new mines of silver and gold.

Later England wanted to sail to the East on open sea to avoid the hostile fortresses of Africa. This entailed a closer study of the motion of the moon. Opposing the traditional view that the moon is moving across the sky Newton instead suggested that the moon is falling towards the earth just as an apple. Newton thus, discovered that the behavior of physical objects is determined by a physical will to change, a natural force called gravity. Unlike the will of the metaphysical Lord this physical will was predictable through calculations, once the art of change-calculations was developed and given a name, differential and integral calculus.

Thus, change-calculations, or calculus, became the basis of modern physics and other natural sciences. And thus, calculus became the basis of the Enlightenment period in the 18th century rebelling against the will of the masters by claiming that man was his own master fully able to choose what was best. The Enlightenment installed two democracies, one in the US and one in France, as well as the modern industrial state based upon the applications of modern science. This changed the western economy from being based on silver to being based on production and knowledge.

So, in the early modern time the West saw two golden periods, the renaissance and the Enlightenment. And these two periods both created and applied new mathematics: ‘Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights’ (Kline 1972: 399).

The two typical trade questions of the Renaissance were that of proportionality ‘if 3 kg cost 5$, what is the price for 7 kg, and what will 12$ buy?’; and that of renting ‘20 days at 5$/day totals? $’.

Because of Italy's immense wealth also money became a commodity that could be rented. But if the rent was not paid back also the rent had to be rented crating the compound interest '20 days at 0.5%/day total ? $'

In England Newton described the movement of the moon as a fall towards the earth just like the fall of an apple. A falling object is described by its velocity, which is a per-number meters per seconds, m/s. If the velocity is constant then the total distance can be calculated by a simple multiplication: 5 seconds at 3 m/s totals 5*3 = 15 meters.

But in the case of a falling object the velocity is not constant, but constant accelerated, giving rise to a question that cannot be solved by simple multiplication: 5 seconds at 3 m/s changing to 4 m/s total ? meters. So, Newton was faced with the question: How to add variable per-numbers?

To sum up the two rich periods of the early modern world were connected with the mathematical development of per-numbers where the question 'how do we add per-numbers' were posed in different connections. In renaissance Italy the addition of per-numbers occurred within trade, both at prices as $/kg-numbers, at rent as $/day-numbers and at interest as %/day-numbers. And in the Enlightenment the addition of per-numbers occurred within physics at velocity as m/s-numbers.

Thus, the Cinderella-principle has led to the discovery of a forgotten hidden mathematics, the addition of per-numbers. Now it is time to use the imagination to set up a school curriculum on addition of per-numbers so that the school can tell the grand narratives about the growth and success of mathematics in the Renaissance and in the Enlightenment by making the addition of per-numbers the core of the curriculum in secondary school.

According to Kronecker and Russell we will try to build a set-free, fraction-free and function-free mathematics on the basis of a thing that exists with necessity, repetition in time.

7 Building Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with Many in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4

4. Multiplicity can be counted in icons producing a stack of, e.g., T = 3 4s = 3*4. The process 'from T take away 4' can by iconised as 'T-4'. The repeated process 'from T take away 4s' can by iconised as 'T/4, a 'per-number'. So, the count&stack calculation T = (T/4)*4 is a prediction of the result when counting T in 4s to be tested by performing the counting.

5. A calculation T=3*4= 12 predicts the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 litres = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T	= 7 kg = (7/2)*2kg = (7/2)*6 litres = 21 litres	T	= 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	T	= 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal's length is predicted by the Pythagorean theorem a^2 + b^2 = c^2, and its angles are predicted by re-counting the sides in diagonals: a = $a/c*c$ = sin$A*c$, and b = $b/c*c$ = cos$A*c$.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:

C = diameter * n * sin(180/n) = diameter * π.

9. Stacks can be added by removing overloads:

T = 38 + 29 = 3ten 8 + 2ten 9 = 5ten 17 = 5ten 1ten 7 = (5+1)ten 7 = 6ten 7 = 67

10. Per-numbers can be added after being transformed to stacks. Thus, the $/day-number a is multiplied with the day-number b before being added to the total $-number T: $T2 = T1 + a*b$.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated and reversed addition of per-numbers leads to integration and differentiation:

Repeated addition of per-numbers	Reversed addition of per-numbers
$T2 = T1 + a*b$	$T2 = T1 + a*b$
$T2 - T1 = + a*b$	$a = (T2-T1)/b$
$\Delta T = \sum a*b$	$a = \Delta T/\Delta b$
$\Delta T = \int y*dx$	$a = dy/dx$

Only in the case of adding constant per-numbers as a constant interest of, e.g., 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers:

4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121,6%

So, a Kronecker-Russel Many-based mathematics can be summarised as a 'count&add-laboratory' adding to predict the result of counting totals and per-numbers, in accordance with the original meaning of the Arabic word 'algebra', reuniting.

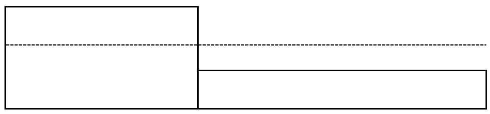
	Constant	Variable
Totals *m, s, kg, $*	$T = a*n$ $T/n = a$	$T2 = T1 + a*n$ $T2-T1 = a*n$
Per-numbers *m/s, $/kg, $/100$ = %*	$T = a\text{^}n$ $\sqrt[n]{T} = a$ $\quad$ $\log_a T = n$	$T2 = T1 + \int a*dx$ $dT/dx = a$

The Count&Add-Laboratory

8 Bringing the Multiplicity-Based Mathematics to the Classroom

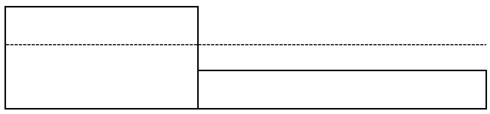
The elementary school introduces the practice of counting Many by bundling and stacking to be predicted by per-numbers, thus, making the students acquainted with the geometrical representation of per-numbers as the height of a stack. Also simple additions as T = 3 4s + 2 5s = ? 9s are carried out both by recounting and by prediction thus, realizing that mathematics is our language of prediction:

T = 3 4s+2 5s = 3*4+2*5 = (3*4+2*5)/9*9 = 2 4/9 *9



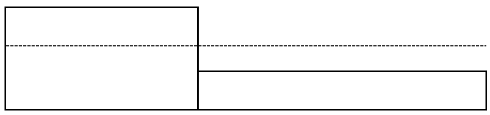
In secondary school addition of per-numbers follows the same pattern within trade:

T = 4kg at 3$/kg + 5kg at 2$/kg = 9kg at 2 4/9$/kg



And so does addition of per-numbers within physics:

T = 4s at 3m/s + 5s at 2m/s = 9s at 2 4/9m/s



This also applies if the m/s-number is locally constant and not just piecewise constant, i.e., if ε and δ changes places in the formal definition of constancy:

A variable y is globally constant c	$\forall\varepsilon>0$: $\lvert y-c \rvert < \varepsilon$ all over
A variable y is piecewise constant c	$\exists\delta>0\ \forall\varepsilon>0$: $\lvert y-c \rvert < \varepsilon$ in the interval δ
A variable y is locally constant c (continuous)	$\forall\varepsilon>0\ \exists\delta>0$: $\lvert y-c \rvert < \varepsilon$ in the interval δ

Thus, a general pattern occurs: Per-numbers are added by the area under its curve. Since any smooth curve is locally constant it can be approximated by a series of stacks to be summed by integration: $\int y*dx \approx \sum y*\Delta x$. However, if $y*\Delta x$ can be written as the change of another variable z ($y*\Delta x = \Delta z$, or $y = \Delta z/\Delta x$) then the sum can be predicted since the sum of single changes = the total change = Terminal-number – Initial-number:

$\sum y*\Delta x = \sum \Delta z = \Delta z = z2 - z1$.

This relation does not depend on the size of the change, so also $\int y*dx = \int dz = z2 - z1$.

Therefore, the time has come to study the behavior of changing variables by going back to the two addition tasks of the Renaissance:

5 days at 3 \$/day = 5*3 = 15 \$
5 days at 3 %/day is 5*3, i.e., 15 %, + *CI* = simple interest + compound interest

The compound interest *CI* can be calculated from the total interest *TI*:

$TI = n*I + CI$, where $1 + TI = (1+i)^n$.

Thus, the compound interest CI is what makes exponential change and linear change different. Or in other words, if we only consider the simple interest and neglect the compound interest then exponential change becomes linear. And since the compound interest first matters in the long run, there is no big harm done by neglecting it in the short run. This 'neglecting the compound interest in the short run' is called differential calculus. In differential calculus a smooth curve is considered locally linear having a locally constant per-number or slope:

The *y*-curve is locally linear if its per-number $\Delta y/\Delta x$ is locally constant $c = dy/dx$	$\forall\varepsilon>0\ \exists\delta>0:\ \lvert(\Delta y/\Delta x) - c\rvert < \varepsilon$ in the interval δ

The change of a stack can be studied geometrically

i	*i*	*i*^2
1	1	*i*
	1	*i*

Thus, differential calculus means neglecting the upper right corner in a changing stack.

So, in differential calculus $(1+dx)^2 = 1+2*dx$, and $(1+dx)^3 = 1+3*dx$ as seen below:

dx	*dx*	2**dx*^2
1	1	2**dx*
	1	2**dx*

In general $(1+dx)^n = 1+n*dx$.

If $y = x^n$, then a change in x, dx, produces a change in y, dy, and

$y + dy = (x+dx)^n = (x*(1+dx/x))^n = x^n*(1+dx/x)^n = y*(1+n*dx/x) = y + n*y*dx/x$

So, $dy = n*y*dx/x$, or $dy/dx = n*y/x = n*x^n/x = n*x^{(n-1)}$

Now we are able to predict the result of adding variable per-numbers through a calculation:

5 sec. at 3m/sec increasing to 4 m/sec total $\int_0^5 (3+\frac{4-3}{5}x)\,dx = \int_0^5 (3+0.2x)\,dx = ?$ m

Since $d/dx\ (3x+0.1x^2) = 3+0.2x$ we get that $d\ (3x+0.1x^2) = (3+0.2x)\ dx$, so

$$\int_0^5 (3+0.2x)\ dx = \int_0^5 d(3x+0.1x^2) = \Delta(3x+0.1x^2) = (3\cdot 5+0.1\cdot 5^2) - 0 = 17.5 \text{ m}$$

9 Multiplicity-Based Mathematics: A Difference in the Classroom

Kronecker-Russell mathematics, Many-based stack-mathematics, Enlightenment-mathematics makes a difference in the Danish pre-calculus classroom (Tarp 2003) and in teacher education in Eastern Europe (Zybartas et al 2001) and in Africa (Tarp 2002). Thus, by being a Cinderella-difference it brings a successful conclusion to the skeptical Cinderella study.

10 Conclusion

The purpose of this study was to apply skeptical Cinderella research to solve the relevance problem in the mathematics classroom. First skepticism was used to identify a non-democratic tradition hiding its alternatives. In this case fractions were identified bringing 'killer-mathematics' into the classroom killing the relevance of mathematics by being only valid inside the classroom and not outside, thus, transforming mathematics to mathematism. Then Cinderella thinking was used. First to travel in time and discover a forgotten alternative to fractions, per-numbers. Then to use sociological imagination to imagine alternative ways to bring the grand narratives of the Renaissance and the Enlightenment to the classroom in the form of a Kronecker-Russell Many-based mathematics.

Finally, this difference turned out to be a Cinderella-difference making a difference. Thus, the skeptical Cinderella study has been successful by offering a hidden alternative to policy makers wanting to solve the relevance paradox of mathematics education.

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Per-Number Calculus, A Postmodern Skeptical Fairy Tale Study

To solve the relevance paradox in mathematics education this paper uses postmodern sceptical fairy tale research to look for new ways to teach calculus in the school. A renaming of 'calculus' to 'adding per-numbers' allows us to think differently about the reality 'sleeping' behind our words, and all of a sudden we see a different calculus taking place both in elementary school, middle school and high school. Being a 'Cinderella-difference' by making a difference when tested this postmodern calculus offers to the classroom an alternative to the thorns of traditional calculus.

How to Find a Hidden User-Friendly Calculus

The background of this study is the worldwide enrolment problem in mathematical based educations and teacher education (Jensen et al 1998). And 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994: 371). In mathematics education 'calculus is difficult' is a widespread observation, and many different efforts have been made to penetrate the hawthorn of calculus with limited success (see, e.g., Steen 1986).

This paper reports on a radical different approach using a postmodern 'sceptical fairy tale research' seeing humans as bewitched by words. This seduction can be counteracted by counter-seduction renaming 'echo-words' by asking 'Sleeping Beauty questions' like:

Calculus did not call itself calculus, so calculus can also be called something else? Mathematics did not name itself, so mathematics can also be named differently?

Counter-seduction is based upon a simple observation, the 'number&word-paradox': Placed between a ruler and a dictionary a thing can point to a number, but not to a word; thus, a thing can falsify a number-statement about its length in the laboratory, but not a word-statement about its name in the library;, i.e., numbers carry reliable information inducing and validating research; words carry debate, i.e., interpretations, which if presented as research become seduction, to be met with counter-seduction searching for hidden 'Cinderella-differences' making a difference (Tarp 2003).

A Different Name for Calculus

According to the number&word-paradox calculus did not call itself 'calculus', so calculus could also be called something else; and this different name might be an eye-opener to a different approach. So, let us free ourselves from the seduction of the library and go to the laboratory to see what kind of questions calculus is dealing with and arose from.

The historical library said 'the moon moves among the stars'. Newton freed himself from this seduction with a counter-formulation saying that the moon falls to the earth just as an apple, both drawn by the same will to change, the same force called gravity.

Unlike the unpredictable will to rule of the metaphysical Lord this physical will to change was predictable through calculations, once the art of change-calculations was developed and given a name, differential and integral calculus.

Newton used multiplication to solve the velocity-problem '5 sec *at* 2 m/s totals ? m': T = 5*2 = 10 m. But gravity changes the velocity thus, changing the problem to, e.g., '5 sec *at* 2 m/s increasing to 4 m/s total ? m', which cannot be solved by a simple multiplication. Thus, Newton was faced with the following problem: We know how to add constant per-numbers, how do we add variable per-numbers?

So, the lesson from the laboratory is that calculus arose from and is dealing with addition of variable per-numbers. Hence 'calculus' can be renamed to 'addition of per-numbers'. This renaming allows us to think differently by translating the question 'how to teach calculus' to 'how to teach addition of per-numbers'.

Mathematics and Mathematism

If we say that research is library-statements induced from and validated by reliable laboratory data then we can distinguish between ‘mathematics’ being validated in the laboratory and ’mathematism’ being validated only in the library.

An example of mathematics is the statement that T = 2*3 = 6 claiming that 2 3s can be recounted as 6 1s. This claim is easily verified in the lab: *** *** -> * * * * * *.

An example of mathematism is the statement that T = 2+3 = 5, a claim that has countless counter-examples in the lab:

T = 2 m + 3 cm = 203 cm, T = 2 weeks + 3 days = 17 days, T = 2 tens + 3 ones = 23

That seduction by mathematism is costly is witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. The units must be alike and put outside a parenthesis to give a valid addition inside. Thus, 20% + 10% = 30% only if taken of the same total. Else the ‘fraction-paradox’ applies:

Inside the classroom	20/100 + 10/10 = = = 20% + 10% =	30/100 = 30%
Outside the classroom	20% + 10% = or =	32% in the case of compound interest b% (10<b<20) in the case of the total average

Brought to the classroom mathematism introduces ‘killer-mathematics’ only valid inside the classroom where it is applied to kill the relevance of mathematics.

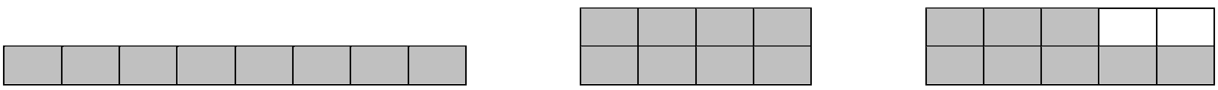
Another example of mathematism and killer-mathematics is the modern function concept: If ‘f(x) = x+2’ means ‘let f(x) be a place-holder for the calculation ‘x+2’ having x as a variable number’ then f(3) = 5 means that 5 is a calculation with 3 as a variable number, which is a double syntax error (Tarp 2002).

So, to fulfil the social and ethical obligation to deliver lab-mathematics and not lib-mathematism, a different seduction-free mathematics has to be created, not from the library, but from the laboratory. One example of such a ‘deconstructed’ mathematics is a Kronecker-Russell Many-based mathematics as is seen in the appendix.

Lab-Based Mathematics

In the laboratory Many is organised through counting taking away bundles of, e.g., 4s to produce a stack of, e.g., T = 2 4s = 2*4. The process ‘from T take away 4’ can be iconised as ‘T-4’. The repeated process ‘from T take away 4s’ can by iconised as ‘T/4’, which is a called a ‘per-number’ counting how many times we have a 4. So, numbers are never abstract entities but parts of a stack, as a height, 2, or as a base, 4.

Thus, the ‘recount-equation’ T = (T/4)*4 is a prediction of the result of recounting T in 4s to be tested by performing the recounting T = 1*8 = ?*4, or T = ?*5.



T = 1*8 = (8/4)*4 = 2*4 = (8/5)*5 = 1 3/5 * 5

Counting in tens per-numbers or ‘recounters’ become decimals and percentages:

T = 32*1 = (32/10)*10 = 3 2/10*10 = 3.2*10 = (32/100)*100 = 0.32*100 = 32%*100

Per-numbers are like numerators, not of the abstract fractions of the fraction-paradox, but of ‘fractions-of’ always being multiplied with their base before they are added.

Also, with units per-numbers are recounters recounting one unit by another unit:

40 $ per 2 kg = 40 per 2 $ per kg = 40/2 $/kg = 20 $/kg = rate.

Thus, the question '5 kg = ? $' is answered by recounting the number (5 kg = (5/2)*2 kg = (5/2)*40 $ = 100 $); or by recounting the unit ($ = ($/kg)*kg = 20*5 = 100).

Thus, the question 'how to add per-numbers' occurs in three different forms:

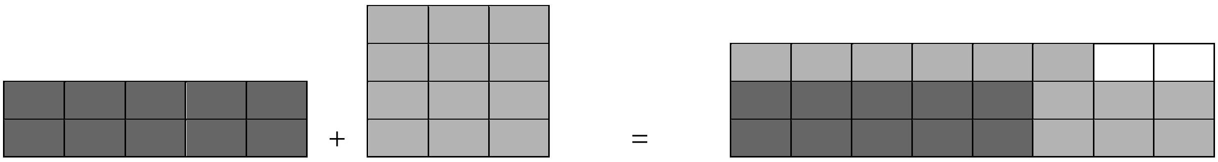
Primary school: *T* = 2 5s + 4 3s = ? 8s

Middle school: *T* = 5 kg at 2 $/kg + 3 kg *at* 4 $/kg = 8 kg at ? $/kg

Secondary school: *T* = 5 sec at 2 m/s increasing to 4 m/s = 5 sec at ? m/s

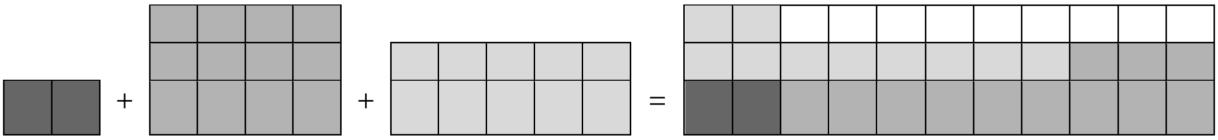
Calculus in Primary School

Adding two stacks as 2 5s and 4 3s is done by a recounting predicted by a recount-equation



T = 2 5s + 4 3s = 2*5 + 4*3 = (2*5 + 4*3)/8*8 = 2 6/8 *8

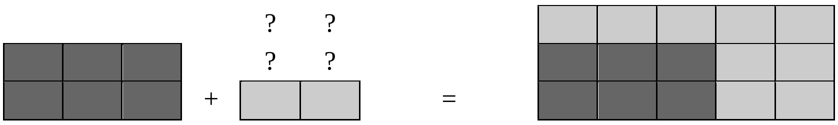
Repeated addition of stacks is later called integration:



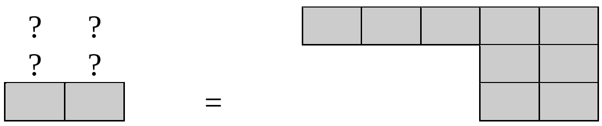
T = 1 2s + 3 4s + 2 5s = 1*2 + 3*4 + 2*5 = (1*2 + 3*4 + 2*5)/11*11 = 2 2/11*11

So, the lesson from the primary school count&add-laboratory is that per-numbers can be added by adding the stacks in which they are the heights.

The addition process can be reversed by asking 2 3s+? 2s = 3 5s:



The answer can be obtained by removing the 2 3s from the 3 5s and then count the remaining 9 in 2s as (9/2)*2 = 4 1/2 * 2. Thus, ? = 4 1/2



Or the answer can be predicated by a reversed calculation, later called differentiation. In this way solving equations becomes another name for a reversed calculation.

2 3s	+ ? 2s	=	3 5s	The question
2*3	+ *x**2	=	3*5	The equation
	*x**2	=	3*5-2*3 = 9	The 2 3s are removed from the 3 5s leaving 9
	*x**2	=	(9/2)*2 = 4 1/2 * 2	The 9 is recounted as 2s
	x	=	4 1/2	The answer

Per-Numbers in Middle School

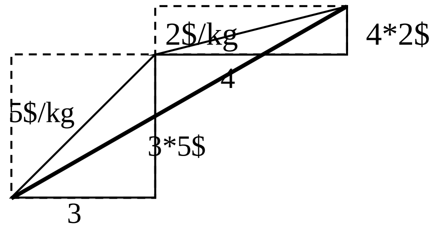
In middle school per-numbers occur in Renaissance trade as price-numbers 4 $/kg or rent-numbers 4 $/day, i.e., as rates counting the number of $ per kg or day. Again the per-number is the height of a stack, or the slope of the diagonal in a change-triangle.

And again per-numbers are added as stacks: The $/kg-number *b* is multiplied with the day-number n before being added to the total $-number *T* = *T*1 + *n***b*.

