Decolonizing 1D Mathema-tism into 2D Many-math

Count & Add in Time & Space

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Introduction

Teaching mathematics online is different from teaching it offline in a classroom. So, we may ask what else could be different, its goal, its teaching, its learning, and math itself? Traditionally, the goal of math education is seen as learning to master math to later master Many. So, a difference could be to see the goal of math education as learning to master Many directly to indirectly learning math on the road, at least the core math as displayed on a calculator: digits, operations, and equations.

Traditionally these all occur as products in space, so a difference could be to see them as processes in time by letting outside Many precede inside math. And indeed, the math core becomes different when created as tales about Many displayed as rectangular stacks of bundles on a plastic ten-by-ten bundle-bundle board, a BBBoard.

To see if a 'process-based' or 'Many-first' or 'Many-math' education will make a difference to the traditional 'product-based' 'Math-first' 'set-Math', micro-curricula are designed using bundle-counting to bring outside totals inside as flexible bundle-numbers with units.

Here both digits and operations emerge as icons: digits when uniting sticks, and operations when division pushes away bundles that multiplication lifts onto a stack that subtraction pulls away so the unbundled may be included as decimals, fractions, or negatives. Once counted, a unit may be changed by recounting predicted by a calculator. Here recounting from tens to icons and vice versa leads to equations, and to tables displayed as the stack left when removing the two surplus stacks above and to the right from the full bundle-bundle on a BBBoard. Here recounting in two physical units leads to per-numbers bridging the two units by recounting in the per-numbers and becoming fractions with like units. Here recounting the sides and the diagonal in a stack leads to trigonometry before geometry.

Finally, once counted, and recounted, totals may add on-top after recounting has provided like units, or next-to as areas as in integral calculus that becomes differential calculus when reversed. As operators needing numbers to become numbers, per-numbers and fractions also add by their areas since they need to be multiplied to unit-numbers before adding.

So, outside totals inside appear in an 'Algebra Square' where unlike and like unit-numbers and per-numbers are united by addition and multiplication, and by integration and power. And later again split by the reverse operations, subtraction, and division, and by differentiation and root or logarithm.

Once process-based Many-first Many-math micro curricula have been designed, they may be tested in online education, as well as in special education to see if a BBBoard may 'Bring Back Brains' excluded by the 'Math-first' education. But there will be no concert without first designing a score. So detailed testing is not addressed here but left to others to perform.

This booklet is an extension of a paper presented at the CTRAS 2023 June Conference.

Allan Tarp, summer 2023

Videos:

"Online math opens for a communicative turn in number language education.", https://www.youtube.com/watch?v=36tan-gGjJg,

"Flexible Bundle Numbers Develop the Childs Innate Mastery of Many", https://youtu.be/z_FM3Mm5RmE.

Background

How different can online mathematics education be from offline? To answer this question, we look at its content: a textbook expressing its educational goal, a teacher showing a way to the goal, and learners supposed to reach the goal. So, we ask if it possible to point to a different goal, a different teaching, and a different learning in online mathematics education.

As to the educational goal of mathematics education, it is typically seen as mastery of inside mathematics by itself, and as a means to later master the outside fact Many. Here a difference could be to search for a direct way to master Many that automatically leads to mastery of core mathematics along the road. In that case, mastery of Many will precede mastery of math, or, in other words, mastery of outside existence will precede mastery of inside essence.

This difference resonates with philosophical Existentialism holding that outside existence precedes inside essence (Sartre, 2007). And it resonates with the original meaning of the word 'mathematics' meaning 'knowledge' in Greece where it was used as a common word for their four areas of knowledge about Many: Many by itself, in space, in time, and in time and space, also called arithmetic, geometry, music and astronomy. Together the four formed the quadrivium that with the trivium consisting of grammar, logic and rhetoric constituted classical education according to Plato (Russell, 1945).

As to learning, two psychological schools differ on their views. Vygotsky expresses a social constructivism by holding that learning means adapting to inside institutionalized essence mediated by well-educated teachers (Vygotsky, 1986). In contrast, Piaget expresses a radical constructivism by holding that learning means adapting to outside existence through personal meetings and communication with peers and guiding teachers (Piaget, 1971). Changing the goal from mastery of inside math to mastery of outside Many thus means a shift from a Vygotskian to a Piagetian view.

As to teaching, two sociological schools differ in their view on how strict math education should be institutionalized. European structure-based sociology sees a population as needing strong self-sustaining institutions as necessary means to provide social order and welfare, which implies clear goals and detailed curricula in education as well as a well-organized teacher education (Weber, 1930). In contrast to this, American actor-based pragmatic sociology sees the population as self-sustaining individuals developing flexible social structures through symbolic interaction, which implies outside goals directing the curricula in education with teachers as guides (Menand, 1997). This pragmatism warns against goal displacements in institutions tempted to invent obstacles towards the goal to secure their own existence and growth, thus themselves becoming the goal using the original goal as a means (Bauman, 1990).

As to the core of mathematics itself, its Many-based and set-based and versions give opposite answers to the question "Which comes first, examples or abstractions?". So, where set-based math derives its concepts as examples of abstractions, Many-based finds the math core on a basic calculator with buttons to the digits, the operations and to an equation sign, and with a display for multidigit numbers and maybe the calculations producing them. However, the digits and operations occur as end-products in space only, we don't see them created through processes in time.

We see the 5-digit as one symbol, we do not see how it may be formed by uniting five sticks into one icon. Likewise, we see the total $T = 234$ as one number without units, we don't see the three numberings using bundle-counting to find the number of singles, bundles, and bundles-of-bundles, i.e., we don't see the polynomial including the units, $T = 234 = 2*BB + 3*B + 4*1$.

We see the order of operations as natural: addition, subtraction, multiplication, division, and power, all building on the basic claim that "2+1 = 3 always"; and from there defining multiplication as repeated addition, power as repeated multiplication, and subtraction and division as reversed addition and multiplication, as well as root and logarithm as reversed 'powering'.

However, with units, 2 weeks + 1 day is 15 days. So, adding numbers without their units may fold outside the classroom if the units are not like, whereas multiplication always holds since in the product 2x3, 3 is the unit. Therefore, to avoid teaching invalid ‘mathematism’ (Tarp, 2018), units must always be included inside also. So, multiplication must precede addition.

Counting fingers in 3s, 3 bundles become 1 bundle-of-bundles, 1 BB or 1 B^2. And ten bundles become 1 bundle-of-bundles, also called 1 hundred, when counting in tens. So, when bringing an outside total inside by bundle-counting, power becomes the first operation, not the last.

Counting 9 sticks in 2s we push-away the bundles with a broom called division, to be stacked by a lift called multiplication. After which we use a rope called subtraction to pull-away the stack to look for unbundled singles to be placed on-top of the stack and seen as a decimal number, 9 = 4B1 2s, or as a fraction when it also is counted in 2s, 9 = 4 ½ 2s, or with a negative number showing what is pulled away from the following stack, 9 = 5B-1 2s. Only now, once counted, and recounted, stacks may finally be added next-to or on-top.

So, apparently from a process-view, the operation order is the opposite: first power, then division, then multiplication, then subtraction, and finally next-to and on-top addition after decimals, fractions and negative numbers occur when placing the unbundled on-top of the bundles.

Furthermore, calculations seldom include the outside total that is calculated. So, number-language sentences seldom are full sentences with an outside subject, a verb, and an inside predicate as are word-language sentences. We say ‘6x7’ instead of including the outside total by saying ‘T = 6x7’. And we see 6x7 as 42 always instead of de-modeling (Tarp, 2020) it outside as the 6 7s they are, which may or may not be recounted in tens as 4 bundle 2 tens written inside as 4B2 tens or 4.2 tens, or shortly as 42 if leaving out both the unit and the decimal point in the final product.

De-modeling allows solving inside problems as outside totals. Asking “What is the product of 2½ and 3 1/3?”, this inside product can be de-modeled as an outside total T to create the number-language sentence, T = 2½ * 3 1/3. From this story about T we may conclude that with 2 and 3 times as many Ts we get 2*3*T = 5*10, so T = 50/6 = 8 2/6.

The above observations suggest an immediate answer to our question. Online math may be different from off-line math if seen as outside processes to be acted in time instead of as an inside end-product in space to be partly memorized.

Outside totals are typically counted in tens inside. So, we may develop a number-language by describing things and actions on a ten-by-ten Bundle-Bundle-Board, a BBBoard, where rubber bands split the bundles vertically and horizontally to create stacks of bundles with a total of T dots. Which will then be the subject in a number-language sentences where the BBBoard allows an outside total of 6 7s to becomes the inside sentence T = 6x7, or T = 6*7, also called a formula or a function. Plastic BBBoards are cheap to produce to schools, where the students also can draw BBBoards on squared paper.

Figure 1. A total of 6 7s shown on a ten-by-ten Bundle-Bundle BBBoard

But is this ‘mastery of Many before math’ a difference that makes make a difference and thus an example of difference-research (Tarp, 2018)? To test this, we first design some micro-curricula (MC) to tell tales about things and actions on a BBBoard inspired by Tarp (2001, 2018, 2019, 2022)

So, we see that philosophy and psychology and sociology and mathematics itself all suggest differences in math education. Here, choosing existentialism and radical constructivism and actor-based sociology allow returning to the original Greek meaning of mathematics as a natural science about the outside fact Many as it occurs in the word-language as the plural form of words; and in space as fingers, and in time as counting sequences among others.

Teaching numbering instead of numbers thus may create a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as had foreign language education in the 1970s as a replacement of teaching language as an example of its grammar (Chomsky, 1965). In his book ‘Explorations in the function of language’ Halliday (1973, p. 7) defines a functional approach to language in the following way:

> A functional approach to language means, first of all, investigating how language is used: trying to find out what are the purposes that language serves for us, and how we are able to achieve these purposes through speaking and listening, reading and writing. But it also means more than this. It means seeking to explain the nature of language in functional terms: seeing whether language itself has been shaped by use, and if so, in what ways - how the form of language has been determined by the functions it has evolved to serve.

Likewise, Widdowson (1978) adopts a “communicative approach to the teaching of language (p. ix)” allowing more students to learn a less correct language to be used for communication about outside things and actions.

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MC01. Digits are icons uniting sticks

The goal is to experience that digits may be formed by uniting the number of sticks that they represent, and to see that these digits are close to the digits on the calculator.

The action is to unite two sticks into one two-cion, three sticks into one three-icon, etc. Other concrete materials could be cars, dolls, spoons etc. Finally, a bending ruler may be used to create the digits.

We see that with Roman numbers, 3 1s is not united into 1 3s; and that Roman numbers use the letter V as a symbol for one hand, and the letter X as a symbol for two hands.

We don't have an icon for ten since bundling in tens makes ten a two-digit number counting both the number of bundles, and the number of unbundled singles:

T = ten = 1Bundle 0 = 1B0, or T = 10 if leaving out the unit.

When including the units, we don't need a place value system since

T = 23 = 2B3; and T = 456 = 4BB 5B 6.

MC02. Operations are icons created by Bundle-counting and re-counting

The goal is to experience how the three outside counting processes, pushing away bundles, stacking bundles, and pulling away stacks may inside be iconized as division, multiplication, and subtraction.

As concrete material serves sticks and snap cubes.

To recount 8 in 2s, 4 times we push away 2s to get the answer T = 8 = 4 2s.

Iconizing 'push away' by a broom, /, called division, the action 'from 8 push away 2s' may be iconized as '8/2' giving 4 on a calculator thus giving an inside prediction of an outside action.

Iconizing 'stacking up' by a lift, x, called multiplication, the action '4 times stack 2s' may be iconized as '4x2', or as '4*2' giving 8 on a calculator, again allowing an inside prediction of an outside action.

Pushing away and stacking combines in a 'recount-formula' (Tarp, 2018),

8 = (8/2)*2, or T = (*T*/*B*)**B* with unspecified numbers, saying that T contains *T*/*B* *B*s.

Iconizing 'pull away' by a rope, –, called subtraction used to find unbundled singles, the action 'from 9 pull away 4 2s' may be iconized as '9-4*2' giving 1 on a calculator inside as expected outside.

Placed on top of the stack, the unbundled may be seen as a decimal number, 4B1, or as a fraction when counted in 2s also, 4 1/2, or creating a negative number showing in time what is pulled away from the next bundle, or what is missing in space for another bundle, 5B-1. So, with 9 – 4*2 = 1,

T = 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s.

Iconizing 'uniting' by a cross showing two directions, next-to and on-top, +, called addition, the action 'uniting 2 3s and 4 5s' may be iconized as '2x3 + 4x5' giving 26 on a calculator if united in tens, and 3B2 8s if added next-to, and 5B1 5s if added on-top.

Pulling away and adding combines in a 'restack-formula' (Tarp, 2018), 8 = (8-2)+2, or T = (*T*-*B*)+*B* with unspecified numbers, saying that T may split in *B* and what is left when *B* is pulled away, *T*-*B*.

MC03. Bundle-counting in icons

The goal is to use an in-time bundle-counting sequence with icon-bundles to represent outside in-space examples of Many as an inside union of singles, bundles, bundle-of-bundles, etc.

As concrete material serves fingers, sticks and snap-cubes.

• First we count ten fingers in 4s. We begin in time by raising them one by one beginning with the little finger.

0B1, 0B2, 0B3, 0B4 or 1B0, moving the fingers to see the difference between 4 1s and 1 4s, called a bundle, B.

And then, 1B1, 1B2, 1B3, 1B4 or 2B0, again moving the fingers to see the difference between 8 1s, 1B4, and 2B0.

And finally, 2B1, 2B2.

So, a total of ten may be counted in 4s as 2 bundles and 2, written shortly as

T = 2B2 4s, or T = 22 4s, if leaving out the unit.

We notice that we need 2 to have 3 bundles, so we may also write T = 3B less2 = 3B-2 4s.

So, with 'flexible bundle-counting' we have three ways to count ten in 4s: underload, normal, and overload:

T = ten = 3B-2 = 2B2 = 1B6 4s.

We now count ten in 4s in space on a BBBoard with a rubber-band showing the 4-bundles.

• Then we count the ten fingers in 3s in the same way, first in time then in space.

0B1, 0B2, 0B3 or 1B0, moving the fingers to see the difference between 3 1s and 1 3s.

And then, 1B1, 1B2, 1B3 or 2B0, again moving the fingers to see the difference between 6 1s, 1B3 and 2B0.

And then, 2B1, 2B2, 2B3 or 3B0, again moving the fingers to see the difference between 6 1s, 1B6, 2B3 and 3B0.

And finally, 3B1 or 4B-2 since we need to pull away 2 from 4B0 to have 3B1.

So, with 'flexible bundle-counting' we have four ways to count ten in 3s:

T = ten = 4B-2 = 3B1 = 2B4 = 1B7 3s.

However, when bundling in 3s, 3 bundles become 1 bundle-of-bundles, 3B = 1BB = 1B^2, which suggests that power should be the first operation:

T = ten = 3B1 = 1BB 0B 1 3s, or T = 101 3s, if leaving out the units.

We now count ten in 3s in space on a BBBoard with a rubber-band showing the 3-bundles.

We notice that a bundle-of-bundles, BB, becomes a square.

• Finally, we count the ten fingers in 2s, in pairs.

We begin by seeing how 5 fingers may be counted in 2s in space.

