Master Many

to later

Master Math

An Opportunity for
an EXISTENCE-based Mathematics
using Flexible Bundle-Numbers with Units

The Canceled Curriculum Chapter
in the ICME Study 24
School Mathematics Curriculum Reforms:
Challenges, Changes and Opportunities

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http://mathecademy.net/appendix-to-curriculum-study-ICMI-24/

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“The Same Mathematics Curriculum for All Students” is the canceled chapter.

“A New Curriculum - But for Which of the 2x2 Kinds of Mathematics Education” is an essay based on observations and reflections at the ICMI Study 24 Curriculum Conference.

“A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many”, my paper to the conference, is on page 317 in the proceedings: https://www.mathunion.org/fileadmin/ICMI/ICMI%20studies/ICMI%20Study%2024/ICMI%20Study%2024%20Proceedings.pdf

Preface

A curriculum for a class is like a score for an orchestra. Follow it, and the result will be a perfect performance. In music perhaps, but not always in a class.

It begins so well. Textbooks follow curricula, and teachers follow the textbooks supposed to mediate perfect learning. But, as shown in international tests, this does not always take place for all learners. But then, other scores may be more successful? Well, with few variations, scores seem to teach the same in the same way: numbers, operations, calculations, formulas, and forms. Why is there so little room for improvisation as in jazz?

So, with the transformation of modern society into a postmodern version, the time has come to ask: How about jazzing up the curricula to allow children's quantitative competences and talents to blossom?

As a curriculum architect using difference research to uncover hidden differences that may make a difference, I warmly greeted the announcement of ICMI study 24 with the title 'School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities'. I was especially excited about including opportunities, which would allow hidden differences to be noticed and perhaps tested. And I jumped for joy with the acceptance of my paper 'A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many.'

At the conference I was asked to contribute writing a report on part B2 asking 'How are mathematics content and pedagogical approaches in reforms determined for different groups of students (for e.g. in different curriculum levels or tracks) and by whom?'

The deadline was end June 2019, but shortly before I was told that this part would be canceled and not appear in the report. Still, I finished my contribution and sent it in. But as expected, it has not been included. Consequently, I have chosen to publish it as an appendix to the ICMI 24 study.

To me, the question is 'Why can't we have the same curriculum for all students?', which of course leads on to the more general question 'What is mathematics education?'

Asking the Three Grand Educational Theories

A question we may redirect to the three grand educational theories, philosophy, sociology and psychology, by asking 'What is existence and essence in mathematics?', 'What does the common institution education do to individuals?', and 'What is the relationship between inside representations and outside reality in mathematics?'

In philosophy, a core question is the relationship between 'outside' existence and 'inside' essence. To existentialists existence precedes essence as formulated by Sartre, or, as formulated by Heidegger: In a judging sentence, trust the subject, it exists, but doubt the predicate, it is a chosen construction. Wanting to protect the second Enlightenment French republic from hidden patronization, poststructuralism took this warning further with Derrida recommending predicates to be deconstructed, and with Foucault warning against ruling 'truth regimes' forcing constructed diagnoses upon humans to force them into cures at normalizing institutions.

As to mathematics meaning knowledge, ancient Greece chose it as a common name for their four knowledge areas: astronomy, music, geometry and arithmetic describing the physical fact Many in time and space, in time, in space, and by itself. And together forming the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric. Today, mathematics typically is a common name for geometry and algebra both indicating the outside existence rooting them: in Greek, geometry means to measure earth, and in Arabic, algebra means to reunite.

Although born as existence-based, mathematics never developed as a natural science about Many in time and space since both Greek and Roman numbers both missed the advantage of only bundling singles and bundles, as did the Hindu-Arabic numbers coming to Europe in the Renaissance.

Instead, the Greeks developed an axiomatic deductive Euclidean geometry well suited to practice logic. So, for centuries, mathematics was a science about essence, essence-based mathematics'. Which was even intensified when around 1900, the abstract concept 'set' was spreading all over mathematics allowing also algebra to be presented in an axiomatic deductive by defining its concepts from above as examples of abstractions instead of from below as abstractions from examples. It culminated with the New Math movement in the 1960s, defining a function as a subset of a set-product where first-component identity implies second-component identity instead of using the original Euler definition where a function was a common name for calculations with both known and unknown numbers.

The English-speaking world soon went back to basics and to the set-free essence-based math curricula. The rest of the world developed new set-based curricula.

What is needed are curricula in existence-based math returning to the original Greek meaning where mathematics is a natural science about Many in time and space.

So, we need curricula that presents mathematics as an existentialist outside referring natural science aiming for mastery of Many instead as a pure self-referring science that is taught and learned for its own sake, and where outside applications are left to others to teach later.

As to sociology looking at how humans cooperate to improve their lives, constructing common concepts and languages allows sharing information, and constructing common institutions as means to reach common goals allows additional time to reach individual goals also. Although seemingly increasing productivity, sociology still warns against potential institutional goal displacements where a means becomes the goal instead by realizing that not reaching the original goal is an effective way to secure survival and growth to the institution itself. And as to institutions treating humans, Foucault warns against education becoming instead a 'pris-pital' combining disciplinary power-techniques of a prison and a hospital by forcing the treated to return to the same room again and again, and by forcing a diagnose upon them as ignorant to be cured by self-referring treatments: 'you know no mathematics, so we teach you mathematics'. To avoid self-reference, an institution must always have a well-formulated outside goal.

So, in mathematics education, to avoid self-reference, mastery of Many is the outside goal that generated mathematics as a means to reach it. And essence-based mathematics and existence-based mathematics should compete as to which is more effective in reaching this outside goal. So, curricula in the latter need to be designed and tested. And it may be that mastery of Many could become not only the goal but also a means to later also master essence-based mathematics if needed or wanted.

Psychology looks at how the human mind constructs inside representations of outside things and actions. This actualizes the core questions of philosophy and sociology: Should the outside precede the inside, or vice versa? Should institutionalized representations precede individual, or vice versa? As to education, this has created two different schools: behaviorism using motivation to make teaching more effective, and constructivism focusing on learning through individual constructions.

So, in mathematics education in the 1980s, Skinner behaviorism was replaced by two kinds of constructivism, a social version mediating institutionalized representations by a teacher that should be well educated, and having Vygotsky as its prime theorist, thus making learners choose between serving or rejecting the mediated truth regime. And a radical version allowing representations to be constructed and negotiated with peers by meeting the outside subject of the sentence with the teacher in the background supplying concrete material with guiding questions, and having Piaget as its prime theorist, thus allowing learners constructing knowledge as ground theory researchers.

The Time has Come for a Curriculum in Existence-Based Mathematics

To make visible hidden opportunities I decided that my paper should focus on curricula based upon existence-based mathematics, thus siding philosophically with existentialism. Sociologically, I sided with the necessity of avoiding a goal displacement by formulating the goal of mathematics education as did the ancient Greeks to be mastery of Many. And psychologically, I sided with radical constructivism by allowing knowledge to be constructed individually by meeting the outside subjects of mastery of Many, the total and the box, to be inside described by predicates in the form of numbers, calculations and formulas helped by guiding questions.

Seeing 3-year-old children describe four fingers as ‘That is two twos’, shortened to ‘T = 2 2s’ indicates that, when adapting to Many before school, children use bundle-numbers with units inside, outside represented by a 2-by-2 box. This resonates with the fact that the sentences T = 2 is meaningless without a unit. So, an existence-based math curriculum must use two-dimensional bundle-numbers instead of the traditional one-dimensional line-numbers describing cardinality without units.

To transform an outside unstructured total into a box and a bundle-number, we must specify the unit by asking, e.g., ‘How many bundles of 2s in a total of 9?’ To get an answer, first we push away bundles. Iconizing the broom by a division sign allows a calculator predict the result by showing ‘9/2 = 4.some’. Next, we stack the bundle into a box. Iconizing the lift by a multiplication sign allows a calculator predict the result by showing ‘4x2 = 8’ or ‘4*2 = 8.’ Then we pull away the box to look for unbundled singles. Iconizing the rope by a subtraction sign allows a calculator predict the result by showing ‘9 – 4*2 = 1.’

Placing the unbundled on-top of the stack, it may be reported by a decimal number, or a fraction if counted in the bundles, or a negative number showing what is needed for yet another bundle thus creating a ‘flexible bundle-number’:

T = 9 = 4Bundle1 2s = 4B1 2s = 4.1 2s = 4 ½ 2s = 5.-1 2s = 3B3 2s = 2B5 2s.

Recounting ten fingers in 3s allows meeting also the bundles of bundles:

T = ten = 3B1 3s = 1 BB0B1 3s = 1BB1B-2 3s.

Flexible bundle-numbers ease calculations:

3 * 45 = 3 * 4B5 = 12B15 = 13B5 = 135, and 135 /3 = 13B5 /3 = 12B15 /3 = 4B5 = 45

Once counted, totals may be recounted in another unit using the recount-formula coming from recounting 8 in 2s as 8 = (8/2)*2, or T = (T/B)*B with unspecified numbers, a direct way to the first of the two columns of mathematics, proportionality or linearity. And to equations where u*2 = 8 is solved by recounting 8 in 2 as 8 = (8/2)*2, so u = 8/2.

Thus, a total with icon-units as, e.g., 2 3s, may be recounted in tens, and vice versa. And it may be squeezed into a square with the square root as the unit, allowing quadratic equations to be easily solved by turning a playing card a quarter round before being placed on-top of another card.

And double-counting a physical quantity as apples leads to per-numbers as p = 2$/5kg allowing changing units by recounting in the per-number, 15 kg = (15/5)*5kg = (15/5)*2$ = 6$.

With like units, per-numbers become fractions: p = 25$/100$ = 25/100 = 25%.

So, per-numbers and fractions are not numbers, but operators needing numbers to become numbers.

Finally, halved by its diagonal, a box splits into two right triangles. Here recounting the sides mutually creates the trigonometry formulas, height = (height/base) * base = tan A * base, etc. As well as a formula for pi, pi = n * tan(180/n) for n big.

And again, once counted, totals as 2 3s and 4 5s may add on-top after recounting makes the units like; or next-to by their areas, a direct way to the second of the two columns of mathematics, integral calculus, as well as to differential calculus when reversed.

And also occurring when adding piecewise constant per-numbers as 2kg at 3$/kg + 4kg at 5$/kg. Here the unit-numbers add directly while the per-numbers must be multiplied to areas before adding. This again exemplifies integral calculus, which also appears when adding locally constant per-numbers by the area under the per-number graph.

And also occurring when writing out fully a number as we say it, e.g., T = 345 = 3*BB + 4*B + 5*1, showing the four ways to unite into a total: on-top addition, multiplication, power and next-to box addition, also called integral calculus. Including also the reverse operation splitting a total: subtraction, division, root and logarithm, and differentiation, allows setting up the ‘algebra square’:

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	$\mathbf{T} = a^b$ $\sqrt[b]{T} = a \qquad log_a(T) = b$

Figure 01. *The 'algebra-square' shows the four ways to unite or split numbers.*

Furthermore, the bundle-counting polynomial formula, T = a*x^2 + b*x + c, also shows the different forms of constant change: proportionality, T = b*x; linearity, T = b*x + c; exponential, T = a*k^x; power, T = a*x^k; and accelerated change, T = a*x^2 + b*x + c

Consequently, in an existence-based mathematics curriculum, the two basic columns of mathematics, linearity and calculus, occur at once in grade 1. So, of course it is possible to have one curriculum for all student. All you have to do is replace essence-based mathematics with existence-based mathematics. All we need is a curriculum for the latter.

So that is what my paper should be about.

Aarhus, Denmark, July 2021, Allan Tarp

The Same Mathematics Curriculum for All Students

To offer mathematics to all students, parallel tracks often occur from the middle of secondary school. The main track continues with a full curriculum, while parallel tracks might use a reduced curriculum leaving out, e.g., calculus; or they might contain a different kind of mathematics meant to be more relevant to students by including more applications. However, an opportunity here presents itself for designing the same curriculum for all students no matter which track they may choose. As number-language, why not let mathematics follow the communicative turn that took place in language education in the 1970s by prioritizing its connection to the outside world higher than its inside connection to its grammar? We will consider examples of all three curricula options.

A Need for Curricula for all Students

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in mathematics education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 year since 1969. Likewise, funding has increased as seen, e.g., by the creation of a National Center for Mathematics Education in Sweden. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to 2012, causing the Organisation for Economic Co-operation and Development (OECD, 2015) to write the report 'Improving Schools in Sweden' describing its school system as 'in need of urgent change':

PISA 2012, however, showed a stark decline in the performance of 15-year-old students in all three core subjects (reading, mathematics and science) during the last decade, with more than one out of four students not even achieving the baseline Level 2 in mathematics at which students begin to demonstrate competencies to actively participate in life. (p. 3)

Other countries also experience declining PISA results; and in high performing countries not all students are doing well.

Addressing the Need

By saying 'All students should study mathematics in each of the four years that they are enrolled in high school.' the US National Council of Teachers of Mathematics (2000, p. 18) has addressed the need for curricula for all students in their publication 'Principles and Standards for School Mathematics'. In the overview the Council writes

We live in a mathematical world. Whenever we decide on a purchase, choose an insurance or health plan, or use a spreadsheet, we rely on mathematical understanding (..) In such a world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence opens doors to productive futures. A lack of mathematical competence closes those doors. (..) everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics. Students need to learn a new set of mathematics basics that enable them to compute fluently and to solve problems creatively and resourcefully. (p. 1)

In this way the Council points out that it is important to master 'mathematical competence', i.e., to understand and do mathematics to solve problems creatively and to compute fluently. This will benefit the personal life, the workplace, as well as further study leading to productive futures.

Consequently, the Council has included in the publication a curriculum that 'is mathematically rich providing students with opportunities to learn important mathematical concepts and procedures with understanding'. This in order to 'provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.'

The publication includes a set of standards: 'The Standards for school mathematics describe the mathematical understanding, knowledge, and skills that students should acquire from prekindergarten to grade 12.' The five standards present goals in the mathematical content areas of number and operations, algebra, geometry, measurement and data analysis and probability. (..) Together, the standards describe the basic skills and understandings that students will need to function effectively in the twenty-first century' (p. 2)

In the chapter 'Number and operations', the Council writes

> Number pervades all areas of mathematics. The other four content standards as well as all five process standards are grounded in number. Central to the number and operation standard is the development of number sense. Students with number sense naturally decompose numbers (..) For example, children in the lower elementary grades can learn that numbers can be decomposed and thought about in many different ways - that 24 is 2 tens and 4 ones and also two sets of 12. (p. 7)

In the chapter 'The Curriculum Principle', the Council writes

> A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades (..) for teachers at each level to know what mathematics their students have already studied and will study in future grades. (p. 3, 4)

All in all, the Council points to the necessity of designing a curriculum that is relevant in students' 'personal life, in the workplace and in further study' and that is coherent at the same time to allow teachers to know 'what mathematics their students have already studied and will study in future grades'.

Coherence and Relevance

So, in their publication, the National Council of Teachers of Mathematics stresses the importance of coherence and relevance. To allow teachers follow a prescribed curriculum effectively, and to allow students build upon what they already know, it must be 'well articulated across the grades'. And, to have importance for students a curriculum must be relevant by supplying them with 'the basic skills and understandings that students will need to function effectively in the twenty-first century'.

With 'cohere' as a verb and 'relevant' as a predicate we can ask: 'to what does this curriculum cohere, and to what is it relevant?' As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.'

We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that 'to cohere' means 'to hold together firmly as parts of the same mass'. As to synonyms for cohere, it lists: 'accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.' And as to antonyms, it lists: 'differ (from), disagree (with).'

In the same dictionary, the word 'relevant' means 'having significant and demonstrable bearing on the matter at hand'. As to synonyms for relevant, it lists: 'applicable, apposite, apropos, germane, material, pertinent, pointed, relative.' And as to antonyms, it lists: 'extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.'

If we accept the verb 'apply' as having a meaning close to the predicate 'relevant', we can rephrase the above analysis question using verbs only: 'to what does this curriculum cohere and apply?'

Metaphorically, we may see education as increasing skills and knowledge by bridging individual start levels to a common end level described by institutional goals. So, we may now give a first definition of an ideal curriculum: 'To apply to a learning process as relevant, a curriculum coheres to the individual start levels and to the end goal, which again coheres with the need expressed by the society funding the educational institution.'

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

How to make the bridge cohere with the individual start levels in a class?
How to make the end level cohere to goals expressed by the society?
How to make the end level cohere to goals expressed by the learners?
How to make the bridge cohere to previous and following bridges?
How to make the bridge (more) passable?

Then specific choices for mathematics education follow these general choices.

Parallel Tracks to the Main Curriculum, Examples

In their publication chapter Grades 9 through 12, the National Council of Teachers of Mathematics discusses to the possibility to introduce parallel courses in the high school.

In secondary school, all students should learn an ambitious common foundation of mathematical ideas and applications. This shared mathematical understanding is as important for students who will enter the workplace as it is for those who will pursue further study in mathematics and science. All students should study mathematics in each of the four years that they are enrolled in high school.

Because students' interests and inspirations may change during and after high school, their mathematics education should guarantee access to a broad spectrum of careers and educational options. They should experience the interplay of algebra, geometry, statistics, probability and discrete mathematics.

> High school mathematics builds on the skills and understandings developed in the lower grades. (..) High school students can study mathematics that extends beyond the material expected of all students in at least three ways. One is to include in the curriculum material that extends the foundational material in depth or sophistication. Two other approaches make use of supplementary courses. In the first students enroll in additional courses concurrent with those expected of all students. In the second, students complete a three-year version of the shared material and take other mathematics courses. In both situations, students can choose from such courses as computer science, technical mathematics, statistics, and calculus. Each of these approaches has the essential property that all students learn the same foundation of mathematics but some, if they wish, can study additional mathematics. (p. 18-19)

The Council thus emphasizes the importance of studying 'mathematics in each of the four years that they are enrolled in high school'. This the council sees as feasible if implementing one or more of three options allowing students to 'study mathematics that extends beyond the material expected of all students'. Some students may want to study 'material that extends the foundational material in depth or sophistication'. Others may want to take additional courses cohering to the college level, especially calculus. Others may want to take additional courses relevant to their daily life or a workplace. We will now look at two examples of that both including examples of finite mathematics, a subject that is normally outside a standard high school curriculum.

For all Practical Purposes, Introduction to Contemporary Mathematics

In the US, the Consortium for Mathematics and its Applications (COMAP) has worked out a material called ' For all practical purposes' (COMAP, 1988). In its preface, the material presents itself as

> (..) an introductory mathematics course for students in the liberal arts or other nontechnical curricula. The course consists of twenty-six half-hour television shows, the textbook, and this Telecourse guide. This series shows mathematics at work in today's world. (..) For all practical purposes aims to develop conceptual understanding of the tools and language of mathematics and the ability to reason using them. We expect most students will have completed elementary algebra and some geometry in high school. We do not assume students will be pursuing additional courses in mathematics, at least none beyond the introductory level. (p. iii)

As to content, the material has five parts (p. v - vi)

Part one focuses on graph theory and linear programming illustrated with network as scheduling and planning. It includes an overview show and four additional shows called street smarts: street networks; trains, planes and critical paths; juggling machines: scheduling problems; juicy problems: linear programming.

