Teaching Mathematics as Communication, Trigonometry Comes Before Geometry, and Probably Makes Every Other Boy an Excited Engineer

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ABSTRACT

Before 1970 both foreign language and mathematics were hard to learn because the two taught grammar before language. Then a turn took place in foreign language education allowing students to learn it through communication. Mathematics education never had a similar turn, so it is still hard for many. Therefore, this paper asks if it is possible to learn mathematics as communication. We see that three different kinds of mathematics are taught, pre-setcentric, setcentric and post-setcentric. And that the three grand theories disagree as to which to recommend. Being inspired by the fact that children communicate about the physical fact Many with two-dimensional box- and bundle-numbers with units, a curriculum is designed where trigonometry is rooted in a mutual recounting of the three sides in a box halved by its diagonal. So, the answer is: Yes, core mathematics can be learned as communication about boxes since it is directly connected to counting and recounting Many in boxes and bundles.

Keywords: Engineer, Geometry, Math, STEM, Trigonometry.

1. MATHEMATICS, EASY OR HARD OR NECESSARY?

Mathematics is easy, or is it? Well, not according to numerous PISA test results. So apparently mathematics is hard. But is mathematics then hard by nature or by choice? The ancient Greek sophists warned that in social matters we should tell nature from choice to avoid being patronized by choice masked as nature.

So, let us look closer at the typical rationality behind mathematics education: "Of course, mathematics is a core subject in education because of its many important social applications. And, of course, you cannot apply mathematics before you have learned it. So, once inside you master mathematics, you will be able to master the outside fact Many!"

But if I don't master mathematics, will I then have to give up on mastering Many? Or, since mastering Many seems to be the end goal with mathematics as a means, we could ask the following question: Is it possible to master Many before mastering what the school call mathematics? If so, then afterwards, those finding it important may try to master mathematics itself, while others focus on the STEM subjects probably wanting to become technicians or engineers.

So, to answer our research question "Can also mathematics be learned as communication?" we begin by looking at the background of mathematics education.

2. BACKGROUND

We, the people, have holes in the head as have other animals. The holes serve to meet our basic needs for food for our stomachs, oxygen for our lungs, and information for our brain. To share what we need, we form societies where human societies are especially efficient since, when standing up, our forelegs became arms with hands to grasp while exchanging sounds for what is grasped and shared, thus allowing us to share information also through communication in a common language that we speak, and maybe also read and write. Using a keyboard, we observe that it contains letters, digits and operations allowing us to communicate in our two languages, the word-language where letters unite to words that unite to sentences typically containing a subject, a verb, and a predicate as 'these are apples'; and a number-language where digits unite to numbers that together with operations unite to formulas or number-language sentences, also containing a subject, a verb, and a predicate as 'the total is four apples' shortened to '$T = 4$'.

Adapting to our surroundings we learn to speak our word-language before school that then teaches us to read and write. Later it teaches grammar, the meta-language that describes the language that describes the world.

We also learn some number-language before school, but here the school starts teaching it from the beginning as a subject called mathematics, which is the grammar of the number-language. Mathematics thus teaches grammar before language as did foreign language education until around 1970 when it took a turn and began teaching communication before grammar [1]. Number-language education never made the same turn, so let us try to see if that is possible: In mathematics education, can we teach language before grammar? Can mathematics be learned as communication?

3. HOW WELL-DEFINED IS MATHEMATICS?

In ancient Greece, the Pythagoreans used mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: arithmetic, geometry, music and astronomy [2], seen by the Greeks as knowledge about Many by itself, in space, in time, and in time and space. Together they form the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric [3].

The Arabic conquest of the silver mines in Spain made Europe descent into a dark middle age with mathematics frozen because of the rigidity of Euclidean geometry and the inability of Roman numbers to perform multiplication [4]. Later, silver found in German Harz financed the Renaissance reopening the trade with silk and pepper from India through Arab middle men bringing Hindu-Arabic bundle-bundle numbers, algebra, and trigonometry to Europe. Later again, the search for a trade route on open sea around Africa let Newton to invent calculus as a means to add locally constant per-numbers.

With astronomy and music as independent knowledge areas, today mathematics should be a common label for calculus, trigonometry as well as the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. Algebra thus contains the four ways to unite as shown when writing out fully the total $T = 342 = 3*B\text{^}2 + 4*B + 2*1$, i.e. 3 bundles of bundles and 4 bundles and 2 unbundled ones, i.e. 3 boxes. Here we see that we unite by using on-top addition, multiplication, power and next-to addition, called integration, each with a reverse splitting operation: subtraction, division, root and logarithm, and differentiation [5].

Operations unite/ split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$T = a + n$ $T - n = a$	$T = a*n$ $T/n = a$
Per-numbers m/s, $/100$ = %	$T = \int f\,dx$ $dT/dx = f$	$T = a\text{^}b$ $b\sqrt{T} = a$ $loga(T) = b$

So, as a label, mathematics has no existence itself, only its content has; and in Europe, Germanic countries taught counting and reckoning in primary school and arithmetic and geometry in the lower secondary school until about 50 years ago when the Greek 'many-math' rooted in Many was replaced by the 'New Math'. Here the invention of the concept Set created a Setcentric [6] 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts.

However, 'well-defined' meant defining by self-reference, i.e., defining concepts top-down as examples of abstractions instead of bottom-up as abstractions from examples. And, by looking at the set of sets not belonging to itself, Russell showed that self-reference leads to the classical liar paradox 'this sentence is false', being false if true and true if false: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo–Fraenkel Set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating concrete examples from abstract concepts.

In this way, Set transformed grounded mathematics into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside classrooms where adding numbers without units as '2 + 3 IS 5' meets counter-examples as 2weeks + 3days is 17days; in contrast to '2*3 = 6' stating that 2 3s can always be re-counted as 6 1s.