T = 5 = 1B3 = 2B1 = 3B-1 2s, and T = 5 = 1BB 0B 1 2s = 101 2s if leaving out the units.

Putting the two hands together, we see that ten can be counted in 2s as

T = ten = 2BB 0B 2 = 1BBB 0BB 1B 0 2s, or T = 1010 2s, if leaving out the units.

With power, we get a polynomial T = ten = 2*B^3 + 1*B 2s.

We now count ten fingers in 2s, in time:

0B1, 0B2 or 1B0, 1B1, 1B2 or 2B0 or 1BB 0B 0, etc., until 1BBB 0BB 0B 1, 1BBB 0BB 1B 0.

We now count ten in 2s in space on a BBBoard with a rubber-band showing the 2-bundles.

Using snap cubes instead we see that a BBB is a cube.

MC04. Bundle-counting in tens

The goal is to use an oral bundle-counting sequence with ten-bundles to represent outside examples of Many as an inside union of singles, bundles, bundle-of-bundles, etc.

As concrete material serves a BBBoard with a rubber band above the first row.

The action is to include both singles and bundles when counting the dots horizontally in time as

0B1, 0B2, …, 0B9, 0Bten or 1B0 or 10. And then, after moving up the rubber band,

1B1, 1B2, …, 1B9, 1Bten or 2B0 or 20, etc., until

9B1, 9B2, …, 9B9, 9Bten or tenB0 or 1BB0B0 or 100.

Again, we see that a bundle-of-bundles is a square.

Bundling in tens, a BundleBundle, BB or B^2, is called hundred, and a BundleBundleBundle, BBB or B^3 is called thousand, and BBBB =B^4 is called ten thousand, or Wan in Chinese. Likewise, BBBBBB = B^6 is called a million.

Beginning again, we now count what is missing for another full bundle:

1B-9, 1B-8, …, 1B-1, 1B0 or 10. And then, after moving up the rubber band,

2B-9, 2B-8, …, 2B-1, 2B0 or 20, etc., until

1BB-9, 1BB-8, …, 1BB-1, 1BB0 or 1BB0B0 or 100

Including also overloads and underloads, we can now practice ‘flexible bundle-counting’ as, e.g.,

T = 67 = 6B7 = 5B17 = 7B-3 tens.

Flexible bundle-numbers may ease standard calculations so we will not need to carry or to borrow.

MC05. Recounting in another unit

The goal is to see how a calculator may predict the result of changing a unit to another unit.

As concrete material serves a BBBoard with a rubber bands, and a calculator.

On a BBBoard or with fingers we see that a total of 2 3s may be recounted as 1B2 4s.

To predict this result on a calculator we first enter ‘2*3/4’ to predict how many 4s may be pushed away form 2 3s. The answer is ‘1.some’. To find the unbundled ‘some’ we pull away 1 4s from the 2 3s, predicted by entering ‘2x3-1x4’ that gives the answer ‘2’. So the calculator predicts that the result is ‘1B2 4s’ as expected: T = 2 3s = 1B2 4s.

In this way we may see that a total of one dozen may be recounted in different units as e.g.

T = 12*1 = 6*2 = 4*3 = 3*4 = 2*6 = 1*12

Marking the upper-right dot on a BBBoard we see that the marks follow a bended curve called a hyperbola. That also occurs when on a BB square on paper we recount 10 1s in 2s, 3s, etc.

Recounting 6 2s to 4 3s show that increasing the stack’s base from 2 to 3 will decrease its height from 6 to 4. This connection between the height and the base in a stack is called inverse proportionality.

MC06. Recounting tens in icons gives equations

The goal is to experience how recounting from tens to icons is another word for an equation, that may be solved, of course, by recounting.

As concrete material serves a BBBoard with a rubber band to show the bundles, snap-cubes, and a calculator.

Asking “8 is how many 2s” may be rephrased to “8 is u 2s” using letters for unspecified or unknown numbers. This then may be shortened to an equation, 8 = u*2, solved by recounting 8 in 2s as 8 = (8/2)*2, so that the solution u = 8/2 is found when isolating the unknown number by moving the known number to opposite side with opposite sign.

MC07. Recounting icons in tens gives rectangles and multiplication tables

The goal is to recount icon-stacks in tens both in time and space to create the multiplication tables.

As concrete material serves a BBBoard with two vertical rubber bands, one to the left showing the actual stacks in space, and one to the right showing a stack of 1s as a line to count on in time.

By accepting totals to be written as underloads also, we may count from 1 to ten as 0B1, 0B2, 0B3, 0B4, 0B5, 1B-4, 1B-3, 1B-2, 1B-1, 1B0. So we only focus on the tables for the numbers from 1 to 5.

• First to the left we mark the stack of 2s and count it in 2s as 1, 2, 3, 4, and 5 2s.

Then we recount them in tens as 0B2, 0B4, 0B6, 0B8, 1B0.

Then to the right we jump with 2-steps and say the same sequence.

Finally, we memorize by folding fingers one by one five times.

Likewise with 3 and 4 and 5 as the bundle-size.

With the numbers 6-9 we place an extra vertical rubber band after 5.

With 6 or B-4 as the bundle-size, again first we count the stack in 6s as 1, 2, 3, 4, 5 6s. Then we recount in tens. First with underloads as 1B-4, 2B-8, 3B-12, 4B-16, 5B-20. Then without as 0B6, 1B2, 1B8, 2B4, 3B0. And finally as 6, 12, 18, 14, 30 to be memorized.

Likewise with 7, 8 and 9 as the bundle-size.

• Now, to recount a total of 7 6s into tens we use two rubber bands to show that 7x6 = (B-3) x (B-4). Then we get the total by removing a 3xB stack horizontally and a Bx4 stack vertically, and then adding the upper right 3x4 stack that was removed twice:

T = 7*6 = (B-3) * (B-4) = 10B – 3B – 4B + 3x4 = 3B + 1B2 = 4B2 = 42.

Here we see that negative multiplied with negative gives plus.

• Now we recount 6 78s in tens. On the BBBoard, the horizontal rubber band is at 6 and the vertical band illustrates 7B8. So, to the left we get 6*7B = 42B. To the right we get 6*8 = 48 = 4B8, so to avoid overload we include the 4 Bundles to the left as 46B. The total then is 46B8 or 468.

It may be quicker to write it on one line with an overload:

T = 6 78s = 6*78 = 6x7B8 = 42B48 = 46B8 = 468.

• Now we recount 34 78s in tens.

On the BBBoard, the horizontal rubber band illustrates 3B4 the vertical illustrates 7B8. This gives the four parts 21BB, 24B, 28B, and 3B2. Here the bundles add up to 24 + 28 + 3 giving 55B or 5BB5B. So, the total is T = 26BB 5B 2, or T = 2652.

Again, it may be quicker to write it on one line with overloads and using the FOIL (First, Outside, Inside, Last) method:

T = 34 78s = 34*78 = 3B4 * 7B8 = (3B + 4) * (7B + 8) = 21BB + 24B + 28B + 32 = 21BB + 55B + 2 = 26BB + 5B + 2 = 2652.

A quicker way may be to write the numbers under each other. Multiplying down the gives the BBs and the unbundled, and cross-multiplication gives the Bs.

3B 4

7B 8

• Long division may also be eased by using overloads:

78 /6 = 7B8 /6 = 6B18 /6 = 1B3 = 13

To do the division 2652 / 34, we notice that 7*34 = 238 makes 265 = (265-238) + 238 = 27 + 238; and that 8*34 = 272 makes 265 = (265-272) + 272 = -7 + 272, so

2652 / 34 = 265B 2 /34 = (238+27)B 2 /34 = 238B 272 /34 = 7B8 = 78, or

2652 / 34 = 265B 2 /34 = (272-7)B2 /34 = 272B-68 /34 = 8B-2 = 7B8 = 78

MC08. Bundle-Bundles are squares

The goal is to count the squares occurring as bundle-bundles.

As concrete material serves a BBBoard with two rubber bands to show the squares.

• With 5 as the bundle-size, the 5*5 square contains 2 5s twice and 1 5s, i.e., 2B5 or 25.

With 4 as the bundle-size, the 4*4 square comes from twice removing 1 5s from 25, and then adding the upper right 1*1 corner removed twice, so the total is 15 +1 or 16.

With 3 as the bundle-size, the 3*3 square comes from twice removing 2 5s from 25, and then adding the upper right 2*2 corner removed twice, so the total is 5 + 4 or 9.

With 2 as the bundle-size, the 2*2 square comes from three times removing 3 5s from 25, and then adding the upper right 3*3 corner removed three times, so the total is -5 + 9 or 4, which may also be seen directly.

With B-1 as the bundle-size, the 9*9 square comes from twice removing 1 bundle, and then adding the upper right 1*1 corner removed twice, so the total is 8B1 or 81.

With B-2 as the bundle-size, the 8*8 square comes from twice removing 2 bundles, and then adding the upper right 2*2 corner removed twice, so the total is 6B4 or 64.

With B-3 as the bundle-size, the 7*7 square comes from twice removing 3 bundles, and then adding the upper right 3*3 corner removed twice, so the total is 4B9 or 49.

With B-4 as the bundle-size, the 6*6 square comes from twice removing 4 bundle, and then adding the upper right 4*4 corner removed twice, so the total is 2B16 or 3B6 or 36.

So, the first 9 square numbers are 1, 4, 9, 16, 25, 36, 49, 64, and 81.

We notice that the end digits occur again in square numbers.

• We may also find the square numbers by adding instead of by removing:

With 2 as the bundle-size, the 2*2 square contains a total *T* = 2*2 = 4.

With 3 as the bundle-size, the 3*3 square comes from extending the 2*2 square vertically and horizontally, and then add the upper right corner, so the total is *T* = 3*3 = 2*2 + 2*2 + 1 = 9.

With 4 as the bundle-size, the 4*4 square comes from e*tending the 3*3 square vertically and horizontally, and then add the upper right corner, so the total is *T* = 4*4 = 3*3 + 2*3 + 1 = 16.

This may be carried on until the 9*9 square.

Here we notice, that changing a 3*3 with 1 will give the changes 2*3 if we neglect the upper right corner. Likewise with the other squares. So apparently, we have a general rule:

Changing a *n***n* square with 1, the change is 2**n* almost if we neglect the upper right corner. This rule returns later when we look more closely at change formulas and find that the change of *x*^2 is 2**x*.

MC09. Three square formulas

The goal is to look at three special square formulas. As concrete material serves a BBBoard with rubber bands to show the stacks.

• To recount a square as 7 7s we use two rubber bands to show that 7*7 = (B-3) * (B-3). We now get the total by removing a horizontal 3*B stack and a vertical B*3 stack, and by adding the upper right 3*3 stack that was removed twice:

T = 7*7 = (B-3) * (B-3) = B*B – 3B – 3B + 3*3 = B^2 – 2*3B + 3^2.

This applies to all numbers, so we have the first square formula:

(B-*n*) * (B-*n*) = B^2 – 2**n**B + *n*^2.

• Now, to recount the square 7 7s as 7*7 = (5+2) * (5+2), we use four rubber bands to see that the total is four stacks:

T = (5+2) * (5+2) = 5*5 + 5*2 + 2*5 + 2*2 = 5^2 + 2*5 + 2^2

This applies to all numbers, so we now have the second square formula:

(B+n) * (B+n) = B^2 + 2**n**B + *n*^2.

• Now, to recount the stack 7 3s as 7*3 = (5+2) * (5-2), we use four rubber bands to see four stacks. To reduce 7 3s to 5 3s we pull away 2 3s and turn it to extend the 5 3s to the square 5 5s with 2 2s pulled away in the upper right corner:

T = (5+2) * (5-2) = 5*5 – 5*2 + 2*5 – 2*2 = 5^2 – 2^2.

This applies to all numbers, so we have the third square formula:

(B+*n*) * (B-*n*) = B^2 – *n*^2.

• We notice that (B-1)*(B+1) = B^2 – 1 = B^2 almost, if we neglect the upper right corner.

With 1/B = n, this becomes

(1-*n*)*(1+*n*) = 1^2 = 1, almost, if we neglect the corner *n*^2 = (1/B)^2 = (1/10)^2 = 1/100 = 0.01

So, 1-*n* = 1/(1+*n*) almost, and 1+*n* = 1/(1-*n*) almost, or

So, 1/1.1 = 0.9 almost, and 1/0.9 = 1.1 almost.

MC10. Recounting stacks as squares gives square roots to solve quadratics

The goal is to experience how a stack may be recounting to fit between two squares; and how a square always contains two smaller squares and two like stacks that may solve quadratics.

As concrete material serves a BBBoard. First, we use two bands to show the rectangle and two to show the square. Then we use two bands to show the two squares and the two stacks.

• Wanting to square a rectangle, we may ask: "How to change 6 3s into a BB-square?"

We see, that recounting 6 3s in 4s and 5s make it fit between a 4*4 and a 5*5 square, and that it is closer to 4*4.

Here, the overflow is 6*3-4*4 = 2 = 2*1, so 1 has to added to the 4s as 1 = (1/4)*4 = ¼ 4s = 0.25.

A guess then would be that the 6 3s may be recounted as a 4.25*4.25 square. If so, we say that 4.25 is the square root of 6*3, iconized as half a stack, √(6*3). A little less since we need a something for the upper right corner. And a calculator shows that the square root is 4.243.

As to 8 3s, we see, that it is closer to 5*5. Here, we miss 5*5-8*3 = 1 = 2*½, so ½ has to be subtracted from 5 as ½ = (½/5)*5 = 0.1 5s.

A guess then would be that the 6 3s may be recounted as a 4.9*4.9 square, so that 4.9 may be close to the square root of 6*3. A little less since again we need a little for the upper right corner. A calculator shows that the square root is 4.90.

• Wanting to split a square in smaller squares, the BBBoard shows that with B = 7, the (B+3) square splits up into a B*B square and a 3*3 square and two 3*B rectangles:

(B+3) * (B+3) = B*B + 3*3 + 2 * (3*B), or (B+3)^2 = B^2 + 6B + 9

With B = 7 we get B^2 + 6B + 9 = 100, or B^2 + 6B - 91 = 0, a so-called 'quadratic'.

• Reversing this process, we may want to solve the quadratic B^2 + 6B – 91 = 0, with B unknown.

So, since 6 = (6/2)*2 = 3*2, on a (B+3)*(B+3) BBBoard we use two rubber bands to show the 3*3 square in the upper right corner and the B*B square in the lower left corner as well as the two 3*B rectangles that combine into a big square (B+3)^2:

$(B+3)^2 = B^2 + 6*B + 9$

To include -91, we now rewrite 9 = -91 + 100, so

$(B+3)^2 = B^2 + 6*B + 100 – 91 + 100$, or with $B^2 + 6*B – 91 = 0$,

$(B+3)^2 = 100 = 10^2$, so

B+3 = 10 or B+3 = -10, so

B = 7 or B = -13

We were lucky, since some quadratics cannot be solved as $B^2 + 6B + 10 = 0$, where 9 becomes 10-1, so that $(B+3)^2 = B^2 + 6*B + 10 -1$ becomes $(B+3)^2 = -1$ that is not possible.