Part two deals with statistics and probability illustrated with collecting and deducing from data. It includes an overview show and four additional shows called behind the headlines: collecting data; picture this: organizing data; place your bets: probability; confident conclusions: statistical inference.

Part three focuses on social choice, fair division and game theory illustrated by different voting systems and conflict handling. It includes an overview show and four additional shows called the impossible dream: election theory; more equal than others: weighted voting; zero-sum games: games of conflict; prisoner's dilemma: games of partial conflict.

Part four focuses on using geometry, the classical conic sections, shapes for tiling a surface, geometric growth in finance in and in population, and measurement. It includes an overview show and four additional shows called how big is too big: scale and form; it grows and grows: populations; stand up conic: conic sections; it started in Greece: measurement.

Part five focuses on computer algorithms. It includes an overview show and four additional shows called rules of the games: algorithms; counting by two's: numerical representation; creating a cde: encoding information; moving picture show: computer graphics.

The video sections are available on YouTube.

A Portuguese Parallel High School Curriculum

Portugal followed up on the COMAP initiative. In his paper called 'Secondary mathematics for the social sciences' (Silva, 2018), Jaime Silva describes how the initiative inspired an innovative two-year curriculum for the Portuguese upper secondary school.

As to the background, Silva writes

> There are two recurring debates about the mathematics curriculum in secondary schools, especially in countries like Portugal where compulsory education goes till the 12th grade. First, should all students study mathematics (not necessarily the same) or should the curriculum leave a part of the students "happy" without the mathematics "torture"? Second, should all students study the same "classic" mathematics, around ideas from differential and integral calculus with some Geometry and some Statistics?
>
> When the 2001 revision (in great part in application today) of the Portuguese Secondary School curriculum was made (involving the 10th, 11th and 12th grades) different kinds of courses were introduced for the different tracks (but not for all of them) that traditionally existed. Mathematics A is for the Science and Technology track and for the Economics track and is a compulsory course. Mathematics B is for the Arts track and is an optional course. Mathematics Applied to the Social Sciences (MACS) is for the Social Sciences track and is an optional course. The Languages track was left without mathematics or science. Later the last two tracks were merged and the MACS course remained optional for the new merged track. The technological or professional tracks could have Mathematics B, Mathematics for the Arts or Modules of Mathematics (3 to 10 to be chosen from 16 different modules, depending on the professions). (p. 309)

As to the result of debating a reform in Portugal, Silva writes

> When, in 2001, there was a possibility to introduce a new Mathematics course for the "Social Sciences" track, for the 10th and 11th grade students, there were some discussions of what could be offered. The model chosen was inspired in the course "For All Practical Purposes" (COMAP, 2000) developed by COMAP because it "uses both contemporary and classic examples to help students appreciate the use of math in their everyday lives". As a consequence, a set of independent chapters, each one with some specific purpose, was chosen for this syllabus, that included 2 years of study, with 4.5 hours of classes per week (normally 3 classes of 90 minutes each). The topics chosen were: 10th grade Decision Methods: Election Methods, Apportionment, Fair Division; Mathematical Models:

> Financial models, Population models Statistics (regression); 11th grade Graph models, Probability models, Statistics (inference). (p. 310)

As to the goal of the curriculum, Silva writes

> The stated goal of this course is for the students to have "significant mathematical experiences that allow them to appreciate adequately the importance of the mathematical approaches in their future activities". This means that the main goal is not to master specific mathematical concepts, but it is really to give students a new perspective on the real world with mathematics, and to change the students view of the importance that mathematical tools will have in their future life. In this course it is expected that the students study simple real situations in a form as complete as possible, and "develop the skills to formulate and solve mathematically problems and develop the skill to communicate mathematical ideas (students should be able to write and read texts with mathematical content describing concrete situations)". (p. 310)

As to the reception of the curriculum, Silva writes

> This was a huge challenge for the Portuguese educational system because most of these topics had never been covered before, and most teachers did not even study Graph Theory at University. Election Methods, Apportionment and Fair Division were of course completely new to everybody. The reception was good from the part of the Portuguese Math Teacher Association APM, as it considered that *"the methodologies and activities suggested in the MACS program promote the development of the skills of social intervention, of citizenship and others"*. The reception from the scientific society SPM was rather negative because they considered the syllabus did not have enough mathematical content. (p. 310-311)

As to the present state of the curriculum, Silva writes

> After 15 years there is no thorough evaluation of how the course is run in practice in the schools, or which is the real impact on the further education or activities of the students that studied "Mathematics Applied to the Social Sciences". In Portugal there is no institution in charge of this type of work and evaluations are done on a case by case basis. All Secondary Schools need to do selfevaluations but normally just compare internal statistics to national ones to see where they are in the national scene. In the reports consulted there was no special mention to the MACS course and so we have the impression that the MACS course entered the normal Portuguese routine in Secondary School. (p. 315)

So as to a parallel track to the traditional curriculum, the National Council of Teachers of Mathematics suggests that including a different kind of mathematics might be an option, e.g., finite mathematics. In the US this idea was taken up by the Consortium for Mathematics and its Applications (COMAP) working out a material including a textbook and a series of television shows to show 'mathematics at work in today's world'. Part of this material was also included in a parallel curriculum in Portugal called 'Mathematics Applied to the Social Sciences' (MACS) offering to Portuguese students also to study mathematics in each of their high school years, as the National Council of Teachers of Mathematics recommends.

Precalculus, Typically the last Mandatory Curriculum

This chapter looks at the part of a mathematics curriculum called precalculus, typically being the first part that is described in a parallel curriculum since it contains operations as root and logarithm that is not considered part of a basic mathematics algebra curriculum. First, we look at an example of a traditional precalculus curriculum. The we ask what could be an ideal precalculus curriculum, and illustrates it with two examples. In the next chapter, we look at a special case, a Danish precalculus curriculum that has served both as a parallel and a serial curriculum during the last 50 years.

A Traditional Precalculus Course

An example of a traditional precalculus course is found in the Research and Education Association book precalculus (Woodward, 2010). The book has ten chapters. Chapter one is on sets, numbers, operations and properties. Chapter two is on coordinate geometry. Chapter three is on fundamental algebraic topics as polynomials, factoring and rational expressions and radicals. Chapter four is on solving equations and inequalities. Chapter five is on functions. Chapter six is on geometry. Chapter 7 is on exponents and logarithms. Chapter eight is on conic sections. Chapter nine is on matrices and determinants. Chapter ten is on miscellaneous subjects as combinatorics, binomial distribution, sequences and series and mathematical induction.

Containing hardly any applications or modeling, this book is an ideal survey book in pure mathematics at the level before calculus. Thus, internally it coheres with the levels before and after, but by lacking external coherence it has only little relevance for students not wanting to continue at the calculus level.

An Ideal Precalculus Curriculum

In their publication, the National Council of Teachers of Mathematics writes 'High school mathematics builds on the skills and understandings developed in the lower grades. (p. 19)'

But why that, since in that case high school students will suffer from whatever lack of skills and understandings they have from the lower grades?

Mathe-matics, Meta-math, and Mathe-matism

Furthermore, what kind of mathematics has been taught? Was it 'grounded mathematics' abstracted bottom-up from its outside roots, or 'ungrounded mathematics' or 'meta-math' exemplified top-down from inside abstractions, maybe becoming 'meta-matism' if mixed with 'mathe-matism' (Tarp, 2018) true inside but seldom outside classrooms as when adding without units?

As to the concept 'function', Euler saw it as a bottom-up abstracted name for 'standby calculations' containing specified and unspecified numbers. Later meta-math defined a function top-down as an example of a subset in a set-product where first-component identity implies second-component identity. However, as in the word-language, a function may be seen as a number-language sentence containing a subject, a verb and a predicate allowing its value to be predicted by a calculation (Tarp, 2018).

As to fractions, meta-math defines them as quotient sets in a set-product created by the equivalence relation that (a,b) ~ (c,d) if cross multiplication holds, a*d = b*c. And they become mathe-matism when added without units so that 1/2 + 2/3 = 7/6 despite 1 red of 2 apples and 2 reds of 3 apples gives 3 reds of 5 apples and cannot give 7 reds of 6 apples. In short, outside the classroom, fractions are not numbers, but operators needing numbers to become numbers.

As to solving equations, meta-math sees it as an example of a group concepts applying the associative and commutative law as well as the neutral element and inverse elements thus using five steps to solve the equation 2*u = 6, given that 1 is the neutral element under multiplication, and that ½ is the inverse element to 2.

2*u = 6, so (2*u)*½ = 6*½, so (u*2)*½ = 3, so u*(2*½) = 3, so u*1 = 3, so u = 3.

However the equation 2*u = 6 can also be seen as recounting 6 in 2s using the recount-formula 'T = (T/B)*B' present all over mathematics as the proportionality formula thus solved in one step:

2*u = 6 = (6/2)*2, giving u = 6/2 = 3.

Thus, a lack of skills and understanding may be caused by being taught inside-inside meta-matism instead of grounded outside-inside mathematics.

Using Sociological Imagination to Create a Paradigm Shift

As a social institution, mathematics education might be inspired by sociological imagination, seen by Mills (1959) and Bauman (1990) as the core of sociology. Thus, it might lead to shift in paradigm (Kuhn, 1962) if, as number-language, mathematics would follow the communicative turn that took place in language education in the 1970s (Halliday, 1973; Widdowson, 1978) by prioritizing its connection to the outside world higher than its inside connection to its grammar

So why not try designing a fresh-start precalculus curriculum that begins from scratch to allow students gain a new and fresh understanding of basic mathematics, and of the real power and beauty of mathematics, its ability as a number-language for modeling to provide an inside prediction about an outside situation? Therefore, let us try to design a precalculus curriculum through, and not before its outside use.

Restarting from Scratch with Grounded Outside-Inside Mathematics

Let students see how outside degrees of Many are iconized by inside digits with as many strokes as it represents, five strokes in the 5-icon etc. Let students see that after nine we count by bundling creating icons for the counting operations as well, where division iconizes a broom pushing away the bundles, where multiplication iconizes a lift stacking the bundles into a block and where subtraction iconizes a rope pulling away the block to look for unbundles ones, and where addition iconizes placing blocks next-to or on-top of each other.

Let students see that an outside block of 2 3s becomes an inside calculation 2*3 and vice versa. Let students see the commutative law by turning and a*b block, and see the distributive law by splitting a into c and d, and see the associative law by turning an a*b*c box.

Let students see that both the word- and the number-language use full sentences with a subject, a verb, and an object or predicate, abbreviating ‘the total is 2 3s’ to ‘T = 2*3’

Let students enjoy flexible bundle-numbers where decimals and fractions negative and numbers are created to describe the unbundle ones placed next-to or on-top of the block, thus allowing 5 to be recounted in 3s as T = 5 = 1B2 = 1.2 B = 1 2/3 B = 2B-1.

Let student see, that recounting in other units may be predicted by the recount-formula ‘T = (T/B)*B’ saying ‘From the total T, T/B times, B may be pushed away’. Let students see that where the recount-formula in primary school recounts 6 in 2s as 6 = (6/2)*2 = 3*2, in secondary school the same task is formulated as solving the equation u*2 = 6 as u*2 = 6 = (6/2)*2 giving u = 6/2, thus moving 2 to the opposite side with the opposite calculation sign.

Let students see the power of ‘flexible bundle-numbers’ when the inside multiplication 7*8 = (B-3)*(B-2) = BB-2B-3B+6 = 5B6 = 56 may be illustrated on an outside ten by ten block, thus showing that of course minus times minus must give plus since the 2*3 corner has been subtracted twice.

Let students see that double-counting in two units create per-numbers as 2$ per 3kg, or 2$/3kg. To bridge the units, we simply recount in the per-number: Asking ‘6$ = ?kg’ we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and asking ‘9kg = ?$’ we recount 9 in 3s: T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$.

And, that double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Thus finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Taking 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so 3/4 = 75%.

And, that double-counting sides in a block halved by its diagonal creates trigonometry: a = (a/c)*c = sinA * c; b = (b/c)*c = cosA * c; a = (a/b)*b = tanA * b. With a circle filled from the inside by right triangles, this also allows phi to be found from a formula: circumference/diameter = $\pi \approx$ n*tan(180/n) for n large.

And, how recounting and double-counting physical units create per-numbers and proportionality all over STEM, Science, Technology, Engineering and mathematics: kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter; meter = (meter/second) * second = velocity * second; force = (force/square-meter) * square-meter = pressure * square-meter.

Also, let students see how a letter like x is used as a placeholder for an unspecified number; and how a letter like f is used as a placeholder for an unspecified calculation formula. Writing ‘y = f(x)’ means that the y-number can be found by specifying the x-number in the f-formula. Thus, specifying f(x) = 2 + x will give y = 2+3 = 5 if x = 3, and y = 2+4 = 6 if x = 4.

Algebra and Geometry, Always Together, Never Apart

Let students enjoy the power and beauty of integrating algebra and geometry.

First, let students enjoy seeing that multiplication creates blocks with areas where 3*7 is 3 7s that may be algebraically recounted in tens as 2.1 tens. Or, that may be geometrically transformed to a square u^2 giving the algebraic equation u^2 = 21, creating root as the reverse calculation to power, u = √21. Which may be found approximately by locating the nearest number p below u, here p = 4, so that u^2 = (4+t)^2 = 4^2 + 2*4*t + t^2 = 21.

Neglecting t^2 since t is less than 1, we get 4^2 + 2*4*t = 21, which gives t = $\frac{21 - 4^2}{4*2}$, or t = $\frac{N - p^2}{p*2}$, if p is the nearest number below u, where u^2 = N.

As an approximation, we then get √N ≈ p + t = p + $\frac{N - p^2}{p*2}$; or √N ≈ $\frac{N + p^2}{p*2}$, if p^2 < N < (p+1)^2

Then let students enjoy the power and beauty of predicting where a line geometrically intersects lines or circles or parabolas by algebraically solving two equations with two unknowns, also predicted by a computer software.

A Number Seen as a Multiple Numbering

Let students see the number 456 as what it really is, not three ordered digits obeying a place-value system, but three numberings of bundles-of-bundles, bundles, and unbundled ones as expressed in the number-formula, formally called a polynomial: T = 456 = 4*B^2 + 5*B + 6*1, with B = ten.

Let students see that a number-formula contains the four different ways to unite, called algebra in Arabic: addition, multiplication, repeated multiplication or power, and block-addition or integration. Which is precisely the core of traditional mathematics education, teaching addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school.

Including the units, students see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $\quad log_a(T) = b$

Figure 01. *The 'algebra-square' shows the four ways to unite or split numbers.*

Let students see calculations as predictions, where 2+3 predicts what happens when counting on 3 times from 2; where 2*5 predicts what happens when adding 2$ 5 times; where 1.02^5 predicts what happens when adding 2% 5 times; and where adding the areas 2*3 + 4*5 predicts how to add the per-numbers when asking '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?'

Solving Equations by Reversed Calculation Moving Numbers to Opposite Side

Let students see that

- the subtraction '5-3' as the unknown number u that added with 3 gives 5, u+3 = 5, thus seeing an equation solved when the unknown is isolated by moving numbers 'to opposite sign with opposite calculation sign'; a rule that applies also to the other reversed operations:
- the division u = 5/3 is the number u that multiplied with 3 gives 5, u*3 = 5
- the root u = 3√5 is the factor u that applied 3 times gives 5, u^3 = 5, making root a 'factor-finder'
- the logarithm u = log3(5) is the number u of 3-factors that gives 5, 3^u = 5, making logarithm a 'factor-counter'.

Let students see multiple calculations reduce to single calculations by unhiding 'hidden bracket' where 2+3*4 = 2+(3*4) since with units, 2+3*4 = 2*1+3*4 = 2 1s + 3 4s. This will prevent solving the equation 2+3*u = 14 as 5*u = 14 with u = 14/5, by allowing the hidden bracket to be shown: 2+3*u = 14, so 2+(3*u) = 14, so 3*u = 14-2, so u = (14-2)/3, so u = 4 to be verified by testing: 2+3*u = 2+(3*u) = 2+(3*4) = 2+12 = 14.

Let students enjoy singing a ‘Hymn to Equations’: ‘Equations are the best we know, they’re solved by isolation. But first the bracket must be placed, around multiplication. We change the sign and take away, and only u itself will stay. We just keep on moving, we never give up; so feed us equations, we don’t want to stop!’

Let students build confidence in rephrasing sentences, also called transposing formulas or solving letter equations as, e.g., T = a+b*c, T = a-b*c, T = a+b/c, T = a-b/c, T = (a+b)/c, T = (a-b)/c, etc.; as well as formulas as, e.g., T = a*b^c, T = a/b^c, T = a+b^c, T = (a-b)^c, T = (a*b)^c, T = (a/b)^c, etc.

Let student place two playing cards on-top with one turned a quarter round to observe the creation of two squares and two blocks with the areas u^2, b^2/4, and b/2*u twice if the cards have the lengths u and u+b/2. Which means that (u + b/2)^2 = u^2 + b*u + b^2/4. So, with a quadratic equation saying u^2 + b*u + c = 0, the first two terms disappear by adding and subtracting c:

(u + b/2)^2 = u^2 + b*u + b^2/4 = (u^2 + b*u + c) + b^2/4 – c = 0 + b^2/4 – c = b^2/4 – c.

Now, moving to opposite side with opposite calculation sign, we get the solution

(u + b/2)^2 = b^2/4 – c

u + b/2 = $\pm\sqrt{\text{b^2/4 – c}}$

u = -b/2 $\pm\sqrt{\text{b^2/4 – c}}$

The Change Formulas

Finally, let students enjoy the power and beauty of the number-formula, containing also the formulas for constant change: T = b*x (proportional), T = b*x + c (linear), T = a*x^n (elastic), T = a*n^x (exponential), T = a*x^2 + b*x + c (accelerated).

If not constant, numbers change: constant change roots precalculus, predictable change roots calculus, and unpredictable change roots statistics using confidence intervals to ‘post-dict’ what we cannot ‘pre-dict’.

Combining linear and exponential change by n times depositing a$ to an interest rate r%, we get a saving A$ predicted by a simple formula, A/a = R/r, where the total interest rate R is predicted by the formula 1+R = (1+r)^n. Such a saving may be used to neutralize a debt Do, that in the same period has changed to D = Do*(1+R).

The formula and the proof are both elegant: in a bank, an account contains the amount a/r. A second account receives the interest amount from the first account, r*a/r = a, and its own interest amount, thus containing a saving A that is the total interest amount R*a/r, which gives A/a = R/r.

Precalculus Deals with Constant Change

Looking at the algebra-square, we thus may define the core of a calculus course as adding and splitting into changing per-numbers creating the operations integration and its reverse, differentiation. Likewise, we may define the core of a precalculus course as adding and splitting into constant per-numbers by creating the operation power and its two inverse operations, root and logarithm.

