Bundle- and Per-numbers Replacing the Number Line will Free the Magic of Numbers from its ‘No-unit’ Greenhouse

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Abstract

Outside, addition folds but multiplication holds, since factors are units while addition presupposes like units. This creates two paradigms in mathematics, an outside ‘unit’ paradigm, and an inside ‘no-unit’ paradigm making mathematics a semi-greenhouse. To make mathematics a true science with valid knowledge, we ask what mathematics can grow from bundle-numbers with units, being areas instead of points on a number line. Concretely constructed, digits become number-icons with as many sticks as they represent, and operations become counting-icons for pushing, lifting and pulling away bundles to be added next-to or on-top. Recounting 8 in 2s creates a recount-formula, T = (T/B) x B, saying that T contains T/B Bs. By changing units, it occurs as proportionality formulas in science; it solves equations; and it shows that per-numbers and fractions, T/B, are not numbers, but operators needing numbers to become numbers. Fractions, decimals, and negative numbers describe how to see the unbundled. Recounting sides in a box halved by its diagonal allows trigonometry to precede plane and coordinate geometry. Once counted, total may add on-top after recounting makes the units the same; or next-to addition by adding areas as in integral calculus, which also occurs when adding per-numbers. So, mathematics created outside the ‘no-units’ greenhouse is the same as inside, only the order is different, and all is linked directly to outside things and actions making it easier to be applied. And, with multiplication preceding it, addition only occurs as integral calculus, unless inside brackets with like units.

Keywords: arithmetic, equation, proportionality, trigonometry, calculus

Chapter 01. The Two Paradigms in Mathematics Education

The necessity of numbers and calculations as social and individual tools makes them educational tasks in school. Writing the book ‘Mathematics as an Educational Task’, Freudenthal (1973) succeeded in giving university mathematics monopoly by claiming that mastering mathematics is the only way to later mastering Many. So, from day one in school, or even in preschool, children are forced to learn its foundation, the one-dimensional number line were 1+1 = 2, despite the fact that, with units, this is seldom the case.

But, numerous international tests together with more than 50 years of mathematics education research following the first International Congress on Mathematics Education, ICME, in 1969 have shown that mathematics is hard to learn. Consequently, more learning must come from more research mediated by more facilitators to more educated teachers so they can be more successful with more hard-working students helped by more advanced technology. In short, send more money.

Difference research (Tarp, 2018) instead asks the Cinderella question: are there hidden alternative ways to master Many that evades the hard-to-learn ‘no-unit’ mathematics? What if children already learn to master Many from adapting to it, can school then develop this into mastering a mathematics that may be different from the school version? So, we ask:

What mathematics can grow from the mastery of Many children develop before school?

How children adapt to Many can be observed from preschool children. Asked "How old next time?", a 3year old will say "Four" and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, 'That is not four, that is two twos', thus describing what exists in space and time: bundles of 2s, and 2 of them.

So, children adapt to Many by using 2-dimensional bundle-numbers with units. And they use full sentences as in the word-language with a subject, 'that', and a verb, 'is', and a predicate, '2 2s', which shortened transforms a number-language sentence into a formula 'T = 2 2s'.

However, the number line does not include units. Instead, school teaches a mathematics that is built upon the assumption that 1+1 = 2 unconditionally. And that thereby fails to meet the basic condition of a science: its statements must not be falsified outside. Which '1+1 = 2' typically is when including units: 1 week + 1 day = 8 days, 1 km + 1mm = 1 km, etc.

Thus, where school works with one-dimensional line numbers without units, children work with two-dimensional area numbers with units. So, there seems to be two paradigms in mathematics. The first is a ruling 'no-unit' paradigm that makes mathematics a semi-greenhouse since outside, addition folds and multiplication holds: 2x3 = 6 simply states that 2 3s can be recounted to 6 1s. The other paradigm is an opposite 'unit-paradigm' where numbers always carry units.

Inside the 'no-unit' greenhouse, outside sentences as T = 2 3s are shortened to 2, thus leaving out the subject, and the unit. This is called an inside modeling of the outside world. The reverse then is called an outside 'de-modeling' or reifying of an inside statement (Tarp, 2020).