T = T1 + T2 = 3 kg at 5 $/ kg + 4 kg at 2 $/ kg = (3+4) kg at (∑*n***b*)/(3+4) $/ kg

Adding per-numbers can take place in tables, or by adding slopes in triangles:

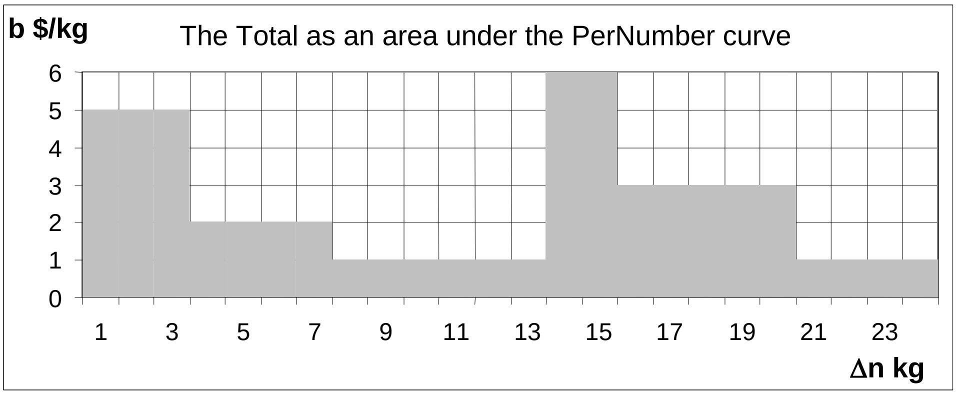
n kg at	*b* $/kg =	*n***b* $ = *T*
3 kg at 4 kg at 7 kg at	5 $/kg = 2 $/kg = *x* $/kg =	3 * 5 = 15 $ 4 * 2 = 8 $ 7 * *x* = ∑a*b= 23= (23/7)*7 $ *x* = 23/7= 3 2/7 $



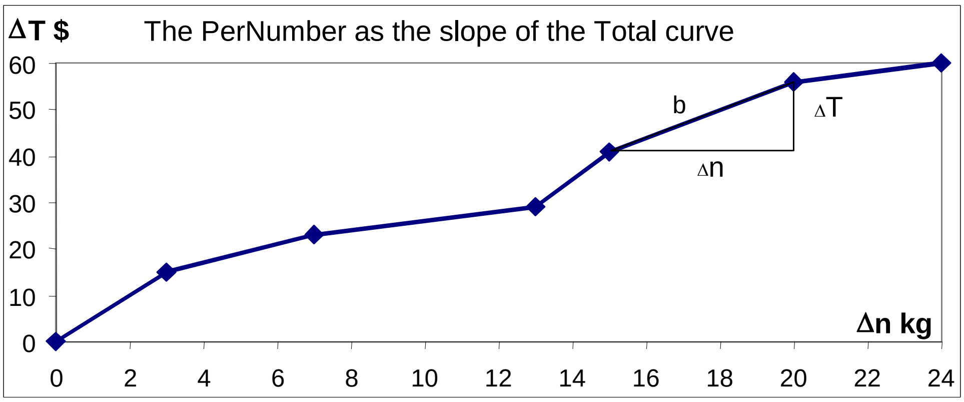
The table can be supplemented with two columns showing the added values of the kg-number Δn, of the $-number ΔT, and of the per-number Σ *b* occurring when re-counting or differentiating ΔT in Δns: Δ*T* = (ΔT/Δn) * Δn = *b* * Δn; as in this example where a teashop is adding different amounts with different prices to create a blending.

Δn kg	*b* $/kg	*Δn***b* = *ΔT*		*Σ Δn* = *Δn*	*Σ ΔT* = *ΔT*	Σ *b* $/kg = *ΔT*/*Δn*
3 kg at	5$/kg =	3 * 5 = 15		3	15$	15/3 = 5.0
4 kg at	2$/kg =	4 * 2 = 8		7	23$	23/7 = 3.3
6 kg at	1$/kg =	6 * 1 = 6		13	29$	29/13 = 2.2
2 kg at	6$/kg =	2 * 6 = 12		15	41$	41/15 = 2.7
5 kg at	3$/kg =	5 * 3 = 15		20	56$	56/20 = 2.8
4 kg at	1$/kg =	4 * 1 = 4		24	60$	60/24 = 2.5

When plotting the per-number *b* $/kg against *Δn* kg in a coordinate system the total $-number is the sum of the stacks, i.e., the area under the per-number curve.



When plotting *ΔT* against *Δn* in a coordinate system the curve shows both the added kg-number *Δn*, the added total *ΔT*, and the single per-numbers *b* = *ΔT*/*Δn* as the slopes.



Thus, from blending tea in a shop we learn that:

The Total is the area under the per-number curve predicted by an integration formula:

$\Delta T = \sum$ \$/kg*kg $= \sum b*\Delta n$ adding the per-number stacks.

The per-number is the slope of the Total curve predicted by a differentiation formula:

$b = \Delta\$/\Delta kg = \Delta T/\Delta n$ recounting the ΔT in Δns.

Adding stacks implies that the initial stack is changed by 3 new stacks:

$\Delta(b*a) = \Delta b*a+b*\Delta a+\Delta b*\Delta a \approx \Delta b*a + b*\Delta a$ neglecting the small upper right stack.

Δb	$\Delta b*a$	$\Delta b*\Delta a$
b	$b*a$	$b*\Delta a$
	a	Δa

To get the percent-change we divide by the initial number b*a:

$$\frac{\Delta(b*a)}{b*a} = \frac{\Delta b*a}{b*a} + \frac{b*\Delta a}{b*a} + \frac{\Delta b*\Delta a}{b*a}, \text{ or } \frac{\Delta(b*a)}{b*a} = \frac{\Delta b}{b} + \frac{\Delta a}{a} + \frac{\Delta b}{b}*\frac{\Delta a}{a} \approx \frac{\Delta b}{b} + \frac{\Delta a}{a}$$

Thus, $\Delta\%(b*a) \approx \Delta\%b + \Delta\%a$.

This change-formula is widely used in social science:

If the total production increases by 2.5% and the population increases by 3.2% then the income per capita decreases by 3.2% - 2.5% = 0.7%.

That repeated addition and reversed addition of per-numbers leads to integration and differentiation follows from the formula for adding per-numbers as stacks:

$T2$	$= T1 + a*b$	$T2$	$= T1 + a*b$
$T2 - T1$	$= + a*b$	$T2 - T1$	$= + a*b$
ΔT	$= + a*b$	$\Delta T/a$	$= b$
ΔT	$= \sum a*b$	$\Delta T/\Delta a$	$= b$
ΔT	$= \int y*dx$	dy/dx	$= b$

Until now the variation of the per-numbers has been arbitrary. It can however also be predictable:

In a trade the following discount is offered: The price is 10\$/kg for the first kg. Then the price is reduced with 0.5\$ for each extra kg until 11 kg, after which the price stays constant. The total cost for 11 kg then can be calculated by the sum

$S = 10 + 9.5 + 9 + 8.5 + ... + 5 = ?$

This sum can be calculated by a spreadsheet or it can be predicted through a calculation leading to the study of arithmetic growth giving the result

$S = n*(an + a1)/2 = 11*(5+10)/2 = 82.5$

In another trade the following discount is offered: The price is 10\$/kg for the first kg. Then the price is reduced with 10% for each extra kg until 11 kg, after which the price stays constant. The total cost for 11 kg then can be calculated by the sum

S = 10 + 10*0.9 + 10*0.9^2 + 10*0.9^3 + ... + 10*0.9^10 = ?

This sum can be calculated by a spreadsheet or it can be predicted through a calculation leading to the study of geometric growth giving the result

$S = 10*(1-a\text{^}n)/(1-a) = 10*(1-0.9\text{^}11)/(1-0.9) = 68.6$

As in elementary school reversing addition of per-numbers leads to equations that now are sat up in a more formal way in calculation tables

4 kg at 3 \$/kg+ 5 kg ? \$/kg = 18 \$, or $4*3 + 5*x = 18$

x = ?	$4*3 + 5*x$	= 18	The problem
	$(4*3) + (5*x)$	= 18	Add the invisible parenthesis
	$12 + (5*x)$	= 18 = (18-12) + 12	Calculate and restack 18
	$5*x$	= 6 = (6/5)*5	Calculate and recount 6
	x	= 6/5	The solution as a calculation
	x	= 1.2	The solution as a calculated number
Test:	*4*3 + 5*1.2*	*= 18*	Insert the solution
	18	*= 18*	Calculate and check

Per-Numbers in High School

In high school per-numbers occur in Enlightenment physics as, e.g., a velocity of 2 meter/second. If the per-number is constant again it can be represented as a stack with the per-number as the height, or as the slope of the Total curve.

However, in many cases the per-number is not constant. Thus, in the case of falling apples and planets the per-number is increasing continuously.

If it had been constant then we could have used our knowledge about per-numbers from middle school. So, the question arises: Can a continuous change be regarded as constant?

To answer this question, we can set up a formal definition of constancy by saying that a variable y is equal to a constant c if the numerical difference between y and c is less than a number ε for all positive numbers ε.

This definition of global constancy is recycled in the definition of piecewise constancy; and in the definition of local constancy, which is called continuity:

A variable y is globally constant c if	$\forall\varepsilon>0$: $/y-c/ < \varepsilon$ in all intervals
A variable y is piecewise constant c if	$\exists\delta>0\ \forall\varepsilon>0$: $/y-c/ < \varepsilon$ in the interval δ
A variable y is locally constant c if	$\forall\varepsilon>0\ \exists\delta>0$: $/y-c/ < \varepsilon$ in the interval δ

So, our knowledge about constant per-numbers still applies: Per-numbers are added by the area under the per-number curve.

Since a smooth curve is locally constant its area ΔA can be approximated by a sum of many stacks:

$\Delta A \approx \sum b*\Delta n$. b is a per-number $b = \Delta T/\Delta n$, so $b*\Delta n = \Delta T$, which gives

$\Delta A \approx \sum b*\Delta n = \sum \Delta T = \Delta T = T2 - T1$

since a sum of single changes = the total change = terminal-number – initial-number

This relation does not depend on the size of the change, so it is also valid for small local changes where we use the symbols '∫' and 'd' for '∑' and 'Δ':

$\Delta A = \sum b*\Delta n = \sum \Delta T = \Delta T = T2 - T1$ -> $\Delta A = \int b*dn = \int dT = \Delta T = T2 - T1$.

Thus, we can substitute the change-symbol in our middle school change-formula:

$$\frac{\Delta(f*g)}{f*g} \approx \frac{\Delta f}{f} + \frac{\Delta g}{g} \quad -> \quad \frac{d(f*g)}{f*g} = \frac{df}{f} + \frac{dg}{g}$$

This formula applies to powers of x:

$T = f*g = x*x = x$^2 $\frac{dT}{T} = \frac{dx}{x} + \frac{dx}{x} = 2*\frac{dx}{x}$ $\frac{dT}{dx} = 2*\frac{T}{x} = 2*\frac{x\text{^}2}{x} = 2*x$	$T = x*x*x*x*x = x$^5 $\frac{dT}{T} = \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} + \frac{dx}{x} = 5*\frac{dx}{x}$ $\frac{dT}{dx} = 5*\frac{T}{x} = 5*\frac{x\text{^}5}{x} = 5*x\text{^}(5\text{-}1)$

Substituting $f*g = h$ we get $f = h/g$:

$$\frac{dh}{h} = \frac{d(h/g)}{h/g} + \frac{dg}{g} \quad \text{giving} \quad \frac{d(h/g)}{h/g} = \frac{dh}{h} - \frac{dg}{g}$$

Now we are able to predict the result of adding variable per-numbers through a calculation as in the classical case of a falling body:

$$\text{5 sec. at 2m/sec increasing to 4 m/sec total} = \int_0^5 (2 + \frac{4\text{-}2}{5}x)\,dx = \int_0^5 (2 + 0.4x)\,dx = \text{? m}$$

Since $d/dx\ (2x + 0.2x\text{^}2) = 2 + 0.4x$ we get that $d(2x + 0.2x\text{^}2) = (2 + 0.4x)\ dx$, so

$$\int_0^5 (2 + 0.4x)\,dx = \int_0^5 d(2x + 0.2x\text{^}2) = \Delta_0^5\ (2x + 0.2x\text{^}2) = (2\cdot5 + 0.2\cdot5\text{^}2) - 0 = 15\text{ m}$$

Thus, in the case of a constant acceleration g the velocity v and position s after t seconds can be predicted by the calculations

$v = vo + t$ seconds at g (m/s)/s = $vo + g*t$

$$s = so + t\text{ seconds at } v\text{ m/s} = so + \int_0^t v\,dt = so + \int_0^t (vo+g*t)\,dt = so + vo*t + ½*g*t\text{^}2$$

At the surface of the earth the acceleration from gravitation is locally constant, but further out in space it decreases. So, let us return to Newton's laboratory to see how he was able to perform the historical calculation that predicted how the moon and the planets fall under the will of gravity.

The social impact of this calculation cannot be overestimated.

By moving the authority from the scriptures in the library to the rulers in the laboratory this calculation created a standard for a modern lab-based research induced from and validated in the laboratory; which in turn created the Enlightenment scepticism that changed of our world from the autocracy of two superior Lords to a democracy building upon self-government and science.

First Brahe used a lifetime to collect data about the motion of the planets.

Then Kepler used these data to conclude that the planets are moving in ellipses with the sun in the focus; and in such a way that the ratio between the cube of the average radius r and the square of the period T is a constant, T^2= $c*r$^3.

This makes the acceleration proportional to 1/r^2 since $v= (2*\pi*r)/T$:

$a = (2*\pi*v)/T = (2*\pi/T)*(2*\pi/T)*r = (2*\pi)\text{^}2*(1/T)\text{^}2*r = (4*\pi\text{^}2/c)*(1/r\text{^}2)$

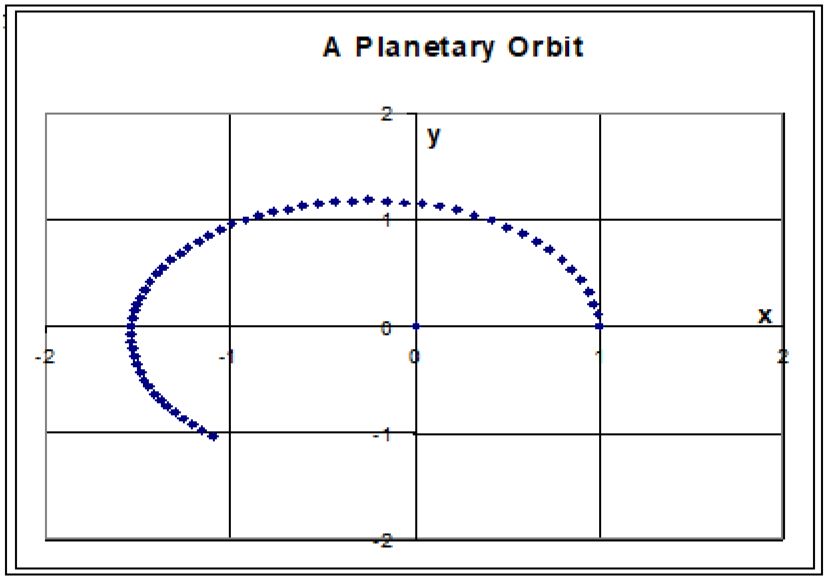
The library said that the force is proportional to the velocity. Newton neglected this and chose instead to test the counter-hypothesis that the force is proportional to the *change* of velocity, i.e., to the acceleration, i.e., to 1/r^2.

To test the predictions from this hypothesis we set up a system of 8 quantities with their corresponding change-equations:

	Initial numbers		Change-numbers		Terminal numbers	
Acceleration	*a* =	1			*a* =	1/*r*^2
Horizontal acceleration	*ax* =	1			*ax* =	-*x*/*r*^3
Vertical acceleration	*ay* =	0			*ay* =	-*y*/*r*^3
Horizontal velocity	*vx* =	0	*Δ vx* =	*ax***Δt*	*vx* =	*vx* + *Δ vx*
Vertical velocity	*vy* =	1.1	*Δ vy* =	*ay***Δt*	*vy* =	*vy* + *Δ vy*
Horizontal position	*x* =	1	*Δ x* =	*vx***Δt*	*x* =	*x* + *Δ x*
Vertical position	*y* =	0	*Δ y* =	*vy***Δt*	*y* =	*y* + *Δ y*
Radius	*r* =	1			*r* =	√(*x*^2+*y*^2)

Today Newton could make a spreadsheet do the calculations and the plotting. The result validates his hypothesis since the orbit is an ellipse obeying Kepler's laws:

Sun:	0	0		*Δt*:	0,1					
ax	*ay*	*Δvx*	*vx*	*Δvy*	*vy*	*Δx*	*x*	*Δy*	*y*	*r*
			0		1,10		1		0	1,00
-1,00	0,00	-0,10	-0,10	0,00	1,10	-0,01	0,99	0,11	0,11	1,00
-1,00	-0,11	-0,10	-0,20	-0,01	1,09	-0,02	0,97	0,11	0,22	0,99
-0,99	-0,22	-0,10	-0,30	-0,02	1,07	-0,03	0,94	0,11	0,33	0,99
-0,95	-0,33	-0,10	-0,39	-0,03	1,03	-0,04	0,90	0,10	0,43	1,00
-0,91	-0,43	-0,09	-0,49	-0,04	0,99	-0,05	0,85	0,10	0,53	1,00
-0,85	-0,52	-0,08	-0,57	-0,05	0,94	-0,06	0,80	0,09	0,62	1,01
-0,77	-0,60	-0,08	-0,65	-0,06	0,88	-0,06	0,73	0,09	0,71	1,02
-0,69	-0,67	-0,07	-0,72	-0,07	0,81	-0,07	0,66	0,08	0,79	1,03
-0,60	-0,73	-0,06	-0,78	-0,07	0,74	-0,08	0,58	0,07	0,86	1,04
-0,51	-0,76	-0,05	-0,83	-0,08	0,66	-0,08	0,50	0,07	0,93	1,06
-0,42	-0,79	-0,04	-0,87	-0,08	0,58	-0,09	0,41	0,06	0,99	1,07
-0,34	-0,81	-0,03	-0,90	-0,08	0,50	-0,09	0,32	0,05	1,04	1,09
-0,25	-0,81	-0,02	-0,93	-0,08	0,42	-0,09	0,23	0,04	1,08	1,10
-0,17	-0,80	-0,02	-0,95	-0,08	0,34	-0,09	0,13	0,03	1,11	1,12



Does the Difference Make a Difference?

At an African teacher college, the instructors had problems with the traditional ε–δ definition of a limit. All textbooks taught it in the same way so the instructors thought it could not be taught otherwise until we asked the 'Sleeping Beauty question': Limits and continuity did not name themselves so they can also be called something else?

Searching for hidden differences, we realized that by interchanging the ε and δ the definition defines a different kind of constancy as described above. So, we tested the three constancy definitions in the laboratory, i.e., in the classrooms. After the double-lesson the student teachers were asked how they found the ε–δ definition of limits before and after this new introduction on a scale from –2 to 2 covering very bad, bad, neutral, good and very good. Among the 76 answers the average scores were –0.9 before and 0.7 after, giving an average difference of 1.5 in favor of the new one. 3 students preferred the old definition to the new and 73 students preferred the new definition to the old.

Also, the differential calculus approach above was tested in a double lesson. Afterwards the student teachers were asked to express their opinion as above as to whether this approach should be introduced at the teacher college, and whether both the traditional and the alternative approach should be introduced at the secondary schools. Among the 117 answers the average scores were 1.2 and 0.9. Also, the integral calculus approach above was tested with some success (Tarp 2005).

Adding per-numbers in primary school has been introduced at an East European teacher College with a positive result (Zybartas et al 2001).

So, by being a Cinderella-difference making a difference in the mathematics classroom the adding per-numbers approach constitutes a user-friendly alternative to traditional calculus, which was the aim of this postmodern sceptical fairy tale study.

Where is the Function, the Limit and the Computer?

Addition of per-numbers is a calculus that is set-free, fraction-free, function-free, limit-free, computer-free, etc. Thus, this postmodern calculus lacks many ingredients of modern calculus, and seems to be more like a pre-modern 'Enlightenment-calculus' from when calculus was born in the absence of both functions and limits. However, to be considered part of the Enlightenment period is just an advantage since mathematics seems to blossom in the laboratory and wither with the winter breeze of rigor in the library:

> The enormous seventeenth-century advances in algebra, analytic geometry, and the calculus; the heavy involvement of mathematics in science, which provided deep and intriguing problems; the excitement generated by Newton's astonishing successes in celestial mechanics; and the improvement in communications provided by the academies and journals all pointed to additional major developments and served to create immense exuberance about the future of mathematics. (..) The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 398-99)

When teaching modern calculus computers make a positive difference (see, e.g., Tall in Biehler et al). In postmodern calculus computers play a different role computing the 'mega-sums' that cannot be added manually: By renaming a 'differential equation' to a 'change equation', a computer elegantly solves any differential equation by recursion saying that '$y := y+dy$' as exemplified by the planetary orbits about. Thus, the computer is not used to improve the teaching of a calculation form made obsolete by the computer, but to get access to the quantitative literature that improved our history by enabling us to predict and install changes in our conditions.

Conclusion

To solve the relevance paradox in mathematics education this paper used a postmodern sceptical fairy tale research to rename 'calculus' to 'adding per-numbers' allowing us to see that calculus takes place both in elementary school, middle school and high school. When tested in the laboratory of teacher education this hidden alternative proved to be a Cinderella-difference making a difference. This raises a political question: What is most important, to learn how to add per-numbers, or to learn to read calculus books? What is most essential, to be prepared for a future in the laboratory, or in the library? Once the policy makers have decided this, we can answer the question: Shall we continue to use our institution to make many students calculus dropouts, or shall we allow all students to learn how to add per-numbers?

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Appendix. A Kronecker-Russell Multiplicity-Based Mathematics

1. Repetition in time exists and can be experienced by putting a finger to the throat.

2. Repetition in time has a 1-1 correspondence with Many in space (1 beat <->1 stroke).

3. Multiplicity in space can be bundled in icons with 4 stokes in the icon 4 etc.: IIII -> 4.

4. Multiplicity can be counted in icons producing a stack of, e.g., *T* = 3 4s = 3*4. The process 'from *T* take away 4' can by iconized as '*T*-4'. The repeated process 'from T take away 4s' can by iconized as '*T*/4, a 'per-number'. So, the 'recount-equation' *T* = (*T*/4)*4 is a prediction of the result when counting *T* in 4s to be tested by performing the counting and stacking:

T = 8 = (8/4)*4 = 2*4, *T* = 8 = (8/5)*5 = 1 3/5 * 5.

5. A calculation *T* = 3*4 = 12 predicts the result when recounting 3 4s in tens and ones.

6. Multiplicity can be re-counted: If 2 kg = 5 $ = 6 liters = 100 % then what is 7 kg? The result can be predicted through a calculation recounting 7 in 2s:

T = 7 kg = (7/2)*2kg = (7/2)*6 liters = 21 liters	*T* = 7 kg = (7/2)*2kg = (7/2)*100 % = 350 %	*T* = 7 kg = (7/2)*2kg = (7/2)*5 $ = 17.50 $

7. A stack is divided into triangles by its diagonal. The diagonal's length is predicted by the Pythagorean theorem a^2 + b^2 = c^2, and its angles are predicted by recounting the sides in diagonals: a = $a/c*c$ = sin$A*c$, and b = $b/c*c$ = cos$A*c$.

8. Diameters divide a circle in triangles with bases adding up to the circle circumference:

C = diameter * *n* * sin(180/*n*) = diameter * π.

9. Stacks add by removing overloads (predicted by the 'restack-equation' *T = (T-B)+B)*:

T = 38+29 = 3ten8+2ten9 = 5ten17 = 5ten1ten7 = 6ten7 = 67

10. Per-numbers can be added after being transformed to stacks. Thus, the $/day-number *b* is multiplied with the day-number n before being added to the total $-number T: T2 = T1 + n*b.

2days at 6$/day + 3days at 8$/day = 5days at (2*6+3*8)/(2+3)$/day = 5days at 7.2$/day

1/2 of 2 cans + 2/3 of 3 cans = (1/2*2+2/3*3)/(2+3) of 5 cans = 3/5 of 5 cans

Repeated addition of per-numbers -> integration	Reversed addition of per-numbers -> differentiation
*T*2 = *T*1 + *n**b* *T*2 - *T*1 = + *n**b* ΔT = $\sum n*b$ ΔT = $\int b*dn$	*T*2 = *T*1 + *n**b* *(T*2-*T*1)/*n* = *b* $\Delta T/\Delta n$ = *b* *dT/dn* = *b*

Only in the case of adding constant per-numbers as a constant interest of, e.g., 5% the per-numbers can be added directly by repeated multiplication of the interest multipliers:

4 years at 5 % /year = 21.6%, since 105%*105%*105%*105% = 105%^4 = 121.6%.

Conclusion. A Kronecker-Russell Many-based mathematics can be summarized as a 'count&add-laboratory' adding to predict the result of counting stacks and per-numbers, in accordance with the original meaning of the Arabic word 'algebra', reuniting.

