Online math may create a communicative turn in number-language education also

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Video: https://www.youtube.com/watch?v=36tan-gGjJg, and
https://www.youtube.com/watch?v=4EPqjz8evd4

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Abstract

Teaching mathematics online is different from teaching it offline in a classroom. So, we may ask what else could be different? On a calculator we see the core of mathematics: digits, multidigit numbers, operations, and equations. But they all occur as products in space, not as processes in time. So, maybe teaching mathematics 'process-based' instead of 'product-based' is different by letting outside Many precede inside Math? And indeed, the math core becomes different when created from tales about Many displayed as rectangles on a concrete ten-by-ten bundle-bundle board, a BBBoard. To see if a 'process-before-product' or 'Many-before-Math' education will make a difference, micro-curricula are designed to bring outside totals inside by bundle-counting creating flexible bundle-numbers with units: ones, bundles, bundle-of-bundles, etc., that all are included in oral counting sequences. Here digits arise as icons when uniting sticks. Here operations arise as icons as well: division pushes away bundles that multiplication lifts onto a stack that subtraction pulls away so the unbundled may be included as decimals, fractions, or negatives. Once counted, a unit may be changed by recounting. Here recounting from tens to icons leads to equations, and when reversed, to tables displayed as the rectangle left when removing the two surplus rectangles from the full bundle-bundle on a BBBoard. Here recounting in two physical units leads to per-numbers bridging the two units and becoming fractions with like units. Here recounting the sides and the diagonal in a rectangle leads to trigonometry before geometry. Finally, once counted and recounted, totals may add on-top after recounting has provided like units, or next-to as areas as in integral calculus that becomes differential calculus when reversed. As operators needing numbers to become numbers, per-numbers and fractions also add by their areas since they need to be multiplied to unit-numbers before adding. So, outside totals inside appear in an 'Algebra Square' where unlike and like unit-numbers and per-numbers are united by addition and multiplication, and by integration and power. And later again split by the reverse operations, subtraction and division, and by differentiation and root or logarithm. Once these curriculum scores have been designed, they may be played in online education, as well as in special education to see if a BBBoard may 'Bring Back Brains' excluded by the 'Math-before-Many' education. But there will be no concert without first designing a score. So actual testing is not addressed here but left to others to perform.

Background

Teaching mathematics online and remotely is different from teaching it offline in a classroom. So, we may ask what else could be different? On a calculator we see the core of mathematics: digits, multidigit numbers, operations, and equations. But they all occur as products in space, not as processes in time. So, maybe teaching mathematics 'process-based' instead of 'product-based' is different by letting outside Many precede inside Math? Especially since a 'product-based math' typically implies the top-down 'abstractions-before-examples' view that 'of course, mathematics first must be learned before it later may be applied', which makes mastery of inside mathematics a necessary means to reach the end goal, mastery of outside Many. In contrast, a 'process-based math' resonates with existentialist philosophy holding that existence precedes essence and thus implies the bottom-up 'examples-before-abstractions' view that "of course, examples should install abstractions instead of being installed by them", which makes mastery of outside Many the direct goal that automatically implies a basic mastery of mathematics later.

Asking a similar question created a communicative turn in foreign language education in the 1970s (Widdowson, 1978). So, online math may create a communicative turn in number-language education as well.

To look for differences we may use 'Difference Research' (Tarp, 2018) that, searching for a difference making a difference, asks "May a different input give a different output?"

Asking 'Could mathematics be different' we find its core on a basic calculator with digits and operations and an equation sign on buttons, and multidigit numbers on the display. However, the digits and operations occur as end-products only, we don't see the process creating them.

We see the 5-digit as one symbol in space, we do not see how it may be formed in time by uniting five sticks or strokes into one icon. Likewise, we see the total $T = 234$ as one number without units. So, we don't see the three numberings using bundle-counting to find the number of singles, bundles, and bundles-of-bundles, i.e., we don't see the polynomial with the units $T = 234 = 2*BB + 3*B + 4*1$.

We see the operations order as natural: addition, subtraction, multiplication, division, and power. Because they build on the basic claim that "2+1 = 3 always". However, with units, 2 weeks + 1 day is 15 days. So, adding numbers without units often folds outside the classroom, whereas multiplication always holds since, in the product 2x3, 3 is the unit. Therefore, to teach valid mathematics instead of invalid 'mathematism' (Tarp, 2018), units must always be included inside.

Counting totals in tens, a bundle of bundles first occurs with three-digit numbers, and without stressing that hundred is just another name for bundle-bundle. Instead, counting fingers in 2s, 2 becomes a Bundle, 4 a Bundle-of-Bundles, a BB or a B^2. And 8 becomes a BBB, or a B^3. So, when bundle-counting an outside total with units, power becomes the first operation, not the last.

Calculations never includes the outside total that is calculated. So instead of saying '$T = 6x7$', we only say '6x7'. And we see 6x7 as 42 always instead of de-modeling (Tarp, 2020) it outside as 6 7s that may or may not be recounted in tens as 4 bundle 2 tens written inside as 4B2 tens or 4.2 tens, or shortly as 42 if leaving out both the unit and the decimal point. Nor do we see or perform 6x7 as an outside process where a total is recounted in 7s by 6 times pushing away the bundles with a broom called division, to be stacked by a lift called multiplication. After which a rope called subtraction pulls away the stack to look for unbundled singles to be placed on-top of the stack as a decimal number, as 9 = 4B1 2s. Only now, once counted and recounted, stacks may finally be added on-top or next-to. So, apparently from a process-before-product view, the operation order is the opposite.

These observations suggest an immediate answer to our question. Mathematics does seem to be different when seen as a process to be acted in time instead of as an end-product in space to be memorized.

This resonates with the philosophy called existentialism that, by holding that existence precedes essence (Sartre, 2007), suggests that outside existence should install inside essence instead of being installed by it. Or in other words, we should treat and teach mathematics as a number-language describing the outside fact Many, already occurring in the word-language as the plural form of words.

Also, this will allow solving inside problems by de-modeling them as outside totals. Thus asking "What is the product of 2½ and 3 1/3?", this inside product could be demodeled as an outside total T, thus creating the number-language sentence, $T = 2½ * 3\ 1/3$, with an outside subject, a verb and an inside predicate as in word-language sentences. From this story about T we may conclude that with 2 and 3 times as many Ts we get $2*T = 5 * 3\ 1/3$, and $6*T = 5*10 = 50$, so $T = 50/6 = 8\ 2/6$.

Outside totals are typically counted in tens inside. So, we may develop a number-language by describing things and actions on a ten-by-ten Bundle-Bundle-Board, a BBBoard, where rubber bands split the bundles vertically and horizontally to create rectangles with a total of T pegs. Which will then be the subject in a number-language sentences where the BBBoard allows an outside total of 2 3s to becomes the inside sentence $T = 2x3$, also called a formula or a function. Plastic BBBoards are cheap to manufacture and give to schools, where the students also can use BBboards on squared paper.