• Now we may want to solve the quadratic $B^2 - 6B +5 = 0$

Since 6 = (6/2)*2 = 3*2, on a BBBoard we use two rubber bands to show the 3*3 square in the upper right corner and the (B-3)*(B-3) square in the lower left corner as well as the two 3*B stacks that combine into a big square B^2: $B^2 = (B-3)^2 + 6*B + 9$.

On the BBBoard we see that the $(B-3)^2$ square is left if we from the B^2 square pull away a 3*B stack twice and then add the 3^2 square that was removed twice:

$(B-3)^2 = B^2 – 6*B + 9$

To include 5, we now split 9 = 5+4, so

$(B-3)^2 = B^2 – 6*B + 5 + 4$, or with $B^2 – 6*B + 5 = 0$,

$(B-3)^2 = 4 = 2^2$, so

B-3 = 2 or B-3 = -2, so

B = 5 or B = 1

Again, we are lucky, since $B^2 – 6B + 10 = 0$ cannot be solved, where 9 becomes 10-1, so that

$(B-3)^2 = B^2 + 6*B + 10 -1$ becomes $(B-3)^2 = -1$ that is not possible.

• To add squares, the BBBoard shows that adding the squares 7*7 and 3*3 along the diagonal gives a 10*10 square except for two 3*7 stacks. Instead we could transform the 3*3 square into two u*7 stacks where u*7*2 = 3*3, or u*14 = 9 = (9/14)*14, so that u = 9/14 .

A guess then would be that the 7*7 and 3*3 squares add as a (7+9/14) * (7+9/14) or 7.643 * 7.643 square giving 58.4133 which becomes the wanted 49+9 = 58 if we pull away the 9/14 * 9/14 square that was added twice, 58.4133 – 0.4133 = 58.

With the root we find the exact answer, $\sqrt{58}$ = 7.62, which turn out to be the length of the raising diagonal in the 3*7 stack, as we can see by cutting out four examples of this stack on paper.

The first stack is placed on the upper left stack on the BBBoard, the next is turned and placed at the upper end, the third is again turned before placed at the end, and likewise the fourth. We now see that the diagonals form a square that is a full 10*10 square where 4 times we pull-away a half square, a total of 2 stacks. Removing the paper stacks, we see that this is also the case when adding the 3*3 and the 7*7 squares along the diagonal.

Thus, we guess that in a stack, its diagonal square is the sum of the squares of its sides. This is called the Pythagoras rule named after a Greek philosopher.

MC11. Recounting in physical units gives per-numbers

The goal is to experience how recounting or double-countingin two physical units creates per-numbers bridging the two to answer questions as “if 3kg costs 4$, then 15kg costs ?$”, and “?kg costs 12$”.

A physical total T may be double-counted both as 3kg and as 4$, T = 3 kg = 4 $, thus giving two per-numbers, 3kg/4$, and 4$/3kg (the price).

The two units are then bridged by recounting in the per-number

So, 15 kg may by recounted in 3s as 15kg = (15/3)*3kg = (15/3)*4$ = 20 $.

Likewise, 12$ = (12/4)*4$ = (12/4)*3kg = 9 kg.

Alternatively, the units may be recounted:

$ = ($/kg)*kg = (4/3)*15 = 20; and kg = (kg/$)*$ = (3/4)*12 = 9

MC12. Recounting in the same unit gives fractions

The goal is to experience how per-numbers become fractions when recounted in the same unit.

As concrete material serves a ten-by-ten BBBoard with two parallel rubber bands to show the whole and the part.

If a whole contains a part, they have the same unit. In this case the per-number becomes a fraction without units. Still, we may use the units 'p' and 'w' for the part and the whole.

To get the fraction 3/5 of 20$ thus means to get 3p/5w of a 20$ whole. Recounting in the per-number thus gives 20w = (20/5)*5w = (20/5)*3p = 12p, or 12$ of 20$.

To get the fraction 3/5 of 100 thus means to get 3p/5w of a 100 whole. Recounting in the per-number thus gives 100w = (100/5)*5w = (100/5)*3p = 60p, or 60 of 100, written as 60%.

To ask "20$ is what percentage of 80$" means asking about the fraction 20/80 of 100. Or we may introduce a new unit 80$ = 100% to see that 20$ = (20/80)*80$ = (20/80)*100% = 40%.

To add 10% to 200$ we introduce the per-number 200$/100%. After the addition the total is

T = 100% + 10% = 110% = (110/100)*100% = (110/100)*200$ = 220$.

So, adding 10% means multiplying with 110%, and adding 10% 5 times means multiplying with 110%^5 = 161.1% thus giving 50% plus 11.1% additional, also called compound interest.

MC13. Recounting the stack sides gives trigonometry before geometry

The goal is to experience how per-numbers in a stack with a diagonal lead to trigonometry.

As concrete material serves a BBBoard on paper, and a protractor.

• On a BBBoard, we mark a 5*10 stack with height 5 and base 10 and recount the height in the base:

height = (height/base) * base = tangent Angle * base, shortened to

$h = (h / b) * b = \tan A * b = \tan A\ b$s,

This gives the formula tangent A = height / base, or $\tan A = h/b$, or $\tan A = 5/10$ in our case.

A protractor shows that the angle A is a little above 25 degrees. Testing this we get tan 25 = 0.466. The reverse tan-button 'tan^-1' gives the precise result, tan^-1(0.5) = 26.6 degrees.

Using the words 'run' and 'rise' instead of 'base' and 'height', we get the diagonal's slope-formula:

$\tan A$ = rise/run.

• The word 'tangent' is used since the height will be a tangent in a circle with center in A, and with the base as its radius. This gives a formula for the circumference since a circle contains many right triangles leaving the center. In a circle with radius 1, h recounts in r as $h = (h/1) * 1 = \tan A$.

A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 1.8 100s = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, h, are almost identical. So, half the circumference, called π, is

$\pi = 100 * h = 100 * \tan 1.8 = 100 * \tan (180/100) = 3.1426$

This gives a formula for the number π:

$\pi = \tan (180/n) * n$, for n sufficiently large.

We also see that in a circle with radius r, the circumference is $2*\pi*r$, and the area is $\pi*r$^2, or $\pi/4* d$^2 where d is the circle's diameter.

So, a d-circle takes up almost 80% of the space inside the surrounding d-square.

• In a 3*4 stack the diagonal is 5. If we recount the height and the base in the diagonal, we get the per-numbers sine and cosine:

height = (height/diagonal) * diagonal = sine Angle * diagonal, shortened to

$h = (h / d) * d = \sin A * d = \sin A\ d$s,

This gives the formula sin A = height / diagonal, or sin $A = h/d$, or sin A = 3/5 in our case.

Likewise, cos A = base / diagonal, or sin $A = b/d$, or cos A = 4/5 in our case.

MC14. Adding next-to and on-top gives calculus and proportionality

The goal is to experience how totals may add next-to or on-top once counted and recounted.

As concrete material serves a BBBoard with rubber bands, as well as snap-cubes.

• We ask "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added next-to as 8s?" We see that next-to addition means adding by areas where multiplication precedes addition. This is called integral calculus.

The recount-formula predicts the result.

T1 + T2 = 2 3s + 4 5s = ((2*3+4*5)/8)*8 = (24 + 2)/8 8s = 3 2/8 8s

We now ask the reverse question "If T1 = 2 3s and T2 add next-to as T = 4 7s, what is T2?" We find the answer by removing the initial stack and recounting the rest in 4s. So now subtraction precedes division, which is natural as reversed integration, also called differential calculus.

The recount-formula predicts the result.

T1 + T2 = 2 3s + T2 = 4*7, so

T2 = ((4*7 – T1)/4)*4 = (4*7 – 2*3)/4 4s = (20+2)/4 4s = 5 2/4 4s

• We ask "With T1 = 2 3s and T2 = 4 5s, what is T1+T2 when added on-top as 5s?" We see that to add on-top the units must be made the same, so with T1 = 2 3s = 1B1 5s, T1+T2 = 5B1 5s, as predicted by the recount-formula:

T1 + T2 = 2 3s + 4 5s = ((2*3+4*5)/5)*5 = (25 + 1)/5 5s = 5 1/5 5s

We now ask the reverse question "If T1 = 2 3s and T2 as some 5s add to T = 4 5s, what is T2?" We find the answer by removing the initial stack and recounting the rest in 5s. So again, subtraction precedes division as in differential calculus.

The recount-formula predicts the result:

T1 + T2 = 2 3s + T2 = 4*5, so

T2 = ((4*5 – T1)/5)*5 = (4*5 – 2*3)/5 5s = (20+2)/4 4s = 5 2/4 4s.

Writing T1+T2 = T, we get T2 = T – T1 = ΔT = ΔT/4 4s using the Greek letter delta to indicate a difference. Here the per-number ΔT/4 is called a difference quotient.

• To solve the equation 2 + 3*x = 14, the units should be included as 2*1 + 3*x = 14*1, or a hidden bracket is used: 2 + (3*x) = 14 = (14-2) + 2 after splitting.

So, 3*x = 14-2 that recounts into (14-2)/3*3, which gives the solution x = (14-2)/3 or x = 4.

MC15. Adding and subtracting one-digit numbers

The goal is to experience how one-digit numbers may be added and subtracted.

As concrete material serves a BBBoard with rubber bands, as well as snap-cubes.

On a BBBoard we see that '7-5 = 2' since pulling away 5 from 7 leaves 2. And that '5-7 = -2' since pulling away 7 from 5 needs 2 to give the result '0B-2'.

On a BBBoard we see that when placed on-top, '3+5 = 2B2 3s = 2B-2 5s', which both may be recounted as 8 1s. Likewise, we see that 6+8 = 2B2 6s = 2B-2 8s', which both may be recounted as 1B4 tens, or 14.

MC16. Adding per-numbers gives calculus

The goal is to experience how per-numbers (and fractions) first must be multiplied to unit-numbers before adding.

As concrete material serves a BBBoard with rubber bands.

We ask "2kg of 3$/kg + 4kg of 5$/kg = 6kg of what?" We see that the unit-numbers 2 and 4 add directly whereas the per-numbers 3 and 5 add by areas since they must first transform to unit-numbers by multiplication, creating the areas.

Here, the per-numbers are piecewise constant. Later, asking 2 seconds of 4m/s increasing constantly to 5m/s leads to finding the area in a 'locally constant' (continuous) situation defining local constancy by special numbers called epsilon and delta.

We now ask the reverse question "2kg of 3$/kg + 4kg of what = 6kg of 5$/kg?" We see that unit-numbers 6 and 2 subtract directly whereas the per-numbers 5 and 3 subtract by areas since they must first transform into unit-number by multiplication, thus creating areas. Later, in a 'locally constant' situation, subtracting per-numbers is called differential calculus.

MC17. Adding unspecified letter-numbers

The goal is to experience how letter-numbers only add with common units.

In the letter-number $T = 3ab$ the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, T may be seen as 3 abs, or as (3a) bs, or as (3b) as.

To avoid being confused by the 's' we will omit it, so $T = 3ab = 3 * ab = 3a * b$ or $3b * a$.

Since totals need a common unit to add, this must be first found:

$T = 3ab + 4ac = 3b * a + 4c * a = (3b+4c) * a$

$T = 2ab$^2 $+ 4bc = ab * 2b + 2c * 2b = (ab+2c) * 2b$

MC18. The Algebra Square

The goal is to experience how an 'Algebra Square' shows the four ways to unite and split-into like and unlike unit-numbers and per-numbers.

Counting ten fingers in 3s gives T = 1BundleBundle 1 3s = 1*B^2 + 0*B + 1, thus exemplifying a general bundle-formula $T = a$*x^2 + b*x + c, called a polynomial, showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration.

Including the units, we see there can be only four ways to unite numbers: addition and multiplication unite unlike and like unit-numbers, and integration and power unite unlike and like per-numbers.

Reversely, subtraction and division split into unlike and like unit-numbers, and differentiation and root or logarithm split into unlike and like per-numbers.

We may call this beautiful simplicity 'the algebra square' in accordance with the Arabic meaning of the word algebra, to reunite.

Operations unite/ *split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	$T = a + n$ $T - n = a$	$T = a * n$ $T/n = a$
Per-numbers m/s, $/100$ = %	$T = \int f\, dx$ $dT/dx = f$	$T = a$^b $b\surd T = a \quad loga(T) = b$

Figure 2. The Algebra Square uniting or splitting into the four number-types shows that equations are solved by moving to opposite side with opposite sign.

MC19. A coordinate system coordinates algebra and geometry.

The goal is to experience how a coordinate-system connects algebraic calculations and geometrical forms.

As concrete material serves a BBBoard with rubber bands and pins, as well as a 10*10 paper square.

From the lower left point on a BBBoard we may travel in the horizontal *x*-direction with an 'out-number' *x*, and in the vertical *y*-direction with an 'up-number' *y*. In this way a point has two coordinates telling how far out and how far up it is placed. Here, the lower left point is (0,0). If the numbers are negative, instead we go back or down.

• From (0,0) we may take '1 out, 2 up' steps to reach the coordinate points (1,2), (2,2*2), (3,2*3), etc. We see that in any coordinate point (*x*,*y*), *y* = 2**x*, which is called the formulas for the linear curve.

Likewise, from (0,9) we may take '1 out, 1 down' steps to reach the points (1,9-1), (2,9-2), (3,9-3), etc. In this case the line formula is *y* = 9-*x*.

Asking where the two lines intersect, the geometry shows that this takes place in the point (3,6).

An algebra calculation gives the same result: Since the intersection point belongs to both lines, the coordinates obey both formulas, *y* = 2**x* and *y* = 9 – *x*, so 2**x* = 9-*x*. This equation is solved by moving to opposite side with opposite sign:

2**x* = 9-*x*, so 2**x* + *x* = 9, so 3**x* = 9, so *x* = 9/3 = 3, so *y* = 2*3 = 6.

Geometry and algebra thus agrees, the intersection point has the coordinates (*x*,*y*) = (3,6).

• On a BBBoard, two rubber bands show 6 7s. An additional vertical rubber band is placed between 4 and 5 to signal that its *x*-number is unspecified.

Now, the diagonal in the '10 7s' rectangle represent '7 out, 10 up' steps as well as '*x* out, *y* up' step.

But since the angle A from horizontal is the same in the two trips we have tan A = *y*/*x* = 10/7. This gives the equation *y*/*x* = 10/7 that may be solved by moving to opposite side with opposite sign:

y/*x* = 10/7, so *y* = 10/7**x*.

The intersection point between the diagonal and the horizontal rubber line then may be found by equating the two *y*-formulas, *y* = 10/7**x* and *y* = 6, again solved by moving to opposite side with opposite sign.

10/7**x* = 6, so *x* = 6*7/10 = 4.2, so *y* = 10/7 *4.2 = 6.

The intersection point thus has the coordinates (*x*,*y*) = (4.2,6).

Likewise, we see that the diagonal in the '6 10s' rectangle has the formula *y* = 6/10**x*.

• On a BBBoard, a line passes through the points (2,1) and (4,5), which are connected by '4-2 out, 5-1 up' steps that may also be called the change-steps for *x* and *y*, written with the Greeks letter, delta, as Δ*x* = 4-2, and Δ*y* = 5-1.

Likewise, the steps from (2,1) to an unspecified point on the line has $\Delta x = x$-2, and $\Delta y = y$-1.