ADDING	Constant	Variable
Stacks m, s, kg, $	*T* = *n**b* *T/n* = *b*	*T*2 = *T*1 + *n**b* *T*2-*T*1 = *n**b*
Per-numbers m/s, $/kg, $/100$ = %	*T* = *b*^*n* $\sqrt[n]{T} = b$ $\log_b T = n$	*T*2 = *T*1 + $\int b*dn$ *dT/dn* = *b*

The Count&Add-Laboratory

FunctionFree PerNumber Calculus

Calculus did not call itself 'calculus'. So calculus could also be called something else. Thus, the non-action word 'calculus' could be reworded to the action-word 'adding per-numbers' - *and then take place from K – 12.*

Background

At a talk in the late 1970's Freudenthal described the didactical expert as a reflective practitioner. This inspired the author to use the mathematics classroom as a laboratory to develop different ways to teach secondary mathematics at the pre-calculus and the calculus level. Later as a research student the author used the inspiration from postmodern thinking to develop a new action focused methodology called skeptical Cinderella research.

The theoretical framework of Cinderella research is institutional skepticism as it appeared in the Enlightenment and was implemented in its two democracies; the American in the form of pragmatism, symbolic interactionism, and grounded theory; and the French in the form of post-structuralism and postmodernism.

Cinderella research sees the pragmatic and postmodern skepticism towards our most basic institution, words, validated by a simple observation 'the pencil-paradox': Placed between a ruler and a dictionary a thing can point to numbers, but not to words - thus, a thing can falsify a number-statement, but not a word-statement. A number is an ill written icon showing the degree of Many (there are 4 strokes in the number sign 4, etc.); a word is a sound made by a person and recognized in some groups and not in others. Numbers carry valid conclusions based upon reliable data, i.e., research. Words carry interpretations, that if presented as research become seduction (Tarp 2003).

Institutional Skepticism Directed Towards Calculus

Looking at calculus, a Cinderella study will use postmodern scepticism to note: Calculus did not call itself 'calculus', so calculus could also be called something else. And then use pragmatism to suggest a different word grounded on what calculus is doing. Thus, the name 'adding variable per-numbers' respects calculus' grounding question: '5 seconds at 3m/s increasing to 4m/s total ? m'. And pre-calculus' grounding questions: '5 days at 3$ (3%) total ? $ (%).

Also, the name 'adding variable per-numbers' respects that the original meaning of the Arabic word 'algebra' is 'reuniting':

Algebra: Reuniting	**Constant**	**Variable**
Totals m, s, kg, $	$\boldsymbol{T = a*n}$ $T/n = a$	$\boldsymbol{T2 = T1 + a*n}$ $T2\text{-}T1 = a*n$
Per-numbers m/s, $/kg, $/100$ = %	$\boldsymbol{T = a\text{^}n}$ $n\sqrt{T} = a \quad \log_a T = n$	$\boldsymbol{T2 = T1 + \int a*dx}$ $dT/dx = a$

With 'adding variable per-numbers' as a parallel name for 'calculus' we can now take a skeptical view at the calculus traditions. The first thing we observe is the absence of the function-concept in the above grounding question, opposed to its presence in the modern tradition. This raises a Cinderella-question:

Is there a neglected different postmodern calculus that might be a Cinderella difference by making the prince dance (i.e., making the students learn)?

Postmodern thinking would use concept archaeology to look for an answer in history and study the social construction of calculus and the function concept. Doing this we find that calculus was constructed before 1700, and the function-concept after 1700, e.g.,

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1748)

We see that Euler defines the function as an abstraction separating two kinds of expressions, calculations without and with a ‘variable quantity’. Thus, instead of saying ‘calculus performs calculations on expressions with variable quantities’, he can just say ‘calculus performs calculations on functions’. However, today this function-concept has been turned upside down by defining a function as an example of an abstraction: ‘A function is an example of a relation between two sets.’

Thus, we are faced with two different kinds of mathematics, a historical one defining a concept as an abstraction from examples, and a modern one defining a concept as an example from an abstraction. To distinguish we can introduce the name ‘meta-matics’ for the modern set-based mathematics. Introducing new names allows you to introduce new actions. If we ask ‘will it be possible to introduce calculus in primary school?’ the answer right away would be: ‘Of course not since calculus builds upon functions, which belongs to upper secondary school!’ If instead we ask ‘will it be possible to introduce addition of per-numbers in primary school?’ the answer would be: ‘Yes, of course it will!’

Primary School

Adding Per-numbers: Integration

Multiplicity is counted in stacks, e.g., 3 4s. 3*4 is 3 4s, and only 12 1s if recounted in tens. 3*4 can be recounted in 3s as 4*3, i.e., 4 3s. Or 3*4 can be recounted in 5s as predicted by the ‘recount-equation’:

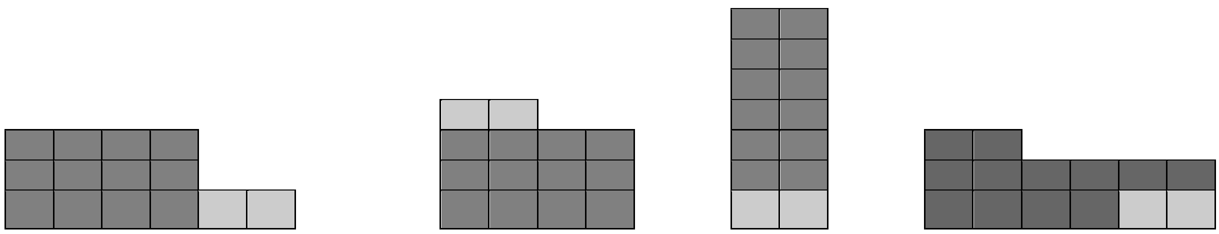
T = (T/5)*5: *T* = (3*4/5)*5 = 2*5+2*1 = 2*5 + 2/5*5 = 2 2/5*5 = 2.2*5.

Two stacks (a stock) can be added to 1 stack: *T* = 3 4s + 1 2s = 3*4 + 1*2 = ?

Added ‘in space’ as 4s: *T* = 3*4 + 1*2 = 3 2/4*4 = 3.2*4

Added ‘in space’ as 2s: *T* = 3*4 + 1*2 = 7*2

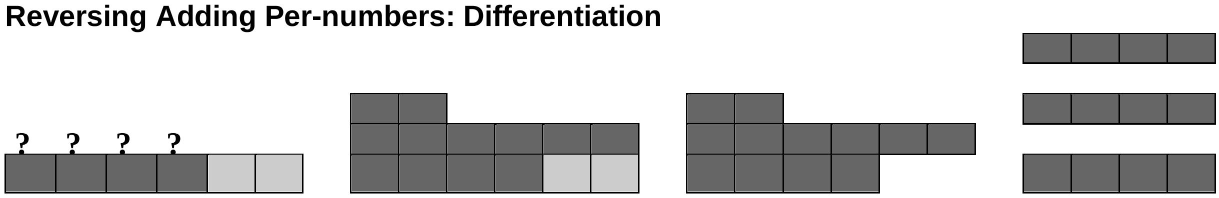
Added ‘in time’ as 6s: *T* = 3*4 + 1*2 = 2 2/6 * 6



T = 3*4 + 1*2 = 3 2/4*4 = 7*2 = 2 2/6 * 6

Recount-predictions: (1*2)/4*4=2/4*4, (3*4)/2*2=7*2, (3*4+1*2)/6*6=2 2/6*6

Reversing Adding Per-numbers: Differentiation



T = ?*4 + 1*2 = 2 2/6 * 6 T-1*2 (T-1*2)/4 *4

Recount-prediction:

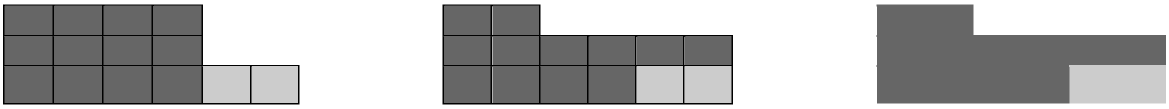
?**n* + *T*1 = *T*2, ? = *(T*2-*T*1*)/n* *Take away 4: T-4* *Take away 4s: T/4*

Middle School

Adding Per-numbers: Integration

4kg at 3\$/kg = 4*3 = 12\$
<u>2kg at 1\$/kg = 2*1 = 2\$</u>
6kg at *x*\$/kg = 6**x* = 14\$ = 14/6*6\$

x = 14/6 \$/kg = 2 2/6 \$/kg = 2.3 \$/kg

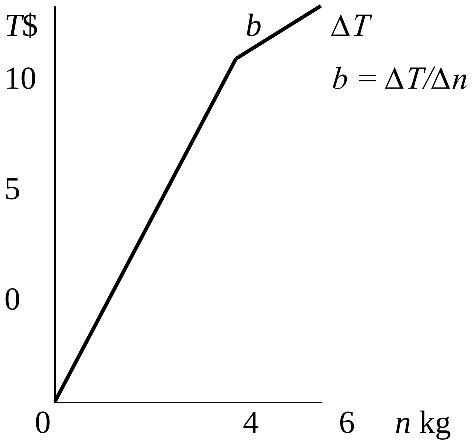


4kg at 3\$/kg+ 2kg at 1\$/kg = 6kg at 2 2/6 \$/kg = 6kg at 2.3 \$/kg

Reversing Adding Per-numbers: Differentiation

4kg at ?\$/kg = 4*x = 4*x\$
<u>2kg at 1\$/kg = 2*1 = 2\$</u>
6kg at 14/6\$/kg = 4*x + 2= 14\$= 14 –2 +2 (the 'restack-equation' T= T–B+B)

x = (14-2)/4 = 3 = (T2-T1)/n = ΔT/Δn



Thus, from adding per-numbers we learn that:

The Total is the area under the per-number curve predicted by an integration formula:

ΔT = ∑ \$/kg*kg = ∑ b*Δn

adding the per-number stacks.

The per-number is the slope of the Total curve predicted by a differentiation formula:

b = Δ\$/Δkg = ΔT/Δn recounting the ΔT in Δns.

High School

The 3 Per-number Problems:

P1. The total of 5 days at 3\$/day is ?\$

P2. The total of 5 days at 3%/day is ?%

P3. The total of 5 days at 3\$/day increasing to 4\$/day is ?\$

The 3 Per-number Solutions:

S1. The total of 5 days at 3\$/day is 5*3\$ = 15\$

The total of n days at 3\$/day is n*3\$: T = n*3 *Linear change*

S2. The total of 5 days at 3%/day is 5*3%+0.9%=15.9% since (1+3%)^5=1.159

The total of n days at 3%/day is (1+3%)^n: T = (1+3%)^n *Exponential change*

Total interest I = simple interest n*i + compounded interest CI

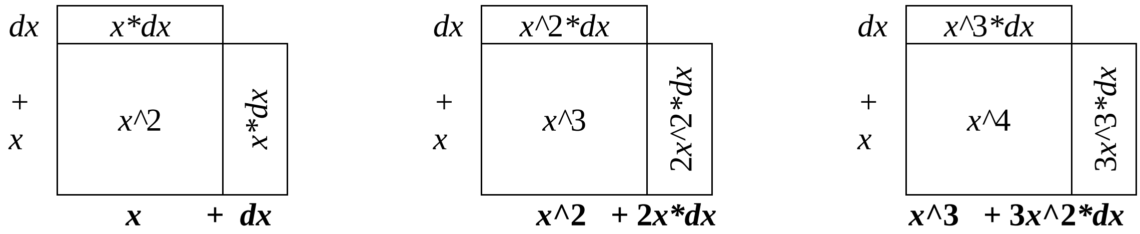
Compounded interest is what keeps exponential change from being linear.

So, locally, exponential change is almost linear since the CI can be neglected at small interests:

The total of 5 days at 0.3%/day ≈ 5*0.3% = 1.5% since (1+0.3%)^5 = 1.01509 ≈ 1.015

Total interest I ≈ simple interest n*I; or (1+I) = (1+i)^n ≈ 1 + n*i

Neglecting compounded interest (or the upper right corner of the change stack) is called differential calculus. So, in differential calculus the non-linear is considered locally linear.



In differential calculus *d(x^2) = 2*x*dx, or d/dx(x^2) = 2x; d(x^3) = 3*x^2*dx,*

or *d/dx(x^3)= 3*x^2; d(x^4)= 4*x^3*dx, or d/dx(x^4) = 4*x^3* etc.

So, *d(x^n) = n*x^(n-1)*dx, or d/dx (x^n) = n*x^(n-1).*

S3. Now we are able to predict the result of adding variable per-numbers through integration using that $\int dT = \Delta T$ since the sum $\int$ of single changes dT is the total change ΔT no matter the size of the changes:

5 sec. at 3m/sec increasing to 4 m/sec total $\int_0^5 (3+\frac{4-3}{5}x)\,dx = \int_0^5 (3+0.2x)\,dx = ?$ m

Since $d/dx\,(3x + 0.1x^2) = 3 + 0.2x$ we get that $d(3x + 0.1x^2) = (3 + 0.2x)\,dx$, so

$$\int_0^5 (3+0.2x)\,dx = \int_0^5 d(3x+0.1x^2) = \Delta(3x+0.1x^2) = (3\cdot5+0.1\cdot5^2) - 0 = 17.5 \text{ m}$$

FunctionFree PerNumber PreCalculus makes a difference in the classroom (Tarp 2003).

For details on FunctionFree PerNumber Calculus in primary, middle and high school please consult Zybartas et al. 2004, Tarp 2004 a, and Tarp 2004 b.

Conclusion

Modern calculus is turned upside down by being based upon functions that is based upon sets. This transformation of historical mathematics into unhistorical 'meta-matics' creates learning problems in the classroom, and a relevance paradox 'formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994). By renaming calculus to 'adding variable per-numbers', post-modern function-free calculus solves the relevance paradox by respecting the historical roots of pre-modern calculus.

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Pastoral Calculus Deconstructed

Calculus becomes pastoral calculus killing the interest of the student by presenting limit- and function- based calculus as a choice suppressing its natural alternatives. Anti-pastoral sophist research searching for alternatives to choices presented as nature uncovers the natural alternatives by bringing calculus back to its roots, adding and splitting stacks and per-numbers.

The background

Pre-modern Enlightenment mathematics presented mathematics as a natural science. Exploring the natural fact Many, it established its definitions as abstractions from examples, and validated its statements by testing deductions on examples. Inspired by the invention of the set-concept, modern mathematics turned Enlightenment mathematics upside down to become 'metamatics' that by defining its concepts as examples of abstractions, and proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring.

However, a self-referring mathematics soon turned out to be an impossible dream. With his paradox on the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox 'this statement is false' being false when true and true when false:

Definition: $M = \{A \mid A \notin A\}$. Statement: $M \in M \Leftrightarrow M \notin M$.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn't be proven consistent since they will always contain statements that can neither be proved nor disproved.

Being still without an alternative, the failing modern mathematics creates big problems to mathematics education as, e.g., the worldwide enrolment problems in mathematical based educations and teacher education (Jensen et al, 1998); and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al, 1994, p. 371).

To design an alternative, mathematics should return to its roots guided by a new kind of research able at uncovering hidden alternatives to choices presented as nature.

Anti-Pastoral Sophist Research

Ancient Greece saw a fierce controversy between two different forms of knowledge represented by the sophists and the philosophers. The sophists warned that to protect democracy, people needed to be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. The philosophers argued that patronization is the natural order since everything physical is an example of meta-physical forms only visible to the philosophers educated at Plato's academy, who then should become the natural patronizing rulers.

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment period: when an apple obeys its own will, people could do the same and replace patronization with democracy. Two democracies were installed: one in US, and one in France, now having its fifth republic.

In France, sophist warning is kept alive in the postmodern thinking of Derrida, Lyotard and Foucault warning against pastoral patronizing categories, discourses and institutions presenting their choices as nature (Tarp 2004). Derrida recommends that pastoral categories be 'deconstructed'. Lyotard recommends the use of postmodern 'paralogy' research to invent alternatives to pastoral discourses. And Foucault uses the term 'pastoral power' to warn against institutions legitimizing their patronization with reference to categories and discourses basing their correctness upon choices claimed to be nature.

In descriptions, numbers and words are different as shown by the ‘number & word dilemma’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘pencil’, thus, being able itself to falsify its number but not its word, which makes numbers nature and words choices, becoming pastoral if suppressing their alternatives; meaning that a thing behind a word only shows part of its nature through a word, needing deconstruction to show other parts.

Thus, anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘In this case, what is nature and what is pastoral choice presented as nature?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the method of natural research developed in the other Enlightenment democracy, the American; and resonating with Piaget’s principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions.

The Nature of Numbers

Feeling the pulse of the heart on the throat shows that repetition in time is a natural fact; and adding one stick and one stroke per repetition creates physical and written multiplicity in space.

A collection or total of, e.g., eight sticks can be treated in different ways. The sticks can be rearranged to an eight-icon 8 containing the eight sticks, written as 8. The sticks can be collected to one eight-bundle, written as 1 8s. The sticks can be ‘decimal-counted’ in 5s by bundling & stacking, bundling the sticks in 5s and stacking the 5-bundles in a left bundle-cup and stacking the unbundled singles in a right single-cup. When writing down the counting-result, cup-writing gradually leads to decimal-writing where the decimal separates the bundle-number from the single-number:

8 = 1 5s + 3 1s = 1]3 = 1.3 5s

So, the nature of numbers is that any total can be decimal-counted by bundling & stacking and written as a decimal number including its unit, the chosen bundle-size.

Since ten is chosen as a standard bundle-size, no icon for ten exists making ten a very special number having its own name but not its own icon. This has big technical advantages as shown when comparing the Arabic numbers with the Roman numbers, that has a special ten-icon X, but where multiplication as XXXIV times DXXVIX is almost impossible to do. But without its own icon ten creates learning problems if introduced to early. So, to avoid installing ten as a cognitive bomb in young brains, the core of mathematics should be introduced by using 1digit numbers alone (Zybartas et al 2005).

Also, together with choosing ten as the standard-bundle size, another choice is made, to leave out the unit of the stack thus, transferring the stack-number 2.3 tens to what is called a natural number 23, but which is instead a choice becoming pastoral by suppressing its alternatives. Leaving out units might create ‘mathematism’ (Tarp 2004) true in the library where 2+3=is true, but not in the laboratory where countless counterexamples exist: 2weeks+3days = 17days, 2m+3cm = 203cm etc.

The Nature of Operations

Operations are icons describing the process of counting by bundling & stacking.

The division-icon ‘/2’ means ‘take away 2s’, i.e., a written report of the physical activity of taking away 2s when counting in 2s, e.g., 8/2 = 4. The multiplication-icon ‘4*’ means ‘(stacked) 4 times’, i.e., a written report of the physical activity of stacking 2-bundles 4 times, T = 4*2

Subtraction ‘- 2’ means ‘take away 2’, i.e., a written report of the physical activity of taking away the bundles to see what rests as unbundled singles, e.g., R = 9 – 4*2. And addition ‘+2’ means ‘plus 2’, i.e., a written report of the physical activity of adding 2 singles to the stack of bundles either as singles or as a new stack of 1s making the original stack a stock of, e.g., T = 2*5 + 3*1, alternatively written as T = 2.3 5s if using decimal-counting.

Thus, the full process of ‘re-counting’ or ‘re-bundling’ 8 1s in 5s can be described by a ‘re-count or re-bundle formula’ containing three operations, together with a ‘rest formula’ finding the rest:

T = (8/5)*5 = 1*5 + 3*1 = 1.3*5   since the rest is *R* = 8 – 1*5 = 3.

All the recount formula *T = (T/B)*B* says is: the total T is first counted in, then stacked in Bs.

This recount formula cannot be used with ten as the bundle-size since we cannot ask a calculator to calculate *T* = (8/ten)*ten. However, this is no problem since the moment ten is chosen as the standard bundle-size, the operations take on new meanings. Now recounting any stack in tens is not done anymore by the recounting formula but by simple multiplication. To re-bundle 3 8s in tensinstead of writing *T* = (3*8)/10*10 = 2.4 * 10, we simply write

T = 3*8 = 24.

The Nature of Formulas

Using the recount formula, the counting result can be partly predicted on a calculator where 9/4 = 2.some. This predicts that recounting 9 in 4s will result in 2 4-bundles and some singles. The number of singles can be predicted by the rest formula *R* = 9 – 2*4 = 1.

So, (9/4)*4 is 2.1 4s.

Thus, the calculator becomes a number-predictor using calculation for predictions. This shows the strength of mathematics as a language for number-prediction able to predict mentally a number that later is verified physically in the ‘laboratory’. Historically, this enabled mathematics to replace pastoral belief with prediction, and to become the language of the natural sciences.

The Nature of Equations

The statement 4 + 3 = 7 describes a bundling where 1 4-bundle and 3 singles are re-bundled to 7 1s. The equation x + 3 = 7 describes the reversed bundling asking what is the bundle-size that together with 3 singles can be re-bundled to 7 1s. Obviously, we must take the 3 singles away from the 7 1s to get the unknown bundle-size: *x* = 7 – 3. So, technically, moving a number to the other side changing its calculation sign solves this equation:

If *x* + 3 = 7, then *x* = 7 – 3.

The statement 2.1* 3 = 7 describes a bundling where 2.1 3-bundles are re-bundled to 7 1s. The equation *x* * 3 = 7 describes the reversed bundling asking how 7 1s can be re-bundled to 3s. Using the re-bundling procedure and formula, the answer is *T* = 7 = (7/3)*3, i.e., *x* = 7/3. Again technically, moving a number to the other side changing its calculation sign solves this equation: If x * 3 = 7, then *x* = 7/3.

The statement 2 * 3 + 1 = 7 describes a bundling where 2 3-bundles and 1 single are re-bundled in 1s. The equation x * 3 + 1 = 7 describes the reversed bundling asking how 7 1s can be re-bundled in 3s leaving 1 unbundled. Obviously, we first take the single unbundled away, 7 – 1, and then bundle the rest in 3s, giving the result *x* = (7-1)/3. Again technically, moving a number to the other side changing its calculation sign solves this equation:

If x * 3 + 1 = 7, then *x* = (7-1)/3.

The statement 2 * 3 + 4 * 5 = 4.2 * 6 describes a bundling where 2 3-bundles and 4 5-bundles are re-bundled in 6s. The equation 2 * 3 + x * 5 = 4.2 * 6 describes the reversed bundling asking how 4.2 6s can be re-bundled to two stacks, 2 3s and some 5s. Obviously, we first take the 2 3s away, and then bundle the rest in 5s. Again technically, moving a number to the other side changing its calculation sign solves this equation:

If 2 * 3 + x * 5 = 4.2 * 6, then *x* = (4.2 * 6 – 2 * 3)/5

The Nature of Calculus

The statement 2 * 3 + 4 * 5 = 3.2 * 8 describes a bundling where 2 3-bundles and 4 5-bundles are re-bundled in the united bundle-size 8s. This is 1digit integration. The equation 2 * 3 + x * 5 = 3.2 * 8 describes the reversed bundling asking how 3.2 8s can be re-bundled to two stacks, 2 3s and some 5s. This is a 1digit differential equation solved by performing 1digit differentiation:

If 2 * 3 + x * 5 = 4.2 * 6, then x = (4.2 * 6 – 2*3)/5 = (T – T1)/5 = ΔT/5

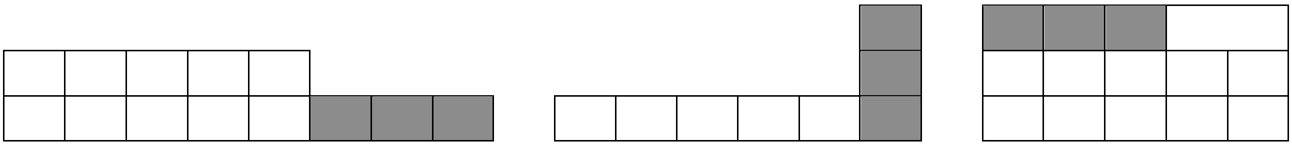
The Nature of Fractions

Once ten has been chosen as the standard bundle-size, the operations take on new meanings. Now recounting any stack in tens is not done anymore by the re-counting formula but by simple multiplication. To re-bundle 3 8s in tens instead of writing T = (3*8)/10*10 = 2.4 * 10, we simply write T = 3*8 = 24. Now tables are practiced for re-bundling 2s, 3s, 4s etc. in tens.