But will this difference make a difference? To test this, we first design some micro-curricula (MC) where we describe things and actions on a Bundle-Bundle BBBoard. The curricula are inspired by the four Tarp papers from 2001 and 2018 - 2022.

Figure 1. A total of 6 7s shown on a Bundle-Bundle BBBoard

MC01. Digits are icons uniting sticks

The goal is to experience that digits may be formed by uniting the number of sticks that they represent, and to see that these digits are like the digits on the calculator.

The action is to unite two sticks to one two-cion, three sticks to one three-icon, etc. Other concrete materials could be cars, dolls, spoons etc. Finally, a bending ruler may be used to create the digits.

We see that with Roman numbers, 3 1s is not united into 1 3s; and that Roman numbers use the letter V as a symbol for a hand, and the letter X as a symbol for two hands.

We don't have an icon for ten since bundling in tens makes ten a two-digit number counting both the number of bundles, and the number of unbundled singles:

T = ten = 1B0, or *T* = 10 if leaving out the unit.

When including the units, we don't need a place value system.

MC02. Operations are icons created by Bundle-counting and recounting

The goal is to experience how the three outside counting processes, pushing away bundles, stacking bundles, and pulling away stacks may inside be iconized as division, multiplication, and subtraction.

As concrete material serves sticks and snap cubes.

To recount 8 in 2s, we push away 2s 4 times to get the answer *T* = 8 = 4 2s.

Iconizing 'push away' by a broom, /, called division, the action 'from 8 push away 2s' may be iconized as '8/2' giving 4 on a calculator.

Iconizing 'stacking up' by a lift, x, called multiplication, the action '4 times stack 2s' may be iconized as '4x2' giving 8 on a calculator.

Pushing and stacking combines in a 'recount-formula' (Tarp, 2018),

8 = (8/2)x2, or *T* = (*T*/*B*)x*B* with unspecified numbers, saying that *T* contains *T*/*B* *B*s.

Iconizing 'pull away' by a rope, –, called subtraction used to find unbundled singles, the action 'from 9 pull away 4 2s' may be iconized as '9-4x2' giving 1 on a calculator as expected.

Placed on top of the stack, the unbundled may be seen as a decimal number or as a fraction when counted in 2s also, or creating a negative number showing what is pulled away form the next bundle:

9 = 9 – 4x2 = 1, so 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s.

Iconizing 'uniting' by a cross showing two directions, +, called addition, the action 'uniting 2 3s and 4 5s' may be iconized as '2x3 + 4x5' giving 26 on a calculator if united in tens, and 5B1 5s if added on-top, and 3B2 if added next-to.

Pulling and adding combines in a 'restack-formula' (Tarp, 2018),

8 = (8-2)+2, or $T = (T\text{-}B)+B$ with unspecified numbers, saying that T-B is left when B is pulled away and placed next to, or simply that pulling away B allows T to split in T-B and B.

MC03. Bundle-counting in icons

The goal is to use an oral bundle-counting sequence with icon-bundles to represent outside examples of Many as an inside union of singles, bundles, bundle-of-bundles, etc.

First, we count the ten fingers in 4s by raising them one by one beginning with the little finger.

0B1, 0B2, 0B3, 0B4 or 1B0, moving the fingers to see the difference between 4 1s and 1 4s, called a bundle, B.

1B1, 1B2, 1B3, 1B4 or 2B0, again moving the fingers to see the difference between the two.

2B1, 2B2.

So, a total of ten may be counted in 4s as 2 bundles and 2, written shortly as

T = 2B2 4s, or T = 22 4s, if leaving out the unit.

We notice that we need 2 to have 3 bundles, so we may also write T = 3B less2 = 3B-2 4s.

So, with 'flexible bundle-counting' we have three ways to count ten in 4s: underload, normal, and overload:

T = ten = 3B-2 = 2B2 = 1B6 4s.

The counting is now repeated on the pegboard with a rubber-band showing the 4-bundles.

Now, we count the ten fingers in 3s by raising them one by one beginning with the little finger.

0B1, 0B2, 0B3 or 1B0, moving the fingers to see the difference between the two.

1B1, 1B2, 1B3 or 2B0, again moving the fingers to see the difference between the two.

2B1, 2B2, 2B3 or 3B0, again moving the fingers to see the difference between the two.

3B1 or 4B-2 since we need 2 to have 4B0.

So, with 'flexible bundle-counting' we have four ways to count ten in 3s:

T = ten = 4B-2 = 3B1 = 2B4 = 1B7 3s.

However, when bundling in 3s, 3 bundles become 1 bundle-of-bundles, 3B = 1BB = 1B^2, which suggests that power should be the first operation:

T = ten = 3B1 = 1BB 0B 1 3s, or T = 101 3s, if leaving out the units.

The counting is now repeated on the pegboard with a rubber-band showing the 3-bundles.

We notice that a bundle-of-bundles BB becomes a square.

Now, we count the ten fingers in 2s, pairs.

We begin by seeing how 5 fingers may be counted in 2s.

T = 5 = 1B3 = 2B1 = 3B-1 2s, and T = 5 = 1BB 0B 1 2s.

Putting the two hands together, we see that ten can be counted in 2s as

T = ten = 2BB 0B 2 = 1BBB 0BB 1B 0 2s, or T = 1010 2s, if leaving out the units.

With power, we get a polynomial T = ten = 2*B^3 + 1*B 2s

The counting is now repeated on the pegboard with a rubber-band showing the 2-bundles.

MC04. Bundle-counting in tens

The goal is to use an oral bundle-counting sequence with ten-bundles to represent outside presences of Many as an inside union of singles, bundles, bundle-of-bundles, etc.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a rubber band above the first row.

The action is to include both singles and bundles when counting the pegs horizontally as

0B1, 0B2, …, 0B9, 0Bten or 1B0 or 10. And then

1B1, 1B2, …, 1B9, 1Bten or 2B0 or 20, moving up the rubber band.

This continues until

9B1, 9B2, …, 9B9, 9Bten or tenB0 or 1BB0B0 or 100.

Bundling in tens, a BundleBundle BB is called hundred, BBB = B^3 is called thousand, and BBBB =B^4 is called ten thousand, or Wan in Chinese. Likewise, BBBBBB = B^6 is called a million.

Alternatively, instead we may count what is missing for another full bundle:

1B-9, 1B-8, …, 1B-1, 1B0 or 10. And then

2B-9, 2B-8, …, 2B-1, 2B0 or 20, moving up the rubber band.

