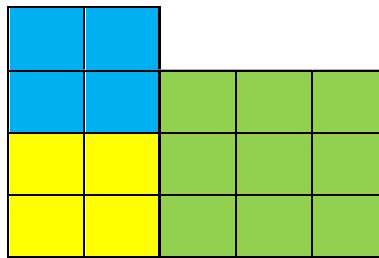


Bundle-Bundle-Math

cancels MatheMatism to Decolonize Mathematics

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What I see?

I see 1 **2s**, 1 **B 2s**

I see 2 **2s**, 1 **BB 2s**

I see 4 **2s**, 2 **BB 2s**, 1 **BBB 2s**

I see 1 **3s**, 1 **B 3s**

I see 3 **3s**, 1 **BB 3s**

And I see 4 **2s** plus 3 **3s** added next-to as **3B2 5s** or **4B-3 5s** or **3 2/5 5s**

- as an example of *Integral Calculus adding areas*

Content

Abstract	1
How valid is mathematics?	2
From Many to bundle-numbers with units, for teachers.....	3
Micro-curricula, for learners.....	7
MC01. Digits as icons in space, IIIII = 5.....	7
MC02. Tally-counting in time, ***** = IIII I	7
MC03. Bundle-counting in time with units: 0B1, ..., 0B5 or 1B0, 3 3s = 1BB	8
MC04. Bundles counted in space with over- and underloads, 5 = 1B3 = 2B1 = 3B-1 2s.....	9
MC05. Splitting, 8 = (8-2)+2	9
MC06. Recounting, 8 = (8/2)x2.....	10
MC07. Including the unbundled, 8 = (8/3)*3 = 2B2 = 2 2/3 = 3B-1 3s	10

MC08. Recounting in squares, $6 \text{ 4s} = 1 \text{ BB ?s}$	11
MC09. Recounting in another icon, $3 \text{ 4s} = ?5$	11
MC10. Recounting from tens to icons, $2 \text{ tens} = ? 7s$	12
MC11. Recounting from icons to tens, $6 \text{ 7s} = ? \text{ tens}$	12
MC12. Recounting in another physical unit creates per-numbers, $3\$/5\text{kg}$	13
MC13. With the same unit, per-numbers become fractions, $3\$/5\$ = 3/5$	13
MC14. Recounting a stack's sides gives trigonometry, $\text{rise} = (\text{rise/run}) * \text{run} = \tan A * \text{run}$	14
MC15. Adding next-to or on-top, $T = 2 \text{ 3s} + 4 \text{ 5s} = ? \text{ 8s}$; $T = ? \text{ 3s}$; $T = ? \text{ 5s}$	15
MC16. Subtracting and adding single digit numbers, $8 + 6 = 1\text{B}2 + 1\text{B}0 = 2\text{B}2 \text{ 6s}$	16
MC17. Adding per-numbers and fractions by integral calculus.....	16
MC18. Adding and subtracting Bundle-Bundle squares	17
MC19. Adding unspecified letter-numbers	18
MC20. Change in time.....	19
MC21. Bundle-numbers in a coordinate system.....	19
MC22. Games Theory and damage control.....	20
MC23. Simple board games.....	22
The Algebra Square.....	22
Fact and fiction and fake, the three genres of number-models	22
Modeling and de-modeling.....	23
Three footnotes.....	26
Teacher education.....	26
How different is the difference?	27
Overview over the differences between Essence- and Existence-math.....	28
Conclusion.....	29
References	29
Appendix.....	30
Unit-number tasks.....	30
Per-number tasks.....	31
Mechanics.....	32
The Economic Flow Diagram.....	33
Meeting many in a STEM context.....	35

Abstract

Looking at four fingers held together two by two, we see four fingers, the essence. But, before school, children see what exists, bundles of twos in space, and two of them when counted in time. So, we may ask how mathematics may be taught to children if using their own two-dimensional bundle-numbers with units instead of the school's one-dimensional line-numbers without units. In other words, we may ask how children may learn mathematics by working with existence instead of listening to essence. Here we use the two core concepts of philosophical Existentialism holding that existence precedes essence. This will mean that counting precedes adding since outside totals must first be counted to be added later inside. In this 'Many-math' approach, mathematical concepts are re-rooted in outside existing examples instead of being defined as examples itself inside. Now tens, hundreds and thousands become bundles, bundle-bundles, and bundle-bundle-bundles, as does 2, 4 and 8 when counting in twos instead. Here one-dimensional lines on a ruler are replaced by two-dimensional rectangles on a ten-by-ten Bundle-Bundle Board, a BBBoard, containing the outside existing subjects that is linked to inside essence predicates in a number-language sentence as in a word-language sentence. Here units are always included in counting sequences as 0Bundle1, 0B2, ..., 1B0. Here digits become icons with as many sticks as they represent. Here also operations become icons created in the counting process. Division is a broom to push-away bundles to be stacked by a multiplication lift before pulled-away by a subtraction rope to find the unbundled that are included on-top as decimals, fractions or negative numbers. Here recounting in another unit creates the recount formula $T = (T/B) \times B$ saying that T contains T/B Bundles. Here recounting tens in icons creates equations solved when recounting moves a number to the 'opposite side with opposite sign'. Here recounting icons in tens leads to early algebra when 6×7 becomes $(B-4) \times (B-3)$ placed on a BBBoard and found by pulling-away the top 4B and side 3B, and adding the 4×3 pulled away twice. Here bundle-bundles allow rectangular stacks to be recounted in squares with the square root as the side. Here recounting in a different physical unit creates per-numbers as $4\$/5\text{kg}$ bridging the units by recounting, and with like units becoming fractions. Here mutual recounting the sides in a stack halved by its diagonal leads to trigonometry before geometry. And now, after being counted and recounted, stacks may finally add on-top after recounted has made the units like, or next-to as areas, i.e., as integral calculus becoming differential calculus when reversed, also used to add per-numbers and fractions that must be multiplied to unit-numbers to add. Squares add as the square formed by their mutual Bottom-Top line. All in all, there are four ways to unite the world's four number-types. Addition and multiplication add unlike and like unit-numbers, where integration and power add unlike and like per-numbers. Together with their opposite operations, subtraction, division, differentiation, and the factor-finding root or the factor-counting logarithm they form an 'Algebra square' that is named after the Arabic word 'Algebra', meaning to reunite. And that now is the number-language allowing us to tell inside number-tales about outside totals using the same three genres, fact and fiction and fake, as does the word-language.



A ten-by-ten Bundle-Bundle Board, A BBBoard with

- 6 7s,
- 4 tens,
- ten 3s
- 4 3s

So, $6 \times 7 = (B-4) \times (B-3) = (10 - 4 - 3) \times B + 4 \times 3 = 3B12 = 4B2 = 42$

How valid is mathematics?

The school teaches that $2+3 = 5$ and that $2 \times 3 = 6$. But are they both valid outside? 2 bundles of 3s may always be recounted as 6 1s, but 2 weeks + 3 days is 17 days. So, even if both hold inside the school, outside ‘multiplication holds, but addition folds.’

Mathematics that adds numbers without units may be called ‘mathematism’, true inside but seldom outside the school, whereas mathematics that add numbers with units may be called ‘Many-math’, using bundle-numbers with units as 2 3s and 4 5s that may be added next-to as 8s, or on-top after shifting the units. But adding areas and shifting units are called ‘calculus’ and ‘proportionality’, the core of mathematics. Where normally they come very late, here they occur in the first lesson.

But, before totals can be added they must be counted or recounted. Counting 8 in 2s, we push-away bundles of 2s to be lifted into a stack of 4 2s, which may be iconized by a broom and as a lift so that $8 = (8/2) \times 2$, or $T = (T/B) \times B$ with T and B for Total and Bundle. This linear ‘proportionality’ recounting-formula to shift units now occurs in the first lesson when we bundle-count with units.

Also, it solves equations where ‘ $u \times 2 = 8$ ’ is asking ‘How many 2s in 8?’, which is answered by ‘ $u = 8/2$ ’ since 8 is recounted in 2s as above, thus simply moving ‘to opposite side with opposite sign’.

Recounting 8 in 3s, we meet subtraction as a rope to pull-away the stack to find 2 unbundled that are included on-top of the stack as a decimal, $8 = 2B2$ 3s, or as a fraction when also counted in 3s as $2 = (2/3) \times 3$, $8 = 2 \frac{2}{3}$ 3s, or with a negative number telling how much is needed for an extra bundle or pulled-away from this, $8 = 3B-1$ 3s. Here we may even see the 3 bundles of 3s as one bundle of bundles, one bundle-bundle, 1BB, so that $8 = 1BB \ 0B -1$ 3s, where the bundle-bundle is a square.

Which makes you wonder if any rectangular bundle-number may be recounted in squares with the square root as its side, and if squares can add as squares, e.g., as the square created by their mutual Bottom-Top line. In that case, calculus will be easy since areas then can add as squares.

Counting before adding thus leads to rectangular and squared bundle-numbers with units; and to decimals, fractions, and negative numbers; and to solving equation by recounting; and to proportionality needed to make units like when adding on-top; and to calculus when adding next-to as areas that again may be added to one square.

So, with Many-math’s ‘counting before adding’ we have learned most mathematics almost before we begin.

This will please the fourth of the 17 UN Sustainable Development Goals that defines quality education as ‘ensuring inclusive and equitable quality education and promoting lifelong learning opportunities for all.’ And where the subgoal 4.6 wants to “By 2030, ensure that all youth and a substantial proportion of adults, both men and women, achieve literacy and numeracy”.

To meet this goal, we therefore replace ‘mathematism’ with ‘Many-math’. To make the difference between the two more clear we may use the basic philosophical question: What comes first, existence or essence, what is in the world or what we think about it?

So, we ask: “What makes children learn mathematics, listening to essence, or working with existence?” Where ‘mathematism’ chooses the former, Many-math chooses the latter to

develop a number-language by working with Many as it exists in time and space as repetition and multiplicity, and in the word-language as plurals.

So finally, the number-language may have its communicative turn to be learned by telling tales about things and actions in space and time, just as the word-language had around the 1970's as described in H.G. Widdowson's book 'Teaching Language as Communication'.

From Many to bundle-numbers with units, for teachers

"That is not four, that is two twos". Said a 3year old child when asked "How many years next time?"; and when seeing four fingers held together two by two. This statement will change mathematics education forever since, as educated, essence is all we see. But as uneducated, the child sees what exists, bundles of twos in space, and two of them when counted in time. The number 'two' thus exist both in space and in time.

In space, 2 exist as 2s, a space number, a bundle of 2s, a 2-bundle, which can be united by a 3-bundle. Either horizontally to a (2+3) bundle, a 5-bundle, or vertically to a stack of 2B1 2s or to 2B-1 3s with B for bundle.

Here 1 plus 1 does not add up to 1 since the units are not the same, whereas one 2-bundle + one 2-bundle may both be two 2-bundles or one 4-bundle, but not 2 4-bundles.

In time, 2 exists together with the unit that was counted, i.e., as 2 units, a time-number, or a counting-number. So, 2+3 is 5 only with like units. Without units, a counting-number is an operator to be multiplied with a unit to become a total that can be added with another total if the units are the same, or after the units are made the same by recounting the two totals in the same unit.

So, as space-numbers, 2+3 is 5, whereas as time-numbers their units decide how they add.

The child thus opens our eyes for a different mathematics that, freed from its present essence-bounds, may return to its original identity as a natural science about Many in space and time. And that allows us to develop a natural number-language when communicating about existence instead of essence.

If existence comes before essence, then counting comes before adding, which is new since normally we are given the numbers to add. First, as educators we ourselves now investigate the consequence of counting before adding. Then we transform our discoveries into a sequence of micro-curricula.

Looking at five fingers we see that the inside essence 'five' outside may exist in different forms with each their label.

In space five may exist as five ones, or as a bundle of one fives that can be rearranged as an icon with the number of sticks it represents. Or, if counted in twos, five may exist as one bundle and three, 1B3, as 2B1, or as 3B-1 needing 1 to become an extra bundle, or even as 1BB 0B 1 since two twos held together exist as a bundle of bundles, one bundle-bundle, 1BB, or 1B-square becoming a square with cubes.

In time the five fingers may appear one by one. Here the bundles should be included as a unit in the counting sequence: 0B1, 0B2, 0B3, 0B4, 0B5. Or, if counting in 2s: 0B1, 0B2 or 1B0, 0B3 or 1B1, 0B4 or 1B2 or 2B0 or 1BB 0B 0, and finally 0B5 or 1B3 or 2B1 or 1BB 0B 1.

So, when counting and adding Many, we will no longer use a ruler's 1dimensional line-numbers without units as 5 and 42. Instead we now will use 2dimensional bundle-numbers with units as 0B5, and 4B2 that exist as rectangular or square totals on a ten-by-ten bundle-bundle pegboard, a 'BBBoard'. Which allows learning mathematics indirectly when

formulating inside tales about outside totals as, e.g., 6 7s existing on the BBBoard limited by two rubber bands, and that may be recounted in tens as a total of four tens and two, shortened to 'T = 4B2' tens. This number-language sentence or formula contains an outside subject linked to an inside predicate, just as does a word-language sentences as 'This is a table'.

Flexible bundle-numbers with units allow the same total, forty-two, to be recounted with over-load or under-load so that $T = 42 = 4B2 = 3B12 = 5B-8$, which makes unneeded the place value system. As well as carrying and borrowing since now $17 + 28 = 1B7 + 2B8 = 3B15 = 4B5 = 45$, and $57 - 28 = 5B7 - 2B8 = 3B-1 = 2B9 = 29$.

