Bundle-Bundle-numbers with Units may make Children stay Numerate

Allan Tarp, the MATHeCADEMY.net, Denmark, February 2024

Content

Abstract

Looking at four fingers held together two by two, we see four fingers, the essence. But, before school, children see what exists, bundles of twos in space, and two of them when counted in time. So, we may ask how mathematics may be taught to children if using their own two-dimensional bundle-numbers with units instead of the school's one-dimensional line-numbers without units. In other words, we may ask how children may learn mathematics by working with existence instead of listening to essence. Here we use the two core concepts of philosophical Existentialism holding that existence precedes essence. This will mean that counting precedes adding since outside totals must first be counted to be added later inside. In this 'Many-math' approach, mathematical concepts are re-rooted in outside existing examples instead of being defined as examples itself inside. Now tens, hundreds and thousands become bundles, bundle-bundles, and bundle-bundle-bundles, as does 2, 4 and 8 when counting in twos instead. Here one-dimensional lines on a ruler are replaced by two-dimensional rectangles on a ten-by-ten Bundle-Bundle Board, a BBBoard, containing the outside existing subjects that is linked to inside essence predicates in a number-language sentence as in a word-language sentence. Here units are always included in counting sequences as 0Bundle1, 0B2, …, 1B0. Here digits become icons with as many sticks as they represent. Here also operations become icons created in the counting process. Division is a broom to push-away bundles to be stacked by a multiplication lift before pulled-away by a subtraction rope to find the unbundled that are included on-top as decimals, fractions or negative numbers. Here recounting in another unit creates the recount formula T = (T/B)xB saying that T contains T/B Bundles. Here recounting tens in icons creates equations solved when recounting moves a number to the 'opposite side with opposite sign'. Here recounting icons in tens leads to early algebra when 6*7 becomes (B-4)*(B-3) placed on a BBBoard and found by pulling-away the top 4B and side 3B, and adding the 4*3 pulled away twice. Here bundle-bundles allow rectangular stacks to be recounted in squares with the square root as the side. Here recounting in a different physical unit creates per-numbers as 4$/5kg bridging the units by recounting, and with like units becoming fractions. Here mutual recounting the sides in a stack halved by its diagonal leads to trigonometry before geometry. And now, after being counted and recounted, stacks may finally add on-top after recounted has made the units like, or next-to as areas, i.e., as integral calculus becoming differential calculus when reversed, also used to add per-numbers and fractions that must be multiplied to unit-numbers to add. Squares add as the square formed by their mutual Bottom-Top line. All in all, there are four ways to unite the world's four number-types. Addition and multiplication add unlike and like unit-numbers, where integration and power add unlike and like per-numbers. Together with their opposite operations, subtraction, division, differentiation, and the factor-finding root or the factor-counting logarithm they form an 'Algebra square' that is named after the Arabic word 'Algebra', meaning to reunite. And that now is the number-language allowing us to tell tales about outside totals using the same three genres, fact and fiction and fake, as does the word-language.

How valid is mathematics?

The school teaches that '2+3 = 5' and that '2x3 = 6'. But are they both valid outside? 2 bundles of 3s may always be recounted as 6 1s, but 2weeks + 3days is 17days. So, even if both hold inside the school, outside 'multiplication holds, but addition folds.'

Mathematics that adds numbers without units may be called 'mathematism', true inside but seldom outside the school, whereas mathematics that add numbers with units may be called 'Many-math', using bundle-numbers with units as 2 3s and 4 5s that may be added next-to as 8s, or on-top after shifting the units. But adding areas and shifting units are called 'calculus' and 'proportionality', the core of mathematics. Where normally they come very late, here they occur in the first lesson.

But, before totals can be added they must be counted or recounted. Counting 8 in 2s, we push-away bundles of 2s to be lifted into a stack of 4 2s, which may be iconized by a broom and as a lift so that 8 = (8/2)x2, or T = (T/B)xB with T and B for Total and Bundle. This linear 'proportionality' recounting-formula to shift units now occurs in the first lesson when we bundle-count with units.

Also, it solves equations where ‘*u* x 2 = 8’ is asking ‘How many 2s in 8?’, which is answered by ‘*u* = 8/2’ since 8 is recounted in 2s as above, thus simply moving ‘to opposite side with opposite sign’.

Recounting 8 in 3s, we meet subtraction as a rope to pull-away the stack to find 2 unbundled that are included on-top of the stack as a decimal, 8 = 2B2 3s, or as a fraction when also counted in 3s as 2 = (2/3)x3, 8 = 2 2/3 3s, or with a negative number telling how much is needed for an extra bundle or pulled-away from this, 8 = 3B-1 3s. Here we may even see the 3 bundles of 3s as one bundle of bundles, one bundle-bundle, 1BB, so that 8 = 1BB 0B -1 3s, where the bundle-bundle is a square.

Which makes you wonder if any rectangular bundle-number may be recounted in squares with the square root as its side, and if squares can add as squares, e.g., as the square created by their mutual Bottom-Top line. In that case, calculus will be easy since areas then can add as squares.

Counting before adding thus leads to rectangular and squared bundle-numbers with units; and to decimals, fractions, and negative numbers; and to solving equation by recounting; and to proportionality needed to make units like when adding on-top; and to calculus when adding next-to as areas that again may be added to one square.

So, with Many-math’s ‘counting before adding’ we have learned most mathematics almost before we begin.

This will please the fourth of the 17 UN Sustainable Development Goals that defines quality education as ‘ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.’ And where the subgoal 4.6 wants to “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy”.

To meet this goal, we therefore replace ‘mathematism’ with ‘Many-math’. To make the difference between the two more clear we may use the basic philosophical question: What comes first, existence or essence, what is in the world or what we think about it?

So, we ask: “What makes children learn mathematics, listening to essence, or working with existence?” Where ‘mathematism’ chooses the former, Many-math chooses the latter to develop a number-language by working with Many as it exists in time and space as repetition and multiplicity, and in the word-language as plurals.

Finally then, the number-language may have its communicative turn to be learned by telling tales about things and actions in space and time, just as the word-language had around the 1970’s as described in H.G. Widdowson’s book ‘Teaching Language as Communication’.

From Many to bundle-numbers with units.

“That is not four, that is two twos”. Said a 3year old child when asked “How many years next time?” And when seeing four fingers held together two by two. This statement will change mathematics education forever since, as educated, essence is all we see. But as uneducated, the child sees what exists, bundles of twos in space, and two of them when counted in time. The child thus opens our eyes for a different mathematics that, freed from its present essence-bounds may return to its original identity as a natural science about Many in space and time: and allow us to develop a natural number-language when communicating about existence instead of essence.

If existence comes before essence, then counting comes before adding, which is new since normally we are given the numbers to add. First, as educators we ourselves now investigate the consequence of counting before adding. Then we transform our discoveries into a sequence of micro-curricula.

Looking at five fingers we see that the inside essence ‘five’ outside may exist in different forms with each their label.

In space five may exist as five ones, or as a bundle of one fives that can be rearranged as an icon with the number of sticks it represents. Or, if counted in twos, five may exist as one bundle and three, 1B3, as 2B1, or as 3B-1 needing 1 to become an extra bundle, or even as 1BB 0B 1 since two twos held together exist as a bundle of bundles, one bundle-bundle, 1BB, or 1B-square becoming a square with snap-cubes.

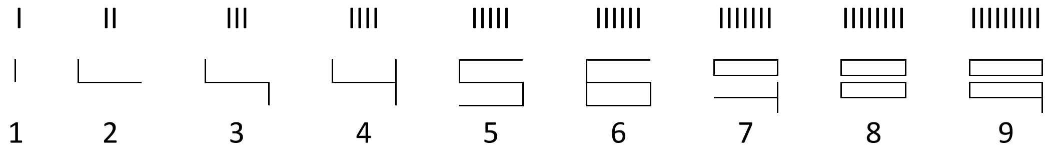

Figure 1. A digit as an icon with the number of sticks as it shows, four in the 4-icon, etc.