Adding 7% to 300$ means ‘adding’ the change-factor 107% to 300$ changing it to 300*1.07 $. Adding 7% n times thus changes 300$ to T = 300*1.07^n $, leading to the formula for change with a constant change-factor, also called exponential change, T = b*a^n. Reversing the question, this formula entails two equations.

The first equation asks about an unknown change-percent. Thus, we might want to find which percent that added ten times will give a total change-percent 70%, or, formulated with change-factors, what is the change-factor, a, that applied ten times gives the change-factor 1.70. So here the job is ‘factor-finding’ which leads to defining the tenth root of 1.70, i.e., 10√1.70, as predicting the factor, a, that applied 10 times gives 1.70: If a^10 = 1.70 then a = 10√1.70 = 1.054 = 105.4%. This is verified by testing: 1.054^10 = 1.692. Thus, the answer is ‘5.4% is the percent that added ten times will give a total change-percent 70%.’

We notice that 5.4% added ten times gives 54% only, so the 16% remaining to 70% is the effect of compound interest adding 5.4% also to the previous changes.

Here we solve the equation a^10 = 1.70 by moving the exponent to the opposite side with the opposite calculation sign, the tenth root, a = 10√1.70. This resonates with the 'to opposite side with opposite calculation sign' method that also solved the equations a+3 = 7 by a = 7-3, and a*3 = 20 by a = 20/3.

The second equation asks about a time-period. Thus, we might want to find how many times 7% must be added to give 70%, 1.07^n = 1.70. So here the job is factor-counting which leads to defining the logarithm log1.07(1.70) as the number of factors 1.07 that will give a total factor at 1.70: If 1.07^n = 1.70 then n = log1.07(1.70) = 7,84 verified by testing: 1.07^7.84 = 1.700.

We notice that simple addition of 7% ten times gives 70%, but with compound interest it gives a total change-factor 1.07^10 = 1.967, i.e., an additional change at 96.7%-70% = 26.7%, explaining why only 7.84 periods are needed instead of ten.

Here we solve the equation 1.07^n = 1.70 by moving the base to the opposite side with the opposite calculation sign, the base logarithm, n = log1.07(1.70). Again, this resonates with the 'to opposite side with opposite calculation sign' method.

An ideal precalculus curriculum could 'de-model' the constant percent change exponential formula T = b*a^n to outside real-world problems as a capital in a bank, or as a population increasing or radioactive atoms decreasing by a constant change-percent per year.

De-modeling may also lead to situations where the change-elasticity is constant as in science multiplication formulas wanting to relate a percent change in T with a percent change in n.

An example is the area of a square T = s^2 where a 1% change in the side s will give a 2% change in the square, approximately:

With To = s^2, T1 = (s*1.01)^2 = s^2*1.01^2 = s^2* 1.0201 = To*1.0201.

Once mastery of constant change-percent is established, it is possible to look at time series in statistical tables asking, e.g., 'How has the unemployment changed over a ten-year period?' Here two answers present themselves. One describes the average yearly change-number by using the constant change-number formula, T = b+a*n. The other describes the average yearly change-percent by using a constant change-percent formula, T = b*a^n. These average numbers would allow setting up a forecast predicting the yearly numbers in the ten-year period, if the numbers were predictable. However, they are not, so instead of predicting the number with a formula, we might 'post-dict' the numbers using statistics dealing with unpredictable numbers, but still trying to predict a plausible interval by describing the unpredictable random change by nonfictional numbers, median and quartiles, or by fictional numbers, mean and standard deviation.

Calculus Deals with Adding Per-Numbers by Their Areas

Likewise, real-world phenomena as unemployment occur in both time and space, so unemployment may also change in space, e.g., from one region to another. This leads to double-tables sorting the workforce in two categories, region and employment status, also called contingency tables or crosstabs. The unit-numbers lead to percent-numbers within each of the categories. To find the total employment percent, the single percent-numbers do not add, they must be multiplied back to unit-numbers to find the total percent. However, once you multiply you create an area, and adding per-numbers by areas is what calculus is about, thus here introduced in a natural way through double-tables from statistical materials.

An example: in one region 10 persons have 50% unemployment, in another, 90 persons have 5% unemployment. To find the total, the unit-numbers can be added directly to 100 persons, but the percent-numbers must be multiplied back to numbers: 10 persons have 10*0,5 = 5 unemployed; and 90 persons have 90*0,05 = 4.5 unemployed, a total of 5+4.5 unemployed = 9.5 unemployed among 100 persons, i.e., a total of 9.5% unemployment, also called the weighted average. Later, this may be renamed to Bayes formula for conditional probability.

Calculus as adding per-numbers by their areas may also be introduced through mixture problems asking, e.g., '2kg at 3$/kg + 4kg at 5$/kg gives 6kg at how many $/kg?' Here, the unit-numbers 2 and 4 add directly, whereas the per-numbers 3 and 5 must be multiplied to unit-numbers before being added, thus adding by their areas.

Modeling in Precalculus

Furthermore, the entry of graphing calculators allows authentic modeling to be included in a pre-calculus curriculum thus giving a natural introduction to the following calculus curriculum as well.

Regression translates a table into a formula. Here a two data-set table allows modeling with a degree1 polynomial with two algebraic parameters geometrically representing the initial height, and a direction changing the height, called the slope or the gradient. And a three data-set table allows modeling with a degree2 polynomial with three algebraic parameters geometrically representing the initial height, and an initial direction changing the height, as well as the curving away from this direction. And a four data-set table allows modeling with a degree3 polynomial with four algebraic parameters geometrically representing the initial height, and an initial direction changing the height, and an initial curving away from this direction, as well as a counter-curving changing the curving.

With polynomials above degree1, curving means that the direction changes from a number to a formula, and disappears in top- and bottom points, easily located on a graphing calculator, that also finds the area under a graph in order to add piecewise or locally constant per-numbers.

The area A from x = 0 to x = x under a constant per-number graph y = 1 is A = x; and the area A under a constant changing per-number graph y = x is A = ½*x^2. So, it seems natural to assume that the area A under a constant accelerating per-number graph y = x^2 is A = 1/3*x^3, which can be tested on a graphing calculator.

Now, if adding many small area strips y*Δx, the total area A = Σ y*Δx is always changed by the last strip. Consequently, ΔA = y*Δx, or ΔA/Δx = y, or dA/dx = y, or A' = y for very small changes.

Reversing the above calculations then shows that if A = x, then y = A' = x' = 1; and that if A = ½*x^2, then y = A' = (½*x^2)' = x; and that if A = 1/3*x^3, then y = A' = (1/3*x^3)' = x^2.

This suggest that to find the area under the per-number graph y = x^2 over the distance from x = 1 to x = 3, instead of adding small strips we just calculate the change in the area over this distance.

This makes sense since adding many small strips means adding many small changes, which gives just one final change since all the in-between end- and start-values cancel out:

$$\int_1^3 y * dx = \int_1^3 dA = \Delta_1^3 \text{ A} = \Delta_1^3 \left(\frac{1}{3} * x^3\right) = \text{end} - \text{start} = \frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$$

On the calculus course we just leave out the area by renaming it to a 'primitive' or an 'antiderivative' when writing

$$\int_1^3 y * dx = \int_1^3 x^2 * dx = \Delta_1^3 \left(\frac{1}{3} * x^3\right) = \text{end} - \text{start} = \frac{1}{3} * 3^3 - \frac{1}{3} * 1^3 = 9 - \frac{1}{3} \approx 8.67$$

A graphing calculator show that this suggestion holds. So, finding areas under per-number graphs not only allows adding per-numbers, it also gives a grounded and natural introduction to integral and differential calculus where integration precedes differentiation just as additions precedes subtraction.

From the outside, regression allows giving a practical introduction to calculus by analysing a road trip where the per-number speed is measured in five second intervals to respectively 10 m/s, 30 m/s, 20 m/s 40 m/s and 15 m/s. With a five data-set table we can choose to model with a degree4 polynomial found by regression. Within this model we can predict when the driving began and ended, what the speed and the acceleration was after 12 seconds, when the speed was 25m/s, when acceleration and braking took place, what the maximum speed was, and what distance is covered in total and in the different intervals.

Another example of regression is the project 'Population versus food' looking at the Malthusian warning: If population changes in a linear way, and food changes in an exponential way, hunger will eventually occur. The model assumes that the world population in millions changes from 1590 in 1900 to 5300 in 1990 and that food measured in million daily rations changes from 1800 to 4500 in the same period. From this 2- line table regression can produce two formulas: with x counting years after 1850, the population is modeled by Y1= 815*1.013^x and the food by Y2= 300 + 30x. This model predicts hunger to occur 123 years after 1850, i.e., from 1973.

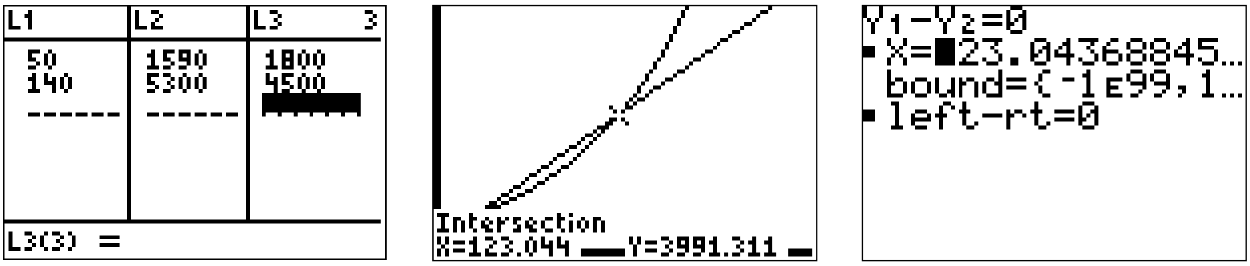

Figure 02. A Malthusian model of population and food levels

An example of an ideal precalculus curriculum is described in a paper called ‘Saving Dropout Ryan With a Ti-82’ (Tarp, 2012). To lower the dropout rate in precalculus classes, a headmaster accepted buying the cheap TI-82 for a class even if the teachers said students weren’t even able to use a TI-30. A compendium called ‘Formula Predict’ (Tarp, 2009) replaced the textbook. A formula’s left-hand side and right-hand side were put on the y-list as Y1 and Y2 and equations were solved by ‘solve Y1-Y2 = 0’. Experiencing meaning and success in a math class, the students put up a speed that allowed including the core of calculus and nine projects.

Besides the two examples above, the compendium also includes projects on how a market price is determined by supply and demand, on how a saving may be used for paying off a debt or for paying out a pension. Likewise, it includes statistics and probability used for handling questionnaires to uncover attitude-difference in different groups, and for testing if a dice is fair or manipulated. Finally, it includes projects on linear programming and zero-sum two-person games, as well as projects about finding the dimensions of a wine box, how to play golf, how to find a ticket price that maximizes a collected fund, all to provide a short practical introduction to calculus.

With the increased educational interest in STEM, modeling also allows including science-problems as, e.g., the transfer of heat taking place when placing an ice cube in water or in a mixture of water and alcohol, or the transfer of energy taking place when connecting an energy source with energy consuming bulbs in series or parallel; as well as technology problems as how to send of a golf ball to hit a desired hole, or when to jump from a swing to maximize the jumping length; as well as engineering problems as how to build a road inclining 5% on a hillside inclining 10%.

Furthermore, precalculus allows students to play with change-equations, later called differential equations since change is calculated as a difference, $\Delta T = T2 – T1$. Using a spreadsheet, it is fun to see the behavior of a total that changes with a constant number or a constant percent, as well as with a decreasing number or a decreasing percent, as well as with half the distance to a maximum value or with a percent decreasing until disappearing at a maximum value. And to see the behavior of a total accelerating with a constant number as in the case of gravity, or with a number proportional to its distance to an equilibrium point as in the case of a spring.

So, by focusing on uniting and splitting into constant per-numbers, the ideal precalculus curriculum has constant change-percent as its core. This will cohere with a previous curriculum on constant change-number or linearity; as well as with the following curriculum on calculus focusing on uniting and splitting into locally constant per-numbers, thus dealing with local linearity. Likewise, such a precalculus curriculum is relevant to the workplace where forecasts based upon assumptions of a constant change-number or change-percent are frequent. This curriculum is also relevant to the students’ daily life as participants in civil society where tables presented in the media are frequent.

Two Curriculum Examples Inspired by an Ideal Precalculus Curriculum

An example of a curriculum inspired by the above outline was tested in a Danish high school around 1980. The curriculum goal was stated as: ‘the students know how to deal with quantities in other school subjects and in their daily life’. The curriculum means included:

1. Quantities. Numbers and Units. Powers of tens. Calculators. Calculating on formulas. Relations among quantities described by tables, curves or formulas, the domain, maximum and minimum, increasing and decreasing. Graph paper, logarithmic paper.

2. Changing quantities. Change measured in number and percent. Calculating total change. Change with a constant change-number. Change with a constant change-percent. Logarithms.

3. Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

4. Trigonometry. Calculation on right-angled triangles.

5. Free hours. Approximately 20 hours will elaborate on one of the above topics or to work with an area in which the subject is used, in collaboration with one or more other subjects.

Later, around year 2000, another version was designed but not tested. The curriculum goal was stated as: 'the students develop their number-language so they can participate in social practices involving quantitative descriptions of change and shape.' The curriculum means included

1. Numbers and calculations. Quantities and qualities. Number-language, word-language, meta-language. Unit-numbers and per-numbers, and how to calculate their totals. Equations as predicting statements. Forwards and reverse calculations.

2. Change calculations. Change measuring change with change-number and change-percent and index-number. Calculation rules for the change of a sum, a product and a ratio.

3. Constant change. Change with a constant change-number. Change with a constant change-percent. Change with both.

4. Unpredictable change. Fractals, mean and deviation, 95% confidence interval. Binomial distribution approximated by a normal distribution.

Distributed quantities. Number and percent. Graphical descriptors. Average. Skewness of distributions. Probability, conditional probability. Sampling, mean and deviation, normal distribution, sample uncertainty, normal test, X ^ 2 test.

5. Trigonometry. Dividing and measuring earth. Calculation the sides and angles in a triangle.

06. Precalculus in the Danish parallel high school, a case study

In the post-war era, the Organization for Economic Co-operation and Development (OECD) called for increasing the population knowledge level, e.g., by offering a second chance to take a high school degree giving entrance to tertiary education. In Denmark in 1966, this resulted in creating a two-year education called 'Higher preparation exam' as a parallel to the traditional high school. Two levels of two-years mathematics courses were included, a basic precalculus course for those who did not choose the calculus course.

The 1966 Curriculum

The precalculus curriculum came from leaving out small parts of the calculus curriculum, thus being an example of a reduced curriculum.

The goal of the calculus course stated it should 'supply students with knowledge about basic mathematical thinking and about applications in other subject areas, thus providing them with prerequisites for carrying through tertiary education needing mathematics.'

The goal of the precalculus course was reduced to 'supplying students with an impression of mathematical thinking and method and to mediate mathematical knowledge useful also to other subject areas.'

So, where the calculus curriculum has to cohere and be relevant to tertiary education needing mathematics, the precalculus course is a parallel curriculum meant to be relevant to the students themselves and to other high school subjects.

The content of the precalculus curriculum had five sections.

The first section contained basic concepts from set theory as sets, subsets, complementary set, union, intersection, product, difference. The function concept. Mapping into an on a different set, one-to one mapping, inverse mapping (inverse function), composite mappings. The calculus curriculum added nothing here.

Section two contained concepts from abstract algebra: Composition rules. The associative law. The commutative law. Neutral element. Inverse element. The group concept with examples. Rules for operations on real numbers. Numeric value. Here the calculus curriculum added the distributive law, the concept of a ring and a field, the ring of whole numbers as well as quotient classes. The calculus curriculum added nothing here.

Section three contained equations and inequalities. Examples on open statements in one or two variables. Equations and inequalities of degree one and two with one unknown. Equations and inequalities with the unknown placed inside a square root or a numeric sign. Simple examples of Equations and inequalities of degree one and two with two unknowns. Graphical illustration. The calculus curriculum added nothing here.

Section four contained basic functions. The linear function in one variable. A piecewise linear function. The second-degree polynomial. The logarithm function with base ten, the logarithmic scale, the calculator stick, the use of logarithm tables. Trigonometric functions, tables with functions values. Calculations on a right-angled triangle using trigonometric functions. Here the calculus curriculum added rational functions in one variable, exponential functions, and the addition formulas and logarithmic formulas in trigonometry.

Section five contained combinatorics. The multiplication principle. Permutations and combinations. Here the calculus curriculum added probability theory, probability field, and examples of probability based upon combinatorics.

Finally, the calculus curriculum added a section about calculus.

The new set-based mathematics coming into education around 1960 inspired the 1966 precalculus curriculum thus cohering with the university mathematics at that time, but it was not especially relevant to the students. Many had difficulties understanding it and they often complained about seeing no reason for learning it or why it was taught.

In my own class, I presented it as a legal game where we were educating us to become lawyers that could convince a jury that we were using lawful methods to solving equations in one of two different methods by referring to the relevant paragraphs in the law. The first method was the traditional one used at that time way by moving numbers to the opposite side with opposite calculation sign, now legitimized by the theorem that in a group the equation a*u = b has as a solution a^-1*b. The second method was a new way with many small steps where, for each step, you have to refer to laws for associativity, and commutativity etc.; and, where a group contained exactly the paragraphs needed to use this method. Once seen that way, the students found it easy but boring. However, they accepted since they needed the exam to go on, and we typically finished the course in half time allowing time for writing a script for a movie to be presented at the annual gala party.

So, all in all, the 1963 curriculum was coherent with the next step, calculus, and with the university math view at that time, set-based; but it was mostly irrelevant to the students.

The 1974 Curriculum

The student rebellion in 1968 asked for relevance in education, which led to a second precalculus in 1974 revision. Here the goal was stated as ‘giving the students a mathematical knowledge that could be useful to other subjects and to their daily life, as well as an impression of mathematical methods thinking’. Now the curriculum structure was changed from a parallel one to a serial one where all students took the precalculus course and some chose to continue with the calculus course afterwards just specifying in its curriculum what was needed to be added.

The 1974 precalculus curriculum now had four sections.

The first section contained concepts from set theory and logic and combinatorics. Set, subset; solution set to an open statement, examples on solving simple equations and inequalities in one variable; the multiplication principle, combinations.

Section two contained the function concepts: Domain, function value, range; injective function; monotony intervals; inverse function, composite function.

Section three contained special functions; graphical illustration. A linear function, a piecewise linear function, an exponential function; examples of functions defined by tables; coordinate system, logarithmic paper.

Section four contained descriptive statistics. Observations described by numbers; frequency and their distribution and cumulated distribution; graphical illustration; statistical descriptors.

Section five described probability and statistics. A random experiment, outcome space, probability function, probability field; sampling; binomial distribution; binomial testing with zero hypothesis, critical set, significance level, single and double-sided test, failure of first degree.

Section six was called ‘Free lessons’. 20m lessons are to be used for studying details in one of the above sections, or together with one or more other school subjects to work with an area applying mathematics.