Since the angle A is the same in the two change-triangles, tan $A = \Delta y/\Delta x$ = (5-1)/(4-2), or (y-1)/(x-2) = 4/2 = 2.

Solving this equation for y gives y-1 = 2*(x-2), or y = 2*x – 4 +1, or y = 2*x-3.

• Also, we see that for the descending diagonal in the '3 tens' rectangle the angle A from horizontal seen from the point (10,0) may be found in two ways, as tan $A = \Delta y/\Delta x$ = (y-0)/(10-x), and as 10/3. This gives the equation y/(10-x) = 10/3 that may be solved by moving to opposite side with opposite sign:

y/(10-x) = 10/3, so y = 10/3*(10-x), so y = -10/3*x + 100/3.

Likewise, looking at the descending diagonal in the '6 tens' rectangle, its formula is

y/(10-x) = 6/10, so y = 6/10*(10-x), so y = -6/10*x + 100/6.

The four diagonals together with the horizontal y = 6 line now form a star with five intersection points with coordinates that may be found both by algebraic calculations and by geometrical inspection.

• On a BBBoard we want to find a parabola y = a*x^2 + b*x + c, that passes through the three points (x,y) = (1,2), (3.6) and (4,5). First, we use two rubber bands to move the lower left corner to the point (1,2). From here the parabola now passes through (p,q) = (0,0), (2,4) and (3,3) and the formula is reduced to q = a*p^2 + b*p where q = y-2 and p = x-1, and c = 0. Inserting the two points gives 2 equations with 2 unknowns:

4*a + 2*b = 4 and 9*a + 3*b = 3.

With u = 2*b, 3*b = (3/2)*2b = (3/2)*u. This changes the two equations to

4*a + u = 4 and 9*a + 3/2*u = 3.

From the first we get u = 4 – 4*a that inserted in the second gives

9*a + 3/2*(4-4*a) = 3, or 9*a + 6 -6*a = 3, or 3*a = -3, or a = -1.

The first equation now gives 4*-1 + u = 4, or u = 8, or 2*b = 8, or b = 8/2 = 4.

So, the formul*a* is q = -p^2 + 4*p, or y-2 = -(x-1)^2 + 4*(x-1), or y = -x^2 + 6*x – 3.

• On a BBBoard showing 6 7s, a triangle is formed by the three lines connecting the points (0,0) and (7,10) and (10,7). Typically, we want to find the 7 important triangle numbers, its area, its three angels and its three sides.

We see that these 7 numbers may be found indirectly by looking at the three half rectangles that is pulled away from the triangle's wrapping rectangle.

In the lower pull-away half-rectangle we see that the angle is predicted by the formula tan A = 6/10, which on a calculator gives A = 31.0 degrees. And that the area is ½*6*10 = 30. And that the length of the diagonal d is found by squaring: d^2 = 10^2 + 6^2 = 136, giving d = √136 = 11.7.

• On a BBBoard, twice rolling 2 dices may suggest we go to the two points (3,6) and (4,3) that then constitute one side in a square. We now may find the area of the square and the intersection point of the two diagonals. We notice that in the slopes of the sides the out- and up-number change places, and one changes the sign also.

• Optimizing income under constraints (also called 'Linear Programming'). At a fair, a class sells caps and shirts. They may buy at most 6 boxes with caps and 4 boxes with shirts that each cost 1unit. Their budget is 8units, and their income is 1unit per shirt-box and 2 units per cap-box. How can they optimize the income?

On a BBBoard a horizontal and a vertical rubber band shows the limit on the shirts and on the caps. A line connecting (0,8) and (8,0) shows the budget-line not to be passed. A line connecting (0,10) and (5,0) shows the 10unit income-line that is moved to the right until (6,2) where the first constraint will be violated. So the calls should buy 6 boxes with caps and 2 boxes with shirts, which will give them an income at 2*6 + 1*2 or 14 units.

MC20. Change in time: Growth and decay

The goal is to experience how a total may change in time in various ways. As concrete material serves a BBBoard with rubber bands, as well as a 12x12 paper square.

• On a BBBoard two rubber bands mark a total T1 as a S*S stack with S = 9. If both sides change by one, T1 will change to T2. We use the Greek letter delta for the change that is left when pulling away T1 from T2, ΔT = T2 – T1. Here the change will be S*1 + 1*S = 2*S if we neglect the upper right corner, so we may write ΔT = 2*S almost, or Δ(S^2) = 2*S almost, if S changes with 1, ΔS = 1.

If we count the change with T as the unit, we get the change per-number or change-percent P:

P = ΔT/T1 = 2*S/(S*S) = 2/S, almost

So, with S = 9, P = 2/9 = 22%. And with S = 8, P = 2/8 = 25%

• With different units vertically and horizontally the 9*9 stack becomes a H*B stack with height H and base B. If we write their changes as ΔH and ΔB, we get Δ(H*B) = ΔH*B + H*ΔB almost. Writing this as per-numbers gives

Δ(H*B)/(H*B) = (ΔH*B + H*ΔB)/(H*B) almost = ΔH/H + ΔB/B, almost.

So, if I work H hours for a salary at D $/hour, my income will be C = H*D. If I work 5% more and the salary increases with 3%, my capital will increase with 5%+3% = 8% almost. And to keep my income unchanged I could work 3% less almost.

• If the sides in the S*S stack change with a small number n, ΔS = n, then Δ(S^2) = 2*S*Δ(S) almost, which gives the change per-number Δ(S^2)/Δ(s) = 2*S almost.

If for very small numbers, we ignore the upper right corner we simple write d(S^2)/dS = (S^2)’ = 2*S and call this per-number a differential quotient or a derivative or a gradient.

So, with T as a *x***x* stack we have ΔT = Δ(*x*^2) = 2**x**Δ*x* almost if *x* changes with one.

And, with a very small change in *x*, d*x*, dT/d*x* = d(*x*^2)/d*x* = (*x*^2)’ = 2**x*.

• Likewise, with T as a *x***x***x* cube we see that T changes with three boxes *x*^2*d*x* boxes almost if *x* changes with one so that ΔT = Δ(*x*^3) = 3**x*^2*Δ*x* almost.

And, with a very small change in *x*, d*x*, dT/d*x* = d(*x*^3)/dx = (*x*^3)’ = 3**x*^2.

We find the same result in a H*B stack where H = x^2 and B = x so t that H*B = x^2*x = x^3. Here

Δ(x^3)/(x^3) = Δ(x^2)/(x^2) + Δx/x = (2*x*Δx)/(x^2) + Δx/x = 3*Δx/x almost, so that

Δ(x^3) = (3*Δx/x)*x^3 = 3*x^2*Δx almost, or Δ(x^3)/Δx = 3*x^2 almost

Or again, with a very small change in *x*, d*x*, dT/d*x* = d(*x*^3)/dx = (*x*^3)’ = 3**x*^2.

Likewise, with *x*^4 as a H*B stack where H = *x*^3 and H = *x*, we find that

d(*x*^4)/d*x* = (*x*^4)’ = 4**x*^3, and (*x*^*n*)/dx = (*x*^*n*)’ = *n***x*^(*n*-1) if we keep on.

• If in a T = HxB stack we want to keep the total T constant, ΔT = 0 then implies that ΔH/H + ΔB/B = 0 almost, or that ΔH/H = - ΔB/B almost.

On a BBBoard we get the same result. Two vertical and two horizontal rubber bands show the original and the changed H*B stack. If we want the upper left stack, ΔH*B, to be like H*ΔB, we must neglect the upper right corner, ΔH*ΔB.

Writing $y = 1/x$ instead of $H = T/B$ we get $\Delta y/y = - \Delta x/x$, or $\Delta y/\Delta x = y/x = (1/x)/x = 1/(x^2)$ almost, or with very small changes,

If $y = 1/x$, then $dy/dx = d/dx(1/x) = 1/(x^2)$

An unspecified number is written as a letter, x, and an unspecified formula with x as an unspecified number is written as $f(x)$, which is also called a function.

Writing out fully with units, the number 345 becomes $T = 3*B^2 + 4*B + 5$. Using y an x instead of T and B, this basic number-formula is called a polynomial, $y = 3*x^2 + 4*x + 5$. Here 5 is the level-number, 4 is the change-number that may be positive or negative if the curve goes up or down, and 3 is the curvature-number that may be positive or negative if the curve curves up or down.

The basic number-formula shows the basic change-formulas: Proportional change, $y = 4*x$; constant or linear change $y = 4*x+5$; constantly changing or accelerated change $y = 3*x^2 + 4*x + 5$; power change $y = 3*x^2$; and exponential change $y = 3*2^x$.

• We first model the orbit of a ball sent away with an angle. A constant up-number will give a line that goes up or down or horizontal. But here gravity makes the up-number decrease so the line curves down as a bended line called a parabola.

We choose the initial angle A determined by $\tan A = 6$. From (0,0) we assume that the ball takes a '1 out, 5 up' step followed by a '1 out, 3 up' and a '1 out, 1 up', etc., to reach the points (1,5), (2,8), (3, 9), (4,8), (5, 5), (6,0). Since $y = 0$ for $x = 0$ and for $x = 6$, the formula may contain the two factors $(x-0)$ and $(6-x)$, so a guess could be $y = a*x*(6-x)$. In the point (1,5) this formula becomes an equation, $5 = a*1*(6-1)$, or $5 = a*5$, solved by $a = 1$.

So, the parabola formula may be $y = 1*x*(6-x)$, or

$y = -x^2 + 6*x$.

This formula holds when tested on the other points:

$8 = -2^2 + 6*2$, or $8 = -4 + 12$, or $8 = 8$, etc.

We find that with 4 as the first up-number, the orbit formula will be $y = -x^2 + 5*x$, etc.

The height after 5 steps is found by the equation

$y = -5^2+6*5 = -25+30 = 5$.

The height 8 is reached after x steps found by the equation

$8 = -x^2+6*x$, or $x^2-6*x+8 = 0$, solved by $x = 2$ and $x = 4$.

The height 10 is never reached since there are no solutions to the equation:

$10 = -x^2+6*x$, or $x^2-6*x+10 = 0$.

Instead, that top-point is found in the middle at $x = 6/2 = 3$, giving

$y = -3^2+6*3$, or $y = 9$.

To see if it breaks through a roof with the formula $y = 12-x$, we equate the two ys and get the equation

$12 – x = -x^2 + 6*x$, or $x^2 - 7x + 12 = 0$, that is solved for $x = 3$ and $x = 4$.

To find the top-point we can also use change- calculations. When x changes with 1, 6*x changes with 6, and x^2 changes with 2*x as we saw in MC07, so the turning point is found by the change-equation $0 = -2*x + 6$, giving $x = 3$.

• We now model the beginning monthly income of a business trying to establish itself at a market. We use the formula $y= x^2-6x+9$ where the steps form a parabola curving up when passing the points (0,9), (1,4), (2, 1), (3,0), (4,1), (5,4), and (6,9).

Later the monthly income will change its curvature from up to down until it reaches a maximum level. So, from x = 3 we use a different model that contains the up-numbers 0, 1, 2, 3, 2, 1, 0, 0. This gives a 'logistical' s-shaped curve describing growth with saturation. When prompted, AI may give a formula for this curve as

$y = 9/(1+25*2\text{^}(-1.9*x))$

We see that the up-numbers form a hill. When prompted, AI may give a formula for this curve as

$y = 3/(2\text{^}(0{,}44*(x-3)\text{^}2))$

• A cats and mice cohabitation on an island is an example of a predator-prey model where cats eat mice. We expect a cycle in time since many cats and many mice leads to many cats and few mice, which leads to few cats and few mice, which leads to few cats and many mice, which leads to many cats and many mice once again.

In a model we assume that a mice-population at 7 and 2 will make the cat-population change with 7-5 and 2-5 respectively. Likewise, a cat-population at 7 and 2 will make the mice-population change with 5-7 and 5-2 respectively. We see that initial populations at the level 5 will give a stable model. Here we assume that the initial populations for the cats and the mice are 8 and 1 respectively. The following period the two populations will then be 8 + (1-5) = 4, and 1 + (5-4) = 2 respectively.

Continuing, we see that the cat population will change as 8, 4, 1, 2, 6, 9, 8; and that the mice population will change as 1, 2, 6, 9, 8, 4, 1. This allows the points (8,1), (4,2), etc., to be marked on a BBBoard, showing a cycle continuing again and again. Different initial numbers will give different cycles.

• We now model cyclic movements up and down as observed in nature with day and night, with summer and winter, and with tide in an ocean. A cyclic movement may be created by the up-numbers +2, +1, +0, -1, -2, -2, -1, +0, +1 +2.

Beginning at the point (0,5), AI may be prompted to give a formula for this curve as

$y = 5+3*\sin\ (0.63*x)$.

• Saving money may take place at home or in a bank. At home the terminal capital c after n months will be $c = b + a*n$, where b is the initial capital, and a is the change-number per month. In a bank, the terminal capital after n months will be $c = b*(1+r)\text{^}n$ where r is the change-percent per month.

Combining the two in a bank, the terminal capital C may be found by the formula $C/a = R/r$ where R is the total interest rate including the compound interest, $1+R = (1+r)\text{^}n$. This capital may be used as an installment plan to pay out a debt D that has grown to $E = D*(1+R)$ in the same period.

• If an interest rate at 100% is split in 12 portions the total interest is found from the equation $1+R = (1+1/12)\text{^}12 = 2.613$ so that $R = 1.613 = 161.3\% = 100\%$ plus 61.3% as additional compound interest. This leads to the Euler number $e = (1+1/n)\text{^}n = 2.7183$ for n large, which shows that the additional compound interest cannot surpass 71.8% when splitting up 100%.

• Biological populations typically grow exponentially with a constant periodical rate, which provides a constant doubling time. This may be shown on a BBBoard where the vertical numbers are in tens. Beginning with ¼, a doubling sequence will be

¼, ½, 1, 2, 4, 8, 16, 32, 64, 128.

Alternatively, a capital may decrease by always taking away half of what is left. Beginning with 8 this gives a halving sequence 8, 4, 2, 1, ½, ¼ .

This exponential decay may be recognized as a mirror of the exponential growth.

• A total of 1 12s may be recounted as 2 6s, 3 4s, 4 3s, 6 2s, and 12 1s. The upper right corner then travels along a hyperbola showing inverse proportionality between the height and the width.

MC21. Distributions in time, probability

The goal is to experience how an unpredictable experiment with two outcomes may still be predictable after many repetitions.

• The Pascal triangle. From the (0,0) dot on a BBBoard we go right to the neighbor (1,0) and (1,1) and from here we continue in the same way. Thus, to (2,0) and (2,1) and (2,2) the numbers of routes are 1 and 2 and 1. And, in column 3 the numbers of routes are 1 and 3 and 3 and 1. In column 4 the numbers of routes are 1 and 4 and 6 and 4 and 1. In column 5 the numbers of routes are 1 and 5 and 10 and 10 and 5 and 1. Etc. These numbers are called the Pascal Triangle numbers.

• The binomial distribution. On a BBBoard with two rubber bands showing a 4x4 box we again go right as in the Pascal Triangle. So, the out-number is always 1, but now the up-number is 1 only if a dice shows one of the two Win-numbers 5 or 6, and 0 if it shows one of the four Lose-numbers 1, 2, 3, 4. So here the probability for winning is the per-number 2/6 or 1/3. And that the probability for losing is 4/6 or 2/3.

