As operators, per-numbers are multiplied before adding as areas

Allan Tarp

MATHeCADEMY.net, Denmark; Allan.Tarp@gmail.com

Abstract

In our matter-with-forces-creating-motion nature quantities often occur with constant per-numbers as is the case in our human made economy. Education thus must ask how to teach per-numbers. So, after identifying examples of per-numbers, this paper reflects on their nature, and on how to teach them at different levels including the perspectives from the three grand theories. A recommendation follows a discussion between the math-before-many tradition and its reversed alternative.

Observing per-numbers

The gravity force first condenses stars as highly accelerated atoms where collisions between electrons and anti-electrons, 'small big bangs', produce light radiating energy, joule, into the space, thus pumping in joule per second per square meter into the third planet allowing water to exist in all three forms, solid and liquid and gas, with different density, kilogram per cubic meter, all depending on the local temperature. This allows gravity create cycles of water that after evaporation on low warm places is pushed up to condense again on high cool places, to be pulled down to cold rocks. Here freezing water increases its density and pulverizes some rocks into sand and minerals dissolved in water with a concentration, moles per liter. The sand follows the water down in rivers creating flatland between the rocks and the sea. Here greens cells use the light's joules per second to replace oxygen with water thus transforming carbon dioxide molecules into carbon hydrate molecules storing some joule per kg, or joule per mole. The green cells' batteries allow grey cells release the joule again by the opposite process. By removing oxygen from green cells decomposing under water, black cells transform carbon hydrate into oil or gas or coal with different joules per kg. This allows creating power plants were the pressure difference between a heated container transforming water into steam and a cooled container condensing steam to water again allows a steam-windmill to create an electrical current carrying joules to machines needing joules per second, delivered by enough electrons per second coming from a source supplying joules per electron.

The human condition thus is the fact that I need my daily bread, and that I need to produce it myself, directly or indirectly by exchanging products on a market. So, I need joules per day which I get by producing items per day for myself, or for exchanging on a market where different items the cost per item differs, and where the exchange may be directly or indirectly via dollars. Yearly, my incoming dollars are taxed with some dollars per 100 dollars to allow the state to transfer dollars to persons unable to work themselves.

In short, humans are part of three cycles: an ecological cycle with green cells, an economic cycle with other producers, and a communicative cycle with a word-language combining letters to words and qualitative sentences, and with a number-language combining digits to numbers that together with operations form predicting formulas for quantitative totals, which are united or split, and that may be constant or changing.

As to uniting totals, multiplication and addition unite constant and changing unit-numbers, and power and integration unite constant and changing per-numbers, that become fractions or percentages with like units, both being operators needing numbers to become numbers.

The four ways to unite totals, multiplication and addition and power and area-addition, are seen in the polynomial way to multi-count a given total as numbers of unbundled, of bundles, of bundle-bundles, etc., as T = 345 = 3 *BB*s and 4 *B*s and 5 1s = 3*B^2 + 4*B + 5*1 where B stands for bundles, and *BB* stands for bundle-bundles. A bundle typically is ten, although the Romans bundled in 5s, and the Vikings bundled in scores or twenties, meaning two tens.

As to splitting totals, division and subtraction split into changing and constant unit numbers, and root or logarithm and differentiation split into constant and changing per-numbers.

Here we focus on integration and differentiation, together called calculus, created to unite, or to split into, changing per-numbers that may by piecewise constant or locally constant (continuous).

Reflecting on per-numbers

Per-numbers as 4$ per 5 kg, or 4$/5kg, or 4/5 $/kg, bridge $- and kg-numbers by recounting them in the per-number using the recount-formula $T = (T/B)*B$ (Tarp, 2018):

T = 20$ = (20/4)*4$ = (20/4)*5kg = 25kg

Per-numbers also occur in mixture-problems as 2kg at 3$/kg + 4 kg at 4$/5kg = (2+4)kg at ? $/kg. The unit-numbers add directly. But as operators, the per-numbers are multiplied to become unit-numbers before adding as areas to give (2*3+4*5)$ per (2+4)kg. Here, the per-numbers are piecewise constant changing from 3 to 4 after 2 kg.