With units, 2digit numbers without may be postponed since with 6s as the unit, $6+9 = (1B0 + 1B3) 6s = 2B3 6s$, and with 9s as the unit, $6+9 = (1B-3 + 1B0) 9s = 2B-3 9s = 1B6 9s$.

When bundle-counting outside totals we find that not only digits but also the operations are icons, but in the reverse order.

Power now is the first operation we meet as a bundle-bundle hat when counting in 3s will change 9 into 3 3s, a bundle of bundles, a bundle-bundle, a BB, or a B^2 , that on a BBBoard is a square where 2 3s is a rectangle that may be transformed into almost a square with the rectangle's square root as the side by moving half the excess from the top to the side.

Division and multiplication then follow as a broom and a lift to push-away and stack bundles. Here, *recounting* in 2s will change 8 into $(8/2) \times 2$, or $T = (T/B) \times B$ telling that the total T contains T/B Bundles. This proportionality 'recount-formula' is used all over to shift units. Also, it solves multiplication equations as ' $u \times 2 = 8$ ' asking "How many 2s in 8?" which of course is found by recounting 8 in 2s as $8 = (8/2) \times 2$, so that the solution is $u = 8/2$ found by moving 'to opposite side with opposite sign'. This follows the formal definition: $8/2$ is the number u that multiplied with 2 gives 8, so if $u \times 2 = 8$ then $u = 8/2$. So, the balancing method solving equations by doing the same to both sides now disappears.

Subtraction now follows as a rope to pull-away the stack to locate unbundled singles, thus *splitting* the total in two, $T = (T-B) + B$, the 'split-formula'. Finally in the end, addition is a cross showing the two ways to unite stacks, next-to and on-top.

The split-formula solves addition equations as ' $u + 2 = 8$ ' asking "What is the number that with 2 added becomes 8?" which of course is found when splitting 8 by pulling-away the 2 that was added, $8 = (8-2) + 2$, so that the solution is $u = 8-2$, again found by moving 'to opposite side with opposite sign'. Also this follows the formal definition: $8-2$ is the number u that with 2 added 2 gives 8, so if $u + 2 = 8$ then $u = 8-2$. So here also the balancing method is not needed.

Recounting 8 in 3s, a calculator may inside predict the outside result. Entering '8/3' gives '2.more', and entering '8-2x3' gives '2' unbundled. This prediction is validated when outside pushing-away 3s from 8 twice. Included on-top of the bundles, the unbundled becomes decimals if writing $T = 2B2 3s$; or fractions if counted in bundles also, $T = 2 \frac{2}{3} 3s$, or replaced by a negative number, $T = 3B-1 3s$, telling what is missing in space for an extra bundle, or what was pulled-away in time from it. So, counting the unbundled leads to decimals, fractions, and negative numbers, $8 = 2B2 = 2 \frac{2}{3} B = 3B-1 3s$.

Recounting from one icon-unit to another may be predicted by a calculator. To inside predict the answer to the outside question " $2 3s = ? 4s$ ", entering ' $2 \times 3/4$ ' gives '1.more'. So, the unbundled are found when pulling-away 1 4s predicted by entering ' $2 \times 3 - 1 \times 4$ ' giving '2'. The calculator thus predicts that $2 3s = 1B2 4s$, which may be validated outside.

Recounting from tens to icons when asking ‘How many 6s in 24?’ leads to the equations ‘ $u \cdot 6 = 24$ ’ solved by $u = 24/6$ since 24 recounts in 6s as ‘ $24 = (24/6) \cdot 6$ ’, so the solution again follows the ‘opposite side & sign’ rule.

Recounting from icons to tens when asking ‘How many tens in 6 7s?’ leads to early algebra when placed on a BBBoard as $(B-4) \cdot (B-3)$, which is left when we pull-away the top 4B and the side 3B and add the 4 3s pulled away twice, so $(B-4) \cdot (B-3) = B \cdot B - 4B - 3B - 4 \cdot 3 = (10-4-3)B + 4 \cdot 3 = 3B0 + 1B2 = 4B2 = 42$, which clearly shows that minus times minus must be plus. Also, the four rectangles on the BBBoard illustrates the FOIL-method, First, Outside, Inside, Last; only here, Outside and Inside has changed place.

Recounting rectangles in squares, we may ask “How to square 6 4s by finding its square root as its side?” Moving half the excess from the top to the side gives the first guess as 5 5s. Then, to fill out the 1x1 top right corner we pull-away a slice, u , from the top and side. Here, $2 \cdot 4 \cdot u = 1$, or $u = 1/8$, so $4 \frac{7}{8} = 4.88$ is our second guess, which is close to the calculator’s answer, 4.90. Since two squares add as the square created by their Bottom-Top BT line we now have a way to add rectangular areas.

A split square may also be used to solve quadratics: In an $(u+3)$ -square, a vertical and horizontal dividing line will split a BBBoard into two squares, u^2 and 3^2 , as well as two $3 \cdot u$ rectangles. So $(u+3)^2 = u^2 + 3^2 + 2 \cdot 3 \cdot u = u^2 + 6 \cdot u + 9$. Now, with the quadratic $u^2 + 6 \cdot u + 8 = 0$ they all disappear except $9-8 = 1$. So $(u+3)^2 = 1$, which gives -2 and -4 as the two solutions. If, on the other hand, $u^2 + 6 \cdot u + 10 = 0$, then $9-10 = -1$, and then the equation $(u+3)^2 = -1$ has no solution.

Also, a split square with $y = x^2$ may show that, with dy and dx as small changes of y and x , dy will be $2 \cdot x \cdot dx$ if we neglect the tiny-tiny upper right corner. So, with $y = x^2$, $dy/dx = 2 \cdot x$.

Recounting in different physical units, e.g., from \$ to kg, $\$ = (\$/\text{kg}) \cdot \text{kg}$, we get a ‘per-number’, $\$/\text{kg}$, to connect the units: with $3\$/5\text{kg}$, $12\$ = (12/3) \cdot 3\$ = (12/3) \cdot 5\text{kg} = 20\text{kg}$. With like units, per-numbers become fractions or percentages, $3\$/5\$ = 3/5$, and $3\$/100\$ = 3/100 = 3\%$.

Recounting mutually the sides in a rectangle split by its diagonal, the per-numbers are called trigonometry that connects the sides and the angles., e.g., $\text{rise} = (\text{rise}/\text{run}) \cdot \text{run} = \text{tangent-Angle} \cdot \text{run}$. Here the tangent-Angle describes the steepness or the rise-ability of the diagonal. In an x - y coordinate system a curve may be generated by a formula $y = f(x)$. Here the curve between two close neighbor points is almost a diagonal in a rectangle, and since the run and the rise here are changes in x and in y , Δx and Δy , the tangent-Angle here describes the steepness of the curve as the per-number $\Delta y/\Delta x$ called the local slope of the curve.

Once counted and recounted, totals may finally be added on-top or next-to.

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like. Adding 2 3s and 4 5s as 8s means adding areas, which is called integral calculus. Reversing the process asking, e.g., “2 3s and how many 5s total 4 8s” is called differential calculus since you must find the difference between the two known totals before recounting it in 5s, $(T2-T1)/5$, or $\Delta T/5$.

Adding 2kg at $3\$/\text{kg}$ and 4 kg at $5\$/\text{kg}$, the unit numbers 2kg and 4kg add directly whereas the per-numbers $3\$/\text{kg}$ and $5\$/\text{kg}$ first must be multiplied to unit-numbers before adding, thus adding as areas, i.e., as integral calculus. Likewise with fractions where 1red of 2 apples plus 2red of 3 gives 3red of 5, and of course not 7red of 6 apples as taught by ‘mathematism’. Per-numbers and fractions and digits thus are not numbers, but operators needing a number to become a number.

Here the per-numbers are piecewise constant, but they may also be locally constant as in the case of a falling object with an increasing meter/second number. Calculus thus occurs three times, as next-to addition of stacks in primary school, as adding piecewise constant per-numbers in middle school's mixture problems, and as adding locally constant per-numbers in high school where the tiny area-strips are written as changes, $p \cdot dx = dA$, to profit from the fact that adding many changes makes all middle changes disappear leaving only the total change of A from the start to the end.

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives $106\%^{10}$ or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

Looking inside my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bend the two outer fingers. So, $1/3$ of the Ls are bent, and $1/2$ of the Rs. Does that mean that $1/3$ of the bent are Ls? No, $1/2$ is. So, in cross-tables we must also see fractions as operators needing numbers to become numbers.

The first months the children thus meet the core of mathematics: functions, equations, proportionality, trigonometry, and calculus.

As well as the four operations that unite unlike and like unit-numbers and per-numbers: addition, multiplication, integration, and power, seen in an 'Algebra-square' that is named after the Arabic word 'algebra' meaning 'to reunite'.

And that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm.

Operations unite/ <i>split Totals in</i>	Unlike	Like
Unit-numbers m, s, kg, \$	$T = a + n$ $T - n = a$	$T = a * n$ $T/n = a$
Per-numbers m/s, \$/100\$ = %	$T = \int f dx$ $dT/dx = f$	$T = a^b$ $b\sqrt[T]{a} \quad \log_a(T) = b$

Figure 01. The Algebra square shows how to unite and split our four number-types, and how to solve equations by moving 'to opposite side with opposite sign'.

Once we know how to count and recount totals, and how to unite and split the four number-types, we can now actively use this number-language to produce tales about numbering and numbers, and about totaling and totals in space and time. This is called modeling. As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then, and what-then models, or room, rate, and risk models. Fact models talk about the past and present and only needs to have the units checked. Fiction models talk about the future and needs to be supplemented with alternative models built upon alternative assumptions. And fake models typically add without units, e.g., when claiming that '2+3 = 5' always despite 2weeks + 3days = 17days, thus transforming mathematics to 'mathematism'.

Micro-curricula, for learners

Based on our investigation we may now design a sequence of micro-curricula, MC, for developing a number-language by working with totals existing as outside things and actions on a BBBoard that may be supplied with centimeter cubes placed on-top of the BBBoard.

MC01. Digits as icons in space, $IIII = 5$

The total here exists as sticks to be rearranged at a table and reported by a drawing on paper. The 'T=?' question is answered in two ways, as a collection of single ones, $IIIII$, or as one bundle of ones, HHH , that may be rearranged into an icon, 5, called a digit containing the number of sticks that it represents if written in a less sloppy way, and thus somewhat like the digits on a calculator. Each time the folding ruler is folded to look like the icon.

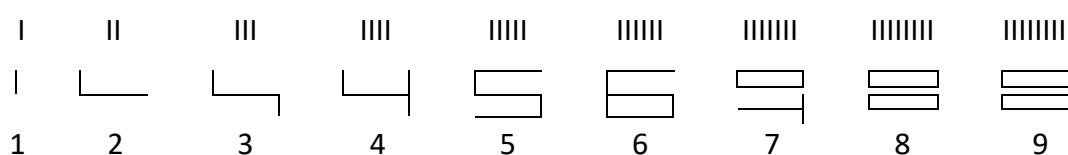


Figure 02. A digit as an icon with the number of sticks as it represents, four in the 4-icon, etc.

If we count in tens, ten sticks are replaced by one stick in a different color or material to allow more bundles to be rearranged as icons also so that 67 means 6-Bundle-7, 6B7, called 6-ten-7. Ten thus needs no icon since it becomes 'one bundle and no unbundled', written as $T = 1B0$ tens, or $T = 1.0$ tens, or $T = 10$ if leaving out the unit and the decimal point.

Example. One stick is one single. An extra stick added to 1 stick gives two singles that may unite to one 2-icon. And so on. An extra stick added to 8 sticks gives nine singles that may unite to one 9-icon. And an extra stick added to 9 sticks gives ten singles that instead of uniting to one ten-icon is bundled together as one bundle replaced by one stick of a different color or a different material and written as 1B0 since there are no singles left. An extra stick added to ten sticks gives eleven singles that may unite to one bundle and 1 single left, which made the Vikings call eleven 'one left' and written as 1B1. Likewise with twelve that the Vikings called 'two left'. There is no 'three left' because of the ancient counting method "one, two, many". So, from 3 we specify both the bundles and the singles. Zero is iconized as a looking glass not finding anything. The name 'twenty' comes from the Vikings' 'twende ti'.

Skill building. Roll some dice twice (physically or virtually) to get the number of bundles and unbundled singles. Then phrase and report the number. So, with 3 and 5, say three-bundle-five, three-ten-five, and thirty-five; and finally write $T = 3B5 = 35$.

End test. Roll some dice twice an extra time.

MC02. Tally-counting in time, $***** = IIII$

The total here exists as sticks to be moved one by one on a table and reported by strokes on paper, as well as some dice. The 'T=?' question is answered by tally-counting the total in time reported as some 5-bundles and some unbundles singles, e.g., $T = 2B1\ 5s$.

Example. In a sentence, count the e's and the a's.

Skill building. Some dice is rolled a dozen times to show Even (1 2s, 2 2s or 3 2s) or Odd (1, 3, 5). The tally counting in 5s is reported with two totals e.g., $W = 1B4$, and $L = 0B3$, giving a total $T = 2B2$, or 1B7, or 3B-3. And giving the difference $D = 1B1$.

End test. Roll some dice an extra time.

MC03. Bundle-counting in time with units: 0B1, ..., 0B5 or 1B0, 3 3s = 1BB

The total here exists as lines or stacks on a BBBoard. The ‘T=?’ question is answered in time by moving the finger along the pegs with a counting sequence that by including the bundle as a unit makes the place value system unneeded. First, we count in lines, then in stacks marked by a vertical rubber band.

