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From a colonized to a decolonized mathematics, from 8 to 2 competences, from non-unit to unit-numbers

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Keywords: Decolonization, mathematics, competence, curriculum, numeracy.

Asking a 3year-old child "How many years next time?" shows a need to decolonize mathematics. The child reacts to 4 fingers held together 2 by 2 "That is not 4, that is 2 twos". Adults only see inside essence, four, but the child sees outside existence, bundles of 2s in space as units when counted in time.

Comparing the claims, "1+2 = 3" and "3x4 = 12", we see that without units, outside examples as "1week+2days = 9days" falsify the first claim. The second claim includes the unit by predicting that 3x4 as 3 4s outside may be re-counted as 1.2 tens. So, multiplication makes mathematics a natural science that becomes decolonized once the colonization by non-unit numbers has ended.

To decolonize a colonized non-unit mathematics with 8 competences, only 2 competences are needed, count and add: counting and re-counting bring outside totals inside to be added or split depending on how they occur, as like or unlike unit or per-numbers. A counting sequence in 3s always include the units: unbundled, bundles, and bundle-of-bundles that become squares on a Bundle-Bundle Pegboard, a BBBoard, where unit-numbers are tiles. Counting before adding changes both the order and the identity of the operations. Power is in bundle-bundles, division is a broom pushing away bundles, multiplication a lift stacking them, subtraction a rope pulling away the stack to find unbundled that placed on-top of the stack becomes decimals, fractions, or negatives, e.g., 9 = 4B1 = 4 ½ = 5B-1 2s. And addition shows the two ways to add stacks, next-to by areas as integral calculus, or on-top after the units are made like by the linearity of a recount-formula showing that when re-counting 8 in 2s, 8 = 4 x 2 = (8/2) x 2, or T = (T/B) x B with T and B for the total and the bundle.

Recounting ten-bundles in digit-bundles creates equations solved by moving to opposite side with opposite sign: *u*x2 = 12 = (12/2) x 2, so *u* = 12/2. Recounting digits in tens gives the tables and early algebra on a BBBoard: 6x7 = (B-4) x (B-3) = 10B-4B-3B+4x3 (taken away twice) = 3B12 = 4B2 = 42. Recounting 6 4s as a bundle-bundle creates square roots and solves quadratics. Recounting $ in kg creates a per-number, e.g., 4$ per 5kg that bridges the units, and that becomes fractions with like units, and that multiplied to $ add as areas, i.e., as integral calculus. Probably all have learned this core math after primary school. Secondary school then is for the communicative turn using the number-language to write literature in its three genres, fact, fiction, and fake, as in the word-language.

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Respecting the child's innate number sense, is that Woke-math?

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Keywords: Mathematics curriculum, decolonization, numeracy, arithmetic, calculus

Woke means respecting multiple identities.

Woke-math respects flexible bundle-numbers for a total instead of imposing a linear number upon it. Five fingers thus exist both as 5 1s, as 1 5s, as 1Bundle3 2s, as 2B1 2s, and as 3B-1 2s. Woke-math occurs when asked "How many years next time?", a 3year child reacts to holding fingers together 2 and 2 by saying "That is not 4, that is 2 2s." So, child sees what exists in space and time, bundles of 2s in space, and 2 of them in time when counted. Woke-math thus builds on the philosophy called existentialism holding that outside existence precedes inside constructed essence. As 'multiplication-math', Woke-math is a natural science where 3x4 = 12 states that 3 4s may be recounted in 1B2 tens. As 'addition-math', traditional math is 'mathematism' where 2+1=3 is falsified by, e.g., 2pairs+1=5.

We ask: *How to design a decolonized curriculum using 2dimensional bundle-numbers with units?*

Bundle-counting ten fingers in 3s gives power as the first operation.

0B1, 0B2, 0B3 or 1B0 or 2B-3, …, 2B2, 2B3 or 3B0 or 1BB0B0 or 1(B^2)0B0, 1(B^2)0B1 or 101.

Bundle-counting with over-loads and under-loads gives negative numbers easing calculations.

5 = 1B3 = 2B1 = 3B-1 2s. 4+58 = 4+5B8 = 5B12 = 6B2 = 62. 4x68 = 4x6B8 = 24B32 = 27B2 = 272.

Bundles in space give digits as icons, and numbers as tiles on a 10-by-10 Bundle-Bundle-Board.

There are four strokes in a 4-icon, five in a 5-icon, etc. And, 2 3s is a 2x3 tile; and a BB is a square.

Operations are icons also.

8 – 2: From 8, pull-away 2 (a rope). 8/2: From 8, push-away 2s (a broom). 4x2: 4 times lift 2s (a lift).

Recounting between icons may be predicted by a calculator.