The Diversity of Numerical Communication Shown by Six Ways to Solve Proportionality Questions

The need to change units has made the two proportionality questions the core outside asked questions. With a uniform motion where the distance 2meter needs 5second, the two questions then go from meter to second and the other way, e.g. Q1: "7 meters need how many seconds?", and Q2: "How many meters is covered in 12 seconds?"

● Europe used the 'Regula de Tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2m takes 5s, 7m takes ?s' to get to the answer (7*5/2)s = 17.5s.
Then we ask, Q2: '5s gives 2m, 12s gives ?m' to get to the answer (12*2)/5s = 4.8m.

2 new methods came, 'find the unit', and cross multiplication in an equation expressing like proportions or ratios:

● Q1: 1m takes 5/2s, so 7m takes 7*(5/2) = 17.5s.
Q2: 1s gives 2/5m, so 12s gives 12*(2/5) = 4.8m.
● Q1: $2/5 = 7/x$, so $2*x = 7*5$, $x = (7*5)/2 = 17.5$.
Q2: $2/5 = x/12$, so $5*x = 12*2$, $x = (12*2)/5 = 4.8$.
● Alternatively, we may recount in the 'per-number' 2m/5s coming from 'double-counting' the total *T*.
Q1: T = 7m = (7/2)*2m = (7/2)*5s = 17.5s;
Q2: T = 12s = (12/5)*5s = (12/5)*2m = 4.8m.
Or, we may simply recount the units:
● sec = (sec/m)*m = 5/2*7 = 17.5
m = (m/sec)*sec = 2/5*12 = 4.8
● Set introduced modeling with linear functions to show the relevance of abstract algebra's group theory: Let us define a linear function $f(x) = c*x$ from the set of m-numbers to the set of s-numbers, having as domain DM = $\{x \in R \mid x>0\}$. Knowing that $f(2) = 5$, we set up the equation $f(2) = c*2 = 5$ to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: $c*2 = 5$, $(c*2)*½ = 5*½$, $c*(2*½) = 5/2$, $c*1 = 5/2$, $c = 5/2$. With $f(x) = 5/2*x$, the inverse function is $f\text{-}1(x) = 2/5*x$. So with 7m, $f(7) = 5/2*7 = 17.5$s; and with 12s, $f\text{-}1(12) = 2/5*12 = 4.8$m.

4. GRAND THEORY LOOKS AT MATH EDUCATION

With mathematics education as a human activity, we might want to see how the three basic human sciences look at it, sociology focusing on 'they', philosophy focusing on 'it', and psychology focusing on 'I'.

Sociology

Two kinds of sociology exist with different views on the role of institutions as means to reach a common goal. Seeing a great potential in institutions to create human welfare, Europe has developed a structural sociology [7]. Seeing institutions as constraining human behavior, North America has developed an agency-based sociology called pragmatism [8].

Sociologically, mathematics education is a double institution containing two social constructions: a text communicating something about something, and a mediation of this text. However, as a rational institution with a goal, it shares the danger of becoming an 'iron cage' [9] with a 'goal displacement' [10] seeing its own survival and growth as the real goal, and the original goal as a means that as long as it is not reached, will secure the continuing existence and possibly growth of the institution. In short, as long as the number-language children develop before school is neglected, and as long as mathematics is considered a hard subject, mathematics education will be needed, and so will research in mathematics education.

As to mathematics education, Europe favors the setcentric New Math, rejected by North America going back to the pre-setcentric version of traditional school mathematics. Here mathematics as communication may offer a post-setcentric alternative that may be attractive to North America as a completely new paradigm [11].

Philosophy

Two kinds of philosophy exist with different views on what is more important, outside ontological existence or its inside epistemological constructed representations. European rationalism favors the platonic view that what exists physical is but examples of metaphysical forms only accessible to philosophers educated at the Plato academy. Hence epistemology precedes ontology. Anglo-Saxon empiricism favors the view of existentialism that existence precedes essence [12].

As to mathematics education, rationalism favors setcentric mathematics, whereas empiricism favors the pre-setcentric version or the existentialist post-setcentric version [5].

Psychology

Psychology focuses on how the human brain adapt to the outside world. Two kinds of psychology exist with different views on what is more important to adapt to: what exits in the outside world, or what is inside constructed as representations. Here Europe favors Vygotsky [13] arguing that the goal of education is that learners adapt to institutionalized knowledge mediated by teachers with respect to what learners already knows, their individual zones of proximal developments, which puts a big responsibility on the teacher to be able to practice a differentiated teaching.

In contrast, North America favors Piaget [14] arguing that the goal of education is that learners adapt to the outside world by creating individual schemata used to assimilate the world to, or to be accommodated when meeting resistance from the world or from peers.

As to mathematics education, psychology has created two forms of constructivism, a Vygotsky-based social constructivism favored in Europe; and a Piaget-based radical constructivism favored in North America.

5. A CONTEMPORARY MATH CURRICULUM

The numbers and operations and the equal sign on a calculator suggest that mathematics education should be about the results of operating on numbers, e.g., that 2+3 = 5. This offers a 'natural' curriculum with one-dimensional linear multidigit numbers obeying a place-value system; and with operations where addition is the base with subtraction as the reversed operation, where multiplication is repeated addition with division as the reversed operation, and where power is repeated multiplication with the factor-finding root and the factor-counting logarithm as the reversed operations.

Reverse operations may create new numbers asking for additional education about the results of operating on these numbers. Subtraction thus creates negative numbers where 2 - (-5) = 7. Division creates fractions, decimals and percentages where 1/2 + 2/3 = 7/6. And root and log create numbers as √2 and log 3 where √2*√3 = √6, and where log 100 = 2.

Using letters for unspecified numbers leads to additional education about the results of operating on such numbers, e.g., that $(a+b) * (a–b) = a$^2 – b^2.

Geometry teaches about points, lines, angles, polygons, circles, and areas. Later, geometry and algebra are coordinated in coordinate geometry. To be followed by a right-angled triangle relating the sides and angles with trigonometrical operations as sine, cosine and tangent where sin(60) = √3/2.

