# MODELING EASED BY DEMODELING AND REROOTING 

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#### Abstract

Modeling should motivate mathematics education, but is not always that easy. Could this be different? Difference research searching for differences making a difference suggests that inside concepts may be de-rooted from the outside world by getting different names and meanings. So de-modeled they may retake their original roles allowing mathematics to again become a number-language communicating about the outside world. The search found children's own flexible bundle-numbers with units, that allow counting and recounting to precede adding. This leads directly to the core of mathematics, using proportionality and calculus to re-unite changing and constant unit-numbers and per-numbers.


## DOES MATHEMATICS MODELING HAVE TO BE SO DIFFICULT?

Inspired by the first International Congress on Mathematical Education, ICME-1, I joined the student revolt in 1969 to secure that mathematics would no more be taught without linking it to its use through modeling. I was allowed to write a master thesis on modeling where I chose Game Theory. In 1974 I published my first textbook "Mathematical Growth Models" showing how calculus is modeling predictable non-constant change. And at the ICME-3 I presented a poster "Mathematics, a collection of arbitrary theoretical structures, or model-building of the real world" as well as a short oral address in English and French "Mathematics, an integral part of the real world". From 1975 I worked with Mogens Niss for three years at Roskilde University. We were both interested in modeling the Danish macro-economic cycle. Niss preferred the actual government model where the mathematics was so complicated that it could not be addressed in high school. I saw the model, not as 'since-then' fact model as in physics, but as an 'if-then' fiction model based upon assumptions that could be different. And here Ockham's razor says with two different models to explain the same, you should prefer the simpler one. So, I worked out a simple linear model that could be used in high school (Tarp, 2001). Niss stayed at the university and I returned to the high school and joined a group that succeeded changing the precalculus curriculum so that polynomials of first and second degree were replaced by linear and exponential functions so modeling could enter the classroom. I was allowed to test a special curriculum showing how statistical tables with categories divided into subcategories and changing over time may be modelled by statistics, linear and exponential change (Tarp, 2021), which allowed all students to pass the exam successfully. But the standard textbooks still presented pure mathematics to be learned before it could be applied. Linear and exponential functions were presented as examples of the function concept that was presented as an example of a set-relation. So instead of presenting an abstract concept though its examples, it was presented as an example of a more abstract concept. This was difficult to many students and resulted in so poor exam results, that it was suggested to remove precalculus as a mandatory class at the reform in 2005.

When my students asked for examples, I chose saving money by adding 5 \$ per week at home or 5\% per year in a bank. "Why can't we call this growth by adding and by multiplying?" the students asked.

This made a difference. So, I used deconstruction (Derrida, 1991) to develop a Difference Research searching for differences making a difference (Tarp, 2018). And the government accepted my advice that precalculus should stay as a mandatory subject, but the function concept should be replaced with variables as in in physics and economics, so that we write $y=b+a * x$ instead of $f(x)=b+a * x$. However, modeling still was difficult to many students. So, I turned to primary school to see if deconstruction by listening to children would make a difference here also. I found that the children see four fingers held together two and two, not as 4 , but as two 2 s, thus using bundle-numbers with units for what exists, bundles of 2 s in space, and 2 of them when counted in time. I described the potentials of deconstruction and bundle-numbers in several MrAlTarp YouTube videos, and in 10+15+16 contributions to the ICME 10-12 (Tarp, 2012). For ICME 13 I wrote 9 papers but was allowed only 1. For the ICMI Study 24 on curriculum I designed several micro curricula where the math core was rerooted and renamed by the process of counting and recounting before adding (Tarp, 2018, 2020). Also, I had designed a teacher education academy in 'ManyMath', MATHeCADEMY.net, Count \& Add in Time \& Space. Typically, modeling is eased by demodeling and re-rooting when tested in math labs, libraries and private education. So, the time has come for others to perform a large-scale testing.

## Demodeling: from the inside to the outside and back

Modeling goes from the outside to the inside and back. Demodeling does the opposite by going from the inside to the outside and back. So demodeling begins with the core of inside mathematics as seen on a calculator: digits, operations, brackets, multidigit numbers, decimal point, and an equation sign. And then asks the question "What outside things and actions have rooted these inside concepts?"