4 5s = ? 6s. Enter '4x5/6' gives '3.more'. Enter '4x5 – 3x6' gives '2'. Answer: 4 5s = 3B2 6s.

Recounting 8 in 2s gives a 'recount-formula'.

8/2 = 4, so 8 = 4x2, or 8 = (8/2)x2, or Total T = (T/B)xB, a 'ReCountFormula' or proportionality.

Recounting 9 in 2s gives the decimal numbers, negative numbers, and fractions.

Count or calculate gives 9 = 4B1. Recount 1 in 2s gives 1 = (1/2)x2. So, 9 = 4.1 = 5B-1 = 4 ½ 2s.

Recounting from tens to icons gives equations.

How many 2s in 8? *u*x2 = 8, but 8 = (8/2)x2, so *u* = 8/2. Method: 'opposite side & opposite sign'.

Recounting from icons to tens gives tables and early algebra on a BBBoard.

Show 6x7 on a BBBoard. Pull away the top- and side-stack, add upper right corner pulled-away twice. 6 7s = ?tens. 6 7s = 6x7 = (B-4)x(B-3) = 10B –4B –3B + 4x3 = 3B + 1B2 = 4B2 = 42. And –x– = +.

Recounting to Bundle-Bundles gives squares, square roots, and quadratics.

Squaring a rectangle: Move half the excess to the top and side, and we almost have the square-root: Squaring a 6x4 rectangle, (6-4)/2 = 1. So 6x4 ≈ (6-1)x(4+1). Pull away the 1x1 corner gives 5.9 close to the calculator prediction 5.899. A (u+3) square has two squares, u^2 and 3^2, and two 3u-tiles, totalling u^2+6u+9. If u^2+6u+8 = 0, then (u+3) squared is 1 squared, which gives the solutions.

Recounting between units gives a per-number 4$ per 5kg, bridging the two units by recounting.

<u>12$ = ? kg</u>. 12$ = (12/4)x4$= (12/4)x5kg = 15kg. <u>20kg = ? $</u>. 20kg = (20/5)x5kg = (20/5)x4$ = 16$.

With like units, per-numbers become fractions.

My share is 3$ per 5$, 3$/5$ = 3/5, how much of 200$? 200$ = (200/5)x5$ gives (200/5)x3$ = 120$.

Recounting the sides in a rectangle with height *H*, length *L*, and diagonal *D* gives trigonometry.

Height = (Height/Length) x Length = tangent (Angle) x Length. And H = (H/D) x D = sine(A) x D. A small height is almost a circle arc. So, if the length is 1, the semicircle length = $\pi \approx n$ x tan(180/n).

Adding next to and reversed gives primary school calculus.

Adding 2 3s and 4 5s next-to means adding areas, called integration, and differentiation if reversed.

Adding on-top gives proportionality.

Adding 2 3s and 4 5s on-top, the units first must be made the same by recounting.

Adding per-numbers and fractions gives middle school calculus.

In the bill 2kg at 3$/kg + 4kg at 5$/kg, the unit numbers add directly, but the per-numbers must be multiplied before being added. So, per-numbers add by areas, i.e., by integral calculus.

Adding like per-numbers gives power calculations.

Adding 5% means to multiply by 105%. 10 times give 105%^10 = 162.9%, or 50% + 12.9% extra. 10 times ?% gives 50%, in u^10=150%, the factor-finding root predicts u= 10√150% = 1.041 or 4.1%. ?times 5% gives 80%, in 105%^u=180%, the factor-counting logarithm predicts u=log1.05(1.80)=12. 100% split in n parts gives (1+1/n)^n, which for n large gives 2,718…, called the Euler-number, e.

Adding one-digit numbers as bundles: 6+9 = 2B3 6s = 2B-3 9s.

Adding letter-numbers by their common unit: 2$abcd$ + 3ace = 2bd*ac + 3e*ac = (2bd+3e)*ac.

Finally, we reach the 'Algebra Square', where, in Arabic, Algebra means to re-unite.

Unlike/like unit-numbers are united by addition/multiplication; and split by subtraction/division. Unlike/like per-numbers are united by integration/power; and split by differentiation/root & log.

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To master or not to master math before Many, that is the question

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It seems evident that the goal of mathematics education must be to learn mathematics; which once learned inside has many important outside applications that allow humans to master Many. It is only a shame that mathematics education is so difficult that it produces many 'slow learners'. Therefore, grand theory asks: With mastery of outside existence as the end goal, and with mastery of an inside institutionalized essence as a subgoal, a goal-displacement may be prevented by exchanging the two. To get an answer we observe that the mastery children develop when adapting to Many before school leads to 12 micro-curricula so completely different from the tradition that they create a Kuhnian paradigm-shift within mathematics education as radical as the change from a flat to a round earth.