As described by Kuhn (1962), to have a career, a researcher must work inside the ruling paradigm or 'truth regime' (Foucault, 1995) where it is more important to include library references than to perform laboratory testing. A paradigm shift first comes when somebody ignores the existing literature, and use outside validation as the only criteria for quality.

One answer to the above question is given by Tarp (2018, 2020). This answer will now be brought to a more detailed level to design micro-curricula to be tested in design research (Bakker, 2018). Because of space limitations, some micro-curricula designs are left out here.

Chapter 02. Counting in Time with Sequences

Before designing, we reflect on how a row with many sticks is worded by a counting sequence with different names until we reach the bundle, after which a reuse typically takes place when multi-counting the singles, the bundles, the bundles-of-bundles, etc., In the end we get a final total as T = 345 = 3 bundles-of-bundles & 4 bundles & 5 unbundled.

Many occurs in time and space. Repetitions in time may be represented in space by a row of sticks. In space, a lot may be rearranged in a row that is transformed into a total by repeating pushing away one item at a time, and at the same time wording the actual total.

Counting outside the 'no-units' greenhouse we always include units when describing what exists. So, we say '0 bundle 1' instead of just '1'.

Number-names become flexible when allowing 'overloads' as 'twenty-nine, twenty-ten, twenty-eleven'. And, when allowing 'underloads' counting 'bundle-less2, bundle-less1, bundle' instead of '8, 9, ten'.

Overloads and underloads allow flexible bundle-numbers with units to outside de-model inside unit-free numbers:

T = 38 = 3Bundle8 = 2Bundle18 = 4Bundle less2, or short

T = 38 = 3*B*8 = 2*B*18 = 4*B*-2, and

T = 347 = 3*BB*4*B*7 = 2*BB*14*B*7 = 1*BB*23*B*17, or

T = 347 = 3*BB*4*B*7 = 3*BB*5*B*-3 = 4*BB*-6*B*7.

Bundling in tens, it takes a while before meeting the bundle-of-bundles, *BB*. But, counting in 3s will allow meeting 9 as 1 *BB* since ten fingers then become a total of

T = ten = 3Bundle1 = 3*B*1 = 1*BB*1 3s, or more precisely

T = ten = 1*BB*0*B*1 3s, modeled inside the 'no-unit' greenhouse as 101.

Including the units thus makes place values needless, or easier to understand.

Based upon the above reflections we may now formulate a micro-curriculum to be tested.

Micro Curriculum 02. Different Counting Sequences Using Flexible Bundle-numbers

The goal is to see inside numbers as short models for what exists outside, totals of unbundled singles, bundles, bundle-of-bundles, etc.

The means is to count a total in flexible bundle-numbers using different bundle-sizes; and to outside de-model inside shortened numbers.

Exemplary guiding tasks may be: "Count five fingers in 2s in different ways using overloads and underloads", "Count ten fingers in 5s", "Count ten fingers in 3s".

Material may be the ten fingers (or twelve if including the arms), the finger-parts, sticks, stones, snap-cubes, strokes on a paper, and a western, Chinese or Japanese abacus.

Chapter 03. Counting in Space and Time with Icons for Digits and Bundling

Before designing, we reflect on how in space, four ones may be rearranged as one four-icon called a 4-digit. And that the same is almost the case with the other digits also. So basically, a digit is an icon with as many sticks or strokes as it represents if written less sloppy (Tarp, 2018).

Bundling in tens, ten has no icon, since ten 1s is 1 ten-bundle and no unbundled:

T = ten = 1Bundle0 tens = 1*B*0 tens, modeled inside the 'no-unit' greenhouse as 10.

Once created, digits may be used when recounting a total, *T*, in a new unit, e.g.,

T = 8 = 4 2s, or *T* = 8 = 2 4s, or *T* = 9 = 3 3s

In time, we recount a total by bundling or grouping, i.e., by repeating pushing away bundles or groups. But which is the better word to use, to bundle or to group?

Humans may be grouped according to different qualities naming the groups: males, females, etc. Repeating grouping groups according to other qualities as, e.g., age will give a smaller number of members in the sub-groups.

Sticks do not have individual qualities to name different groups. Instead, bundling allows a bundle of 1s to be transformed into 1 bundle that becomes a unit for repeated counting; where again a bundle of bundles may be transformed into a new unit, 1 bundle-of-bundles containing more members than the bundle.