This continues until

1BB-9, 1BB-8, …, 1BB-1, 1BB0 or 1BB0B0 or 100

Including also overloads and underloads, we can now practice ‘flexible bundle-counting’ as

T = 67 = 6B7 = 5B17 = 7B-3 tens

MC05. Recounting tens in icons gives equations

The goal is to experience how recounting from tens to icons is another word for an equation, that may be solved by, of course, recounting.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a rubber band to show the bundles, as well as snap-cubes, and a calculator.

Asking “8 is how many 2s” may be rephrased to “8 is u 2s” using letters for unspecified or unknown numbers. This then may be shortened to an equation, 8 = ux2, solved by recounting 8 in 2s as 8 = (8/2)x2, so that the solution u = 8/2 is found when isolating the unknown number by moving the known number to opposite side with opposite sign.

Here we see that when decreasing the bottom in a stack the height increases.

MC06. Recounting icons in tens gives rectangles and multiplication tables

The goal is to translate an oral icon-counting sequence to ten-counting, e.g., translating the sequence 1, 2, 3, 4, 5 4s into 4, 8, 12, 26, 20.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a vertical rubber band to show the stack of bundles.

With 2 as the bundle-size, first we count vertically in 2s as 1, 2, 3, 4, 5 2s. Then we count in tens as 2, 4, 6, 8, 10. Then we count horizontally but letting a finger slide over the pegs. Finally, we memorize by folding fingers.

Likewise with 3, 4 and 5 as the bundle-size. Each time we see that increasing the bundle-size will decrease the height. It is like water that sinks when the bottom is increased.

With 6 or B-4 as the bundle-size, again first we count vertically in 6s as 1, 2, 3, 4, 5 6s. Then we count in tens as 1B-4, 2B-8, 3B-12, 4B-16, 5B-20, or 0B6, 1B2, 1B8, 2B4, 3B0, or 6, 12, 18, 14, 30.

Likewise with 7, 8 and 9 as the bundle-size.

Finally, to translate 7 6s into tens we use two rubber bands to show that 7x6 = (B-3) x (B-4). So, we get the total by removing 3 horizontal bundles and 4 vertical bundles and then add the upper right rectangle 3 4s that was removed twice:

T = 7x6 = (B-3) x (B-4) = 10B – 3B – 4B + 3x4 = 3B + 1B2 = 4B2 = 42.

Here we see that negative multiplied with negative gives plus.

Now we recount 6 78s in tens

T = 6 78s = 6x78 = 6x7B8 = 42B48 = 46B8 = 468

On the pegboard, the horizontal rubber band is a 6 and the vertical illustrates 7B8.

Now we recount 36 78s in tens

T = 36 78s = 36x78 = 3B6 x 7B8 = 21BB (3x8+6x7)B 48 = 21BB (2B4+4B2)B 48 = 21BB (6B6)B 48 = (21+6)BB (6+4) B 8 = 27BB 10B 8 = 28BB0B8 = 2808

On the pegboard, the horizontal rubber band illustrates 3B6 the vertical illustrates 7B8. This gives the four parts 21BB, 24B, 42B, and 4B8, or 21BB, 2BB4B, 4BB2B, and 4B8, adding up to 27BB10B8 or 28BB0B8 or 2808.

A quick way is to write the numbers under each other. Multiplying down the gives the BBs and the unbundled, and cross-multiplication gives the Bs.

MC07. Bundle-Bundles are squares

The goal is to count the squares occurring as bundle-bundles.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with two rubber bands to show the squares.

With 2 as the bundle-size, the 2x2 square contains a total of 4, *T* = 2x2 = 4.

With 3 as the bundle-size, the 3x3 square comes from adding a vertical and a horizontal 3-line, and then remove the upper right corner added twice, so the total is 9, *T* = 3x3 = 9.

With 4 as the bundle-size, the 4x4 square comes from adding a vertical and a horizontal 4-line, and then remove the upper right corner added twice, so the total is 16, *T* = 4x4 = 16.

With 5 as the bundle-size, the 5x5 square comes from adding a vertical and a horizontal 5-line, and then remove the upper right corner added twice, so the total is 25, *T* = 5x5 = 25.

The first 5 squares may be memorized by folding fingers at the same time.

With B-1 as the bundle-size, the 9x9 square comes from twice removing 1 bundle, and then add the upper right corner removed twice, so the total is 8B1 or 81.

With B-2 as the bundle-size, the 8x8 square comes from twice removing 2 bundles, and then add the upper right corner removed twice, so the total is 6B4 or 64.

With B-3 as the bundle-size, the 7x7 square comes from twice removing 3 bundles, and then add the upper right corner removed twice, so the total is 4B9 or 49.

With B-4 as the bundle-size, the 6x6 square comes from twice removing 4 bundle, and then add the upper right corner removed twice, so the total is 2B16 or 3B6 or 36.

We notice, that with square numbers, the end digit occurs again.

MC08. Recounting rectangles as squares gives square roots to solve quadratics

The goal is to experience how a stack may be squared by fitting it between two squares; and how a square always contains two smaller squares and two like rectangles that may solve quadratics.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with rubber bands. First, we use two bands to show the rectangle and two to show the square. Then we use two bands to show the two squares and the two rectangles.

Wanting to square a rectangle, we may ask: "How to change 6 3s into a BB square?"

We see, that 6 3s as a square will fit between a 4x4 and 5x5 square and that it is closer to 4x4.

The overflow here is 6x3-4x4 = 2 = 2x1, so 1 has to added to 4the 4s as 1 = (1/4)x4 = ¼ 4s = 0.25.

So, a guess would be that the 6 3s may be transformed into a 4.25x4.25 square, so that 4.25 may be close to the square root of 6x3. A little less since we need a little for the upper right corner.

A calculator shows that the square root is 4.243.

As to 8 3s, we see, that it is closer to 5x5. The missing here is 5x5-8x3 = 1 = 2x½, so ½ has to be subtracted from 5 as ½ = (½/5)x5 = 0.1 5s.

So, a guess would be that the 6 3s may be transformed into a 4.9x4.9 square, so that 4.9 may be close to the square root of 6x3. A little less since again we need a little for the upper right corner.

A calculator shows that the square root is 4.90.

Wanting to split a square in less squares, the BBBoard shows that with B = 7, the (B+3) square splits up into a BxB square and a 3x3 square and two 3xB rectangles:

(B+3)x(B+3) = BxB + 3x3 + 2 x (3xB), or (B+3)^2 = B^2 + 6B + 9

With B = 7 we get B^2 + 6B + 9 = 100, or B^2 + 6B - 91 = 0, a so-called 'quadratics'.

Turning the process around, we may want to find B by solving the quadratic B^2 + 6B – 91 = 0, with B unknown.