The second 1974 curriculum thus maintains a basis of set-theory but leaves out the abstract algebra. As to functions, it replaces the second-degree polynomial with the exponential function. Here trigonometry is excluded to be included in the calculus curriculum.

The combinatorics section is to great extent replaced by descriptive statics.

Finally, the section has been added with quite detailed probability theory and testing theory within statistics.

All in all, the coherence with the university set-based mathematics has been softened by leaving out abstract algebra and second-degree polynomial. Instead of introducing a first-degree polynomial together with a second-degree polynomial, the former now is introduced as a linear function together with the exponential function allowing modelling outside change with both a constant change-number and a constant change-percent. This makes the curriculum more relevant to the students individually as well as to other high school subjects as required by the goal statement.

The quite detailed section on testing theory was supposed to make the curriculum more relevant to students but the degree of detail make it fail to do so by drowning in quite abstract concepts.

The 1990 Curriculum

As the years passed on it was observed that the free hours were used on trigonometry, and on savings and instalments, the first cohering with the following calculus course, the latter highly relevant to many students, and at the same time combining linear and exponential change, the core of the curriculum. This let to designing an alternative curriculum around 1990 to choose instead of the standard curriculum if wanted.

The 1990 curriculum did not change the goal but included the following subjects

1) Numbers, integers, rational and real numbers together with their calculation rules. Number sets. Calculations with power and root.

2) Calculations including percent and interest rates: Average percent, index number, weighed average. Simple and compound interest, saving and installments.

3) Geometry and trigonometry. Similar triangles. Right triangles. Calculations on sides and angles.

4) Functions. The function concept, domain, functional values, range, monotony. Various ways to define a function. Elementary functions as linear, piecewise linear and exponential growth and decay. Coordinate system. Examples of simple equations and inequalities including the functions mentioned above.

5) Probability and statistics. A stochastic experiment. Discrete stochastic variables, probability distribution, mean value, binomial distribution, observation sets described graphically, representation by statistical descriptors, examples of a normal distribution, normal distribution paper.

6) Calculation aids. Pocket calculator, formulas, tables, semi logarithmic paper, normal distribution paper.

The 2005 Curriculum

Then a major reform of the Danish upper secondary high school was planned for 2005. As to precalculus, it was inspired by the entry of graphing calculators and computer assisted systems allowing regression to transform tables into formulas, thus allowing realistic modeling to be included.

Now the goal defined the competences students should acquire:

The students can

• handle simple formulas and translate between symbolic and natural language and use symbolic language to solve simple problems with a mathematical content.

- apply simple statistical models for describing a given data set, pose questions based upon the model and sense what kind of answers are to be expected and knows how to formulate conclusion in a clear language.
- apply relations between variables to model a given data set, can make forecasts, and can reflect on them and their domain of relevance
- describe geometrical models and solve geometrical problems
- produce simple mathematical reasoning
- demonstrate knowledge about mathematical methods, applications of mathematics, and examples of cooperation between mathematics and other sciences, as well as its cultural and historical development
- apply information technology for solving mathematical problems

The means include

The hierarchy of operations, solving equations graphically and with simple analytical methods, calculating percent and interest rates, absolute and relative change

Formulas describing direct and inverse proportionality as well as linear, exponential ad power relations between variables

Simple statistical methods for handling data sets, graphical representation of statistical materials, simple statistical descriptors

Ratios in similar triangles and trigonometry used for calculations in arbitrary triangles.

xy-plot of data sets together with characteristics of linear, exponential and power relations, the use of regression.

Additional activities for 25 lessons are examples of mathematical reasoning and proofs, modeling authentic data sets, examples of historical mathematics.

The 2017 Curriculum

Then in 2017 e new reform was made to inspire more students to continue with the calculus level by moving some subjects to the precalculus level:

- interpreting the slope of a tangent as a growth rate in a mathematical model
- combinatorics, basic probability theory and symmetrical probability space
- the function concept and characteristics of linear, exponential and power functions and their graphs
- graphical handling of a quadratic function, and the logarithm functions and their characteristics
- graphical determination of a tangent, and monotony intervals, as well as finding extrema values in a closed interval
- prime characteristics at mathematical models and simple modelling using the functions above alone or in combination.

Relevance and Coherence

The 1966 had internal coherence with the previous and following curriculum, but with the emphasis on abstract algebra, there was little external coherence. It was indirectly relevant to students wanting later to take a calculus course but only little relevant to the daily life of students

The 1972 curriculum took the consequence and changed from a parallel curriculum to a serial curriculum so that it had internal coherence to the calculus curriculum, and by replacing quadratics with exponential functions, it obtained an external relevance to change calculations with a constant change-number or a constant change-percent. Also, including a considerable amount of probability gave coherence to eternal testing situations, however these were not part of student daily life, so they didn’t add to the relevance for students. However, including the free lessons allowed the students to choose areas that they found relevant, in this case interest rates and saving and installment calculations as well as trigonometry.

The 1990 curriculum was inspired by this and re-included trigonometry and interest rates while at the same time reducing probability a little.

The 2005 reform was informed by the occurrence of competence concept as well as the advances in calculation technology. Her the function concept was replaced by variables to make it cohere more with external applications in science and economics and daily life. Now the probability was gone, so this curriculum showed coherence and relevance to external appliers and to the student's daily life as well for other school subjects. It was close to the ideal precalculus curriculum.

The 2017 reform was inspired by the wish to motive more to continue with a calculus course, so part of this was moved down to the precalculus level, making the two levels cohere better, however the things imported had little relevance to the students' daily life.

A Refugee Camp Curriculum

The name 'refugee camp curriculum' is a metaphor for a situation where mathematics is taught from the beginning and with simple manipulatives. Thus, it is also a proposal for a curriculum for early childhood education, for adult education, for educating immigrants and for learning mathematics outside institutionalized education. It considers mathematics a number-language parallel to our word-language, both describing the outside world in full sentences, typically containing a subject and a verb and a predicate. The task of the number-language is to describe the natural fact Many in space and time, first by counting and recounting and double-counting to transform outside examples of Many to inside sentences about the total; then by adding to unite (or split) inside totals in different ways depending on their units and on them being constant or changing. This allows designing a curriculum for all students inspired by Tarp (2018) that focuses on proportionality, solving equations and calculus from the beginning, since proportionality occurs when recounting in a different unit, equations occur when recounting from tens to icons, and calculus occurs when adding block-numbers next-to and when adding per-numbers coming from double-counting in two units.

Talking about 'refugee camp mathematics' thus allows locating a setting where children do not have access to normal education, thus raising the question 'What kind and how much mathematics can children learn outside normal education especially when residing outside normal housing conditions and without access to traditional leaning materials?'. This motivates another question 'How much mathematics can be learned as 'finger-math' using the examples of Many coming from the body as fingers, arms, toes and legs?'

So the goal of 'refugee camp mathematics' is to learn core mathematics through 'Finger-math' disclosing how much math comes from counting the fingers.

Focus 01. Digits as Icons with as Many Outside Sticks and Inside Strokes as They Present

Activity 01. With outside things (sticks, cars, dolls, animals), many ones are rearranged into one many-icon with as many things as it represents. Inside, we write the icon with as many strokes as it represents. Observe that the actual digits from 1 to 9 are icons with as many strokes as they represent if written less sloppy. A discovery glass showing nothing is an icon for zero. When counting by bundling in tens, ten become '1 Bundle, 0 unbundled' or 1B0 or just 10, thus needing no icon since after nine, a double-counting takes place of bundles and unbundled.

Focus 02. Counting Ten Fingers in Various Ways

Activity 01. Double-count ten fingers in bundles of 5s and in singles

● Outside, lift the finger to be counted; inside say '0 bundle 1, 0B2, 0B3, 0B4, 0B5 or 1B0. Then continue with saying '1B1, …, 1B5 or 2B'. ● Outside, look at the fingers not yet counted; inside say '1 bundle less4, 1B-3, 1B-2, 1B-1, 1B or 1B0. Then continue with saying '2B-4, …, 2B or 2B0'. ● Outside, show the fingers as ten ones. ● Outside, show ten fingers as 1 5s and 5 1s; inside say 'The total is 1Bundle5 5s' and write 'T = 1B5 5s'. ● Outside, show ten fingers as 2 5s; inside say 'The total is 2Bundle0 5s' and write 'T = 2B0 5s'.

Activity 02. Double-count ten fingers in bundles of tens and in singles

● Outside, lift the finger to be counted; inside say '0 bundle 1, 0B2, 0B3, …, 0B9, 0Bten, or 1B0'. ● Outside, look at the fingers not yet counted; inside say '1 bundle less9, 1B-8, …, 1B-2, 1B-1, 1B or 1B0.

Activity 03. Counting ten fingers in bundles of 4s using 'flexible bundle-numbers'.

● Outside, show the fingers as ten ones, then as one tens. ● Outside, show ten fingers as 1 4s and 6 1s; inside say 'The total is 1Bundle6 4s, an overload' and write 'T = 1B6 4s' ● Outside, show ten fingers as 2 4s and 2 1s; inside say 'The total is 2Bundle2 4s, a standard form' and write 'T = 2B2 4s'. ● Outside, show ten fingers as 3 4s less 2; inside say 'The total is 3Bundle, less2, 4s, an underload' and write 'T = 3B-2 4s'.

Activity 04. Counting ten fingers in bundles of 3s using 'flexible bundle-numbers'.

● Outside, show ten fingers as 1 3s and 7 1s; inside say 'The total is 1Bundle7 3s, an overload' and write 'T = 1B7 3s'. ● Outside, show ten fingers as 2 3s and 4 1s; inside say 'The total is 2Bundle4 3s, an overload' and write 'T = 2B4 3s'. ● Outside, show ten fingers as 3 3s and 1 1s; inside say 'The total is 3Bundle1 3s, a standard form' and write 'T = 3B1 3s'. ● Outside, show ten fingers as 4 3s less 2; inside say 'The total is 4Bundle, less2, 3s, an underload' and write 'T = 4B-2 3s'.

Activity 05. Counting ten fingers in bundles of 3s, now also using bundles of bundles.

● Outside, show ten fingers as 3 3s (a bundle of bundles) and 1 1s; inside say 'The total is 1BundleBundle1 3s' and write 'T = 1BB1 3s'. Now, inside say 'The total is 1BundleBundle 0 Bundle 1 3s' and write 'T = 1BB 0B 1 3s'. Now, inside say 'The total is 1BundleBundle 1 Bundle, less2, 3s' and write 'T = 1BB 1B -2 3s'.

Focus 03. Counting Ten Sticks in Various Ways

The same as Focus 02, but now with sticks instead of fingers.

Focus 04. Counting Ten Cubes in Various Ways

The same as Focus 02, but now with cubes, e.g., centi-cubes or Lego Bricks, instead of fingers. When possible, transform multiple bundles into 1 block, e.g., 2 4s = 1 2x4 block; inside say 'The total is 1 2x4 block' and write 'T = 2B0 4s.'

Focus 05. Counting a Dozen Finger-parts in Various Ways

Except for the thumps, our fingers all have three parts. So, four fingers have three parts four times, i.e., a total of T = 4 3s = 1 dozen finger-parts.

Focus 05 is the same as focus 02, but now with a dozen finger-parts instead of ten fingers.

Focus 06. Counting a Dozen Sticks in Various Ways

Focus 06 is the same as focus 03, but now with a dozen sticks instead of ten.

Focus 07. Counting a Dozen Cubes in Various Ways

Focus 07 is the same as focus 04, but now with a dozen cubes instead of ten.

Focus 08. Counting Numbers with Underloads and Overloads.

Activity 01. Totals counted in tens may also be recounted in under- or overloads.

● Inside, rewrite T = 23 as T = 2B3 tens, then as 1B13 tens, then as 3B-7tens. ● Try other two-digit numbers as well. ● Inside, rewrite T = 234 as T = 2BB3B4 tens, then as T = 2BB 2B14, then as T = 2BB 4B-6. Now rewrite T = 234 as T = 23B4, then as 22B14, then as 24B-6. Now rewrite T = 234 as T = 3BB-7B4, then as 3BB-6B-6. ● Try other three-digit numbers as well.

Focus 09. Operations as Icons Showing Pushing, Lifting and Pulling

Activity 01. Transform the three outside counting operations (push, lift and pull) into three inside operation-icons: division, multiplication and subtraction.

● Outside, place five sticks as 5 1s. ● Outside, push away 2s with a hand or a sheet; inside say 'The total 5 is counted in 2s by pushing away 2s with a broom iconized as an uphill stroke' and write 'T = 5 = 5/2 2s'. ● Outside, rearrange the 2 2s into 1 2x2 block by lifting up the bundles into a stack; inside say 'The bundles are stacked into a 2x2 block by lifting up bundles iconized as a lift' and write 'T = 2 2s = 2x2'. ● Outside, pull away the 2x2 block to locate unbundled 1s; inside say 'The 2x2 block is pulled away, iconized as a rope' and write 'T = 5 – 2x2 = 1'.

Five counted in 2s:

I I I I I (push away 2s) II II I (lift to stack) $\begin{matrix}\text{II}\\ \text{II}\end{matrix}$ I (pull to find unbundles ones) $\begin{matrix}\text{II}\\ \text{II}\end{matrix}$ I

Focus 10. The Inside Recount-Formula T = (T/B)xB Predicts Outside Bundlecounting Results

Activity 01. Use a calculator to predict a bundle-counting result by a recount-formula T = (T/B)xB, saying 'from T, T/B times, B is pushed away', thus using a full number-language sentence with a subject, a verb and a predicate.

● Outside, place five cubes as 5 1s. ● Outside, push away 2s with a 'broom'; inside say 'Asked '5/2', a calculator answers '2.some', meaning that 2 times we can push ways bundles of 2s. ● Outside, stack the 2s into one 2x2 stack by lifting; inside say 'We lift the 2 bundles into one 2x2 stack, and we write T = 2 2s = 2x2 ● Outside, we locate the unbundled by, from 5 pulling away the 2x2 block; inside we say 'Asked '5-2x2', a calculator answers '1'. We write T = 2B1 2s and say 'The recount-formula predicts that 5 recounts in 2s as T = 2B1 2s, which is tested by recounting five sticks manually outside.'

Activity 02. The same as activity 01, but now with 4 3s counted in 5s, 4s and 3s.

Focus 11. Discovering Decimals, Fractions and Negative Numbers.

Activity 01. When bundle-counting a total, the unbundled can be placed next-to or on-top.

● Outside, chose seven cubes to be counted in 3s. ● Outside, push away 3s to be lifted into a 2x3 stack to be pulled away to locate one unbundled single. Inside use the recount-formula to predict the result, and say 'seven ones recounts as 2B1 3s' and write T = 2B1 3s. ● Outside, place the single next-to the stack. Inside say 'Placed next-to the stack the single becomes a decimal-fraction '.1' so now seven recounts as 2.1 3s' and write T = 2.1 3s. ● Outside, place the single on-top of the stack. Inside say 'Placed on-top of the stack the single becomes a fraction-part 1 of 3, so now seven recounts as 2 1/3 3s' and write T = 2 1/3 3s. Now, inside say 'Placed on-top of the stack the single becomes a full bundle less 2, so now seven recounts as 3.-2 3s' and write T = 3.-2 3s. Finally, inside say 'With 3 3s as 1 bundle-bundle of 3s, seven recounts as 1BB-2 3s.'

Activity 02. The same as activity 01, but now with first 2 then 3 until a dozen counted in 3s.

Activity 03. The same as activity 01, but now with first 2 then 3 until a dozen counted in 4s.

Activity 04. The same as activity 01, but now with first 2 then 3 until a dozen counted in 5s.

Focus 12. Recount in a New Unit to Change Units, Predicted by the Recount-Formula

Activity 01. When bundle-counting, all numbers have units that may be changed into a new unit by recounting predicted by the recount-formula.

● Outside, chose 3 4s to be recounted in 5s. ● Outside, rearrange the block in 5s to find the answer T = 3 4s = 2B2 5s. Inside use the recount-formula to predict the result, and say 'three fours recounts as 2B2 5s' and write T = 3 4s = 2B2 5s = 3B-3 5s = 2 2/5 5s. Repeat with other examples as, e.g., 4 5s recounted in 6s.

Focus 13. Recount from Tens to Icons

Activity 01. A total counted in tens may be recounted in icons, traditionally called division.

● Outside, chose 29 or 2B9 tens to be recounted in 8s. ● Outside, rearrange the block in 8s to find the answer T = 29 = 3B5 8s and notice that a block that decreases its base must increase its height to keep the total the same. Inside use the recount-formula to predict the result, and say 'With the recount-formula, a calculator predicts that 2 bundle 9 tens recounts as 3B5 8s' and write T = 29 = 2B9 tens = 3B 5 8s = 4B-3 8s = 3 5/8 8s. Repeat with other examples as, e.g., 27 recounted in 6s.

* Now, inside reformulate the outside question 'T = 29 = ? 8s' as an equation using the letter u for the unknown number, u*8 = 24, to be solved by recounting 24 in 8s: T = u*8 = 24 = (24/8)*8, so that the unknown number is u = 24/8, attained by moving 8 to the opposite side with the opposite sign. Use an outside ten-by-ten abacus to see that when a block decreases its base from ten to 8, it must increase its height from 2.4 to 3. Repeat with other examples as, e.g., 17 = ? 3s.

Focus 14. Recount from Icons to Tens

Activity 01. Oops, without a ten-button, a calculator cannot use the recount-formula to predict the answer if asking 'T = 3 7s = ? tens'. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. Use an outside ten-by-ten abacus to see that when a block increases its base from 7 to ten, it must decrease its height from 3 to 2.1.

Activity 02. Use 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Consequently 'less less 1' means adding 1.

Focus 15. Double-Counting in Two Physical Units

Activity 01. We observe that double-counting in two physical units creates 'per-numbers' as, e.g., 2$ per 3kg, or 2$/3kg. To bridge units, we recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg; and T = 9kg = (9/3)*3kg = (9/3)*2$ = 6$. Repeat with other examples as, e.g., 4$ per 5days.

Focus 16. Double-Counting in the Same Unit Creates Fractions

Activity 01. Double-counting in the same unit creates fractions and percent as 4$/5$ = 4/5, or 40$/100$ = 40/100 = 4%. Finding 40% of 20$ means finding 40$ per 100$ so we re-count 20 in 100s: T = 20$ = (20/100)*100$ giving (20/100)*40$ = 8$. Finding 3$ per 4$ in percent, we recount 100 in 4s, that many times we get 3$: T = 100$ = (100/4)*4$ giving (100/4)*3$ = 75$ per 100$, so ¾ = 75%. We observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Repeat with other examples as, e.g., 2$/5$.

Focus 17. Mutually Double-Counting the Sides in a Block Halved by its Diagonal

Activity 01. Recount sides in a block halved by its diagonal? Here, in a block with base b, height a, and diagonal c, recounting creates the per-numbers: a = (a/c)*c = sinA*c; b = (b/c)*c = cosA*c; a = (a/b)*b = tanA*b. Use these formulas to predict the sides in a half-block with base 6 and angle 30 degrees. Use these formulas to predict the angles and side in a half-block with base 6 and height 4.