So 'bundle' may be preferred to grouping since it reflects more precisely what exists and happens in space and time outside the 'no-unit' greenhouse.

We observe that we recount a total of 8 ones in 2s by from 8 pushing away bundles of 2s four times. This may be written with an uphill stroke iconizing a broom pushing away the bundles, called division.

By showing that 8/2 = 4, a calculator predicts that "from 8, we push 2, max 4 times".

Using snap-cubes, we can now stack the 4 twos as a 4-by-2 stack or box. This may be written with a cross iconizing a lift lifting away the 2-bundles 4 times, called multiplication.

By showing that 4 x 2 = 8, a calculator predicts that "4 times lifting 2s gives 8".

However, since the 4 came when pushing 2s away from 8, we may now write 8 = 4 x 2 as 8 = (8/2) x 2, or using letters for unspecified numbers

$T = (T/B) \times B$ saying: "From *T*, *T* push away *B* times we lift *B*."

Since this 'recount formula' changes the units, it is also called a proportionality formula, a core formula used all over mathematics, science and economics.

We observe that 1 is left unbundled if recounting 9 in 2s. The action 'from 9, pull away 4 2s' may be written with a horizontal stroke iconizing a rope pulling away the stack, called subtraction.

By showing that 9 – 4x2 = 1, a calculator predicts: "From 9, pull 4 2s, leaves 1".

Placed on-top of the stack, the unbundled may be described as an underload,

T = 9 = 4*B*1 2s = 5*B*-1 2s.

Or, it may be separated by the bundles by a decimal point,

T = 9 = 4*B*1 2s = 4.1 2s.

Or, recounted in 2s as 1 = (1/2) x 2, it may be seen as a part or fraction of a full bundle,

T = 9 = 4 ½ 2s.

Based upon the above reflections we may now formulate a micro-curriculum to be tested.

Micro Curriculum 03. Iconizing Digits and Recounting

The goal is to see digits as number-icons with as many sticks as they represent; and to see the operations division, multiplication, and subtraction as operation-icons for pushing, lifting and pulling away bundles; and to see that when changing units, the result may be predicted by a recount formula saying that a total T contains T/B units B.

The means is to rearrange many sticks, cars or dolls into 1 icon; and to recount a row of snap-cubes using a playing card to push away bundles to be lifted into a stack, that is pulled away by a rope or a rubber band; and to let a calculator predict the recounting result before carrying it out.

Exemplary guiding tasks may be: "There seems to be four sticks in the symbol for four so that digits are not symbols as are letters, but icons with as many sticks as they represent. Does this apply to the other digits also?", "Recount first 8, then 9, then ten fingers or sticks or snap-cubes in 2s, 3s and 4s", "Predict and test the result of recounting 2 3s in 4s; 4 2s in 3s; 3 4s in 2s. Likewise with 3 4s in 5s."

Material may be the ten fingers (or twelve if including the arms), the finger-parts, sticks, stones, snap-cubes, strokes on a paper, a western, Chinese or Japanese abacus, and a folding ruler.

Chapter 04. Recounting in and from Tens Solves Equations and Introduce Algebra

Before designing, we reflect on recounting between icons and tens.

Recounting from tens to icons means asking, e.g., “How many 2s in 8?”.

Using *u* for the unknown number, this may be written as an equation “*u* x 2 = 8”.

But, since 8 can be recounted in 2s as 8 = (8/2) x 2, we see that *u* = 8/2.

So, the equation *u* x 2 = 8 is solved by *u* = 8/2, i.e., by moving the known number to the opposite side with the opposite calculation sign. After solving an equation, the answer must be tested in the original equation:

With *u* = 8/2 = 4, *u* x 2 = 4 x 2 = 8 as expected.

The ‘opposite side & sign’ method resonates with the formal definition for division inside the ‘no-unit’ greenhouse. Here 8/2 is what multiplied with 2 gives 8: if 8/2 = *u* then *u* x 2 = 8.

Later we will see that it resonates with the formal definition for other operations, so basic equations are solved by moving to opposite side with opposite calculation sign:

u + 2 = 8	*u* x 2 = 8	*u* ^2 = 8	2^*u* = 8
u = 8 – 2	$u = \frac{8}{2}$	$u = \sqrt[2]{8}$	$u = \log_2 8$

Figure 1. Solving equations by moving to opposite side with opposite calculation sign

Recounting from icons to tens means asking, e.g., “How many tens are 6 8s?”.