So, since 6 = (6/2)*2 = 3*2, on a (B+3)x(B+3) BBBoard we use two rubber bands to show the 3x3 square in the upper right corner and the BxB square in the lower left corner as well as the two 3xB rectangles that combine to a big square (B+3)^2:

(B+3)^2 = B^2 + 6*B + 9

To include -91, we now rewrite 9 = 100 – 91, so

(B+3)^2 = B^2 + 6*B + 100 - 91,

But, since B^2 + 6*B – 91 = 0, we get

(B+3)^2 = 100 = 10^2, so

B+3 = 10 or B+3 = -10, so

B = 7 or B = -13

We were lucky, since some quadratics cannot be solved as B^2 + 6B + 10 = 0, where 9 becomes 10-1, so that (B+3)^2 = B^2 + 6*B + 10 -1 becomes (B+3)^2 = -1 that is not possible.

Now we may want to solve the quadratics B^2 - 6B +5 = 0

So, since 6 = (6/2)*2 = 3*2, on a BBBoard we use two rubber bands to show the 3x3 square in the upper right corner and the (B-3)x(B-3) square in the lower left corner as well as the two 3xB rectangles that combine to a big square B^2: B^2 = (B-3)^2 + 6*B + 9.

On the BBBoard we see that the (B-3)^2 square is left if we from the B^2 square pull away a 3xB rectangle twice and then add the 3^2 square that was removed twice:

(B-3)^2 = B^2 – 6*B + 9

To include 5, we now split 9 = 5+4, so

(B-3)^2 = B^2 – 6*B + 5 + 4

But, since B^2 – 6*B + 5 = 0, we get

(B-3)^2 = 4 = 2^2, so

B-3 = 2 or B-3 = -2, so

B = 5 or B = 1

Again were lucky, since B^2 - 6B +10 = 0 cannot be solved, where 9 becomes 10-1, so that

(B-3)^2 = B^2 + 6*B + 10 -1 becomes (B-3)^2 = -1 that is not possible.

The BBBoard shows that adding the 7x7 and the 3x3 square along the diagonal gives a 10x10 square except for two 3x7 rectangles (or tiles). Instead we could transform the 3x3 square into two ux7 tiles where ux7x2 = 3x3, or ux14 = 9 = (9/14)x14, so that u = 9/14 = 0,64.

So, we guess that the 7x7 and 3x3 squares add as a 7.64x7.64 square giving 58.4 which becomes the wanted 49+9 = 58 if we pull away the 0.64x0.64 square that was added twice, 58.4 – 0,4 = 58. With the root we find the exact answer, V 58 = 7.62, which turn out to be the length of the raising diagonal in the 3x7 tile, as we can see by cutting out four examples of this tile on paper.

The first paper-tile is placed on the upper left tile on the BBBoard, the next is turned and placed at the upper end, the third is again turned before placed at the end, and likewise the fourth. We now see that the diagonals form a square that is a full 10x10 square where 4 times we pull-away a half square, a total of 2 tiles. Removing the paper tiles, we see that this is also the case when adding the 3x3 and the 7x7 squares along the diagonal. Thus, we guess that in a tile, its diagonal square is the sum of the squares of its sides. This is called the Pythagoras rule named after a Greek philosopher.

MC09. Recounting in physical units gives per-numbers

The goal is to experience how recounting in two physical units creates per-numbers bridging the two to answers proportionality questions as "if 3kg costs 4$, then 15kg costs ?$, and ?kg costs 12$".

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with a rubber band placed on a paper to show the two units.

Recounting 3kg as 4$ gives two per-numbers 4$/3kg (the price), and 3kg/4kg shown by a vertical rubber band as 3 on the board, and as 4 below on the paper. On a peg board 15 kg may by squeezed to 3s by recounting, 15kg = (15/3)x3kg = 15/3 times 3kg, so we also have 15/3 times 4$, or 20 $.

Likewise, 12$ = (12/4)x4$ = (12/4)x3kg = 9 kg.

MC10. Recounting in the same unit gives fractions

The goal is to experience how per-numbers become fractions when recounting in the same unit.

As concrete material serves a ten-by-ten bundle-of-bundles pegboard with two parallel rubber bands to show the whole and the part.

The BBBoard shows that getting a share of 3$ per 5$ means getting the fraction 3 per 5, or 3/5.

And that getting 3 per 5 means getting 6 per 10, 9 per 15, etc.

And that to get 3 per 5 of 20, we recount 20 in 5s as 20 = (20/5)*5, so we get (20/5)*3 = 12 as seen on the BBBoard.

MC11. Recounting rectangle-sides gives trigonometry before geometry

The goal is to experience how per-numbers in a rectangle split by its diagonal leads to trigonometry.

Recounting the height in the base, height = (height/base) * base = tangent Angle * base, shortened to

$h = (h / b) * b = \tan A * b = \tan A\ b$s,

This gives the formula tangent A = height / base, or $\tan A = h/b$.

Using the words run and rise instead of base and height, we get the diagonal's slope-formula:

$\tan A$ = rise/run.

The word 'tangent' is used since the height will be a tangent in a circle with center in A with the base as its radius. This gives a formula for the circumference since a circle contains many right triangles leaving the center. In a circle with radius 1, h recounts in r as $h = (h/1) * 1 = \tan A$.

A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 1.8 100s = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, h, are almost identical. So, half the circumference, called π, is

$\pi = 100 * h = 100 * \tan 1.8 = 100 * \tan (180/100) = 3.1426$

This gives a formula for the number π:

$\pi = \tan (180/n) * n$, for n sufficiently large.

We also see that in a circle with radius r, the circumference is $2*\pi*r$, and the area is $\pi*r$^2, or $\pi/4* d$^2 where d is the circle's diameter.

So, a d-circle to takes up almost 80% of the space inside the surrounding d-square.

MC12. Adding next-to and on-top gives calculus and proportionality

The goal is to experience how, once counted and recounted, totals may add in two different ways, next-to and on-top.

As concrete material serves a ten-by-ten Bundle-Bundle BBBoard with rubber bands, as well as snap-cubes.

We ask "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?" We see that next-to addition geometrically means adding by areas, so multiplication precedes addition. Algebraically, the recount-formula predicts the result. Next-to addition is called integral calculus.

We now ask the reverse question "If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?" We find the answer by removing the initial stack and recounting the rest in 3s. So now subtraction precedes division, which is natural as reversed integration, also called differential calculus.

We ask "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 3s; and as 5s?" We see that on-top addition means changing units by using the recount-formula. Thus, on-top addition typically applies proportionality.

We now ask the reverse question "If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?" We find the answer by removing the initial stack and recounting the rest in 5s. So again, subtraction precedes division as in differential calculus.

MC13. Adding per-numbers gives calculus

The goal is to experience how per-numbers (and fractions) first must be multiplied to unit-numbers before being added.

We ask "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" We see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas. Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by epsilon and delta.

We now ask the reverse question "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" We see that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, creating the areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

MC14. Adding unspecified letter-numbers

The goal is to experience how letter-numbers add by first finding their common unit.