Keywords: Elementary school mathematics, special education, numeracy, arithmetic, calculus

Children's 2D bundle-numbers with units replaced by 1D line-numbers without

Children talk vividly about Many before school. But then they stop doing so in school, and risk being excluded and sent to special education. An hour inside a classroom tells you why. The students no more talk about Many, instead a textbook mediated by a teacher teaches them about what they must learn first to communicate later: multidigit 1D line-numbers that obey a place value system, and that are added without units. Later, also fractions are added without units, thus disregarding the fact that both digits and fractions are not numbers, but operators needing numbers to become numbers. The textbook thus presents a 'science', which despite 2weeks + 1day is 15days builds on the belief that 2+1 is 3 always, and thus should be called 'mathematism' true inside but seldom outside (Tarp, 2018).

Grand theory looks at mathematics education

Within philosophy, Existentialism holds that existence precedes essence so that in a sentence, the existing subject outweighs any chosen predicate (Marino, 2004). So, 'Many' should be seen onto-logically, what it is in itself, instead of epistemologically, how some may perceive and verbalize it.

Within psychology, Piaget sees learning as adapting to outside existence, whereas a Vygotsky sees learning as adapting to inside institutionalized socially constructed essence.

Foucault (1995) points to 'pastoral power' that may be installed by mixing action words, verbs, and judgement words, predicates, when diagnosing humans. Not being a verb, 'math' should be replaced by 'number' in the diagnose "you don't know how to math, so we who know will now teach you". If not replaced, sociologically a school becomes a 'pris-pital-barrack' forcing constant return to the same room to be cured from the self-referring diagnose 'inability to math' under order. Also within sociology, a structure-agent debate discusses if humans should obey institutionalised essence or allow this to be constantly negotiated between peers. Here, a Weberian viewpoint (1930) may ask if SET is a rationalization gone too far by leaving Many de-enchanted and leaving learners in an 'iron cage'. A Baumanian viewpoint (1990) suggests that, by monopolizing the road to mastery of Many, traditional math has created a 'goal displacement' making the institutionalized means a goal instead. So, the word 'mathematics' must leave goal descriptions to avoid a meaningless self-reference.

The tradition sees Many as an example of 1D linear cardinality always able to absorb one more thus built on the belief that ‘2+1 = 3 period’. In contrast, humans see Many as a union of 2D stacks coming from numbering singles, bundles, bundles of bundles, etc., e.g., $T = 345 = 3*BB + 4*B + 5*1$. The tradition sees mastery of math as its primary goal. A difference thus could see mastery of Many as the goal to be reached directly by using the bundle-numbers with units that children bring to school and that allows learning core math on the way. And could make the tradition more meaningful by de-modelling it (Tarp, 2020). As to differences, Difference Research (Tarp, 2018) searching for differences making a difference may design micro curricula (MC) to be tested with Design Research.

Counting Many with bundles, children deserve a bundle-number curriculum

Asked “How many years next time?”, a 3year old typically will say “Four” and show 4 fingers; but will react strongly to 4 fingers held together 2 by 2, “That’s not four, that’s two twos”, thus describing what exists in space and in time: bundles of 2s as totals in space, and two of them when counted in time as a sequence. Inside, children thus adapt to outside totals by using two-dimensional bundle-numbers with units. Also, they use full sentences as in the word-language with a subject ’that’, and a verb ‘is’, and a predicate ‘2 2s’ which abbreviated demodels a formula as a number-language sentence ‘T = 2 2s’ shown with a vertical and a horizontal rubber band on a 2D ten-by-ten pegboard, a ‘BundleBundleBoard’, or ‘BBBoard’, that now will replace the 1D number-line. So, we ask:

What math comes from a question-guided curriculum using a BBBoard and the 2D bundle-numbers with units that children bring to school to develop their number-language and mastery of Many?

MC01. Demodeling digits as icons

The tradition presents both digits and letters as symbols. A difference is letting students build digits as icons with as many sticks or strokes as they represent (Tarp, 2018) to see that inside icons link directly to outside degrees of Many; and that 5 ones differ from 1 fives; and that ten has no icon since as a bundle ten becomes 1-unit-0, or 10. 2-digit numbers thus are two numberings of bundles and of unbundled singles. A guiding question may be “There seems to be 5 strokes in a 5-digit if written less sloppy. Is this also the case with other digits?” Outside material may be sticks, a folding ruler, cars, dolls, a BBBoard, etc. Discussing why numbers after ten have no icon leads on to bundle-counting.