In a formula, changing the input x will change the output y, making y a function of x, $y = f(x)$, using f for an unspecified formula. Relating the two changes creates an operation on calculations called differentiation, also creating additional education about the results of operating on calculations, e.g., that $(f*g)'/(f*g) = f'/f + g'/g$. And with a reverse operation, integration, again creating additional education about results of operating on calculations, e.g., that ∫6*x^2dx = 2*x^3 + c, where c is an arbitrary constant.

Having taught inside how to operate on numbers and calculations, its outside use may then be shown as inside-outside applications, or as outside-inside modelling transforming an outside problem into an inside problem transformed back into an outside solution after being solved inside. This introduces quantitative literature, also having three genres as has the qualitative: fact, fiction and fiddle [15].

How Children Communicate About Many

How to master Many may be learned from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists, and with units: bundles of 2s, and 2 of them. Children thus use two-dimensional bundle-numbers inside as representation for outside boxes, which resonates nicely with the box-structure of Hindu-Arabic numbers, bout not with the one-dimensional cardinal numbers that schools teach.

Children also use box-numbers when talking about snap cubes as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 extra'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 2x2s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as T = 7 = 1B5 = 2B3 = 3B1 = 4B1; or even as 1BB3 or 1BB1B1. Also, children love to count in 3s, 4s, and in hands.

6. POST SETCENTRIC MATHEMATICS

Since children already communicate about Many before school, why not let mathematics education develop instead of reject the core mastery of Many that children bring to school?

Digits as Icons

Many exists in space as a total, and in time as counting. A total of 4 sticks may be rearranged as 1 4-icon containing 4 sticks, likewise with the other digits. So, we may say that a digit is a number-icon containing as many sticks or strokes as it represents if written less sloppy. The icon for zero is a magnifying glass finding nothing. Ten is a special number with its own name but with no icon. That is because we count by bundling in tens, which makes ten a double counting of bundles and singles, where 10 is short for 1B0, 1bundle and no unbundled. So, where letters are mere symbols unconnected to their sounds, digits are icons communicating directly about their degree of Many.

Bundle-Counting Makes Operations Icons Also

Counting a total of 9 in 2s, first we push away 2s, iconized by a division-broom, 9/2. The we stack the bundles 4 times, iconized by a multiplication-lift, 4x2 or 4*2. Then we pull away the stack to look for unbundled singles iconized by a subtraction-rope, 9 – 4*2. Finally, we use an addition-cross to iconize the two ways to unite boxes, on-top or next-to.

Outside, counting a total of 9 in 2s means boxing 9 as 4 2s with an unbundled single on-top. Inside we report this as T = 9 = 4B1 2s or T = 4.1 2s using a decimal point to separate the bundles from the unbundled. Or as T = 9 = 4 ½ 2s indicating that the single should also be counted in bundles. Or as T = 9 = 5B-1 2s indicating that 1 is needed for another bundle. In this way, decimals, fractions and negative numbers are connected to how the unbundled singles are treated and seen.

Bundling in tens, we may see 1 as a bundle of ten parts and write 9/4 as 2.5, or 0.9 tens/4 = 0.25 tens.

Recounting Creates a Recount-Formula

Recounting 8 in 2s by 8/2 times pushing away 2s may be written as a 'recount-formula' 8 = (8/2)*2, or $T = (T/B)*B$ with unspecified numbers, saying 'From T, T/B times, B is pushed away.' This formula may also be called a proportionality formula used to change number units or physical units. Thus double-counting apples as 4$ and 5kg creates a 'per-number' 4$/5kg as a bridge between the units. So 12$ quickly change unit from $ to kg by recounting in the per-number:

T = 12$ = (12/4)*4$ = (12/4)*5kg = 15kg.

Likewise, the recount-formula may be used to solve equations occurring when asking 'How many 2s in 8' or 'u*2 = 8' immediately solved by u = 8/2 as seen by recounting 8 in 2s:

u*2 = 8 = (8/2)*2, giving u = 8/2.

Solving equations by moving a number to the opposite side with the opposite calculation sign works for all operations:

$u + 2 = 8$	$u*2 = 8$	u^8 = 2	2^u = 8
$u = 8 – 2$	$u = 8/2$	$u = 8\sqrt{2}$	u = log2(8)

Totals May be Boxed, Ten'ed, or Squared

So, an outside total is boxed in the given unit. Thus 4 5s is reported inside as T = 4B0 5s. We may want to squeeze the box into tens, which of course decreases its height since it increases the base, T = 4 5s = 2 tens. If we want to squeeze it in 8s, inside we ask the question 'u*8 = 20' which is an equation easily solved by recounting 20 in 8s as u = 20/8.

Wanting to square the 4x5 box, the answer is a √20 square. To find √20 we observe that in a 4+t square, removing the 4 square leaves 20 – 4x4 = 4 shared by the two 4xt boxes, giving t = 0.5. A little less since we neglect the t square. Inside, a calculator confirms that √20 = 4.472.

In geometry, intersection points between lines and circles leads to quadratic equations as x^2 + 6x + 8 = 0, easily solved when rotating the upper of two x by x+3 tiles to create a square with sides x+3, which is zero apart from the 1 left in the upper 3-by-3 square after 8 is removed.

In total, (x + 3)^2 = 0 + 1 = 1 = √1^2,
so x+3 = ±1, giving x = - 2 and x = - 4.

Trigonometry: Recounting a Box Halved by its Diagonal

So, when counting and recounting, the core outside object is a rectangular form that may be called a stack, a tile, a block, or a box, with base, b or AC, and height, h or CB. Here the diagonal, d or AB, splits the box in two like right angled triangles, which may also be called 'half-boxes'. Alternatively, we may say that we have boxed the line AB so it becomes a diagonal with a base and the height, which may also be called its floor and wall, or horizontal and vertical shadow, or run and rise. We will not use the Greek name 'cathetus' since it does not distinguish between the two sides.