With no ten-button on a calculator, we cannot use the recount formula. But multiplication gives the result directly, only without units and decimal point: *T* = 6 8s = 6 x 8 = 48 = 4.8 tens

On a peg-board or a squared paper we may set up a ten-by-ten square with 6 and 8 on the sides written with underloads as *B*-4 and *B*-2. We see that the 6 x 8 stack is left when from the *B* x *B* stack we pull away a 4 x *B*, and a *B* x 2 stack, and add the 4 x 2 stack pulled away twice.

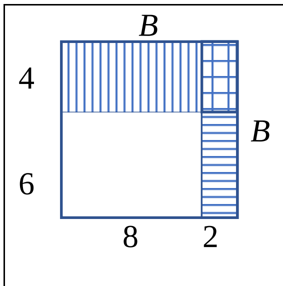
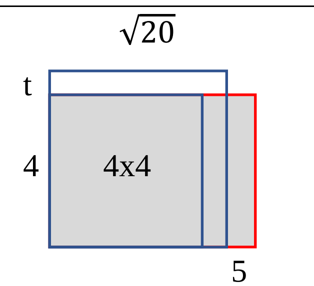
	*T* = 6 x 8 = (*B*-4) x (*B*-2) = *BB* – 2*B* – 4*B* + 8 = 4*B*8 = 48	$T = \binom{1B-4}{1B-2}$ = 1*BB* – 2*B* – 4*B* + 8 = 10*B* – 6*B* + 8 = 4*B*8 = 48	

Figure 2. Multiplying Numbers as Binomials; and squaring a 4x5 stack.

Multiplying two ‘less-numbers’ horizontally thus creates a FOIL-rule: First, Outside, Inside, Last. Multiplying them vertically creates a cross-multiplication rule: First multiply down to get the bundle-of-bundles and the unbundled, then cross-multiply to get the bundles. A short rule is: multiplying with less-numbers, subtract their sum and add their product (to 100).

A stack changes form with the unit, so to hold the same total, increasing the base will decrease the height, and vice versa.

A special form is a square. Wanting to square a 4-by-5 stack, its side is called $\sqrt{20}$, using lines to iconize the square. To find $\sqrt{20}$ we see that removing the 4-square leaves 20 – 4x4 = 4 shared by the two 4 x *t* stacks in a 4 + *t* square, giving *t* = 0.5. A little less since we neglect the *t*-square in the upper right corner. A calculator predicts that $\sqrt{20}$ = 4.472.

Chapter 05. Recounting in Physical Units Creates Per-numbers

Before designing, we reflect on recounting in physical units.

Inside the 'no-unit' greenhouse, shifting physical units is seen as an outside application of proportionality where division allows finding the unit cost to answer the two question types: "With 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?"

Outside the 'no-unit' greenhouse, recounting a total that is already counted in one physical unit creates a 'per-number' as 3$/2kg, used to easily shift units by recounting:

T = 6 kg = (6/2) x 2 kg = (6/2) x 3 $ = 9$; and

T = 15$ = (15/3) x 3$ = (15/3) x 2kg = 10 kg.

Alternatively, we may recount the units:

$ = ($/kg) x kg = (3/2) x 6 = 9; and

kg = (kg/$) x $ = (2/3) x 15 = 10.

Per-numbers occur all over science and mathematics:

meter = (meter/sec) x sec = speed x sec,

kg = (kg/cubic-meter) x cubic-meter = density x cubic-meter;

energy = (energy/sec) x sec = Watt x sec;

y-change = (*y*-change/*x*-change) x *x*-change = slope x *x*-change, or $\Delta y = (\Delta y/\Delta x) \times \Delta x$

With like units, per-numbers become fractions: 1$/4$ = ¼.

Inside, the 'no-unit' greenhouse teaches fractions as division: 1/4 of 12 = 12/4. Outside, a fraction is a per-number counting both the part and the total, ¼ is 1-part per 4-total.

Chapter 06. Trigonometry Recounts the Sides in a Box Halved by its Diagonal

Before designing, we reflect on recounting the sides in a box halved by its diagonal.