## Demodeling digits and multidigit numbers

Digits and letters may both be seen as symbols. But digits may also be seen as icons with as many sticks as they represent, five sticks in the five-icon, etc., if written 'less sloppy'. A sequence of digits may be seen as one multidigit number obeying a place-value system with ones, tens, hundreds, etc., and seldom with the word 'bundle-of-bundles' used for 'hundred'. But a multidigit number may also be seen as rooted in several numberings of unbundled, bundles, bundles of bundles, etc. (Tarp, 2018).

Recounting a total $T$ of ten in 3 s we get $T=3$ Bundles 1 , or $T=3 B 1$, or $T=1 B B 0 B 1=1 B^{\wedge} 20 B 1$ since 3 bundles is 1 bundle-of-bundles. So, bundling bundles roots power, and bundle-counting totals roots polynomials, $\mathrm{T}=345=3 * \mathrm{~B}^{\wedge} 2+4 * \mathrm{~B}+5^{*}$. And it also roots functions as number-language sentences with an outside subject, a verb, and an inside predicate as in word-language sentences.

To bundle-count a total, bundles are pushed away and lifted into a stack to be pulled away to look for unbundles singles. These actions root division as an icon for a push-away broom, multiplication as a lift, subtraction as a pull-away rope, and addition showing two ways to unite stacks, on-top or next-to. Placed on-top of the stack, the unbundled may be seen as a decimal number, or as a fraction when counted in bundles also, or described by what has been pulled away in time from the next bundle, or what is missing in space for another bundle. Recounting 9 in 2 s , the end result may thus be written as $T=9=4 \mathrm{~B} 12 \mathrm{~s}=41 / 22 \mathrm{~s}=5 \mathrm{~B}-12 \mathrm{~s}$ (an underload), or with an overload, $\mathrm{T}=3 \mathrm{~B} 32 \mathrm{~s}$, and $\mathrm{T}=2 \mathrm{~B} 52 \mathrm{~s}$. As to the process, to recount 8 in 2 s we push-away 2 s to be stacked as 42 s , which may be written as $8=(8 / 2) \times 2=8 / 22 \mathrm{~s}$, or $T=(T / B) \times B=T / B B \mathrm{~s}$ with unspecified numbers.

By changing the unit, this recount-formula roots the proportionality formula $T=a^{*} b$ recounting $T$ in $b s$. Examples may be meter $=$ meter $/ \mathrm{sec}^{*} \mathrm{sec}$, recounting a distance in seconds, or $\$=\$ / \mathrm{kg} * \mathrm{~kg}$, recounting dollars in weight, thus creating 'per-numbers' as meter/sec, $\$ / \mathrm{kg}$, etc. Or part = part/whole * whole, recounting a part in wholes and becoming fractions with like units. In time, terminal $=$ terminal/initial * initial recounts the end-value in start-values.

A rectangle has base, $b$, height, $h$, and diagonal, $d$, raising an angle, $A$. Here, mutual recounting roots trigonometry: $h=(h / b)^{*} b=\tan (A)^{*} b, h=(h / d)^{*} d=\sin (A)^{*} d$, and $b=(b / d)^{*} d=\cos (A)^{*} d$. In half a radius 1 circle, splitting the circumference in $n$ parts gives the pi-number $\pi=n^{*} \tan (180 / n)$ for $n$ big.

Recounting between icons and tens root equations and early algebra.
Recounting from tens to icons, we ask "How many 2 s in 8 ?" This roots equations solved by recounting 8 in 2 s : $u^{*} 2=8=(8 / 2) * 2$, so $u=8 / 2$ from pushing-away 2 s from 8 , showing that an equation is solved by reversing the process, i.e., by moving a number to the opposite side with the opposite sign. This follows the formal definition: $8 / 2$ is the number $u$ that multiplied with 2 gives $8, u * 2=8$. 'To opposite side with opposite sign' may be rooted outside, while the inside balancing method is derived from abstract algebra concepts as group, inverse and neutral elements, associativity and commutativity.