Focus 18. Adding Next-to

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added next-to as 8s?' Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Since 3*5 is an area, adding next-to in 8s means adding areas, called integral calculus. Asking a calculator, the two answers, '2.some' and '5', predict the result as 2B5 8s.

Focus 19. Reversed Adding Next-to

Activity 01. With T1 = 2 3s and T2 adding next-to as T = 4 7s, what is T2?' Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Asking '3 5s and how many 3s total 2B6 8s?', using sticks will give the answer 2B1 3s. Adding or integrating two stacks next-to each other means multiplying before adding. Reversing integration then means subtracting before dividing, as shown in the gradient formula

$y' = \Delta y/t = (y2 - y1)/t$.

Focus 20. Adding On-top

Activity 01. With T1 = 2 3s and T2 = 3 5s, what is T1+T2 when added on-top as 3s; and as 5s?' Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Adding on-top in 5s, '3 5s + 2 3s = ? 5s?', re-counting must make the units the same. Asking a calculator, the two answers, '4.some' and '1', predict the result as 4B1 5s.

Focus 21. Reversed Adding On-top

Activity 01. With T1 = 2 3s and T2 as some 5s adding to T = 4 5s, what is T2?' Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting. Reversed addition is called backward calculation or solving equations.

Focus 22. Adding Tens

Activity 01. With T1 = 23 and T2 = 48, what is T1+T2 id added as tens?' Recounting removes an overload: T1+T2 = 23 + 48 = 2B3 + 4B8 = 6B11 = 7B1 = 71.

Focus 23. Subtracting Tens

Activity 01. 'If T1 = 23 and T2 add to T = 71, what is T2?' Here, recounting removes an underload: T2 = 71 – 23 = 7B1 – 2B3 = 5B-2 = 4B8 = 48; or T2 = 956 – 487 = 9BB5B6 – 4BB8B7 = 5BB-3B-1 = 4BB7B-1 = 4BB6B9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, so -(-1) = +1.

Focus 24. Multiplying Tens

Activity 01. 'What is 7 43s recounted in tens?' Here the learning opportunity is that also multiplication may create overloads: T = 7*43 = 7*4B3 = 28B21 = 30B1 = 301; or 27*43 = 2B7*4B3 =8BB+6B+28B+21 =8BB34B21 =8BB36B1 = 11BB6B1 = 1161, solved geometrically in a 2x2 block.

Focus 25. Dividing Tens

Activity 01. 'What is 348 recounted in 6s?' Here the learning opportunity is that recounting a total with overload often eases division: T = 348 /6 = 34B8 /6 = 30B48 /6 = 5B8 = 58; and T = 349 /6 = 34B9 /6 = 30B49 /6 = (30B48 +1) /6 = 58 + 1/6.

Focus 26. Adding Per-Numbers

Activity 01. '2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?' Here we see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

Activity 02. Two groups of voters have a different positive attitude to a proposal. How to find the total positive attitude?

● Asking '20 voters with 30% positive + 60 voters with 10% positive = 80 voters with ? positive.' Here we see that the unit-numbers 20 and 40 add directly whereas the per-numbers 30% and 10% add by areas since they must first transform to unit-numbers by multiplication, creating the areas.

Focus 27. Subtracting Per-Numbers

Activity 01. '2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?' Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

Focus 28. Adding Differences

Activity 01. Adding many numbers is time-consuming, but not if the numbers are changes, then the sum is simply calculated as the change from the start to the end-number.

● Write down ten numbers vertically. The first number must be 3 and the last 5, the rest can be any numbers between 1 and 9. In the next column write down the individual changes 'end-start'. In the third column add up the individual changes along the way. Try to explain why the result must be 5-3 regardless of the in-between numbers.

● Draw a square with side n. Let n have a small positive change t. Show that the square will change with two next blocks when disregarding the small txt square. This shows that the change in an n*n square is 2*n*t, so if we want to add arears under a y = 2*n curve we must add very many small areas y*t = 2*n*t. However, since each may be written as a change in a square, we just have to find the change of the square from the start-point to the end-point. That is how integral calculus works.

Focus 29. Finding Common Units

Activity 01. 'Only add with like units, so how add T = 4ab^2 + 6abc?'. Here units come from factorizing: T = 2*2*a*b*b + 2*3*a*b*c = 2*b*(2*a*b).

Focus 30. Finding Square Roots

Activity 01. A 7x7 square can be recounted in tens as 4.9 tens. The inverse question is how to transform a 6x7 block into a square, or in other words, to find the square root of 4.2 tens. A quick way to approach a relevant number is to first find two consecutive numbers, p and p+1, that squared are too low and too high. Then the an approximate value for the square root may be calculated by using that if p^2 < N < (p+1)^2, then √N ≈ $\frac{\mathrm{N+\ p^\wedge 2}}{p*2}$.

Summing Up

A curriculum for a refugee camp assumes that the learners have only the knowledge they acquire outside traditional education. The same is the case for street children living outside traditional homes; and for nomadic people always moving around.

However, a refugee camp curriculum might also be applied in a traditional school setting allowing the children to keep on to the two-dimensional block numbers they bring to school allowing them to learn core mathematics as proportionality, equations, functions and calculus in the first grade, thus not needing parallel curricula later on.

So, the need for parallel curricula after grade 9 is not there by nature, but by choice. It is the result of disrespecting the mastery of many children bring to school and force them to adopts numbers as names along a number line, and force them to add numbers that are given to them without allowing them to find them themselves by counting, recounting and double-counting.

Do We Really Need Parallel Curricula?

Why do we need different curricula for different groups of students? Why can't all students have the same curriculum? After all, the word-language does not need different curricula for different groups, so why does the number-language?

Both languages have two levels, a language level describing the 'outside' world, and a grammar level describing the 'inside' language. In the word-language, the language level is for all students and includes many examples of real-world descriptions, both fact and fiction. Whereas grammar level details are reserved for special students.

Could it be the same with the number-language, teaching the language level to all students including many examples of fact and fiction? And reserving grammar level details to special students?

Before 1970, schools taught language as an example of its grammar (Chomsky, 1965). Then a reaction emerged in the so-called 'communicative turn' in language education. In his book 'Explorations in the function of language' Halliday (1973, p. 7) defines a functional approach to language in the following way:

A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a 'communicative approach to the teaching of language (p. ix)' allowing more students to learn a less correct language to be used for communication about outside things and actions. Thus, in language teaching the communicative turn changed language from being inside grammar-based to being outside world-based. However, this version never made it to the sister-language of the word-language, the number-language. So, maybe it is time to ask how mathematics will look like if

- instead of being taught as a grammar, it is taught as a number-language communicating about outside things and actions.
- instead of learned before its use, it is learned through its use
- instead of learning about numbers, students learn how to number and enumerate, and how to communicate in full sentences with an outside subject, a linking verb, and an inside predicate as in the word- language.

After all, the word language seems more voluminous with its many letters, words and sentence rules. In contrast, a pocket calculator shows that the number language contains ten digits together with a minor number of operations and an equal sign.

And, where letters are arbitrary signs, digits are close to being icons for the number they represent, 5 strokes in the 5-icon etc. (Tarp, 2018)

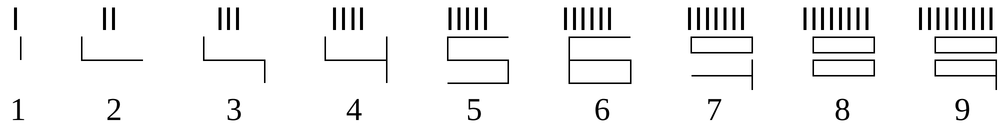

Figure 03. Digits as icons with as many sticks as they represent.

Furthermore, also the operations are icons describing how we total by counting unbundled, bundles, bundles of bundles etc. Here division iconizes pushing away bundles to be stacked, iconized by a multiplication lift, again to be pulled away, iconized by a subtraction rope, to identify unbundled singles that are placed next-to the stack iconized by an addition cross, or by a decimal point; or on-top iconized by a fraction or a negative number.

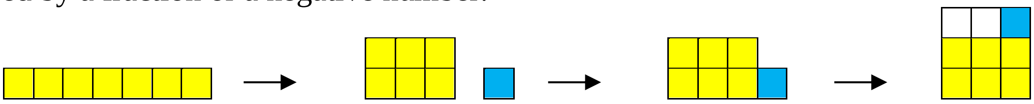

Figure 04. Seven counted as 2 3s & 1 or 2*B*1 3s, and 2.1 3s, and as 2 1/3 3s or 3.-2 3s.

The operations allow predicting counting by a recount-sentence or formula ‘T = (T/B)*B’ saying that ‘from T, T/B times, B can be taken away’, making natural numbers as bundle- or block numbers as, e.g., T = 3B2 4s or T = 3*4+2.

And, using proportionality to change the unit when two blocks need the same unit to be added on-top, or next-to in a combined unit called integral calculus.

So, it seems as if early childhood education may introduce core mathematics as proportionality, solving equations, and integral calculus, thus leaving footnotes to later classes who can also benefit from the quantitative literature having the same two genres as the qualitative literature, fact and fiction.

Thus, there is indeed an opportunity to design a core curriculum in mathematics for all students without splitting it up in tracks. But, only if the word- and the number-language are taught and learned in the same way by describing outside things and actions in words and in numbers coming from counting and adding.

So, why not introduce a paradigm shift by teaching the number-language and the word-language in the same way through its use, and not before, thus allowing both languages being taught in the space between the inside language and the outside world.

Why keep on teaching the number-language in the space between the language and its meta-language or grammar, which makes the number-language more abstract, leaving many educational challenges unsolved despite close to half a century of mathematics education research.

Why not begin teaching children how to number, and stop teaching children about numbers and operation to be explained and learned before they can be applied to the outside world.

Why not accept and develop children’s already existing ‘many-sense’, instead of teaching them the eight different aspects of what is called ‘number-sense’ described by Sayers and Andrews (2015) that after reviewing research in the Whole Number Arithmetic domain created a framework called foundational number sense (FoNS) with eight categories: number recognition, systematic counting, awareness of the relationship between number and quantity, quantity discrimination, an understanding of different representations of number, estimation, simple arithmetic competence and awareness of number patterns.

And, why not simply let children talk about counting and adding constant and changing unit-numbers and per-numbers using full sentences with a subject, a verb, and a predicate; instead of teaching them the eight different components of what is called ‘mathematical competencies’ (Niss, 2003), thus reducing their numbers from eight to two: count and add (Tarp, 2002)?

So maybe we should go back to the mother Humboldt university in Berlin and reflect on Karl Marx thesis 11 written on the staircase: ‘Die Philosophen haben die Welt nur verschieden interpretiert; es kömmt drauf an, sie zu verändern.’ (The philosophers have only interpreted the world, in various ways. The point, however, is to change it.)

Conclusion

Let us return to the dream of the National Council of Teachers of Mathematics, to 'provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.' Consequently, 'everyone needs to be able to use mathematics in his or her personal life, in the workplace and in further study. All students deserve an opportunity to understand the power and beauty of mathematics.' Furthermore, let us also accept what the council write about numbers: 'Number pervades all areas of mathematics.'

So let us look for a curriculum that allows the students to understand and use and numbers, and see how far such a curriculum can carry all students without splitting into parallel tracks.

Now, what does it mean to understand a number like 456?

Is the ability to say that the three digits obey a place-value system where, from right to left, the first digits is ones, then tens, then hundred, then thousands, then, oops no-name unless we use the Chinese name wan, then no-name, then million, then no-name, then no-name, then billions or milliards, etc. Names and lack of names that give little meaning to children where only few understand why ten has its own name but not its own icon but has two digits as 10.

On the other hand, is it the ability to understand that of course ten becomes 10 since it is short for '1 bundle and no singles'? And, that it would have been 20 had we counted in bundles of 5s instead as they do on an eastern abacus, where the two digits 10 then would be used for the bundle size 5.

And that ten is just another word for bundle, and hundred for bundle-bundle, i.e., 2 times bundling; and thousand for bundle-bundle-bundle, i.e., or bundling 3 times, etc. where we never end in a situation with no name. Isn't it both power and beauty to transform an unorganized total into a repeated bundling with the ability that only the decimal point moves if you change the number of bundling, T = 32.1 tens = 3.21 tentens, which is not the case with romans bundling where 3 tens is 6 fives. The romans didn't stick to bundling bundles since they bundled in both fives and tens and fifties but not in 5 5s, i.e., in 25s. Power and beauty comes from bundle bundles only.

Consequently, to understand the number 456 is to see it, not as one number, but as three numberings of a total that has been bundled 0 times, bundled 1 times, bundled 2 times, etc. And to read the total as 4 bundled 2 times and 5 bundled once and 6 not bundled, or as 4 bundle-bundles and 5 bundles and 6 unbundles singles. And to write the total as T = 4BB 5B 6. And to allow the same total to be recounted with an underload as T = 4BB 6B -4, or with an overload as T = 45B 6 = 4BB 56; or as T = 45B -4 if combining overload and underload.

This understanding allows an existing unorganized total become a number-language sentence connecting the outside subject T to an inside calculation, T = 4*B^2 + 5*B^1 + 6*B^0.

Which again is an example, or specification, of an unspecified number-formula or polynomial T = a*x^2 + 5*x + 6.

The power and beauty of the number-formula is manifold. It shows four ways to unite: power, multiplication, addition and next-to block addition also called integration. By including the units, we realize that there are only four types of numbers in the world as shown in the algebra-square above, constant and changing unit-numbers and per-numbers, united by precisely these four ways: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers.

Furthermore, we observe that splitting a total into parts will reverse uniting parts into a total, meaning that all uniting operations have reverse operations: subtraction and division split a total into changing and constant unit-numbers; and differentiation and root & logarithm split a total in changing and constant per-numbers. This makes root a factor-finder, and logarithm a factor-counter, and differentiation a finder of per-numbers.

And, if we use the word 'equation' for the need to split instead of unite, we observe that solving an equation means isolating the unknown by moving numbers to the opposite side with opposite calculation sign. Furthermore, using variables instead of digits we observe that the number-formula contains the different formulas for constant change as shown above.

As to a non-constant change, there are two kinds. Predictable change roots calculus as shown by the algebra-square; and unpredictable change roots statistics to instead 'post-dict' numbers by a mean and a deviation to be used by probability to pre-dict a confidence interval for unpredictable numbers.

Thus the 'power and beauty' of mathematics resides in the number-formula, as does the ability 'to use mathematics in students' personal life, in the workplace and in further study'. So, designing a curriculum based upon the number-formula will 'provide our students with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.'

Furthermore, a number-formula based curriculum need not split into parallel curricula until after calculus, i.e., until after secondary education.

So, one number-language curriculum for all is possible, as it is for the word-language. Thus, it is possible to allow all students to learn about the four ways to unite and the five ways to split a total.

The most effective way to design a curriculum for all students is to adopt the curriculum designed refugee camp from the beginning since it accepts and develops the number-language children bring to school. Presenting figures and operations as icons, it bridges outside existence with inside essence. All four uniting methods occur in grade one when counting and recounting in different units, and when adding totals next-to and on-top. It respects the natural order of operations by letting division precede multiplication and subtraction, thus postponing addition until after counting, recounting and double-counting have taken place. It introduces the core recounting-formula expressing proportionality when changing units from the beginning, which allows a calculator to predict inside an outside recounting result. By connecting outside blocks with inside bundle-writing, geometry and algebra are introduced as Siamese twins never to part. Using flexible bundle-numbers connects inside decimals, fractions and negative numbers to unbundled leftovers placed next-to or on-top the outside block. It introduces solving equations when recounting from tens to icons. It introduces per-numbers and fractions when double counting in units that may be the same or different. And, it introduces trigonometry before geometry when double-counting sides in a block halved by its diagonal.

Another option is to integrate calculus in a precalculus course by presenting integral calculus before differential calculus, which makes sense since until now inverse operations are always taught after the operation, subtraction after addition etc. Consequently, differential calculus should wait until after it has been motivated by integral calculus that is motivated by adding changing per-numbers in trade and physics, and by adding percent in statistical double-tables.

In their publication, the National Council of Teachers of Mathematics writes 'High school mathematics builds on the skills and understandings developed in the lower grades. (p. 19)' If this has to be like that then high school education will suffer from lack of student skills and misunderstandings; and often teachers say that precalculus is the hardest course to teach because of a poor student knowledge background.

So, we have to ask: Can we design a fresh-start curriculum for high school that integrates precalculus and calculus? And indeed, it is possible to go back to the power and beauty of the number-formula as described above, and build a curriculum based upon the algebra-square. It gives an overview of the four kinds of numbers that exist in the outside world, and how to unite or split them. It shows a direct way to solve equations based upon the definitions of the reverse operations: move to opposite side with opposite calculation sign.

Furthermore, it provides 2x2 guiding questions: how to unite or split into constant per-numbers, as needed outside when facing change with a constant change-factor? And how to unite or split into changing per-numbers that are piecewise or locally constant, as needed outside when describing, e.g., the motion with a changing velocity of a falling object.

As a reverse operation, differential calculus is a quick way to deliver the change-formula that solve the integration problem of adding the many area-strips coming from transforming locally constant per-numbers to unit-numbers by multiplication. Also, by providing change-formulas, differential calculus can extend the formulas for constant change coming from the number-formula.

An additional extension comes from combining constant change-number and change-percent to one of the most beautiful formulas in mathematics that is too often ignored, the saving-formula, A/a = R/r, a formula that is highly applicable in individual and social financial decisions.

Working with constant and changing change also raises the question what to do about unpredictable change, which leads directly into statistics and probability.

So, designing and implementing a fresh-start integrated precalculus and calculus curriculum will allow the National Council of Teachers of Mathematics to have their dream come through, so that in the future high schools can provide all students 'with the best mathematics education possible, one that enables them to fulfil personal ambitions and career goals.'

As a number-language, mathematics is placed between its outside roots and its inside meta-language or grammar. So, institutionalized education must make a choice: should the number-language be learned through its grammar before being applied to outside descriptions; or should it as the word-language be learned through its use to describe the outside world? I short, shall mathematics education tach about numbers and operations and postpone applications till after this has been taught? Or shall mathematics education teach how to number and how to use operations to predict a numbering result thus teaching rooting instead of applications?

Choosing the first 'inside-inside' option means connecting mathematics to its grammar as a 'meta-math' defining concepts 'from above' as top-down examples from abstractions instead of 'from below' as bottom-up abstractions from examples. This is illustrated by the function concept that can be defined from above as an example of a set-product relation where first component identity implies second-component identity, or from below as a common name for 'stand-by' calculations containing unspecified numbers.