In the letter-number $T = 3ab$ the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, T may be seen as 3 *ab*s, or as (3*a*) *b*s, or as (3*b*) *a*s. To avoid being confused by the 's' we will omit it, so $T = 3ab = 3 * ab = 3a * b$ or $3b * a$.

Since totals need a common unit to add, this must be first found:

$T = 3ab + 4ac = 3b * a + 4c * a = (3b+4c) * a$

$T = 2ab\text{^}2 + 4bc = ab * 2b + 2c * 2b = (ab+2c) * 2b$

MC15. The Algebra Square

The goal is to experience how an 'Algebra Square' shows the four ways to unite and split-into constant and changing unit-numbers and per-numbers.

Counting ten fingers in 3s gives T = 1BundleBundle 1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula $T = a*x\text{^}2 + b*x + c$, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration; in accordance with the Arabic meaning of the word algebra, to reunite.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite changing and constant unit-numbers, and integration and power unite changing and constant per-numbers. We might call this beautiful simplicity 'the algebra square'.

Operations unite/ split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	$T = a + n$ $T - n = a$	$T = a * n$ $T/n = a$
Per-numbers m/s, $/100$ = %	$T = \int f\,dx$ $dT/dx = f$	$T = a\text{^}b$ $b\surd T = a \quad loga(T) = b$

Figure 2. The Algebra Square Reunite into Total to be Split into the Four Number-Types

MC16. Algebra and geometry coordinated.

The goal is to experience how a coordinate-system connects algebra calculations and geometry forms. As concrete material serves a Bundle-Bundle BBBoard with rubber bands and pins, as well as a 0-by-10 square on paper.

From the lower left point on a BBBoard we may travel in the horizontal x-direction with an 'out-number' x, and in the vertical y-direction with an 'up-number' y. In this way a point has two coordinates telling how far out and how far up it is placed. Here, the lower left point is (0,0).

If we from (0,0) take a '1 out, 2 up' trip we reach the points (1,2), (2,4), (3,6), etc. We see that y and x are linked by the formula $y = 2*x$ which is called the formulas for the line we travel along.

Likewise, if we from (0,9) take a '1 out, 1 down' trip we reach the points (1,8), (2,7), (3,6), etc. In this case the line formula is $y = 9\text{-}x$.

Asking where the two lines intersect, the geometry forms show that this takes place in the point (3,6).

An algebra calculation gives the same result: Since the intersection point belongs to both lines, the coordinates obey both formulas, $y = 2*x$ and $y = 9 - x$, so $2*x = 9\text{-}x$. This equation is solved by moving to opposite side with opposite sign:

$2*x = 9\text{-}x$, so $2*x + x = 9$, so $3*x = 9$, so $x = 9/3 = 3$, so $y = 2*3 = 6$.

The intersection point thus has the coordinates $(x,y) = (3,6)$.

On a BBBoard, two rubber bands show 6 7s as a 6-by-7 rectangle. An additional vertical rubber band is placed between 4 and 5 to signal that its x-number is unspecified.

Now, the diagonal in the '10 7s' rectangle represent a '7 out, 10 up' trip as well as a 'x out, y up' trip.

But since the angle A from horizontal is the same in the two trips we have tan A = $y/x = 10/7$. This gives the equation $y/x = 10/7$ that may be solved by moving to opposite side with opposite sign:

$y/x = 10/7$, so $y = 10/7*x$.

The intersection point between the diagonal and the horizontal rubber line then may be found by equating the two y-formulas, $y = 10/7*x$ and $y = 6$, again solved by moving to opposite side with opposite sign.

$10/7*x = 6$, so $x = 6*7/10 = 4.2$, which gives $y = 10/7 *4.2 = 6$.

The intersection point thus has the coordinates $(x,y) = (4.2,6)$.

Likewise, we see that the diagonal in the '6 10s' rectangle has the formula $y = 6/10*x$.

Also, we see that for the descending diagonal in the '3 tens' rectangle the angle A from horizontal seen from the point (10,0) may be found in two ways, as tan A = $y/(10-x)$, and as 10/3. This gives the equation $y/(10-x) = 10/3$ that may be solved by moving to opposite side with opposite sign:

$y/(10-x) = 10/3$, so $y = 10/3*(10-x) = -10/3*x + 100/3$.

Likewise, looking at the descending diagonal in the '6 tens' rectangle, its formula is

$y/(10-x) = 6/10$, so $y = 6/10*(10-x) = -6/10*x + 100/6$.

The four diagonals together with the horizontal $y = 6$ line now form a star with five intersection points with coordinates that may be found both by algebra calculations and by geometry inspection.

On a BBBoard showing 6 7s, a triangle is formed by the three lines connecting the points (0,0) and (7,10) and (10,7). Typically, we want to find the 7 important triangle numbers, its area, its three angels and its three sides.

We see that these 7 triangle numbers may be found indirectly by looking at the three half rectangles that is pulled away from the triangle's wrapping rectangle.

In the lower pull-away half-rectangle we see that the angle is predicted by the formula tanA = 6/10, which on a calculator gives A = 31.0 degrees. And that the area is ½*6*10 = 30. And that the length of the diagonal d is found by squaring: d^2 = 10^2 + 6^2 = 136, giving d = V136 = 11.7.

Following the line with the formula $y = 0.1*x^2 + 1$ we see that the line bends upwards and becomes a curve instead, called a parabola. It intersects the diagonal $y = x$ in two points that may be calculated by equating the two formulas:

$0.1*x^2 + 1 = x$, or $x^2 + 10 = 10x$, or $x^2 -10x + 10 = 0$

This quadratics is solved by $x = 1.1$ and $x = 8.9$.

The two intersections points thus have the coordinates (1.1, 1.1) and (8.9, 8.9).

From (0,0) we may take a trip with a constant x-number 1, but with a falling up-number y beginning with 4 and ending with -4. We then reach the points (1,4), (2,7), (3,9), (4,10), (5,10), (6,9), (7,7), (8,4), and (9,0). Again, we get a parabola, but now curving downwards. Since the y-number is 0 when x is 0 and 9, its formula could contain the two factors x and $(x-9)$, $y = a*x*(x-9)$. Here we have three unspecified numbers, but in the point (1,4) we only have one:

$4 = a*1*(1-9)$, or $4 = a*(-8)$, so $a = 4/(-8) = -½$

The formula of the parabola then is

$y = -½*x*(x-9)$, or $y = -½*x\^2 + 4½ x$

The two intersection points with the line $y = 3+x$ may now be found from inspection to be (1,4) and (6,9). Or from calculations by equating the two formulas:

$-½*x\^2 + 4½ x = 3+x$, that after changing units to -½s gives

$x\^2 – 9*x = -6 – 2*x$, or $x\^2 – 7*x + 6 = 0$ with the solutions $x = 1$ and $x = 6$.