We see that the track WWW with 3 Wins has the probability 1/3*1/3*1/3 = 1/27. Likewise, 2 Wins may occur in 3 different ways according to the Pascal Triangle, WWL, WLW, and LWW. Since each has the same probability the total probability for 2 Wins is 3*(1/3*1/3*2/3) = 6/27. Likewise, 1 Wins has the probability 3*(1/3*2/3*2/3) = 12/27, and 0 Wins has the probability 2/3*2/3*2/3 = 8/27.

With a 1/3 chance for winning, we expect to win once in 3 tries. And this outcome also has the highest probability, 12/27. But the probability that this will not happen is even higher, 15/27.

We see that the average outcome is 3*1/27 + 2*6/27 + 1*12/27 + 0*8/27 = 27/27 or 1 as we expected, which is the same as 3*1/3, the number of plays multiplied with the probability of winning.

To test the validity of our predictions we may roll 3 dices 27 times.

MC22. Distributions in space, statistics

The goal is to experience how an outcome that cannot be pre-dicted may instead be post-dicted by statistics giving a clue to the next outcome.

• A BBBoard has a horizontal rubber band after 6. Some dice is rolled ten times and the results are marked as heights on the board. A horizontal pin may be placed to the right showing how the average height is changing as the experiment goes on. In time, the results could be 5, 6, 5, 1, 1, 2, 4, 4, 4, 1.

To see the results in space, the BBBoard is turned a half round to show the results as 3 1s, 1 2s, 0 3s, 3 4s, 2 5s and 1 6s. The average height is calculated the sum divided by ten, 33/10 = 3.3, and a rubber band is placed at this height.

That the average of all the outcomes is 3.3 means that if like they would have been 3.3. But they are not like, many deviates from the average, some are below, and some are above. So, we can now find the average deviation from the average. Here the outcomes below the average deviate with 3*(3.3-1) + 1*(3.3-2) + 0*(3.3-3) = 8.2, and the outcomes above deviate with 3*(4-3.3) + 2*(5-3.3) + 1*(6-3.3) = 8.2, so the average deviation is 16.4/10 = 1.6. Approximately since the tradition calculates in a slightly different way to 1.8.

• Rolling some dice, the outcome cannot be predicted as a number, but as an interval that is found by many repetitions of the above experiment.

In our experiment, the results varied between 1 and 6 with 1 and 4 as the most frequent result and with 4 as the median in the middle. Had they all been like they would have been 3.3, but they deviate. Had the deviations been like they would have been 1.6. So, from our statistics we could assume that the next outcome with a high degree of probability will be in the interval from 3.3-1.6 to 3.3+1.6, i.e., roughly between 2 to 5, where 6/10 is situated in our case. Which is close to the prior prediction that 4/6 = 2/3 should be here.

In a binomial experiment with two outcomes with probabilities p and $1-p$, and n repetitions, there are precise formulas for the mean, m, and the deviation, d:

$m = n*p$, and $d^2 = n*p*(1-p)$

• A questionnaire typically contains answers to questions, e.g., age-group, A and B, and opinion, Yes and No. In time the answers could be (A,Y), (B,N), (A,Y), (B,Y), (A,N), (A,Y), and (A,N). In space, the 7 results can be presented in a 2x2 cross-table showing with unit-numbers that 3 answered A and Y, 2 answered A and N, 1 answered B and Y, and 1 answered B and N. This unit-number table allows several per-number to be created:

The part of A, p(A) = 5/7; and the part of B, p(B) = 2/7

The part of Y, p(Y) = 4/7; and the part of N, p(N) = 3/7

The part of A&Y, p(A&Y) = 3/7; and the part of A&N, p(A&N) = 2/7

The part of B&Y, p(B&Y) = 1/7; and the part of B&N, p(B&N) = 1/7

Among A, the part of Y is p(Y|A) = 3/5; and among A, the part of N is p(N|A) = 2/5

Among B, the part of Y is p(Y|B) = 1/2; and among B, the part of N is p(N|B) = 1/2

Among Y, the part of A is p(A|Y) = 3/4; and among Y, the part of B is p(B|Y) = 1/4

Among N, the part of A is p(A|N) = 2/3; and among N, the part of B is p(B|N) = 1/3

A typical is mistake is to go directly from one per-number to another:

Claim: "Among A, 3 of 5 says Yes, so 3 of 5 saying Yes comes from A".

No, of the 4 saying Yes 3 comes from A.

We must be careful when adding or comparing per-numbers. I not taken of the same total they should first be changed to unit-numbers before becoming new per-numbers.

A shortcut to this is expressed in Bayes formula allowing a direct transformation between per-numbers without first passing through the unit-numbers, p(A|Y)*p(Y) = p(Y|A)*p(A), simply using the per-numbers p(A) and p(Y) instead of the unit-numbers n(A) and n(Y).

P(A|Y)*4/7 = 3/5*5/7, so P(A|Y) = ¾.

• Are the two groups A and B different? Apparently, since in A, 3/5 or 60% votes Yes, whereas in B only ½ or 50% vote Yes. But with only one more No in A the groups would be like, so can we trust the result? To make it more trustworthy we could ask more people and set up the hypothesis that maybe the total average of Yes, 4/7 or 57% would be the right answer to which the two groups deviate a little because of randomness in choosing respondents.

So, we now test the hypothesis that the two groups are like with a Yes percentage at 57%. The questionnaires then become repetitions of a binomial experiment with 57% probability for winning. If we repeat this experiment 100 times, the mean will become $m = n*p$ = 100*57% = 57, and twice the standard deviation will be $2*d = 2*\sqrt{(100*0.57*0.43)} = 9.9 \approx 10$.

Advanced calculation shows that there is only 5% probability to deviate more than this double standard deviation, so such an outcome will reject the hypothesis. Outcomes from 47 to 67 will then be accepted. And here the outcomes in group A and B are 3/5 of 100 or 60 and ½ of 100 or 50 which are both acceptable.

However, with 1000 repetitions the numbers will be m = 570, and $2*d$ = 31. Here, outcomes from 539 to 601 will then be accepted. And now the outcome in group A is 3/5 of 1000 or 600, which is acceptable. But the outcomes in group B is 1/2 of 1000 or 500, which is not acceptable. So, in this case the hypothesis is rejected, which gives the conclusion that the groups are different.

MC23. Simple board games

The goal is to experience how different games may take place on a BBBoard.

• A race track. A 4x4 lawn is placed in the middle of a BBBoard. The start- and end-line goes from (5,0) to (5,3). A trip may change zero or one unit in the horizontal and vertical direction. You may touch but not cross the interior or exterior boundaries. If you do so you restart with 0 speed as close to the crossing point as possible. You may cross your opponent's track, but not end in the same point. The race may be repeated with different lawn shapes.

• Survival. You begin in (5,5). You roll a dice and move the number to the right if the number is even, else to the left. You roll again and now move the number up if the number is even, else down. You may touch but not cross the boundary. How many steps can you survive?

• Vertical race. A rubber band splits a BBBoard vertically in the middle. Two players each have three bricks placed at level one. They roll a dice and pick a brick to move the number upwards, and down again if there is a surplus. The winner is the first to have all three bricks at level ten.

MC24. Modeling and de-modeling

The goal is to experience how formulas calculating y from x form curves that expresses change in time, and how totals in space may be split in parts that each then becoming a percentage of the total.

• Modeling means solving an outside problem inside with four steps. First an outside problem is translated to an inside problem. Then the inside problem leads to an inside solution that then is translated to an outside solution, that finally is evaluated to see if another cycle is needed.

A typical example is mixture problems. the outside problems here may ask "2kg at 3$ per kg and 4kg at 5$ per kg total what?" The inside problem places the second information under the first ready to add. The inside solution my then simply add all numbers, which leads to the outside solution "2kg at 3$ per kg and 4kg at 5$ per kg total 6 kg at 8$/kg". This model is not accepted, so another cycle is needed. This time the per-numbers are multiplied to unit numbers before adding, which leads to the outside solution "2kg at 3$ per kg and 4kg at 5$ per kg total 6 kg at 26$/6kg". This model is accepted.

• De-modeling is the opposite process: It means solving an inside problem outside with four steps. First an inside problem is translated to an outside problem, then the outside problem leads to an outside solution that then is translated to an inside solution, that finally is evaluated to see if another cycle is needed.

A typical example is uniting fractions.

Adding fractions as ½ + 2/3 only has meaning when taken of the same unit, $u = (u/6)*6 = k*6$, where $k = u/6$, and $6 = 2*3$

$T = (½ + 2/3)*u = (½ + 2/3)*6*k = (3+4)*k = 7*k = 7*u/6 = 7/6*u$,

So, in this case, ½ + 2/3 = 7/6.

MC25. The Three Tales: Fact, Fiction and Fake

The goal is to experience that as qualitative literature, quantitative literature also has the genres, fact and fiction and fake when modeling real world situations (Tarp, 2001).

Fact is 'since-then' calculations using numbers and formulas to quantify and to predict predictable quantities as, e.g., 'since the base is 4 and the height is 5, then the area of the rectangle is $T = 4*5 = 20$'. Fact models can be trusted once the numbers, and the units, and the formulas, and the calculation have been checked. Special care must be shown with units to avoid adding meters and inches as in the case of the failure of the 1999 Mars-orbiter. In statistics, describing a collection of numbers with a median is an example of fact. Models from science typically are fact models holding whenever a falsification has been tried.

Fiction is 'if-then' calculations using numbers and formulas to quantify and to predict unpredictable quantities as, e.g., 'if the unit-price is 4 and we buy 5, then the total cost is $T = 4*5 = 20$'. Fiction models build upon assumptions that must be complemented with scenarios based upon alternative assumptions before a choice is made. In statistics, describing a collection of numbers with a mean is an example of fiction assuming that all numbers are like. Economical models typically are fiction assuming that the average behavior in the past will continue in the future is used to create forecasts.

Fake is 'what-then' calculations using numbers and formulas to quantify and to predict unpredictable qualities as, e.g., 'Let us close hospitals since it is cheaper to rest in a graveyard'. Fake models build on dubious assumptions and should be replaced by a political process containing debate before decision. Fake is also adding numbers without units, both unit-numbers and per-numbers. In statistics, describing a collection of numbers with a mean is an example of fake if it is meaningless to assume that all numbers are like: "Students in grade 1 and 9 go in grade 5 in average."

MC26. Game theory

The goal is to experience how damage control may be modeled by Game Theory.

Game theory looks, e.g., at a 2x2 zero-sum game where two players A and B have two strategies each resulting in four different payments from B to A, where A could be nature choosing blindly.

The payment from B to A could be 8 or 2 if B chooses strategy B1 and A chooses strategy A1 or A2. Else the payment is 4 and 6. This 2x2 game may be shown at a BBBoard split by two rubber bands.

By choosing B1 the risk to B is 8, and by choosing B2 the risk is 6. If both players mix the strategies with equal weight or probability ½ the average result will 5 as ¼ of the sum. But with A1 chosen 3 times of 4 the average result will be ¾*(8+4) + ¼*(2+6) = 11.

So maybe B should also mix the strategies with the probabilities x and (1-x) respectively.

If A choses A1, a mixed strategy for B will give the total payment $T = x*8 + (1-x)*4 = 4x+4$. On a BBBoard with 1 and 10 as the maximum x- and y-number, this payment line may be show as a rubber-band connecting (0,4) and (1,8). Here the minimum payment is 4 to the left with $x = 0$. If A choses A2 the total payment will be $T = x*2 + (1-x)*6= -4x+6$, shown as a rubber-band connecting (0,6) and (1,2).

Here the minimum payment is 2 to the right with x = 1.

We see that the payment may vary between 4 and 6 at $x = 0$, and between 2 and 8 at $x = 1$. But we also see that the payment stays constant 5 at the intersection point found by equating the two formulas, $4x+4 = -4x+6$, giving $8*x = 2$, or $x = 2/8 = ¼$. So, B will not pay more than 5 if mixing the strategies with the probabilities ¼ and ¾, e.g., by tossing two coins. Can A do the same?

If B choses B1, a mixed strategy for A will give the total payment $T = x*8 + (1-x)*2 = 6x+2$. On the BBBoard this payment line may be show as a rubber-line connecting (0,2) and (1,8). Here the maximum payment is 8 to the right with x = 1. If B choses B2 the total payment will be $T = x*4 + (1-x)*6= -2x+6$, show as a rubber-band connecting (0,6) and (1,4).

Here the maximum payment is 6 to the left with $x = 0$.

But we again we see that the payment stays constant 5 at the intersection point where $x = 1/2$. So, A will get at least 5 by mixing the strategies with the probabilities ½ and ½, e.g., by tossing one coin.

So, this game has the value 5, to be paid from A to B before each game to make it fair. Alternatively, the four payment numbers should be reduced with 5 each from 8, 2, 4, 6 to 3, -3, -1, 1.

Had the payments been 2, 8, 4, 6, the value would be 6, which is a 'saddle-point' going down to the A-side and up to the B-side, so that no player would like to change strategy. To A, the strategy A2 here gives the 'max-min' value, and to B the strategy B2 gives the 'min-max' value, which here are the same. So both players here choose a 'safety first' principle.

MC27. The Three footnotes

The goal is to experience the content of three calculation laws.

The commutative law: The order does not matter, a*b = b*a

The distributive law: When adding, like units may be bracket out, a*c + b*c = (a+b)*c

The associative law: Bracket may be moved at will, a*(b*c) = (a*b)*c

On a BBBoard two rubber bands mar 6 3s. Turning the board a quarter round we have 3 6s thus illustrating that 6*3 = 3*6.

A third rubber band split the 6 3s in 4 3s and 2 3s to illustrate that 4*3 + 2*3 = (4+2)*3.

With snap-cubes 2 3s 4 times gives a Total of (2*3)*4. Turning it over, twice we have 3 4s, thus illustrating that (2*3)*4 = 2*(3*4),

MC28. Math with playing cards

The goal is to show how math formulas may be discovered by working with ordinary playing cards. Some formulas are limited by the fact that cards only have positive numbers, so the question if the formulas also apply to negative numbers may be partly answered by testing. The instruction may be found at http://mathecademy.net/wp-content/uploads/2020/01/Math-with-playing-cards.pdf

01. The little, medium and big Pythagoras with 3, 4 and 5 playing cards.
02. PI with three playing cards.
03. Proportionality with the 2 playing cards.
04. Product rules with 2-4 playing cards.
05. The quadratic equation with 2 playing cards.
06. Change by adding and multiplying with playing cards.
07. The saving formula with 9 playing cards.
08. The change of a product with 3 playing cards.
09. Integral- and differential calculus with 2 playing cards.
10. Differentiating sine and cosine with 3 playing cards.
11. Topology with 6 playing cards.

Teacher education in CATS: Count & Add in Time & Space

The goal is to experience how teacher education may be different to obtain a communicative turn in number-language education.

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows mastery of Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged in countries choosing experiential (Kolb, 1984) learning curricula as, e.g., in Vietnam.

The MATHeCADEMY.net offers a free one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e., both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = $(T/B)*B$. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

Discussing the difference

We asked what could be different in online education as compared with offline education.