MC02. Demodeling counting sequences by always including bundles, and bundles-of-bundles

Using a place value system, the tradition never uses units in counting sequence. A difference is to practice bundle-counting in tens, fives, and threes, and always include the units. In this way students may see that including bundles in number-names prevents mixing up 31 and 13. And they may also be informed that the strange names ‘eleven’ and ‘twelve’ are Viking names meaning ‘one left’ and ‘two left’, and that the name ‘twenty’ has stayed unchanged since the Vikings said ‘tvende ti’; The Viking tradition saying ‘three-and-twenty’ instead of ‘twenty-three’ was used in English for many years. Now it stops after 20. Now only Danes still counts in scores: 80 = 4 scores, 90 = half-5 scores. A guiding question may be “Always count what exists when bundle-counting in tens, in 5s and in 3s.” Outside material may be fingers, sticks, cubes, and a BBBoard. Including the arm as an extra finger we can count to twelve, called a dozen.

First, we count a dozen in 5s (hands): 0B1, …, 0B4, 0B5 or 1B0, 1B1, …, 1B5 or 2B0, 2B1, 2B2.

Then we count a dozen in 3s (triplets): 0*B*1, …, 0*B*3 or 1*B*0, …, 1*B*3 or 2*B*0, …, 2*B*3 or 3*B*0, 3*B*1, ...

Counting cubes in 3s, 3 bundles is 1 quadratic bundle-of-bundles or 1*BB* in writing, so we repeat: We count a dozen in 3s: 0*B*1, …, 2*B*3 or 3*B*0 or 1*BB*0*B*0, 1*BB*0*B*1, 1*BB*0*B*2, 1*BB*0*B*3 or 1*BB*1*B*0.

Counting fingers in 2s gives a total of ten as 1*BBB* 0*BB* 1*B* 0. So, power becomes the first operation.

Then we count in tens, again including the bundles: 0*B*1, …, 0*B*8, 0*B*9, 0*B*10 or 1*B*0, 1*B*1, 1*B*2. Finally, we bundle-count in tens from 0 to 111.

MC03. Demodeling multi-digit numbers with units, underloads, and overloads

Obeying place values, the tradition silences the units when writing ‘two hundred and fifty-seven’ as plain 257. A difference may be inspired by the Romans using ‘underloads’ when writing ‘four’ as ‘five less one’, IV; and by overloads when children use ‘past-counting’: ‘twenty-nine, twenty-ten, twenty-eleven’. A guiding question may be “Let us count with underloads missing for the next bundle. And with overloads saying ‘twenty-eleven’.” Outside material may be sticks, cubes, and a BBBoard.

First, we notice that five fingers can be counted in pairs in three different ways.

T = 5 = I I I I I = ~~II~~ I I I = 1*B*3, overload
T = 5 = I I I I I = ~~II~~ ~~II~~ I = 2*B*1, normal
T = 5 = I I I I I = ~~II~~ ~~II~~ ~~II~~ = 3*B*-1, underload

Using fingers and arms, first we count a dozen with underloads: 0*B*1 or 1*B*-9, 0*B*2 or 1*B*-8, …, 0*B*9 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-9, 1*B*2 or 2*B*-8.

Then in 5s (hands): 0*B*1 or 1*B*-4, 0*B*2 or 1*B*-3, …, 0*B*4 or 1*B*-1, 1*B*0, 1*B*1 or 2*B*-4, …, 2*B*2 or 3*B*-3. And in 3s (triplets): 0*B*1 or 1*B*-2, 0*B*2 or 1*B*-1, 0*B*3 or 1*B*0, 1*B*1 or 2*B*-2, …, 3*B*3 or 4*B*0 or 1*BB*1*B*0.

Cup-counting with a cup for bundles, and for bundles-of-bundles: *T* = 1]1]0 = 4]0 = 3]3 = 2]6 = 1]9.

Then we count in tens from 1 to 111, using ‘past-counting’:

… 1*B*9, 1*B*10, 1*B*11 or 2*B*1, 2*B*2, …, 2*B*11 or 3*B*1, …, 9*B*9, 9*B*10, 9*B*11 or 10*B*1 or 1*BB*0*B*1.

Counting in tens, we may also use ‘flexible bundle-numbers’ with overloads and with underloads:
T = 38 = 3*B*8 = 2*B*18 = 1*B*28 = 4*B*-2 = 5*B*-12.

Traditional carrying and borrowing become useless when using demodeling with units instead:
T = 65 + 27 = 6B5 + 2B7 = 8B12 = 9B2 = 92, and T = 65 – 27 = 6B5 – 2B7 = 4B-2 = 3B8 = 38.