The two intersections points thus have the coordinates (1, 4) and (6, 9).

MC17. Numbers in time and space: Change and distribution

The goal is to experience how formulas calculating *y* from *x* form curves that expresses change in time, and how totals in space may be split in parts that each then becoming a percentage of the total.

An unspecified number becomes a letter, and so does an unspecified formula, $y = f(x)$, also called a function.

Writing out fully with units, the number 345 becomes $T = 3*B\^2 + 4*B + 5$. Using *y* an *x* instead of *T* and *B*, this basic number-formula is called a polynomial, $y = 3*x\^2 + 4*x + 5$. Here 5 is called the level-number, 4 is called the change-number that may be positive or negative if the curve goes upwards or downwards, and 3 is called a curvature-number that may be positive or negative if the curve curves upwards or downwards.

The basic number-formula also shows the basic change-formulas: Proportional change, $y = 4*x$; constant or linear change $y = 4*x+5$; constantly changing or accelerated change $y = 3*x\^2 + 4*x + 5$; power change $y = 3*x\^2$; and exponential change $y = 3*2\^x$.

Saving money in a bank combines a constant change-number, *a*, with a constant change-percent, *r*. Here the total input-amount, *A*, and the total input-percent, *R*, are the same when recounted in the single numbers respectively: $A/a = R/r$.

Formulas typically is used to predict future behavior, either with certainty or with probabilities. If that is not possible, instead statistics is used to describe past behavior with a mean and a deviation to predict that with 95% probability the next occurrence will fall within the interval created by the mean plus or minus twice the deviation.

Distributing a total in two different categories leads to cross tables that may be shown with numbers or with percentages. Again, as per-numbers, percents must first be multiplied to unit-numbers before being added. So, care must be shown when describing cross tables with percentages: On my hand I have 2 right R-fingers and 3 left L-fingers. One R and two L carry a green dot. So, we may say “among the green, 1/3 is R” but not “among the R, 1/3 is green” since ½ is green among the R.

MC18. The Three Tales: Fact, Fiction and Fake

The goal is to experience that as qualitative literature, quantitative literature also has the genres, fact and fiction and fake when modeling real world situations (Tarp, 2001).

Fact is ‘since-then’ calculations using numbers and formulas to quantify and to predict predictable quantities as, e.g., ‘since the base is 4 and the height is 5, then the area of the rectangle is $T = 4*5 = 20$’. Fact models can be trusted once the numbers, and the units, and the formulas, and the calculation have been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter.

Fiction is ‘if-then’ calculations using numbers and formulas to quantify and to predict unpredictable quantities as, e.g., ‘if the unit-price is 4 and we buy 5, then the total cost is $T = 4*5 = 20$’. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made.

Fake is ‘what-then’ calculations using numbers and formulas to quantify and to predict unpredictable qualities as, e.g., ‘Let us close hospitals since it is cheaper to rest in a graveyard’. Fake models build on dubious assumptions and should be replaced by a political process containing debate before decision.

MC19. Teacher education in CATS: Count & Add in Time & Space

The goal is to experience how teacher education may be different to obtain a communicative turn in number-language education.

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows mastery of Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental (Kolb, 1984) learning curricula as, e.g., in Vietnam.

The MATHeCADEMY.net offers a free one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e., both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by $T = (T/B)$*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Discussing the difference

We asked what could be different in online education as compared with offline education.

Typically, mathematics is taught as an end-product from a textbook. So, a difference would be to teach mathematics as a process where things in space are acted upon in time. This would make mathematics a number-language with sentences containing an outside subject, a verb, and an inside predicate as in word-language sentences. As the outside thing to talk about we choose a ten-by-ten bundle-bundle board, a BBBoard, where outside totals appear as rectangles within two rubber bands showing the bundle-number and the counting number.

A process-based mathematics contains numerous differences.

● Here, numbers are no longer traditional line-numbers. Now they are rectangular stack-numbers with both a unit-number and a counting-number. Here changing units leads directly to the recount-formula expressing proportionality, to equations solved be recounting, and to per-numbers adding by their areas just as stack-numbers do when adding next-to. So here the three core parts of mathematics, proportionality and equations and calculus, appear from the beginning, and not in the middle and the end of the K-12 curriculum.

● Here, digits are no longer symbols with names to memorize and to learn how to read and write. Now they are icons built in time by uniting five sticks or strokes in one five-icon, etc. Of course, the icons are not exactly identical to the textbook symbols, but they are close. And, and they give a concrete meaning to the two-digit number 10 by writing it with units, 1 Bundle 0, or 1B0, after having quickly collected all ten sticks into one bundle with none left behind. And then followed by 1B1, 1B2, called 'one-left' and 'two-left' by the Nordic Vikings.

● Here, the counting sequence is no longer a sequence of names like the sequence of months in a year or cities on a railway. Now a combination of digits and units occurs when performing a multiple counting of singles, bundles, bundles-of-bundles etc. This changes one three-digit number to three numberings, and thus makes obsolete the place-value system. So here, 7 changes to 0B7, 45 to 4B5, and 678 to 6BB7B8. And accepting underloads and overloads leads to flexible bundle-numbers with units allowing 48 to be written both as 5B-2, 4B8, and 3B18 tens, which eases standard calculations by making carrying obsolete.

● Here, power is no longer the last operation. Now, power occurs as the first operation since counting ten fingers in 3s allows meeting a bundle-of-bundles, a BB or B^2, since 9 is 3B0, or 1BB0B0, or 100 3s. And counting in 2s allows meeting a bundle-of-bundle-of-bundles, a BBB or B^3, since 8 is 4B0 or 1BBB0BB0B0, or 1000 2s.

● Here, the order of operations is no longer add-subtract-multiply-divide. Now it is the exact opposite. With the process-based math, the first task is to bring an outside total inside by counting it in bundles. So, the first process is to push-away bundles, which may be iconized as a broom so that the outside action 'from 8, push-away 2s' inside becomes the calculation 8/2. The next task is to lift the bundles into a stack, which may be iconized as a lift so that the so that the outside action '4 times lift 2s' inside becomes the calculation 4x2. The next task is to pull-away the stack to look for unbundled singles, which may be iconized as a rope so that the outside action 'from 9, pull-away 8' inside becomes the calculation 9 – 8. Included on-top of the stack, the unbundled may be seen as a decimal number, 4B1 2s, or counted in the unit, 4 ½ 2s, or described at what is pulled-away from the top bundle, 5B-1 2s. Finally, after brought inside as stacks outside totals may now be added next-to as areas like integral calculus, or on-top after recounting has made the units like.

So here, the order of operations is different as the precise opposite. First power comes when bundling bundles, then division pushes-away bundles, then multiplication lift the bundles, then subtraction pulls-away the stack to allow the unbundled to also join the stacks, which finally may add next-to or on-top.