In time the five fingers may appear one by one. Here the bundles should be included as a unit in the counting sequence: 0B1, 0B2, 0B3, 0B4, 0B5. Or, if counting in 2s: 0B1, 0B2 or 1B0, 0B3 or 1B1, 0B4 or 1B2 or 2B0 or 1BB 0B 0, and finally 0B5 or 1B3 or 2B1 or 1BB 0B 1.

So, when *counting* and *adding* Many, we will no longer use a ruler's 1dimensional line-numbers without units as 5 and 42. Instead we now will use 2dimensional bundle-numbers with units as 0B5, and 4B2 that exist as rectangular or square totals on a ten-by-ten bundle-bundle pegboard, a 'BBBoard'. Which allows learning mathematics indirectly when formulating inside tales about outside totals as, e.g., 6 7s existing on the BBBoard limited by two rubber bands, and that may be recounted in tens as a total of four tens and two, shortened to 'T = 4B2' tens. This number-language sentence or formula contains an outside subject linked to an inside predicate, just as does a word-language sentences as 'This is a table'.

Flexible bundle-numbers with units allow the same total, forty-two, to be recounted with over-load or under-load so that T = 42 = 4B2 = 3B12 = 5B-8, which makes unneeded the place value system. As well as carrying and borrowing since now 17 + 28 = 1B7 + 2B8 = 3B15 = 4B5 = 45, and 57 – 28 = 5B7 – 2B8 = 3B-1 = 2B9 = 29.

With units, 2digit numbers without may be postponed since with 6s as the unit, 6+9 = (1B0 + 1B3) 6s = 2B3 6s, and with 9s as the unit, 6+9 = (1B-3 + 1B0) 9s = 2B-3 9s = 1B6 9s.

When bundle-counting outside totals we find that not only digits but also the operations are icons, but in the reverse order.

Power now is the first operation we meet as a bundle-bundle hat when *counting* in 3s will change 9 into 3 3s, a bundle of bundles, a bundle-bundle, a BB, or a B^2, that on a BBBoard is a square where 2 3s is a rectangle that may be transformed into almost a square with the rectangle's square root as the side by moving half the excess from the top to the side.

Division and multiplication then follow as a broom and a lift to push-away and stack bundles. Here, *recounting* in 2s will change 8 into (8/2)x2, or T = (T/B)xB telling that the total T contains T/B Bundles. This proportionality 'recount-formula' is used all over to shift units. Also, it solves multiplication equations as '*u**2 = 8' asking "How many 2s in 8?" which of course is found by recounting 8 in 2s as 8 = (8/2)*2, so that the solution is *u* = 8/2 found by moving 'to opposite side with opposite sign'. This follows the formal definition: 8/2 is the number *u* that multiplied with 2 gives 8, so if *u**2 = 8 then *u* = 8/2. So, the balancing method solving equations by doing the same to both sides now disappears.

Subtraction now follows as a rope to pull-away the stack to locate unbundled singles, thus *splitting* the total in two, T = (T–B)+B, the 'split-formula'. Finaly in the end, addition is a cross showing the two ways to unite stacks, next-to and on-top.

The split-formula solves addition equations as '*u*+2 = 8' asking "What is the number that with 2 added becomes 8?" which of course is found when splitting 8 by pulling-away the 2 that was added, 8 = (8-2)+2, so that the solution is *u* = 8-2, again found by moving 'to opposite side with opposite sign'. Also this follows the formal definition: 8-2 is the number *u* that with 2 added 2 gives 8, so if *u*+2 = 8 then *u* = 8-2. So here also the balancing method is not needed.

Recounting 8 in 3s, a calculator may inside predict the outside result. Entering '8/3' gives '2.more', and entering '8-2x3' gives '2' unbundled. This prediction is validated when outside pushing-away 3s from 8 twice. Included on-top of the bundles, the unbundled becomes decimals if writing T = 2B2 3s; or fractions if counted in bundles also, T = 2 2/3 3s, or replaced by a negative number, T = 3B-1 3s, telling what is missing in space for an extra bundle, or what was pulled-away in time from it. So, counting the unbundled leads to decimals, fractions, and negative numbers, 8 = 2B2 = 2 2/3 B = 3B-1 3s.

Recounting from one icon-unit to another may be predicted by a calculator. To inside predict the answer to the outside question "2 3s = ? 4s", entering '2*3/4' gives '1.more'. So, the unbundled are found when pulling-away 1 4s predicted by entering '2*3 – 1*4' giving '2'. The calculator thus predicts that 2 3s = 1B2 4s, which may be validated outside.

Recounting from tens to icons when asking 'How many 6s in 24?' leads to the equations 'u*6 = 24' solved by u = 24/6 since 24 recounts in 6s as '24 = (24/6)*6', so the solution again follows the 'opposite side & sign' rule.

Recounting from icons to tens when asking 'How many tens in 6 7s?' leads to early algebra when placed on a BBBoard as (B-4)*(B-3), which is left when we pull-away the top 4B and the side 3B and add the 4 3s pulled away twice, so (B-4)*(B-3) = B*B – 4B – 3B -- 4*3 = (10-4-3)B + 4*3 = 3B0 + 1B2 = 4B2 = 42, which clearly shows that minus times minus must be plus. Also, the four rectangles on the BBBoard illustrates the FOIL-method, First, Outside, Inside, Last; only here, Outside and Inside has changed place.

Recounting rectangles in squares, we may ask "How to square 6 4s by finding its square root as its side?" Moving half the excess from the top to the side gives the first guess as 5 5s. Then, to fill out the 1x1 top right corner we pull-away a slice, u, from the top and side. Here, 2*4*u = 1, or u = 1/8, so 4 7/8 = 4.88 is our second guess, which is close to the calculator's answer, 4.90. Since two squares add as the square created by their Bottom-Top BT line we now have a way to add rectangular areas.

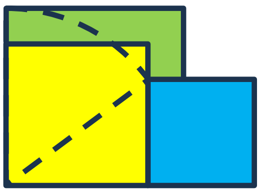

Figure 2. Two squares add as the square created by their mutual BT Bottom-Top line.

A split square may also be used to solve quadratics: In an (u+3)-square, a vertical and horizontal dividing line will split a BBBoard into two squares, u^2 and 3^2, as well as two 3*u rectangles. So (u+3)^2 = u^2 + 3^2 + 2*3*u = u^2 + 6 *u + 9. Now, with the quadratic u^2 + 6*u + 8 = 0 they all disappear except 9-8 = 1. So (u+3)^2 = 1, which gives -2 and -4 as the two solutions.

Also, a split square with y = x^2 may show that, with dy and dx as small changes of y and x, dy will be 2*x*dx if we neglect the tiny-tiny upper right corner. So, with y = x^2, dy/dx = 2*x.

Recounting in a different physical units, e.g., from \$ to kg, \$ = (\$/kg)*kg, we get a 'per-number', \$/kg, to connect the units: with 3\$/5kg, 12\$ = (12/3)*3\$ = (12/3)*5kg = 20kg. With like units, per-numbers become fractions or percentages, 3\$/5\$ = 3/5, and 3\$/100\$ = 3/100 = 3%.