Counting 5 fingers in 3s we ask, “What do we have here?” to emphasize that we focus on existence instead of essence. We cannot count one finger as ‘one’ since 1 3s is 3 1s and we only have one. Instead, we count ‘0 bundle 1, 0B2, 0B3 or 1B0’ since 3 1s is 1 bundle with no unbundled left.

Counting 5 fingers in 2s we notice that four fingers are 1B2, but also 2B0, and 1BB0B0 since 2 2s is a bundle of bundles, a bundle-bundle, a BB, that is a square, as is 3 3s, 4 4s etc.

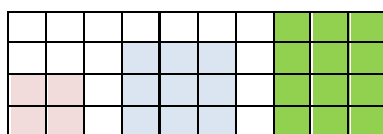


Figure 03. 2 2s, and 3 3s, and 4 4s as bundle-bundle squares

Counting the five fingers on a hand, their essence ‘5’ may exist in various ways:

$$T = 1\ 5s = 1B1\ 4s = 1B2\ 3s = 1B3\ 2s = 2B1\ 2s = 1BB\ 0B\ 1\ 2s = 5\ 1s$$

Counting ten fingers in 3s, we get 0B1, 0B2, 0B3 or 1B0, 1B1, ..., 2B3 or 3B0, 3B1.

T = 3B1 = 1BB0B1 3s, since 3 3s are a bundle of bundles, a bundle-bundle, a BB, a square.

Counting ten fingers in 2s we notice that 8 as 2BB 0B 0 is 1BBB 0BB 0B 0. So, we may also write ten as 1BBB 0BB 1B 0 2s, or as 1010 if leaving out the units.

Finally, when counting hundred on the BBBoard we finish with 1BB 0B 0:

0B1, 0B2, ..., 0B9, 0Bten or 1B0, 1B1, ..., 9B8, 9B9, 9Bten or tenB0 or 1BB0B0.

1BB0B 0	1BB0B 1	1BB0B 2	1BB0B 3	1BB0B 4	1BB0B 5	1BB0B 6	1BB0B 7	1BB0B 8	1BB0B 9	1BB0B 0
10B0	10B1	10B2	10B3	10B4	10B5	10B6	10B7	10B8	10B9	10B10
9B0	9B1	9B2	9B3	9B4	9B5	9B6	9B7	9B8	9B9	9B10
8B0	8B1	8B2	8B3	8B4	8B5	8B6	8B7	8B8	8B9	8B10
7B0	7B1	7B2	7B3	7B4	7B5	7B6	7B7	7B8	7B9	7B10
6B0	6B1	6B2	6B3	6B4	6B5	6B6	6B7	6B8	6B9	6B10
5B0	5B1	5B2	5B3	5B4	5B5	5B6	5B7	5B8	5B9	5B10
4B0	4B1	4B2	4B3	4B4	4B5	4B6	4B7	4B8	4B9	4B10
3B0	3B1	3B2	3B3	3B4	3B5	3B6	3B7	3B8	3B9	3B10
2B0	2B1	2B2	2B3	2B4	2B5	2B6	2B7	2B8	2B9	2B10
1B0	1B1	1B2	1B3	1B4	1B5	1B6	1B7	1B8	1B9	1B10
0B0	0B1	0B2	0B3	0B4	0B5	0B6	0B7	0B8	0B9	0B10

Figure 04. Counting from zero to 109 with units.

Skill building. Count a dozen and a score in 5s, 4s, 3s, and 2s.

End test. Count 30 in 3s.

MC04. Bundles counted in space with over- and underloads, $5 = 1B3 = 2B1 = 3B-1$ 2s

The total here exists as fingers and sticks. The ‘T=?’ question is answered in space by ‘flexible bundle counting’ that allows unbundled to stay unbundled as an overload, and that allows borrowing extra sticks to fill up an extra bundle with an underload. Using flexible bundle-numbers with units makes carrying and borrowing unneeded.

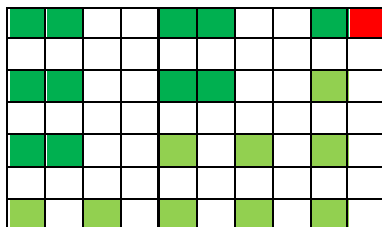


Figure 05. Five fingers may be recounted in 2s as 0B5, or 1B3, or 2B1, or 3B-1

Five fingers may be recounted in 5s as 0B5 (an overload) or 1B0 or 2B-5 (an underload).

Five fingers may be recounted in 4s as 0B5 or 1B1 or 2B-3.

Five fingers may be recounted in 3s as 0B5 or 1B2 or 2B-1.

Ten fingers may be recounted in tens as $\frac{1}{2}B$ from 1 to ten:

$\frac{1}{2}B-4, \frac{1}{2}B-3, \frac{1}{2}B-2, \frac{1}{2}B-1, \frac{1}{2}B0, \frac{1}{2}B1, \frac{1}{2}B2, \frac{1}{2}B3, \frac{1}{2}B4, \frac{1}{2}B5$ or 1B0 as ten.

This may ease standard calculations.

$$T = 6+3 = \frac{1}{2}B1 + \frac{1}{2}B-2 = 1B-1 = 0B9 = 9$$

$$T = 6 + 7 = \frac{1}{2}B1 + \frac{1}{2}B2 = 1B3 = 13$$

$$T = 8-3 = \frac{1}{2}B3 - \frac{1}{2}B-2 = 0B5 = 5, \text{ thus showing that } -(-2) = +2$$

$$T = 4*7 = 4*\frac{1}{2}B2 = 2B8 = 28$$

Skill building. The action is repeated with nine fingers arranged counted in 5s, 4s, 3s, and 2s. And with 2-digit numbers, e.g., $67 = 6B7 = 7B-3 = 5B17$. Then with cubes, and with sticks.

End test. The action is repeated on a BBBoard, or on an abacus.

MC05. Splitting, $8 = (8-2)+2$

The total here exists as a line of pegs on a BBBoard. The ‘T=?’ question is answered by ‘pulling-away’ a bundle hidden under cubes.

A total of 1 8s is split by pulling-away 2. To pull-away just once may be iconized by a rope, -, so that ‘8-2’ means ‘from 8 pull-away 2’ in time, or ‘from 8 pulled-away 2’ in space.

The original 8 now is split in 8-2 and 2 so that $8 = (8-2) + 2$. Here addition is iconized by a cross showing the two directions we can add, next-to or on-top so that ‘4+2’ means ‘4 with 2 added’. With T for the total and B for the bundle this ‘split-formula’ may be written as $T = (T-B)+B$.

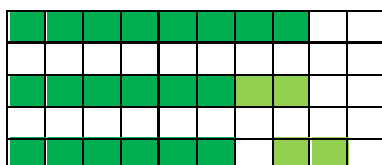


Figure 06. A total of 8 split in two parts by pulling-away 2, so $8 = (8 - 2) + 2$

Splitting may be used to solve equations coming from reversed actions.

The question “What is the number that with 2 added gives 8” may be shortened to an equation with a letter for the unknown number, ‘ $u+2 = 8$ ’. Of course, the number is found by reversing the action and pull-away the number that was originally added, so $u = 8-2$, which also comes from splitting 8, ‘ $u+2 = 8 = (8-2) + 2$ ’. So, we see that the solution is found by moving “to the opposite side with the opposite sign”. Also, it follows the formal definition of subtraction: $8-2$ is the number u that added to 2 gives 8, or if $u+2 = 8$ then $u = 8-2$.

Skill building. The action is repeated with fingers, sticks, cubes, and an abacus. The action is repeated with other numbers, e.g., $9 = (9 - 3) + 3$.

End test. Pick two numbers.

MC06. Recounting, $8 = (8/2) \times 2$

The total here exists as lines of pegs on a BBBoard. The ‘T=?’ question is answered by ‘pushing-away’ bundles hidden under cubes.

A total of 1 8s is recounted in 2s by 4 times pushing-away 2s. To pull-away more times may be iconized by a broom, /, so that ‘ $8/2$ ’ means ‘from 8 push-away 2s’ in time, and ‘from 8 pushed-away 2s’ in space.

With the pushed-away 2s arranged in a stack, 8 contains 2s 4 times or $8/2$ times so that $8 = 4 \times 2$, or $8 = (8/2) \times 2$. Here multiplication is iconized by a lift so that ‘ 4×2 ’ or ‘ $4 * 2$ ’ means ‘4 times stacking 2s’. With T for the total and B for the bundle this ‘recount-formula’ may be written as $T = (T/B) \times B$.

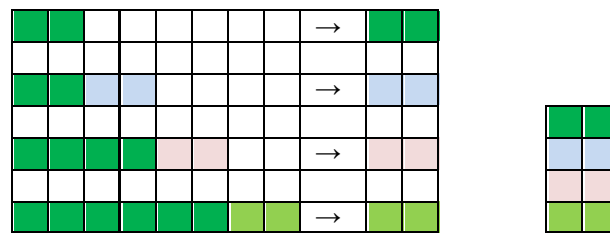


Figure 07. A total of 8 is recounted by pushing-away 2s and lifted into a stack, so $8 = (8/2) \times 2$

Recounting may be used to solve equations coming from reversing actions: The question “What is the number of 2s in 8” may be shortened to an equation with a letter for the unknown number, “ $u * 2 = 8$ ”. Of course, the number is found by reversing the action and push-away the 2s that was originally united, so $u = 8/2$, which also comes from recounting 8, ‘ $u * 2 = 8 = (8/2) * 2$ ’. So, we see that the solution is found by moving “to the opposite side with the opposite sign”. Also, it follows the formal definition of division: $8/2$ is the number u that multiplied with 2 gives 8, or if $u * 2 = 8$ then $u = 8/2$.

Skill building. The action is repeated with 12 counted in 2s and 3s using a finger to hide a bundle.

End test. 18 counted in 2s, and in 3s.

MC07. Including the unbundled, $8 = (8/3) * 3 = 2B2 = 2 \frac{2}{3} = 3B-1 \text{ 3s}$

The total here exists as cubes. The ‘T=?’ question is answered by pushing-away bundles to a stack that is then pulled-away to find the unbundles that then are included on-top of the stack.

Recounting 8 in 3s, first 2 times we push-away 3s, then we pull-away the stack of 2 3s and find 2 unbundled that are placed on-top of the stack. Here they may be seen as singles in a bundle described by a decimal number, $8 = 2B2 \text{ 3s}$, or as a fraction part when also recounted in 3s as $2 = (2/3) * 3$, $8 = 2 \frac{2}{3} \text{ 3s}$. Or we may write $8 = 3B-1$ to show that in space 1 is missing in the next bundle, or that in time 1 is pulled-away from it.

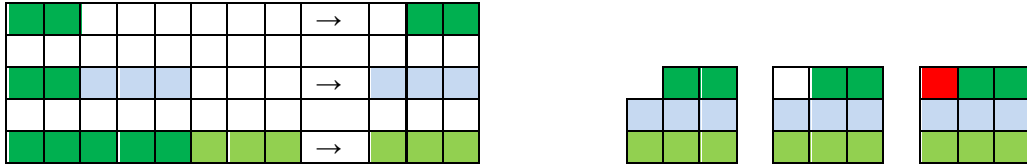


Figure 08. Unbundled become decimals, fractions or less-numbers, $8 = 2B2 = 2 \frac{2}{3} = 3B-1 \frac{3}{3}$
 With ten as the bundle-number, unbundled occur in the same way:

$$T = 4B3 = 4 \frac{3}{10} = 5B-7 \text{ tens.}$$

Skill building. The action is repeated on a BBBoard with 11 counted in 3s and 4s using cubes or fingers to hide a bundle.

End test. Recount 8 in 5s, and in 3s.

MC08. Recounting in squares, $6 \text{ 4s} = 1 \text{ BB ?s}$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by working on the upper right corner occurring when half the excess is move from the top to the side to give a first guess about the square root. The rectangle may be shown with rubber bands or with cubes.

If we want to square the total $T = 6 \text{ 4s}$ we move half of the excessing 2 4s from the top to the side to get a 5×5 square, and an unfilled square in the upper right corner that we try to fill with a rectangular $4*u$ slice of the top and the side. Here u is found by the equation $2*u*4 = 1$, or $8*u = 1$, giving $u = 1/8 = 0.125$, and $5-0.125 = 4.875$ as our next guess. However, now there is too much in the corner, so we repeat the process, or consult a calculator showing that the correct answer is $\sqrt{(6*4)} = 4.90$, which is very close to our third guess.

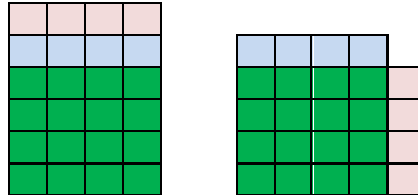


Figure 09. Recounting 6 4s by moving half the excess to the side to try to get a 5×5 square
 To find the quadratic numbers we see that 5 5s comes from 4 4s by adding 4 twice and 1 for the top right corner. So, with 4 4s as 16, 5 5s is $16 + 4 + 4 + 1 = 25$. In this way we may predict the square numbers to be 1, 4, 9, 16, 25, 36, 49, 84, 91 and 100.

And we see that a BB square increases with $2B+1$ when B increases with 1.

Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. Square 9 5s. Solve the quadratic equation $x^2 + 8*x + 12 = 0$

MC09. Recounting in another icon, $3 \text{ 4s} = ?5$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered on a BBBoard and predicted on a calculator.

With rubber bands on a BBBoard we see that 3 4s may be recounted as 2B2 5s. This may be predicted by a calculator. To find how many 5s there is in 3 4s we enter “ $3*4/5$ ”. The answer is ‘2.more’.