With multiplication taking over there is no more need for division in re-bundling and recounting. So, division takes on a new meaning in 'per-numbers': If 2 kg costs 8 $, then the unit-price is 8$ per 2kg, i.e., 8$/2kg = 8/2 $/kg. Thus, if 4kg cost 5$, the guide-equation '4kg = 5$' is used when re-counting the actual kg-number in 4s, and re-counting the actual $-number in 5s:

10kg = (10/4)*4kg = (10/4)* 5$ = 12.5$, and 18$ = (18/5)*5$ =(18/5)*4 kg = 14.4kg.

Division is also part of fractions, originally occurring if instead of placing 3 singles besides the existing stack of 5-bundles, the 3 singles are bundled as a 5-bundle and put on top of the 5-stack giving a stack of T = 2*5 + (3/5)*5 = 2 3/5 *5 = 2 3/5 5s.



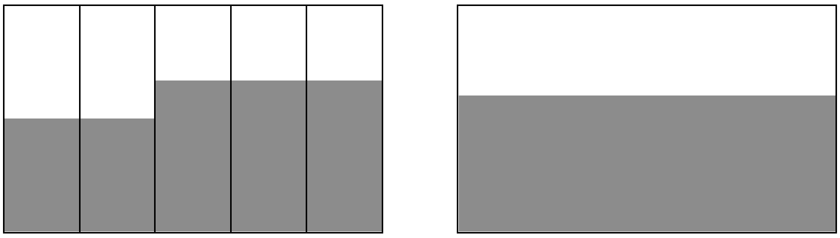
T = 2*5 + 1*3 = 2.3 *5 = 2 3/5 * 5

When adding fractions, it is important to reintroduce the units to avoid scaring the learners with mathematism as when performing the following 'fraction test' the first day of secondary school:

The teacher:	The students:
Welcome to secondary School! What is 1/2 + 2/3?	1/2 + 2/3 = (1+2)/(2+3) = 3/5
No. The correct answer is: 1/2 + 2/3 = 3/6 + 4/6 = 7/6	But 1/2 of 2 cokes + 2/3 of 3 cokes is 3/5 of 5 cokes! How can it be 7 cokes out of 6 cokes?
If you want to pass the exam then 1/2 + 2/3 = 7/6!	

That seduction by mathematism is costly is witnessed by the US Mars program crashing two probes by neglecting the units cm and inches when adding. So, to add numbers the units must be included, also when adding fractions. And adding fractions f is basically integration:

T = 1/2 *2 + 2/3 *3 = Σ (f*Δx) later to become $\int f\,dx$

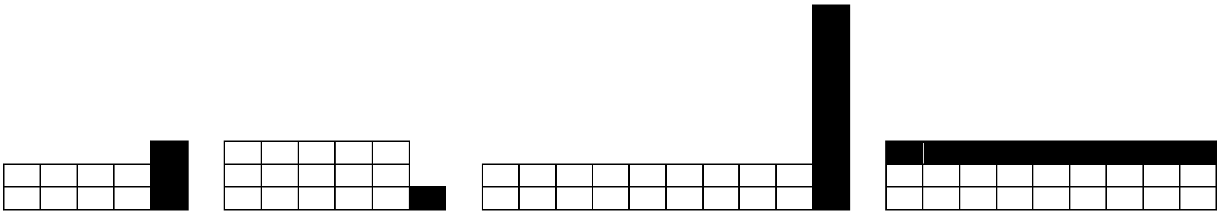


T = 1/2 *2 + 2/3 *3 = 3/5 *5

Primary School Calculus

In primary school integration means integrating 2 stacks into one where the bundle-size is the sum of the stacks' bundle-sizes. Thus, a typical integration problem is 2.3 4s + 3.1 5s = ? 9s.

Manually, integration means placing the 2 bundle-stacks next-to each other; then placing the two 1-stacks on top of each other; then moving any unbundled to the 1-stack; and finally move any bundles from the 1-stack to the bundle-stack.

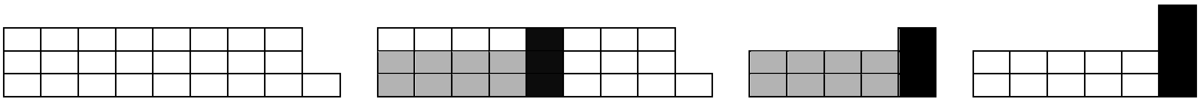


2.3 4s + 3.1 5s = 2.9 9s = 3.0 9s

Reversing the integration of two stacks becomes differentiation. Thus, a typical differentiation question is 2.3 4s + ? 5s = 3.1 8s

Manually, differentiation means taking away a stack of bundles and a stack of 1s from a bigger stack; and then recount the rest in the given bundle size. Removing stacks reports this process:

? = (3.1 8s - 2.3 4s)/5 * 5, later to be written as a differential quotient $(T - T1)/5 = \Delta T/5$.



3.1 8s = 2.3 4s + ? 5s = 2.3 4s + 2.4 5s

Middle School Calculus

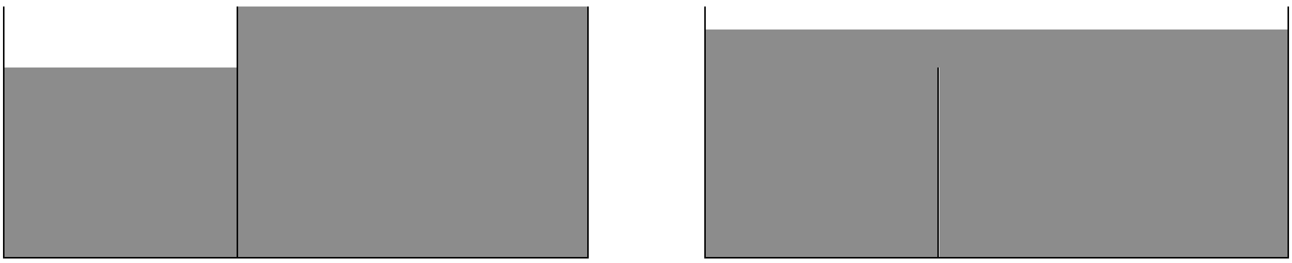
In middle school integration means integrating two fractions or per-numbers into one. Thus, a typical integration problem is 1/2 of 2 + 2/3 of 3 = ? of 5; and 10% of 2 + 40% of 3 = ? of 5; and 2kg at 6$/kg + 3kg at 9$/kg = 5kg at ? $/kg.

Manually, integration means drawing next to each other two rectangular pools and then find the average water-level if the separating wall is removed. Note-writing reports this process:

2 kg at 6 $/kg = 2*6 = 12 $

3 kg at 9 $/kg = 3*9 = 27 $

5 kg at ? x/kg = 5*x = 39 $, so x = 39/5 = 7.8 $/kg



1/2 of 2 + 2/3 of 3 = ? of 5

Reversing the integration of two pools becomes differentiation. Thus, a typical differentiation question is 2kg at 5$/kg + 3kg at ?$/kg = 5kg at 6 $/kg.

Manually, differentiation means drawing two rectangular pools next to each other and then finding the resulting water-level of the right pool after water is pumped from the left pool.

Mentally, combined subtraction and division reports the manual process:

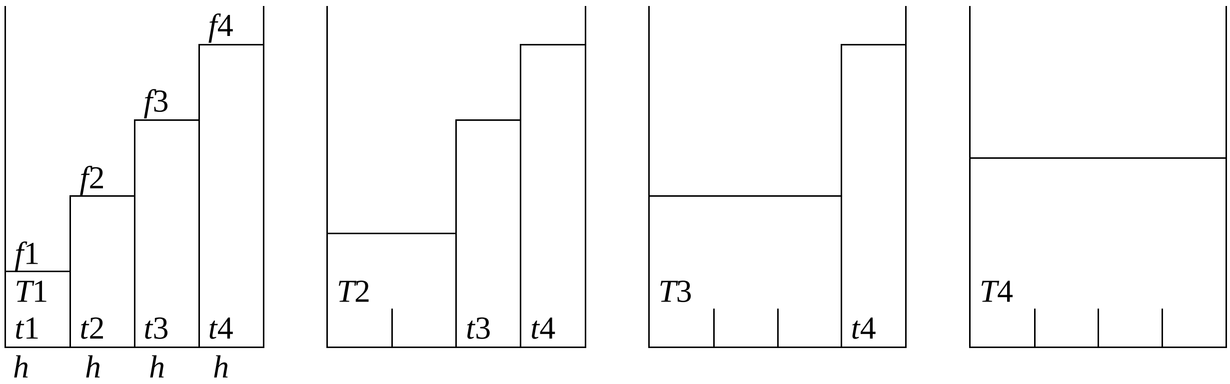
2*5 + 3*x = 5*6 , x = (5*6 – 2*5)/3 , later to be written as $(T - T1)/5 = \Delta T/5$.

High School Calculus

In high school, integration means integrating many per-numbers into one. Thus, a typical integration problem is: 7 seconds at 2 m/s increasing to 4 m/s totals 7 seconds at ? m/s in average.

Manually, integration means drawing next to each other many rectangular micro-pools with width h and water level described by a formula f; and then finding the formula describing the water level when the walls are removed one by one.

Mentally, cumulating describes the manual process: The volume of pool1 is $t1 = f1*h$. The total volume then is $T4 = T3 + t4 = t1 + t2 + t3 + t4 = \Sigma ti = \Sigma fi * h$

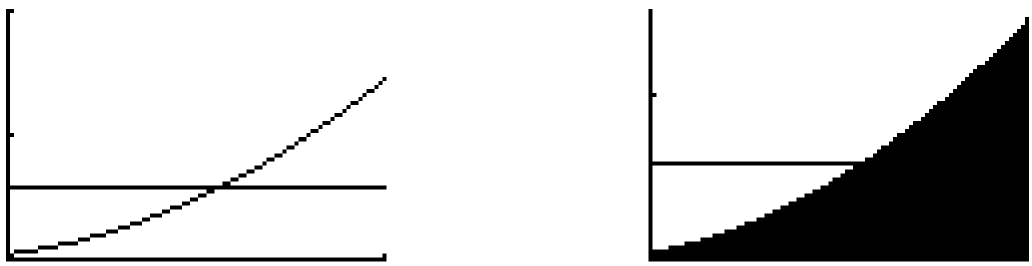


Reversing the integration of pools becomes differentiation. Thus, a typical differentiation question is: 5 seconds at 1.82 m/s in average + 0.1 second at ? m/s totals 5.1 seconds at 1.84 m/s in average.

Manually differentiation means drawing two rectangular pools next to each other and then finding the resulting water-level of the right pool after water is pumped from the left pool.

Mentally combined subtraction and division reports the manual process: $T3 + f4*h = T4$,

$f4 = (T4 – T3)/h$, now to be written as $\Delta T/h = dT/dx$ *if* $h = dx$ is micro-small.



In the case of many micro-pools a Graphical Display Calculator (GDC) can do the drawing.

The Choices of Pastoral Calculus

Modern calculus still believes in the existence of sets despite of Russell's paradox. Instead of introducing calculus as integrating per-numbers in middle school it waits for algebra to define the real numbers. Then the concept of a limit can be given its ε–δ definition by calculus believing this gives a precise meaning to the term 'micro-small'; but forgetting that δ is the micro-number giving the level of exactness described by ε.

So, instead of working with micro-numbers, modern calculus presents both the derivative and the integral as examples of the concept limit, which creates big problems to learners. Also, the so-called fundamental theorem of calculus by it naming alone is presented as a deep insight where instead it is a mere banality as shown below.

The Natural Alternatives of Anti-pastoral Calculus

To uncover natural alternatives to the choices of modern calculus, becoming pastoral by suppressing its alternatives, anti-pastoral sophist research sees calculus as a natural science exploring the natural fact Many rearranged by bundling and stacking. In this approach the roots of calculus is found to be adding two stacks in a combined bundle-size, as shown above. Later calculus also applies when adding per-numbers as $/kg or m/s. Here integration means predicting the area under the per-number graph to predict the total $-number or m-number; and differentiation means predicting the gradient on the total $-graph or m-graph to predict the per-number.

As to finding gradient formulas given a total formula, and finding area-formulas given a per-number formula, the method of natural science can be used involving observation, induction and testing predictions.

First the GDC's gradient calculator *dy/dx* is validated if its predictions are verified and never falsified on examples with known gradient. Thus, no matter where the *dy/dx*-number is calculated on the graph $y = 2.7\,x + 4$, the answer is the expected $dy/dx = 2.7$.

Now a table can be set up for the relation between x and the gradient-number *dy/dx* on the graph $y = x^2$. Using regression, it turns out that a gradient-formula $dy/dx = 2x$ can be induced and used for deducing predictions that all are verified. In this way the experimental method of natural science can be used to find the different gradient-formulas.

Next the GDC's area calculator $\int f(x)dx$ is validated if its predictions are verifying and never falsified on examples with known area. Thus, no matter where the area-number is calculated on the graph $y = 2$, the answer agrees with the known formula $A = 2*(x2\text{-}x1)$.

Now a table can be set up for the relation between x and the area-number under the graph $y = x^2$ from 0 to x. Using regression, it turns out that an area-formula $A = x^3/3$ can be induced and used for deducing predictions that are all verified. In this way the experimental method of natural science can be used to find the different area-formulas.

As to finding the area-number using the fundamental theorem of Calculus, as simple observation shows that the following statement cannot be falsified:

The total change = the cumulated step-change = y end – y start, or $\Delta y = \Sigma\Delta y = y2 - y1$.

This statement does not depend upon the size or number of changes, so it also applies for small micro-changes:

$\Delta y = \int dy = y2 - y1$, or $\Delta y = \int y'\,dx = y2 - y1$ if dy is re-counted in dxs as $dy = dy/dx * dx$.

So, when calculating the area under an f-graph, if f can be re-written as a change-formula $f(x) = dy/dx$, then the area $\int f(x)\,dx$ can be written as $\int dy$ and calculated as the difference $y2 - y1$.

Number y	Step-change Δy	Cumulated step-change $\Sigma\Delta y$	Total change $\Delta y = y2 - y1$
2			
5	3	3	3
4	-1	2	2
9	5	7	7

Conclusion

Modern mathematics finds it natural to postpone calculus until the end of high school or the beginning of university, wanting to present it as metamatics, i.e., as an example of the higher abstractions as sets, functions, real numbers and limits; in spite of the fact that historically calculus was developed before these abstractions. However, this choice turns out to be a pastoral choice suppressing its natural alternatives uncovered by anti-pastoral sophist research searching for alternatives to choice, presented as nature. The natural alternative is to introduce calculus in primary school as adding stacks in united bundle-size, and to reintroduce calculus in middle school as adding per-number and fractions with units, and finally using the methods of natural science to make high school calculus limit-free.

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Calculus Grounded in Adding Per-numbers

Mathematics education is an institution claiming to provide the learner with well-proven knowledge about well-defined concepts applicable to the outside world. However, from a skeptical postmodern perspective wanting to tell nature from choice, three questions arise: Are the concepts grounded in nature or forcing choices upon nature? How can ungrounded mathematics be replaced by grounded mathematics? What are the roots of calculus?

The Background

The Enlightenment period treated mathematics as a natural science. Grounded in the natural fact Many, it induced its definitions as abstractions from examples, and validated its statements by testing deductions on examples (Kline, 1972). Using the concept set, modern mathematics turned Enlightenment mathematics upside down to a purely deductive 'metamatics' that by defining its concepts as examples of abstractions, and by proving its statements as deductions from meta-physical axioms, needs no outside world and becomes entirely self-referring. However, self-referring mathematics was soon proven contradictory. Being false when true and true when false, the classical liar-paradox 'this statement is false' inspired Russell to formulate a paradox about the set of sets not belonging to itself: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise, without using self-reference it is impossible to prove that a proof is a proof, as shown by Gödel.

To avoid becoming metamatics, mathematics must return to its roots, the natural fact Many, guided by a contingency research looking for hidden alternatives to choices presented as nature.

Postmodern Contingency Research

Skepticism towards hidden patronization is at the root of postmodern thinking as formulated in Lyotard's definition 'Simplifying to the extreme, I define postmodern as incredulity toward metanarratives (Lyotard, 1984: xxiv).'

In ancient Greece the need for patronization was debated between philosophers and sophists arguing that in a democracy the people must be enlightened to tell nature from choice to prevent patronization by choices presented as nature. To the philosophers, choice didn't exist since the physical is examples of metaphysical forms, only accessible to philosophers educated at Plato's academy, thus, obliged to patronize ordinary people (Russell, 1945). Later the academy was transformed into Christian monasteries, again changed into academies after the Reformation.

Brahe, Kepler and Newton rebelled against the patronization of clerical knowledge by bringing the authority from the library back to the laboratory. Based upon Brahe's observations, Kepler formulated a heliocentric hypothesis that could not be tested before Newton showed that universal gravitation makes both moons and apples fall to the earth following their own calculable will. When apples follow their own will instead of that of the patronizor, humans could do the same and replace autocratic patronization with enlightenment-based democracy.

Thus, the Enlightenment century created two republics, an American and a French. The US still has its first republic using pragmatism and grounded theory to exert skepticism towards philosophical claims. The French now has its fifth republic being turned over repeatedly by the German neighbors, which has caused France to develop a skeptical thinking warning against hidden patronization (Tarp, 2004).

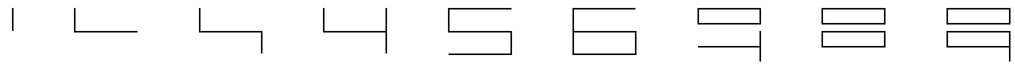
Derrida thus, uses the terms 'logocentrism' and 'deconstruction' to warn against patronizing words installing instead of labeling what they describe. Lyotard uses the term 'postmodern' to warn against patronizing sentences stating political nstead of natural correctness. Foucault uses the term 'pastoral power' to warn against patronizing institutions promising humans salvation from abnormalities installed by 'scientific' discourses. And Bourdieu uses the term 'symbolic violence' to label patronizing education that allows only the mandarin-class to acquire knowledge capital. In short, postmodern thinking means skepticism towards hidden patronization of ungrounded words, sentences and institutions. In the case of mathematics this means skepticism towards mathematical concepts, statements and education.

Based upon the sophist and French warning against hidden patronization, a research paradigm can be created called postmodern ‘contingency research’ deconstructing patronizing choices presented as nature by uncovering hidden alternatives. To keep categories, discourses and institutions non-patronizing they should be grounded in nature using Grounded Theory (Glaser et al, 1967), the natural research method developed in the US enlightenment democracy; and resonating with Piaget’s principles of natural learning (Piaget, 1970). Confronted with a tradition wanting mathematics to provide learners with well-proven knowledge about well-defined concepts applicable to the outside world (e.g., NCTM, 2000), contingency research instead looks for grounded concepts and verifiable statements that will enlighten the learner about the roots of mathematics, the natural fact Many.

Recounting as the Root of Primary School Mathematics

To deal with the natural fact Many we iconize and bundle. 1.order counting rearranges sticks to form an icon. Thus, the five-icon 5 contains five sticks if written in a less sloppy way. In this way icons are created for numbers until ten, the only number with a name, but without an icon.

I II III IIII IIIII IIIIII IIIIIII IIIIIIII IIIIIIIII



1 2 3 4 5 6 7 8 9

2.order counting bundles in icon-bundles and 3.order counting bundles in tens, the only number with its own name but without an icon.

With 2.order counting a total of 7 can be bundled in 3s as T = 2 3s and 1, and placed in a left bundle-cup and in a right single-cup. In the bundle-cup a bundle is traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using cup-writing 2]1, then using decimal-writing to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III I -> III III] I -> ▌▌] I -> II] I -> 2]1 -> 2.1 3s

The counting result can be predicted by a ‘recount-formula’ $T = (T/B)*B$ telling that the total T is counted in Bs by taking away bs T/B times. Thus, recounting a total of T = 3 6s in 7s, the prediction says T = (3*6/7) 7s. Using a calculator, we get the result 2 7s and some leftovers that can be found by the ‘rest-formula’ $T = (T–B) + B$ telling that $T–B$ is what is left when B is taken away: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. So, the combined prediction says T = 3*6 = 2*7 + 4*1 = 2.4 7s.

This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

Once counted, totals can be added. However, two kinds of addition exist: Soft addition next-to and hard addition on-top.

Soft addition next-to adds $T1$ = 2 3s and $T2$ = 3 4s into T = 2.4 7s. Thus, the totals are added by juxtaposing their stacks, i.e., by adding the areas of the stacks. In this way soft addition next-to is the root of integration also using areas when adding

To perform hard addition on-top, the units must be the same, so the 3s must be recounted in 4s or vice versa. With $T1$ = 2 3s = 1.2 4s, $T1$ + $T2$ = 4.2 4s. And with $T2$ = 3 4s = 4 3s, $T1$ + $T2$ = 6 3s. In this way recounting and changing units becomes the root of proportionality.

With 3.order counting in tens only hard addition on-top is possible. Thus, skipping 2.order counting means also skipping the roots of calculus and proportionality

Per-numbers as the Natural Roots of Middle School Mathematics

Also, in middle school the root of mathematics is recounting the natural fact Many, now occurring different places in the outside world, e.g., in time and space and in economy with units as seconds, minutes, cm, m, m^2, m^3, liters, $, £ etc.

Recounting creates fractions when, e.g., counting 3 1s in 5s as T = 3 = 0.3 5s = (3/5)*5, which shows that 3/b = 0.3 for any bundle-size b; and 'per-numbers' when a quantity is double-counted, e.g., both as 2$ and as 5kg thus, containing 2$ per 5kg, or 2$/5kg or 2/5 $/kg.

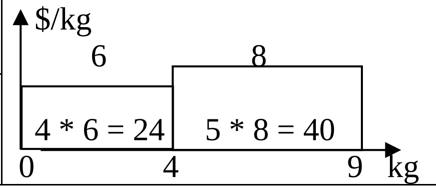
Again the recount-formula predicts recounting results when asking '6$ = ?kg', or '14kg = ?$:

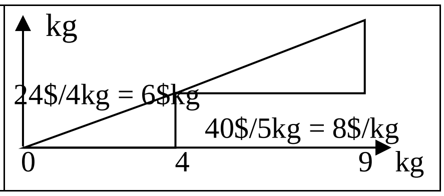
T = 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg; and T = 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$,

Or kg = kg/$ * $ = 5/2 * 6 = 15, and $ = $/kg * kg = 2/5 * 14 = 5.6.

In primary school soft addition next-to adds stacks by integrating bundles, e.g., 2 3s + 4 5s = ? 8s. Now integration adds per-numbers when adding two double-counted quantities, asking, e.g., 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg. This question can be answered by using a table or a graph.

4kg at 6$/kg	= 4 * 6 = 24
5kg at 8$/kg	= 5 * 8 = 40
9kg at ?$/kg	= 9 * ? = 64
	? = 64/9





Using a graph, we see that integration means finding the area under a per-number graph; and opposite that the per-number is found as the gradient on the total-graph.

Change as the Natural Roots of High School Mathematics

In high school, the natural root of mathematics is recounting change: Being related by a formula y = f(x), how will a change in x, Δx, affect the change of y, Δy? Here the recount-equation gives the change-formula

Δy = (Δy/Δx)*Δx, or dy = (dy/dx)*dx = y'dx for small micro-changes.

In trade, when the volume increases from 0 to x kg, the initial cost b increases to the final cost y = b + a*x if the cost increases a$/kg. This is called linear change or ++ change.

Here Δy/Δx = a.

In a bank, when the years increase from 0 to x, the initial capital b increases to the final capital y = b* (1+r)^x if the capital increases with r %/year. This is called exponential change or +* change since we add 7% by multiplying with 107% = 1 + 7%.

Here Δy/Δx = r*y.

In geometry, when the side-length is 3-doubled from 2 to 6, the area of a square y = x^2 is 3-doubled twice from 4 to 36. This is called potential change or ** change, in general given as y = b*x^a. Here Δy/y = a*Δx/x., where a is called the elasticity.

The x-change Δx, might be a micro-change, dx. On a calculator we observe that, approximately, 1.001^5 = 1.005, 1.001^9 = 1.009, √1.001 = 1.0005 and 1.001^-3 = 0.997.