Recounting mutually the sides in a rectangle split by its diagonal, the per-numbers are called trigonometry that connects the sides and the angles., e.g., rise = (rise/run)*run = tangent-Angle*run. Here the tangent-Angle describes the steepness or the rise-ability of the diagonal. In an x-y coordinate system a curve may be generated by a formula $y = f(x)$. Here the curve between two close neighbor points is almost a diagonal in a rectangle, and since the run and the rise here are changes in x and in y, Δx and Δy, the tangent-Angle here describes the steepness of the curve as the per-number $\Delta y/\Delta x$ called the local slope of the curve.

Once *counted* and *recounted*, totals may finally be *added* on-top or next-to.

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like. Adding 2 3s and 4 5s as 8s means adding areas, which is called integral calculus. Reversing the process by asking, e.g., "2 3s and how many 5s total 4 8s" is called differential calculus since you must find the difference between the two known totals before recounting it in 5s, ($T2$-$T1$)/5, or $\Delta T/5$.

Adding 2kg at 3$/kg and 4 kg at 5$/kg, the unit numbers 2kg and 4kg add directly whereas the per-numbers 3$/kg and 5$/kg first must be multiplied to unit-numbers before adding, thus adding as areas, i.e., as integral calculus. Likewise with fractions where 1red of 2 apples plus 2red of 3 gives 3red of 5, and of course not 7red of 6 apples as taught by 'mathematism'. Per-numbers and fractions and digits thus are not numbers, but operators needing a number to become a number.

Here the per-numbers are piecewise constant, but they may also be locally constant as in the case of a falling object with an increasing meter/second number. Calculus thus occurs three times, as next-to addition of stacks in primary school, as adding piecewise constant per-numbers in middle school's mixture problems, and as adding locally constant per-numbers in high school where the tiny area-strips are written as changes, $p*dx = dA$, to profit from the fact that adding many changes makes all middle changes disappear leaving only the total change of A from the start to the end.

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives 106%^10 or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

Looking inside my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bend the two outer fingers. So, 1/3 of the Ls are bent, and ½ of the Rs. Does that mean that 1/3 of the bent are Ls? No, ½ is. So, in cross-tables we must also see fractions as operators needing numbers to become numbers.

The first months the children thus meet the core of mathematics: functions, equations, proportionality, trigonometry, and calculus. As well as the four operations that unite unlike and like unit-numbers and per-numbers: addition, multiplication, integration, and power, seen in an 'Algebra-square' that is named after the Arabic word 'algebra' meaning 'to reunite'. And that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm.

Operations **unite**/ *split Totals in*	Unlike	Like
Unit-numbers m, s, kg, $	$\boldsymbol{T = a + n}$ $T - n = a$	$\boldsymbol{T = a * n}$ $T/n = a$
Per-numbers m/s, $/100$ = %	$\boldsymbol{T = \int f\, dx}$ $dT/dx = f$	$\boldsymbol{T = a\hat{}b}$ $b\sqrt{T} = a \quad loga(T) = b$

Figure 3. The Algebra square shows how to unite and split the four number-types, and how to solve equations by moving to opposite side with opposite sign.

Once we know how to count and recount totals, and how to unite and split the four number-types, we can now actively use this number-language to produce tales about numbering and numbers, and about totaling and totals in space and time. This is called modeling. As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then, and what-then models, or room, rate, and risk models. Fact models talk about the past and present and only needs to have the units checked. Fiction models talk about the future and needs to be supplemented with alternative models built upon alternative assumptions. And fake models typically add without units, e.g., when claiming that ‘2+3 = 5’ always despite 2weeks + 3days = 17days, thus transforming mathematics to ‘mathematism’.

Based on our investigation we may now design a sequence of micro-curricula, MC, for developing a number-language by working with things and actions on a BBBoard that may be supplied with centi-cubes on-top.

MC01. Digits as icons in space, IIIII = 5

The total here exists as sticks to be rearranged at a table and reported by a drawing on paper. The ‘T=?’ question is answered in two ways, as a collection of single ones, I I I I I, or as one bundle of ones, ~~IIII~~, that may be rearranged into an icon, 5, called a digit containing the number of sticks that it represents if written in a less sloppy way, and thus somewhat like the digits on a calculator. Each time the folding ruler is folded to look like the icon.

If we count in tens, ten sticks are replaced by one stick in a different color or material to allow more bundles to be rearranged as icons also so that 67 means 6-Bundle-7, 6B7, called 6-ten-7.Ten thus needs no icon since it becomes ‘one bundle and no unbundled’, written as T = 1B0 tens, or T = 1.0 tens, or T = 10 if leaving out the unit and the decimal point.

Example. One stick is one single. An extra stick added to 1 stick gives two singles that may unite to one 2-icon. And so on. An extra stick added to 8 sticks gives nine singles that may unite to one 9-icon. And an extra stick added to 9 sticks gives ten singles that instead of uniting to one ten-icon is bundled together as one bundle replaced by one stick of a different color or a different material and written as 1B0 since there are no singles left. An extra stick added to ten sticks gives eleven singles that may unite to one bundle and 1 single left, which made the Vikings call eleven ‘one left’ and written as 1B1. Likewise with twelve that the Vikings called ‘two left’. There is no ‘three left’ because of the ancient counting method “one, two, many”. So, from 3 we specify both the bundles and the singles. Zero is iconized as a looking glass not finding anything. The name ‘twenty’ comes from the Vikikgs’ ‘tvende ti’.

Skill building. Roll some dice twice (physically or virtually) to get the number of bundles and unbundled singles. Then phrase and report the number. So, with 3 and 5, say three-bundle-five, three-ten-five, and thirty-five; and finally write T = 3B5 = 35.

End test. Roll some dice twice an extra time.

MC02. Tally-counting in time, ****** = ~~IIII~~ I

The total here exists as sticks to be moved one by one on a table and reported by strokes on paper, as well as some dice. The ‘T=?’ question is answered by tally-counting the total in time reported as some 5-bundles and some unbundles singles, e.g., T = 2B1 5s.

Example. In a sentence, count the e’s and the a’s.

Skill building. Some dice is rolled a dozen times to show Even (1 2s, 2 2s or 3 2s) or Odd (1, 5, 5). The tally counting in 5s is reported with two totals e.g., W = 1B4, and L = 0B3, giving a total T = 2B2, or 1B7, or 3B-3. And giving the difference D = 1B1.

End test. Roll some dice an extra time.

MC03. Bundle-counting in time as a sequence with units: 0B1, 0B2, …, 0B5 or 1B0, 3 3s = 1BB

The total here exists as lines or stacks on a BBBoard. The ‘T=?’ question is answered in time by moving the finger along the pegs with a counting sequence that by including the bundle as a unit makes the place value system unneeded. First, we count in lines, then in stacks marked by a vertical rubber band.

Counting 5 fingers in 3s we ask, “What do we have here?” to emphasize that we focus on existence instead of essence. We cannot count one finger as ‘one’ since 1 3s is 3 1s and we only have one. Instead, we count ‘0 bundle 1, 0B2, 0B3 or 1B0’ since 3 1s is 1 bundle with no unbundled left.

Counting 5 fingers in 2s we notice that four fingers are 1B2, but also 2B0, and 1BB0B0 since 2 2s is a bundle of bundles, a bundle-bundle, a BB, that is a square, as is 3 3s, 4 4s etc.

Counting the five fingers on a hand, their essence ‘5’ may exist in various ways:

T = 1 5s = 1B1 4s = 1B2 3s = 1B3 2s = 2B1 2s = 1BB 0B 1 2s = 5 1s

Counting ten fingers in 3s, we get 0B1, 0B2, 0B3 or 1B0, 1B1, … , 2B3 or 3B0, 3B1. T = 3B1 3s

Counting ten fingers in 2s we notice that 8 as 2BB 0B 0 is 1BBB 0BB 0B 0. So, we may also write ten as 1BBB 0BB 1B 0 2s, or as 1010 if leaving out the units.