In the case of a falling object, instead the per-number is locally constant changing each moment. Here, areas under the per-number graph, $A = \Sigma\, p(x)*\Delta x$ approximates better and better the smaller the moment is chosen, only giving more areas to add. However, since the multiplied per-number is a difference describing a change of the area, $p(x)*\mathrm{d}x = \mathrm{d}A$, addition makes all middle terms disappear leaving only one difference between the terminal and the initial number. This motivates supplementing integration with differentiation solving differential equations as dA = x^2 dx, finding a formula for the area under the per-number graph $p(x)$ = x^2. Looking at the tiny shadows of a rectangular $p*q$ book we find the never falsified formula, $\mathrm{d}(p*q) = \mathrm{d}p*q + p*\mathrm{d}q$, so that d($x$^2) = $\mathrm{d}(x*x) = 2*x*\mathrm{d}x$

Reflecting on how to teach calculus

Using their own bundle-numbers with units, children quickly master ‘primary school calculus’ where integration occur in questions as ‘2 3s + 4 5s = ? 8s’ where multiplication precedes addition. In in the reverse question ‘2 3s + ? 5s = 4 8s’, subtraction precedes division as in differentiation where their hands first pull away the initial stack, 2 3s, before counting the rest by pushing way 5s.

In middle school, per-numbers may be introduced physically as bridging plastic *S*- and *C*-letters given that 3 *S*-letters have the same value as 5 *C*-letters, which gives the per-number 3*S*/5*C*. Recounting in the per-number thus gives T = 12S = (12/3)*3S = (12/3)*5C = 20C. This leads on to traditional proportionality questions with per-numbers as 3$/5kg, and 3m/5sec, and 3£/5$. To be followed by fractions introduced as per-numbers with like units, 3$/5$ is 3/5. Again we use recounting to see that 3/5 of 20$ means 3$/5$ of 20$, so 20$ = (20/5)*5$ gives (20/5)*3$ = 12 $.

Adding per-numbers occur in mixture-problems as '2kg at 3$/kg + 4 kg at 4$/5kg = 6kg at ? $/kg' and its reversed version. Likewise with adding fractions in problems as '2$, of which 3/4 + 6$, of which 4/5 = 8$, of which ?' and its reverse. So typically, fraction must not add without units.

Multiplying per-numbers and fractions leads to statistics and to Bayes formula: On my hand I have 1 right-finger and 4 left-fingers where the right and one left finger are bent. So, 100% of my right-fingers and 25% of my left-fingers are bent, but only 20% of the fingers are bent. Among the bent, 50% are left-fingers, but among the left-fingers only 25% are bent.

In high school, calculus focus on adding locally constant per-numbers.

Modelling with per-numbers

Seeing calculus as per-number addition allows it to be part of modelling already in middle school, since a CAS tool can show areas under per-number graphs as well as locate turning points on graphs. With integration as more frequent than differentiation in modeling it should come first.

Reflecting on limits

Where a simple interest is added to the initial loan, a compounded interest is also added to former interest amounts. So, a periodical rate, *r*, gives the total simple rate, *R*, where 1+*R* = (1+*r*) ^*n*.

If a yearly rate of 100% split in 12 parts and added 12 times we get *R* = (1+1/12)^12 – 1 = 1.613, showing 61.3% in additional rate. This however has a limit since (1+1/*n*)^*n* can come arbitrarily close to but not exceed 2.718 called *e*. A solver thus shows that (1+1/*n*)^*n* = 2.712 for *n* = 234.

Inside a circle with radius 1 there are many right triangles with the long side from the center to the circle. Splitting 180 degrees in *n* parts, the height of the triangle is sin(180/*n*), so the circumference of a half circle is close to *n**sin(180/*n*) that is 3.1411 for *n* = 100. Again, we have a limit since *n**sin(180/*n*) can come arbitrarily close to but not exceed 3.1416 called pi, π.