From this a hypothesis can be made saying that, approximately, (1+dx)^n = 1 + ndx, allowing numerous predictions to be tested as 1.0002^4 = 1.0008 etc., all being verified by the calculator.

If y = x^n, then

(x + dx)^n = (x(1+dx/x))^n = x^n(1+ndx/x) = x^n + ndx*x^n/n = y + ndx*x^(n-1) = y+dy,

so dy/dx = n*x^(n-1).

With micro-changes *dx*, the area under a graph f(x) can be split into micro-strips with area height *width = f**dx*. The total area from a to *b* is then found by summing up the micro-strips, written as

$$A = \int_a^b f\,dx = \int_a^b dF = F(b) - F(a)$$ if *f dx* can be written as a micro-change dF, since summing up single changes gives a total change = terminal number – initial number, no matter their sizes.

Technology helps solve any change-equation dF = *f dx* by calculating the change dF that added to the initial F-value gives the terminal F-value, becoming the initial F-value in the next period.

Conclusion

The tradition presents calculus as an example of a limit process, thus, introducing limits and continuity before the derivative. A grounded alternative generalizes primary school's soft addition of stacks in combined bundle-sizes, and middle school's adding fractions with units, to adding per-numbers with units; and introduces the terms continuous and differentiable as what they really are: foreign words for locally constant and locally linear in contrast to piecewise constant and piecewise linear.

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Saving Dropout Ryan With A Ti-82

To lower the dropout rate in pre-calculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' replaced the textbook. A formula's left-hand and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by 'solve Y1-Y2 = 0'. Experiencing meaning and success in a math class, the learners put up a speed that allowed including the core of calculus and nine projects.

The Task: Reduce the Dropout Rate!

The headmaster asked the mathematics teachers: "We have too many pre-calculus dropouts. What can we do?" I proposed buying the cheap TI-82 graphical calculator, but the other teachers rejected this proposal arguing that students weren't even able to use a simple TI-30. Still, I was allowed to buy this calculator for my class allowing me to replace the textbook with a compendium emphasizing modeling with TI-82.

Concepts: Examples of Abstractions, or Vice Versa

Enlightenment mathematics was as a natural science exploring the natural fact Many (Kline, 1972) by grounding its abstract concepts in examples, and by using the lack of falsifying examples to validate its theory. But after abstracting the set-concept, mathematics was turned upside down to modern mathematics or 'metamatism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathematism' true in the library, but not in the laboratory, as, e.g., 2+3 = 5, which has countless counterexamples: 2m+3cm = 203 cm, 2weeks+3days = 17 days etc. Being self-referring, this modern mathematics did not need an outside world. However, a self-referring mathematics turned out to be a self-contradiction. With his paradox on the set of sets not belonging to itself, Russell proved that sets implies self-reference and self-contradiction as known from the classical liar-paradox 'this statement is false' being false when true and true when false: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise Gödel proved that a well-proven theory is a dream since it will always contain statements that can be neither proved nor disproved.

In spite of being neither well-defined nor well-proven, mathematics still teaches metamatism. This creates big problems to mathematics education as shown, e.g., by 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles.

Contingency Research Unmasks Choices Presented as Nature

Alternatively, mathematics could return to its roots, Many, guided by contingency research uncovering hidden patronization by discovering alternatives to choices presented as nature.

Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists warned that to practice democracy, the people must be enlightened to tell choice from nature in order to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since to them all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should be given the role as natural patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment: when an apple obeys its own will, people could do the same and replace patronization with democracy.

Two democracies were installed: one in the US still having its first republic; and one in France, now having its fifth republic. German autocracy tried to stop the French democracy by sending in an army. However, a German mercenary was no match to a French conscript aware of the feudal consequence of defeat. So, the French stopped the Germans and later occupied Germany. Unable to use the army, the German autocracy instead used the school to stop enlightenment in spreading from France. As counter-enlightenment, Humboldt used Hegel philosophy to create a patronizing line-organized Bildung school system based upon three principles: To avoid democracy, the people must not be enlightened; instead, romanticism should install nationalism so the people sees itself as a 'nation' willing to fight other 'nations', especially the democratic ones; and the population elite should be extracted and receive 'Bildung' to become a knowledge-nobility for a new strong central administration replacing the former blood-nobility unable to stop the French democracy.

As democracies, EU still holds on to line-organized education instead of changing to block-organized education as in the North American republics allowing young students to uncover and develop their personal talent through individually chosen half-year knowledge blocks.

In France, the sophist warning against hidden patronization is kept alive in the post-structural thinking of Derrida, Lyotard, Foucault and Bourdieu. Derrida warns against ungrounded words installing what they label, such word should be 'deconstructed' into labels. Lyotard warns against ungrounded sentences installing political instead of natural correctness. Foucault warns against institutionalized disciplines claiming to express knowledge about humans; instead, they install order by disciplining both themselves and their subject. And Bourdieu warns against using education as symbolic violence to monopolize the knowledge capital for a knowledge-nobility (Tarp, 2004).

Thus, contingency research does not refer to, but questions existing research by asking 'Is this nature or choice presented as nature?' To prevent patronization, categories should be grounded in nature using Grounded Theory (Glaser et al, 1967), the method of natural research developed in the first Enlightenment democracy, the American, and resonating with Piaget's principles of natural learning (Piaget, 1970).

The Case of Teaching Math Dropouts

Being our language about quantities, mathematics is a core part of education in both primary and secondary education. Most parents accept the importance of learning mathematics, but many students fail to see the meaning in doing so. Consequently, special core math courses for dropouts are developed.

Traditions of Core Precalculus Courses for Dropouts

A typical core course for math dropouts is halving the content and doubling the text volume. So, in a slow pace the students work their way through a textbook once more presenting mathematics as a subject about numbers, operations, equations and functions applied to space, time, mass and money. To prevent spending time on basic arithmetic, a TI-30 calculator is handed out without instruction.

As to numbers, the tradition focuses on fractions and how to add fractions.

Then solving equations is introduced using the traditional balancing method isolating the unknown by performing identical operations to both sides of the equation. Typically, the unknown occurs on both sides of the equation as $2x + 3 = 4x – 5$; or in fractions as $5 = 40/x$.

Then relations between variables are introduced using tables, graphs and functions with emphasis on the linear function $y = f(x) = b+a*x$. In a traditional curriculum, a linear function is followed by the quadratic function. But a core course might instead go on to the exponential function $y = b*a$^x. To avoid solving its equations the solutions are given as formulas.

Problems in Traditional Core Courses

The intention of a traditional core course is to give a second chance to learners having dropped out of the traditional math course. However, from a skeptical viewpoint trying to avoid hidden patronization by presenting choice as nature, several questions can be raised.

As to numbers, are fractions numbers or calculations that can be expressed with as many decimals as we want, typical asking for three significant figures? Is it meaningful to add fractions without units as shown by the fraction-paradox above?

As to equations, is the balancing method nature or choice presented as nature? The number $x = 8\text{-}3$ is defined as the number x that added with 3 gives 8, $x + 3 = 8$. This can be restated as saying that the equation $x + 3 = 8$ has the solution $x = 8\text{-}3$, suggesting that the natural way to solve equations is the 'move to opposite side with opposite sign' method. This method can be applied to all cases of reversed calculation:

$x + 3 = 15$	$x*3 = 15$	x^3 = 125	3^x = 243
$x = 15 – 3$	$x = 15/3$	$x = 3\sqrt{}125$	x = log3(243)

Figure 1. Equations solved by the 'opposite side & sign' method of inverse calculation

Defining a function as an example of a set-relation, is that nature or choice presented as nature?

In a formula as $y = a+b$ all numbers might be known.

If one number is unknown, we have an equation to be solved, e.g., $5 = 3+x$, if not already solved, $x = 3+5$. With two unknown numbers we have a function as in $y = 3+x$, or a relation as in $3 = x+y$ that can be changed into the function $y = 3 – x$. So, a function is just a label for a formula with two unknown variables.

Giving solution formulas to the exponential equations x^3 = 125 and 3^x = 243, is that nature or choice presented as nature? Since equation is just another name for inverse calculation, using the inverse operations root and log is the natural way to solve exponential equations.

The prime goal of education is that learners adapt to the outside world by proper actions. An action as 'Peter eats apples' is a three-term sentence with a subject, a verb and an object. Thus, mathematics education should be described in this way. The learner is the subject, the object is the natural fact Many, and the verb is how we deal with Many: we totalize expressing the total as a formula, e.g., T = 345 = 3*10^2 + 4*10 + 5*1 showing that totalizing means counting and adding bundles, that all numbers carry units, and that there are four ways to add: +, *, ^ and integration adding unlike and like unit-numbers and like and unlike per-numbers.

Totalizing can also be called algebra if using the Arabic word for reuniting. Not being a verb, mathematics could be renamed to 'totalizing', 'counting and adding' or reckoning.

Designing a Grounded Math Core Course

Real word problems translate to formulas by modeling, or triangulation in the case of forms. So, to adapt to the outside world, mathematics education has as its prime goal that persons learn how to totalize, i.e., how to count and add, how to model and how to triangulate.

A traditional core course seems to be filled with examples of choices presented as nature. This leads to the question: is it possible to design an alternative core course based upon nature instead of choices presented as nature? In other words, what would be the content of a core course in pre-calculus if grounded in the roots of mathematics, the natural fact Many?

Mathematics as a Number-Language Using Predicting Formulas

As to the nature of the subject itself, mathematics is a number-language that together with the word-language allows users to describe quantities and qualities in everyday life. Thus, a calculator is a typewriter using numbers instead of letters. A typewriter combines letters to words and sentences. A calculator combines digits to numbers that combined with operations become formulas. Thus, formulas are the sentences of the number-language.

A difference between the word-language and the number-language is that sentences describe whereas formulas predict the four different ways of uniting numbers:

Addition predicts the result of uniting unlike unit-numbers: uniting 4$ and 5$ gives a total that is predicted by the formula $T = a+b = 4+5 = 9$

Multiplication predicts the result of uniting like unit-numbers: uniting 4$ 5 times gives a total that is predicted by the formula $T = a*b = 4*5 = 20$.

Power predicts the result of uniting like per-numbers: uniting 4% 5 times gives a total that is predicted by the formula $1+T = a$^$b = 1.04$^5 = 1.217, i.e., $T = 0.217 = 21.7\%$.

Integration predicts the result of uniting unlike per-numbers: uniting 2kg at 7$/kg and 3kg at 8$/kg gives 5 kg at *T*$/5kg where *T* is the area under the $/kg per-number graph, $T = \Sigma$ p*Δx.

Solving Equations with a Solver

As shown above, inverse operations solve equations, as do the TI-82 using a solver. Equations as $2+x = 6$ always has a left-hand side and a right-hand side that can be entered on the calculators *Y*-list as *Y*1 and *Y*2. So, any equation has the form *Y*1 = *Y*2, or *Y*1 – *Y*2 = 0 that only has to be entered to the solver once. After that, solving equations just means entering its two sides as *Y*1 and *Y*2. Using graphs, *Y*1 and *Y*2 becomes two curves having the same values at intersection points.

If one of the numbers in a calculation is unknown, then so is the result. A table describes a formula with two unknowns by answering the question 'if *x* is this, then what is *y*?' Graphing a table allows the inverse question to be addressed by reading from the y-axis.

Producing Formulas with Regression

Once a formula is known, it produces answers by being solved or graphed. Real world data often come as tables, so to model real world problems we need to be able to set up formulas from tables. Simple formulas describe levels as, e.g., cost = price*volume. Calculus formulas describe predictable change where pre-calculus describes constant change.

If a variable *y* begins with the value *b* and *x* times changes by a number a, then $y = b+a*x$. This is called linear change and occurs in everyday trade and in interest-free saving.

If *y* begins with the value *b* and *x* times changes by a percentage r, then *y = b*(1+r)^x = b*a^x* since adding 5% means multiplying with 105% = 100%+5%= 1.05. This is called exponential change and occurs when saving money in a bank and when populations grow or decay.

In the case of linear and exponential change, a two-line table allows regression on a TI-82 to find the two constants *b* and *a*.

Multi-line tables can be modeled with polynomials, the formula used to describe numbers. Thus, a 3-line table might be modeled by a degree 2 polynomial *y = b + a*x + c*x*^2 including also a bending-number *c*; and a 4-line table by a degree 3 polynomial *y = b + a*x + c*x*^2 + *d*x*^3 including also a counter-bending number *d*, etc.

Graphically, a degree 2 polynomial is a bending line, a parabola; and a degree 3 polynomial is a double parabola. The top and the bottom of a bending line as well as its zeros can be found directly by graphical methods of the TI-82.

Fractions as Per-numbers

Fractions are rooted in per-numbers: 3$ per 5 kg = 3$/5kg = 3/5 $/kg. To add per-numbers they first must be changed to unit numbers by being multiplied with their units:

3 kg at 4 $/kg + 5 kg at 6 $/kg = 8 kg at (3*4 + 5*6)/8 $/kg

Geometrically this means that the areas under their graph add per-numbers.

Her again TI-82 comes in handy when calculating areas under graphs. Areas can be calculated also in the case where the graph is not horizontal but a bending line, representing the case when the per-number is changing continuously as, e.g., in a falling body: 3 seconds at 4 m/s increasing to 6 m/s totals 15 m because of a constant acceleration.

Models as Quantitative Literature

Using TI-82 as a quantitative typewriter able to set up formulas from tables and to answer both *x*- and *y*-questions, it becomes possible to include models as quantitative literature.

All models share the same structure: A real world problem is translated into a mathematical problem that is solved and translated back into a real-world solution to be evaluated.

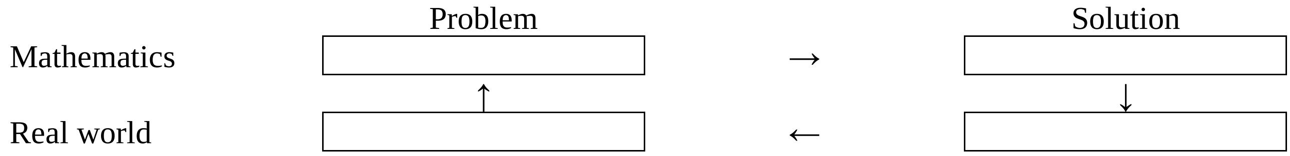


Figure 2. The four phases of mathematical modeling

The project ‘Population versus food’ looks at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2-line table regression can produce two formulas: with x counting years after 1850, the population is modeled by $Y1 = 815*1.013^x$ and the food by $Y2 = 300 + 30x$. This model predicts hunger to occur 123 years after 1850, i.e., from 1973.

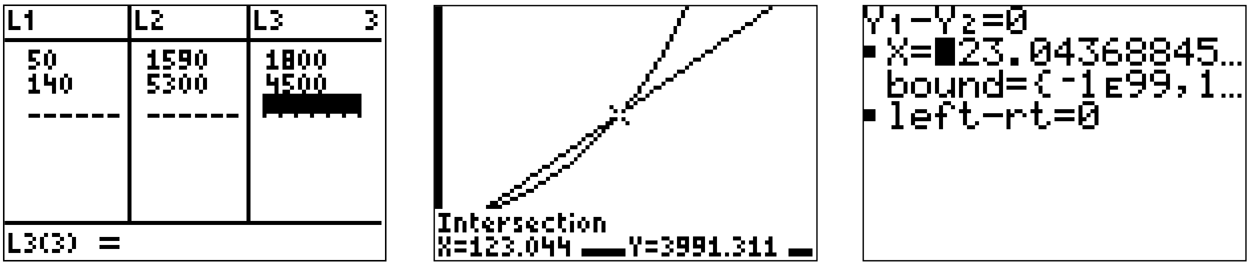


Figure 3. A Malthusian model of population and food levels

The project ‘Fundraising’ finds the revenue of a fundraising assuming all students will accept a free ticket, that 100 students will buy a 20$ ticket and that no one will buy a 40$ ticket. From this 3-line table the demand is modeled by a degree 2 polynomial $Y1 = .375*x^2 – 27.5*x + 500$. Thus, the revenue formula is the product of the price and the demand, $Y2 = x*Y1$. Graphical methods shows that the maximum revenue will be 2688 $ at a ticket price of 12$.

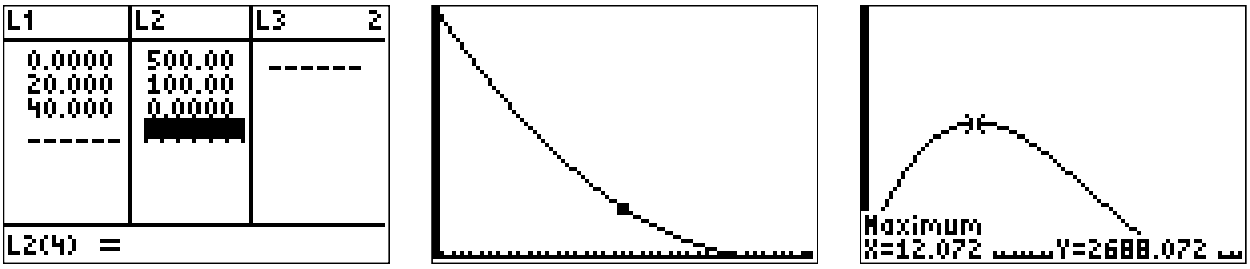


Figure 4. Modeling the optimal ticket price in a fundraising

In the project ‘Driving with Peter’ his velocity is measured five times. A model can answer many questions, e.g., when was Peter accelerating? And what distance did Peter travel in a given time interval? From a 5-line table the speed can be modeled by the degree 4 polynomial $Y1 = -0.009x^4 + 0.53x^3 - 10.875x^2 + 91.25x - 235$. Visually, the triple parabola fits the data points. Graphical methods shows that a minimum speed is attained after 14.2 seconds; and that Peter traveled 115 meters from the 10 th to the 15 th second.

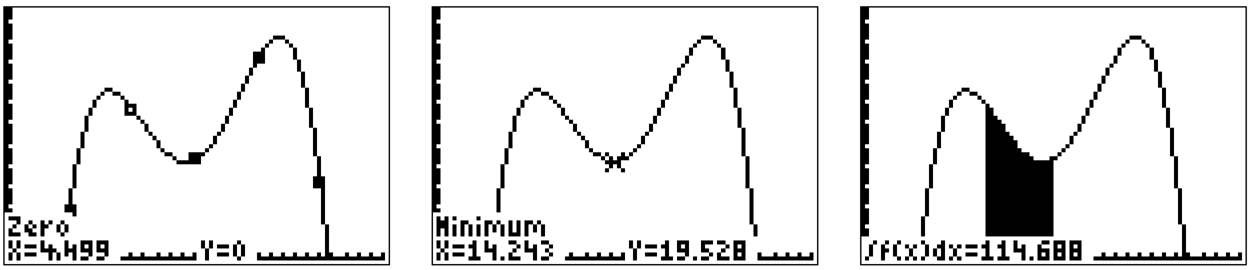


Figure 5. Modeling a car in motion

Six other projects were included in the course. The project ‘Forecasts’ modeled a capital growing constantly in three different ways: linear, exponential and potential. The project ‘Determining a Distance’ uses trigonometry to predict the distance to an inaccessible point on the other side of a river. The project ‘The Bridge’ uses trigonometry to predict the dimensions of a simple expansion bridge over a canyon. The project ‘Playing Golf’ predicts the formula for the orbit of a ball that has to pass three given points: a starting point, an ending point and the top of a hedge. The project ‘Saving and Pension’ predicts the size of a ten years monthly pension created by a thirty years monthly payment of 1000$ at an interest rate of 0.4% per month. And the project ‘The Takeover Attempt’ predicts how much company A has spent buying stocks in company B given an oscillating course described by a 4-line table.

Introducing the Three Genres of Quantitative Literature

Qualitative and quantitative literature has three genres: fact, fiction and fiddle (Tarp, 2001).

Fact models quantify and predict predictable quantities, as, e.g., ‘What is the area of the walls in this room?’ In this case the calculated answer of the model is what is observed. Hence calculated values from a fact models can be trusted.

A fact model can also be called a since- then model or a room-model. Most models from basic science and economy are fact models.

Fiction models quantify and predict unpredictable quantities, as, e.g., ‘My debt will soon be paid off at this rate!’ Fiction models are based upon assumptions and its solutions should be supplemented with predictions based upon alternative assumptions or scenarios.

A fiction model can also be called an if-then model or a rate-model. Models from basic economy assuming variables to be constant or predictable by a linear formula are fiction models.

Fiddle models quantify qualities: ‘Is the risk of this road high enough to cost a bridge?’ Many risk-models are fiddle models. The basic risk model says: Risk = Consequence * probability.

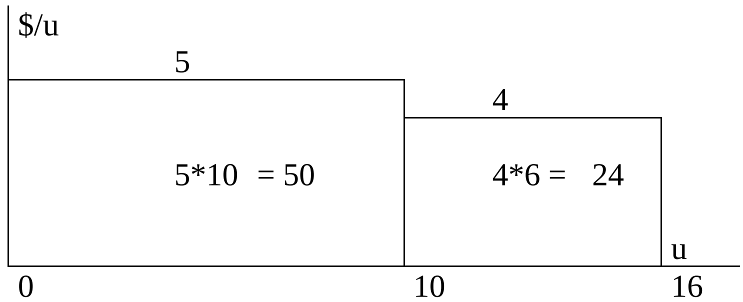
Statistics can provide probabilities for casualties, but if casualties are quantified, it is much cheaper to stay in a cemetery than in a hospital, pointing to the solution ‘no bridge’.

Fiddle models should be rejected asking for a word description instead of a number description.

Introducing the Core of Calculus as Adding Per-Numbers

As an introduction to calculus the students looked at discounts: The price 5 $/u goes down to 4 $/u when buying more than 10 units, what is the price when buying 16 units?

10 units at 5 $/u gives 10*5 = 50 $
6 units at 4 $/u gives 6*4 = 24 $
16 units at 9 $/u gives 16*9 = 74 $
?? How do we add per-numbers??



The problem is that where unit-numbers are added directly, per-numbers are added as areas under the per-number graph, i.e., as $\Sigma\ p*\Delta x$, written by TI.82 as $\int p\ dx$.

So, if a per-number p is constant, the total cost T for buying 5 units is $T = p*5$. And if the per-number p is a formula, the total cost T for buying 5 units is the area under the p-graph.

Completing the Algebra Project

Seeing the use of integration as adding per-numbers, the students enjoyed having completed the reuniting project of algebra since now they were able to add all four number types:

Operations unite/split into	Unlike	Like
Unit-numbers *m, s, kg, \$*	***T = a + b*** *T – b = a*	***T = a*b*** *T/b = a*
Per-numbers *m/s, \$/kg, \$/(100\$) = %*	***T = ∫ p dx*** *dT/dx = p*	***T = a^b*** *b√T = a* *loga(T) = b*

The 'algebra-square' has 4 ways to unite, and 5 to split totals

Students Asking for Proofs

To stop fellow students mocking them by saying that the class was not on mathematics but on reckoning, the students asked for sophisticated proofs. Four were sufficient.

Depositing *n* times the interest *r*(a/r) = a*\$ of *a/r* \$ to an account makes this a saving account containing the total interest *A = R*(a/r)*. Consequently *A/a = R/r* where 1+*R* = (1+*r*)^*n*.

In a triangle *ABC* with no angle above 90, the outside squares of the sides are divided in rectangles by the heights. Projections show that the two rectangles containing *C* have the same area, *a*b**cos*C*. This gives *c*^2 = *a*^2 + *b*^2 – 2**a*b**cos*C*, or *c*^2 = *a*^2 + *b*^2 if *C* is 90.

Adding 100% interest in n parts n times gives the growth multiplier *m* = (1+1/*n*)^*n*. For *n* sufficiently big, *m* = 2.7182818 = *e*. Thus, compounding 100% can add at most 71.8%.

Inscribing a symmetrical *n*-radius star in a unit circle gives n intersection points, from which tangents create a polygon with circumference *c* = 2**n**tan(180/*n*). For *n* big, *c* gets close to that of the unit-circle, 2*π. Hence $\pi = n$*tan(180/*n*) = 3.14159 for *n* sufficiently big.