Finally, when counting hundred on the BBBoard we finish with 1BB 0B 0:

0B1, 0B2, … ,0B9, 0Bten or 1B0, 1B1, … , 9B8, 9B9, 9Bten or tenB0 or 1BB0B0.

To find the quadratic numbers we see that 5 5s comes from 4 4s by adding 4 twice and 1 for the top right corner. So, with 4 4s as 16, 5 5s is 16 + 4 + 4 + 1= 25. In this way we may predict the square numbers to be 1, 4, 9, 16, 25, 36, 49, 84, 91 and 100. And we see that a BB square increases with 2B+1 when B increase with 1.

Skill building. Count a dozen and a score in 5s, 4s, 3s, and 2s.

End test. Count 30 in 3s.

MC04. Bundles counted in space with over- and underloads, 5 = 1B3 = 2B1 = 3B-1 2s

The total here exists as fingers and sticks. The ‘T=?’ question is answered in space by ‘flexible bundle counting’ that allows unbundled to stay unbundled as an overload, and that allows borrowing extra sticks to fill up an extra bundle with an underload. Using flexible bundle-numbers with units makes carrying and borrowing unneeded.

Five fingers may be recounted in 5s as 0B5 (an overload) or 1B0 or 2B-5 (an underload).

Five fingers may be recounted in 4s as 0B5 or 1B1 or 2B-3.

Five fingers may be recounted in 3s as 0B5 or 1B2 or 2B-1.

Five fingers may be recounted in 2s as 0B5 or 1B3 or 2B1 or 3B-1.

Ten fingers may be recounted in tens as ½B from 1 to ten:

½B-4, ½B-3, ½B-2, ½B-1, ½B0, ½B1, ½B2, ½B3, ½B4, ½B5 or 1B0 as ten.

This may ease standard calculations.

T = 6+3 = ½B1 + ½B-2 = 1B-1 = 0B9 = 9

T = 6 + 7 = ½B1 + ½B2 = 1B3 =13

T = 8-3 = ½B3 – ½B-2 = 0B5 = 5, thus showing that -(-2) = +2

T = 4*7 = 4*½B2 = 2B8 = 28

Skill building. The action is repeated with nine fingers arranged counted in 5s, 4s, 3s, and 2s,

The action is repeated with two-digit numbers, e.g., 67 = 6B7 = 7B-3 = 5B17.

Then everything is repeated with snap-cubes, and with sticks.

End test. The action is repeated on a BBBoard, or on an abacus.

MC05. Splitting, 8 = (8-2)+2

The total here exists as a line of pegs on a BBBoard. The ‘T=?’ question is answered by ‘pulling-away’ a bundle hidden under cubes.

A total of 1 8s is split by pulling-away 2. To pull-away just once may be iconized by a rope, -, so that ‘8-2’ means ‘from 8 pull-away 2’ in time, or ‘from 8 pulled-away 2’ in space.

The original 8 now is split in 8-2 and 2 so that 8 = (8-2) + 2. Here addition is iconized by a cross showing the two directions we can add, next-to or on-top so that ‘4+2’ means ‘4 with 2 added’. With T for the total and B for the bundle this ‘split-formula’ may be written as T = (T–B)+B.

It is used to solve equations coming from reversed actions.

The question “What is the number that with 2 added gives 8” may be shortened to an equation with a letter for the unknown number, ‘u+2 = 8’. Of course, the number is found by reversing the action and pull-away the number that was originally added, so u = 8-2, which also comes from splitting 8, ‘u+2 = 8 = (8-2) + 2’. So, we see that the solution is found by moving “to the opposite side with the opposite sign”. Also, it follows the formal definition of subtraction: 8-2 is the number u that added to 2 gives 8, or if u+2 = 8 then u = 8-2.

Skill building. The action is repeated with fingers, sticks, cubes, and an abacus.

The action is repeated with other numbers, e.g., 9 = (9-3)+3.

End test. Pick two numbers.

MC06. Recounting, 8 = (8/2)x2

The total here exists as lines of pegs on a BBBoard. The ‘T=?’ question is answered by ‘pushing-away’ bundles hidden under cubes.

A total of 1 8s is recounted in 2s by 4 times pushing-away 2s. To pull-away more times may be iconized by a broom, /, so that ‘8/2’ means ‘from 8 push-away 2s’ in time, and ‘from 8 pushed-away 2s’ in space.

With the pushed-away 2s arranged in a stack, 8 contains 2s 4 times or 8/2 times so that 8 = 4*2, or 8 = (8/2)x2. Here multiplication is iconized by a lift so that ‘4x2’ or ‘4*2’ means ‘4 times stacking 2s’. With T for the total and B for the bundle this ‘recount-formula’ may be written as T = (T/B)xB.

It is used to solve equations coming from reversing actions: The question “What is the number of 2s in 8” may be shortened to an equation with a letter for the unknown number, “u*2 = 8”. Of course, the number is found be reversing the action and push-away the 2s that was originally united, so u = 8/2, which also comes from recounting 8, ‘u*2 = 8 = (8/2)*2’. So, we see that the solution is found by moving “to the opposite side with the opposite sign”. Also, it follows the formal definition of subtraction: 8/2 is the number u that multiplied with 2 gives 8, or if u*2 = 8 then u = 8/2.

Skill building. The action is repeated with 12 counted in 2s and 3s using a finger to hide a bundle.

End test. 18 counted in 2s, and in 3s.

MC07. Including the unbundled, 8 = (8/3)*3 = 2B2 = 2 2/3 = 3B-1 3s

The total here exists as a collection of snap cubes. The ‘T=?’ question is answered by pushing-away bundles to a stack that is then pulled-away to find the unbundles that then are included on-top of the stack.

Recounting 8 in 3s, first 2 times we push-away 3s, then we pull-away the stack of 2 3s and find 2 unbundled that are placed on-top of the stack. Here they may be seen as part of a bundle described by a decimal number, 8 = 2B2 3s, or as a fraction when also recounted in 3s as 2 = (2/3)*3, 8 = 2 2/3 3s. Or we may write 8 = 3B-1 to show that in space 1 is missing in the next bundle, or that in time 1 is pulled-away from it.

With ten as the bundle-number, unbundled occur in the same way: T = 4B3 = 4 3/10 = 5B-7 tens.

Skill building. The action is repeated on a BBBoard with 11 counted in 3s and 4s using cubes or fingers to hide a bundle.

End test. Recount 8 in 5s, and in 3s.

MC08. Recounting in squares, 6 4s = 1 BB

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered by working on the upper right corner occurring when half the excess is move from the top to the side to give a first guess about the square root. The rectangle may be shown with rubber bands or with snap cubes.

If we want to square the total T = 6 4s we move half of the excessing 2 4s form the top to the side to get a 5 x 5 square, and an unfilled square in the upper right corner that we try to fill with a rectangular 4**u* slice of what we moved where u is found by the equation 2**u**4 = 1, or 8**u* = 1, giving *u* = 1/8 = 0.125, and 5-0.125 = 4.875 as our guess. However, now there is too much in the corner, so we repeat the process, or consult a calculator showing that the correct answer is √(6*4) = 4.90, which is very close to our guess.

To solve a quadratic equation we see that on a BBBorad a T = (*x*+3)*(*x*+3) square has four parts, two squares *x*^2 and 3^2, and two stacks 2*3**x*, so that T = *x*^2 + 6**x* + 9. The quadratic equation *x*^2 + 6**x* + 8 = 0 then makes all the square go away except for 9-8 = 1. So (*x*+3)^2 = 1, which gives two solutions , *x*= -2 and *x* = -4. We Also see that the quadratic equation '*x*^2 + 6**x* + 10 = 0' has no solutions since here '(*x* + 3)^2 = -1'.

Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. Square 9 5s.

MC09. Recounting to another icon, 3 4s = ?5

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered on a BBBoard and predicted on a calculator.

With rubber bands on a BBBoard we see that 3 4s may be recounted as 2B2 5s. This may be predicted by a calculator. To find how many 5s there is in 3 4s we enter "3*4/5". The answer is '2.more'. To find them we pull-away the stack of 2 5s by entering '3*4-2*5' that gives the answer '2'. So, the calculator predicts that 3 4s recount as 2B2 5s, which is validated on the BBBoard.

Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. 4 5s = ?6s.

MC10. Recounting from tens to icons, 2 tens = ? 5s

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered by recounting.

With rubber bands on a BBBoard we see that 2 tens may be recounted as 4B0 5s. This may be predicted by a calculator. To find how many 5s there is in 2 tens by recounting we enter "20/5". The answer is '4'. So, the calculator predicts that 3 4s recount as 2B2 5s, which is validated on the BBBoard. Alternatively, the question "How many 5s in 20?" leads to the equation *u**5 = 20 that is solved moving to opposite side with opposite sign so that again *u* = 20/5, or *u* = 4.

We notice that decreasing the bundle will increase the height. To study this closer we recount 1 dozen in 6s, 4s, 3, 2s, and 1s and place a dot in the upper right corner each time. The points then form a curve called a hyperbola.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 4 tens = ?8s.

MC11. Recounting from icons to tens, 6 7s = ? tens

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by finding what we must pull-away from the bundle-bundle.

Recounting from icons to tens apparently is another name for the multiplication tables. With rubber bands on a BBBoard we see that 6 7s is left if from the ten bundles we pull-away 4 top and 3 side bundles, and add the upper right 4 3s that we pull-away twice:

T = 6 7s = 6*7 = (10 – 4 – 3)*B + 4 3s = 3B + 1B2 = 4B2 = 42.

This leads to early algebra if instead we write:

T = 6 7s = 6*7 = (B – 4) * (B – 3) = BB – 4*B – 3*B + 3*4

Here we see that minus times minus must be plus.

So, a quick way to find the answer is to add and multiply the less-numbers and subtract the first and add the latter. With 4 and 3 as the less numbers here, we quickly learn to say:

“Less (4+3) bundle (4*3)”, gives “Less 7 bundle 12” gives 3 bundle 12, or 4 bundle 2, or 42.

We may also write B – 4 and B – 3 on top of each other and then multiply down and across. Or we may use the FOIL method: First, Outside, Inside, Last.

Multiplying the two-digit numbers 23*46 as 2B3 4B6s, a vertical and a horizontal rubber band between the bundles and the singles allows a BBBoard to show the four stacks 2B 4Bs, and 2B 6s, below the 3 4Bs, and the 3 6s. With overloads, they add up to 8BB 24 B 18, or to 10BB 5B 8 = 1058 without.

This process may be reversed when asking ‘1058 = ? 46s’. First 1058 is written with an overload as 10BB5B8 = 8BB25B8. Since 4B*2B = 8BB, the 2B contributes 2B*6 to the Bs. The remaining 13B8 may be rewritten as 12B18, which divided by 3 gives 4B6. So, the answer is 1058 = 23 46s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 4 tens = ?8s.

MC12. Recounting to another physical unit creates per-numbers, 4$/5kg

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by changing the unit in the per-number rectangle.

Recounting a physical total T as 4$ and 3 kg gives a ‘per-number’ 4$/3kg called the price and marked as a 4x3 rectangle on a BBBoard.

The question “15kg = ?$” then is answered by recounting in the per-number: 15kg = (15/3)*3kg = (15/3)*4$ = 20 $.

Or we may introduce a new unit to make the digits like: 4$ = (4/3)*3$ = m*3$ = 3m$ with the multiplier *m* = 4/3. So, 15 kg = 15*m*$ = 15*4/3$ = 20$. On a BBBoard the new unit now allows reading the 4x3 rectangle as a 20x15 rectangle

Alternatively, the units may be recounted:

$ = ($/kg)*kg = (4/3)*15 = 20; and kg = (kg/$)*$ = (3/4)*12 = 9

Or we may equate the per-numbers: kg/$ = *u*/15 = 4/3. Moving to opposite side with opposite sign we then get 3**u* = 4*15, or *u* = 4*15/3 = 20.

Skill building. The action is repeated with other numbers.

End test. With 5$/2kg, 12kg = ?$, and ?kg = 12$.

MC13. With the same unit, per-numbers become fractions, 4$/5$ = 4/5

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered by changing the unit in the per-number rectangle.

If a whole contains a part, they have the same unit. In this case the per-number becomes a fraction without units. Still, we may use the units 'p' and 'w' for the part and the whole.

To get the fraction 3/5 of 20$ thus means to get 3p/5w of a 20$ whole. Recounting in the per-number thus gives 20w = (20/5)*5w = (20/5)*3p = 12p, or 12$ of 20$.

To get the fraction 3/5 of 100 thus means to get 3p/5w of a 100 whole. Recounting in the per-number thus gives 100w = (100/5)*5w = (100/5)*3p = 60p, or 60 of 100, written as 60%.

To ask "20$ is what percentage of 80$" means asking about the fraction 20/80 of 100. Or we may introduce a new unit 80$ = 100% to see that 20$ = (20/80)*80$ = (20/80)*100% = 40%.

To add 10% to 200$ we introduce the per-number 200$/100%. After the addition the total is

T = 100% + 10% = 110% = (110/100)*100% = (110/100)*200$ = 220$.

So, adding 10% means multiplying with 110%, and adding 10% 5 times means multiplying with 110%^5 = 161.1% thus giving 50% plus 11.1% additional, also called compound interest.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. With 2p/5w, 10p = ?w, and ?p = 20w, and 2p/5w = ?%.

MC14. Recounting a stack's sides gives trigonometry before geometry, rise = (rise/run)*run = tanA*run

The total here exists as a rectangular bundle-number on a BBBoard. Here the 'T=?' question is answered by changing the unit in the per-number rectangle.

On a BBBoard, we mark a 3x4stack as a rectangle with height 3 and base 4. If we recount the height and the base in the diagonal, we get the per-numbers sine and cosine:

height = (height/diagonal) * diagonal = sine Angle * diagonal, shortened to

$h = (h/d) * d = \sin A * d = \sin A\ d$s,

This gives the formula sin A = height / diagonal, or sin A = h/d, or sin A = 3/5 in our case.

Likewise, cos A = base / diagonal, or sin A = b/d, or cos A = 4/5 in our case.

height = (height/base) * base = tangent Angle * base, shortened to

$h = (h / b) * b = \tan A * b = \tan A\ b$s,

This gives the formula tangent A = height / base, or tan A = h/b, or tan A = 5/10 in our case.

A protractor shows that the angle A is a little above 25 degrees. Testing this we get tan 25 = 0.466. The reverse tan-button 'tan^-1' gives the precise result, tan^-1(0.5) = 26.6 degrees.

Using the words 'run' and 'rise' instead of 'base' and 'height', we get the diagonal's slope-formula:

tan A = rise/run.

Here the tangent-number describes the steepness or the rise-ability of the diagonal. In a x-y coordinate system a curve may be generated by a formula $y = f(x)$. Here the curve between two close neighbor points is a diagonal in a rectangle, and since the run and the rise are changes in x, Δx, and in y, Δy, the tangent-number here describes the steepness of the curve as the per-number $\Delta y/\Delta x$ called the local slope of the curve.