To find them we pull-away the stack of 2 5s by entering ‘ $3*4-2*5$ ’ that gives the answer ‘2’. So, the calculator predicts that 3 4s recount as 2B2 5s, which is validated on the BBBoard.

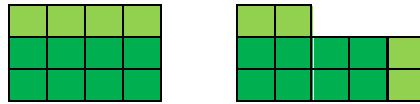


Figure 10. A total of 3 4s recounted in 5s manually. and predicted by a calculator
Skill building. The action is repeated on a BBBoard with other rectangular numbers.

End test. $4 \times 5 = ?6s$.

MC10. Recounting from tens to icons, 2 tens = ? 7s

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by recounting.

With rubber bands on a BBBoard we see that 2 tens may be recounted as 2B6 7s. This may be predicted by a calculator. To find how many 7s there is in 2 tens by recounting we enter ‘20/7’. The answer is ‘2.more’ found by pulling away the stack predicted by ‘20-2*7’ giving ‘6’. So, the calculator predicts that 2 tens recount as 2B6 7s, which is validated on the BBBoard. Alternatively, the question “How many 7s in 20?” leads to the equation $u \times 7 = 20$ that is solved moving to opposite side with opposite sign so that again $u = 20/7$, or $u = 2 \frac{6}{7}$.



Figure 11. A total of 2 tens recounted in 7s manually, and predicted by a calculator

We notice that decreasing the bundle will increase the height. To study this closer we recount 1 dozen in 6s, 4s, 3s, 2s, and 1s and place a dot in the upper right corner each time. The points then form a curve called a hyperbola.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. $4 \text{ tens} = ?8s$.

MC11. Recounting from icons to tens, 6 7s = ? tens

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by finding what we must pull-away from the bundle-bundle.

Recounting from icons to tens apparently is another name for the multiplication tables. With rubber bands on a BBBoard we see that 6 7s is left if from the ten bundles we pull-away 4 top and 3 side bundles, and add the upper right 4 3s that we pull-away twice:

$$T = 6 \text{ 7s} = 6 \times 7 = (10 - 4 - 3) \times B + 4 \text{ 3s} = 3B + 1B2 = 4B2 = 42.$$

This leads to early algebra if instead we write:

$$T = 6 \text{ 7s} = 6 \times 7 = (B - 4) \times (B - 3) = BB - 4 \times B - 3 \times B + 3 \times 4$$

Here we see that minus times minus must be plus.

So, a quick way to find the answer is to add and multiply the less-numbers and subtract the first and add the latter. With 4 and 3 as the less-numbers here, we quickly learn to say:

“Less (4+3) bundle (4*3)”, or “Less 7 bundle 12”, or “3 bundle 12”, or “4 bundle 2”, or “42”.

We may also write $B - 4$ and $B - 3$ on top of each other and then multiply down and across. Or we may use the FOIL method: First, Outside, Inside, Last.

	$ \begin{aligned} T &= 6 * 7 \\ &= (B-4) * (B-3) \\ &= BB - 4B - 3B + 4*3 \\ &= 3B12 \\ &= 4B2 \\ &= 42 \end{aligned} $	$ \begin{aligned} T &= \begin{pmatrix} 1B & -4 \\ 1B & -3 \end{pmatrix} \\ &= 1BB - 4B - 3B + 4*3 \\ &= 10B - 7B + 1B2 \\ &= 3B12 = 42 \end{aligned} $	$ \begin{aligned} T &= \begin{pmatrix} 2B & +3 \\ 4B & +6 \end{pmatrix} \\ &= 8BB + 12B + 12B + 18 \\ &= 8BB + 24B + 18 \\ &= 10BB 5B 8 \\ &= 1058 \end{aligned} $
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Figure 12. $6*7$ is left when pulling-away $4B$ and $3B$ and adding the 4 3 s pulled away twice. Multiplying the two-digit numbers $23*46$ as $2B3$ $4B6$ s, a vertical and a horizontal rubber band between the bundles and the singles allows a BBBoard to show the four stacks $2B$ $4B$ s, and $2B$ 6 s, below the 3 $4B$ s, and the 3 6 s. With overloads, they add up to $8BB$ 24 B 18 , or to $10BB$ $5B$ $8 = 1058$ without.

This process may be reversed when asking ' $1058 = ? 46$ s'. First 1058 is written with an overload as $10BB$ $5B$ $8 = 8BB$ $25B$ 8 . Since $4B*2B = 8BB$, the $2B$ contributes $2B*6$ to the 25 B s. The rest $13B$ 8 may be rewritten as $12B$ 18 , which recounted in 3 s gives $4B6$. So, the answer is $1058 = 23$ 46 s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. 7 8 s = ? tens.

MC12. Recounting in another physical unit creates per-numbers, $3\$/5kg$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ' $T=?$ ' question is answered by changing the unit in the per-number rectangle.

Recounting a physical total T as 3% and 5 kg gives a 'per-number' $3\$/5kg$ called the price and marked as a $3x5$ rectangle on a BBBoard.

The question " $20kg = ?\%$ " then is answered by recounting in the per-number:

$$20kg = (20/5)*5kg = (20/5)*3\$ = 12 \$.$$

On a BBBoard the counting sequences now are $5, 10, 15, 20kg$, and $3, 6, 9, 12\%$ since the per-number here has changed from $3/5$ to $12/20$.

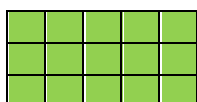


Figure 13. The per-number $3\$/5kg$ show as 3 5 s, or 6 10 s, or 9 15 s, or 12 20 s, etc.

Or we may introduce a new unit to make the digits like: $3\$ = (3/5)*5\$ = n*5\$ = 5n\$$ with the new $n = 3/5$. So, 20 kg = $20n\$ = 20*3/5\$ = 12\$$.

Alternatively, the units may be recounted:

$$\$ = (\$/kg)*kg = (3/5)*20 = 12$$

Or we may equate the per-numbers: $\$/kg = u/20 = 3/5$. Moving to opposite side with opposite sign we then get $5*u = 3*20$, or $u = 3*20/5 = 12$.

Skill building. The action is repeated with other numbers.

End test. With $5\$/2kg$, $12kg = ?\%$, and $?\$ = 12\%$.

MC13. With the same unit, per-numbers become fractions, $3\$/5\$ = 3/5$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ' $T=?$ ' question is answered by changing the unit in the per-number rectangle.

If a whole contains a part, they have the same unit. In this case the per-number becomes a fraction without units. Still, we may use the units ‘p’ and ‘w’ for the part and the whole.

To get the fraction 3/5 of 20\$ thus means to get 3p/5w of a 20\$ whole. Recounting in the per-number thus gives $20w = (20/5)*5w = (20/5)*3p = 12p$, or 12\$ of 20\$.

To get the fraction 3/5 of 100 thus means to get 3p/5w of a 100 whole. Recounting in the per-number thus gives $100w = (100/5)*5w = (100/5)*3p = 60p$, or 60 of 100, written as 60%.

To ask “20\$ is what percentage of 80\$” means asking about the fraction 20/80 of 100. Or we may introduce a new unit $80\$ = 100\%$ to see that $20\$ = (20/80)*80\$ = (20/80)*100\% = 40\%$.

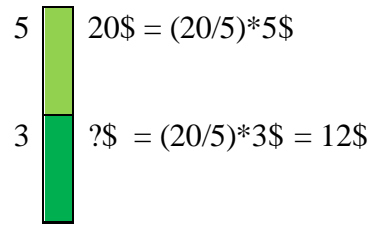


Figure 14. A fraction column with per-numbers to the left and unit-numbers to the right
 To add 10% to 200\$ we introduce the per-number 200\$/100%. After the addition the total is $T = 100\% + 10\% = 110\% = (110/100)*100\% = (110/100)*200\$ = 220\$$.

So, adding 10% means multiplying with 110%, and adding 10% 5 times means multiplying with $110\%^5 = 161.1\%$ thus giving 50% plus 11.1% extra, also called compound interest.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. With $2p/5w$, $10p = ?w$, and $?p = 20w$, and $2p/5w = ?\%$.

MC14. Recounting a stack’s sides gives trigonometry, $\text{rise} = (\text{rise}/\text{run}) * \text{run} = \tan A * \text{run}$
 The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘T=?’ question is answered by changing the unit in the per-number rectangle.

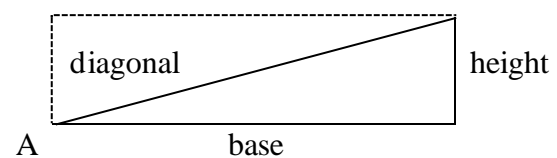


Figure 15. A stack with a base and a height and a diagonal

On a BBBoard, we mark a 3x4stack as a rectangle with height 3 and base 4. If we recount the height and the base in the diagonal, we get the per-numbers sine and cosine:

height = (height/diagonal) * diagonal = sine Angle * diagonal, shortened to

$$h = (h/d) * d = \sin A * d = \sin A ds,$$

This gives the formula $\sin A = \text{height} / \text{diagonal}$, or $\sin A = h/d$, or $\sin A = 3/5$ in our case.

Likewise, $\cos A = \text{base} / \text{diagonal}$, or $\sin A = b/d$, or $\cos A = 4/5$ in our case.

height = (height/base) * base = tangent Angle * base, shortened to

$$h = (h/b) * b = \tan A * b = \tan A bs,$$

This gives the formula $\text{tangent } A = \text{height} / \text{base}$, or $\tan A = h/b$, or $\tan A = 5/10$ in our case.

A protractor shows that the angle A is a little above 25 degrees. Testing this we get $\tan 25 = 0.466$. The reverse tan-button ‘tan⁻¹’ gives the precise result, $\tan^{-1}(0.5) = 26.6$ degrees.

Using the words ‘run’ and ‘rise’ instead of ‘base’ and ‘height’, we get the diagonal’s slope-formula: $\tan A = \text{rise}/\text{run}$. Here the tangent-number describes the steepness of the diagonal.

In a x-y coordinate system a curve may be generated by a formula $y = f(x)$. Here the curve between two close neighbor points is a diagonal in a rectangle, and since the run and the rise are changes in x , Δx , and in y , Δy , the tangent-number here describes the steepness of the curve as the per-number $\Delta y/\Delta x$ called the local slope of the curve.

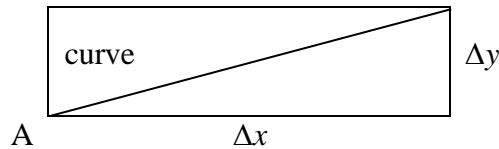


Figure 16. A curving curve is linear locally with small changes in x and y

The word ‘tangent’ is used since the height will be a tangent in a circle with center in A , and with the base as its radius. This gives a formula for the circumference since a circle contains many right triangles leaving the center. In a circle with radius 1, $h = \tan A$.

A half circle is 180 degrees that split in 100 small parts as $180 = (180/100) * 100 = 1.8 \text{ } 100\text{s} = 100 \text{ } 1.8\text{s}$. With A as 1.8 degrees, the circle and the tangent, h , are almost identical.

Half the circumference in a circle with radius 1 is called π , and $\pi = 100 * h = 100 * \tan 1.8 = 100 * \tan (180/100) = 3.1416$. This gives a formula for the number π : $\pi = \tan (180/n) * n$, for n large enough.

We also see that in a circle with radius r , the circumference is $2 * \pi * r$, and the area is $\pi * r^2$, or $\pi/4 * d^2$ where d is the circle’s diameter. So, a d -circle takes up almost 80% of the space inside the surrounding d -square.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. Add a 4-square and a 6-square as a square.

MC15. Adding next-to or on-top, $T = 2 \text{ } 3\text{s} + 4 \text{ } 5\text{s} = ? \text{ } 8\text{s}$; $T = ? \text{ } 3\text{s}$; $T = ? \text{ } 5\text{s}$

The total here exists as a rectangular bundle-number on a BBBoard. Here the ‘ $T=?$ ’ question is answered by recounting.

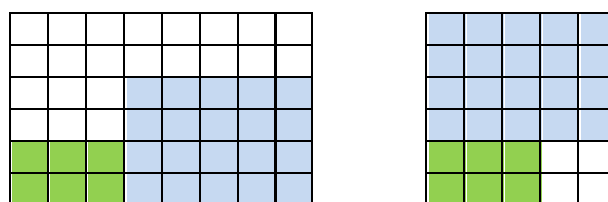


Figure 17. Two totals $2 \text{ } 3\text{s}$ and $4 \text{ } 5\text{s}$ added next-to, and added on-top

Adding $2 \text{ } 3\text{s}$ and $4 \text{ } 5\text{s}$ as 8s means adding areas, which is called integral calculus. Reversing the process by asking ‘ $2 \text{ } 3\text{s}$ and how many 5s total $4 \text{ } 8\text{s}$ ’ is called differential calculus because you find the difference between the known totals before recounting it in 5s , $(T_2 - T_1)/5$, or $\Delta T/5$.

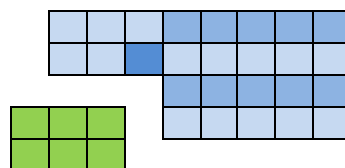


Figure 18. Next-to addition is reversed when asking $2 \text{ } 3\text{s} + ? \text{ } 5\text{s} = 4 \text{ } 8\text{s}$

Adding 2 3s and 4 5s as 3s or 5s, first recounting must make the units like To recount 2 3s in 5s, first we enter '2*3/5' giving '1.more', then we enter '2*3-1*5' giving '1', so 2 3s is 1B1 5s, which give the total 1B1 5s + 4B0 5s = (1B1 + 4B0) 5s = 5B1 5s.

Skill building. The action is repeated on a BBBoard with other numbers.