Overload	Underload	Overload	Overload
65 + 27	65 – 27	7 x 48	336 /7
6 *B* 5 + 2 *B* 7	6 *B* 5 – 2 *B* 7	7 x 4 *B* 8	33 *B* 6 /7
8 *B* 12 9 *B* 2	4 *B* -2 3 *B* 8	28 *B* 56 33 *B* 6	28 *B* 56 /7 4 *B* 8
92	38	336	48

Figure 1: Doing math using flexible bundle-numbers with units

MC04. Demodeling solving equations to reversing formulas

Reducing the outside fact 'the Total is 3 4s' to the inside statement '3*4 = 12' the tradition silences both the subject and the verb. And forces the total to accept the identity '1.2 tens', even leaving out the unit and the decimal point. A difference is to use full sentences with an existing outside subject, a verb, and an inside chosen predicate. And to emphasize that a formula is an inside prediction of an outside action. The sentence "$T = 5*6 = 30$" thus inside predicts that an outside total 5 6s existing on a BBBoard may be re-counted as 3 tens on a number-line thus losing its identity.

A guiding question may be. "Let us talk math with full sentences about the totals we count and how."

Pulling-away 2 from a total T will leave 'from T pull-away 2' iconized by a rope called subtraction: Before we had T. After we have T-2 and 2, so $T = (T – 2) + 2$, or $T = (T – B) + B$ in general.

We call this formula a 're-stack formula' since, with the total as a stack, we may pull away the bundle from the top and place it next-to as its own stack.

Outside asking "Adding 2 to what gives 5?", inside becomes "? + 2 = 5" in writing. Using the letter u for the unknown number, this becomes an equation '$u + 2 = 5$', easily solved outside by pulling away the 2 that was added, described inside by restacking the 5: $u + 2 = 5 = (5 – 2) + 2$, so $u = 5 – 2$.

So inside, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the definition of the number '5-2': "5 minus 2 is the number u that with 2 added gives 5".

We count and re-count in bundles. Re-counting 8 1s in 2s, we use '/' to iconize a broom pushing away 2s. So '8/2 = 4' is an inside prediction for the outside action 'From 8, push-away 2s, 4 times.'

Having been pushed away, the 2-bundles are stacked. This is iconized by an 'x' for lifting the bundles, so '4x2 = 8' is an inside prediction for the outside action '4 times stacking 2s gives 8 1s.'

Re-counting 8 in 2s gives a 'recount formula' 8 = (8/2) x 2, outside showing a stack with the height 8/2 and the width 2, and with the total 8 as the area. So, with outside totals as inside areas, totals add by areas, called integral calculus. With unspecified numbers it says: $T = (T/B) \times B$, or $T = (T/B) * B$, simply stating that when recounting a total T in Bs, T contains T/B Bs.

Outside asking "How many 2s in 8", inside is the equation '? * 2 = 8', or '$u * 2 = 8$' easily solved outside by pushing away 2s, described inside by recounting 8 in 2s: $u * 2 = 8 = (8/2) * 2$, so $u = 8/ 2$.

Again, an equation is solved by moving a number to the opposite side with the opposite sign. Also, we see the formal definition of '8/2': "8 divided by 2 is the number u that multiplied with 2 gives 8.

MC05. Demodeling decimals, fractions and negatives as names for the unbundled singles

Without bundling, the tradition cannot talk about the unbundled singles. A difference is to see them in three different ways when placed on-top of the stack of bundles. A guiding question may be "How to see the unbundled singles?". Outside materials may be cubes and a BBBoard.

Before recounting 9 in 2s outside, inside we let a calculator predict the result: Entering 9/2 gives '4.more' predicting that '9 contains 4 2s, and more' that are found by outside pulling away the 4 2s, predicted inside by entering '9 – 4 * 2' giving 1. So, inside, the calculator predicts that 9 recounts as 4*B*1 2s, which is also observed outside.

Recounting 9 cubes in 2s, the unbundled can outside be placed on-top of the stack. Inside it may be described by a decimal point separating the bundles from the unbundled: T = 4B1 2s = 4.1 2s. Likewise, when counting in tens: T = 4B2 tens = 4.2 tens = 4.2 * 10 = 42.

Seen Outside as a part of a bundle, inside we can count it in bundles as a ‘fraction’, 1 = (1/2) * 2 = 1/2 2s; or we can count what is missing in a full bundle, 1 = 1B-1 2s.

Again, we see the flexibility of bundle-numbers: T = 4B1 2s = 4 ½ 2s = 4.1 2s = 5.-1 2s.

Likewise, when counting in tens: T = 4B2 tens = 4 2/10 tens = 4.2 tens = 5.-8 tens.

MC06. Demodeling multiplication tables and equations as changing number units

Always counting in tens, the tradition never asks how to change number units. A difference is to change from one icon-unit to another, from icons to tens, or from tens to icons, or into a square.

A guiding question may be “How to change number units?”. Outside materials may be a BBBoard.