The word ‘tangent’ is used since the height will be a tangent in a circle with center in A, and with the base as its radius. This gives a formula for the circumference since a circle contains many right triangles leaving the center. In a circle with radius 1, h recounts in r as h = (h/1) * 1 = tan A.

A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 1.8 100s = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, h, are almost identical. So, half the circumference, called π, is

$\pi = 100 * h = 100 * \tan 1.8 = 100 * \tan (180/100) = 3.1426$

This gives a formula for the number π:

$\pi = \tan (180/n) * n$, for n sufficiently large.

We also see that in a circle with radius r, the circumference is $2*\pi*r$, and the area is $\pi*r$^2, or $\pi/4*$ d^2 where d is the circle’s diameter.

So, a d-circle takes up almost 80% of the space inside the surrounding d-square.

Squares may be added as the square of their mutual Bottom-Top line, see figure 2 above. Pythagoras has given name to this rule.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. Add a 4-swauer and a 6-square as a square.

MC15. Adding next-to or on-top, 2 3s + 4 5s = ? 8s, 2 3s + 4 5s = ? 5s

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by recounting.

Adding 2 3s and 4 5s as 8s means adding areas, which is called integral calculus. Reversing the process by asking “2 3s and how many 5s total 4 8s” is called differential calculus because you find the difference between the known totals before recounting it in 5s, (T2-T1)/5, or ΔT/5.

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like To recount 2 3s in 5s, first we enter ‘2*3/5’ giving ‘1.more’, then we enter ‘2*3-1*5’ giving ‘1’, so 2 3s is 1B1 5s, which give the total 1B1 5s + 4B0 5s = (1B1 + 4B0) 5s = 5B1 5s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 3 4s + 6 5s = ? 9s. And 3 4s + 6 5s = ? 4s. And 3 4s + 6 5s = ? 4s. And 3 2s + ? 5s = 4 6s.

MC16. Subtracting and adding single digit numbers, 8 + 6 = 1B2 + 1B0 = 2B2 6s

The total here exists as lines of pegs on a BBBoard. The ‘T=?’ question is answered by using rubber bands to mark the bundles.

With a subtraction as ‘8-6=?’, a rubber band marks 8 on a BBBoard, and fingers hide the 6 that is pulled-away, so 8-6 = 2.

With an addition as ‘8+6=?’, two rubber bands marks 8 and 6 on two BBBoard parallel lines to show that the sum may exist in two ways, as 2B2 6s, or as 2B-2 8s.

Here, using half bundles, 5s, will easy recounting in tens since 6+8 = ½B1 + ½B3 = 1B4 = 14.

Multidigit numbers may be added and subtracted with an over- or an under-load, which makes carrying and borrowing unneeded.

T = 36 + 47 = 3B6 + 4B7 = 7B13 = 8B3 = 83

T = 86 – 37 = 8B6 – 3B7 = 5B-1 = 4B9 = 49

Skill building. The action is repeated with other one-digit and two-digit numbers.

End test. 9 – 7 = ?, 9+7 = ?, T = 38+46 = ?; T = 82 – 54 = ?

MC17. Adding per-numbers and fractions by integral calculus

The total here exists as a rectangular bundle-number on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

Adding "2kg at 3$/kg and 4 kg at 5$/kg total what?" the unit numbers 2kg and 4kg add directly whereas the per-numbers 3$/kg and 5$/kg first must be multiplied to unit-numbers before adding, thus added as areas, i.e., as integral calculus. Here the per-numbers are piecewise constant, but they may also be locally constant as in the case with a falling object having an increasing meter/second number.

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives 106%^10 or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

Before adding, fractions must also be multiplied to unit, numbers. So, with apples, 1red of 2 plus 2red of 3 gives 3red of 5, and of course not 7red of 6 as taught by 'mathematism'.

Looking inside my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bent the two outer fingers. So, 1/3 of the Ls are bent, and ½ of the Rs. Does that mean that 1/3 of the bent are Ls? No, ½ is. So, in a cross table we cannot go from the percentages in one direction to those in the other direction without first reconstruing the unit number table. This is called the Bayes-principle.

Skill building. The action is repeated with other numbers.

End test. 3kg at 4$/kg and 5 kg at 6$/kg total what?"

MC18. Adding unspecified letter-numbers

The total here exists as a rectangular bundle-number on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

In the letter-number $T = 3ab$ the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, T may be seen as 3 *ab*s, or as (3*a*) *b*s, or as (3*b*) *a*s. To avoid being confused by the 's' we will omit it, so $T = 3ab = 3 * ab = 3a * b$ or $3b * a$.

Since totals need a common unit to add, this must be first found:

$T = 3ab + 4ac = 3b * a + 4c * a = (3b+4c) * a$

$T = 2ab$^2 $+ 4bc = ab * 2b + 2c * 2b = (ab+2c) * 2b$

Skill building. The action is repeated with other numbers and letters.

End test. T = $4ab^{2}d + 8bcd$

MC19. Change in time

The total here exists as dots on a BBBoard. The 'T=?' question is answered by transferring the results to a squared paper and connect the dots with a curve.

In time, a total grows by being added or multiplied by a number, called addition-growth and multiplying-growth, or linear and exponential growth.

Addition-growth: Final number = Initial number + growth-number * growth times, or shortly,

$T = B + a*n$. The number a is also called the slope.

Multiplying-growth: Final number = Initial number * growth-factor ^ growth times, or shortly,

$T = B * a$^n, since 200$ + 5% = (200*105%) $, so here the number a is 1 + interest rate.

Combined growth (savings in a bank): Here we have that $A/a = R/r$, where A is end-dollars, a is the period-dollars, R is the end-rate, r is the period-rate, and $1+R = (1+r)$^n, where n is the number of periods.

100% split in n parts will give the Euler number $e = (1+1/n)$^n for n large.

Changing the growth-number constantly will give a quadratic growth with a parabola curving upwards or downwards if the change increases or decreases.

Changing the curvature constantly will give cubic growth with a double parabola with curvature and counter-curvature.

Decreasing the growth-factor constantly will give logistic saturation growth with a hill-curve in infections. Confusion between exponential and saturation growth can cause unnecessary damage.

MC20. Bundle-numbers in a coordinate system

The total here exists as dots on a BBBoard. The 'T=?' question is answered by rubber bands as lines on the BBBoard.

The bundle-number 'y xs' with a height y and a width x may be called a 'changing bundle-number'. Here $y = 2*x$ gives a rising and $y = 9-x$ a falling bundle-number.

Marking the top right corners we get two lines. To inside predict the outside intersection point we equate the two heights, $2*x = 9 – x$. Moving to opposite side with opposite sign we get $3*x = 9$, and $x = 9/3 = 3$, which makes $y = 2*3 = 6$. So, the prediction is that the two bundle-numbers become like as 3 6s, which is validated on the BBBoard where the first dot now is 0 instead of 1.

In the rising bundle-number its total T will increase since here the height increases with increasing width. In the falling bundle-number this is not the case since the height decreases with increasing width. Here the total is T = $y*x = (9-x)*x = 9x – x$^2. Setting up a table with x = 1, 2, …, 9 we see that first T increases and then T decreases; and that T tops as 20 for x = 4 and x = 5; and that T = 20.25 for x = 4.5.

Even if a bundle-number is rising, its rise may be falling, so its marked corners will lay on a bending line called a parabola where $y = b*x + a*x$^2. Passing through the points (x,y) = (1,6) and (2,10) we find that 10 = b*2+ a*4, and 6 = b*1 + a*1, or 12 = b*2 +a*2.

We now equate the two equations for b*2: 10 – a*4 = 12 – a*2. Moving to opposite side with opposite sign we get 10 – 12 = a*4 – a*2, or -2 = a*2, or -1 = a. With 6 = $b+a$, this gives b = 7.