Testing the Core Course

The students expressed surprise and content with the course. Their hand-ins were on time. And the course finished before time giving room for additional models from classical physics: vertical falling balls, projectile orbits, colliding balls, circular motion, pendulums, gravity points, drying wasted whisky with ice cubes and supplying bulbs with energy.

At the written and the oral exam, for the first time at the school, all the students passed. Some students wanted to move on to a calculus class, other were reluctant arguing that they had already learned the core of calculus so they didn't need an extra class to study engineering.

Reporting Back to the Headmaster

The headmaster expressed satisfaction, but the teachers didn't like setting aside the textbook and its traditional mathematics. To encourage the teachers, the headmaster ordered the TI-82 to be bought for all pre-calculus classes.

Discussing the Result

Discussing the result with teachers, an argument was that by not following the textbook, the students don't learn mathematics, only reckoning. My answer was that in the first place, the textbook does not teach mathematics but metamatism, so what dropouts reject is metamatism, not mathematics. Furthermore, teachers should respect the Nuremberg sentences from 1946: You can't just follow orders, you must evaluate the consequences to your clients. Your order is to teach mathematics, but it must be useful and not harmful to the learners. So, skepticism should reformulate the orders, e.g., to: the goal of mathematics education is to adapt learners to the natural fact Many by totalizing developing basic competences in how to count, add, model and triangulate; and how to cooperate with calculation technology able to perform both forward and backward calculation and to illustrate calculations with tables and graphs. Another argument was that by using technology, the students would not understand mathematics. My answer was that world problems can't be translated into formulas without understanding how the different operations are defined and used for forward and backward calculation. However, just as multiplication is useful to speed up addition, a graphical calculator should be allowed to speed up modeling so the human brain can be used to formulate questions and evaluate answers.

Discussing the result with researchers, a typical argument was that this work does not build on traditional theory in mathematics education research, e.g., theory about concept formation. My answer was that in most cases research is describing education in metamatism, not in mathematics. And in many cases research produces political instead of natural correctness as shown by the 'pencil paradox': Placed between a ruler and a dictionary, a '17 cm long pencil' can point to '15', but not to 'knife', so being itself able to falsify its number but not its word means that numbers and words produce natural and political correctness respectively. Only contingency research can produce natural correctness by uncovering hidden alternatives to choices presented as nature. Consequently, contingency research is very effective as action research assisting an actor in a field to implement change as in this case. But contingency research is banned from discourse protecting EU universities as predicted by Lyotard (1984).

Conclusion: How to Make Losers Users

In order to give dropout students in mathematics an extra chance it has good meaning to create a course boiling the mathematics content down to its core. However, to be successful, the core should be grounded in its roots, the natural fact Many. Numbers should be presented as polynomials to show the four operations uniting numbers according to Algebra's reuniting project. Also, direct and inverse operations should be presented as means to predict the united total and its parts. In this way the core of basic algebra becomes solving equations with the move and change sign method, or with the solver of a graphical calculator. And the core of pre-calculus becomes regression, enabling tables to be translated to formulas that can be processed when entered into the y-list of the TI82. Thus, grounding mathematics in its natural roots and including a graphical calculator provides ordinary students with a typewriter that can be used to model and predict the behavior of real-world quantities (Tarp 2009). In this way a TI-82 develops competences in human-technology cooperation: Humans get the data and the questions, and technology provides the answers. Traditional metamatism makes losers of Ryan and other boys. Replacing metamatism with grounded mathematics and a TI-82 will not only save them, it will also install in them a confidence and a belief that they can become successful engineers cooperating with technology instead of being math dropouts. Thus, learning technology based grounded mathematics in an educational system that is transformed from being line-organized to block-organized will transform boys from being dropouts to engineers, which again will help the EU solve its present economic crisis.

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PerNumbers Replace Proportionality, Fractions and Calculus

Increased research can lead to decreasing PISA math results as In Sweden. A goal/means confusion might be the cause. Grounded as a means to an outside goal, mathematics becomes a natural science about the physical fact Many. This ManyMatics differs from the school's MetaMatism, mixing MetaMatics, defining its concepts as examples from internal abstractions, with MatheMatism, true inside but not outside the class. Replacing proportionality, fractions and calculus with per-numbers will change math from goal to means.

Decreasing PISA Performance, a Result of a Goal/Means Confusion?

Being highly useful to the outside world has made mathematics a core of education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased witnessed, e.g., by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA level in mathematics decrease from 509 in 2003 to 478 in 2012, far below the OECD average at 494. This has made OECD write a report describing the Swedish school system as being 'in need of urgent change' (OECD, 2015).

Created to enable students cope with the outside world, schools consist of subjects that are described by goals and means with the outside world as the goal and the subjects as the means. However, a goal/means confusion might occur where the subjects become the goals and the outside world a means.

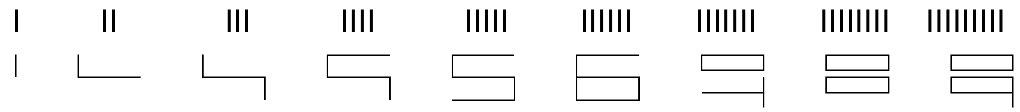
A goal/means confusion is problematic since while there is only one goal there are many means that can be replaced if not leading to the goal, unless an ineffective means becomes a goal itself, leading to a new discussing about which means will best lead to this false goal; thus preventing looking for alternative means that would more effectively lead to the original goal.

So, we can ask: Does mathematics education build on a goal-means confusion seeing math as the goal and the outside world as a means? For a grounded answer (Glaser 1967) we reformulate the question: How will mathematics look like if built as a means for proper real-world actions?

Mathematics is not an action word itself, but so are its two main activities, geometry and algebra, meaning to measure earth in Greek, and to reunite numbers in Arabic. Thus, mathematics is an answer to the two basic questions of mankind: How to divide the earth we live on, and the many goods it produces? (Tarp 2012). So, what we really ask is: Which actions will enable us to deal with the physical fact Many as it exists in space and in time?

Mathematics as a Natural Science about Many

To deal with Many we count and add. To count we stack icon-bundles. To iconize five, we bundle five ones to one fives to be rearranged as one five-icon 5 with five sticks if written in a less sloppy way. We create icons until ten since we do not need an icon for the bundle-number as show when counting in fives: one, two, three, four, bundle, one bundle one, one bundle two etc.



1 2 3 4 5 6 7 8 9

With Icons we count by bundling a total in icon-bundles. Thus, a total T of 7 1s can be bundled in 3s as T = 2 3s and 1. Now we place two sticks in a left bundle-cup and one stick in a right single-cup to write the total in 'algebra-form'. Here the cup-content is described by an icon, first using 'cup-writing' 2]1, then using 'decimal-writing' with a decimal point to separate the bundles from the unbundled, and including the unit 3s, T= 2.1 3s.

Alternatively, we can use the plastic letters, *b* for a bundle and *C* for a bundle of bundles.

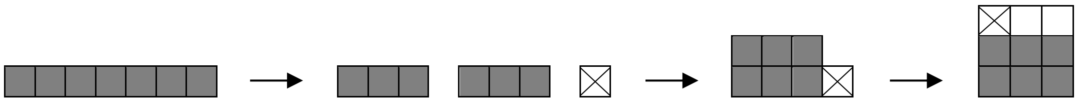
IIIIIII → III III I → II] I → 2]1 → 2.1 3s or BB 1 → *C* 1

A calculator can predict the counting result. To count in 3s we take away 3s, iconized as '/3' showing the broom wiping away the 3s several times. Building a stack of 2 3s we iconize as '2x3' showing a jack used to lift the 3s. And '-2x3' iconizes the trace coming from taking away 2 3s to look for unbundled. These three operations are called division, multiplication and subtraction respectively. Entering '7/3' the answer is '2.some'. To find the unbundled we take away the 2 3s by asking '7 – 2x3'. From the answer '1' we conclude that 7 = 2.1 3s.

7 / 3	2.some
7 – 2 x 3	1

Thus, a total T is counted in 3s by taking away 3 T/3 times. This can be written as a 're-count formula' *T* = (T/3)x3 or as *T* = (T/B)xB if re-counting T in bs. Taking away a stack *b* to be placed next-to the unbundled T-b can be written as a 're-stack formula' T= (T-B)+B.

To write the total in 'geometry-form' we use squares or LEGO blocks or an abacus to stack the 2 3-bundles on-top of each other with an extra stack of unbundled 1s next-to or on-top, thus describing the total as a decimal number 2.1 3s, or as a fraction number 2 1/3 3s counting the unbundled 1 in 3s.



T = 7 = 2 3s & 1 = 2.1 3s = 2 1/3 3s

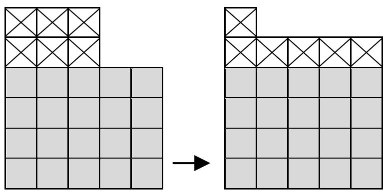
DoubleCounting creates PerNumbers Bridging Units

A physical quantity can be counted in different units, e.g. as 4kg or as 5$. This creates the 'per-number' 4kg/5$ = 4/5 kg/$. To shift from one unit to another we simply recount in the part of the per-number that has the same unit:

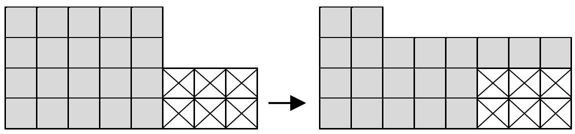
7 kg = ? $	8 $ = ? kg
7 kg = (7/4) x 4 kg = (7/4) x 5 $ = 8.75 $	8 $ = (8/5) x 5 $ = (8/5) x 4 kg = 6.4 kg

Adding Totals

Once Counted, totals can be added on-top or next-to. To add on-top, the units must be changed to be the same, typically by recounting one total in the other total's unit. Adding next-to is called integrating areas.



On-Top: 4 5s + 2 3s = 5.1 5s



Next-to: 4 5s + 2 3s = 3.2 8s

Next-to addition is also used when adding piecewise constant per-numbers:

4 kg at 5 $/kg + 2 kg at 3 $/kg = (4x5 + 2x3) $ = Σ (per-number x quantity)

Or when adding locally constant (continuous) per-numbers:

6 kg at 5 $/kg decreasing to 3 $/kg = $\int_0^6 (5 + \frac{3-5}{6} u)\, du$

Global, piecewise and local constancy all express the fact that y is a constant k if the distance between the two can be made arbitrarily small:

- y is globally constant k if V ε > 0 : I y – k I < ε.
- y is piecewise constant kc if Э *C* so V ε > 0 : I y – kc I < ε inside C.

● y is locally constant yo if V ε > 0 Э *C* : I y – yo I < ε inside C.

Reversing Addition, or Solving Equations

Reversing addition, we ask, e.g., '2+? = 8'. Restacking 8 as (8-2)+2 we get the answer 8-2. Reversing multiplication, we ask, e.g., '2x? = 8'. Recounting 8 in 2s as (8/2)x2 we get the answer 8/2. We see that solving equations means moving numbers to the opposite side with the opposite sign.

OnTop		NextTo
2 + ? = 8 = (8-2) + 2	**2 x ? = 8** = (8/2) x 2	**2 3s + ? 5s = 3 8s**
? = 8-2	? = 8/2	? = (3 8s – 2 3s)/5

Reversing adding next-to, we ask, e.g., '2 3s + ? 5s = 3 8s'. To find what was added we take away the 2 3s and count the rest in 5s. Combining subtraction and division in this way is called reversed integration or differentiation.

The Algebra Project: the Four Ways to Add

Meaning 'to re-unite' in Arabic, the 'Algebra-square' shows that with variable and constant unit-numbers and per-numbers there are four ways to unite numbers into a total and five ways to split a total: addition/subtraction unites/splits-into variable unit-numbers, multiplication/division unites/splits-into constant unit-numbers, power/root&log unites/splits-into constant per-numbers and integration/differentiation unites/slits into variable per-numbers.

Uniting/*splitting*	Variable	Constant
Unit-numbers	***T* = a + n**, *T – a = n*	***T* = a x n**, *T/n = a*
Per-numbers	***T* = ∫ a dn**, *dT/dn = a*	***T* = a^n**, *loga(T) = n, n√T = a*

School only counts in tens writing 2.3 tens as 23 thus leaving out the unit and misplacing the decimal point. So, icon-counting must take place in preschool.

Writing 345 as 3x10^2 + 4x10 + 5x1, i.e., as areas placed next-to each other, again shows that there are four ways to unite, and that all numbers have units.

ManyMatics versus MatheMatism and MetaMatics

Built as a natural science about the physical fact Many, mathematics becomes ManyMatics dealing with Many by counting and adding as shown by the Algebra-square and in accordance with the Arabic meaning of algebra.

With counting and adding Many as outside goal, a proper means would teach icon-counting and on-top and next-to addition in grade one. However, only ten-counting occurs. And addition takes place without including units claiming that 2+3 IS 5 in spite of counterexamples as 2 weeks + 3 days = 17 days. So, what is taught in primary school is not ManyMatics leading to proper actions to deal with Many, but what could be called 'MatheMatism' true inside but not outside a class thus making itself a goal not caring about outside world falsifications.

A counting result can be predicted by a re-count and a re-stack formula. So, formulas as means to real world number-prediction should be a core subject in secondary school. However, here a formula is presented as an example of a function, again being an example of a set-relation where first-component identity implies second-component identity. So, what is taught in primary school is not ManyMatics leading to the ability to predict numbers, but what could be called 'MetaMatics' presenting concepts from the inside as examples of abstractions instead of from the outside as abstractions from real world examples; and becoming 'MetaMatism' when mixed with MatheMatism.

So yes, a goal-means confusion exits in mathematics education seeing MetaMatsm as the goal and real world as applications and means; and claiming that ‘of course mathematics must be learned before it can be applied’. To lift this confusion the outside world must again be the goal and ManyMatics the means. Testing examples will show if this can turnaround the PISA-results.

Proportionality or Linearity

Linearity is a core concept in mathematics, defined by MetaMatism as a function f obeying the criterion $f(x+y) = f(x)*f(y)$. The function $f(x) = a*x$ is linear since $f(x+y) = a*(x+y) = a*x + a*y = f(x) + f(y)$. This ‘proportionality function’ is applied to the outside world when solving a ‘3&4&5-problem’: ‘If 3 kg cost 4 $ then 5 kg cost ? $’. Asking ‘5 kg = ? $’ shows that the ‘3&4&5-problem’ is an example of a more general ‘change-unit problem’ as e.g. ‘5 £ = ? $’.

Historically, the outside goal ‘to change-units’ has created different means. The Middle Ages taught ‘Regula Detri’, the rule of three: The middle number is multiplied with the last number and then divided by the first number.

The industrial age introduced a two-step rule: First go to the unit by dividing 4 by 3, then multiply by 5. Having learned how to solve equations in secondary school, a proportion can be set up equalizing two ratios: 3/4 = 5/u. Now cross-multiplication leads to the equation 3xu = 4x5 with the solution u = 4x5/3. As shown above, the per-number 3$/4kg offers a fifth alternative finding the answer by recounting 5 in 3s: T = 5$ = (5/3)x3 $ = (5/3)x4 kg.

So, the action ‘to change unit’ can be attained by five different means, all to be part of teacher education in order to create a turnaround in the PISA results.

Fractions

Defining everything as examples of sets, MetaMatism sees fractions as what is called ‘rational numbers’, defined as equivalence sets in the set-product of ordered pairs of integers created by an equivalence relation making (a,b) equivalent to (c,d) if cross multiplication holds: $axd = bxc$.

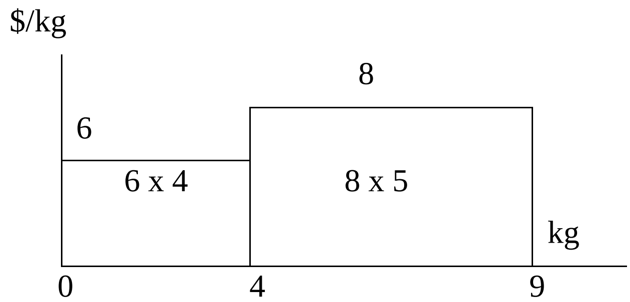
In primary school fractions come after division, the last of the basic operations. Unit fractions come in geometry as parts of pizzas or chocolate bars; and in algebra as parts of a total: 1/4 of the 12 apples is 12/4 apples. To find 4/5 of 20, first 1/5 of 20 is found by dividing with 5 and then the result is multiplied by 4. Then it is time for decimals as tenths, and percentages as hundredths. Then similar fractions occur when adding or removing common factors in the numerator and the denominator.

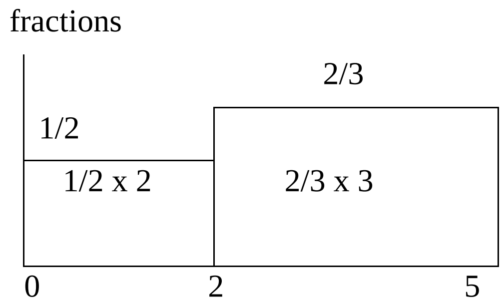
When including units, fractions respect the outside goal ‘to divide something’. Excluding units, adding fractions becomes MateMatism as shown by the ‘fraction paradox’: 1/2 + 2/3 is 7/6 inside a classroom, but can be 3/5 outside where 1 red of 2 apples plus 2 red of 3 total 3 red of 5 and certainly not 7 of 6.

From outside examples, per-numbers become fractions, 4kg/5$ = 4/5 kg/$. And, as per-numbers, fractions add by integrating the areas under their graph:

4kg at 6$/kg + to 5kg at 8$/kg = 9 kg at (6x4 + 8x5)/9 $/kg.

2 of which 1/2 + 3 of which 2/3 = 5 of which (1/2 x 2 + 2/3 x 3)





Integration

Adding variable per-numbers by integrating blocks, integration is one of the four ways to add as shown by the Algebra-square. So, integration should not be postponed to late secondary school but be part of primary school when adding icon-blocks next-to and when integrating areas under fraction graphs.

Also, integration should be taught before differentiation and before functions, since what we integrate (and differentiate into) is per-numbers, not functions.

Conclusion

To see if mathematics education has a goal/means confusion we asked: How will mathematics look like if built as a means for proper real-world actions? Or more precisely: Which actions will enable us to deal with the physical fact Many as it exists in space and in time?

To deal with Many, first we count, then we add. But first rearranging Many create icons. Counted in icon-bundles, a total transform into a stack of unbundled, bundles, bundles of bundles etc., i.e., into a decimal number with a unit. The basic operations, / and x and –, iconize the three counting operations: to take away bundles, to stack bundles and to take away a stack. Double-counting in different units create per-numbers used to bridge the units.

Once counted, totals can be added on-top or next to; and addition can be reversed by inventing reverse operations as shown in the Algebra-square.

Constructed as abstractions from the physical fact Many, ManyMatics prevents a goal/means confusion in mathematics education seeing the outside world as applications of MetaMatism, a mixture of MetaMatics defining concepts as examples from abstractions instead of as abstractions from examples, and MatheMatism with statements that are true inside but not outside a classroom.

Recommendations

So, to improve PISA results, mathematics education must teach actions enabling students to deal with the physical fact Many. Making mathematics a means and the outside world the goal prevents a goal/means confusion to occur. Consequently, mathematics education must teach ManyMatics abstracted from the outside world as a natural science about Many. And it must reject self-referring MateMatism containing concepts based internally instead of externally, and neglecting outside falsification of inside correctness.

In primary school, recounting in different icons should precede adding on-top and next-to. And double-counting create the per-numbers allowing the two units to be bridged without waiting for proportionality. To avoid nonsense, fractions must be added as per-numbers by integrating areas thus introducing primary school calculus as the fourth way to unite numbers. In this way everybody will be able to deal with Many by applying the full Algebra-square.

The MATHeCADEMY.net is designed to teach teachers to teach mathematics as ManyMatics as illustrated by its many MrAlTarp videos on YouTube.

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Primary, Middle and High School Calculus

To improve PISA results, institutional skepticism rethinks mathematics education to uncover hidden alternatives to choices institutionalized as nature. Rethinking calculus uncovers a preschool calculus where icon-blocks add next-to, and a middle school calculus adding piecewise constant per-numbers, preparing for adding locally constant per-numbers in high school integral calculus.

Background

Institutionalized education typically has mathematics as a core subject in primary and secondary school. To evaluate the success of mathematics education, OECD arranges PISA studies on a regular basis. Here increased funding of mathematics education research should improve PISA results. However, the opposite seems to be the case in Scandinavia as witnessed by the latest PISA study and by the OECD report 'Improving Schools in Sweden' (OECD 2015). This might be an effect of teaching a label, mathematics instead of what it labels, geometry and algebra; and of hiding the Arabic meaning of algebra: to reunite variable and constant unit-numbers and per-numbers by addition, multiplication, power and integration, all allowed in preschool icon-counting before primary school institutionalizes the monopoly of ten-counting. Based upon existentialist philosophy this paper asks: What is existentialist calculus and what difference will it make?

Institutional Skepticism

The ancient Greek sophists recommended enlightenment to avoid hidden patronization by choices presented as nature. Inspired by this, institutional skepticism combines the skepticism of existentialist and postmodern thinking. The 1700 Enlightenment century created two republics, one in North America and one in France. In North America, the sophist warning against hidden patronization is kept alive by American pragmatism, Symbolic interactionism and Grounded theory (Glaser et al 1967), the method of natural research resonating with Piaget's principles of natural learning (Piaget 1970) and with the Enlightenment principles for research: observe, abstract and test predictions. In France, the sophist skepticism is found in the poststructuralist thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, discourses, and education presenting their patronizing choices as nature (Lyotard 1984).

Building on the work of Kierkegaard, Nietzsche and Heidegger, Sartre defines existentialism by saying that to existentialist thinkers 'existence precedes essence, or (..) that subjectivity must be the starting point' (Marino 2004: 344). Focusing on the three classical virtues Truth, and Beauty and Goodness, Kierkegaard left truth to the natural sciences and argued that to change from a person to a personality the individual should stop admiring essence created by others and instead realize their own existence through individual choices and actions. Furthermore he showed violent resistance against institutionalized Christianity in the form of Christendom, seen also by Nietzsche as imprisoning people in moral serfdom. To break out he hoped that someday we will see a '*redeeming* man (..) whose isolation is misunderstood by the people as if it were flight *from* reality – while it is only his absorption, immersion, penetration *into* reality, so that (..) he may bring home the redemption of this reality: its redemption from the curse that the hitherto reigning ideal has laid upon it.' (Marino 2004: 186-187).

Arendt carried Heidegger's work further by dividing human activity into labor and work both focusing on the private sphere, and action focusing on the political sphere creating institutions that should be treated with care to avoid the banality of evil by turning totalitarian. (Arendt 1963).

Calculus as Essence

Textbooks see mathematics as a collection of well-proven statements about well-defined concepts, defined from above as examples from abstractions instead of from below as abstractions from examples. The invention of the set-concept allowed mathematics to be self-referring. By looking at the set of sets not belonging to itself, Russell showed that self-reference led to the classical liar paradox 'this sentence it false' being false if true and true if false:

If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$.

The Zermelo–Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus, becoming meaningless by its inability to separate concrete examples from abstract essence. That institutionalized education still teaches self-referring mathematics instead of grounded algebra and geometry can be seen as an example of ‘symbolic violence’ used as an exclusion technique to keep today’s knowledge nobility in power (Bourdieu 1977).

In set-based mathematics, differential calculus precedes integral calculus. First a function is defined as an example of a set-relation where first component-identity implies second-component identity. Then the limit concept is used to define a continuous and a differentiable function as well as its derivative. Theorems are presented for the derivative of specific functions and for their combinations. Applications include natural science and function monotony. Integral calculus begins by defining a primitive and presenting the theorem that the area-function to a function *f* is a primitive to f. This allows presenting the main theorem of calculus: A definite integral of a function is found by subtracting the values of a primitive evaluated in the two endpoints.