Asking ‘3 4s = ? 5s’, we inside predict the result by entering on a calculator the 3 4s as 3*4, to be counted in 5s by dividing by 5. The answer ‘2.more’ predicts that 3 4s contains 2B &more 5s. To find the unbundled singles, outside we pull away the 2 fives from the 3 4s; inside we predict this by entering ‘3*4 – 2*5’. The answer ‘2’ then predicts that 3 4s may be recounted as 2 5s & 2, or 2B2 5s.

Asking “40 = ? 5s”, we predict by solving the equation “$u * 5 = 40$” by recounting 40 in 5s: $u * 5 = 40 = (40/5) * 5$, so $u = 40/5$.Changing units also change the form of the height and width of the stack.

Asking “6 8s = ? tens”, or “6 * 8 = ?”, we inside predict the result by looking at a ten-by-ten square with 6 and 8 as B-4 and B-2 on the sides. We then see that the 6*8 stack is left when from the B*B stack we pull away a B*2 and a 4*B stack, and then add the 4*2 stack that was pulled away twice.

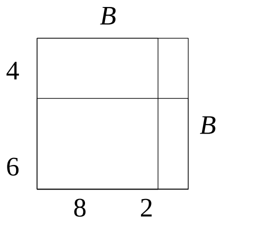	$T = 6 * 8$ $= (B\text{-}4) * (B\text{-}2)$ $= BB – 2B – 4B + 4*2$ $= 4B8 = 48$	$T = \begin{pmatrix}1B & -4\\ 1B & -2\end{pmatrix}$ $= 1BB – 2B – 4B + 4*2$ $= 10B – 6B + 8$ $= 4B8 = 48$	$T = \begin{pmatrix}6B & +4\\ 8B & +2\end{pmatrix}$ $= 48BB + 12B + 32B + 8$ $= 48BB + 44B + 8$ $= 52BB\ 4B\ 8 = 5248$

Figure 2: Multiplying 6*8 and 64*82 as binomials

Here, 6*8 exists as 6 8s outside. Inside recounting in tens shows that clearly minus times minus is plus. This view is an alternative to the multiplication controversy created by the ‘YouCubed’ website.

Inside, multiplying two ‘less-numbers’ horizontally creates a FOIL-rule: First, Outside, Inside, Last. Multiplying them vertically creates a cross-multiplication rule: First multiply down to get the bundle-of-bundles and the unbundled, then cross-multiply to get the bundles.

MC07. Demodeling proportionality as per-numbers changing physical units

The tradition sees shifting physical units as an application of proportionality. Typically, finding the unit cost will answer questions as “with 2 kg costing 3$, what does 3 kg cost, and what does 6$ buy?” A difference is to use ‘per-numbers’ (Tarp, 2018) coming from double-counting the same total in two units, e.g., T = 3$ = (3$/2kg) * 2kg = p * 2kg, with the per-number p = 3$/2kg, or 3/2 $/kg. A guiding question may be "How to change physical units?". Outside materials may be coloured cubes.

Recounting in the per-number allows shifting units:

T = 6 kg = (6/2)*2 kg = (6/2)*3 $ = 9$; and T = 15$ = (15/3)*3$ = (15/3)*2kg = 10 kg.

Alternatively, we recount the units: $ = ($/kg)*kg = (3/2)*6 = 9; and kg = (kg/$)*$ = (2/3)*15 = 10.

With like units, per-numbers become fractions: 1$/4$ = ¼. The tradition teaches fractions as division: 1/4 of 12 = 12/4. A difference is to see a fraction as a part of a bundle counted in bundles, 1 = (1/4)*4, so 1/4 = 1 part per 4. Finding 3/4 of 12 thus means finding 3parts per 4 of 12 that recounts in 4s as: T = 12 = (12/4) * 4 = (12/4) * 3parts = 9parts, so 3 per 4 is the same as 9 per 12, or 3/4 = 9/12. Likewise, 3/4 of 100 means finding 3 parts per 4 of 100 = (100/4)*4, giving 75parts per 100 or 75%.

MC08. Demodeling square roots as recounting stacks as Bundle-Bundle squares

The tradition postpones squares, square roots and quadratics to upper secondary school. A difference is to see a square as a BB, a Bundle-Bundle. This allows stacks with overloads to be squared. Materials may be books, windows, doors, and a BBBoard, which also may be used to solve quadratics.

Squaring a 6-by-4 stack, its side is called √24, with lines to iconize the square. To find √24, half of the top surplus, ½*(6-4) = 1, is place it next to the side, which gives a 5-by-5 square less the upper right 1-by-1 square, so the side must be reduced by u, where 5*u = ½*1, giving u = 0.1. So √24 = 4.9 almost since we still miss a small upper right square. Inside, a calculator predicts that √24 = 4.899.