So on the parabola, the points (x,y) are connected by the formula $y = 7*x – x$^2. It thus passes through the poits (0,0), (1,6), (2,10), (3,12), (4,12), (5,10), (6,6), and (7,0).

Can the intersection points be predicted between the parabola and the two line above?

Can it be predicted that a falling bundle-number $b-a*x$ will have its maximum at the width $b/(2*a)$?

Skill building. The action is repeated with other rising and falling bundle-numbers to find when they are like and when the falling bundle-numbers tops.

End test. $y = 9-2*x$ and $y = x$.

MC21. Games Theory and damage control

The total here exists as towers of snap-cubes and dots on a BBBoard. The 'T=?' question is answered by rubber bands as lines on the BBBoard.

In a Game Theory 2x2 zero-sum game two players A and B each have 2 strategies resulting in four different payments from B to A. It is called a zero-sum game since one player's gain is the other player's loss.

In some game, if B chooses strategy B1 then the payment to A is 8$ or 2$ if A chooses strategy A1 or A2. And if B chooses strategy B2 then the payment to A is 4$ or 6$ and A chooses strategy A1 or A2. We may show this game by building four towers with snap-cubes.

First, we assume that B chooses strategy B1. If A now p times chooses A2 and n-p times A1 then A's outcome after n rounds will total:

$T = 2*p + 8*(n-p) = 8*n – 6*p = (8*n – 6*p)/n * n = (8 – 6*p/n) * n$.

Which is T1 = 8 – 6*p/n per round, shown on a BBBoard as a line connecting 8 to the left where p is 0, to 2 to the right where p is n.

Next, we assume that B chooses strategy B2. If A now p times chooses A2 and n-p times A1 then A's outcome after n rounds will total

$T = 6*p + 4*(n-p) = 4*n + 2*p = (4*n + 2*p)/n*n = (4 + 2*p/n)*n$.

Which is T2 = 4 + 2*p/n per round, shown on a BBBoard as a line connecting 4 to the left where p is 0, to 6 to the right where p is n.

With p/n as u we find the intersection point by equating the two totals: T1 = T2, or 8 – 6*u = 4 + 2*u, or 8*u = 4 = (4/8)*8, or u = 4/8 = ½ giving T1 = T2 = 5$.

So, if A mixes the strategies 1-to-1by flipping a coin then the average result will be 5$ per round.

Seen from B's side we also get the two lines S1 = 4 + 4*u, and S2 = 6 – 4*u that intersect where 4 + 4*u = 6 – 4*u, or 8u = 2, or u = 2/8, or u = 1/4 giving S1 = S2 = 5$.

So, if B mixes the strategies 1-to-3 by flipping two coins then the average result again will be 5$ per round. 5$ then is called the value of the game, i.e., the amount B must receive per round to make the game fair.

From A's side the 5$ is called the 'maxi-min' value since deviating from it will decrease the value. From B's side the 5$ is called the 'mini-max' value since deviating from it will increase the value.

In a similar game 4$ is changed to 8$. Here the strategy A1 dominates A2 that will always be lower to A. Likewise, the strategy B1 dominates B2 that will always be higher to B. So, here the value of the game is 6. This point is called a saddle point since the payment goes up one way and down the other.

Skill building. The game is repeated with four other payments found, e.g., by rolling some dice.

End test. Replace the four payments 8,2,4,6 with 9,3,5,8.

The Algebra Square

There are two kinds of numbers, unit-numbers, and per-numbers, which can be unlike or like and which may be reunited. The aim of mathematics is therefore not to 'math', because you can't do that, but to act: "Re-Unite Un-like & Like Unit-Numbers & Per-Numbers".

The four operations that unite unlike and like unit-and per-numbers are: addition, multiplication, integration, and power as shown in the Algebra-square above that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm. See figure 3 above.

Fact and fiction and fake, the three genres of number-models

Once we know how to count and recount totals, and how to unite and split the four number-types, unlike and like unit-numbers and per-numbers, we can actively use this number-language to produce tales about numbering and numbers, and about totaling and totals in space and time. This is called modeling.

As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then and what-then models, or room, rate, and risk models.

Fact stories are 'since-then' stories that quantify and predict predictable quantities by using factual numbers and formulas. Typically, they model the past and the present. They need to be checked for correctness and units.

Fiction stories are 'if-then' stories that quantify and predict unpredictable quantities by using assumed numbers and formulas. Typically, they model the future. They need to be supplied with scenarios building on alternative assumptions.

Fake stories are 'what-then' stories that quantify and predict unpredictable qualities by using fake numbers and formulas. Typically, they add without units or hide some numerical consequences. They need to be replaced by word stories.

Modeling and de-modeling

The goal is to experience how formulas calculating y from x form curves that expresses change in time, and how totals in space may be split in parts that each then becoming a percentage of the total.

• Modeling means solving an outside problem inside with four steps. First an outside problem is translated to an inside problem. Then the inside problem leads to an inside solution that then is translated to an outside solution, that finally is evaluated to see if another cycle is needed.

A typical example is mixture problems. the outside problems here may ask "2kg at 3$ per kg and 4kg at 5$ per kg total what?" The inside problem places the second information under the first ready to add. The inside solution my then simply add all numbers, which leads to the outside solution "2kg at 3$ per kg and 4kg at 5$ per kg total 6 kg at 8$/kg". This model is not accepted, so another cycle is needed. This time the per-numbers are multiplied to unit numbers before adding, which leads to the outside solution "2kg at 3$ per kg and 4kg at 5$ per kg total 6 kg at 26$/6kg". This model is accepted.

• De-modeling is the opposite process: It means solving an inside problem outside with four steps. First an inside problem is translated to an outside problem, then the outside problem leads to an outside solution that then is translated to an inside solution, that finally is evaluated to see if another cycle is needed.

A typical example is uniting fractions.

Adding fractions as 1/2 + 2/3 only has meaning when taken of the same unit, $u = (u/6)*6 = k*6$, where $k = u/6$, and 6 = 2*3

$T = (½ + 2/3)*u = (½ + 2/3)*6*k = (3+4)*k = 7*k = 7*u/6 = 7/6*u$,

So, in this case, 1/2 + 2/3 = 7/6.

Three footnotes

The total here exists as rectangular bundle-number on a BBBoard.

The goal is to experience the content of three calculation laws.

The commutative law: The order does not matter, $a*b = b*a$

The distributive law: When adding, like units may be bracket out, $a*c + b*c = (a + b)*c$

The associative law: Bracket may be moved at will, $a*(b*c) = (a*b)*c$

On a BBBoard two rubber bands mark 6 3s. Turning the board a quarter round we have 3 6s thus illustrating that 6*3 = 3*6.

A third rubber band split the 6 3s in 4 3s and 2 3s to illustrate that 4*3 + 2*3 = (4+2)*3.

With snap-cubes 2 3s 4 times gives a Total of (2*3)*4. Turning it over, twice we have 3 4s, thus illustrating that (2*3)*4 = 2*(3*4),

Teacher education

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating predicates by asking the subject itself instead of an instructor. This allows mastery of Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged.

The MATHeCADEMY.net offers a free one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for primary school, and C2, A2, T2 and S2 is for primary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e., both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other's routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by T = (T/B)*B. So, T = 8= (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 2/3*3 = 2.2 3s. Bundling bundles gives a multiple stack, a stock or polynomial: T = 456 = 4BundleBundle + 5Bundle + 6 = 4tenten5ten6 = 4*B^2+5*B+6*1.

How different is the difference?

Digits now are no longer symbols as letters, but icons with as many sticks as they represent. 3 now is called '1B0 3s' or '0B3 tens'. Ten, eleven end twelve now are also called 'one-bundle-zero', 'one-bundle-one', and 'one-bundle two'. And hundred and thousand are also called 'bundle-bundle' and 'bundle-bundle-bundle'.