Typically, mathematics is taught as an end-product from a textbook. So, a difference would be to teach mathematics as a process where things in space are acted upon in time. This would make mathematics a number-language with sentences containing an outside subject, a verb, and an inside predicate as in word-language sentences. As the outside thing to talk about we chose a ten-by-ten bundle-bundle board, a BBBoard, where outside totals appear as stacks within two rubber bands showing the bundle-number and the counting-number.

A process-based mathematics apparently contains numerous differences.

● Here, numbers are no longer one-dimensional line-numbers. Now they are two-dimensional stack-numbers with both a counting-number and a bundle-number serving as the unit. Here changing units leads directly to the recount-formula expressing proportionality, to equations solved by recounting, and to per-numbers adding by their areas just as stacks do when adding next-to. So here the three core parts of math, proportionality and equations and calculus, appear from the beginning, and not in the middle and the end of the K-12 curriculum.

● Here, digits are no longer symbols with names and forms to memorize and to learn to read and write. Now they are icons built in time by uniting five sticks or strokes in one five-icon, etc. Of course, the icons are not identical to the textbook symbols, but they are close. And they give a concrete meaning to the two-digit number 10 by writing it with units, 1 Bundle 0, or 1B0, after having quickly collected all ten sticks into one bundle with none left behind. To be followed by 1B1, 1B2, called ‘one-left’ and ‘two-left’ by the Nordic Vikings also using the word ‘tvende-ti’ for twenty.

● Here, the counting sequence is no longer a sequence of names like the sequence of months in a year or cities on a railway. A multiple counting of singles, bundles, bundles-of-bundles, etc., now produces a sum of digits with units. This changes one three-digit number to three numberings, and thus makes obsolete the place-value system. So here, 7 changes to 0B7, 45 to 4B5, and 678 to 6BB7B8. And accepting underloads and overloads since the outside total is the same leads to flexible bundle-numbers with units allowing 48 to be written both as 5B-2, 4B8, and 3B18 tens, which eases standard calculations by making carrying obsolete.

● Here, power is no longer the last operation. Now, power occurs as the first operation since counting ten fingers in 3s allows meeting 9 as a bundle-of-bundles, a BB or B^2, making 9 into 3B0, or 1BB0B0, or 100 3s. And counting in 2s allows meeting 8 as a bundle-of-bundle-of-bundles, a BBB or B^3, making 8 into 4B0 or 1BBB0BB0B0, or 1000 2s.

● Here, the order of operations is no longer add-subtract-multiply-divide-power. Now it is the exact opposite. With the process-based math the first task is to bring an outside total inside by bundle-counting that includes Bundles-of-bundles as power. The next process is to push-away bundles, which may be iconized as a broom so that the outside action ‘from 8, push-away 2s’ inside becomes the calculation 8/2. The next task is to lift the bundles into a stack, which may be iconized as a lift so that the so that the outside action ‘4 times lift 2s’ inside becomes the calculation 4*2. The next task is to pull-away the stack to look for unbundled singles, which may be iconized as a rope so that the outside action ‘from 9, pull-away 8’ inside becomes the calculation 9 – 8. Included on-top of the stack, the unbundled may be seen as a decimal number, 4B1 2s, or counted in the unit, 4 ½ 2s, or described by what is pulled-away from the next top bundle, 5B-1 2s. Finally, after brought inside as stacks, outside totals may now be added next-to as areas like integral calculus, or on-top after recounting has made the units like, which may be iconized as a cross to show the two directions so that the outside action 'unite 2 3s and 4 5s' inside becomes the calculation $2*3 + 4*5$.

So here, the order of operations is different as the precise opposite. First power comes when bundling bundles, then division pushes-away bundles, then multiplication lift the bundles, then subtraction pulls-away the stack to allow the unbundled to also join the stacks, which finally may add next-to or on-top.

Also, the operations have different meanings, now 8/2 means 8 counted in 2s and not only 8 splits in 2, and now 4*8 means 4 8s and not only 3B2 tens, and now subtraction is a means to allow the unbundled to join the stack and give concrete meaning to decimals, fractions, and negative numbers. Finally, addition has two different meanings leading directly to the core of mathematics, calculus and linearity (proportionality). So here, integral calculus occurs in grade one as next-to addition of stacks, which becomes differentiation when reversing the process. Later it reappears in middle and in high school where per-numbers also add by areas since they must be multiplied to unit-numbers before adding.

Since bundle-numbers have units a core task is to change units. Here division and multiplication immediately are combined in the recount-formula that recounts the total T in Bs as T/B Bs, T = $(T/B)*B$, expressing the proportionality that is visible when observing that inside increasing the base of a stack implies decreasing the height to keep the outside Total unchanged. Likewise, subtraction and addition immediately are combined in a split-formula that splits T in B and $T – B$ by pulling away B from T, 'T = $(T – B) + B$'. In its process-based version, the recount formula is a core formula present all over mathematics and science. In its product-based version it is totally absent.

In textbooks, the first operation typically is addition based on the statement that '2+1=3 always', which allows 2+3 to be defined as 2+1+1+1. However, '2+1=3' is not a number-statement, but a word-statement saying that "after two comes three" useful in time when traveling through a row of places. It is not a statement about uniting two totals in space, since here the units are needed also as, e.g., 2 pairs + 1 triple is 2B1 3s, or 3B1 2s, or 7 1s. Built on the assumption that '2+1=3 always', mathematics becomes 'mathematism' (Trap, 2018) true inside its own self-referring world, but seldom outside in the real world where numbers always carry units, and where "additions often folds while multiplication always holds" since 2*3 = 6 simply states that 2 3s may be recounted as 6 1s. So '2+1=3' should be replaced by '2*1 = 2, 2*2 = 4, etc., which allows differentiating between prime units or numbers as 2, 3, 5, 7, etc., and compound or folded units as 4, 6, 8, 9, etc.

● Here, decimal numbers and fractions and negative numbers no longer must wait until they are introduced as rational and negative numbers in middle school to extend the set of natural numbers. Now they arise at once as different ways to see the unbundled placed on-top of the bundle-stack as described above where 9 = 4B1 2s = 4 ½ 2s = 5B-1 2s.

● Here, counting sequences no longer leave out the units. Now the units are included so that 2 becomes what exists outside, 0-bundle-2, shortened to 0B2. Likewise, 23 becomes 2B3, and 456 becomes 4BB5B6. So here, counting ten fingers in 3s makes 9 a bundle-of-bundles, a BB or B^2, which allows power to be the first operation, and no longer the last. Also, 'over-counting' with an overload is now allowed as in France calling 7B1 for 6B11 since they gave up understanding the Vikings in Normandy saying '1 and half-four' meaning '1 and half the way to 4 twenties'. Likewise is 'less-counting' counting not to ten but from ten as "Bundle less nine, B-8, B-7, …, B-1, 1 Bundle, 2B-9", etc.

● Here, functions are no longer subsets in a set-product where first component identity implies second component identity, and which is postponed to middle school. Now a function is a number-language sentence called a formula with a subject, a verb and a predicate as in word-language sentences. Only here, the inside predicate is a prediction of what happens with the outside subject, so that $T = 2*3 = 6$ is an inside prediction of the outside fact that 2 3s may be recounted as 6 1s.

● Here, equations no longer are equivalent number-names that may be transformed by the same action on both sides, and which is postponed to middle school. Now equations describe reversed processes from the beginning. Here, the addition equation 'u+3 = 7" asks "7 splits in 3 and something?", which of course is found by splitting 7 in 7 = (7 – 3) + 3, so that u = 7-3. This resonates with the formal definition of subtraction: "7-3 is the number n that added to 3 gives 7", or "if n+3 = 7, then n = 7-3". So, we see that the equation is solved by reversing the process by moving a number across the equal sign with the reverse calculation sign.

Here, the multiplication equation 'u*3 = 12" arises when recounting from tens to icons leads to asking "How many 3s in 12?". That of course is found by recounting 12 in 3s as 12 = (12/3)*3 so that u = 12/3. This resonates with the formal definition of division: "12/3 is the number n that multiplied with 3 gives 12", or "if n*3 = 12, then n = 12/3". So, again we see that the equation is solved by reversing the process by moving a number across the equal sign with the reverse calculation sign.

Likewise, reversing power-operations leads to equations as u^3 = 8 solved by the factor-finding root u = 3√8, and to equations as 2^u = 8 solved by the factor-counting logarithm u = log2(8) both resonating with their formal definitions. Finally, integration moves across as differentiation. So now there is no need for the group concept in 'abstract algebra' together with its as communicative and associative laws as well as neutral and inverse elements, which step by step motivates the 'do the same to both sides outside the brackets´ method that is presented fully to teachers but only partwise to students.

● Here, multiplication tables are no longer sequences learned by heart. Now 6*7 is 6 7s in its own right. And it may or may not be recounted in tens. And if so, it may take place on a BBBoard where 6 and 7 are seen as B-4 and B-3 respectively. This transformation of 6*7 into (B-4)*(B-3) introduces early algebra where the BBBoard shows the process where the 6 7s are left when pulling-away from the 10 Bundles first 4 bundles then 3 bundles, and finally adding the 4 3s that have been pulled away twice, thus clearly showing that minus times minus gives plus. So, to get the answer quickly, from 10 you subtract (3+4), see them as bundles, and add 3*4. All you have to memorize is the lower left corner on the BBBoard where you build up of from 2 2s to 5 2s, then from 2 3s to 5 3s, then with 4s and finally with 5s. Multiplication on a BBBoard thus offers a middle way in 'the multiplication war' (Economist 2021).

● Here, squares are no longer one of the numbers in the multiplication tables. Now they are Bundle-Bundles on a BBBoard that may be built by always adding an extra line horizontally and vertically and subtracting the upper right corner-number that was added twice. So now squares form a sequence of their own right that is quickly learned. And here the square root no longer must wait until irrational numbers are introduced in middle school. Now they arise from a simple outside question "How to square a rectangle without changing its total?" The BBBoard shows that you simply move half the overload from vertical to horizontal position or vice versa. The answer then needs to be reduced a little so have a little for the upper right corner as well.

Finally, here quadratics no longer must wait to high school. Now they appear on a (x+a)*(x+a) board when two rubber bands show the two squares, x^2 and a^2, as well as the two a*x rectangles that all disappear if x^2 + 2ax + b = 0, only leaving a^2 – b as (x+a)^2 after being squared.

● Here, proportionality no longer must wait to middle school. Now it appears when recounting a total in two units creates per-numbers bridging the units by recounting in the per-number: with 4kg per 5$, to answer the question '12 kg = ?$', we simply recount 12 in 4s: 12 kg = (12/4)*4kg = (12/4)*5$ = 15$. Likewise, with answering the inverse question '?kg = 20$'. Or we may just recount the units: $ = ($/kg)*kg = (5/4)*12 = 15.

● Here, fractions no longer are rational numbers that can add without units. Instead, fractions like per-numbers are operators needing numbers to become numbers before adding. So, we must always ask "fractions of what?" Counting red apples provides an example. Here ½ of 2 apples plus 2/3 of 3 apples gives (1+2)/(2+3) of 5 apples, i.e., 3/5 of 5 apples, and not (1*3+ 2*2) of 2*3 apples, i.e., 7/6 of 6 apples. So ½ + 2/3 is 7/6 only if they are taken of the same total, just as 2+1 is 3 only if they have the same unit. Addition therefore only has meaning inside a bracket with the common units outside, as expressed in the distributive law, $p*r + q*r = (p+q)*r$. Shortening or enlarging fractions now simply means removing or adding a common unit to both numbers.

● Here, trigonometry no longer must wait to high school where plane geometry as well as coordinate geometry has already been taught. Now it appears as per-numbers coming from recounting the sides in a stack split by its diagonal. This means that concrete examples from all STEM areas may be introduced very early.

● Here, coordinate geometry no longer comes after plane geometry. Instead, it serves its name by coordinating algebraic calculations with standard geometrical forms which may or may not later be studied by themselves in plane geometry.

● Here, addition no longer is the first operation to teach. Instead, it is the last since counting and recounting must first bring outside totals inside to be added in two different ways, next-to by their areas, or on-top after recounting has made the units like. So, when finally introduced with units, addition and its reverse leads directly to the core of mathematics, calculus and proportionality. And, with totals occurring as flexible bundle numbers with units, overloads and underloads make carrying and borrowing unneeded.

● Here, calculus no longer must wait to be taught to few the last year of school. Now it is taught to all the first year when totals are added next-to as areas. In middle school it is again taught to all when adding piecewise constant per-numbers and when adding fractions in cross tables. So, in principle, also in high school calculus can be taught to all as addition of locally constant per-numbers from science. Especially if the natural order is followed where integration comes before differentiation invented to recount areas as changes to allow many changes to be added as one change only from the initial to the end number since all middle terms cancel out.

● Here, algebra no longer is an abstract activity looking for pattens. Instead, algebra takes on its original Arabic meaning, to reunite like and unlike unit-and per-numbers as shown in the Algebra-square that also shows that operations are reversed by moving to opposite side with opposite calculation sign.

● Here, modeling no longer must obey the parole "Of course, math must be learned, before it can be applied". As a number-language with sentences assigning inside numbers and calculations to outside totals, modeling occurs always. But, as in the word-language, it is also important to evaluate if the sentences express fact, fiction or fake. So, these three genres now become an important part of using the number-language to be skeptical towards any quantitative statements or literature.

Testing the difference

Being very costly, changing expensive textbooks and long-term teacher education makes testing the validity of ManyMath difficult in traditional education, except for where it becomes difficult, e.g., when teaching division and fractions and letter calculations. But process-based 'ManyMath' may be tested outside the main track: in preschool, special education, home schooling, adult education, migrant or refugee education, or where students choose between different half-year blocks instead of having multi-year compulsory lines forced upon them.

As to teacher training, the MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows Mastering Many with ManyMath to be tested and developed worldwide in small scale design studies (Bakker, 2018) ready to be enlarged in countries choosing experiental learning curricula as, e.g., in Vietnam (Kolb, 1984).

Search questions about 'bundle-numbers' and 'recounting' may be given to small groups of four preschool children to get ideas about how to design a full first-generation curriculum. Asked "How old next time?” a three-year-old will say four showing four fingers, but the child will react to seeing the fingers held together two by two: “That is not four. That is two twos!” The child thus describes what exists, bundles of 2s in space, and 2 of them when counted in time. Likewise, counting a total of 8 sticks in bundles of 2s by pushing away 2s, a 5-year-old easily accepts iconizing this as 8 = (8/2)*2 using a stroke as an icon for a broom pushing away bundles, and a cross as an icon for a lift stacking the bundles. And laughs when seeing that a calculator confirms or predict this before carrying it out. And do not mind shortening ‘Total’ and ‘Bundle’ to T and B to get the recount formula ‘$T = (T/B)*B$’ saying “T contains B, T/B times.”

Conclusion

Inspired by the difference between teaching mathematics on-line outside the classroom and teaching it off-line inside we asked what else could be different. This led to a possible answer: a difference could be to teach mathematics as a process in time instead of as a product in space. Or, in other words, a difference could be to teach mathematics as an inside number-language about outside quantities parallel to the inside word-language about outside qualities. Or again in other words, a difference could be to let mastery of the outside existence, Many, precede mastery of the inside essence, math.