End test. $3\ 4s + 6\ 5s = ?\ 9s$. And $3\ 4s + 6\ 5s = ?\ 4s$. And $3\ 4s + 6\ 5s = ?\ 5s$. And $3\ 2s + ?\ 5s = 4\ 6s$.

MC16. Subtracting and adding single digit numbers, $8 + 6 = 1B2 + 1B0 = 2B2\ 6s$

The total here exists as lines of pegs on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

With a subtraction as ' $8 - 6 = ?$ ', a rubber band marks 8 on a BBBoard, and fingers hide the 6 that is pulled-away, so $8 - 6 = 2$.

With an addition as ' $8+6=?$ ', two rubber bands marks 8 and 6 on two BBBoard parallel lines to show that the sum may exist in two ways, as 2B2 6s, or as 2B-2 8s.



Figure 19. Adding 6 and 8 as 2B2 6s, or as 2B-2 8s, or as $2*\frac{1}{2}B + 1 + 3$

Here, using half bundles, 5s, will easy recounting in tens since

$$6+8 = \frac{1}{2}B1 + \frac{1}{2}B3 = 1B4 = 14.$$

Multidigit numbers may be added and subtracted with an over- or an under-load, which makes carrying and borrowing unneeded.

$$T = 36 + 47 = 3B6 + 4B7 = 7B13 = 8B3 = 83$$

$$T = 86 - 37 = 8B6 - 3B7 = 5B-1 = 4B9 = 49$$

$$T = 4*67 = 4*6B7 = 24B28 = 26B8 = 268$$

$$T = 268 / 4 = 26B8 / 4 = 24B28 / 4 = 6B7 = 67$$

Skill building. The action is repeated with other one-digit and two-digit numbers.

End test. $9 - 7 = ?$, $9+7 = ?$, $T = 38+46 = ?$; $T = 82 - 54 = ?$

MC17. Adding per-numbers and fractions by integral calculus

The total here exists as a rectangular bundle-number on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

Asking "2kg at 3\$/kg and 4 kg at 5\$/kg total what?" the unit numbers 2kg and 4kg add directly whereas the per-numbers 3\$/kg and 5\$/kg first must be multiplied to unit-numbers before adding, thus added as areas, i.e., as integral calculus. Here the per-numbers are piecewise constant, but they may also be locally constant as in the case with a falling object having an increasing meter/second number. Before adding, fractions must also be multiplied to unit, numbers. So, with apples, 1red of 2 plus 2red of 3 gives 3red of 5, and of course not 7red of 6 as taught by 'mathematism'.



Figure 20. Per-numbers add as areas and fractions add with units, both as integral calculus

Adding like per-numbers is predicted by power where, e.g., 6% 10 times gives $106\%^{10}$ or 179%, i.e., the expected 60% plus additional 19%, and where 6% 20 times gives 321%, i.e., the expected 120% plus additional 201% showing the benefit of pensions.

	B	Ⓕ	
Left	1	2	3
Right	1	1	2
Total	2	3	5

	B	Ⓕ	
Left	1/3	2/3	1
Right	1/2	1/2	1
Total	-	-	-

	B	Ⓕ	
Left	1/2	2/3	-
Right	1/2	1/3	-
Total	1	1	-

Figure 21. In cross tables, per-numbers must pass through the unit-numbers

Looking at my right hand I see 3 fingers to the left, the Ls, and 2 fingers to the right, the Rs. I bent the two outer fingers. So, $1/3$ of the Ls are bent, and $1/2$ of the Rs. Does ‘ $1/3$ of the Ls are bent’ mean that ‘ $1/3$ of the bent are Ls’? No, $1/2$ is. So, in a cross table we cannot go from the per-numbers in one direction to those in the other direction without going through the unit-number table. This is called the Bayes-principle.

Skill building. The action is repeated with other numbers.

End test. 3kg at 4\$/kg and 5 kg at 6\$/kg total what?”

MC18. Adding and subtracting Bundle-Bundle squares

The total here exists as bundle-bundle number squares on a BBBoard. The ‘T=?’ question is answered by using rubber bands or cubes to mark the bundles.

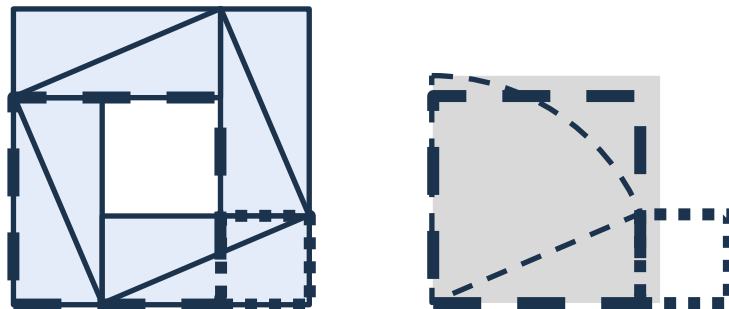


Figure 22. The two squares add as the square formed by the mutual Bottom-Top BT line

On a BBBoard we place four 7 3s so they form a ten-by-ten square that inside contains two squares, 7 7s and 3 3s as well as two stacks. But it also contains one square formed by the diagonals in the stacks as well as four half stacks. So, the two squares add as the square formed by the mutual Bottom-Top BT line thus having the length as the square-root of the sum, i.e., $\sqrt{7^2 + 3^2} = 7.62$.

So, in this stack, adding the height and the bundle as squares gives the square of the diagonal. This rule is named by the ancient Greek thinker, Pythagoras.

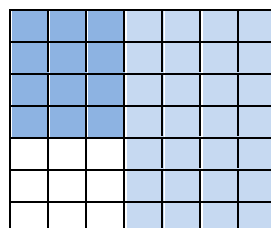


Figure 23. On a BBBoard we see that $7^2 - 3^2 = 7*(7-3) + (7-3)*3 = (7+3)*(7-3)$

On a BBBoard we pull-away the a 3-by-3 square from a 7-by-7 square. This leaves 7 (7-3)s and (7-3) 3s that may be turned to 3 (7-3)s totaling (7+3) (7-3)s.

So $7^2 - 3^2 = (7+3)(7 - 3)$.

Adding squares may also be involved when solving a quadratic equation. On a BBBoard we see that $T = (x+3)(x+3)$ is a square with four parts, two squares x^2 and 3^2 , and two stacks $2 \cdot 3 \cdot x$, so that $T = x^2 + 6 \cdot x + 9$. The quadratic equation $x^2 + 6 \cdot x + 8 = 0$ then makes the whole square go away except for $9-8 = 1$. So $(x+3)^2 = 1$, which gives two solutions, $x = -2$ and $x = -4$ that hold when tested:

$(-2)^2 + 6 \cdot (-2) + 8 = 4 - 12 + 8 = 0$, and $(-4)^2 + 6 \cdot (-4) + 8 = 16 - 24 + 8 = 0$

The quadratic equation ' $x^2 + 6 \cdot x + 10 = 0$ ' has no solutions since here ' $(x + 3)^2 = -1$ '.

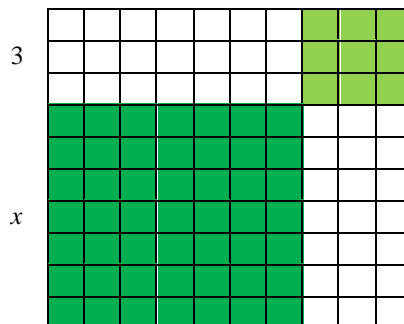


Figure 24. An $(x+3)$ -square contains an x -square and a 3 -square and two $3 \cdot x$ stacks

Alternatively, we can rewrite the equation $x^2 + 6 \cdot x + 8 = 0$, first as

$x^2 + 2 \cdot 3 \cdot x + 3^2 - 3^2 + 8 = 0$, then as

$(x+3)^2 - 9 + 8 = 0$, then as

$(x+3)^2 = 9 - 8 = 1$ again with the solutions $x = -2$ and $x = -4$ that hold when tested:

$(-2)^2 + 6 \cdot (-2) + 8 = 4 - 12 + 8 = 0$ $(-4)^2 + 6 \cdot (-4) + 8 = 16 - 24 + 8 = 0$

Skill building. The action is repeated with other numbers.

End test. Add 3 3s and 4 4s as a bundle-bundle square. Solve the equation $x^2 + 8 \cdot x + 12 = 0$

MC19. Adding unspecified letter-numbers

The total here exists as a rectangular bundle-number on a BBBoard. The 'T=?' question is answered by using rubber bands to mark the bundles.

In the letter-number $T = 3ab$ the multiplication sign is invisible, and the letters stands for unspecified numbers. Since any factor may be a unit, T may be seen as $3 abs$, or as $(3a) bs$, or as $(3b) as$. To avoid being confused by the 's' we will omit it, so $T = 3ab = 3 \cdot ab = 3a \cdot b$ or $3b \cdot a$.

Since totals need a common unit to add, this must be first found:

$T = 3ab + 4ac = 3b \cdot a + 4c \cdot a = (3b+4c) \cdot a$

$T = 2ab^2 + 4bc = ab \cdot 2b + 2c \cdot 2b = (ab+2c) \cdot 2b$

Skill building. The action is repeated with other numbers and letters.

End test. $T = 4ab^2d + 8bcd$

MC20. Change in time

The total here exists as dots on a BBBoard. The 'T=?' question is answered by transferring the results to a squared paper and connect the dots with a curve.

In time, a total grows by being added or multiplied by a number, called addition-growth and multiplying-growth, or linear and exponential growth.

Addition-growth:

Final number = Initial number + growth-number * growth times, or shortly, $T = B + a*n$.

The number a is also called the slope.

Multiplying-growth:

Final number = Initial number * growth-factor ^ growth times, or shortly, $T = B * a^n$, since $200\$+5\% = (200*105\%) \$$, so here a is $1+\text{interest rate} = 100\% + 5\% = 105\%$.

Combined growth (savings in a bank):

Here we have that $A/a = R/r$, where A is end-dollars, a is the period-dollars, R is the end-rate, r is the period-rate, and $1+R = (1+r)^n$, where n is the number of periods.

100% split in n parts will give the Euler number $e = (1+1/n)^n = 2.718$ for n large enough.

Changing the growth-number constantly will give a quadratic growth with a parabola curving upwards or downwards if the number increases or decreases.

Changing the curvature constantly will give cubic growth with a double parabola with curvature and counter-curvature.

Decreasing the growth-factor constantly give logistic saturation growth with a hill-curve in infections. Confusing exponential and saturation growth can cause unnecessary damage.

MC21. Bundle-numbers in a coordinate system

The total here exists as dots on a BBBoard. The 'T=?' question is answered by rubber bands as lines on the BBBoard.

The bundle-number 'y xs' with a height y and a width x may be called a 'changing bundle-number'. Here $y = 2*x$ gives a rising and $y = 9-x$ a falling bundle-number.

Marking the top right corners we get two lines. To inside predict the outside intersection point we equate the two heights, $2*x = 9 - x$. Moving to opposite side with opposite sign we get $3*x = 9$, and $x = 9/3 = 3$, which makes $y = 2*3 = 6$. So, the prediction is that the two bundle-numbers become like as 3 6s, which is validated on the BBBoard where the first dot now is 0 instead of 1.

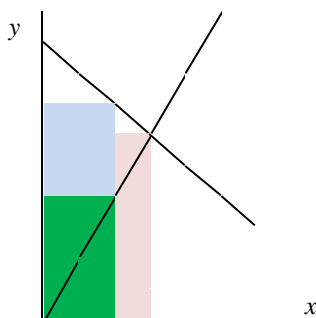


Figure 25. In an x-y coordinate system bundle-number stacks may rise and fall

In a rising bundle-number its total T will increase since here the height increases with increasing width. In the falling bundle-number this is not the case since the height decreases with increasing width. Here the total is $T = y*x = (9-x)*x = 9x - x^2$.

Setting up a table with $x = 0, 1, 2, \dots, 9$ we see that first T increases and then T decreases; and that T tops as 20 for $x = 4$ and $x = 5$; and that $T = 20.25$ for $x = 4.5$.

x	0	1	2	3	4	5	6	7	8	9
y	0	8	14	18	20	20	18	14	8	0

In general, even if a bundle-number is rising, its rise may be falling, so its marked corners will lay on a bending line called a parabola where $y = b*x + a*x^2$.

Passing through the points $(x,y) = (1,6)$ and $(2,10)$ we find that $10 = b*2 + a*4$, and $6 = b*1 + a*1$, or $12 = b*2 + a*2$. We now equate the two equations for $b*2$: $10 - a*4 = 12 - a*2$.

Moving to opposite side with opposite sign we get $10 - 12 = a*4 - a*2$, or $-2 = a*2$, or $-1 = a$. With $6 = b + a$, this gives $b = 7$.

So on the parabola, the points (x,y) are connected by the formula $y = 7*x - x^2$. It thus passes through the points $(0,0)$, $(1,6)$, $(2,10)$, $(3,12)$, $(4,12)$, $(5,10)$, $(6,6)$, and $(7,0)$.

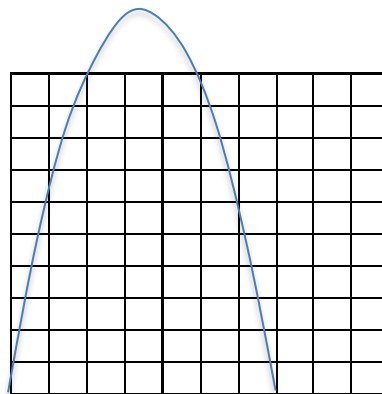


Figure 26. Passing through $(0,0)$ and $(1,6)$ and $(2,10)$, the parabola formula is $y = 7*x - x^2$

Can the intersection points be predicted between the parabola and the two line above?