Choosing the inside-inside 'mathematics-as-metamatics' option means teaching about numbers and operations before applying them. Here numbers never carry units but become names on a number-line; here numbers are added by counting on; and the other operations are presented as inside means to inside tasks: multiplication as repeated addition, power as repeated multiplication, subtraction as inverse addition, and division as inverse multiplication. Here fractions are numbers instead of operators needing numbers to become numbers. Here adding numbers and fractions without units leads to 'mathe-matism', true inside classrooms where 2+3 is 5 unconditionally, but seldom outside classrooms where counterexamples exist as, e.g., 2weeks + 3days is 17days or 2 3/7 weeks. Here geometry and algebra occur independently and before trigonometry. Here primary and lower secondary school focus on addition, subtraction, multiplication and division with power and root present as squaring and square roots, thus leaving general roots and logarithm and trigonometry to the different tracks in upper secondary school where differential calculus is introduced before integral calculus, if at all.

Choosing the inside-outside 'mathematics-as-manymath' option means to teach digits as icons with as many strokes as they represent. And to also teach operations as icons, rooted in the counting process where division wipes away bundles to be stacked by multiplication, again to be removed by subtraction to identify unbundled singles. This will allow giving a final description of the total using a full sentence with a subject, a verb and a predicate predicted by the recount-formula T = (T/B)*B, e.g., T = 2Bundle 1 3s = 2.1 3s = 2 1/3 3s thus including decimal numbers and fractions in a natural number. Here a double description of Many as an outside block and an inside bundle-number allows outside geometry and inside algebra to be united from the start. Once counted, totals can be recounted. First in the same unit to create overloads and underloads introducing negative numbers. Then between icon- and ten-bundles introducing the multiplication table and solving equations. Then double-counting in two units creates per-numbers becoming fractions with like units. Finally, recounting the sides in a block halved by its diagonal will root trigonometry before geometry, that integrated with algebra can predict intersection points. Then follows addition and reversed addition in its two versions, on-top or next-to. On-top addition calls for recounting the totals in the same unit, thus rooting proportionality. And next-to addition means adding blocks as areas, thus rooting integral calculus. Reversed addition roots equations and differential calculus. Per-numbers are added as operators including the units, thus rooting integral calculus, later defined as adding locally constant per-numbers. Thus, this option means that the core of mathematics is learned in primary school allowing ample of time in secondary school to enjoy the number-language literature by examining existing models or producing models yourself. And it means that only one curriculum is needed for all students as in the word-language. Furthermore, the root and use of calculus to add changing per-numbers is easily introduced at the precalculus level when adding ingredients with different per-numbers and when adding categories in statistics with different percent.

And, the fact that the difficulty by adding many numbers disappears when the numbers can be written as change-numbers since adding up any number of small changes total just one change from the start- to the end-number. Which of course motivates differential calculus.

Consequently, there is no need for a parallel curriculum to the traditional, since everybody can learn calculus in a communicative way. Of course, one additional optional course may be given to look at all the theoretical footnotes.

To offer a completely different kind of mathematics as graph theory and game theory and voting theory risks depriving the students of the understanding that mathematics is put in the world as a number-language that use operations to predict the result of counting, recounting and double-counting. A language that only needs four operations to unite parts into a total, and only five operations to split a total into parts.

Without calculus in the final high school curriculum, students may not understand how to add per-numbers and might add them as unit-numbers instead of as areas; and this will close many 'doors to productive futures' as the US National Council of Teachers of Mathematics talks about.

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A New Curriculum - But for Which of the 2x2 Kinds of Mathematics Education

An Essay on Observations and Reflections at the ICMI Study 24 Curriculum Conference

As part of institutionalized education, mathematics needs a curriculum describing goals and means. There are however two kinds of mathematics: essence-based and existence-based; and there are two kinds of education: multi-year mandatory lines and half-year self-chosen blocks. Thus, there are 2x2 kinds of mathematics education to choose from before deciding on a specific curriculum; and if changing, shall the curriculum stay within the actual kind or change to a different kind? The absence of federal states from the conference suggests that curricula should change from national multi-year macro-curricula to local half-year micro-curricula; and maybe change to existence-based mathematics.

Coherence and Relevance in the School Mathematics Curriculum

The International Commission on Mathematical Instruction, ICMI, has named its 24th study 'School mathematics Curriculum Reforms: Challenges, Changes and Opportunities'. Its discussion document has 5 themes among which theme B, 'Analysing school mathematics curriculum for coherence and relevance' says that 'All mathematics curricula set out the goals expected to be achieved in learning through the teaching of mathematics; and embed particular values, which may be explicit or implicit.'

So, to analyze we use the verb 'cohere' and the predicate 'relevant' when asking: 'to what does this curriculum cohere and to what is it relevant?' As to the meaning of the words 'cohere' and 'relevant' we may ask dictionaries.

The Oxford Dictionaries (en.oxforddictionaries.com) writes that 'to cohere' means 'to form a unified whole' with its origin coming from Latin 'cohaerere', from co- 'together' + haerere 'to stick'; and that 'relevant' means being 'closely connected or appropriate to what is being done or considered.' We see, that where 'cohere' relates to states, 'relevant' relates to changes or processes taking place.

The Merriam-Webster dictionary (merriam-webster.com) seems to agree upon these meanings. It writes that 'to cohere' means 'to hold together firmly as parts of the same mass'. As to synonyms for cohere, it lists: 'accord, agree, answer, check, chord, coincide, comport, conform, consist, correspond, dovetail, fit, go, harmonize, jibe, rhyme (also rime), sort, square, tally.' And as to antonyms, it lists: 'differ (from), disagree (with).'

In the same dictionary, the word 'relevant' means 'having significant and demonstrable bearing on the matter at hand'. As to synonyms for relevant, it lists: 'applicable, apposite, apropos, germane, material, pertinent, pointed, relative.' And as to antonyms, it lists: 'extraneous, immaterial, impertinent, inapplicable, inapposite, irrelative, irrelevant, pointless.'

If we accept the verb 'apply' as having a meaning close to the predicate 'relevant', we can rephrase the above analysis question using verbs only: 'to what does this curriculum cohere and apply?'

Seeing education metaphorically as bridging an individual start level for skills and knowledge to a common end level described by goals and values, we may now give a first definition of an ideal curriculum: 'To apply to a learning process as relevant and useable, a curriculum coheres to the start and end levels for skills and knowledge.'

This definition involves obvious choices, and surprising choices also if actualizing the ancient Greek sophist warning against choice masked as nature. The five main curriculum choices are:

How to make the bridge cohere with the individual start levels in a class?

How to make the end level cohere to goals and values expressed by the society?

How to make the end level cohere to goals and values expressed by the learners?

How to make the bridge cohere to previous and following bridges?

How to make the bridge (more) passable?
Then specific choices for mathematics education follow these general choices.

Goals and Values Expressed by the Society

In her plenary address about the 'OECD 2030 Learning Framework', Taguma shared a vision:

> The members of the OECD Education 2030 Working Group are committed to helping every learner develop as a whole person, fulfil his or her potential and help shape a shared future built on the well-being of individuals, communities and the planet. (..) And in an era characterised by a new explosion of scientific knowledge and a growing array of complex societal problems, it is appropriate that curricula should continue to evolve, perhaps in radical ways (p. 10).

Talking about learner agency, Taguma said:

> Future-ready students need to exercise agency, in their own education and throughout life. (..) To help enable agency, educators must not only recognise learners' individuality, (..) Two factors, in particular, help learners enable agency. The first is a personalised learning environment that supports and motivates each student to nurture his or her passions, make connections between different learning experiences and opportunities, and design their own learning projects and processes in collaboration with others. The second is building a solid foundation: literacy and numeracy remain crucial. (p. 11)

By emphasizing learner's individual potentials, personalised learning environment and own learning projects and processes, Taguma seems to indicate that flexible half-year micro-curricula may cohere better with learners' future needs than rigid multi-year macro-curricula. As to specifics, numeracy is mentioned as one of the two parts of a solid foundation helping learners enable agency.

Different Kinds of Numeracy

Numeracy, however, is not that well defined. Oxford Dictionaries and Merriam-Webster agree on saying 'ability to understand and work with numbers'; whereas the private organization National Numeracy (nationalnumeracy.org.uk) says 'By numeracy we mean the ability to use mathematics in everyday life'.

The wish to show usage was also part of the Kilpatrick address, describing mathematics as bipolar:

> I want to stress that bipolarity because I think that's an important quality of the school curriculum and every teacher and every country has to deal with: how much attention do we give to the purer side of mathematics. The New Math thought that it should be entire but that didn't work really as well as people thought. So how much attention do we give to the pure part of mathematics and how much to the applications and how much do we engage together. Because it turns out if the applications are well-chosen and can be understood by the children then that helps them move toward the purer parts of the field. (p. 20)

After discussing some problems caused by applications in the curriculum, Kilpatrick concludes:

> If we stick with pure mathematics, with no application, what students cannot see, 'when will I ever use this?', it's not surprising that they don't go onto take more mathematics. So, I think for self-preservation, mathematicians and mathematics educators should work on the question of: how do we orchestrate the curriculum so that applications play a good role? There is even is even a problem with the word applications, because it implies first you do the mathematics, then you apply it. And actually, it can go the other way. (p. 22)

So, discussing what came first, the hen or the egg, applications or mathematics, makes it problematic to define numeracy as the ability to apply mathematics since it gives mathematics a primacy and a monopoly as a prerequisite for numeracy. At the plenary afterwards discussion, I suggested using the word 're-rooting' instead of 'applying' to indicate that from the beginning, mathematics was rooted in the outside world as shown by the original meanings of geometry and algebra: ‘to measure earth’ in Greek and ‘to reunite’ in Arabic.

Mathematics Through History

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy, seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the ‘quadrivium’ recommended by Plato as a general curriculum together with ‘trivium’ consisting of grammar, logic and rhetoric.

Today, mathematics typically is a common name for geometry and algebra both indicating the outside existence rooting them: in Greek, geometry means to measure earth, and in Arabic, algebra means to reunite.

Although born as existence-based, mathematics never developed as a natural science about Many in time and space since both Greek and Roman numbers both missed the advantage of only bundling singles and bundles, as did the Hindu-Arabic numbers coming to Europe in the Renaissance. Instead, the Greeks developed an axiomatic deductive Euclidean geometry well suited to practice logic.

So, for centuries, mathematics was a science about essence, ‘essence-based math’. Which was even intensified when around 1900, the abstract concept ‘set’ was spreading all over mathematics, and finally reached the education level as the ‘New Math’, recommended by Bruner (1962) arguing that a subject should have the same form at the educational and scientific level.

Here a wish for exactness and unity created a essence-based ‘setcentric’ (Derrida, 1991) ‘meta-math’ as a collection of ‘well-proven’ statements about ‘well-defined’ concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox ‘this sentence is false’, being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

Setcentrism thus changed classical grounded existence-based ‘many-math’ into a self-referring essence-based ‘meta-matism’, a mixture of meta-math and ‘mathe-matism’ true inside but seldom outside a classroom where adding numbers without units as ‘1 + 2 IS 3’ meets counter-examples as, e.g., 1week + 2days is 9days.

The introduction of the setcentric New Math created different reactions. Inside the United States it was quickly abandoned with a ‘back-to-basics’ movement. Outside it was implemented at teacher education, and in schools where it gradually softened. However, it never retook its original form or name, despite, in contrast to ‘mathematics’, ‘reckon’ is an action-word better suited to the general aim of education, to teach humans to master the outside world through appropriate actions.

Different Kinds of Mathematics

So, a curriculum must choose between an existence-based and essence-based mathematics as illustrated by an example from McCallum’s plenary talk. After noting that ‘a particularly knotty area in mathematics curriculum is the progression from fractions to ratios to proportional relationships’ (p. 4), McCallum asked the audience: ‘What is the difference between 5/3 and 5÷3’.

Essence-based mathematics will say that 5/3 is a number on the number-line reached by taking 5 steps of the length coming from dividing the unit in 3 parts; and that 5÷3 means 5 items shared between 3.

In its modern version, essence-based mathematics will say that 5/3 is a rational number defined as an equivalence class in the product set of integers, created by the equivalence relation (a,b) eq. (c,d) if cross-multiplication holds, axd = bxc; and, with 1/3 as the inverse element to 3 under multiplication, 5÷3 should be written as 5x1/3, i.e., the as the solution to the equation 3xu = 5, found by applying and thus legitimizing abstract algebra and group theory; thus finally saying goodbye to the Renaissance use of a vertical line to separate addends from subtrahends, and a horizontal line to separate multipliers from divisors.

Existence-based mathematics (Tarp, 2018) sees essence-based setcentric mathematics as meta-matism hiding the original Greek meaning of mathematics as a science about Many. In this 'Many-math', 5/3 is a per-number coming from double-counting in different units (5$/3kg), becoming a fraction with like units (5$/3$ = 5/3). Here per-numbers and fractions are not numbers but operators needing a number to become a number (5/3 of 3 is 5, 5/3 of 6 is 10); and 5÷3 means 5 counted in 3s occurring in the 'recount-formula' recounting a total *T* in bundles of 3s as *T* = (*T*/3)x3, saying 'from *T*, *T*/3 times, 3 can be taken away'. This gives flexible numbers: *T* = 5 = 1B2 3s = 1.2 3s = 1 2/3 3s = 2B-1 3s = 2.-1 3s, introduced in grade one where bundle-counting and re-counting in another unit precedes adding, and where recounting from tens to icons, *T* = 2.4 tens = ? 6s, leads to the equation *T* = *u*x6 = 24 = (24/6)x6 solved by recounting. In existence-based mathematics, per-numbers, fractions, ratios and proportionality melt together since double-counting in two units gives per-numbers as ratios, becoming fractions with like units. And here proportionality means changing units using the recount-formula to recount in the per-number: With 5$/3kg, 'how much for 20$?' is found by re-counting 20 in 5s: *T* = 20$ = (20/5)x5$ = (20/5)x3kg = 12 kg. Likewise, if asking 'how much for 15 kg?'

Different Kinds of Education

As to education, from secondary school there is a choice between multi-year lines and half-year blocks. At the discussion after the Kilpatrick plenary session, I made a comment about these two educational systems, which mas a lady from the United States say I was misinforming since in the states Calculus required a full year block. Together with other comments in the break, this made me realize that internationally there is little awareness of these two different kinds of educational systems. So here is another example of what the Greek sophists warned against, choice masked as nature.

Typically, unitary states have one multi-year curriculum for primary and lower secondary school, followed by parallel multi-year curricula for upper secondary and tertiary education. Whereas, by definition, federal states have parallel curricula, or even half-year curricula from secondary school as in the United States.

At the conference, the almost total absence of federal states as Germany, Canada, the United States and Russia seems to indicate that the problems reside with multi-year national curricula, becoming rigid traditions difficult to change. While federal competition or half-year blocks creates flexibility through an opportunity to try out different curricula.

Moreover, as a social institution involving individual constraint, education calls for sociological perspectives. Seeing the Enlightenment Century as rooting education, it is interesting to study its forms in its two Enlightenment republics, the North American from 1776 and the French from 1789. In North America, education enlightens children about their outside world, and enlightens teenagers about their inside individual talent, uncovered and developed through self-chosen half-year blocks with teachers teaching only one subject in their own classrooms.

To protect its republic against its German speaking neighbors, France created elite schools, criticized today for exerting hidden patronization. Bourdieu thus calls education 'symbolic violence', and Foucault points out that a school is really a 'pris-pital' mixing power techniques from a prison and a hospital, thus raising two ethical issues: On which ethical ground do we force children and teenagers to return to the same room, hour after hour, day after day, week after week, month after month for several years? On which ethical ground do we force children and teenagers to be cured from self-referring diagnoses as, e.g., the purpose of mathematics education is to cure mathematics ignorance? Issues, the first Enlightenment republic avoids by offering teenagers self-chosen half-year blocks; and by teaching, not mathematics, but algebra and geometry referring to the outside world by their original meanings.

Different Kinds of Competences

As to competences, new to many curricula, there are at least three alternatives to choose among. The European Union recommends two basic competences, acquiring and applying, when saying that 'Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge.'

At the conference two alternative notions of competences were presented. In his plenary address, Niss recommended a matrix with 8 competences per concept (p. 73). In his paper, Tarp (pp. 317-324) acknowledged that 8 competences may be needed if the goal of mathematics education is to learn present essence-based setcentric university mathematics; but if the goal is to learn to master Many with existence-based mathematics, then only two competences are needed: counting and adding, rooting a twin curriculum teaching counting, recounting in different units and double-counting before adding.

Making the Learning Road More Passable

Once a curriculum is chosen, the next question is to make its bridge between the start and end levels for skills and knowledge more passable. Here didactics and pedagogy come in; didactics as the captain choosing the way from the start to the end, typically presented as a textbook leaving it to pedagogy, the lieutenants, to take the learners through the different stages.

The didactical choices must answer general questions from grand theory. Thus, philosophy will ask: shall the curriculum follow the existentialist recommendation, that existence precedes essence? And psychology will ask: shall the curriculum follow Vygotsky mediating institutionalized essence, or Piaget arranging learning meetings with what exists in the outside world? And sociology will ask: on which ethical grounds are children and teenagers retained to be cured by institutionalized education?

Colonizing or Decolonizing Curricula

The conference contained two plenary panels, the first with contributors from France, China, The Philippines and Denmark, almost all from the northern hemisphere; the second with contributors from Chile, Australia, Lebanon and South Africa, almost all from the southern hemisphere. Where the first panel talked more about solutions, the second panel talked more about problems.

In the first panel, France and Denmark represented some of the world's most centralized states with war-time educational systems dating back to the Napoleon era, which in France created elite-schools to protect the young republic from the Germans, and in Germany created the Humboldt Bildung schools to end the French occupation by mediating nationalism, and to sort out the population elite for jobs as civil servants in the new central administration; both just replacing the blood-nobility with a knowledge-nobility as noted by Bourdieu. The Bildung system latter spread to most of Europe.

Not surprisingly, both countries see university mathematics as the goal of mathematics education ('mathematics is what mathematicians do'), despite the obvious self-reference avoided by instead formulating the goal as, e.g., learning numerical competence, mastery of Many or number-language. Seeing mathematics as the goal, makes mathematics education an example of a goal displacement (Bauman) where a monopoly transforms a means into a goal. A monopoly that makes setcentric mathematics an example of what Habermas and Derrida would call a 'center-periphery colonization', to be decentered and decolonized by deconstruction.

Artigue from France thus advocated an anthropological theory of the didactic, ATD, (p. 43-44), with a 'didactic transposition process' containing four parts: scholarly knowledge (institutions producing and using the knowledge), knowledge to be taught (educational system, 'noosphere'), taught knowledge (classroom), and learned available knowledge (community of study).

The theory of didactic transposition developed in the early 1980s to overcome the limitation of the prevalent vision at the time, seeing in the development of taught knowledge a simple process of elementarization of scholarly knowledge (Chevallard 1985). Beyond the well-known succession offered by this theory, which goes from the reference knowledge to the knowledge actually taught in classrooms (..), ecological concepts such as those of niche, habitat and trophic chain (Artaud 1997) are also essential in it.

Niss from Denmark described the Danish 'KOM Project' leading to eight mathematical competencies per mathematical topic (pp. 71-72).