Inside the 'no-unit' greenhouse, trigonometry is taught after plane and coordinate geometry, and with heavy emphasis on trigonometric identities.

Outside, trigonometry are per-numbers coming from recounting the sides in a height-x-base box halved by a diagonal cut, which produces two like right triangles with horizontal base-side, *b*, a vertical height-side, *h*, and a slant cut-side, *c*.

Recounting the height in the base we get

height = (height / base) x base = tangent *A* x base, shortened to

h = (*h* / *b*) x *b* = tan *A* x *b* = tan *A* *b*s, thus giving the formula tangent A = height / base, or tan $A = h/b$.

Using the words 'run' and 'rise' for 'base' and 'height', we get the formula:

tan A = rise / run, giving the steepness or slope of the diagonal.

The word 'tangent' is used since the height will be a tangent in a circle with centre in A, and with the base as its radius.

Increasing the angle will increase the tangent-number. A calculator has a tangent-button to show the relation between the two: tan 30 = 0.577, and reversely, tan-1(2/3) = 33.7.

Tangent gives a circumference formula since a circle contains many right triangles:

In an h-by-r half-box, h recounts in r as $h = (h/r) \times r = \tan A \times r$.

A half circle is 180 degrees that split in 100 small parts as

180 = (180/100) x 100 = 1.8 100s = 100 1.8s.

With A as 1.8 degrees, the circle and the tangent, h, are almost identical.

So, half the circumference is

$\frac{1}{2}C = 100 \times h = 100 \times \tan 1.8 \times r = 100 \times \tan (180/100) \times r = 3.1426 \times r$

Calling the circumference for $2 \times \pi \times r$, we get a formula for the number π:

$\pi = \tan (180/n) \times n$, for n sufficiently large.

Chapter 07. Once Counted and Recounted, Total can Add Next-to or On-top

Before designing, we reflect on how to add stacks horizontally and vertically.

Inside the 'no-unit' greenhouse, numbers are seen as placed along a one-dimensional number line with addition defined as counting on. Outside, numbers carry units and become 2-dimensional stacks with areas that add next-to or on-top.

Adding 2 3s and 4 5s next-to as 8s means adding or integrating the areas, also called integral calculus. Adding 2 3s and 4 5s on-top, the units must be the same by squeezing one or both stacks, i.e., by recounting one or both.

Again, the recount formula predicts the result on a calculator by entering

$(2 \times 3 + 4 \times 5)/B$, where B can be 3 or 5 or 8.

We see that when adding stacks, multiplication comes before addition.

Adding 20% to 30$, we have two units with the per-number 30$ per 100%. Adding 20% to 100% gives 120% that recounts in 100s as

120% = (120/100) x 100% = (120/100) x 30 $ = 120% x 30$.

So, we add 20% by multiplying with 120%, also called the index-number 120.

Reversing next-to and on-top addition, we may ask "2 3s and how many 5s total 4 8s?", leading to the equation $2 \times 3 + u \times 5 = 4 \times 8$

To find the answer, we pull away the 2 3s from the total T before recounting in 5s,

$u = (T - 2 \times 3) / 5 = \Delta T / 5$

The answer is predicted by asking a calculator: (4 x 8 – 2 x 3) / 5.

Here we used a difference to calculate the change in the total before recounting. The combination of subtraction and division is called differential calculus. That subtraction comes before division is natural since differentiating stacks is the opposite of integrating them where multiplication comes before addition.

Chapter 08. Subtracting and Adding Numbers with like Units

Before designing, we reflect on how to subtract and add numbers with like units.

Inside the 'no-unit' greenhouse, 1digit numbers are drawn serial next-to to find the result by counting on from 6 or 9. Outside, they are drawn parallel on-top showing that

T = 6 + 9 = 2*B*3 6s = 2*B*-3 9s, that both recounts as 1*B*5 tens

2 6s + 3 = 2 x 6 + 3 = 2 x (*B*-4) + 3 = 2*B*-8 + 3 = 12 + 3 = 15, and

2 9s – 3 = 2 x 9 – 3 = 2 x (*B*-1) – 3 = 2*B*-2 – 3 = 18 – 3 = 15

Or, directly with flexible bundle-numbers, *T* = 6 + 9 = *B*-4 + *B*-1 = 2*B*-5 = 15

However, subtraction before addition will allow using the hands to show the result after predicting it with a calculator.