Can it be predicted that a falling bundle-number $b - a*x$ will have its maximum at the width $b/(2*a)$?

Skill building. The action is repeated with other rising and falling bundle-numbers to find when they are like and when the falling bundle-numbers tops.

End test. $y = 9 - 2*x$ and $y = x$.

MC22. Games Theory and damage control

The total here exists as towers of cubes and dots on a BBBoard. The ‘T=?’ question is answered by rubber bands as lines on the BBBoard.

In a Game Theory 2x2 zero-sum game two players A and B each have 2 strategies resulting in four different payments from B to A. It is called a zero-sum game since one player’s gain is the other player’s loss.

In some game, if B chooses strategy B1 then the payment to A is 8\$ or 2\$ if A chooses strategy A1 or A2. And if B chooses strategy B2 then the payment to A is 4\$ or 6\$ and A chooses strategy A1 or A2. We may show this game by building four towers with cubes.

First, we assume that B chooses strategy B1. If A now p times chooses A2 and $n-p$ times A1 then A's outcome after n rounds will total

$$T = 2*p + 8*(n-p) = 8*n - 6*p = (8*n - 6*p)/n * n = (8 - 6*p/n) * n.$$

Which is $T1 = 8 - 6*p/n$ per round, shown on a BBBoard as a line connecting 8 to the left where p is 0, to 2 to the right where p is n .

Next, we assume that B chooses strategy B2. If A now p times chooses A2 and $n-p$ times A1 then A's outcome after n rounds will total

$$T = 6*p + 4*(n-p) = 4*n + 2*p = (4*n + 2*p)/n * n = (4 + 2*p/n)*n.$$

Which is $T2 = 4 + 2*p/n$ per round, shown on a BBBoard as a line connecting 4 to the left where p is 0, to 6 to the right where p is n .

With p/n as u we find the intersection point by equating the two totals: $T1 = T2$, or $8 - 6*u = 4 + 2*u$, or $8*u = 4 = (4/8)*8$, or $u = 4/8 = 1/2$ giving $T1 = T2 = 5\$$.

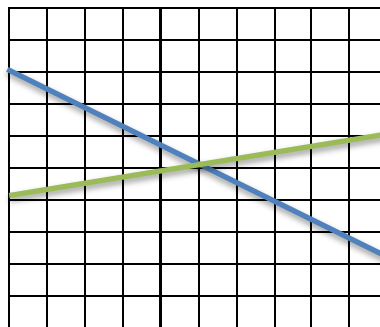


Figure 27. If B choses strategy B1, A will receive between 8 and 2\$, else between 4 and 6\$ So, if A mixes the strategies 1-to-1 by flipping a coin then the average result will be 5\$ per round.

Seen from B's side we also get the two lines $S1 = 4 + 4*u$, and $S2 = 6 - 4*u$ that intersect where $4 + 4*u = 6 - 4*u$, or $8u = 2$, or $u = 2/8$, or $u = 1/4$ giving $S1 = S2 = 5\$$.

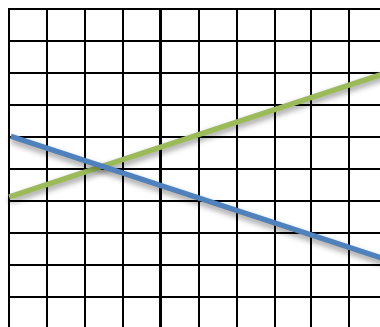


Figure 28. If A chosess strategy A1, A will receive between 4 and 8\$, else between 6 and 4\$ So, if B mixes the strategies 1-to-3 by flipping two coins then the average result again will be 5\$ per round. 5\$ then is called the value of the game, i.e., the amount B must receive per round to make the game fair with no winner or loser in the long run.

From A's side the 5\$ is called the 'maxi-min' value since deviating from it will decrease the value. From B's side the 5\$ is called the 'mini-max' value since deviating from it will increase the value.

In a similar game 4\$ is changed to 8\$. Here the strategy A1 dominates A2 that will always be lower to A. Likewise, the strategy B1 dominates B2 that will always be higher to B. So, here

the value of the game is 6. This point is called a saddle point since the payment goes up one way and down the other.

Skill building. The game is repeated with other payments found, e.g., by rolling some dice.

End test. Replace the four payments 8,2,4,6 with 9,3,5,8.

MC23. Simple board games

The goal is to experience how different games may take place on a BBBoard.

- A race track. A 4x4 lawn is placed in the middle of a BBBoard. The start- and end-line goes from (5,0) to (5,3). A trip may change zero or one unit in the horizontal and vertical direction. You may touch but not cross the interior or exterior boundaries. If you do so you restart with 0 speed as close to the crossing point as possible. You may cross your opponent's track, but not end in the same point. The race may be repeated with different lawn shapes.
- Survival. You begin in (5,5). You roll a dice and move the number to the right if the number is even, else to the left. You roll again and now move the number up if the number is even, else down. You may touch but not cross the boundary. How many steps can you survive?
- Vertical race. A rubber band splits a BBBoard vertically in the middle. Two players each have three bricks placed at level one. They roll a dice and pick a brick to move the number upwards, and down again if there is a surplus. The winner is the first to have all three bricks at level ten.

The Algebra Square

There are two kinds of numbers in the world, unit-numbers, and per-numbers, which may be unlike or like and which may be reunited. The aim of mathematics is therefore not to 'math', because you can't do that, but to act: 'Re-Unite Un-like & Like Unit-Numbers & Per-Numbers'.

The four operations that unite unlike and like unit- and per-numbers are: addition, multiplication, integration, and power as shown in the Algebra-square above that also includes the ways to split a total: subtraction, division, differentiation, as well as the factor-finding root and the factor-counting logarithm.

Operations unite/ <i>split Totals in</i>	Unlike	Like
Unit-numbers m, s, kg, \$	$T = a + n$ $T - n = a$	$T = a * n$ $T/n = a$
Per-numbers m/s, \$/100\$ = %	$T = \int f dx$ $dT/dx = f$	$T = a^b$ $b\sqrt[T]{a} = a$ $\log_a(T) = b$

Figure 29. The Algebra square shows how to unite and split our four number-types, and how to solve equations by moving 'to opposite side with opposite sign'.

Fact and fiction and fake, the three genres of number-models

Once we know how to count and recount totals, and how to unite and split the four number-types, unlike and like unit-numbers and per-numbers, we can actively use this number-language to produce inside tales about outside totals existing in space and time. This is called modeling.

As in the word-language, number-language tales also come in three genres: fact, fiction, and fake models that are also called since-then, if-then and what-then models, or room, rate, and risk models.

Fact stories are ‘since-then’ stories that quantify and predict predictable quantities by using factual numbers and formulas. Typically, they model the past and the present. They need to be checked for correctness and units.

Fiction stories are ‘if-then’ stories that quantify and predict unpredictable quantities by using assumed numbers and formulas. Typically, they model the future. They need to be supplied with scenarios building on alternative assumptions.

Fake stories are ‘what-then’ stories that quantify and predict unpredictable qualities by using fake numbers and formulas. Typically, they add without units or hide alternatives. Here, number stories need to be replaced by word stories.

Modeling and de-modeling

The goal is to experience how formulas calculating y from x form curves that expresses change in time, and how totals in space may be split in parts that each then becoming a percentage of the total.

Modeling means solving an outside problem inside with four steps. First an outside problem is translated to an inside problem. Then the inside problem leads to an inside solution that then is translated to an outside solution, that finally is evaluated to see if another cycle is needed.

A typical example is mixture problems. the outside problems here may ask “2kg at 3\$ per kg and 4kg at 5\$ per kg total what?” The inside problem places the second information under the first ready to add. The inside solution may then simply add all numbers, which leads to the outside solution “2kg at 3\$ per kg and 4kg at 5\$ per kg total 6 kg at 8\$/kg”. This model is not accepted, so another cycle is needed. This time the per-numbers are multiplied to unit numbers before adding, which leads to the outside solution “2kg at 3\$ per kg and 4kg at 5\$ per kg total 6 kg at 26\$/6kg”. This model is accepted.

De-modeling is the opposite process: It means solving an inside problem outside with four steps. First an inside problem is translated to an outside problem, then the outside problem leads to an outside solution that then is translated to an inside solution, that finally is evaluated to see if another cycle is needed.

A typical example is uniting fractions.

Adding fractions as $1/2 + 2/3$ only has meaning when taken of the same unit,

$$u = (u/6)*6 = k*6, \text{ where } k = u/6, \text{ and } 6 = 2*3$$

$$T = (1/2 + 2/3)*u = (1/2 + 2/3)*6*k = (3+4)*k = 7*k = 7*u/6 = 7/6*u,$$

So, in this case, $1/2 + 2/3 = 7/6$.

- We now model the orbit of a ball sent away with an angle. A constant up-number will give a line that goes up or down or horizontal. But here gravity makes the up-number decrease so the line curves down as a bended line called a parabola.

We choose the initial angle A determined by $\tan A = 6$.

From $(0,0)$ we assume that the ball takes a ‘1 out, 5 up’ step followed by a ‘1 out, 3 up’ and a ‘1 out, 1 up’, etc., to reach the points $(1,5), (2,8), (3, 9), (4,8), (5, 5), (6,0)$.

Since $y = 0$ for $x = 0$ and for $x = 6$, the formula may contain the two factors $(x-0)$ and $(6-x)$, so a guess could be $y = a*x*(6-x)$.

In the point $(1,5)$ this formula becomes an equation,

$5 = a \cdot 1 \cdot (6-1)$, or $5 = a \cdot 5$, solved by $a = 1$.

So, the parabola formula may be $y = 1 \cdot x \cdot (6-x)$, or

$$y = -x^2 + 6x.$$

This formula holds when tested on the other points:

$$8 = -2^2 + 6 \cdot 2, \text{ or } 8 = -4 + 12, \text{ or } 8 = 8, \text{ etc.}$$

We find that with 4 as the first up-number, the orbit formula will be $y = -x^2 + 5x$, etc.

The height after 5 steps is found by the equation

$$y = -5^2 + 6 \cdot 5 = -25 + 30 = 5.$$

The height 8 is reached after x steps found by the equation

$$8 = -x^2 + 6x, \text{ or } x^2 - 6x + 8 = 0, \text{ solved by } x = 2 \text{ and } x = 4.$$

The height 10 is never reached since there are no solutions to the equation:

$$10 = -x^2 + 6x, \text{ or } x^2 - 6x + 10 = 0.$$

Instead, that top-point is found in the middle at $x = 6/2 = 3$, giving

$$y = -3^2 + 6 \cdot 3, \text{ or } y = 9.$$

To see if it breaks through a roof with the formula $y = 12-x$, we equate the two y s and get the equation

$$12 - x = -x^2 + 6x, \text{ or } x^2 - 7x + 12 = 0, \text{ that is solved for } x = 3 \text{ and } x = 4.$$

- We now model the beginning monthly income of a business trying to establish itself at a market. We use the formula $y = x^2 - 6x + 9$ where the steps form a parabola curving up when passing the points (0,9), (1,4), (2, 1), (3,0), (4,1), (5,4), and (6,9).

Later the monthly income will change its curvature from up to down until it reaches a maximum level. So, from $x = 3$ we use a different model that contains the up-numbers 0, 1, 2, 3, 2, 1, 0, 0. This gives a 'logistical' s-shaped curve describing growth with saturation. When prompted, AI may give a formula for this curve as $y = 9 / (1 + 25 \cdot 2^{(-1.9 \cdot x)})$

We see that the up-numbers form a hill. When prompted, AI may give a formula for this curve as $y = 3 / (2^{(0.44 \cdot (x-3)^2)})$

- A cats and mice cohabitation on an island is an example of a predator-prey model where cats eat mice. We expect a cycle in time since many cats and many mice leads to many cats and few mice, which leads to few cats and few mice, which leads to few cats and many mice, which leads to many cats and many mice once again.

In a model we assume that a mice-population at 7 and 2 will make the cat-population change with 7-5 and 2-5 respectively. Likewise, a cat-population at 7 and 2 will make the mice-population change with 5-7 and 5-2 respectively. We see that initial populations at the level 5 will give a stable model. Here we assume that the initial populations for the cats and the mice are 8 and 1 respectively. The following period the two populations will then be $8 + (1-5) = 4$, and $1 + (5-4) = 2$ respectively.

Continuing, we see that the cat population will change as 8, 4, 1, 2, 6, 9, 8; and that the mice population will change as 1, 2, 6, 9, 8, 4, 1. This allows the points (8,1), (4,2), etc., to be marked on a BBBoard, showing a cycle continuing again and again. Different initial numbers will give different cycles.

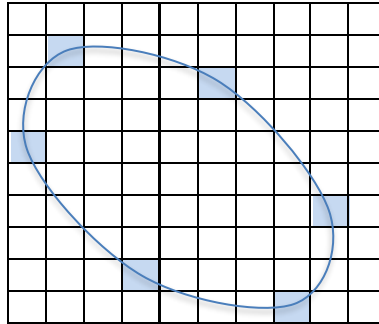


Figure 30. Cats eat mice, so the mice-number decreases, so the cat-number decreases, so the mice-number increases, so the cat number increases, so the mice-number decreases, etc.

- We now model cyclic movements up and down as observed in nature with day and night, with summer and winter, and with tide in an ocean. A cyclic movement may be created by the up-numbers +2, +1, +0, -1, -2, -2, -1, +0, +1 +2.