> The KOM Project took its point of departure in the need for creating and adopting a general conceptualisation of mathematics that goes across and beyond educational levels and institutions. (..) We therefore decided to base our work on an attempt to define and characterise mathematical competence in an overarching sense that would pertain to and make sense in any mathematical context. Focusing - as a consequence of this approach - first and foremost on the *enactment* of mathematics means attributing, at first, a secondary role to mathematical content. We then came up with the following definition of mathematical competence: Possessing *mathematical competence* – mastering mathematics – is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve actual or potential mathematical challenges of any kind. In order to identify and characterise the fundamental constituents in mathematical competence, we introduced the notion of mathematical competencies: A *mathematical competency* is an individual's capability and readiness to act appropriately, and in a knowledge-based manner, in situations and contexts that involve a certain kind of mathematical challenge.

Some of the consequences by being colonized by setcentrism was described in the second panel.

In his paper 'School Mathematics Reform in South Africa: A Curriculum for All and by All?' Volmink from South Africa Volmink writes (pp. 106-107):

> At the same time the educational measurement industry both locally and internationally has, with its narrow focus, taken the attention away from the things that matter and has led to a traditional approach of raising the knowledge level. South Africa performs very poorly on the TIMSS study. In the 2015 study South Africa was ranked 38th out of 39 countries at Grade 9 level for mathematics and 47th out of 48 countries for Grade 5 level numeracy. Also, in the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ), South Africa was placed 9th out of the 15 countries participating in Mathematics and Science – and these are countries which spend less on education and are not as wealthy as we are. South Africa has now developed its own Annual National Assessment (ANA) tests for Grades 3, 6 and 9. In the ANA of 2011 Grade 3 learners scored an average of 35% for literacy and 28% for numeracy while Grade 6 learners averaged 28% for literacy and 30% for numeracy.

After thanking for the opportunity to participate in a cooperative effort on the search of better education for boys, girls and young people around the world, Oteiza from Chile talked about 'The Gap Factor' creating social and economic differences. A slide with the distribution of raw scores at PSU mathematics by type of school roughly showed that out of 80 points, the median scores were 40 and 20 for private and public schools respectively. In his paper, Oteiza writes (pp. 81-83):

> Results, in national tests, show that students attending public schools, close to de 85% of school population, are not fulfilling those standards. How does mathematical school curriculum contribute to this gap? How might mathematical curriculum be a factor in the reduction of these differences? (..) There is tremendous and extremely valuable talent diversity. Can we justify the existence of only one curriculum and only one way to evaluate it through standardized tests? (..) There is a fundamental role played by researchers, and research and development centers and institutions. (..) How do the questions that originate in the classroom reach a research center or a graduate program? '*Publish or perish*' has led our researchers to publish in prestigious international journals, but, are the problems and local questions addressed by those publications?'

The Gap Factor is also addressed in a paper by Hoyos from Mexico (pp. 258-259):

> The PISA 2009 had 6 performance levels (from level 1 to level 6). In the global mathematics scale, level 6 is the highest and level 1 is the lowest. (..) It is to notice that, in PISA 2009, 21.8% of Mexican students do not reach level 1, and, in PISA 2015, the percentage of the same level is a little bit higher

(25.6%). In other words, the percentage of Mexican students that in PISA 2009 are below level 2 (, i.e., attaining the level 1 or zero) was 51%, and this percentage is 57% in PISA 2015, evidencing then an increment of Mexican students in the poor levels of performance. According to the INEE, students at levels 1 or cero are susceptible to experiment serious difficulties in using mathematics and benefiting from new educational opportunities throughout its life. Therefore, the challenges of an adequate educational attention to this population are huge, even more if it is also considered that approximately another fourth of the total Mexican population (33.3 million) are children under 15 years of age, a population in priority of attention'.

As a comment to Volminks remark 'Another reason for its lack of efficacy was the sense of scepticism and even distrust about the notion of People's Mathematics as a poor substitute for the 'real mathematics'' (p. 104), and inspired by the sociological Centre–Periphery Model for colonizing, by post-colonial studies, and by Habermas' notion of rationalization and colonization of the lifeworld by the instrumental rationality of bureaucracies, I formulated the following question in the afterwards discussion: 'As former colonies you might ask: Has colonizing stopped, or is it still taking place? Is there an outside central mathematics that is still colonizing the mind? What happens to what could be called local math, street math, ethno-math or the child's own math?'

Conclusion and Recommendations

Designing a curriculum for mathematics education involves several choices. First pre-, present and existence-based mathematics together with multi-year lines and half-year blocks constitute 3x2 different kinds of mathematics education. Combined with three different ways of seeing competences, this offers a total of 18 different ways in which to perform mathematics education at each of the three educational levels, primary and secondary and tertiary, which may even be divided into parts.

Once chosen, institutional rigidity may hinder curriculum changes. So, to avoid the ethical issues of forcing cures from self-referring diagnoses upon children and teenagers in need of guidance instead of cures, the absence of participants from federal states might be taken as an advice to replace the national multi-year macro-curriculum with regional half-year micro-curricula. At the same time, adopting the post version of setcentric mathematics will make the curriculum coherent with the mastery of Many children bring to school, and relevant to learning the quantitative competence and numeracy desired by society.

And, as Derrida says in an essay called 'Ellipsis' in 'Writing and Difference': 'Why would one mourn for the center? Is not the center, the absence of play and difference, another name for death?'

Postscript: Many-Math, an Existence-based Mathematics for All

As existence-based mathematics, Many-math, can provide numeracy for all by celebrating the simplicity of mathematics occurring when recounting the ten fingers in bundles of 3s:

T = ten = 1B7 3s = 2B4 3s = 3B1 3s = 4B-2 3s.

Or, if seeing 3 bundles of 3s as 1 bundle of bundles,

T = ten = 1BB0B1 3s = 1*B^2 + 0*B + 1 3s, or

T = ten = 1BB1B-2 3s = 1*B^2 + 1*B – 2 3s.

This number-formula shows that a number is really a multi-numbering of singles, bundles, bundles of bundles etc. represented geometrically by parallel block-numbers with units. Also, it shows the four ways to unite: on-top addition, multiplication, power and next-to addition, also called integration. Which are precisely the four ways to unite constant and changing unit- and per-numbers numbers into totals as seen by including the units; each with a reverse way to split totals.

Thus, addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the

Algebra Square', also showing that equations are solved by moving to the opposite side with opposite signs.

Operations **unite**/ *split* Totals in	Changing	Constant
Unit-numbers m, s, kg, $	**T = a + n** $T - n = a$	**T = a*n** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	**T = ∫ f dx** $\frac{dT}{dx} = f$	**T =** a^b $\sqrt[b]{T} = a$ $log_a(T) = b$

Figure 01. *The 'algebra-square' shows the four ways to unite or split numbers.*

An unbundled single can be placed on-top of the block counted in 3s as T = 1 = 1/3 3s, or next-to the block as a block of its own written as T = 1 = .1 3s Writing T = ten = 3 1/3 3s = 3.1 3s = 4.-2 3s thus introduces fractions and decimals and negative numbers together with counting.

The importance of bundling as the unit is emphasized by counting: 1, 2, 3, 4, 5, 6 or bundle less 4, 7 or B-3, 8 or B-2, 9 or B-1, ten or 1 bundle naught, 1B1, …, 1B5, 2B-4, 2B-3, 2B-2, 2B-1, 2B naught.

This resonates with 'Viking-counting': 1, 2, 3, 4, hand, and1, and2, and3, less2, less1, half, 1left, 2left. Here '1left' and '2left' still exist as 'eleven' and 'twelve', and 'half' when saying 'half-tree', 'half-four' and 'half-five' instead of 50, 70 and 90 in Danish, counting in scores; as did Lincoln in his Gettysburg address: 'Four scores and seven years ago …'

Counting means wiping away bundles (called division iconized as a broom) to be stacked (called multiplication iconized as a lift) to be removed to find unbundled singles (called subtraction iconized as a horizontal trace). Thus, counting means postponing adding and introducing the operations in the opposite order of the tradition, and with new meanings: 7/3 means 7 counted in 3s, 2x3 means stacking 3s 2 times. Addition has two forms, on-top needing recounting to make the units like, and next-to adding areas, i.e., integral calculus. Reversed they create equations and differential calculus.

The recount-formula T = (T/B)*B appears all over mathematics and science as proportionality or linearity formula:

- Change unit, T = (T/B)*B, e.g., T = 8 = (8/2)*2 = 4*2 = 4 2s
- Proportionality, $ = ($/kg)*kg = price*kg
- Trigonometry, a = (a/c)*c = sinA*c, a = (a/b)*b = tanA*b, b = (b/c)*c = cosA*c
- STEM-formulas, meter = (meter/sec)*sec = speed*sec, kg = (kg/m^3)*m^3 = density*m^3
- Coordinate geometry, Δy = (Δy/Δx)*Δx = m*Δx
- Differential calculus, dy = (dy/dx)*dx = y' * dx

The number-formula also contains the formulas for constant change:

*T = b*x* (proportional)

*T = b*x + c* (linear)

*T = a*x^n* (elastic)

*T = a*n^x* (exponential)

*T = a*x^2 + b*x + c* (accelerated)

If not constant, numbers change: constant change roots pre-calculus, predictable change roots calculus, and unpredictable change roots statistics, 'post-dicting' what we cannot be 'pre-dicted'.

The General Curriculum Choices of Existence-based Mathematics

Making the curriculum bridge cohere with the individual start levels in a class is obtained by always beginning with the number-formula, and with recounting tens in icons less than ten, e.g., T = 4.2 tens = ? 7s, or u*7 = 42 = (42/7)*7, thus solving equations by moving to opposite side with opposite sign. And by always using full number-language sentences with a subject, a verb and a predicate as in the word language, e.g., T = 2*3. This also makes the bridge cohere to previous and following bridges.

Making the end level cohere to goals and values expressed by the society and by the learners is obtained by choosing mastery as the end goal, not of the inside self-referring setcentric construction of contemporary university mathematics, but of the outside universal physical reality, Many.

Making the bridge passable is obtained by choosing Piagetian psychology instead of Vygotskyan.

Flexible Bundle-Numbers may make Teachers Follow

Changing a curriculum raises the question: will the teachers follow? Here, seeing the advantage of flexible bundle-numbers may make teachers interested in learning more about existence-based mathematics:

Typically, division creates problems to students, e.g., 336/7. With flexible numbers a total of 336 can be recounted with an overload as

T = 336 = 33B6 = 28B56, so 336/7 = 28B56 /7 = 4B8 = 48; or with an underload as

T = 336 = 33B6 = 35B-14, so 336/7 = 35B-14 /7 = 5B-2 = 48.

Flexible numbers ease all operations:

T = 48*7 = 4B8*7 = 28B56 = 33B6 = 336

T = 92 – 28 = 9B2 – 2B8 = 7B-6 = 6B4 = 64

T = 54 + 28 = 5B4 + 2B8 = 7B12 = 8B2 = 82

To learn more about flexible numbers, a group of teachers can go to the MATHeCADEMY.net designed to teach teachers to teach MatheMatics as ManyMatics, a natural science about Many, to watch some of its YouTube videos. Next, the group can try out the 'Free 1day Skype Teacher Seminar: Cure Math Dislike by ReCounting' where, in the morning, a power point presentation 'Curing Math Dislike' is watched and discussed locally, and at a Skype conference with an instructor. After lunch the group tries out a 'BundleCount before you Add booklet' to experience proportionality and calculus and solving equations as golden learning opportunities in bundle-counting and re-counting and next-to addition. Then another Skype conference follows after the coffee break.

To learn more, a group of eight teachers can take a one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for secondary school. For modelling, there is a study unit in quantitative literature.

The course is organized as PYRAMIDeDUCATION where the 8 teachers form 2 teams of 4, choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e., both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material mediates learning by experimenting with the subject in number-language sentences, i.e., the total T. Thus, the material is self-instructing, saying 'When in doubt, ask the subject, not the instructor'.

The material for primary and secondary school has a short question-and-answer format. The question could be: 'How to count Many? How to recount 8 in 3s? How to count in standard bundles?' The corresponding answers would be:

'By bundling and stacking the total T, predicted by

T = (T/B)*B. So, T = 8 = (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s = 3.-1 3s.

Bundling bundles gives multiple blocks, a polynomial:

T = 456 = 4BundleBundle + 5Bundle + 6 = 4*B^2 + 5*B + 6*1.'

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A Twin Curriculum Since Contemporary Mathematics May Block the Road to its Educational Goal, Mastery of Many

Mathematics education research still leaves many issues unsolved after half a century. Since it refers primarily to local theory, we may ask if grand theory may be helpful. Here philosophy suggests respecting and developing the epistemological mastery of Many children bring to school instead of forcing ontological university mathematics upon them. And sociology warns against the goal displacement created by seeing contemporary institutionalized mathematics as the goal needing eight competences to be learned, instead of aiming at its outside root, mastery of Many, needing only two competences, to count and to unite, described and implemented through a guiding twin curriculum.

Poor PISA Performance Despite Fifty Years of Research

Being highly useful to the outside world, mathematics is a core part of institutionalized education. Consequently, research in math education has grown as witnessed by the International Congress on Mathematics Education taking place each 4 years since 1969. However, despite increased research and funding, the former model country Sweden has seen its PISA result decrease from 2003 to significantly below the OECD average in 2012, causing OECD (2015) to write the report ‘Improving Schools in Sweden’. Likewise, math dislike seems to be widespread in high performing countries also. With mathematics and education as social institutions, grand theory may explain this ‘irrelevance paradox’, the apparent negative correlation between research and performance.

Grand Theory

Ancient Greece saw two forms of knowledge, ‘sophy’. To the sophists, knowing nature from choice would prevent patronization by choice presented as nature. To the philosophers, choice was an illusion since the physical is but examples of metaphysical forms only visible to the philosophers educated at Plato's Academy. Christianity eagerly took over metaphysical patronage and changed the academies into monasteries. The sophist skepticism was revived by Brahe and Newton, insisting that knowledge about nature comes from laboratory observations, not from library books (Russell, 1945).

Newton’s discovery of a non-metaphysical changing will lead to the Enlightenment period: When falling bodies follow their own will, humans can do likewise and replace patronage with democracy. Two republics arose, in the United States and in France. The US still has its first Republic, France its fifth, since its German-speaking neighbors tried to overthrow the French Republic again and again.

In North America, the sophist warning against hidden patronization lives on in American pragmatism and symbolic interactionism; and in Grounded Theory, the method of natural research resonating with Piaget’s principles of natural learning. In France, skepticism towards our four fundamental institutions, words and sentences and cures and schools, is formulated in the poststructural thinking of Derrida, Lyotard, Foucault and Bourdieu warning against institutionalized categories, correctness, diagnosed cures, and education; all may hide patronizing choices presented as nature (Lyotard, 1984).

Within philosophy itself, the Enlightenment created existentialism (Marino, 2004) described by Sartre as holding that ‘existence precedes essence’, exemplified by the Heidegger-warning: In a sentence, trust the subject, it exists; doubt the predicate, it is essence coming from a verdict or gossip.

The Enlightenment also gave birth to sociology. Here Weber was the first to theorize the increasing goal-oriented rationalization that dis-enchants the world and creates an iron cage if carried to wide. Mills (1959) sees imagination as the core of sociology. Bauman (1990) agrees by saying that sociological thinking “renders flexible again the world hitherto oppressive in its apparent fixity; it shows it as a world which could be different from what it is now” (p. 16). But he also formulates a warning (p. 84): “The ideal model of action subjected to rationality as the supreme criterion contains an inherent danger of another deviation from that purpose - the danger of so-called goal displacement. (..) The survival of the organization, however useless it may have become in the light of its original end, becomes the purpose in its own right". Which may lead to 'the banality of evil' (Arendt, 1963).

As to what we say about the world, Foucault (1995) focuses on discourses about humans that, if labeled scientific, establish a 'truth regime'. In the first part of his work, he shows how a discourse disciplines itself by only accepting comments to already accepted comments. In the second part he shows how a discourse disciplines also its subject by locking humans up in a predicate prison of abnormalities from which they can only escape by accepting the diagnose and cure offered by the 'pastoral power' of the truth regime. Foucault thus sees a school as a 'pris-pital' mixing the power techniques of a prison and a hospital: the 'pati-mates' must return to their cell daily and accept the diagnose 'un-educated' to be cured by, of course, education as defined by the ruling truth regime.

Mathematics, Stable until the Arrival of SET

In ancient Greece, the Pythagoreans chose the word mathematics, meaning knowledge in Greek, as a common label for their four knowledge areas: geometry, arithmetic, music and astronomy (Freudenthal, 1973), seen by the Greeks as knowledge about Many in space, Many by itself, Many in time, and Many in space and time. Together they formed the 'quadrivium' recommended by Plato as a general curriculum together with 'trivium' consisting of grammar, logic and rhetoric.

With astronomy and music as independent areas, mathematics became a common label for the two remaining activities, geometry and algebra, both rooted in the physical fact Many through their original meanings, 'to measure earth' in Greek and 'to reunite' in Arabic. And in Europe, Germanic countries taught 'reckoning' in primary school and 'arithmetic' and 'geometry' in the lower secondary school until about 50 years ago when they all were replaced by the 'New Mathematics'.

Here a wish for exactness and unity created a SET-derived 'meta-matics' as a collection of 'well-proven' statements about 'well-defined' concepts, defined top-down as examples from abstractions instead of bottom-up as abstractions from examples. But Russell showed that the self-referential liar paradox 'this sentence is false', being false if true and true if false, reappears in the set of sets not belonging to itself, where a set belongs only if it does not: If $M = \{A \mid A \notin A\}$ then $M \in M \Leftrightarrow M \notin M$. The Zermelo-Fraenkel set-theory avoids self-reference by not distinguishing between sets and elements, thus becoming meaningless by not separating abstract concepts from concrete examples.

SET thus transformed classical grounded 'many-matics' into today's self-referring 'meta-matism', a mixture of meta-matics and 'mathe-matism' true inside but seldom outside a classroom where adding numbers without units as '1 + 2 IS 3' meets counter-examples as e.g. 1week + 2days is 9days.

Proportionality Illustrates the Variety of Mastery of Many and of Quantitative Competence

Proportionality is rooted in questions as "2kg costs 5$, what does 7kg cost; and what does 12$ buy?"

Europe used the 'Regula de Tri' (rule of three) until around 1900: arrange the four numbers with alternating units and the unknown at last. Now, from behind, first multiply, then divide. So first we ask, Q1: '2kg cost 5$, 7kg cost ?$' to get to the answer (7*5/2)$ = 17.5$. Then we ask, Q2: '5$ buys 2kg, 12$ buys ?kg' to get to the answer (12*2)/5$ = 4.8kg.

Then, two new methods appeared, 'find the unit', and cross multiplication in an equation expressing like proportions or ratios:

Q1: 1kg costs 5/2$, so 7kg cost 7*(5/2) = 17.5$. Q2: 1$ buys 2/5kg, so 12$ buys 12*(2/5) = 4.8kg. Q1: 2/5 = 7/x, so 2*x = 7*5, x = (7*5)/2 = 17.5. Q2: 2/5 = x/12, so 5*x = 12*2, x = (12*2)/5 = 4.8.