Observing with fingers or snap-cubes that 5 – 3 = 2, we also observe that 5 = 2 + 3, so that 5 = (5 – 3) + 3. With unspecified numbers this gives a 'restack formula' *T* = (*T*–*B*) + *B* saying "From a total *T*, *T*–*B* is left when pulling away *B* to place next-to".

This formula solves addition-equations. Asking "3 added to what gives" 5 may be formulated as an equation, *u* + 3 = 5. Taking away from 5 the 3 we added will give the answer,

u = 5 – 3, which can also be seen when restacking 5:

u + 3 = 5 = (5 – 3) + 3, gives *u* = 5 – 3.

Inside the 'no-unit' greenhouse, a combination of 3 digits is seen as one number obeying a place value principle where the places describe the number of ones, tens, hundreds etc. Seldom, if ever, ten is called 'bundle', and hundred is called 'bundle-of-bundles'.

When adding, numbers are placed in columns, and carrying or borrowing may occur.

Outside, a combination of 3 digits is seen as 3 numberings of unbundled singles, bundles and bundle-of bundles as shown when including the units in the number:

T = 456 = 4*BB*5*B*6, or more formally as a polynomial:

T = 456 = 4 x *B*^2 + 5 x *B* + 6, with *B* = 10

The flexibility of bundle-numbers is handy with subtraction may give an underload:

T = 41 – 18 = 4*B*1 – 1*B*8 = 3*B*-7 = 2*B*3

Overloads may prove handy also:

T = 18 + 23 = 1*B*8 + 2*B*3 = 3*B*11 = 4*B*1, and

T = 336 / 7 = 33*B*6 / 7 = 28*B*56 / 7 = 4*B*8 = 48

T = 2 x 37 = 2 x 3*B*7 = 6*B*14 = 7*B*4

T = 34 x 56 = 3*B*4 x 5*B*6 = 15*BB* (18+20)*B* 24

= 15*BB* 38*B* 24 = 15*BB* 40*B* 4 =19*BB* 0*B* 4 =1904

Or, with stacks that may be extended to any multidigit numbers

	3*B*	4	**34** / **x 56**
	15*BB*	20*B*	5*B*
	18*B*	24	6
15*BB*	38*B*	24	
15*BB*	40*B*	4	
19*BB*	0*B*	4	
			1904

Figure 3. Multiplying Numbers as Binomials

So, outside flexible bundle-numbers do not need inside carrying or place values.

Chapter 09. Adding Per-Numbers and Fractions

Before designing, we reflect on how to add per-numbers and fractions.

Inside the 'no-unit' greenhouse, the concept 'per-number' is not accepted; and fractions are added without units, e.g., 1/2 + 2/3 is 7/6.

Outside, this hold only if the fractions are taken of the same total. It does not hold if the units are the same since 1/2 of a pie plus 2/3 of a pie total 7/12 of two pies; and since 1/2 of 2 apples plus 2/3 of 3 apples total 1+2 of 2+3 apples, i.e., 3/5 of the 5 apples, and not 7 apples of 6 as the 'no-unit' greenhouse teaches.

Outside, fractions are seen as per-numbers coming from recounting in the same unit. And when adding per-numbers we need to be careful with the units.

Asking "What is 2kg at 3$/kg plus 4kg at 5$/kg?" we observe that the unit-numbers 2 and 4 add directly to 6, whereas the per-numbers must be multiplied to unit-numbers before adding. And since multiplication creates areas, per-numbers add by their areas under the per-number-graph, i.e., as integral calculus.

And again, the opposite of integral calculus is called differential calculus asking "2kg at 3$/kg plus 4kg at how many $/kg total 6 kg at 5$/kg?" As before, we subtract the $-number we added before recounting the rest in kg, p = (6 x 5 – 2 x 3) / 4 = Δ$ / Δkg.

Chapter 10. Change by Adding or by Multiplying

Before designing, we reflect on how to change a number by adding or by multiplying.

Inside the 'no-unit' greenhouse, change by adding or multiplying is called arithmetic and geometric sequences that are added as series.

Outside, we see that adding 3$/day to 2$ gives a total of *T* = 2 + 3 x *n* after *n* days. The general formula, *T* = *b* + *a* x *n*, is called change by adding, or linear change.