Also, the operations have different meanings, now 8/2 means 8 counted in 2s and not only 8 split in 2, and now 4x8 means 4 8s and not only 3B2 tens, and now subtraction is a means to allow the unbundled to join the stack and give concrete meaning to decimals, fractions, and negative numbers. Finally, addition has two different meanings leading directly to the core of mathematics, calculus and linearity (proportionality). So here, integral calculus occurs in grade one as next-to addition of stacks, which becomes differentiation when reversing the process. Later it reappears in middle school where per-numbers also add by areas since they must be multiplied to unit-numbers before adding.

Since bundle-numbers have units a core task is to change units. Here division and multiplication immediately are combined in the recount-formula that recounts the total T in Bs as T/B Bs, $T = (T/B)\text{x}B$, expressing the proportionality that is visible when observing that increasing the base of a stack implies decreasing the height to keep the Total unchanged. In its process-based version, the recount formula is a core formula present all over mathematics and science. In its product-based version it is totally absent. Likewise, subtraction and addition immediately are combined in a split-formula that splits T in B and $T - B$ by pulling away B from T, '$T = (T - B) + B$'.

In textbooks, the first operation typically is addition based on the statement that '2+1=3 always', which allows 2+3 to be defined as 2+1+1+1. However, this is not a number-statement, but a word-statement saying that "the follower of two is three" useful in time when traveling through a row of places. It is not a statement about uniting two totals in space, since here the units are needed also as, e.g., 2 pairs + 1 triple is 2B1 3s, or 3B1 2s, or 7 1s. Built on the assumption that '2+1=3 always', mathematics becomes 'mathematism' true inside its own world, but seldom outside in the real world where numbers always carry units, and where "additions often folds while multiplication always holds" since 2x3 = 6 simply states that 2 3s may be recounted as 6 1s.

● Here, decimal numbers and fractions and negative numbers no longer have to wait until they are introduced as rational and negative numbers in middle school. Now they arise at once as different ways to see the unbundled placed on-top of the bundle-stack as described above where 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s.

● Here, counting sequences no longer leave out the units. Now the units are included so that 2 becomes what exists outside, 0-bundle-2, shortened to 0B2. Likewise, 23 becomes 2B3, and 456 becomes 4BB5B6. So here, counting ten fingers in 3s makes 9 a bundle-of-bundles, a BB or B^2, which allows power to be the first operation, and no longer the last. Also, 'past-counting' is now allowed as in France calling 7B1 for 6B11 since they gave up understanding the Vikings in Normandy saying '1 and half-four' meaning '1 and half the way to 4 twenties'. Likewise is 'less-counting' saying "Bundle less nine, B-8, B-7, …, B-1, 1 Bundle, 2B-9, etc.

● Here, functions are no longer subsets in a set-product where first component identity implies second component identity, and which is postponed to middle school. Now a function is a number-language sentence called a formula with a subject, a verb and a predicate as in word-language sentences. Only here, the inside predicate is a prediction of what happens with the outside subject, so that $T = 2x3 = 6$ is an inside prediction of the outside fact that 2 3s may be recounted as 6 1s.

● Here, equations no longer are identical number-names that may be transformed by the same action on both sides, and which is postponed to middle school. Now equations describe reversed processes from the beginning. Here, the addition equation 'u+3 = 7" asks "7 splits in 3 and something?". That of course is found by splitting 7 in 7 = (7 – 3) + 3, so that u = 7-3. This resonates with the formal definition of subtraction: "7-3 is the number n that added to 3 gives 7", or "if n+3 = 7, then n = 7-3". So, we see that the equation is solved when reversing the process by moving a number across the equal sign with the reverse calculation sign.

Here, the multiplication equation 'u*3 = 12" arises when recounting from tens to icons leads to asking "How many 3s in 12?". That of course is found by recounting 12 in 3s as 12 = (12/3)*3 so that u = 12/3. This resonates with the formal definition of division: "12/3 is the number n that multiplied with 3 gives 12", or "if n*3 = 12, then n = 12/3". So, again we see that the equation is solved when reversing the process by moving a number across the equal sign with the reverse calculation sign.

Likewise, reversing power-operations leads to equations as u^3 = 8 solved by the factor-finding root u = 3√8, and to equations as 2^u = 8 solved by the factor-counting logarithm u = log2(8). Finally, integration moves across as differentiation. So now there is no need for the abstract algebra group concepts as communicative and associative laws as well as neutral and inverse elements, which step by step motivates the 'do the same to both sides outside the brackets´ method that is presented fully to teachers but only in parts to students.

● Here, multiplication tables are no longer sequences learned by heart. Now 6x7 is 6 7s in its own right. And it may or may not be recounted in tens. And if so, it takes place on the BBBoard where 6 and 7 are seen as B-4 and B-3 respectively. This transformation of 6x7 into (B-4)x(B-3) introduces early algebra where the BBBoard shows the process where the 6 7s are left when pulling-away from the 10 Bundles, first 4 bundles then 3 bundles and finally adding the 4 3s that have been pulled away twice, thus clearly showing that minus times minus gives plus. So, to get the answer quickly, from 10 you subtract (3+4), see them as bundles, and add 3x4. So, all you have to memorize is the lower left corner on the BBBoard where you build up of from 2 2s to 5 2s, then from 2 3s to 5 3s, then with 4s and finally with 5s. Multiplication on a BBBoard thus offers a middle way in 'the multiplication war' (Economist 2021).

● Here, squares are no longer special numbers in the multiplication tables. Now they are Bundle-Bundles on a BBBoard that may be built by always adding an extra line horizontally and vertically and subtracting the upper right corner-number that was added twice. So now squares form a sequence of their own quickly learned. And here the square root no longer has to wait until irrational numbers are introduced in middle school. Now they arise from a simple question "How to square a rectangle without changing its total?" The BBBoard shows that you simply move half the overload from vertical to horizontal position or vice versa. The answer then needs to be reduced a little so that there is also something for the upper right corner.

Finally, here quadratics no longer must wait to high school. Now they appear on a $(x+a)*(x+a)$ board when two rubber bands show the two squares, x^2 and a^2, as well as the two $a*x$ rectangles that all disappear if $x^2 + 2ax + b = 0$, only leaving $a^2 – b$ as $(x+a)^2$ after being squared.

● Here, proportionality no longer must wait to middle school. Now it appears when recounting a total in two units creates per-numbers bridging the units by recounting in the per-number: with 4kg per 5$, to answer the question '12 kg = ?$', we simply recount 12 in 4s: 12 kg = (12/4)*4kg = (12/4)*5$ = 15$. Likewise, with answering the inverse question '?kg = 20$'. Or we may just recount the units: $ = ($/kg)*kg = (5/4)*12 = 15.