Besides its three sides, a half-box also has three angles counting the turning of the lines in degrees. The turning from the base to the diagonal is called the angle A. The quarter round turning from the base to the height is called the right-angle C, which is counted to 90 degrees since a full round is 360 degrees. With two like triangles in the full box we see that A + B = 90. Trigonometry finds formulas that predict the sides and angles in a half-box.

The sides may be counted in standard units. But they may also be recounted in the box's own units. Recounting the height in the base gives height = (height/base) * base = tangent A * base = tangent A bases, abbreviated to

$h = (h / b) * b$ = tan A * b = tan A bs, thus giving the formula

tangent A = height / base, or tan A = h/b.

The word tangent is used since in a circle with center in A and the base as its radius, the height will be a tangent. This allows two ways to count angles, by degrees or by a tangent number.

Thus outside, in a 1x2 half-box, the angle A is counted on a protractor to 27 degrees; and in a 1x1 half-box, the angle A is counted to 45 degrees; and in a 2x1 half-box, the angle A is counted to 63 degrees. Inside these numbers may be predicted by the tangent formula

A = tan-1(1/2), A = tan-1(1/1), A = tan-1(2/1).

In the same we can create other trigonometry formulas as

sin A = height/diagonal, or sin A = h/d

cos A = base/diagonal, or cos A = b/d.

A Circle as Many Right Triangles

In an h x r half-box with area ½*h* r we look at the circle with center in A and r as its radius. Here $h = (h/r)*r$ = tan A*r .

A half circle is 180 degrees that may split in 100 small parts as

180 = (180/100)*100 = 1.8 100s = 100 1.8s

With A as 1.8 degrees, the length of the circle and the tangent, h, are almost identical. So, with good approximation

Half the circumference $C = 100 * h = 100 * \tan 1.8 * r = 100 * \tan(180/100) * r = 3.1426 * r$

Or better, $C = 1000 * \tan(180/1000) * r = 3.1416 * r$

Calling the circumference for $2 * \pi * r$, we get a formula for the number π: For n sufficiently large, $\pi = \tan(180/n) * n$.

The area of the full circle then is

$A = 2 * ½ * 3.1416 * r * r = 3.1416 * r$^2 $= \pi * r$^2

Adding Sides in a Half-Box

Arranging 4 like half-boxes as a (base + height) square, it contains a diagonal square plus 4 half-boxes. However, it also contains a height square and a base square plus 2 full boxes. Consequently, the height square and the base square add up to the diagonal square as predicted inside by the Pythagorean formula h^2 + b^2 = d^2. So, in right angled triangles, the sides add by their squares.

Adding Two Half-Boxes

Two half-boxes ADB and CDB have the same height DB. They add as two right triangles into a non-right triangle with base AC and with BD as its height, h, and with $AD = p$ and $CD = q$.

From the left half-box we get c^2 = p^2 + h^2. From the right half-box we get h^2 = a^2 – q^2, and $q = a$*cos C. In all we get

c^2 = a^2 + p^2 – q^2 = a^2 + p^2 + q^2 – 2*q^2

With $b = p+q$ we get b^2 = p^2 + q^2 + 2*p*q. This gives

c^2 = a^2 + b^2 – 2*p*q – 2*q^2
= a^2 + b^2 – 2*$(p+q)$*q
= a^2 + b^2 – 2*b*a*cos C.

This formula is called the extended Pythagorean formula or the cosine formula.

Equating the two formulas for the height BD in the two half-boxes, we get

$BD = AB * \sin A = BC * \sin C$, or $BD = c * \sin A = a * \sin C$

This may be rearranged to the sine formula $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

Intersection Line-Line

Extending a 3x4 box, ACBD, with a 3x2 box, $CEFB$, to a 3x6 box, $AEFD$, will allow finding intersection points between the diagonals in the three boxes. To find the height, PQ, of the intersection point between the diagonals AF and CD we notice that the angle A is part of a small triangle, AQP, and a large triangle, AEF, so that

$\tan A = QP/AQ = EF/AE = 3/6$.

Likewise, since angle C is part of a small triangle, CQP, and of a large triangle, CAD, we get $C = QP/CQ = AD/CA = 3/4$.

Equating QP in the two triangles, and observing that with $AQ = u$, $CQ = 4 – u$, we get

$QP = 3/6*AQ = 3/4* CQ$, giving $3/6 * u = 3/4 * (4 – u)$, that is solved by $u = AQ = 12/5 = 2.4$, giving $QP = 3/6*2.4 = 1.2$.

Intersection Line-Circle

In a 3x5 box, $ACBD$, we look at a circle having its center in A and its radius $r = AC$. The intersection point P with the upper base BD and its point Q below on AC may be found by observing that the point is situated on the circle and the upper base line so that

AP^2 = AQ^2 + QP^2, and $QP = 3$. So, 5^2 = AQ^2 + 3^2, giving AQ^2 = 5^2 – 3^2 = 16 and $AQ = \sqrt{16} = 4$.

So, to come from A to P we go out 4 units and up 3 units.

To find the intersection point P with the diagonal AB we see that the point is situated on the circle and the diagonal so that

$\tan A = QP/AQ = 3/5$, and AQ^2 + QP^2 = 5^2. With $AQ = u$, $QP = 3/5 * u$ and u^2 = 5^2 – (0.6 * u)^2

This transforms into a quadratic equation

u^2 + 0.36 u = 5^2, or 1.36 u^2 = 25, or u^2 = 25/1.36 = 18.38

with $u = AQ = \sqrt{18.38} = 4.29$, giving $QP = 3/5*4.29 = 2.57$.

So, to come from A to P we go out 4.29 units and up 2.57 units.

To find the intersection point P with the diagonal CD we observe that the point is situated on the circle and the diagonal. With $AQ = u$ and $QP = v$, we get that

$\tan C = 3/5 = v/(5 + u)$ giving $v = 3/5*(5 + u) = 3 + 0.6u$; and u^2 + v^2 = 5^2, so

u^2 + (3 + 0.6 u)^2 = 25, or 1.36 u^2 + 3.6 u + 9 = 25, or
u^2 + 2.64 u – 11.8 = 0, solved by $u = 2.36$, giving $v = 4.42$

So, to come from A to P in this case, we go back 2.36 units and up 4.42 units.