Recounting from icons and tens, we ask " 67 s is how many tens?" This roots early algebra if allowing underloads: $T=67 s=6 * 7=(B-4) *(B-3)=B B-4 B-3 B+43 s$, as seen on a $B * B$ square where the 6 7 s is left when from ten bundles we pull-away 4 bundles and 3 bundles, and finally add the 43 s that was pulled-away twice.

Once counted, stacks may be added, on-top after recounting provides like units, or next-to as areas thus rooting integral calculus, as well as differential calculus when reversing asks " $23 \mathrm{~s}+? 5 \mathrm{~s}=48 \mathrm{~s}$ ".

Per-numbers are added in mixture problems as " 2 kg at $3 \$ / \mathrm{kg}$ plus 4 kg at $5 \$ / \mathrm{kg}$ give what?". With like units, the unit-numbers 2 and 4 add directly. But per-numbers must be multiplied to unit-numbers before adding as the areas created by the multiplication. So, mixture problems root integral calculus, preceding differential calculus occurring when the problem is reversed.

Inside, outside totals become rectangular stacks as $\mathrm{T}=95 \mathrm{~s}=9^{*} 5$, or squares in the case of bundlebundles, $T=55 \mathrm{~s}=5 * 5$. So, we may ask "How to square a rectangle?", e.g., $T=95 \mathrm{~s}=\mathrm{B} * \mathrm{~B}=\mathrm{B}^{\wedge} 2$ where $B$ is the square root of $45, B=\sqrt{ } 45$, iconized by half a perimeter. Here, half the excessing 45 s is placed to the right to create a $7 * 7$ square except for the upper $2 * 2$ right corner. Again, half of this is placed to the right to create a $65 / 7$-square, since $1 / 2^{*} 4=2=(2 / 7)^{* 7}$. This is close: $(65 / 7)^{\wedge} 2=45.1$.

## Fact, fiction and fake, the $\mathbf{3}$ modeling genres

With mathematics as a number-language modeling outside things in space and actions in time, its quantitative literature needs to be divided into fact, fiction or fake, the same genres used in the wordlanguage for qualitative literature. Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas; and that need to be checked for units and correctness. Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas; and that need to be supplied with scenarios with alternative assumptions. Fake stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas; and that need to be replaced by word stories (Tarp, 2001).

## Discussion and conclusion

What should name a mathematical concept? Its outside root, or its inside relation to other concepts on a lower or higher abstraction level? Should a function be named a 'sentence' using a verb to link an outside subject to an inside calculation? Or a 'standby calculation' with specified and unspecified numbers? Or an example of a 'many-to-one set relation'? In fact, what we here ask is: The goal of math education, is that to learn to master math to later master Many, or the other way around? Holding that existence precedes essence, existentialist philosophy (Sartre, 2007) prefers the latter. And we see that to master Many to later master mathematics by re-rooting often implies a different name and order of de-rooted concepts. Since, to bring outside Many inside, counting and recounting precedes addition, and 2D flexible bundle-numbers with units replace the traditional 1D line-numbers without, and recounting to change units leads to proportionality, $T=(T / B) \times B$, to per-numbers with different physical units, and to fractions with like units. Likewise, recounting between icons and tens leads to equations solved in the original way by moving to 'opposite side with opposite sign', and to early algebra when 67 s is rewritten as $(\mathrm{B}-4)^{*}(\mathrm{~B}-3)$. And, recounting rectangular stacks as squares roots the square root, and to quadratic equations. And in a stack halved by its diagonal, mutual recounting between the sides roots trigonometry thus preceding both plane and coordinate geometry. And once counted and recounted, stacks may be added in two ways, on-top, after recounting has made the units like, or nextto as areas thus rooting integral calculus, and differential when the question is reversed. And, since per-numbers must be multiplied to unit-numbers before adding they also add by areas as integral calculus, facilitated by differential calculus trying to rewrite area-strips as differences so the sum of many differences becomes one difference between the end and start values.

So, demodeling and re-rooting inside de-rooted concepts may ease modeling: Now you don't first learn about inside essence, but learn math directly by manipulating and communicating about (Widdowson, 1978) outside existence, e.g., things and actions on a ten-by-ten Bundle-Bundle-Board (Tarp, 2023).

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