● Here, fractions no longer are rational numbers that can add without units. Instead, fractions like per-numbers are operators needing numbers to become numbers before adding. So, we must always ask "fractions of what?" Counting red apples provides an example. Here ½ of 2 apples plus 2/3 of 3 apples gives (1+2)/(2+3) of 5 apples, i.e., 3/5 of 5 apples, and not (1*3+ 2*2) of 2*3 apples, i.e., 7/6 of 6 apples. So ½ + 2/3 is 7/6 only if they are taken of the same total, just as 2+1 is 3 only if they have the same unit. So addition has only meaning inside a bracket with the common units outside, as expressed in the distributive law, $p*r + q*r = (p+q)*r$. Shortening or enlarging fractions now simply means removing or adding a common unit.

● Here, trigonometry no longer must wait to high school where plane geometry as well as coordinate geometry has been taught. Now it appears as per-numbers coming from recounting the sides in a rectangle split by a diagonal. This means that concrete examples from all STEM areas may be looked at very early.

● Here, coordinate geometry no longer comes after plane geometry. Instead, it serves its name by coordinating algebraic calculations with standard geometrical forms which may or may not later be studied by themselves in plane geometry.

● Here, addition no longer is the first operation to teach. Instead, it is the last since counting and recounting first must bring outside totals inside to be added in two different ways, next-to by their areas, or on-top after recounting has made the units like. So, when finally introduced, addition and its reverse leads directly to the core of mathematics, calculus and proportionality. And, with totals occurring as flexible bundle numbers with units, overloads and underloads make carry and borrow math unneeded.

● Here, functions no longer must wait until late middle school to be defined as a rule that assigns to each element in one set one and only one element in another set, which is the soft version as the one taught in teacher education: a function is a subset in a set-product where first component-identity implies second-component identity. Instead function now occur as number-language sentences with a subject, a verb and a predicate just as in word-language sentences. And taking on the name 'formula' when containing unspecified letter-numbers.

● Here, calculus no longer must wait to be taught to few the last year of school. Now it is taught to all the first year when totals are added next-to as areas. In middle school it is again taught to all when adding piecewise constant per-numbers and when adding fractions in cross tables. So, in principle, it can be taught to all in high school also as a way to add locally constant per-numbers from science. Especially if the natural order is followed where integration comes before differentiation invented to recount areas as changes to allow many changes to be added as one change only from the initial to the end number.

● Here, algebra no longer is an abstract activity looking for pattens. Instead, algebra takes on its original Arabic meaning, to reunite like and unlike unit-and per-numbers as shown in the Algebra-square. Here the rules for calculating on with unspecified letter-numbers come automatically recount totals from icons to tens on a BBBoard.

● Here, modeling no longer must obey the parole “Of course, math must be learned, before it can be applied”. As a number-language with sentences assigning inside numbers and calculations to outside totals, modeling occurs always. Dut, as in the word-language, it is also important to evaluate if the sentences express fact, fiction or fake. So, these three genres now become an important part of using the number-language to be skeptical towards any quantitative literature.

Testing the difference

Being very costly to change expensive textbooks and long-term teacher education makes testing the validity of ManyMath difficult in a traditional education, except for where it is stuck, e.g., division and fractions and letter calculations. But ManyMath may be tested outside the main track: in preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

As to teacher training, the MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows Mastering Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiental learning curricula as, e.g., in Vietnam.

Search questions about ‘bundle-numbers’ and ‘recounting’ may be given to small groups of four preschool children to get ideas about how to design a full first-generation curriculum. Asked “How old next time?” a three-year-old will say four showing four fingers, but the child will react to seeing the fingers held together two by two: “That is not four. That is two twos!” The child thus describes what exists, bundles of 2s, and 2 of them. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)x2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms this independent of the total and the bundle thus giving a formula with unspecified numbers ‘$T = (T/B)\text{x}B$’ saying “T contains B, T/B times.”

Conclusion

Inspired by the difference between teaching mathematics on-line outside the classroom and teaching it off-line inside we formulated the question about what else could be different that led to a possible answer: a difference could be to teach mathematics as a process in time instead of as a product in space. Or, in other words, a difference could be to teach mathematics as an inside number-language about outside quantities parallel to the inside word-language about outside qualities. Or again in other words, a difference could be to let mastery of the outside existence, Many, precede mastery of the inside essence, Mathematics.

And yes, mastery of Math may be preceded by mastery of Many that in return automatically leads to mastery of the Math core. Which allows details to be taught as footnotes to those who may be interested while the rest may focus on using their mastery to work with tales about Many and to discuss which of the three genres they belong to, fact or fiction or fake. And since outside Many are brought inside as total counted and recounted in flexible bundle-numbers with units as shown in the polynomial form of numbers, the basic inside tales about outside Many could be about things and actions on a ten-by-ten Bundle-Bundle Board, a BBBoard. So yes, a difference is to teach mathematics as a process in time instead of as a product in space.

Of course, working on a BBBoard, learners may not learn school or university mathematics in a strict traditional form, but they learn to communicate about Many; and by always including units they avoid the tradition’s mathematism with its fake addition claims only holding inside but often folding outside the classroom.

So, it is possible to have a communicative turn in number-language education like the communicative turn in word-language education that took place in the 1970. Where outside-inside use of outside examples to be described by inside language-sentences was allowed to precede or even replaced by inside-inside use of inside sentences to be described by inside grammar-sentences.

What remains is testing to see if the difference expressed in the above micro-curricula makes a difference. This may be impossible inside traditional education because of heavy cost to change textbooks and to establish in-service teacher training. But it may be possible outside in math labs and in special education where the goal could be to use BBB to obtain BBB, i.e., to use Bundle-Bundle-Boards to Bring Back Brains. But, although designing and testing micro-curricula go together in design research, they should not be mixed: Architects design houses, they do not build them. Likewise with curriculum architects.

And in the end the core question is what right allows traditional education to replace the child's innate concrete number-language with a foreign abstract language? Is this not just another example of using the power of an imported institution to colonize the native brains?

So, a final question could be: Forcing mastery of inside mathematics to precede mastery of outside Many, does that follow the 17 United Nations Sustainable Development Goals? Here goal 4 about quality in education formulates the wish to "ensure inclusive and equitable quality education and promote lifelong learning opportunities for all". And here target 4.6 states the subgoal to "By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy".

A communicative turn in number-language education may allow this development goal to be reached now. Postponing this turn clearly will not and may instead keep widespread innumeracy alive indefinitely. Which of course instead may serve well the wish for ever more funds for teacher education of teachers and researchers as a goal displacement warned against in sociology (Bauman, 1990; Arendt, 1963).

Phenomenologically, it is important to respect and develop the way Many presents itself to children thus providing them with the quantitative competence of a number-language. Teaching numbering instead of numbers thus creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education. The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not learn from children instead?

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