Contact Points Tangent-Circle

In a 3x5 box, $ACBD$, we look at a circle having its center in A and its radius $r = AC$. The line from B to C is a tangent to the circle with the contact point C, and forming a right angle with the radius AC. From the point B, there is however also another tangent to the circle. We observe that the contact point P is situated both on the circle and on the tangent. Turning the horizontal line AC to AP means turning the vertical tangent to the PB direction. Consequently, the two angles CAP and QPB are like. With S as the intersection point between PQ and DB, with $PQ = v$, and with $AQ = u$, we have

$\tan P = SB/SP = (5 – u)/(v – 3) = \tan A = v/u$, giving

$(5 – u) * u = (v – 3) * v$, or

$5u$ – u^2 = v^2 – $3v$, or $5u$ = u^2 + v^2 – $3v$ = 5^2 – $3v$, giving

$u = 5 – 3/5v$, or $u = 5 – 0.6*v$

At the same time, u^2 + v^2 = 5^2, so

$(5 – 3/5v)^2 + v^2 = 5^2$, giving $-6v + 1.36v^2 = 0$, giving $v = 6/1.36 = 4.41$, giving $u = 5 – 0.6*4.41 = 2.35$

So, to come from *A* to the second contact point *P* we go out 2.35 units, and go up 4.41 units.

Perpendicular Lines

In a 3x5 box, *ACBD*, we look at a line departing from *A* perpendicular to the diagonal *CD* intersecting it in S. In the right triangle ASC, tan *A* = *CS/AS*, and tan *C* = *AS/CS*. Consequently, tan *A* * tan *C* = 1. With tan *C* = 3/5, tan *A* = 5/3.

We now look at a circle having its center in *A* and its radius *r* = *AC*. Here *AS* intersects the circle in the point *P*. To find *P* and *Q* we observe that *P* is situated on the line *AP* and on the circle. With *AQ* = *u* and *QP* = *v*, we have

tan $A = 5/3 = v/u$, and $u^2 + v^2 = 5^2$, giving
$(3/5v)^2 + v^2 = 25$, or $1.36v^2 = 25$, giving
$v = 4.287$, and $u = 3/5*4.287 = 2.57$

So, to come from *A* to the intersection point *P* we go out 2.57 units, and go up 4.29 units.

Technical Tasks

In the technical world, forms exist in abundance. Here are but a few examples.

- A six-edged screw head has two diameters. The big one is 27.7 mm. What is the little one?
- An axle-end increases the diameter from 15 mm to 25 mm over 30 mm. What is the angle?
- In a large tube with an internal diameter 10 cm must contain 3 equal tubes so that they touch each other precisely while each of the 3 tubes also touches the inside of the large tube. Calculate the outer diameter of the 3 tubes.

Astronomy

In astronomy you have sight-lines to objects far away in space. These lines may turn a little if they come from different places on the earth, e.g., from where the line is vertical and where it is horizontal. Knowing the diameter of the earth it is possible to calculate the distance from the earth to the sun, and then the diameter of the sun. Likewise, with the moon.

The amount of energy the earth receives from the sun is called the solar constant, 1.367 kW per square meter. However, if the square meter is turned v degrees away from the rays, the energy is reduced by cos v factor.

Triangles in a Coordinate System

In a coordinate system, a point has two coordinates *x* and y communicating how far out and how far up it is situated from a starting point, or how far back and how far down if using negative numbers also.

Two points thus creates a line with a length and a direction that can both be found by boxing the points. Thus the points *A*(2,1) and *C*(7,3) become the diagonal in a box with a horizontal length 7 – 2 = 5, and a vertical height 3 – 1 = 2. Also, the line constitutes a trip from *A* to *C*, called a vector <u>*AC*</u>, where the *x* coordinate has the change $\Delta x = 7 – 2 = 5$, and the y-coordinate has the change $\Delta y = 3 – 1 = 2$, thus giving the vector the coordinates $\underline{AC} = \binom{\Delta y}{\Delta x} = \binom{5}{2}$. The Pythagorean and the tangent formula now allow finding the length of the line *AC* and its direction, also called its slope or gradient, as

$AC = \sqrt{}(\Delta x^2 + \Delta y^2) = \sqrt{}((7-2)^2 + (3-1))^2 = \sqrt{29} = 5.39$; and
$\tan A = \Delta y/\Delta x = (3-1)/(7-2) = 2/5 = 0.4$

Unless situated on the same line, 3 points as *A*(2,1) and *C*(7,3) and *B*(5,7) create a triangle with sides and angles that can be found in two ways, by boxing the triangle, or from its vectors.

Using boxing, the three points create a box with a horizontal length 7 – 2 = 5, and a vertical height 7 – 1 = 6. In this box, the triangle ABC is surrounded by three right triangles. Once these have their sides and angles determined, a simple subtraction gives the angles of the triangle. Likewise, with the area.

Using vectors is simpler. The triangle *ABC* is defined by the two trip vectors $\underline{AC} = \binom{5}{2}$ and $\underline{AB} = \binom{3}{6}$. A special formula called the dot-product formula now allows finding the angle between the two vectors:

$$\underline{AC}*\underline{AB} = \binom{a1}{a2}*\binom{b1}{b2} = a1*b1 + a2*b2 = \underline{|AC|} * \underline{|AB|} * \cos A$$

Likewise, a formula called the cross-product formula allows finding the box area and the angle between two vectors:

$$\underline{AC} \times \underline{AB} = \binom{a1}{a2}\mathrm{x}\binom{b1}{b2} = a1*b2 – a2*b1 = \underline{|AC|} * \underline{|AB|} * \sin A$$

Intersection in a Coordinate System

The three points *A*(2,1) and *C*(7,3) and *B*(5,7) create a triangle that may be boxed by a 6x5 rectangle. A line from a to the upper right corner *M*(7,7) will intersect the line *BC* in a point *G* with the coordinates *u* and *v*. To find it we equate tan *A* found in a lower and an upper triangle

$\tan A = 6/5 = (v – 1)/(u – 2) = (7 – v)/(7 – u)$

This gives to equations with two unknowns that are solved by $u = 5.75$ and $v = 5.5$

$6(u – 2) = 5(v – 1)$, and $6(7 – u) = 5(7 – v)$

A quicker way is to find a formula for the points on the line *AG* by replacing *u* and *v* with *x* and y

$6(x – 2) = 5(y – 1)$, giving $y = 6/5(x – 2) + 1$.