To solve the quadratic equation u^2 + 6u +8 = 0, we look at a (u+3) x (u+3) square divided in four sections, u^2, and 3u twice, and 9 = 8+1. Since u^2 + 6u +8 = 0, (u+3) squared is 1 = 1 squared. Hence u+3 = +1 or -1, so there are two solutions, u = -3+1 = -2, and u = -3-1 = -4.

MC09. Demodeling trigonometry as recounting the sides in a stack halved by its diagonal

The tradition teaches trigonometry after plane and coordinate geometry. A difference is to see trigonometry an example of per-numbers, recounting the sides in a stack halved by its diagonal. A guiding question may be "How to recount the sides in a stack halved by its diagonal?". Outside materials may be tiles, cards, a BBBoard, and books.

Recounting the height in the base, height = (height/base) * base = tangent A * base, shortened to

h = (h/b) * b = tan A * b = tan A bs, thus giving the formula: tangent A = height / base, or tan A = h/b.

This gives a formula for the length of a unit circle containing many right triangles: A half circle is 180 degrees that split in 100 small parts as 180 = (180/100)*100 = 100 1.8s. With A as 1.8 degrees, the circle and the tangent, h, are almost identical. So, the length of a half circle is ½C = 100 * h = 100 * tan 1.8 = 100 * tan (180/100) = 3.1426 ≈ π, where the number π is tan (180/n)*n, for n large enough.

MC10. Once counted and recounted, stacks may be added on-top or next-to

The tradition sees numbers as 1D line-numbers with addition defined as counting on. A difference is to accept children's 2D bundle-numbers that add next-to and on-top. A guiding question may be "How to add 2 3s and 4 5s on-top and next-to?". Materials may be cubes and a BBBoard.

Adding 2 3s and 4 5s on-top, the units must be made the same, outside by squeezing or pulling, inside by recounting to change units. The recount formula predicts the result when entering (2*3+4*5)/*B*, where *B* may be 3 or 5 or 8. With like units, digits add on-top: 8+5 = 2*B*(8-5) 5s = 2*B*(5-8) 8s.

Adding stacks next-to by areas is called integral calculus; and differential calculus if reversed asking "4 5s plus how many 3s gives 5 8s?". Outside, we pull-away 4 5s from the total *T* before recounting in 3s, which is predicted inside by a calculator: (5*8 – 4*5)/3, or $\Delta T/3$. Using a difference to calculate the change in the total, ΔT, before using division to recount in 3s, this is called differential calculus.

Adding BB-squares as a square, its side is the 'lower diagonal' in the major square placed first.

MC11. Adding unspecified letter-numbers

The tradition sees adding letter-numbers as an application of a distributive law. A difference is to find the common unit. In $T = 3ab$ the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, *T* may be seen as 3 *ab*s, or as (3*a*) *b*s, or as (3*b*) *a*s. To avoid confusion the 's' will be omitted, so $T = 3ab = 3 * ab = 3a * b$ or $3b * a$. Since totals need a common unit to add, this must be first found as $T = 3ab + 4ac = 3b *a + 4c *a = (3b+4c) *a = 3b+4c$ *a*s.

MC12. Demodeling integral calculus as adding per-numbers or fractions

Adding numbers without units may be called 'mathematism', true inside but seldom outside where, e.g., 2m + 3cm = 203cm. A difference respects that the recount-formula shows that fractions and per-numbers are not numbers, but operators needing numbers to become numbers before adding. A guiding question may be "What is 2kg at 3$/kg plus 4kg at 5$/kg?" Outside materials may be a BBBoard with rubber bands, vertically placed in the distances 2 and 6, and horizontally in 3 and 5.

Inside we see that unit-numbers add directly. Whereas per-numbers first must be multiplied to become unit-numbers. And since multiplication creates areas, per-numbers add by their areas, i.e., as the area under the per-number curve. And again, adding areas is called integral calculus. Again, the opposite is called differential calculus asking "2kg at 3$/kg plus 4kg at how many $/kg total 6 kg at 5$/kg?" The two connect by the fact that when adding differences, the middle terms disappear leaving only the difference between the end and initial numbers: $(b – a) + (c – b) + (d– c) = d – a$.

We now know all 4 ways to unite parts into a total, and to split a total in parts, the 'Algebra-square' (Tarp, 2018) that respects the Arabic meaning of the word algebra, to re-unite. And that shows how an outside total, *T*, is brought inside by a predicate describing how the total occurs as united or split into the four existing number types, unlike and like unit-numbers, and unlike and like per-numbers.