This gives a formal definition for constancy: *y* is constant *c* it the distance between the two is arbitrarily small, i.e., the distance is less than any positive number epsilon. And for piecewise constancy: *y* is piecewise constant *c* if an interval delta exists such that the distance between the two here is less than any positive number epsilon. Interchanging epsilon and delta then gives a formal definition for local constancy, or continuity: *y* is locally constant *c* if for any positive number epsilon an interval delta exists such that the distance between the two here is less than epsilon.

Including grand theory, philosophy and psychology and sociology

Whitin philosophy, existentialism and essentialism discuss if existence precedes essence or the opposite. Here, the traditional LDI limit-derivate-integral approach going from the abstract to the concrete is an essence-based calculus, where the opposite IDL per-number addition approach is an existence-based calculus abstracting its concepts from concrete examples.

Within psychology, Vygotskian social constructivism sides with essentialism while Piagetian radical constructivism sides with existentialism.

Within sociology, calculus is a socially constructed institution created as a means to reach the goal, addition of per-numbers. But with a 'goal displacement' (Bauman, 1990) where not reaching the goal makes the means the goal ensure a monopoly to mastering calculus as the only way to later master per-numbers, almost an example of 'the banality of evil' (Arendt, 1963). A sociological perspective thus allows discussing the difference between a good and an evil calculus.

Discussing with the tradition

The ICME-13 Topical Survey says on its first page that it has "a particular focus on established research topics associated to limit, derivative and integral". Grand theory sees this LDI as a Vygotskian essence-based approach with a goal displacement ensuring that the end goal, per-number addition, is reached by only few, while many give up or are prevented from learning it. The opposite IDL is seen as a Piagetian existence-based approach learning the goal before a means (Tarp, 2019). If tested and researched the two opposite approaches have opposite effect on the learners, it will be possible to talk about good and evil calculus.

Recommendation

With 'Math before Many', the tradition insists that mastering math is the only way to master Many, the end goal of math education. But an unnoticed direct way exists where children use their own bundle-numbers with units to unite or split into constant and changing unit- and per-numbers, which is the core of mastering Many. This new 'Many before Math' paradigm (Kuhn, 1962) needs to be studied to see if finally, we may have found a mathematics for all, and if it gives a more passable way to later master mathematics if needed. If successful, there is no need for different curricula to different students, only the numbers of theoretical footnotes will differ (Tarp, 2021). This will create the same communicative turn in number-language education as the word-language saw in the 1970s, where using the language became more important than knowing its grammar (Widdowson, 1978). A first step to a calculus deconstruction (Foucault, 1995) may be to treat fractions as per-numbers, and to reverse the order in calculus by rooting it in mixture problems in middle school.

References

Arendt, H. (1963). *Eichmann in Jerusalem, a report on the banality of evil.* London, UK: Penguin Books.

Bauman, Z. (1990). *Thinking sociologically.* Oxford, UK: Blackwell.

Foucault, M. (1995). *Discipline & punish.* New York, NY: Vintage Books.

Kuhn T.S. (1962). *The structure of scientific revolutions.* Chicago: University of Chicago Press.

Tarp, A. (2018). Mastering Many by counting, re-counting and double-counting before adding on-top and next-to. *Journal of Mathematics Education, 11*(1), 103-117.

Tarp, A. (2019). *Calculus adds (globally, piecewise, or locally) constant per-numbers by areas.* Retrieved at http://mathecademy.net/wp-content/uploads/2021/05/Calculus-papers.pdf.

Tarp, A. (2021). *Master Many to later master math, the canceled curriculum chapter in the ICME study 24.* Retrieved at http://mathecademy.net/appendix-to-curriculum-study-icmi-24/.

Widdowson, H. G. (1978). *Teaching language as communication.* Oxford, UK: Oxford Univ. Press.