In the same way we may find the formula for the line through *B* and *C*: $\tan B = 4/2 = (7 – y)/(x – 5)$, giving $y = -2(x – 5) + 7$

Since the intersection point is situated on both lines, we equate their *y*'s

$y = 6/5(x– 2) + 1 = -2(x– 5) + 7$, solved by $x = 5.75$ and $y = 5.5$

In a similar way, intersection points between a line and a circle are quickly predicted by their formulas.

Circular Motion

A stick with unit length rotates around its left endpoint. The right endpoint then moves in a circle if observed from the front. Seen from the side, its height becomes its sine-number. The endpoint moves quickly from height 0 and slows down around height 1 to return to height 0 again. Described in a coordinate system we then see that as the angle increases from 0 through 90 to 180 degrees, the sine-number increases quickly from 1 to 0 and slowly back to 1.

Allowing the rotation to continue after 180 through 270 to 360 degrees, we get a sine curve. Likewise, with a cosine curve where, seen from above, the endpoint decreases slowly from 1 to 0 and the quickly to -1. Seen from the front, the endpoint P's coordinates then will be $P(\cos v, \sin v)$. Sine and cosine curves may be used to predict periodic phenomena as tide, solar height, alternating current, market fluctuations, etc.

7. EXPERIENCING THE POWER OF PREDICTION

The three trigonometry formulas and the Pythagorean formula allow us to experience the power of prediction. To find the diagonal in a 4x5 box we may use a 'formula table' to solve the equation occurring with only one unknown in the formula. The table has two columns, one for the numbers, and one for the formula. On the top line we write the unknown number first and then the formula used to find it. On the next lines we write the known numbers to the left, and to the right we write the equation occurring after the known numbers are inserted in the formula, leaving one unknown to be isolated by moving numbers to the opposite side with opposite calculation sign. In the last line we test to see if the right hand side and the left hand side of the formulas produce the same numbers when inserting all the numbers.

c = ?	c^2 = a^2 + b^2
a = 4	c^2 = 4^2 + 5^2
b = 5	c = √41
	c = 6.40
Test	6.40^2 = 4^2 + 5^2 ??
	41.0 = 41 ☺

On a graphing calculator, entering 'MATHSolver 0 = 4^2 + 5^2 – x^2' gives the same answer, x = 6.40.

Alternatively, c may be found from the two equations tan A = 4/5 giving A = 38.7; and sin 38.7 = 4/c giving c = 6.40.

Now books, tables, windows, doors, tiles, boxes, etc. may be used as examples.

Likewise, rotating a laser pointer 30 degrees form the 4-meter-wide floor allows predicting that the height of the spot on the wall will be 4 * tan 30, or 2.31 cm.

Distance to a Far Away Point
Wanting to predict the vertical distance PC across a river to an inaccessible point P, we use a known baseline, AB, to measure the angles A and B to P. With $AB = m$, $BC = u$, and $PC = v$, the two right angled triangles ACP and BCP give

tan $A = v/(m+u)$, and tan $C = v/u$, giving

u = tan A/(tan B – tan A) * m, and
v = (tan B * tan A)/(tan B – tan A) * m

With AB = 366 cm, angle CAP = 34 degrees, angle CBP = 55 degrees, we get

$PC = v$ = (tan 55 * tan 34)/(tan 55 – tan 34) * 366 = 468.

Various Applications
In the outside world, a line will always be situated in space so it may often be of interest to box it to see how much is in vertical and horizontal direction. There are many such problems. They may be given as projects with open ended questions, or as specific exercises to be solved [16].

Project 1: What is the height of a high thing, e.g., a flagpole?

Project 2: How to move a thing around a corner?

Project 3: How to calculate shortcuts?

Project 4: How do you travel fastest from a point in one area to a point in another area when you move at different speeds in the two areas?

Project 5: How to calculate the different openings when a door opens ajar?

Project 6: How to build a path up a steep mountainside?

Project 7: How steep can a ladder be placed so as not to break glass or ice?

Project 8: How are astronomical distances calculated?

Project 9: How should a triangular bridge over a river be sized?

Project 10: On a swing, when to jump off to get a long jump?

Exercise 1. Determine the height of a high thing (a flagpole) in two different ways: a light way, where we can get close to the thing; and the hard way where we can't. Determine in the same way the width of a wide thing (a house wall or a 'river'). If possible, check the calculations by measuring.

Exercise 2. A corridor with width 2 meters has a corner. A stick is turned around it without lifting. What is the maximum length of the stick? A box with a width of 1 meter is turned around the corner without lifting. What is the maximum length of the box? If possible, try with the other objectives also.

Exercise 3. A corridor with width 3 meters has a corner and continues with a width of 4 meters. A stick is turned around the corner without lifting. What is the maximum length of the stick? A box with a width of 2 meters is turned around the corner without lifting. What is the maximum length of the box?

Exercise 4. A treasure can be found by first traveling 4 meters to the east, then 2.5 meters to the south, and finally 3.1 meter to the west. Find a shortcut to the treasure.

Exercise 5. A treasure can be found by first traveling 4 meters to the east in 32 degrees northerly direction, then 2.5 meters to the south in 68 degrees westerly direction, and finally 3.1 meter to the west in a 14-degree northerly direction. Find a shortcut to the treasure.
Exercise 6. A boat rows over a 50-meter-wide river at a speed of 20 meter/minute. The current in the river is 10 meter/minute. What angle should you choose for landing right across on the opposite side? How long does the trip take?