Operations unite/split Totals in	Unlike	Like
Unit-numbers m, s, kg, $	$T = a + n$ $T – n = a$	$T = a*n$ $T / n = a$
Per-numbers m/s, $/kg, $/100$ = %	$T = \int a*dn$ $dT/dn = a$	T = a^n $\sqrt[n]{T} = a$ $n = \log_a T$

Figure 3: The 4 ways to unite parts into a total, and the 5 ways to split a total into parts

Conclusion and recommendation

We asked what mathematics comes from replacing an essence-based with an existence-based curriculum answering the question 'how many?' by numbering outside totals inside by (re)counting. Digits occur as icons with as many strokes as they represent, thus becoming units when numbering totals existing in time and space with 2D bundle-numbers that are flexible by allowing both overloads and underloads. Bundling bundles also leads to squares and square roots; and to power as the first of the operations; which are icons also, but with different meanings and opposite order. Division now means counting iconized by a broom to push-away bundles. Multiplication is iconized by a lift uniting the bundles in a stack that a subtraction rope pulls-away to find the unbundled, seen as decimals, fractions, or negative numbers on top of the stack. Combined, bundling and stacking create a recount-formula with a per-number that changes units and used all over STEM. Thus, both proportionality and trigonometry occur in year one, as well as formulas and functions as full number-language sentences allowing calculators to predict numbering results. So 8/2 is 8 counted in 2s, and 6*8 is 6 8s, a stack that recounted in tens changes both width and height, and that introduces early algebra when written with underloads: 6*8 = (*B*-4)*(*B*-2) = (10-4-2)**B* + 4*2 = 4*B*8 = 48. Once counted and recounted, totals may add on-top, or next-to by areas as in integral calculus, also used to add per-numbers. An existence-based curriculum will finally allow a communicative turn within the number-language as within the word-language in the 1970s (Widdowson, 1978). Using children's own flexible bundle-number with units thus represents a paradigm shift (Kuhn, 1962) that opens new areas for research and innovation; as well as a self-organized pre- and in-service teacher training asking the subject instead of the instructor as exemplified on the MATHeCADEMY.net website.

The fourth of the 17 UN Sustainable Development Goals defines quality education as 'ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.' We could ask if this is possible if an educational tradition rejects the child's own 2D flexible bundle-numbers with units, and replaces them with inflexible 1D line-numbers without units? So maybe the time has come where mathematics education should stop teaching 'mathematism' to children and instead begin to learn from them how to master Many with their flexible bundle-numbers with units.

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A rejected proposal for a symposium

Niss and Skovsmose and Tarp

The symposia title is “School mathematics, past & present & future, 1974 & 2024 & 2074.”

The objectives of the session are to ask representatives for three Danish research directions grounded before 1980: “School mathematics, how was it in 1974? What has your direction been working with the past 50 years? What does your direction expect to happen in the coming 50 years?”

In 1981 the Danish educational journal ‘Pædagogik (3)’ asked three young scholars born in 1944, 1944 and 1945 to give a status on school mathematics in Denmark. Mogens Niss called his article ‘Mathematics Education in a Society, for the society or for the population?’ Ole Skovsmose called his article ‘Critical mathematics’. And Allan Tarp called his article ‘Mathematics, a quantitative description of the real world.’. The articles may be found at the Royal Danish Library, www.kb.dk, and serves as symposium papers. The symposium also celebrates the three scholars about to turn 80.

Overview and structure of the presentation. There will be three sessions. In the first session each presenter has ten minutes. In the next 30 minutes there will be three two-persons dialogs. In the last session the floor may send in short written contributions to be handled by a moderator.

Scholarly or scientific significance. The presenters represent three aspects on school mathematics. The Niss-direction sees university mathematics as an educational task to quote Freudenthal. So, it holds that the goal of mathematics education is to learn mathematics as defined by the university.

The Skovsmose-direction accepts this but stresses a critical attitude to the real-world examples and applications that are included. So, it holds that the goal of mathematics education is to use mathematics as a tool to unmask injustice, and to work for equality and equity.

The Tarp-direction holds that existence precedes essence, so mastery of Many should precede mastery of math. Children thus should be allowed to develop their innate number language to enable them to unite and split like and unlike unit- and per-numbers to allow a communicative turn in school math.

These directions constitute three stress fields allowing school math to develop in different directions.

The Niss- and Skovsmose-directions will discuss what should be stressed in school math, its closeness to university mathematics to allow students to become math graduates, or its real-world applications allowing the students to become critical citizens.

The Niss- and Tarp-directions will discuss if the goal of math education is to master inside mathematics before it can be applied to master outside Many, or if mastery of Many automatically implies mastery of core math.

The Skovsmose- and Tarp-directions will discuss which grand theories to use in math education, Marxism critical to applications or Existentialism sceptical to core predicates by deconstructing them.