And yes, mastery of math may be preceded by mastery of Many that in return automatically leads to mastery of the Math core. Which allows details to be taught as footnotes to those who may be interested while the rest may focus on using their mastery to work out tales about Many and to discuss which of the three genres they belong to, fact or fiction or fake. And since outside Many are brought inside as a total counted and recounted in flexible bundle-numbers with units as shown in the polynomial form of numbers, the basic inside tales about outside Many could be about things and actions on a ten-by-ten Bundle-Bundle Board, a BBBoard. So yes, teaching math as a process in time instead of as a product in space is a difference.

Of course, working on a BBBoard, learners may not learn school or university mathematics in a strict traditional form, but they learn to communicate about Many; and by always including units they avoid the tradition’s mathematism with its fake addition claims only holding inside but often folding outside the classroom.

So, it is possible to have a communicative turn in number-language education like the communicative turn in word-language education that took place in the 1970. Here outside-inside use of outside examples to be described by inside language-sentences was allowed to precede or even replace inside-inside use of inside sentences to be described by inside grammar.

What remains is testing to see if the difference expressed in the above micro-curricula makes a difference. This may be impossible within traditional education because of heavy cost to change textbooks and to establish in-service teacher training. But it may be possible outside in math labs and in special education where the goal could be to use BBB to obtain BBB, i.e., to use Bundle-Bundle-Boards to Bring Back Brains. But, although designing and testing micro-curricula go together in design research, they should not be mixed: Architects design houses, they do not build them. Likewise with curriculum architecture.

And in the end the core question is what right allows traditional education to replace the child’s innate concrete number-language with a foreign abstract language? Is this not just another example of using the power of an imported institution to colonize the native brains?

So, a final question could be: Forcing mastery of inside mathematics to precede mastery of outside Many, does that follow the 17 United Nations Sustainable Development Goals? Here goal 4 about quality in education formulates the wish to “ensure inclusive and equitable quality education and promote lifelong learning opportunities for all”. And here target 4.6 states the subgoal to “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy”.

A communicative turn in number-language education may allow this development goal to be reached now. Postponing this turn clearly will not and may instead keep widespread innumeracy alive indefinitely. Which of course serves well the wish for ever more funds for researching the difficulties in teaching mathematism, teacher education of teachers and researchers as a goal displacement warned against in sociology (Bauman, 1990; Arendt, 1963).

Ethically and phenomenologically, it is important to respect and develop the way Many presents itself to children to provide them with the quantitative competence of a number-language.

Teaching numbering instead of numbers thus creates a new and different Kuhnian paradigm (1962) that allows mathematics education to have its communicative turn as in foreign language education. The micro-curricula allow research to blossom in an educational setting where the goal of mathematics education is to master outside Many, and where inside schoolbook and university mathematics is treated as grammatical footnotes to bracket if blocking the way to the outside goal, mastery of Many.

To master mathematics may be hard, but to master Many is not. So, to reach this goal, why force upon students a detour over a mountain too difficult for them to climb? If the children already possess mastery of Many, why teach them otherwise? Why not learn from children instead?

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Appendix

Unit-number tasks

Type1.1 Numbers

Two numbers have the sum 72, and one is twice as large as the other. What are the numbers?

Text Number1	*Numbers* x = ?	*ANSWER* *24*	*Equation* x + y = 72
Number2	y = 2*x	*48*	x + 2*x = 72 3*x = 72 x = 72/3 = 24

Type1.2 Money

A pays a bill of 210$ with three types of coins: 1s, 2s and 5s. There are 4 times as many 1s as 2s, and 20 fewer 2s than 5s. How many coins of each type were used?

Text 5s	*Numbers* x = ?	*ANSWER* *30*	*Equation* x*5 + (x-20)*2 + 4*(x-20)*1 = 210
2s 1s	x-20 4*(x-20)	*10* *40*	5*x + 2*x – 40 + 4*x – 80 = 210 11*x = 210 + 120 x = 330/11 x = 30

Type1.3 Age

A is 4 times as old as B. 5 years ago, A was 7 times as old as B. How old are A and B now?

Text B's age now	*Numbers* x = ?	*ANSWER* *10*	*Equation* 7*(x-5) = 4*x – 5
A's age now B's age then A's age then	4*x x - 5 4*x - 5	*40*	7*x – 35 = 4*x – 5 7*x – 4*x = – 5 + 35 3*x = 30 x = 30/3 x = 10

Type1.4 Geometry

A rectangle has a circumference of 224 meters. The length is 4 meters shorter than 3 times the width. What is length and width?

Text Width	*Numbers* x = ? meters	*ANSWER* *29*	*Equation* 2*x + 2*(3*x-4) = 224
Length	3*x-4 meters	*83*	2*x + 6*x – 8 = 224 8*x = 224 + 8 x = 232/8 x = 29

Type1.5 Lever

A, B and C settle on a seesaw, B and C on the same side. They weigh 100kg, 80kg and 40kg respectively. A and B both sit 3 meters from the focal point. Where must C sit for equilibrium?

Text C's meter-tal	*Numbers* x = ?	*ANSWER* *1.5*	*Equation* 100*3 = 80*3 + 40*x
A's contribution B's contribution C's contribution	100*3 80*3 40*x		300 = 240 + 40*x 300 – 240 = 40*x 60/40 = x 1.5 = x

Tasks.

1. Two numbers have the sum 48, and one is twice as large as the other. What numbers are they?
2. Two numbers have the sum 48, and one is three times as large as the other. What numbers are they?
3. A pays a bill of 290 kr. with three types of coins: 1ere, 2ere and 5ere. There are 5 times as many 1s as 2s, and 10 fewer 2s than 5s. How many coins of each type were used?
4. A pays a bill of 200 kr. with three types of coins: 1ere, 2ere and 5ere. There are 3 times as many 1s as 2s, and 20 more 2s than 5s. How many coins of each type were used?
5. A is 5 times as old as B. 4 years ago, A was 6 times as old as B. How old are A and B now?
6. A is 8 times as old as B. 5 years ago, A was 9 times as old as B. How old are A and B now?
7. The circumference of a rectangle is 128 meters. The length is 4 meters longer than 5 times the width. What is length and width?
8. The perimeter of a rectangle is 110 meters. The length is 5 meters shorter than 4 times the width. What is length and width?
9. A, B and C settle on a seesaw, B and C on the same side. They weigh 120kg, 60kg and 50kg respectively. A and B both sit 4 meters from the focal point. Where must C sit for equilibrium?
10. A, B and C settle on a seesaw, B and C on the same side. They weigh 90kg, 70kg and 20kg respectively. A and B both sit 2 meters from the focal point. Where must C sit for equilibrium?

Per-number tasks

In per-number tasks, they must always be converted to unit-numbers before the equation can be established.

Travel

Train1 runs from A to B at a speed of 40 km/h. Two hours later, train2 runs from A to B at a speed of 60 km/h. When does train2 overtake train 1?

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*	
Hours		x = ?	*4*	40*(x+2)	= 60*x
Speed1	40 km/h			40*x + 80	= 60*x
Speed2	60 km/h			80	= 60*x – 40*x = 20*x
Km-number1		40*(x+2) km	*240*	80/20	= x
Km-number2		60*x km	*240*	4	= x

Train1 runs from A to B at a speed of 40 km/h. At the same time, train2 runs from B to A at a speed of 60 km/h. When do the two trains meet when the distance from A to B is 300 km?

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*	
Hours		x = ?	*4*	40*x + 60*x	= 300
Speed1	40 km/h			100*x	= 300*x
Speed2	60 km/h			x	= 300/100
Km-number1		40*x km	*120*	x	= 3
Km-number2		60*x km	*180*		

The same distance takes 3 hours upstream, and 2 hours downstream. What is the speed of the motorboat when the speed of the current is 5 km/h?

Text	*Per-num.*	*Unit numbers*	*ANSWER*	*Equation*	
Speed	x = ? km/h		*25*	km = km/h*h = (x-5)*3	= (x+5)*2
Speed upstream	x – 5 km/h		*20*	3*x-15	= 2*x+10
Speed downstream	x + 5 km/h		*30*	3*x-2*x	= 10+15
Hours		3 hours		x	= 25

Mixture

? Liter 40% alcohol + 3 liters 20% alcohol gives ? liter 32% alcohol

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*	
Liters		x = ? liters	*4.5*	0.4*x + 0.2*3	= 0.32*(x+3)
Liters-number3		x+3 liters	*7.5*	0.4*x + 0.6	= 0.32*x + 0.96
Alcohol1	40%	0.4*x liters		0.4*x - 0.32*x	= 0.96- 0.6
Alcohol2	20%	0.2*3 liters		0.08*x	= 0.36
Alcohol3	32%	0.32*(x+3)	liters	x	= 0.36/0.08 = 4.5

Finance

A invests a gain of 400,000$ in the following way: Some is set at a rate of return at 3% p.a., the rest is put into 8% bonds. How much did he invest in each when the annual dividend is $20,000?

Text	*Per-num*	*Unit numbers*	*ANSWER*	*Equation*	
Bank in a thousand		x = ? $	*240*	3%*x + 8%*(400-x)	= 20
Bonds in thousands		x+3 $	*160*	0.03*x + 32 – 0.08*x	= 20
Interest rate in bank	3%			32 - 20	= 0.08*x – 0.03*x
Interest rate on bonds	8%			12	= 0.05*x
The Bank's contribution		3%*x $		12/0.05	= x
Bonds' contribution		8%*(400-x) $		240	= x

Work

A can dig a trench in 4 hours. B can dig the same trench in 3 hours. How long does it take to dig it together?

Text	*Per-number*	*Unit numbers*	*ANSWER*	*Equation*	
Time		x = ? hours	*12/7*	¼*x + 1/3*x	= 1
A's speed	1/4 trench/h			(¼ + 1/3)*x	= 1
B's speed	1/3 trench/h			7/12*x	= 1
A contributes		¼*x		x	= 12/7
B contributes		1/3*x			

Tasks.

1. Train1 runs from A to B at a speed of 50 km/h. Three hours later, train2 runs from A to B at a speed of 60 km/h. When does train2 overtake train 1?
2. Train1 runs from A to B at a speed of 50 km/h. At the same time, train2 runs from B to A at a speed of 60 km/h. When do the two trains meet when the distance from A to B is 400 km?
3. Same distance 4 hours countercurrent, and 3 hours downstream. What is the speed of the motorboat when the speed of the current is 6 km/h?
4. A can dig a trench in 5 hours. B can dig the same trench in 4 hours. How long does it take to dig it together?
5. A can dig a trench in 6 hours. B can dig the same trench in 3 hours. How long does it take to dig it together?

Mechanics

M1. A ball falls from the top of a skyscraper (air resistance is disregarded). After 0 seconds, the ball is at an altitude of 300 meters. After 5 seconds the ball is in ? meters height. After? seconds is the ball at a height of 0 meters. What is the impact velocity?.

Height after 5 sec: *Time:* *Speed:*

s = ? meter	s = ½*g*t^2		t = ? sek.	s = ½*g*t^2		v = ? m/s	v = g*t
t = 5 sek. g = 9.8 m/s^2 Height = ?	s = ½*9.8*5^2 s = 123.7 meter H = 300-123.7 H = 177.3 m		s = 300 m g = 9.8 m/s^2	2*s/g = t^2 √(2*s/g) = t √(2*300/9.8g) = t 7.82 seconds = t		t = 7.82 sek. g = 9.8 m/s^2	v = 9.8*7.82 v = 76.6 m/s

M2. A ball is shot vertically up at an initial velocity of 30 m/s (air resistance is disregarded). After 5 seconds the ball is in ? meters height. After? seconds is the ball at a height of 40 meters. After? seconds is the ball at maximum height?

Height after 5 sec. *Time to 40 m:*

s = ? meter	s = ½*g*t^2+vo*t		t = ? sek.	s = ½*g*t^2+vo*t
t = 5 sek. g = -9.8 m/s^2 vo = 30 m/s	s = -½*9.8*5^2+30*5 s = 27.5 meter		s = 40 m. g = -9.8 m/s^2 vo = 30 m/s	40 = -4.9*t^2+30*t 4.9*t^2-30*t+40 = 0 t = 1.96 and 4.16 seconds

Rising time until speed = 0 *Rising height:*

t = ? sek.	v = g*t+vo		s = ? meter	s = ½*g*t^2+vo*t
v = 0 m/s g = -9.8 m/s^2 vo = 30 m/s	(v-vo)/g = t (0-30)/(-9.8) = t 3.1 seconds = t		t = 3.1 sek. g = -9.8 m/s^2 vo = 30 m/s	s = -½*9.8*3.1^2+30*3.1 s = 45.9 meter

The height part of the task can also be counted as a task in the conversion of energy from kinetic to potential energy.

Rising height	h = ? meters	Ep = Ek
Rising time Acceleration Initial speed Kinetic energy Potential energy	t = 3.1 sek. g = -9.8 m/s^2 vo = 30 m/s Ek = 1/2*m*v^2 Ep = m*g*h	m*g*h = 1/2*m*v^2 h = 1/2*v^2/g h = 1/2*30^2/9.82 h = 45.8 meters

M3. A 100 kg person performs a Bounty jump from a bridge (air resistance is disregarded). It is 220 meters down. The feet are fixed in a rope of 120 meters, which is fixed in a spring with spring constant k = 100 N/m, corresponding to 10 kg being able to extend the spring 1 m. How far does the person get down? What if the person weighed 150 kg?

Spring dislocation	x = ? metre	Ef = Eb
Fall distance Acceleration Kinetic energy Potential energy Spring energy	d = 120+x g = -9.8 m/s^2 Ek = 1/2*m*v^2 Ep = m*g*h If = 1/2*k*x^2	1/2*k*x^2 = m*g*h x^2 = 2*m*g*h/k x =√(2*m*g*h/k) x = √(2*100*9.82*120/100) x = 48.5 d = 120+48.5 = 168.5 meters

M4. A person swings in a swing (air resistance is disregarded). The swing set is 4 m high and cord length is 3 m. What is the oscillation time? In the extreme position, the fluctuation is 50 degrees. What is the maximum speed? How far is the jump if the takeoff is in the bottom position?