Exercise 7. Construct a straight line and two points on either side of the line. Go from one point to the other with steps equal to 1 shoe length. 1 second corresponds to 1 shoe length in one area and 2 shoe lengths in the other. Find the fastest route. Could the fastest route be figured out? If necessary, use a PC spreadsheet to calculate different routes. A formula applies to the incidence and refraction angle at the point on the boundary line which the fastest route passes. Which?

Exercise 8. Open a door ajar. There are now three openings, one perpendicular to the wall, one parallel to the wall and one parallel to the door. How big are these openings? How much will 10 degrees extra increase these openings by?

Exercise 9. Tip a plate 30 degrees and design a way up that can rise no more than 20 degrees (a hairpin bend). Repeat the exercise with other degree numbers. How much does the gravitational pull of a car increase when the rise of the road increases 10 degrees?

Exercise 10. Place a heavy book on a scale. Tilt it to different positions. What happens to the weight? Could this result be calculated? How much is the pressure against a vertical hand supporting it?

Exercise 11. Construct and load a triangular bridge and control the pressure against the surface by placing the bridge on two scales. Can these numbers be calculated?

Exercise 12. Distances in space cannot be measured, they must be calculated. How is the radius of the earth calculated? How far is it to the moon? What is the radius of the moon? How far is the sun? What is the radius of the sun?

Exercise 13 (difficult). A person sits in a swing that is suspended in ropes that are 3 meters long. Pull out the swing so that it is in height 1 meter above the bottom point. What angle does this correspond to?

If the swing is released, the speed v can be calculated from the formula: v^2 = 19.6 * h, where h is the distance from the maximum height in the outer position. The speed consists of a horizontal and a vertical part. Where do we have to jump off to get a long jump? (After the jump, the horizontal part of the speed will be unchanged, while the vertical part will grow downwards by 9.8 m/s every second.)

8. TRIGONOMETRY IN STEM

STEM is short for Science, Engineering, Technology and Mathematics that all see applications of trigonometry. The subjects are all rooted in a wish to predict the outside nature that we are part of.

To meet we must specify place and time in a nature consisting of heavy things at rest or in motion. So, in general, we see that what exists in nature is interacting matter in space and time.
A falling ball shows nature’s three main ingrediences, matter and force and motion, like the three social factors, humans and will and obedience.

As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when colliding, and gravity pumping motion in and out of matter when moving in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; meaning that the motion has lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases.

But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, core STEM problems could be about cycling water. Heating transforms water from solid to liquid to gas, i.e. from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave, so the lighter box is pushed up by gravity until becoming heavy water by cooling, now pulled down by gravity as rain in mountains and through rivers to the sea. On its way down, a dam can transform falling water into electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to dissolving matter in water; to the orbit of a ball pulled down by gravity; to putting steam and electrons to work in a power plant creating an electrical circuit transporting energy from a source to many consumers.

Heavy things in motion are combined by the momentum = mass * velocity, a multiplication formula as most STEM formulas expressing re-counting by per-numbers:

kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter, where force is the per-number change in momentum per second.

Thus, STEM-subjects are swarming with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Science: Motion
Example 1. Pulled by gravity, a ball in a tube will start running the moment the tube is lifted 20 degrees. However, the gravity pull F must be boxed to see that only a part of it, F * sin 20, goes in the direction of the tube. Leaving the tube with a velocity vector v, this again must be boxed to see that v * sin20 is the vertical velocity and v * cos20 is the horizontal velocity.

Example 2. An inclined gun sends a ping-pong ball upwards. This allows a double-counting between the distance and the time to the top, 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second.

To find its initial speed we turn the gun 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, 2.5 meters and 5 meters and 0,73 seconds. From a folding ruler we see, that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin 45 = cos 45 = 0,707.

The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u * 0,707) * 0,73, solved as u = 9.69 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app.

Compared with the horizontal, the vertical distance is halved, but the speed changes from 9.69 to 9.69 * 0.707 = 6.85. However, the speed squared is halved from 9.69 * 9.69 = 94 to 6.85 * 6.85 = 47.

So horizontally, there is a proportionality between the distance and the speed. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

Example 3. Light traveling from one medium to another with two different velocities $v1$ and $v2$ follows Snell's law of refraction. With angles of incidence $w1$ and refraction $w2$ both measured from the boundary normal, the law says that

$$\frac{\sin w1}{\sin w2} = \frac{v1}{v2}$$

Engineering

Example 1, building a bridge. Over an 8-meter-wide canyon a suspension bridge made of steel is fastened to the points A and G on a cliff where A is 5 meters above G; and to the points B and F on a 3 meter vertical upright placed 1 meter from the edge, and fixed by a wire lifted 30 degrees. We want to determine the length of the 3 beams and the welding point C.

From the right-angled triangles EFB, GFB and FGA we calculate BE, BG and FA. C is found as the intersection point between line BG and line FA.

First in triangle EFB we use the sine-formula to find BE = 7.00. Then in the triangles FGA and GFB we use the Pythagorean formula to find AF = 10.30 and BG = 9.66.

Example 2, how many turns on a steep hill. On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance $BC = u$.

In the triangle BCD, the angle B is 30 degrees, and $BD = u$ * cos(30). With Pythagoras we get

u^2 = CD^2 + BD^2 = CD^2 + u^2 * cos(30)^2, or

CD^2 = u^2(1–cos(30)^2) = u^2 * sin(30)^2.

In the triangle ACD, the angle A is 10 degrees, and $AD = AC$ * cos(10). With Pythagoras we get

AC^2 = CD^2 + AD^2 = CD^2 + AC^2 * cos(10)^2, or

CD^2 = AC^2(1–cos(10)^2) = AC^2 * sin(10)^2.

In the triangle ACB, AB = 1 and $BC = u$, so Pythagoras gives

AC^2 = 1^2 + u^2, or AC = √(1 + u^2).