Multidigit numbers no longer occur without units since with units, 23 now is 2B3 thus making the place value system unneeded.

Calculations with overloads and underloads give bundle-numbers with units a flexibility that makes carrying and borrowing unneeded, e.g., 46+37 = 4B6+3B7 = 7B13 = 8B3 = 83. And 86 – 37 = 8B6 – 3B7 = 5B-1 = 4B9 = 49.

Addition now depends on the units so 2+3 is not 5 by necessity. 2weeks + 3weeks = 5weeks, but 2weeks + 3days = 17days. So, without a unit, 3 does not exist, only with a bundle-unit as, e.g., 0B3 tens, or 1B0 3s, or 1B1 2s, or 1B-1 4s, or 1B-2 5s, etc. So, to add, 2 and 3 must have the same unit, e.g., '2+3' = (1B0 + 1B1) 2s = 2B1 2s, or '2+3' = (1B-1 + 1B0) 3s = 2B-1 3s = 1B2 3s. Likewise with subtraction '9-6' = (1B3 -1B0) 6s = 0B3 6s = '3', or '9-6' = (1B0 -1B-3) 9s = 0B--3 9s = 0B3 9s, showing that minus times minus must be plus.

Also, addition now is not well-defined since 2 3s and 4 5s may be added both on-top after a recounting has made the units like, or next-to by areas as integral calculus.

Multiplication now carries units automatically, and 6*8 is not 48 by necessity. Instead, 6*8 exists as 6 8s that may or may not be recounted in another unit, e.g., in 9s or in tens: 6 8s is 5B3 9s, and 4B8 tens.

Division now is different, since no longer 8/2 primarily means '8 split in 2', but '8 split in 2s' when recounting 8 in 2.

Solving equations now is different. The equation 'u*2 = 8' asks "How many 2s in 8?" which of course is found by recounting 8 in 2s as 8 = (8/2)*2, so that the solution is u = 8/2 that is found by 'moving to opposite side with opposite sign', which follows the formal definition: 8/2 is the number u that multiplied with 2 gives 8, if u*2 = 8 then u = 8/2. So, the balancing method now is unneeded. Thus, no longer equations are seen as two equivalent numbers-names that remain equivalent if the same operation is performed on both. And no longer are they transformed by using the communicative, associative and distributive law; or the two abstract concepts, 1 over 2 as the inverse element to 2, and 0 and 1 as the neutral elements. And we no longer use the neutralizing 'do the same to both sides' weight-method to solve the equation 2*x = 8 by saying:

2*x = 8;   (2*x)*½ = 8*½;   (x*2)*½ = 4; x*(2*½) = 4; x*1 = 4; x = 4

The multiple calculation 2+3*4 no longer is 14 by definition or by the 'PEMDAS' rule. With units, 2+3*4 exits as 2 1s + 3 4s which is (0B2 + 3B0) 4s or 3B2 4s, or 1B4 tens.

The letter-calculation '2*a + 3*a = (2+3)*a' no longer is an example of a distributive law, but an example of having like units.

Proportionality no longer 'go over one', instead a per-number links the two units by recounting: with 4$ per 5kg, or 4$/5kg, 16$ = (16/4)*4$ = (16/4)*5kg = 20 kg.

Fractions no longer are numbers by themselves, instead they are per-number with like units, 3meter/4meter = ¾, 3 meter/100meter = 3/100 = 3%. So finally, per-numbers are accepted along with fractions.

Without units, digits, per-numbers, and fractions are not numbers, but operators needing a number to become a number. So, fractions also need units to add: 1 red of two apples plus 2 red of 3 apples total (1+2) red of (3+4) apples, i.e., ½ + 2/3 = (1+2)/(2+3) = 3/5 in this case, and not 7 red of 6 apples as mathematism teaches.

Trigonometry no longer must wait to after plane and coordinate geometry, since it occurs when mutually recounting the sides in a stack split by its diagonal.

Differential calculus no longer precedes integral calculus since the latter answers the core questions: how to add stacks in grade one, and how to add piecewise and locally constant per-numbers in mixture problems in middle school and high school.

Solving a quadratic equation no longer must wait to secondary school since Bundle-Bundles are squares that lead directly to the question 'how to square a rectangle' that provides a double split square containing the three parts of a quadratic equation.

The amazing simplicity of the Algebra Square will no longer be hidden.

And no longer will models be seen as mere approximations but as tales with three genres, fact and fiction and fake.

Overview over the differences between Essence- and Existence-math

	Essence-math, mathematism	**Existence-math, Many-math**
Digits	Symbols	Icons
345	Place value system	T = 3BB 4B 5, BB = B^2, BBB = B^3
Operations	Functions, order: + - x / ^	Icons, opposite order: ^ / x - +
3 + 4	3 + 4 = 7	Meaningless with no units

3 * 4	3 * 4 = 12	3*4 = 3 4s, *may be recounted to 1.2 tens*
9 = ? 2s	Meaningless, only ten-counting	9 = 3B3 = 5B-2 = 4B1 = 4½ 2s
8 = ? 2s	Meaningless, only ten-counting	8 = (8/2)*2, T = (T/B)*B, proportionality
2*u = 8	(2*u)*½ = 8*½, so (u*2)*½ = 4, so u*(2*½) = 4, so u*1 = 4, so u = 4	2*u = 8 = (8/2)*2, så u = 8/2
6*7 = ?	eh 44, eh 52, eh 42? OK	(B-4)*(B-3) = (10-4-3)B + 12 = 4B2
4kg=5$, 6kg=?	1kg = 5/4$, 6kg = 5/4*6$	6kg = (6/4)*4kg = (6/4)*5$
1/2 + 2/3 = ?	1/2 + 2/3 = 3/6 + 4/6 = 7/6	1/2*2 + 2/3*3 = 3/5*5
2 3s + 4 5s	2*3+4*5 = 6+4*5 = 10*5 ?	2*3 + 4*5 = 3B2 8s, integration
6 + 9 = ?	6 + 9 = 15	2B3 6s = 2B-3 9s = 1B (1+4) = 15
Tangent = ?	Tan = sine/cosine	raise = (raise/run)*run, tan = raise/run

Conclusion

We began by observing the difference between 'mathematism' adding without units and true inside but seldom outside, and 'Many-math' instead using bundle-numbers with units inspired by how the uneducated child sees the outside existing fact Many. We then explored the consequences of letting existence come before essence by letting counting and recounting come before adding. Finaly, we formulated a sequence of micro-curricula on how Many-math may be learned in school by working with things and actions on a 2dimensional Bundle-Bundle-Board. But, will this allow the learner to learn mathematics or even to be numerate as wished by the UN Sustainable Development Goals?

Apparently, different definitions of 'numerate' exist where existence and essence have different order. The English Oxford Dictionary defines it as being "competent in the basic principles of mathematics, esp. arithmetic". In contrast, the American Merriam-Webster dictionary defines it as "having the ability to understand and work with numbers." In their common history, England once colonized America. So, the difference in the definitions is interesting. The former uses the passive term 'being' where the latter uses the active term 'having'. The former connects the definition to the inside essence of mathematics while the latter connects it directly to the outside existence of numbers. The choice thus is: shall existence precede essence as philosophical Existentialism holds, or shall essence be allowed to colonize existence with a 'no-unit regime' to use a phrase of M. Foucault?

Maybe it is time to see if children stay numerate if their own 2D bundle-numbers with units are not colonized by 1D line-numbers without units. Maybe it is finally time for a Kuhnian paradigm in shift in number-language education. So, think things.

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