Swing time	T = ? seconds	T = 2*π* √(l/g)
Cord length Acceleration	l = 3 m g = -9.8 m/s^2	T = 2*π* √(3/9.82) T = 3.47 seconds

Rise height at 50 degree oscillation *Maximum speed at 0 degree oscillation:*

s = ? meters	s = l – l*body v		v = ? m/sec.	Ek = Ep
l = 3 meters v = 50 degrees	s = 3 – 3*cos50 s = 1.07 meters		h = 1.07 m g = 9.8 m/s^2	1/2*m*v^2 = m*g*h v^2 = 2*g*h v =√(2*g*h) v = √(2*9.82*1.07) v = 4.58 meters/seconds

Drop time at 0 degree oscillation *Jump length at 0 degree oscillation:*

t = ? seconds	s = ½*g*t^2		s = ? meter	s = v*t
s = 4-3 = 1 meters g = 9.8 m/s^2	2*s/g = t^2 √(2*s/g) = t √(2*1/9.82) = t 0.45 seconds = t		v = 4.58 m/s t = 0.45 s	s = 4.58*0.45 s = 2.06 meters

The Economic Flow Diagram

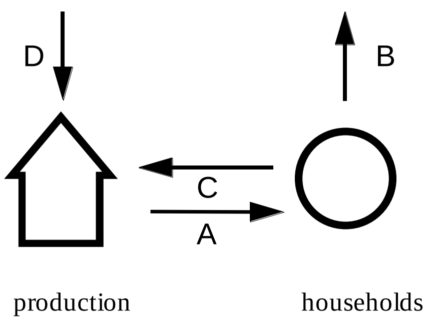
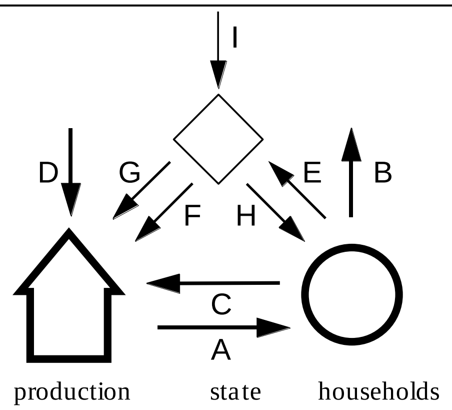
The basic economic flow consists of two sectors, production and private households. We have a number of needs that we meet by producing goods and services for others. In return, we receive an income that we can use to cover our own needs. This creates the basic economic flow consisting of the two money flows: Production creates an income A (wage), which is used for consumption C (food, clothing, etc.), which in turn leads to a new production, which in turn creates new consumption, etc. If income and consumption are in balance, the economic flow is stable. However, there is a drain and a source in the flow: Savings B and investments D. Savings are money that is not spent on consumption. Investment is money used to buy goods that cannot be consumed, e.g. buildings and machinery, etc.

The flow is stable if savings and investments are in balance. If savings are greater than investment, the flow will shrink, resulting in mass unemployment. This was the case after the First World War, when Germany was forced to send money to France as war reparations without France being obliged to buy German goods for the money. This caused the English economist J. M. Keynes to withdraw from the peace negotiations.

And this was the case in the United States during the Great Depression of the 1930s, where investment in stocks fell dramatically after the Great Wall Street crash of 1929, and where savings increased in order to repay the large loans taken out to participate in speculation on the stock market.

An economic flow with a state sector.

Keynes showed how a third public sector can balance a two-sector circuit. The public sector pulls taxes E out of the loop and uses this money to pump money back into the flow through transfer income H to the unemployed, public consumption F (more public employees, etc.) and public investment G (more roads, etc.). The public authorities may borrow loans, which will be repaid when the flow is back in balance.

	
A model of the basic economic flow could contain 4 equations: First trip: 1 Initial consumption Co = 100 2 Initial savings Bo = 20 3 The investment is assumed to be a constant percentage of consumption Do = d*Co 4 Income is consumption plus investment Ao = Co+Do Next trip: 1 Consumption is assumed to be a constant percentage of income C1 = c*Ao 2 Savings are the income that is not consumed B1 = Ao–C1 etc. 2 and 4 are fact equations, 1 and 3 are fiction equations. That is, the model as such is a fiction that should be supplemented with alternative models and scenarios. For example. Could proportionality equations 1 and 3 be replaced by linearity equations: 1 C1 = c*Ao -> C1 = c*Ao + K 4 D = d*C -> D = d*C + L Finally, an intervention can be made that changes the investment rate d from d to d+i D = d*C -> D = (d+i)*C This justifies the introduction of a state sector in the flow. In both cases, the systems of equations can be solved on an Excel spreadsheet:	A model for this 3-sector economic flow with 9 equations: First trip: 1 Initial consumption Co = 100 2 Initial savings Bo = 20 3 Initial investment Do = 20 4 Income is consumption plus investment Ao = Co+Do 5 Initial transfers Ho = 4 Next trip: 1 Taxes are assumed to be a constant percentage of income and transfers E1 = e*(Ao+Ho) 2 Private consumption is assumed to be a constant percentage of availability C1 = c*(Ao+Ho–E1) 3 Savings are the disposable amount that is not consumed: B1 = Ao+Ho–E1–C1 4 Government consump. assumed to be constant F1= constant 5 Public investment assumed to be a constant percentage of the investment gap G1 = g*(B1–Do) 6 Private investment is assumed to be a constant percentage of consumption D1 = d*(C1+F1) 7 The next income is that produced for consumption and investment A1 = Co+Do+Fo+Go 8 Transfers are assumed to be a constant percentage of the employment gap H1 = h*(Ao-A1) 9 Borrowing is the difference between taxes and government expenditure I1 = E1-F1-G1-H1 10 Debt is the summed up borrowing 3, 7 and 9 are fact equations, the rest are fiction equations. That is, the model as such is a fiction that should be supplemented by alternative models and scenarios.

Simulation of the economic cycle with and without a public sector

Change the rates in cells C5-C10
Don't edit the equations

Rates

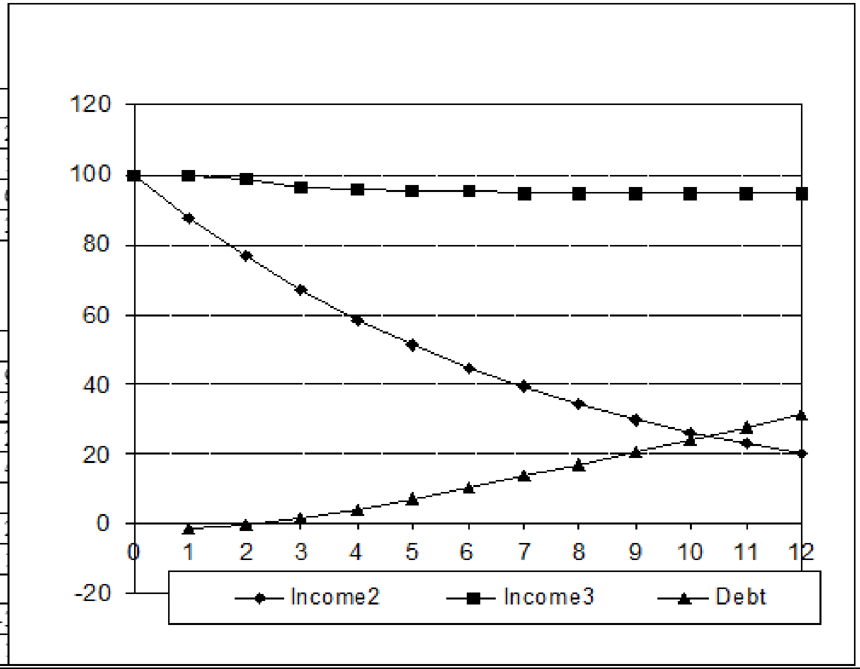
Time	n	0	1	2	3	4	5	6	7	8	9	10	11	12
Private consumption	c	**70%**	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%
PrivateInvestment	d	**25%**	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%
Tax	e	**30%**	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%
Public investment	g	**50%**	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%
Income transfer	h	**50%**	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%
Public consumption	F	**20**	20	20	20	20	20	20	20	20	20	20	20	20

2sector model

Time	*n*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Consumption	C	80	70	61,25	53,59	46,89	41,03	35,9	31,42
Savings	B	20	30	26,25	22,97	20,1	17,59	15,39	13,46
Investment	D	20	17,5	15,31	13,4	11,72	10,26	8,976	7,854
Income2	A	100	87,5	76,56	66,99	58,62	51,29	44,88	39,27

3sector model

Time	*n*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Income3	A	100	100	98,7	96,25	95,85	95,1	94,98	94,75
Transfers	H	4	0	0,65	1,875	2,074	2,449	2,51	2,625
Tax	E		31,2	30	29,81	29,44	29,38	29,27	29,25
Private consumption	C	80	50,96	49	48,68	48,08	47,98	47,8	47,77
Public consumption	F		20	20	20	20	20	20	20
Savings	B	20	21,84	21	20,86	20,61	20,56	20,49	20,47
PrivateInvestment	D	20	17,74	17,25	17,17	17,02	17	16,95	16,94
PublicInvestment	G		10	10	10	10	10	10	10
Borrowing	I		1,2	-0,65	-2,07	-2,637	-3,071	-3,245	-3,378
Debt			-1,2	-0,55	1,52	4,157	7,228	10,47	13,85



Meeting many in a STEM context

OECD (2015b) says: “In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.” STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature’s behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e., tools, motors and computers, that combined to robots help transforming nature into human necessities.

Nature as Things in Motion

To meet, we must specify space and time in a nature consisting of things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature’s three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e., from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

In nature, we count matter in kilograms, space in meters and time in seconds. Things in motion have a momentum = mass * velocity, a multiplication formula as most STEM formulas expressing recounting by per-numbers:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/liter) * liter = molarity * liter
- energy = (energy/kg/degree) * kg * degree = heat * kg * degree
- Δ momentum = (Δ momentum/second) * second = force * seconds
- Δ energy = (Δ energy/meter) * meter = force * meter = work
- energy/sec = (energy/charge) * (charge/sec) or Watt = Volt * Amp.

Thus, STEM-subjects swarm with per-numbers: kg/m^3 (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m^2 (pressure), etc.

Warming and Boiling Water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number 410/70/1.4 Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is 316/0.14 kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving Material in Water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get 100 gram = (100/59)*59 gram = (100/59)*1 mole = 1.69 mole, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 mole/liter, or 0.676 mole/liter.

Building Batteries with Water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H_2O, and carbon dioxide CO_2, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH_4, the energy comes from using 4 oxygen atoms to burn it.

Technology & Engineering: Steam and Electrons Produce and Distribute Energy

A water molecule contains two hydrogen and one oxygen atom weighing 2*1+16 units making a mole of water weigh 18 gram. Since the density of water is roughly 1 kilogram/liter, the volume of 1000 moles is 18 liters. With about 22.4 liter per mole, its volume increases to about 22.4*1000 liters if transformed into steam, which is an increase factor of 22,400 liters per 18 liters = 1,244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, $p*V = n*R*T$, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used.

The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g., steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes. In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to consumers.

An Electrical Circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per 'carrier' (charge-unit). Recounting 2000 in 220 gives (2000/220)*220 = 9.1*220, so we need 9.1 carriers per second, which is called the electrical current counted in Ampere, a per-number double-counting the number of carriers per second. To create this current, the kettle must have a resistance R according to a circuit law 'Volt = Resistance*Ampere', i.e., 220 = Resistance*9.1, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per carrier)*(carrier per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120-Watt bulb, the latter needs twice the energy and current, and consequently has half the resistance of the former, making the latter receive half the energy if connected in series.

How High Up and How Far Out

A spring sends a ping-pong ball upwards. This allows a recounting between the distance and the time to the top, e.g. 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the spring 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, e.g., 2.5 meters, 5 meters and 0,71 seconds. From a folding ruler we see that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor sin45 = cos45 = 0,707. The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: 5 = (u*0,707)*0,71, solved as u = 9.92 meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app. Compared with the horizontal distance, the vertical distance is halved, but the speed changes from 9.92 to 9.92*0.707 = 7.01. However, the speed squared is halved from 9.92*9.92 = 98.4 to 7.01*7.01 = 49.2. So horizontally, the distance and the speed are proportional. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

How to construct a road up a steep hill side

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance BC = u.

In the triangle BCD, the angle B is 30 degrees, and BD = u*cos(30). With Pythagoras we get u^2 = CD^2 + BD^2 = CD^2 + u^2*cos(30)^2, or CD^2 = u^2(1-cos(30)^2) = u^2*sin(30)^2. In the triangle ACD, the angle A is 10 degrees, and AD = AC*cos(10). With Pythagoras we get AC^2 = CD^2 + AD^2 = CD^2 + AC^2*cos(10)^2, or CD^2 = AC^2(1-cos(10)^2) = AC^2*sin(10)^2. In the triangle ACB, AB = 1 and BC = u, so with Pythagoras we get AC^2 = 1^2 + u^2, or AC = √(1+u^2).

Consequently, u^2*sin(30)^2 = AC^2*sin(10)^2, or u = AC*sin(10)/sin(30) = AC*r,
or u = √(1+u^2)*r, or u^2 = (1+u^2)*r^2, or u^2*(1-r^2) = r^2, or u^2 = r^2/(1-r^2) = 0.137, giving the distance BC = u = √0.137 = 0.37.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Jumping from a swing

When I jump from a swing I have the maximum speed at the bottom point. But here the angle with horizontal begins increasing from zero. But now the speed decreases. So a what point should I jump to obtain a maximum length?

The Twelve Math-Blunders

Math-Blunder 1: Teaching Both Numbers and Letters as Symbols
A number is an icon showing a degree of multiplicity. A letter installs a sound to be distinct.

Math-Blunder 2: Teaching 2digit Numbers Before Decimal Numbers
Math grows from counting by bundling & stacking (T = 2 3s = 1 1/5 5s = 1B1 5s) using 1digit numbers only. Using two digit numbers directly, neglecting the unit may make 10 and 100 cognitive bombs.

Math-Blunder 3: Teaching Fractions Before Decimals
In a natural approach both fractions and decimals occur in grade 1 as different ways of accounting for leftovers: fractions on top of the stack, and decimals next to the stack.

Math-Blunder 4: Teaching Addition Before Division
Counting in 3-bundles can be predicted by the 'recount-equation' and division: T = (T/3)*3.
Adding without units leads to 'MatheMatism' true in the library but not in the laboratory.
MatheMatism: 2+3=5 since 2m+3cm=203cm etc.
Mathematics: 2*3=6 since 2 3s is 6 1s.

Math-Blunder 5: Forgetting the Units
Fraction paradox: 10/100 (10%) + 20/100 (20%) = 30/100 (32% or 12% or 18% or …).

Math-Blunder 6: Teaching Fractions Before Integration
Add = integrate: 6kg @ 5$/3kg + 8kg @ 9$/4kg = 6*5/3$ + 8*9/4$ = 28$ = 8(28$/14kg)*14kg = 14 kg at 28$/14kg

Math-Blunder 7: Teaching Proportionality Instead of recounting
'Per-numbers' occurs when recounting in two different units:
If 4kg = 5$, then 12kg = (12/4)*4kg = (12/4)*4kg * 5$ = 3*5$ = 15$. (recounting 12 in 4s)

Math-Blunder 8: Teaching Balancing Instead of Backward Calculation
Forward calculation: 2+3*5 = ? ,
Backward calculation: 2+3*x = 14
Solution: x = (14-2)/3

Math-Blunder 9: Killer Equations Instead of Grounded Equations
2$ plus 3kg at ? $/kg total 14$' leads to '2+3*x=14'.
Killer-equation: 2 + (3 - 4x)/(5x – 6) = 7x – 8/(9x)

Math-Blunder 10: Teaching Geometry Before Trigonometry
Geometry: earth-measuring.
Earth splits into triangles, triangles into right-angled triangles.

Math-Blunder 11: Postponing Calculus
Primary school integration: 2 3s + 4 5s = ? 8s.
Secondary school integration 6 kg at 2 $/kg + 8 kg at 5 $/kg = 14 kg at ? $/kg

Math-Blunder 12: The 5 Meta-Blunders of Math Education
1) Forgetting Prediction.
2) Interchanging Product & Process.
3) Interchanging Goal & Means.
4) Funding Library Research Instead of Laboratory Research.
5) Turning Enlightenment LAB-based Mathematics into Modern LIB-based MetaMatism.

Reality is the root, not an application of Mathematics. So, think things.