Consequently

u^2 * sin(30)^2 = AC^2 * sin(10)^2, or
u = AC * sin(10)/sin(30) = AC * r, or
u = √(1+u^2) * r, or
u^2 = (1+u^2) * r^2, or
u^2 * (1–r^2) = r^2, or
u^2 = r^2/(1–r^2) = 0.137, giving the distance
$BC = u$ = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Technology

Pressure machines typically have a piston at the top of a cylinder with water that evaporated makes the piston go up, and vice versa go down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g., steam locomotives.

The force on the piston then has to be applied to a rotating disk in order to have vertical motion transformed into horizontal motion. So, we need to box the force twice. First to see how much goes in the direction of the pin connecting the piston to the disk. Next, boxing this to see how much goes in the tangent direction to turn the disk.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating lifts the pressure by increasing both the temperature and the number of steam molecules. And vice versa in the cold cylinder where cooling lowers the pressure by decreasing both the temperature and the number of steam molecules that, when condensed to water, are pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to industries and homes.

9. CONCLUSION

We asked if also mathematics can be learned as communication. Well, only if there is something out there to communicate about. And indeed, there is, boxes. They transform a total into a Hindu-Arabic number. And they split in two by their diagonals creating trigonometry formulas when recounting the sides mutually. Or the other way around, boxing a line makes it a diagonal allowing trigonometry formulas predict its horizontal and vertical sides. Which leads to many core STEM examples.

This of course is to be expected given the original meaning of the word geometry and algebra. With geometry meaning earth-measuring in Greek, of course it may be learned as communicating about how to measure earth; and with algebra meaning re-uniting in Arabic, of course it may be learned as communicating about how to unite changing or constant unit- and per-numbers [17].

So, it seems that instead of following the ruling 'truth regime' [18] teaching one-dimensional line numbers instead of two-dimensional box- and bundle-numbers, and teaching plane geometry before trigonometry, we may be able to master outside boxes without learning plane geometry first. And the question is really, do we need to learn it at all? Of course, it has a quality that many theorems about general cases may be proven. Bet is it not far more interesting and important to use formulas to predict specific cases? And as long as no counter-example has been found, the formula works, and therefore it should be valid according North American pragmatism.

And since societies around the world scream for technicians and engineers able to master STEM problems, maybe the time has come to change the status of plane geometry from mandatory to elective for people finding it interesting.

Thus it seem possible that also the number-language should have its communicative turn as had the word-language in the 70s. Accepting the mastery of Many children develop before school will make a box the core outside subject to master, inside described by two-dimensional flexible bundle-numbers allowing fractions, decimals and negative numbers to communicate about what to do with the unbundled singles, and allowing number-language sentences, formulas, to inside predict what will happen outside.

And, experiencing the excitement of working with predicting formulas will dramatically increase the number of students wanting to go on to create a successful future within the STEM subjects.

Therefore, teaching trigonometry before geometry well help boys acquire a self-idenitty as masters of Many, thus improving the world economy to solve poverty and climatic issues. So maybe the time has come to realize that changing from pre-setcentric and setcentric to the child's own post-setcentric mathematics will make the number language a human right in accordance with fourth the United Nation's sustainable development goals, the right to quality education.

10. REFERENCES

[1] H.G. Widdowson, **Teaching Language as Communication**, Oxford Univ. Press, 1978.
[2] H. Freudenthal, **Mathematics as an Educational Task**, Dordrecht-Holland: D. Reidel Publ. Comp., 1973.
[3] B. Russell, **A History of Western Philosophy**, New York: Touchstone Book, 1945.
[4] M. Kline, **Mathematical Thoughts from Ancient to Modern Times**, New York: Oxford Univ. Press, 1972.
[5] A. Tarp, **Mastering Many by Counting, Re-Counting and Double-Counting Before Adding On-Top and Next-To**, Journal of Mathematics Education, vol 11, No. 1, 2018, pp. 103-117.
[6] J. Derrida, **A Derrida Reader**, P. Kamuf (ed.), New York: Columbia Univ. Press, 1991.
[7] E. Durkheim, **The Rules of Sociological Method**, New York: Free Press, 2014.
[8] L. Menand, **Pragmatism**, New York: Vintage Books, 1997.
[9] M. Weber, **The Protestant Ethic and the Spirit of Capitalism**, London UK: Unwin Hyman, 1930.
[10] Z. Bauman, **Thinking Sociologically**, Oxford, UK: Blackwell, 1990.
[11] T.S. Kuhn, **The Structure of Scientific Revolutions**, Chicago: Univ. of Chicago Press, 1962.
[12] J.P. Sartre, **Existentialism is a Humanism**, New Haven, CT: Yale Univ. Press, 2007.
[13] L. Vygotsky, **Thought and Language**, Cambridge MA, MIT press, 1986.
[14] J. Piaget, **Science of Education of The Psychology of The Child**, New York: Viking Compass, 1970.
[15] A. Tarp, **Fact, Fiction, Fiddle - Three Types of Models**, in J. F. Matos & W. Blum & K. Houston & S. P. Carreira (Eds.), Modelling and Mathematics Education: ICTMA 9: Applications in Science and Technology. Proceedings of the 9th International Conference on the Teaching of Mathematical Modelling and Applications (pp. 62-71), Chichester UK: Horwood Publishing, 2001.
[16] A. Tarp, **Geometry from Below,** http://mathecademy.net/geometry-from-below/, 2020, Retrieved January 2021.
[17] A. Tarp, **De-Modeling Numbers, Operations and Equations: From Inside-Inside to Outside-Inside Understanding**, Ho Chi Minh City University of Education Journal of Science, vol 17, No. 3, 2020, pp. 453-466.
[18] M. Foucault, **Discipline and Punish**, New York: Vintage Books, 1995.

Paper as YouTube video:
https://www.youtube.com/watch?v=yEG27dNgCzE

Paper at
http://mathecademy.net/trigonometry-before-geometry/