Calculus as Existence

The Pythagoreans used the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas (Freudenthal 1973). With astronomy and music now as independent subjects, today only the two other activities remain both rooted as natural sciences about the physical fact Many, Geometry meaning to measure earth in Greek, and Algebra meaning to reunite numbers in Arabic and replacing Greek arithmetic.

Meeting Many we ask ‘how many?’ Counting and adding gives the answer as shown by the word algebra, meaning to reunite numbers in Arabic. We count by bundling and stacking as seen when writing a total T in its full form: T = 345 = 3*B^2 + 4*B + 5*1 where the bundle B typically is ten. This shows the four ways to unite: On-top addition unites variable numbers, multiplication constant numbers, power constant factors, and next-to addition, also called integration, unites variable blocks.

As indicated by its name, uniting can be reversed to split a total into parts predicted by the reversed operations, subtraction and division and root & logarithm and differentiation. Likewise, a total can be presented in two forms, an algebraic using place value to separate the singles from the bundles and from the bundle-bundles, and a picture showing three blocks placed next to each other.

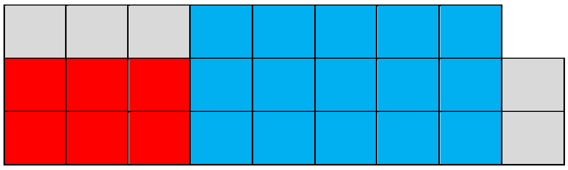
Thus, the root of calculus is next-to addition of blocks which takes place in preschool, in middle school and in high school, but not in primary school where blocks can only be added on-top because of ten-counting monopoly. But before this, preschool allows icon-counting in units below ten by bundling and stacking resulting in a decimal number with the bundles-size as the unit and with a decimal point to separate the unbundled form the unbundled. Thus, a given total might be counted as 3 7s and recounted as 2.5 8s or 4.1 5s. Once counted, totals can be added on-top and next-to. To add 2 3s and 4 5s on-top the units must be the same giving 5.1 5s since 2 3s can be recounted to 1.1 5s. Added next-to as 8s means adding areas, also called integration giving the result is 3.2 8s.

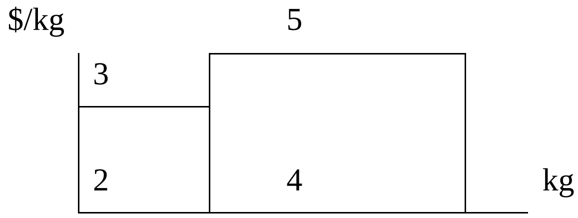
Primary school allows counting in one unit only, tens: But middle school allows several units when counting distances, time, mass, value, etc. The need for changing unit creates per-numbers as 3$/4kg serving as bridges when recounting $s in 3s or kgs in 4s:

15$ = (15/3)*3$ = (15/3)*4kg = 20kg.

Like fractions, per-numbers are not numbers but operators needing a number to become a number. To add, per-numbers must be multiplied to unit-numbers, thus, adding as areas, called integration.

Adding 2kg at 3$/kg and 4kg at 5$/kg gives (2+4)kg at (3*2+5*4)/(2+4) $/kg. We see that unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add as areas under the per-number graph that is piecewise constant in middle school.





Left: Adding 2 3s and 4 5s to 3.2 8s.   Right: Adding 2kg at 3$/kg and 4kg at 5$/kg as areas

In high school, the per-number graph p is locally constant, so the area comes from adding many area strips with the area p**dx* where *dx* is a small change in x.

Adding many numbers is time-consuming unless they are differences:

The numbers *n*1, *n*2, *n*3, …, *n*20 generates the differences (*n*2-*n*1), (*n*3-*n*2),…, (*n*20-*n*19) adding up to n20 – n1 since the middle numbers cancel out. So, to find the total area we try to write the single area $p*dx$ as a difference or a change: $p*dx = dP$, or $dP/dx = p$.

Thus, we need to develop a theory for finding *d/dx* of functions, called differential calculus - and not the other way around. To do so we need a formal definition of constancy of a variable *y*:

y is *constant c* if the distance between y and *c* is arbitrarily small, or in other words, if for all critical distances ε, the distance between them is less than ε.	$\forall \varepsilon > 0 : \lvert y - c \rvert < \varepsilon$
y is *piecewise constant* c_A, if an area *A* exists where y is constant c_A:	$\exists A, \forall \varepsilon > 0 : \lvert y - c_A \rvert < \varepsilon$ in A
y is *locally constant* c_o, if for all critical distances ε there is an area *A* where y is constant c_o	$\forall \varepsilon > 0, \exists A : \lvert y - c_o \rvert < \varepsilon$ in A

Likewise, we call the relation between two variables y and x piecewise or locally linear if the change per-number $\Delta y/\Delta x$ is piecewise or locally constant. Continuous and differentiable are other words for locally constant and locally linear.

Testing in The Classroom

In Denmark the passing limit in calculus has been lowered to about 30% correctness. Typically, a class starts with a traditional fraction course including 'the fraction paradox' where the teacher insists that 1/2 + 2/3 IS 7/6 even if the students protest: counting cokes, 1/2 of 2 bottles and 2/3 of 3 bottles gives 3/5 of 5 as cokes and never 7 cokes of 6 bottles. This produces many dropouts forced to stay in class so the school will not lose government funding. One class continued by presenting fractions as per-numbers added as areas. The students were surprised to have their side of the fraction paradox accepted and to hear about per-numbers for the first time. And going on with integral calculus before differential calculus made most students finish with good results. A report was sent to the teacher's journal and to the ministry of education but stayed unpublished.

Conclusion and Recommendation

To implement its goal, an institution hires civil servants who will also guard it to protect their jobs and careers. To avoid the banality of evil, education should institutionalize only essence with existence behind it guaranteed by using institutional skepticism to uncover pure essence. In calculus, what has existence is next-to addition of icon-counted totals in preschool and of areas created by changing per-numbers to unit-numbers by multiplication in middle school.

So, to improve PISA results, textbooks should include calculus in preschool and middle school; and in high school integral calculus should be presented before and as motivation for its reverse, differential calculus; and as a solution to the question: How to add locally constant per-numbers? In a postmodern vocabulary: Maybe calculus needs to be deconstructed (MrAlTarp 2012).

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Sustainable Adaption to Double-Quantity: From Pre-calculus to Per-number Calculations

Their biological capacity to adapt make children develop a number-language based upon two-dimensional block-numbers. Education could profit from this to teach primary school calculus that adds blocks. Instead, it teaches one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must wait until precalculus has introduced the functions to operate on. This inside-perspective makes both hard to learn. In contrast to an outside-perspective presenting both as means to unite and split into per-numbers that are globally or piecewise or locally constant, by utilizing that after being multiplied to unit-numbers, per-numbers add by their area blocks.

A Need for Curricula for all Students

Being highly useful to the outside world, mathematics is one of the core parts of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as witnessed, e.g., by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'

Traditionally, a school system is divided into a primary school for children and a secondary school for adolescents, typically divided into a compulsory lower part, and an elective upper part having precalculus as its only compulsory math course. So, looking for a change we ask: how can precalculus be sustainably changed?

A Traditional Precalculus Curriculum

Typically, basic math is seen as dealing with numbers and shapes; and with operations transforming numbers into new numbers through calculations or functions. Later, calculus introduces two additional operations now transforming functions into new functions through differentiation and integration as described, e.g., in the ICME-13 Topical Survey aiming to "give a view of some of the main evolutions of the research in the field of learning and teaching Calculus, with a particular focus on established research topics associated to limit, derivative and integral." (Bressoud et al, 2016)

Consequently, precalculus focuses on introducing the different functions: polynomials, exponential functions, power functions, logarithmic functions, trigonometric functions, as well as the algebra of functions with sum, difference, product, quotient, inverse and composite functions.

Woodward (2010) is an example of a traditional precalculus course. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

A Different Precalculus Curriculum

Inspired by difference research (Tarp, 2018) we can ask: Can this be different; is it so by nature or by choice? In their 'Principles and Standards for School Mathematics' (2000), the US National Council of Teachers of Mathematics, NCTM, identifies five standards: number and operations, algebra, geometry, measurement and data analysis and probability, saying that "Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century (p. 2)." In the chapter 'Number and operations', the Council writes: "Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number (p. 7)."

Their biological capacity to adapt to their environment make children develop a number-language allowing them to describe quantity with two-dimensional block- and bundle-numbers. Education could profit from this to teach children primary school calculus that adds blocks (Tarp, 2018). Instead, it imposes upon children one-dimensional line-numbers, claiming that numbers must be learned before they can be applied. Likewise, calculus must be learned before it can be applied to operate on the functions introduced at the precalculus level.

However, listening to the Ausubel (1968) advice "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly (p. vi).", we might want to return to the two-dimensional block-numbers that children brought to school.

So, let us face a number as 456 as what it really is, not a one-dimensional linear sequence of three digits obeying a place-value principle, but three two-dimensional blocks numbering unbundled singles, bundles, bundles-of-bundles, etc., as expressed in the number-formula, formally called a polynomial: T = 456 = 4*B^2 + 5*B + 6*1, with ten as the bundle-size, B.

This number-formula contains the four different ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant 'double-numbers' or 'per-numbers'. We might call this beautiful simplicity 'the algebra square' inspired by the Arabic meaning of the word algebra, to re-unite.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a*n}$ $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	$\boldsymbol{T = \int f\,dx}$ $\frac{dT}{dx} = f$	$\boldsymbol{T = a^b}$ $\sqrt[b]{T} = a \qquad log_a(T) = b$

Figure 01. The 'algebra-square' has 4 ways to unite, and 5 to split totals

Looking at the algebra-square, we thus, may define the core of a calculus course as adding and splitting into changing per-numbers, creating the operations integration and its reverse operation, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power, and its two reverse operations, root and logarithm.

Precalculus, Building on or Re-building?

In their publication, the NCTM writes "High school mathematics builds on the skills and understandings developed in the lower grades (p. 19)."

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they may have from the lower grades?

Furthermore, what kind of mathematics has been taught? Was it 'grounded mathematics' abstracted 'bottom-up' from its outside roots as reflected by the original meaning of 'geometry' and 'algebra' meaning 'earth-measuring' in Greek and 're-uniting' in Arabic?

Or was it 'ungrounded mathematics' or 'meta-matics' exemplified 'top-down' from inside abstractions, and becoming 'meta-matism' if mixed with 'mathe-matism' (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept 'function', Euler saw it as a bottom-up name abstracted from 'standby calculations' containing specified and unspecified numbers. Later meta-matics defined a function as an inside-inside top-down example of a subset in a set-product where first-component identity implies second-component identity.

However, as in the word-language, a function may also be seen as an outside-inside bottom-up number-language sentence containing a subject, a verb and a predicate allowing a value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-matics defines them as quotient sets in a set-product created by the equivalence relation that $(a,b) \sim (c,d)$ if cross multiplication holds, $a*d = b*c$.

And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-matics sees it as an example of a group operation applying the associative and commutative law as well as the neutral element and inverse elements, thus, using five steps to solve the equation $2*u = 6$, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2:

$2*u = 6$, so $(2*u)*½ = 6*½$, so $(u*2)*½ = 3$, so $u*(2*½) = 3$, so $u*1 = 3$, so $u = 3$.

However, $2*u = 6$ can also be seen as recounting 6 in 2s using the recount-formula '$T = (T/B)*B$' (Tarp, 2018), present all over mathematics as a proportionality formula, thus, solved in one step:

$2*u = 6 = (6/2)*2$, giving $u = 6/2 = 3$.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Baumann (1990) as the core of sociology.

Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as a number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar.

So, why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction for an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

A Grounded Outside-Inside Fresh-start Precalculus from Scratch

Let students see that both the word-language and the number-language provide 'inside' descriptions of 'outside' things and actions by using full sentences with a subject, a verb, and an object or predicate, where a number-language sentence is called a formula connecting an outside total with an inside number or calculation, shortening 'the total is 2 3s' to '$T = 2*3$';

Let students see how an outside degree of Many at first is iconized by an inside digit with as many strokes as it represents, five strokes in the 5-icon etc. Later the icons are reused when counting by bundling, which creates icons for the bundling operations as well. Here division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like *f* is used as a placeholder for an unspecified calculation. Writing '$y = f(x)$' means that the y-number is found by specifying the x-number and the f-calculation. Thus, with $x = 3$, and with $f(x) = 2+x$, we get $y = 2+3 = 5$.

Let students see how calculations predict: how 2+3 predicts what happens when counting on 3 times from 2; how 2*5 predicts what happens when adding 2$ 5 times; how 1.02^5 predicts what happens when adding 2% 5 times; and how adding the areas 2*3 + 4*5 predicts adding the 'per-numbers' when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Solving Equations by Moving to Opposite Side with Opposite Sign

Let students see the subtraction '$u = 5-3$' as the unknown number *u* that added with 3 gives 5, $u+3 = 5$, thus, seeing an equation solved when the unknown is isolated by moving numbers 'to opposite side with opposite calculation sign'; a rule that applies also to the other reversed operations:

- the division $u = 5/3$ is the number *u* that multiplied with 3 gives 5, thus, solving the equation $u*3 = 5$
- the root $u = 3\sqrt{}5$ is the factor *u* that applied 3 times gives 5, thus, solving the equation $u\text{^}3 = 5$, and making root a 'factor-finder'
- the logarithm $u = \log3(5)$ is the number *u* of 3-factors that gives 5, thus, solving the equation $3\text{^}u = 5$, and making logarithm a 'factor-counter'.

Let students see multiple calculations reduce to a single calculation by unhiding 'hidden brackets' where 2+3*4 = 2+(3*4) since, with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s.

This prevents solving the equation $2+3*u = 14$ as $5*u = 14$ with $u = 14/5$. Allowing to unhide the hidden bracket we get:

$2+3*u = 14$, so $2+(3*u) = 14$, so $3*u = 14-2$, so $u = (14-2)/3$, so $u = 4$

This solution is verified by testing: $2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14$.

Let students enjoy a 'Hymn to Equations': "Equations are the best we know; they're solved by isolation. But first the bracket must be placed, around multiplication. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so, feed us equations, we don't want to stop!"

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as, e.g.,

$T = a+b*c$, $T = a-b*c$, $T = a+b/c$, $T = a-b/c$, $T = (a+b)/c$, $T = (a-b)/c$, etc.

as well as formulas as, e.g.,

$T = a*b\text{^}c$, $T = a/b\text{^}c$, $T = a+b\text{^}c$, $T = (a-b)\text{^}c$, $T = (a*b)\text{^}c$, $T = (a/b)\text{^}c$, etc.

Let students place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2.

Which means that (u + b/2)^2 = u^2 + b*u + b^2/4.

So, with a quadratic equation saying u^2 + b*u + c = 0, three terms disappear if adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = b^2/4 – c.

Moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c, so u + b/2 = ± $\sqrt{b\text{^}2/4 - c}$, so u = -b/2 ± $\sqrt{\mathrm{b\text{^}2/4 - c}}$

Recounting Grounds Proportionality

Let students see how recounting in another unit may be predicted by the recount-formula‘

*T = (T/B)*B*, saying “From the total *T*, *T/B* times, *B* may be pushed away” (Tarp, 2018).

In primary school this formula recounts 6 in 2s as 6 = (6/2)*2 = 3*. In secondary school the task is formulated as an equation u*2 = 6 solved by recounting 6 in 2s as u*2 = 6 = (6/2)*2 giving u = 6/2, thus, again moving 2 ‘to opposite side with opposite calculation sign’.

Thus, an inside equation u*b = c can be ‘de-modeled’ (Tarp, 2020) to the outside question ‘recount c from ten to bs’, and solved inside by the recount-formula:

*u*b = c = (c/b)*b* giving *u = c/b*.

Let students see how recounting sides in a block halved by its diagonal creates trigonometry:

*a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b.*

And see how filling a circle with right triangles from the inside gives a formula for pi:

Circumference/diameter = $\pi \approx n$*tan(180/n) for n large.

Double-counting Grounds Per-numbers and Fractions

Let students see how double-counting in two units create ‘double-numbers’ or ‘per-numbers’ as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number:

- Asking ‘6$ = ?kg’ we recount 6 in 2s: *T* = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg.
- Asking ‘9kg = ?$’ we recount 9 in 3s: *T* = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

Let students see how double-counting in the same unit creates fractions and percent as

4$/5$ = 4/5, or 40$/100$ = 40/100 = 40%.

To find 40% of 20$ means finding 40$ per 100$, so we re-count 20 in 100s:

T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$.

Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$:

T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

Let students see how double-counting physical units create per-numbers all over STEM (Science, Technology, Engineering and mathematics):

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter;
- meter = (meter/second) * second = velocity * second;
- joule = (joule/second) * second = watt * second

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula,

T = 456 = 4*B^2 + 5*B + 6*1, containing the formulas for constant change:

$T = b*x$ (proportional),

$T = b*x + c$ (linear),

$T = a*x$^n (elastic),

$T = a*n$^x (exponential),

$T = a*x$^2 + $b*x + c$ (accelerated).

If not constant, numbers change. So, where constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics to 'post-dict' what we can't 'pre-dict'; and using confidence for predicting intervals.

Combining linear and exponential change by n times depositing a\$ to an interest percent rate r, we get a saving A\$ predicted by a simple formula, $A/a = R/r$, where the total interest percent rate R is predicted by the formula 1+R = (1+r)^n. This saving may be used to neutralize a debt Do, that in the same period changes to $D = D$o*(1+R).

This formula and its proof are both elegant:

In a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, $r*a/r = a$, and its own interest amount, thus, containing a saving A that is the total interest amount $R*a/r$, which gives $A/a = R/r$.

Precalculus Deals with Uniting Constant Per-Numbers as Factors

Adding 7% to 300\$ means 'adding' the change-factor 107% to 300\$, changing it to 300*1.07 \$. Adding 7% n times thus, changes 300\$ to T = 300*1.07^n \$, the formula for change with a constant change-factor, also called exponential change.

Reversing the question, this formula entails two equations. Asking 600 = 300*a^5, we look for an unknown change-factor. So, here the job is 'factor-finding' which leads to defining the fifth root of 2, i.e., 5√2, found by moving the exponent 5 to opposite side with opposite calculation sign, root.

Asking instead 600 = 300*1.2^n, we now look for an unknown time period. So, here the job is 'factor-counting' which leads to defining the 1.2 logarithm of 2, i.e., log1.2(2), found by moving the base 1.2 to opposite side with opposite calculation sign, logarithm.

Calculus Deals with Uniting Changing Per-Numbers as Areas

In mixture problems we ask, e.g., '2kg at 3\$/kg + 4kg at 5\$/kg gives 6kg at how many \$/kg?' Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before added, thus, adding by areas. So, here multiplication precedes addition.

Asking inversely '2kg at 3\$/kg + 4kg at how many \$/kg gives 6kg at 5 \$/kg?', we first subtract the areas 6*5 – 2*3 before dividing by 4, a combination called differentiation, $\Delta T/4$, thus, meaningfully postponed to after integration.

Statistics Deals with Unpredictable Change

Once mastery of constant change is established, it is possible to look at time series in statistical tables asking, e.g., "How has the unemployment changed over a ten-year period?" Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, $T = b+a*n$. The other describes the average yearly change-percent by using a constant change-percent formula, $T = b*a$^n.

The average numbers allow calculating all totals in the period, assuming the numbers are predictable. However, they are not, so instead of predicting the number with a formula, we might 'post-dict' the numbers using statistics dealing with unpredictable numbers. This, in turn, offers a likely prediction interval by describing the unpredictable random change with nonfictional numbers, median and quartiles, or with fictional numbers, mean and standard deviation.

Calculus as adding per-numbers by their areas may also be introduced through cross-tables showing real-world phenomena as unemployment changing in time and space, e.g., from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add. First, they must multiply back to unit-numbers to find the total percent. However, multiplying creates areas, so per-numbers add by areas, which is what calculus is about.

Modeling in Precalculus Exemplifies Quantitative Literature

Furthermore, graphing calculators allows authentic modeling to be included in a precalculus curriculum thus, giving a natural introduction to the following calculus curriculum, as well as introducing 'quantitative literature' having the same genres as qualitative literature: fact, fiction and fiddle (Tarp, 2001).

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from $x = 0$ to $x = x$ under a constant per-number graph $y = 1$ is $A = x$; and the area A under a constant changing per-number graph $y = x$ is $A = ½*x^2$. So, it seems natural to assume that the area A under a constant accelerating per-number graph $y = x^2$ is $A = 1/3*x^3$, which can be tested on a graphing calculator thus, using a natural science proof, valid until finding counterexamples.

Now, if adding many small area strips $y*\Delta x$, the total area $A = \Sigma\, y*\Delta x$ is always changed by the last strip.

Consequently, $\Delta A = y*\Delta x$, or $\Delta A/\Delta x = y$, or $dA/dx = y$, or $A' = y$ for very small changes.

Reversing the above calculations then shows that if $A = x$, then $y = A' = x' = 1$.

And that if $A = ½*x^2$, then $y = A' = (½*x^2)' = x$.

And that if $A = 1/3*x^3$, then $y = A' = (1/3*x^3)' = x^2$.

This suggest that to find the area under the per-number graph $y = x^2$ over the distance from $x = 1$ to 3 instead of adding small strips we just calculate the change in the area over this distance, later called the fundamental theorem of calculus.

A Literature Based Compendium

An example of an ideal precalculus curriculum is described in 'Saving Dropout Ryan With a Ti-82' (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren't even able to use a TI-30. A compendium called 'Formula Predict' (Tarp, 2009) replaced the textbook. A formula's left-hand side and right-hand side were put on the y-list as $Y1$ and $Y2$ and equations were solved by 'solve $Y1-Y2 = 0$'. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

An Example of a Fresh-start Precalculus Curriculum

This example was tested in a Danish high school around 1980. The curriculum goal was stated as: 'the students know how to deal with quantities in other school subjects and in their daily life'. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

An Example of an Exam Question

Authentic material was used both at the written and oral exam. The first had specific, the second had open questions as the following asking 'What does the table tell?

Agriculture: Number of agricultural farms allocated over agricultural area

	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977
Farms in total	**161142**	**154 694**	**148 512**	**144 070**	**143093**	**141 137**	**137712**	**13424S**	**130 7S3**	**127117**
0,0- 4,9 ha	25 285	23 493	21 533	21623	22123	21872	21093	19915	18 852	17 833
5.0- 9.9-	34 644	32129	30 235	28 404	27693	26 926	26109	25072	24066	23152
10,0-19.9-	48 997	46482	43 971	41910	40850	39501	38261	36 702	35 301	34 343
20.0-29.9-	25670	25 402	25181	24 472	24 195	23 759	23 506	23134	22737	22376
30,0-49.9-	18 505	18 779	18 923	18 705	18 968	18 330	19 095	19 304	10 305	19 408
50,0-99.9-	6 552	6 852	7 076	7 275	7 549	7956	7 847	8247	8 556	8723
100.0 ha and over	1489	1 557	1611	1681	1 715	1791	1801	1871	1934	1882

Figure 02. A table found in a statistical survey used at an oral exam.

Discussion and Conclusion

Asking "how can precalculus be sustainably changed?" an inside answer would be: "By its nature, precalculus must prepare the ground for calculus by making all function types available to operate on. How can this be different?"

An outside answer could be to see precalculus, not as a goal but as a means, an extension to the number-language allowing us to talk about how to unite and split into changing and constant per-numbers. This could motivate renaming precalculus to per-numbers calculations.

In this way, precalculus becomes sustainable by dealing with adding, finding and counting change-factors using power, roots and logarithm. Furthermore, by including adding piecewise constant per-numbers by their areas, precalculus gives a natural introduction to calculus by letting integral calculus precede and motivate differential calculus since an area changes with the last strip, thus, connecting the unit number, the area, with the per-number, the height.

Finally, graphing calculators allows authentic modeling to take place so that precalculus may be learned through its use, and through its outside literature.

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The Power of Bundle- and Per-Numbers Unleashed in Primary School: Calculus in Grade One – What Else?