Also, we see that adding 3%/day to 2$ gives a total of T = 2 x 103%^n after n days since adding 3% means multiplying with 103%. The general formula, $T = b$ x a^n = b x (1+r)^n, is called change by multiplying or exponential change.

Combining the two gives a simple formula for saving, $A/a = R/r$. Here a and r is the per-day input, and A and R is the final output, where 1+R = (1+r)^n.

Reversing change by adding gives an equation as 100 = 20 + 5 x u, easily solved by restacking and recounting:

100 = (100-20) + 20, so u x 5 = 100 – 20 = 80 = (80/5) x 5, so u = 80/5 = 16.

Reversing change by multiplying gives two equations. In the equation 20 = u^5, we need a factor u of which 5 gives 20, predicted by the factor-finding root $\sqrt[5]{20}$ = 1.82. In 20 = 5^u we need the number u of 5-factors that gives 20, predicted by the factor-counting logarithm $\log_5(20)$ = 1.86.

We now know all the ways to unite parts into a total, and to split a total in parts, the 'Algebra-square' (Tarp, 2018):

Operations **unite**/ split Totals in	Changing	Constant
Unit-numbers m, s, kg, $	***T = a + n*** $T - n = a$	***T = a x n*** $\frac{T}{n} = a$
Per-numbers m/s, $/kg, $/100$ = %	***T = ∫a x dn*** $\frac{\mathrm{d}T}{\mathrm{d}n} = a$	***T = a^n*** $\sqrt[n]{T} = a$ $n = \log_a T$

Figure 4. The Ways to Unite Parts into a Total, and to Split a Total into Parts.

Chapter 11. Conclusion, Finally a Communicative Turn in Mathematics Education

It goes without saying that a total must be counted before being added. Where small totals may be glanced directly in space, larger totals need to be counted in time by pushing away 1s, i.e., by dividing by 1. Here the recount formula gives, e.g., 8 = (8/1) x 1 = 8 x 1, which shows that the total's space-number is the same as its counted time-number.

But when counting in 1s, we never meet the bundles-of-bundles since the square, 1 1s, is still 1, whereas the square, 2 2s, is 4. And we may not see that in reality we bundle in tens, and ten-tens, etc., since we may never push away tens, the real unit in our number system as expressed by the full number-formula, the polynomial,

T = 345 = 3 Bundle-bundles + 4 Bundles + 5 = 3B^2+4B+5.

Whereas, if counting in 3s and using the name '1 Bundle 0', for 3, we meet 9 as the bundle-of-bundles square, which may inspire us to use the same name for hundred when counting in tens.

So, we need to count in at least 2s to see the nature of outside bundle-counting. 1 is not a prime unit, as the other prime units that cannot split into new prime units.

The original question may now be answered in the following way:

Children count in bundles, which gives the recount formula, $T = (T/B)\text{x}B$, as a direct key to core mathematics.

By changing units, it creates per-numbers that add by integrating their areas as in calculus, and that opens up to proportionality and linearity, and thus to countless STEM formulas, and to solving their equations. It makes trigonometry precede plane and coordinate geometry. It shows that per-numbers and fractions are not numbers, but operators needing a number to become a number. It shows that counting before adding makes the two basic counting operations, push and stack, division and multiplication, precede subtraction and addition that is ambiguous with its two options, next-to and on-top. It shows our four basic operations as icons for the outside counting actions: push, lift, pull and unite. It shows that negative numbers, decimals and fractions are different ways to count the unbundled. But, most importantly, it shows the power of formulas as predicting number-language sentences making us master many in nature and in society, and in time and space.

So, building on the mastery of Many developed when adapting to many before school, a 'counting before adding' curriculum allows children to outside master the same mathematics as is taught with great difficulties inside the 'no-units' paradigm's greenhouse seeing its very foundation, 1+1=2, fold when meeting units.

So once tested, flexible bundle-numbers with units may also fold the myth "math is hard, and needs more funding." Meaning that we can finally have a communicative turn in number-language education as the foreign language education had in the 1970's (Widdowson, 1978).

Feynman famously asked: "If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generation of creatures, what statement would contain the most information in the fewest words?"

Certainly, the recount formula is a candidate. So why not enlighten humans about it instead of forcing them inside the darkness of a greenhouse without units.

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