Beginning at the point (0,5), AI may be prompted to give a formula for this curve as

$$y = 5 + 3 \cdot \sin(0.63 \cdot x).$$

- Saving money may take place at home or in a bank. At home the terminal capital c after n months will be $c = b + a \cdot n$, where b is the initial capital, and a is the change-number per month. In a bank, the terminal capital after n months will be $c = b \cdot (1+r)^n$ where r is the change-percent per month.

Combining the two in a bank, the terminal capital C may be found by the formula $C/a = R/r$ where R is the total interest rate including the compound interest, $1+R = (1+r)^n$. This capital may be used as an installment plan to pay out a debt D that has grown to $E = D \cdot (1+R)$ in the same period.

- If an interest rate at 100% is split in 12 portions the total interest is found from the equation $1+R = (1+1/12)^{12} = 2.613$ so that $R = 1.613 = 161.3\% = 100\%$ plus 61.3% as additional compound interest. This leads to the Euler number $e = (1+1/n)^n = 2.7183$ for n large, which shows that the additional compound interest cannot surpass 71.8% when splitting up 100%.
- Biological populations typically grow exponentially with a constant periodical rate, which provides a constant doubling time. This may be shown on a BBBoard where the vertical numbers are in tens. Beginning with $\frac{1}{4}$, a doubling sequence will be

$\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32, 64, 128.$

Alternatively, a capital may decrease by always taking away half of what is left. Beginning with 8 this gives a halving sequence $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}.$

This exponential decay may be recognized as a mirror of the exponential growth.

- On a BBBoard showing 6 7s, a triangle is formed by the three lines connecting the points (0,0) and (7,10) and (10,7). Typically, we want to find the 7 important triangle numbers, its area, its three angles and its three sides.

We see that these 7 numbers may be found indirectly by looking at the three half rectangles that is pulled away from the triangle's wrapping rectangle.

In the lower pull-away half-rectangle we see that the angle is predicted by the formula $\tan A = 6/10$, which on a calculator gives $A = 31.0$ degrees. And that the area is $\frac{1}{2} \cdot 6 \cdot 10 = 30$. And that the length of the diagonal d is found by squaring: $d^2 = 10^2 + 6^2 = 136$, giving $d = \sqrt{136} = 11.7$.

- On a BBBoard, twice rolling 2 dices may suggest we go to the two points (3,6) and (4,3) that then constitute one side in a square. We now may find the area of the square and the intersection point of the two diagonals. We notice that in the slopes of the sides the out- and up-number change places, and one changes the sign also.
- Optimizing income under constraints (also called ‘Linear Programming’). At a fair, a class sells caps and shirts. They may buy at most 6 boxes with caps and 4 boxes with shirts that each cost 1 unit. Their budget is 8 units, and their income is 1 unit per shirt-box and 2 units per cap-box. How can they optimize the income?

On a BBBoard a horizontal and a vertical rubber band shows the limit on the shirts and on the caps. A line connecting (0,8) and (8,0) shows the budget-line not to be passed. A line connecting (0,10) and (5,0) shows the 10 unit income-line that is moved to the right until (6,2) where the first constraint will be violated. So the calls should buy 6 boxes with caps and 2 boxes with shirts, which will give them an income at $2*6 + 1*2$ or 14 units.

Three footnotes

The total here exists as rectangular bundle-number on a BBBoard.

The goal is to experience the content of three calculation laws.

The commutative law: The order does not matter, $a*b = b*a$

The distributive law: When adding, like units may be bracket out, $a*c + b*c = (a + b)*c$

The associative law: Bracket may be moved at will, $a*(b*c) = (a*b)*c$

On a BBBoard two rubber bands mark 6 3s. Turning the board a quarter round we have 3 6s thus illustrating that $6*3 = 3*6$.

A third rubber band split the 6 3s in 4 3s and 2 3s to illustrate that $4*3 + 2*3 = (4+2)*3$.

With cubes 2 3s 4 times gives a Total of $(2*3)*4$. Turning it over, twice we have 3 4s, thus illustrating that $(2*3)*4 = 2*(3*4)$,

Teacher education

The MATHeCADEMY.net is designed to provide material for pre- and in-service teacher education using PYRAMIDeDUCATION allowing professional development to take place on the internet in self-controlling groups with eight participants validating internal predicates by asking the outside subject itself instead of an instructor. This allows mastery of Many with ManyMath to be tested and developed worldwide in small scale design studies ready to be enlarged.

The MATHeCADEMY.net offers a free one-year in-service distance education course in the CATS approach to mathematics, Count & Add in Time & Space. C1, A1, T1 and S1 is for the primary school, and C2, A2, T2 and S2 is for the secondary school. Furthermore, there is a study unit in quantitative literature. The course is organized as PYRAMIDeDUCATION where 8 teachers form 2 teams of 4 choosing 3 pairs and 2 instructors by turn. An external coach helps the instructors instructing the rest of their team. Each pair works together to solve count&add problems and routine problems; and to carry out an educational task to be reported in an essay rich on observations of examples of cognition, both re-cognition and new cognition, i.e., both assimilation and accommodation. The coach assists the instructors in correcting the count&add assignments. In a pair, each teacher corrects the other’s routine-assignment. Each pair is the opponent on the essay of another pair. Each teacher pays for the education by coaching a new group of 8 teachers.

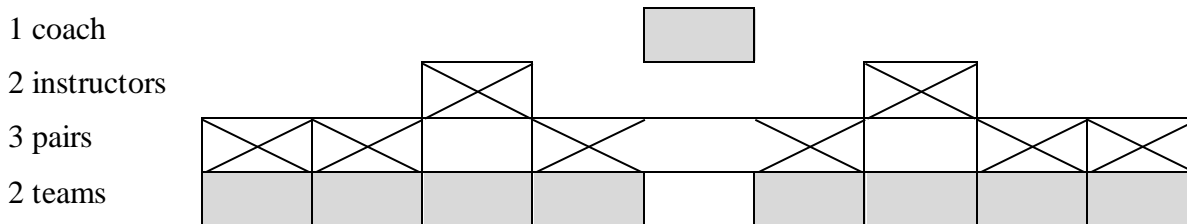


Figure 31. PYRAMIDeDUCATION with 2 teams as 3 pairs and 2 instructors, plus a coach

The material for primary and secondary school has a short question-and-answer format. The question could be: How to count Many? How to recount 8 in 3s? How to count in standard bundles? The corresponding answers would be: By bundling and stacking the total T predicted by $T = (T/B)*B$. So, $T = 8 = (8/3)*3 = 2*3 + 2 = 2*3 + 2/3*3 = 2 \frac{2}{3}*3 = 2B \frac{2}{3} 3s$. Bundling bundles gives a multiple stack, a stock or polynomial: $T = 456 = 4\text{BundleBundle} + 5\text{Bundle} + 6 = 4\text{tente}5\text{ten}6 = 4*B^2 + 5*B + 6*1$. Additional material may be found as MrAlTarp YouTube videos.

How different is the difference?

Digits now are no longer symbols as letters, but icons with as many sticks as they represent. 3 now is called '1B0 3s' or '0B3 tens'. Ten, eleven and twelve now are also called 'one-bundle-zero', 'one-bundle-one', and 'one-bundle two'. And hundred and thousand are also called 'bundle-bundle' and 'bundle-bundle-bundle'.

Multidigit numbers no longer occur without units since with units, 23 now is 2B3 thus making the place value system unneeded. Calculations with overloads and underloads give bundle-numbers with units a flexibility that makes carrying and borrowing unneeded, e.g., $46+37 = 4B6+3B7 = 7B13 = 8B3 = 83$. And $86 - 37 = 8B6 - 3B7 = 5B-1 = 4B9 = 49$.

Addition now depends on the units so $2+3$ is not 5 by necessity. $2\text{weeks} + 3\text{weeks} = 5\text{weeks}$, but $2\text{weeks} + 3\text{days} = 17\text{days}$. So, without a unit, 3 does not exist, only with a bundle-unit as, e.g., 0B3 tens, or 1B0 3s, or 1B1 2s, or 1B-1 4s, or 1B-2 5s, etc. So, to add, 2 and 3 must have the same unit, e.g., ' $2+3$ ' = $(1B0 + 1B1) 2s = 2B1 2s$, or ' $2+3$ ' = $(1B-1 + 1B0) 3s = 2B-1 3s = 1B2 3s$. Likewise with subtraction ' $9-6$ ' = $(1B3 - 1B0) 6s = 0B3 6s = '3'$, or ' $9-6$ ' = $(1B0 - 1B-3) 9s = 0B--3 9s = 0B3 9s$, showing that minus times minus must be plus.

Also, addition now is not well-defined since 2 3s and 4 5s may be added both on-top after a recounting has made the units like, or next-to by areas as integral calculus.

Multiplication now carries units automatically, and $6*8$ is not 48 by necessity. Instead, $6*8$ exists as 6 8s that may or may not be recounted in another unit, e.g., in 9s or in tens: 6 8s is 5B3 9s, and 4B8 tens. Division now is different, since $8/2$ has different meanings in time and space by meaning '8 split in 2 in time', but '8 split in 2s in space' when recounting 8 in 2s.

Solving equations now is different. The equation ' $u*2 = 8$ ' asks "How many 2s in 8?" which of course is found by recounting 8 in 2s as $8 = (8/2)*2$, so that the solution is $u = 8/2$ that is found by 'moving to opposite side with opposite sign', which follows the formal definition: $8/2$ is the number u that multiplied with 2 gives 8, if $u*2 = 8$ then $u = 8/2$. So, the balancing method now is unneeded. Thus, no longer equations are seen as two equivalent numbers-names that remain equivalent if the same operation is performed on both. And no longer are they transformed by using the communicative, associative, and distributive law; or the two abstract concepts, 1 over 2 as the inverse element to 2, and 0 and 1 as the neutral elements. And we no longer use the neutralizing 'do the same to both sides' weight-method to solve the equation $2*x = 8$ saying: $2*x = 8$; $(2*x)*\frac{1}{2} = 8*\frac{1}{2}$; $(x*2)*\frac{1}{2} = 4$; $x*(2*\frac{1}{2}) = 4$; $x*1 = 4$; $x = 4$

The multiple calculation $2+3*4$ no longer is 14 by definition or by the ‘PEMDAS’ rule. With units, $2+3*4$ exists as 2 1s + 3 4s which is (0B2 + 3B0) 4s or 3B2 4s, or 1B4 tens.

The letter-calculation ‘ $2*a + 3*a = (2+3)*a$ ’ no longer is an example of a distributive law, but an example of having like units.

Proportionality no longer ‘go over one’, instead a per-number links the two units by recounting: with 4\$ per 5kg, or $4\$/5\text{kg}$, $16\$ = (16/4)*4\$ = (16/4)*5\text{kg} = 20 \text{ kg}$.

Fractions no longer are numbers by themselves, instead they are per-number with like units, $3\text{meter}/4\text{meter} = 3/4$, $3 \text{ meter}/100\text{meter} = 3/100 = 3\%$. So finally, per-numbers are accepted along with fractions.

Without units, digits, per-numbers, and fractions are not numbers, but operators needing a number to become a number. So, fractions also need units to add: 1 red of two apples plus 2 red of 3 apples total (1+2) red of (3+4) apples, i.e., $1/2 + 2/3 = (1+2)/(2+3) = 3/5$ in this case, and not 7 red of 6 apples as mathematism teaches.

Trigonometry no longer must wait to after plane and coordinate geometry, since it occurs when mutually recounting the sides in a stack split by its diagonal.

Differential calculus no longer precedes integral calculus since the latter answers the core questions: how to add stacks in grade one, and how to add piecewise and locally constant per-numbers in mixture problems in middle school and high school.

Solving a quadratic equation no longer must wait to secondary school since Bundle-Bundles are squares that lead directly to the question ‘how to square a rectangle’ that provides a double split square containing the three parts of a quadratic equation.

The simplicity of the Algebra Square will no longer be hidden. And no longer will models be seen as mere approximations but as tales with three genres, fact and fiction and fake.

Overview over the differences between Essence- and Existence-math

	Essence-math, mathematism	Existence-math, Many-math
Digits	Symbols	Icons
345	Place value system	$T = 3BB \ 4B \ 5$, $BB = B^2$, $BBB = B^3$
Operations	Functions, order: + - x / ^	Icons, opposite order: ^ / x - +
$3 + 4$	$3 + 4 = 7$	Meaningless without units
$3 * 4$	$3 * 4 = 12$	$3*4 = 3 \ 4\text{s}$, may be recounted to 1.2 tens
$9 = ? \ 2\text{s}$	Meaningless, only ten-counting	$9 = 3B3 = 5B-2 = 4B1 = 4\frac{1}{2} \ 2\text{s}$
$8 = ? \ 2\text{s}$	Meaningless, only ten-counting	$8 = (8/2)*2$, $T = (T/B)*B$, proportionality
$2*u = 8$	$(2*u)^{1/2} = 8^{1/2}$, so $(u*2)^{1/2} = 4$, so $u*(2^{1/2}) = 4$, so $u*1 = 4$, so $u = 4$	$2*u = 8 = (8/2)*2$, så $u = 8/2$
$6*7 = ?$	eh 44? eh 52? eh 42? OK	$6*7 = (B-4)*(B-3) = (10-4-3)*B + 4*3 = 3B12 = 4B2 = 42$
$4\text{kg}=5\$, 6\text{kg}=?$	$1\text{kg} = 5/4\$, 6\text{kg} = 6*5/4\$$	$6\text{kg} = (6/4)*4\text{kg} = (6/4)*5\$$
$1/2 + 2/3 = ?$	$1/2 + 2/3 = 3/6 + 4/6 = 7/6$	$1/2*2 + 2/3*3 = 3/5*5$
$2 \ 3\text{s} + 4 \ 5\text{s}$	$2*3+4*5 = 6+4*5 = 10*5 ?$	$2*3+4*5 = 3B2 \ 8\text{s}$ or $5B1 \ 5\text{s}$ (next-to, on-top)
$6 + 9 = ?$	$6 + 9 = 15$	$2B3 \ 6\text{s}$ or $2B-3 \ 9\text{s}$ or $2 \frac{1}{2}B(1+4) = 1B5 \ \text{tens}$
Tangent = ?	tan = sine/cosine	raise = (raise/run)*run, tan = raise/run

Conclusion

We began by observing the difference between ‘mathematism’ adding without units and therefore always true inside but seldom outside, and ‘Many-math’ instead using bundle-numbers with units inspired by how the uneducated child sees the outside existing fact Many. We then explored the consequences of letting existence come before essence by letting counting and recounting come before adding. Finally, we formulated a sequence of micro-curricula on how Many-math may be learned in school by working with things and actions on a 2dimensional Bundle-Bundle-Board. But, will this allow the learner to learn mathematics or even to be numerate as wished by the UN Sustainable Development Goals?