In middle school, fraction, percentage, ratio, rate, and proportion create problems to many students. So, why not teach it in primary school instead where they all may be examples of per-numbers coming from double-counting a total in two units. And bundle-numbers with units is what children develop when adapting to Many before school. Here children love counting, recounting, and double-counting before adding totals on-top or next-to as in calculus, also occurring when adding per-numbers. Why not accept, and learn from the mastery of Many that children possess until mathematics takes it away?

Mathematics is Hard, or is it?

"Is mathematics hard by nature or by choice?" is a core sociological question inspired by the ancient Greek sophists warning against choice masked as nature. That mathematics seems to be hard is seen by the challenges left unsolved after 50 years of mathematics education research presented, e.g., at the International Congress on Mathematics Education, ICME, taking place each 4 year since 1969. Likewise, increased funding used, e.g., for a National Center for Mathematics Education in Sweden, seems to have little effect since this former model country saw its PISA result in mathematics decrease from 509 in 2003 to 478 in 2012, the lowest in the Nordic countries, and significantly below the OECD average at 494. This caused OECD (2015) to write the report 'Improving Schools in Sweden' describing the Swedish school system as being 'in need of urgent change'.

Also, among the countries with poor PISA performance, Denmark has lowered the passing limit at the final exam to around 15% and 20 % in lower and upper secondary school. And, at conferences as, e.g., The Third International Conference on Mathematics Textbook Research and Development, ICMT3 2019, high-ranking countries admit they have a high percentage of low scoring students. Likewise at conferences, discussing in the breaks what is the goal of mathematics education, the answer is almost always 'to learn mathematics'. When asked to define mathematics, some point to schoolbooks, others to universities; but all agree that learning it is important to master its outside applications.

So, we may ask, is the goal of mathematics education to master outside Many, or to first master inside mathematics as a means to later master outside Many. Here, institutionalizing mathematics as THE only inside means leading to the final outside goal may risk creating a goal displacement transforming the means to the goal instead (Bauman, 1990) leading on to the banality of evil (Arendt, 1963) by just following the orders of the tradition with little concern about its effect as to reaching the outside goal. To avoid this, this paper will answer the question about the hardness by working backwards, not from mathematics to Many, but from Many to mathematics. So, here the focus is not to study why students have difficulties mastering inside mathematics, but to observe and investigate the mastery of outside Many that children bring to school before being forced to learn about inside mathematics instead.

Research Method

Difference research searching for differences has uncovered hidden differences (Tarp, 2018c). To see if the differences make a difference, phenomenology (Tarp, 2018a), experiential learning (Kolb, 1984), and design research (Bakker, 2018) may create cycles of observations, reflections, and designs of micro curricula to be tested in order to create a new cycle for testing the next generation of curricula.

Observations and reflections 01

Asked "How old next time?" a three-year-old will say four showing four fingers, but will react to seeing the fingers held together two by two: "That is not four. That is two twos!" The child thus, describes what exists, bundles of 2s, and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)x2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving us a formula with unspecified numbers '$T = (T/B)\text{x}B$' saying "from *T*, *T/B* times, *B* may be pushed away and stacked". Consequently, search questions about 'bundle-numbers' and 'recounting' may be given to small groups of four preschool children to get ideas about how to design a generation one curriculum.

Guiding Questions

The following guiding questions were used: "There seems to be five strokes in the symbol five. How about the other symbols?", "How many bundles of 2s are there in ten?", "How to count if including the bundle?", "How to count if using a cup for the bundles?", "Can bundles also be bundled, e.g., if counting ten in 3s?", "What happens if we bundle to little or too much?", "How to recount icon-numbers in tens?", "How to manually recount 8 in 2s, and recount 7 in 2s?", "What to do if a bundle is not full?", "How to bundle-count seconds, minutes, hours, and days?", "How to double-count lengths in centimeters and inches?", "A dice decided my share in a lottery ticket, how to share a gain?", "Which numbers can be folded in other numbers than 1s?", "Asking how many 2s in 8 may be written as $u*2 = 8$, how can this equation be solved?", "How to recount from tens to icons?", "How to add 2 3s and 4 5s next-to?", "How to add 2 3s and 4 5s on-top?", " 2 3s and some 5s gave 3 8s, how many?", "How to add totals bundle-counted in tens?", "How to subtract totals bundle-counted in tens?", "How to add per-numbers?", "How to enlarge or diminish bundle-bundle squares?", "What happens when recounted stacks are placed on a squared paper?", "What happens when turning or stacking stacks?"

Observations and reflections 02

Data and ideas led to designing Micro Curricula (MC) with guiding questions and answers (Q, A).

MC 01: Digits as Icons

With strokes, sticks, dolls, and cars we observe that four 1s can be bundled into 1 fours that can be rearranged into a 4-icon if written less sloppy. So, for each 4 1s there is 1 4s, or there is 1 4s per 4 1s. In this way, all digits may be iconized, and used as units for bundle-counting (Tarp, 2018b).

MC 02: Bundle-counting Ten Fingers

A total of ten ones occurring as ten fingers, sticks or cubes may be counted in ones, in bundles, or with 'underloads' counting what must be borrowed to have a full bundle. Count ten in 5s, 4s, 3s, 2s.

In 5s with bundles: 0*B*1, …, 0*B*4, 0*B*5 no 1*B*0, 1*B*1, …, 1*B*4, 1*B*5 no 2*B*0.

In 5s with bundles and underloads: 1*B*-4, 1*B*-3, …, 1*B*0, 2*B*-4, … , 2*B*0.

MC 03: Counting Sequences Using Tens and Hundreds

In oral counting-sequences the bundle is present as tens, hundreds, thousands, ten thousand (wan in Chinese) etc. By instead using bundles, bundles of bundles etc. it is possible to let power appear as the number of times, bundles have been bundled thus, preparing the ground for later writing out a multi-digit number fully as a polynomial, $T = 345 = 3BB4B5 = 3*B\text{^}2 + 4*B + 5*1$.

Count 10, 20, 30, …, 90, 100 etc. Then 1*B*, 2*B*, …, 9*B*, ten*B* no 1*BB*.

Count 100, 200, 300, …, 900, ten-hundred no thousand. Then 1*BB*, 2*BB*, …, 9*BB*, ten*BB* no 1*BBB*.

Count 100, 110, 120, 130, …, 190, 200 etc. Then 1*BB*0*B*, 1*BB*1*B*, …, 1*BB*9*B*, 1*BB*ten*B* no 2*BB*0*B*.

A dice shows 3 then 4. Name it in five ways: thirty-four, three-ten-four, three-bundle-four, four-bundle-less6, and forty less 6. Travel on a chess board while saying 1*B*1, 2*B*1, 3*B*1, 3*B*2, …; 3*B*4.

MC 04: Cup-counting and Bundle-bundles

When counting a total, a bundle may be changed to a single thing representing the bundle to go to a cup for bundles, later adding an extra cup for bundles of bundles. Writing down the result, bundles and unbundled may be separated by a bundle-letter, a bracket indicating the cups, or a decimal point.

Q. *T* = two hands, how many 3s?

A. With 1 3s per 3 1s we count 3 bundles and 1 unbundled, and write *T* = 3*B*1 3s = 3]1 3s = 3.1 3s showing 3 bundles inside the cup, and 1 unbundled outside. However, 3 bundles are 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 1*BB*0*B*1 3s = 1]0]1 3s = 10.1 3s with an additional cup for the bundle-bundles.

Q. *T* = two hands, how many 2s?

A. With 1 2s per 2 1s we count 5 bundles, *T* = 5*B*0 2s = 5]0 2s = 5.0 2s. But, 2 bundles is 1 bundle-of-bundles, 1*BB*, so with bundle-bundles we write *T* = 2*BB*1*B*0 2s = 2]1]0 2s = 21.0 2s. However, 2 bundles-of-bundles is 1 bundle-of-bundles-of-bundles, 1*BBB*, so with bundle-bundle-bundles we write *T* = 1*BBB*0*BB*1*B*0 2s= 1]0]1]0 2s = 101.0 2s with an extra cup for the bundle-bundle-bundles.

MC 05: Recounting in the Same Unit Creates Underloads and Overloads

Recounting 8 1s in 2s gives *T* = 4*B*0 2s. We may create an underload by borrowing 2 to get 5 2s. Then *T* = 5*B*-2 2s = 5]-2 2s = 5.-2 2s. Or, we may create an overload by leaving some bundles unbundled. Then *T* = 3*B*2 2s = 2*B*4 2s = 1*B*6 2s. Later, such ‘flexible bundle-numbers’ will ease calculations.

MC 06: Recounting in Tens

With ten fingers, we typically use ten as the counting unit thus, becoming 1*B*0 needing no icon.

Q. *T* = 3 4s, how many tens? Use sticks first, then cubes.

A. With 1 tens per ten 1s we count 1 bundle and 2, and write *T* = 3 4s = 1*B*2 tens = 1]2 tens = 1.2 tens, or *T* = 2*B*-8 tens = 2.-8 tens using flexible bundle-numbers. Using cubes or a pegboard we see that increasing the base from 4s to tens means decreasing the height of the stack. On a calculator we see that 3x4 = 12 = 1.2 tens, using a cross called multiplication as an icon for a lift stacking bundles. Only the calculator leaves out the unit and the decimal point. Often a star * replaces the cross x.

Q. *T* = 6 7s, how many tens?

A. With 1 tens per ten 1s we count 4 bundles and 2, and write *T* = 6 7s = 4*B*2 tens = 4]2 tens = 4.2 tens. Using flexible bundle-numbers we write *T* = 6 7s = 5*B*-8 tens = 5]-8 tens = 5.-8 tens = 3*B*12 tens. Using cubes or a pegboard we see that increasing the base from 7s to tens means decreasing the height of the stack. On a calculator we see that 6*7 = 42 = 4.2 tens.

Q. *T* = 6 7s, how many tens if using flexible bundle-numbers on a pegboard?

A. *T* = 6 7s = 6*7 = (*B*-4) * (*B*-3) = *BB*-3*B*-4*B*+4*3 = 10*B*-3*B*-4*B*+1*B*2 = 4*B*2 since the 4 3s must be added after being subtracted twice.

MC 07: Recounting Iconizes Operations and Creates a Recount-formula for Prediction

A cross called multiplication is an icon for a lift stacking bundles. Likewise, an uphill stroke called division is an icon for a broom pushing away bundles. Recounting 8 1s in 2s by pushing away 2-bundles may then be written as a ‘recount-formula’ 8 = (8/2)*2 = 8/2 2s, or *T* = (*T*/*B*)**B* = *T*/*B* *B*s, saying “From *T*, *T*/*B* times, we push away *B* to be stacked”. Division followed by multiplication is called changing units or proportionality. Likewise, we may use a horizontal line called subtraction as an icon for a rope pulling away the stack to look for unbundled singles.

These operations allow a calculator predict recounting 7 1s in 2s. First entering ‘7/2’ gives the answer ‘3.some’ predicting that pushing away 2s from 7 can be done 3 times leaving some unbundled singles that are found by pulling away the stack of 3 2s from 7. Here, entering ‘7-3*2’ gives the result ‘1’, thus, predicting that 7 recounts in 2s as 7 = 3*B*1 2s = 3]1 2s = 3.1 2s.

Recounting 8 1s in 3s gives a stack of 2 3s and 2 unbundled. The singles may be placed next-to the stack as a stack of unbundled 1s, written as *T* = 8 = 2.2 3s. Or they may be placed on-top of the stack counted in bundles as 2 = (2/3)*3, written as T = 8 = 2 2/3 3s thus, introducing fractions. Or, as T = 8 = 3.-1 3s if counting what must be borrowed to have another bundle.

Q. T = 9, 8, 7; use the recount-formula to predict how many 2s, 3s, 4s, 5s before testing with cubes.

MC 08: Recounting in Time

Counting in time, a bundle of 7days is called a week, so 60days may be recounted as T = 60days = (60/7)*7days = 8B4 7days = 8weeks 4days. A bundle of 60 seconds is called a minute, and a bundle of 60 minutes is called an hour, so 1 hour is 1 bundle-of-bundles of seconds. A bundle of 12hours is called a half-day, and a bundle of 12months is called a year.

MC 09: Double-counting in Space Creates Per-Numbers or Rates

Counting in space has seen many units. Today centimeter and inches are common. 'Double-counting' a length in inches and centimeters approximately gives a 'per-number' or rate 2in/5cm shown with cubes forming an L. Out walking we may go 3 meters each 5 seconds, giving the per-number 3m/5sec. The two units may be bridged by recounting in the per-number, or by physically combining Ls.

Q. T = 12in = ?cm; and T = 20cm = ?in

A1. T = 12in = (12/2)*2in = (12/2)*5cm = 30cm; and

A2. T = 20cm = (20/5)*5cm = (20/5)*2in = 8in

MC 10: Per-numbers Become Fractions

Double-counting in the same unit makes a per-number a fraction. Recounting 8 in 3s leaves 2 that on-top of the stack become part of a whole, and a fraction when counted in 3s:

T = 2 = (2/3)*3 = 2/3 3s.

Q. Having 2 per 3 means having what per 12?

A. We recount 12 in 3s to find the number of 2s:

T = 12 = (12/3)*3 giving (12/3) 2s = (12/3)*2 = 8. So, having 2/3 means having 8/12. Here we enlarge both numbers in the fraction by 12/3 = 4.

Q. Having 2 per 3 means having 12 per what?

A. We recount 12 in 2s to find the number of 3s:

T = 12 = (12/2)*2 giving (12/2) 3s = (12/2)*3 = 18. So, having 2/3 means having 12/18.

Here we enlarge both numbers in the fraction by 12/2 = 6.

MC 11: Per-numbers Become Ratios

Recounting a dozen in 5s gives 2 full bundles, and one bundle with 2 present, and 3 absent: T = 12 = 2B2 5s = 3B-3 5s. We say that the ratio between the present and the absent is 2:3 meaning that with 5 places there will be 2 present and 3 absent, so the present and the absent constitute 2/5 and 3/5 of a bundle.

Likewise, if recounting 11 in 5s, the ratio between the present and the absent will be 1:4, since the present constitutes 1/5 of a bundle, and the absents constitute 4/5 of a bundle. So, splitting a total between two persons A and B in the ration 2:4 means that A gets 2, and B gets 4 per 6 parts, so that A gets the fraction 2/6, and B gets the fraction 4/6 of the total.

MC 12: Prime Units and Foldable Units

Bundle-counting in 2s has 4 as a bundle-bundle. 1s cannot be a unit since 1 bundle-bundle stays as 1. 2 and 3 are prime units that can be folded in 1s only. 4 is a foldable unit hiding a prime unit since 1 4s = 2 2s.

Equal number can be folded in 2s, odd numbers cannot. Nine is an odd number that is foldable in 3s, 9 1s = 3 3s. Find prime units and foldable units up to two dozen.

MC 13: Recounting Changes Units and Solves Equations

Rephrasing the question "Recount 8 1s in 2s" to "How many 2s are there in 8?" creates the equation 'u*2 = 8' that evidently is solved by recounting 8 in 2s since the job is the same:

If u*2 = 8, then u*2 = 8 = (8/2)*2, so u = 8/2 = 4.

The solution u = 8/2 to u*2 = 8 thus, comes from moving a number to the opposite side with the opposite calculation sign. The solution is verified by inserting it in the equation:

u*2 = 4*2 = 8, OK.

Recounting from tens to icons gives equations: "42 is how many 7s" becomes

u*7 = 42 = (42/7)*7.

MC 14: Next-to Addition of Bundle-Numbers Involves Integration

Once recounted into stacks, totals may be united next-to or on-top, iconized by a cross called addition.

To add bundle-numbers as 2 3s and 4 5s next-to means adding the areas 2*3 and 4*5, called integral calculus where multiplication is followed by addition.

Q. Next-to addition of 2 3s and 4 5s gives how many 8s?

A1. T = 2 3s + 4 5s = (2*3+4*5)/8 8s = 3.2 8s; or

A2. T = 2 3s + 4 5s = 26 = (26/8) 8s = 3.2 8s

MC 15: On-top Addition of Bundle-Numbers Involves Proportionality

To add bundle-numbers as 2 3s and 4 5s on-top, the units must be made the same by recounting.

Q. On-top addition of 2 3s and 4 5s gives how many 3s and how many 5s?

A1. T = 2 3s = (2*3/5)*5 = 1.1 5s, so 2 3s and 4 5s gives 5.1 5s

A2. T = 2 3s + 4 5s = (2*3+4*5)/5 5s = 5.1 5s; or T = 2 3s + 4 5s = 26 = (26/5) 5s = 5.1 5s

MC 16: Reversed Addition of Bundle-Numbers Involves Differentiation

Reversed addition may be performed by a reverse operation, or by solving an equation.

Q. Next-to addition of 2 3s and how many 5s gives 3 8s?

A1: Removing the 2*3 stack from the 3*8 stack, and recounting the rest in 5s gives (3*8 – 2*3)/5 5s or 3.3 5s. Subtraction followed by division is called differentiation.

A2: The equation 2 3s + u*5 = 3 8s is solved by moving to opposite side with opposite calculation sign.

u*5 = 3 8s – 2 3s = 3*8 – 2*3, so u = (3*8 – 2*3)/5 = 18/5 = 3 3/5, giving 3.3 5s.

MC 17: Adding and Subtracting Tens

Bundle-counting typically counts in tens, but leaves out the unit and the decimal point separating bundles and unbundled: T = 4B6 tens = 4.6 tens = 46.

Except for e-notation with a decimal point after the first digit followed by an e with the number of times, bundles have been bundled: T = 468 = 4.68e2.

Calculations often leads to overloads or underloads that disappear when re-bundling:

Addition: 456 + 269 = 4BB5B6 + 2BB6B9 = 6BB11B15 = 7BB12B5 = 7BB2B5 = 725.

Subtraction: 456 – 269 = 4BB5B6 – 2BB6B9 = 2BB-1B-3 = 2BB-2B7 = 1BB8B7 = 187

Multiplication: 2 * 456 = 2 * 4BB5B6 = 8BB10B12 = 8BB11B2 = 9B1B2 = 912

Division: 154 /2 = 15B4 /2 = 14B12 /2 = 7B6 = 76

MC 18: Next-to Addition & Subtraction of Per-Numbers and Fractions is Calculus
Throwing a dice 8 times, the outcome 1 and 6 places 4 cubes on a chess board, and the rest 2 cubes. When ordered it may be 5 squares with 2 cubes per square, and 3 squares with 4 cubes per square. When adding, the square-numbers 5 and 3 add as single-numbers to 5+3 squares, but the per-numbers add as stack-numbers, i.e., as 2 5s + 4 3s = (2*5+4*3)/8*8 = 2.6 8s called the average: If alike, the per-numbers would be 2.6 cubes per square.

Thus, per-numbers add by areas, i.e., by integration. Reversing the question to 2 5s + ? 3s total 3 8s then leads to differentiation: 2 5s + ? 3s = 3 8s gives the equation

$2*5 + u*3 = 3*8$, so $u*3 = 3*8 – 2*5$, so $u = (3*8 – 2*5)/3 = 4\ 2/3$, or $u = (T2-T1)/3 = \Delta T/3$

Likewise, with fractions. With 2 apples of which 1/2 is red, and 3 apples of which 2/3 are red, the total is 5 apples of which 3/5 are red. Again, the unit-numbers add as single numbers, and, as per-numbers, the fractions must be multiplied before adding thus, creating areas added by integration.

MC 19: Having Fun with Bundle-Bundle Squares
On a pegboard we see that 5 5s + 2 5s + 1 = 6 6s, and 5 5s – 2 5s + 1 = 4 4s suggesting three formulas:

$n*n + 2*n + 1 = (n+1)*(n+1)$.

And $n*n – 2*n + 1 = (n-1)*(n-1)$.

And $(n–1)*(n+1) = n*n – 1$.

Two $s*s$ bundle-bundles form two squares that halved by their diagonal d gives four half-squares called right triangles. Rearranged, they form a diagonal-square $d*d$.

Consequently, $d*d = 2*s*s$

Four $c*b$ playing cards with diagonal d are placed after each other to form a $(b+c)*(b+c)$ bundle-bundle square. Below to the left is a $c*c$ square, and to the right a $b*b$ square. On-top are 2 playing cards. Inside there is a $d*d$ square and 4 half-cards. Since 4 half-cards is the same as 2 cards, we have the formula $c*c + b*b = d*d$ making it easy the add squares, you just square the diagonal.

MC 20: Having Fun with Halving Stacks by its Diagonal to Create Trigonometry
Halving a stack by its diagonal creates two right triangles. Traveling around the triangle we turn three times before ending up in the same direction. Turning 360 degrees implies that the inside angles total 180 degree, and that a right angle is 90 degrees. Measuring a 5up_per_10out angle to 27 degrees we see that tan(27) is 0.5 approximately. So, the tan-number comes from recounting the height in the base.

MC 21: Having Fun with a Squared Paper
A dozen may be 12 1s, 6 2s, 4 3s, 3 4s, 2 6s, or 1 12s. Placed on a squared paper with the lower left corners coinciding, the upper right corners travel on a bending line called a hyperbola showing that a dozen may be transformed to a 3.5 3.5s bundle-bundle square approximately. Traveling by saying “3up_per_1out, 2up_per_1out, …, 3down_per_1out” allows the end points to follow a parabola. With a per-number 2G/3R, a dozen R can be changed to 2G+9R, 4G+6R, 6G+3R, and 8G. Plotted on a square paper with R horizontally and G vertically will give a line sloping down with the per-number.

MC 22: Having Fun with Turning and Combining Stacks
Turned over, a 3*5 stack becomes a 5*3 stack with the same total, so multiplication-numbers may commute (the commutative law). Adding 2 7s on-top of 4 7s totals (2+4) 7s, 2*7+4*7 = (2+4)*7 (the distributive law). Stacking stacks gives boxes. Thus, 2 3s may be stacked 4 times to the box $T = 4*(2*3)$ that turned over becomes a 3*(2*4) box. So, 2 may freely associate with 3 or 4 (the associative law).

Discussion and Recommendation

This paper asks: what mastery of Many does the child develop before school? The question comes from observing that mathematics education still seems to be hard after 50 years of research; and from wondering if it is hard by nature or by choice, and if it is needed to achieve its goal, mastery of Many.

To find an answer, phenomenology, experiential, and design research is used to create a cycle of observations, reflections, and testing of micro curricula designed from observing the reflections of preschoolers to guiding questions on mastering Many. The first observation is that children use two-dimensional bundle-numbers with units instead of the one-dimensional single numbers without units that is taught in school together with a place value system. Reflecting on this we see that units make counting, recounting, and double-counting core activities leading to proportionality by combining division and multiplication, thus, reversing the order of operations: first division pulls away bundles to be lifted by multiplication into a stack that is pulled away by subtraction to identify unbundled singles that becomes decimal, fractional or negative numbers. And that recounting between icons and tens leads to equations when asking, e.g., 'how many 5s are 3 tens?' And that units make addition ambiguous: shall totals add on-top after proportionality has made the units like, or shall they add next-to as an example of integral calculus adding areas, and leading to differentiation when reversed? Finally, we see that flexible bundle-numbers ease traditional calculations on ten-based numbers.

Testing the micro curricula will now show if mathematics is hard by nature or by choice. Of course, investments in traditional textbooks and teacher education, all teaching single numbers without units, will deport testing to the outskirts of education, to pre-school or post-school; or to special, adult, migrant, or refugee education; or to classes stuck in, e.g., division, fractions, precalculus, etc. All that is needed is asking students to count fingers in bundles. Recounting 8 in 2s thus, directly gives the proportionality recount-formula $8 = (8/2)*2$ or $T = (T/B)*B$ used in STEM, and to solve equations. Likewise, direct and reversed next-to addition leads directly to calculus. Furthermore, testing micro curricula will allow teachers to practice action learning and action research in their own classroom.

Phenomenologically, it is important to respect and develop the way Many presents itself to children thus, providing them with the quantitative competence of a number-language. Teaching numbering instead of numbers thus, creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education (Widdowson, 1978). The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not learn from children instead?

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