SET chose modeling with linear functions to show the relevance of abstract algebra's group theory: Let us define a linear function $f(x) = c*x$ from the set of kg-numbers to the set of $-numbers, having as domain $DM = \{x \in \mathrm{R} \mid x>0\}$. Knowing that $f(2) = 5$, we set up the equation $f(2) = c*2 = 5$ to be solved by multiplying with the inverse element to 2 on both sides and applying the associative law: $c*2 = 5$, $(c*2)*½ = 5*½$, $c*(2*½) = 5/2$, $c*1 = 5/2$, $c = 5/2$. With $f(x) = 5/2*x$, the inverse function is $f\text{-}1(x) = 2/5*x$. So with 7kg, $f(7) = 5/2*7 = 17.5$\$; and with 12\$, $f\text{-}1(12) = 2/5*12 = 4.8$kg.

In the future, we simply 're-count' in the 'per-number' 2kg/5$ coming from 'double-counting' the total *T*. Q1: *T* = 7kg = (7/2)*2kg = (7/2)*5$ = 17.5$; Q2: *T* = 12$ = (12/5)*5$ = (12/5)*2kg = 4.8kg.

Grand Theory Looks at Mathematics Education

Philosophically, we can ask if Many should be seen ontologically, what it is in itself; or epistemologically, how we perceive and verbalize it. University mathematics holds that Many should be treated as cardinality that is linear by its ability to always absorb one more. However, in human number-language, Many is a union of blocks coming from counting singles, bundles, bundles of bundles etc., T = 345 = 3*BB+4*B+5*1, resonating with what children bring to school, e.g. T = 2 5s.

Likewise, we can ask: in a sentence what is more important, that subject or what we say about it? University mathematics holds that both are important if well-defined and well-proven; and both should be mediated according to Vygotskian psychology. Existentialism holds that existence precedes essence, and Heidegger even warns against predicates as possible gossip. Consequently, learning should come from openly meeting the subject, Many, according to Piagetian psychology.

Sociologically, a Weberian viewpoint would ask if SET is a rationalization of Many gone too far leaving Many dis-enchanted and the learners in an iron cage. A Baumanian viewpoint would suggest that, by monopolizing the road to mastery of Many, contemporary university mathematics has created a goal displacement. Institutions are means, not goals. As an institution, mathematics is a means, so the word 'mathematics' must go from goal descriptions. Thus, to cure we must be sure the diagnose is not self-referring. Seeing education as a pris-pital, a Foucaultian viewpoint would ask, first which structure to choose, European line-organization forcing a return to the same cell after each hour, day and month for several years; or the North American block-organization changing cell each hour, and changing the daily schedule twice a year? Next, as prisoners of a 'the goal of math education is to learn math' discourse and truth regime, how can we look for different means to the outside goal, mastery of Many, e.g. by examining and developing the existing mastery children bring to school?

Meeting Many, Children Bundle in Block-Numbers to Count and Share

How to master Many can be learned from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists, and with units: bundles of 2s, and 2 of them.

Children also use block-numbers when talking about Lego bricks as '2 3s' or '3 4s'. When asked "How many 3s when united?" they typically say '5 3s and 3 extra'; and when asked "How many 4s?" they may say '5 4s less 2'; and, placing them next-to each other, they typically say '2 7s and 3 extra'.

Children have fun recounting 7 sticks in 2s in various ways, as 1 2s &5, 2 2s &3, 3 2s &1, 4 2s less 1, 1 4s &3, etc. And children don't mind writing a total of 7 using 'bundle-writing' as *T* = 7 = 1*B*5 = 2*B*3 = 3*B*1 = 4*B*~~1~~; or even as 1*BB*3 or 1*BB*1*B*1. Also, children love to count in 3s, 4s, and in hands.

Sharing 9 cakes, 4 children take one by turn saying they take 1 of each 4. Taking away 4s roots division as counting in 4s; and with 1 left they often say "let's count it as 4". Thus 4 preschool children typically share by taking away 4s from 9, and by taking away 1 per 4, and by taking 1 of 4 parts. And they smile when seeing that entering '9/4' allows a calculator to predict the sharing result as 2 1/4; and when seeing that entering '2*5/3' will predict the result of sharing 2 5s between 3 children.

Children thus master sharing, taking parts and splitting into parts before division and counting- and splitting-fractions is taught; which they may like to learn before being forced to add without units.

So why not develop instead of rejecting the core mastery of Many that children bring to school?

A Typical Contemporary Mathematics Curriculum

Typically, the core of a curriculum is how to operate on specified and unspecified numbers. Digits are given directly as symbols without letting children discover them as icons with as many strokes or sticks as they represent. Numbers are given as digits respecting a place value system without letting children discover the thrill of bundling, counting both singles and bundles and bundles of bundles. Seldom 0 is included as 01 and 02 in the counting sequence to show the importance of bundling. Never children are told that eleven and twelve comes from the Vikings counting '(ten and) 1 left', '(ten and) 2 left'. Never children are asked to use full number-language sentences, *T* = 2 5s, including both a subject, a verb and a predicate with a unit. Never children are asked to describe numbers after ten as 1.4 tens with a decimal point and including the unit. Renaming 17 as 2.-3 tens and 24 as 1*B*14 tens is not allowed. Adding without units always precedes both bundling iconized by division, stacking iconized by multiplication, and removing stacks to look for unbundled singles iconized by subtraction. In short, children never experience the enchantment of counting, recounting and double-counting Many before adding. So, to re-enchant Many will be an overall goal of a twin curriculum in mastery of Many through developing the children's existing mastery and quantitative competence.

A Question Guided Counting Curriculum

The question guided re-enchantment curriculum in counting could be named 'Mastering Many by counting, recounting and double-counting'. The design is inspired by Tarp (2018). It accepts that while eight competencies might be needed to learn university mathematics (Niss, 2003), only two are needed to master Many (Tarp, 2002), counting and uniting, motivating a twin curriculum. The corresponding pre-service or in-service teacher education can be found at the MATHeCADEMY.net. Remedial curricula for classes stuck in contemporary mathematics can be found in Tarp (2017).

Q01, icon-making: "The digit 5 seems to be an icon with five sticks. Does this apply to all digits?" Here the learning opportunity is that we can change many ones to one icon with as many sticks or strokes as it represents if written in a less sloppy way. Follow-up activities could be rearranging four dolls as one 4-icon, five cars as one 5-icon, etc.; followed by rearranging sticks on a table or on a paper; and by using a folding ruler to construct the ten digits as icons.

Q02, counting sequences: "How to count fingers?" Here the learning opportunity is that five fingers can also be counted "01, 02, 03, 04, Hand" to include the bundle; and ten fingers as "01, 02, Hand less2, Hand-1, Hand, Hand&1, H&2, 2H-2, 2H-1, 2H". Follow-up activities could be counting things.

Q03, icon-counting: "How to count fingers by bundling?" Here the learning opportunity is that five fingers can be bundle-counted in pairs or triplets allowing both an overload and an underload; and reported in a number-language sentence with subject, verb and predicate: *T* = 5 = 1Bundle3 2s = 2*B*1 2s = 3*B*-1 2s = 1*BB*1 2s, called an 'inside bundle-number' describing the 'outside block-number'. A western abacus shows this in 'outside geometry space-mode' with the 2 2s on the second and third bar and 1 on the first bar; or in 'inside algebra time-mode' with 2 on the second bar and 1 on the first bar. Turning over a two- or three-dimensional block or splitting it in two shows its commutativity, associativity and distributivity: T = 2*3 = 3*2; T = 2*(3*4) = (2*3)*4; T = (2+3)*4 = 2*4 + 3*4.

Q04, calculator-prediction: "How can a calculator predict a counting result?" Here the learning opportunity is to see the division sign as an icon for a broom wiping away bundles: 5/2 means 'from 5, wipe away bundles of 2s'. The calculator says '2.some', thus predicting it can be done 2 times. Now the multiplication sign iconizes a lift stacking the bundles into a block. Finally, the subtraction sign iconizes the trace left when dragging away the block to look for unbundled singles. By showing '5-2*2 = 1' the calculator indirectly predicts that a total of 5 can be recounted as 2B1 2s. An additional learning opportunity is to write and use the 'recount-formula' $T = (T/B)*B$ saying "From T, T/B times B can be taken away." This proportionality formula occurs all over mathematics and science. Follow-up activities could be counting cents: 7 2s is how many fives and tens? 8 5s is how many tens?

Q05, unbundled as decimals, fractions or negative numbers: "Where to put the unbundled singles?" Here the learning opportunity is to see that with blocks, the unbundled occur in three ways. Next-to the block as a block of its own, written as T = 7 = 2.1 3s, where a decimal point separates the bundles from the singles. Or on-top as a part of the bundle, written as T = 7 = 2 1/3 3s = 3.-2 3s counting the singles in 3s, or counting what is needed for an extra bundle. Counting in tens, the outside block 4 tens & 7 can be described inside as T = 4.7 tens = 4 7/10 tens = 5.-3 tens, or 47 if leaving out the unit.

Q06, prime or foldable units: "Which blocks can be folded?" Here the learning opportunity is to examine the stability of a block. The block T = 2 4s = 2*4 has 4 as the unit. Turning over gives T = 4*2, now with 2 as the unit. Here 4 can be folded in another unit as 2 2s, whereas 2 cannot be folded (1 is not a real unit since a bundle of bundles stays as 1). Thus, we call 2 a 'prime unit' and 4 a 'foldable unit', 4 = 2 2s. So, a block of 3 2s cannot be folded, whereas a block of 3 4s can: T = 3 4s = 3 * (2*2) = (3*2) * 2. A number is called even if it can be written with 2 as the unit, else odd.

Q07, finding units: "What are possible units in T = 12?". Here the learning opportunity is that units come from factorizing in prime units, T = 12 = 2*2*3. Follow-up activities could be other examples.

Q08, recounting in another unit: "How to change a unit?" Here the learning opportunity is to observe how the recount-formula changes the unit. Asking e.g. T = 3 4s = ? 5s, the recount-formula will say T = 3 4s = (3*4/5) 5s. Entering 3*4/5, the answer '2.some' shows that a stack of 2 5s can be taken away. Entering 3*4 – 2*5, the answer '2' shows that 3 4s can be recounted in 5s as 2B2 5s or 2.2 5s.

Q09, recounting from tens to icons: "How to change unit from tens to icons?" Here the learning opportunity is that asking 'T = 2.4 tens = 24 = ? 8s' can be formulated as an equation using the letter u for the unknown number, u*8 = 24. This is easily solved by recounting 24 in 8s as 24 = (24/8)*8 so that the unknown number is u = 24/8 attained by moving 8 to the opposite side with the opposite sign. Follow-up activities could be other examples of recounting from tens to icons.

Q10, recounting from icons to tens: "How to change unit from icons to tens?" Here the learning opportunity is that if asking 'T = 3 7s = ? tens', the recount-formula cannot be used since the calculator has no ten-button. However, it is programmed to give the answer directly by using multiplication alone: T = 3 7s = 3*7 = 21 = 2.1 tens, only it leaves out the unit and misplaces the decimal point. An additional learning opportunity uses 'less-numbers', geometrically on an abacus, or algebraically with brackets: T = 3*7 = 3 * (ten less 3) = 3 * ten less 3*3 = 3ten less 9 = 3ten less (ten less1) = 2ten less less 1 = 2ten & 1 = 21. Follow-up activities could be other examples of recounting from icons to tens.

Q11, double-counting in two units: "How to double-count in two different units?" Here the learning opportunity is to observe how double-counting in two physical units creates 'per-numbers' as e.g. 2$ per 3kg, or 2$/3kg. To answer questions we just recount in the per-number: Asking '6$ = ?kg' we recount 6 in 2s: T = 6$ = (6/2)*2$ = (6/2)*3kg = 9kg. And vice versa, asking '?$ = 12kg', the answer is T = 12kg = (12/3)*3kg = (12/3)*2$ = 8$. Follow-up activities could be numerous other examples of double-counting in two different units since per-numbers and proportionality are core concepts.

Q12, double-counting in the same unit: "How to double-count in the same unit?" Here the learning opportunity is that when double-counted in the same unit, per-numbers take the form of fractions, 3$ per 5$ = 3/5; or percentages, 3 per hundred = 3/100 = 3%. Thus, to find a fraction or a percentage of a total, again we just recount in the per-number. Also, we observe that per-numbers and fractions are not numbers, but operators needing a number to become a number. Follow-up activities could be other examples of double-counting in the same unit since fractions and percentages are core concepts.

Q13, recounting the sides in a block. "How to recount the sides of a block halved by its diagonal?" Here, in a block with base *b*, height *a*, and diagonal *c*, mutual recounting creates the trigonometric per-numbers: $a = (a/c)*c = \sin A*c$; $b = (b/c)*c = \cos A*c$; $a = (a/b)*b = \tan A*b$. Thus, rotating a line can be described by a per-number a/b, or as $\tan A$ per 1, allowing angles to be found from per-numbers. Follow-up activities could be other blocks e.g. from a folding ruler.

Q14, double-counting in STEM (Science, Technology, Engineering, Math) multiplication formulas with per-numbers coming from double-counting. Examples: kg = (kg/cubic-meter)*cubic-meter = density*cubic-meter; force = (force/square-meter) * square-meter = pressure * square-meter; meter = (meter/sec)*sec = velocity*sec; energy = (energy/sec)*sec = Watt*sec; energy = (energy/kg) * kg = heat * kg; gram = (gram/mole) * mole = molar mass * mole; Δ momentum = (Δ momentum/sec) * sec = force * sec; Δ energy = (Δ energy/ meter) * meter = force * meter = work; energy/sec = (energy/charge)*(charge/sec) or Watt = Volt*Amp; dollar = (dollar/hour)*hour = wage*hour.

Q15, navigating. "Avoid the rocks on a squared paper". Four rocks are placed on a squared paper. A journey begins in the midpoint. Two dices tell the horizontal and vertical change, where odd numbers are negative. How many throws before hitting a rock? Predict and measure the angles on the journey.

A Question Guided Uniting Curriculum

The question guided re-enchantment curriculum in uniting could be named 'Mastering Many by uniting and splitting constant and changing unit-numbers and per-numbers'.

A general bundle-formula $T = a*x^2 + b*x + c$ is called a polynomial. It shows the four ways to unite: addition, multiplication, repeated multiplication or power, and block-addition or integration. The tradition teaches addition and multiplication together with their reverse operations subtraction and division in primary school; and power and integration together with their reverse operations factor-finding (root), factor-counting (logarithm) and per-number-finding (differentiation) in secondary school. The formula also includes the formulas for constant change: proportional, linear, exponential, power and accelerated. Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Q21, next-to addition: "With *T*1 = 2 3s and *T*2 = 4 5s, what is *T*1+*T*2 when added next-to as 8s?" Here the learning opportunity is that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus. Follow-up activities could be other examples of next-to addition.

Q22, reversed next-to addition: "If *T*1 = 2 3s and *T*2 add next-to as *T* = 4 7s, what is *T*2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 3s, subtraction precedes division, which is natural as reversed integration, also called differential calculus. Follow-up activities could be other examples of reversed next-to addition.

Q23, on-top addition: "With *T*1 = 2 3s and *T*2 = 4 5s, what is *T*1+*T*2 when added on-top as 3s; and as 5s?" Here the learning opportunity is that on-top addition means changing units by using the recount-formula. Thus, on-top addition may apply proportionality; an overload is removed by recounting in the same unit. Follow-up activities could be other examples of on-top addition.

Q24, reversed on-top addition: "If *T*1 = 2 3s and *T*2 as some 5s add to *T* = 4 5s, what is *T*2?" Here the learning opportunity is that when finding the answer by removing the initial block and recounting the rest in 5s, subtraction precedes division, again called differential calculus. An underload is removed by recounting. Follow-up activities could be other examples of reversed on-top addition.

Q25, adding tens: “With *T*1 = 23 and *T*2 = 48, what is *T*1+*T*2 when added as tens?” Again, recounting removes an overload: *T*1+*T*2 = 23 + 48 = 2*B*3 + 4*B*8 = 6*B*11 = 7*B*1 = 71; or *T* = 236 + 487 = 2*BB*3*B*6 + 4*BB*8*B*7 = 6*BB*11*B*13 = 6*BB*12*B*3 = 7*BB*2*B*3 = 723.

Q26, subtracting tens: “If *T*1 = 23 and *T*2 add to *T* = 71, what is *T*2?” Again, recounting removes an underload: *T*2 = 71 – 23 = 7*B*1 – 2*B*3 = 5*B*-2 = 4*B*8 = 48; or *T*2 = 956 – 487 = 9*BB*5*B*6 – 4*BB*8*B*7 = 5*BB*-3*B*-1 = 4*BB*7*B*-1 = 4*BB*6*B*9 = 469. Since T = 19 = 2.-1 tens, T2 = 19 -(-1) = 2.-1 tens take away -1 = 2 tens = 20 = 19+1, showing that -(-1) = +1.

Q27, multiplying tens: “What is 7 43s recounted in tens?” Here the learning opportunity is that also multiplication may create overloads: *T* = 7*43 = 7*4*B*3 = 28*B*21 = 30*B*1 = 301; or 27*43 = 2*B*7*4*B*3 =8*BB*+6*B*+28*B*+21 =8*BB*34*B*21 =8*BB*36*B*1 = 11*BB*6*B*1 = 1161, solved geometrically in a 2x2 block.

Q28, dividing tens: “What is 348 recounted in 6s?” Here the learning opportunity is that recounting a total with overload often eases division: *T* = 348 /6 = 3*BB*4*B*8 /6 = 34*B*8 /6 = 30*B*48 /6 = 5*B*8 = 58.

Q29, adding per-numbers: “2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?” Here the learning opportunity is that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a ‘locally constant’ (continuous) situation defining constancy by epsilon and delta.

Q30, subtracting per-numbers: “2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?” Here the learning opportunity is that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. In a ‘locally constant’ situation, subtracting per-numbers is called differential calculus.

Q31, finding common units: “Only add with like units, so how to add *T* = 4ab^2 + 6abc?”. Here units come from factorizing: *T* = 2*2*a*b*b* + 2*3*a*b*c* = 2*b**(2*a*b*) + 3*c**(2*a*b*) = 2*b*+3*c* 2*ab*s.

Conclusion

A curriculum wants to develop brains, and colonizing wants to develop countries. Decolonizing accepts that maybe countries and brains can develop themselves if helped by options instead of directions from developed countries and brains. Some prefer a direction-giving multi-year macro-curriculum; others prefer option-giving half-year micro-curricula. Some prefer a curriculum to be a cure prescribing mathematics competencies and literacy; others prefer developing the existing quantitative competence and numeracy, defined by dictionaries as the ability to use numbers and operations in everyday life, thus silencing the word ‘mathematics’ to avoid a hidden continuing colonization. In the transition period between colonizing and decolonizing brains, grand theory has an advice to the ‘irrelevance paradox’ of mathematics education research: accept the brain’s own epistemology to avoid a goal displacement blocking the road to its educational goal, mastery of Many.

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