Apparently, different definitions of ‘numerate’ exist where existence and essence have different order. The English Oxford Dictionary defines it as being “competent in the basic principles of mathematics, esp. arithmetic”. In contrast, the American Merriam-Webster dictionary defines it as “having the ability to understand and work with numbers.”

In their common history, England once colonized America. So, the difference in the definitions is interesting. The former uses the passive term ‘being’ where the latter uses the active term ‘having’. The former connects the definition to the inside essence of mathematics while the latter connects it directly to the outside existence of numbers. The choice thus is: shall existence precede essence as philosophical Existentialism holds, or shall essence be allowed to colonize existence with a ‘no-unit regime’ to use a Foucault-phrase?

Maybe it is time to see if children stay numerate if their own 2D bundle-numbers with units are not colonized by 1D line-numbers without units. Maybe it is finally time for a Kuhnian paradigm in shift in number-language education. Therefore, think things. Or, in the Viking version: “Derfor, tænk ting”.

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Appendix

Unit-number tasks

Type1.1 Numbers

Two numbers have the sum 72, and one is twice as large as the other. What are the numbers?

Text	Numbers	ANSWER	Equation
Number1	$x = ?$	24	$x + y = 72$
Number2	$y = 2 * x$	48	$x + 2 * x = 72$ $3 * x = 72$ $x = 72 / 3 = 24$

Type1.2 Money

A pays a bill of 210\$ with three types of coins: 1s, 2s and 5s. There are 4 times as many 1s as 2s, and 20 fewer 2s than 5s. How many coins of each type were used?

Text	Numbers	ANSWER	Equation
5s	$x = ?$	30	$x * 5 + (x - 20) * 2 + 4 * (x - 20) * 1 = 210$
2s	$x - 20$	10	$5 * x + 2 * x - 40 + 4 * x - 80 = 210$
1s	$4 * (x - 20)$	40	$11 * x = 210 + 120$ $x = 330 / 11$ $x = 30$

Type1.3 Age

A is 4 times as old as B. 5 years ago, A was 7 times as old as B. How old are A and B now?

Text	Numbers	ANSWER	Equation
B's age now	$x = ?$	10	$7 * (x - 5) = 4 * x - 5$
A's age now	$4 * x$	40	$7 * x - 35 = 4 * x - 5$
B's age then	$x - 5$		$7 * x - 4 * x = -5 + 35$
A's age then	$4 * x - 5$		$3 * x = 30$ $x = 30 / 3$ $x = 10$

Type1.4 Geometry

A rectangle has a circumference of 224 meters. The length is 4 meters shorter than 3 times the width. What is length and width?

Text	Numbers	ANSWER	Equation
Width	$x = ?$ meters	29	$2 * x + 2 * (3 * x - 4) = 224$
Length	$3 * x - 4$ meters	83	$2 * x + 6 * x - 8 = 224$ $8 * x = 224 + 8$ $x = 232 / 8$ $x = 29$

Type1.5 Lever

A, B and C settle on a seesaw, B and C on the same side. They weigh 100kg, 80kg and 40kg respectively. A and B both sit 3 meters from the focal point. Where must C sit for equilibrium?

Text	Numbers	ANSWER	Equation
C's meter-tal	$x = ?$	1.5	$100 * 3 = 80 * 3 + 40 * x$
A's contribution	$100 * 3$		$300 = 240 + 40 * x$
B's contribution	$80 * 3$		$300 - 240 = 40 * x$
C's contribution	$40 * x$		$60 / 40 = x$ $1.5 = x$

Tasks.

- Two numbers have the sum 48, and one is twice as large as the other. What numbers are they?
- Two numbers have the sum 48, and one is three times as large as the other. What numbers are they?
- A pays a bill of 290 kr. with three types of coins: 1ere, 2ere and 5ere. There are 5 times as many 1s as 2s, and 10 fewer 2s than 5s. How many coins of each type were used?
- A pays a bill of 200 kr. with three types of coins: 1ere, 2ere and 5ere. There are 3 times as many 1s as 2s, and 20 more 2s than 5s. How many coins of each type were used?
- A is 5 times as old as B. 4 years ago, A was 6 times as old as B. How old are A and B now?
- A is 8 times as old as B. 5 years ago, A was 9 times as old as B. How old are A and B now?
- The circumference of a rectangle is 128 meters. The length is 4 meters longer than 5 times the width. What is length and width?
- The perimeter of a rectangle is 110 meters. The length is 5 meters shorter than 4 times the width. What is length and width?
- A, B and C settle on a seesaw, B and C on the same side. They weigh 120kg, 60kg and 50kg respectively. A and B both sit 4 meters from the focal point. Where must C sit for equilibrium?
- A, B and C settle on a seesaw, B and C on the same side. They weigh 90kg, 70kg and 20kg respectively. A and B both sit 2 meters from the focal point. Where must C sit for equilibrium?

Per-number tasks

In per-number tasks, they must always be converted to unit-numbers before the equation can be established.

Travel

Train1 runs from A to B at a speed of 40 km/h. Two hours later, train2 runs from A to B at a speed of 60 km/h. When does train2 overtake train 1?

Text	Per-number	Unit numbers	ANSWER	Equation
Hours		$x = ?$	4	$40*(x+2) = 60*x$
Speed1	40 km/h			$40*x + 80 = 60*x$
Speed2	60 km/h			$80 = 60*x - 40*x = 20*x$
Km-number1		$40*(x+2)$ km	240	$80/20 = x$
Km-number2		$60*x$ km	240	$4 = x$

Train1 runs from A to B at a speed of 40 km/h. At the same time, train2 runs from B to A at a speed of 60 km/h. When do the two trains meet when the distance from A to B is 300 km?

Text	Per-number	Unit numbers	ANSWER	Equation
Hours		$x = ?$	4	$40*x + 60*x = 300$
Speed1	40 km/h			$100*x = 300*x$
Speed2	60 km/h			$x = 300/100$
Km-number1		$40*x$ km	120	$x = 3$
Km-number2		$60*x$ km	180	

The same distance takes 3 hours upstream, and 2 hours downstream. What is the speed of the motorboat when the speed of the current is 5 km/h?

Text	Per-num.	Unit numbers	ANSWER	Equation
Speed	$x = ?$ km/h		25	$km = km/h * h = (x-5)*3 = (x+5)*2$
Speed upstream	$x - 5$ km/h		20	$3*x - 15 = 2*x + 10$
Speed downstream	$x + 5$ km/h		30	$3*x - 2*x = 10 + 15$
Hours		3 hours		$x = 25$

Mixture

? Liter 40% alcohol + 3 liters 20% alcohol gives ? liter 32% alcohol

Text	Per-number	Unit numbers	ANSWER	Equation
Liters		$x = ?$ liters	4.5	$0.4*x + 0.2*3 = 0.32*(x+3)$
Liters-number3		$x+3$ liters	7.5	$0.4*x + 0.6 = 0.32*x + 0.96$
Alcohol1	40%	$0.4*x$ liters		$0.4*x - 0.32*x = 0.96 - 0.6$
Alcohol2	20%	$0.2*3$ liters		$0.08*x = 0.36$
Alcohol3	32%	$0.32*(x+3)$	liters	$x = 0.36/0.08 = 4.5$

Finance

A invests a gain of 400,000\$ in the following way: Some is set at a rate of return at 3% p.a., the rest is put into 8% bonds. How much did he invest in each when the annual dividend is \$20,000?

Text	Per-num	Unit numbers	ANSWER	Equation
Bank in a thousand		$x = ?$ \$	240	$3%*x + 8%*(400-x) = 20$
Bonds in thousands		$x+3$ \$	160	$0.03*x + 32 - 0.08*x = 20$
Interest rate in bank	3%			$32 - 20 = 0.08*x - 0.03*x$
Interest rate on bonds	8%			$12 = 0.05*x$
The Bank's contribution		$3%*x$ \$		$12/0.05 = x$
Bonds' contribution		$8%*(400-x)$ \$	240	$= x$

Work

A can dig a trench in 4 hours. B can dig the same trench in 3 hours. How long does it take to dig it together?

Text	Per-number	Unit numbers	ANSWER	Equation
Time		$x = ?$ hours	12/7	$1/4*x + 1/3*x = 1$
A's speed	1/4 trench/h			$(1/4 + 1/3)*x = 1$
B's speed	1/3 trench/h			$7/12*x = 1$
A contributes		$1/4*x$		$x = 12/7$
B contributes		$1/3*x$		

Tasks.

- Train1 runs from A to B at a speed of 50 km/h. Three hours later, train2 runs from A to B at a speed of 60 km/h. When does train2 overtake train 1?
- Train1 runs from A to B at a speed of 50 km/h. At the same time, train2 runs from B to A at a speed of 60 km/h. When do the two trains meet when the distance from A to B is 400 km?
- Same distance 4 hours countercurrent, and 3 hours downstream. What is the speed of the motorboat when the speed of the current is 6 km/h?
- A can dig a trench in 5 hours. B can dig the same trench in 4 hours. How long does it take to dig it together?
- A can dig a trench in 6 hours. B can dig the same trench in 3 hours. How long does it take to dig it together?

Mechanics

M1. A ball falls from the top of a skyscraper (air resistance is disregarded). After 0 seconds, the ball is at an altitude of 300 meters. After 5 seconds the ball is in ? meters height. After? seconds is the ball at a height of 0 meters. What is the impact velocity?.

Height after 5 sec:

Time:

Speed:

s = ? meter	$s = \frac{1}{2} * g * t^2$	t = ? sek.	$s = \frac{1}{2} * g * t^2$	v = ? m/s	$v = g * t$
t = 5 sek.	$s = \frac{1}{2} * 9.8 * 5^2$	s = 300 m	$2 * s / g = t^2$	t = 7.82 sek.	$v = 9.8 * 7.82$
g = 9.8 m/s ²	s = 123.7 meter	g = 9.8 m/s ²	$\sqrt{(2 * s / g)} = t$	g = 9.8 m/s ²	v = 76.6 m/s
Height = ?	H = 300 - 123.7 H = 177.3 m		$\sqrt{(2 * 300 / 9.8)} = t$ 7.82 seconds = t		

M2. A ball is shot vertically up at an initial velocity of 30 m/s (air resistance is disregarded). After 5 seconds the ball is in ? meters height. After? seconds is the ball at a height of 40 meters. After? seconds is the ball at maximum height?

Height after 5 sec.

Time to 40 m:

s = ? meter	$s = \frac{1}{2} * g * t^2 + v_0 * t$	t = ? sek.	$s = \frac{1}{2} * g * t^2 + v_0 * t$
t = 5 sek.	$s = -\frac{1}{2} * 9.8 * 5^2 + 30 * 5$	s = 40 m.	$40 = -4.9 * t^2 + 30 * t$
g = -9.8 m/s ²	s = 27.5 meter	g = -9.8 m/s ²	$4.9 * t^2 - 30 * t + 40 = 0$
v ₀ = 30 m/s		v ₀ = 30 m/s	t = 1.96 and 4.16 seconds

Rising time until speed = 0

Rising height:

t = ? sek.	$v = g * t + v_0$	s = ? meter	$s = \frac{1}{2} * g * t^2 + v_0 * t$
v = 0 m/s	$(v - v_0) / g = t$	t = 3.1 sek.	$s = -\frac{1}{2} * 9.8 * 3.1^2 + 30 * 3.1$
g = -9.8 m/s ²	$(0 - 30) / (-9.8) = t$	g = -9.8 m/s ²	s = 45.9 meter
v ₀ = 30 m/s	3.1 seconds = t	v ₀ = 30 m/s	

The height part of the task can also be counted as a task in the conversion of energy from kinetic to potential energy.

Rising height	h = ? meters	Ep = Ek
Rising time	t = 3.1 sek.	$m * g * h = \frac{1}{2} * m * v^2$
Acceleration	g = -9.8 m/s ²	$h = \frac{1}{2} * v^2 / g$
Initial speed	v ₀ = 30 m/s	$h = \frac{1}{2} * 30^2 / 9.82$
Kinetic energy	Ek = $\frac{1}{2} * m * v^2$	h = 45.8 meters
Potential energy	Ep = m * g * h	

M3. A 100 kg person performs a Bounty jump from a bridge (air resistance is disregarded). It is 220 meters down. The feet are fixed in a rope of 120 meters, which is fixed in a spring with spring constant k = 100 N/m, corresponding to 10 kg being able to extend the spring 1 m. How far does the person get down? What if the person weighed 150 kg?

Spring dislocation	x = ? metre	Ef = Eb
Fall distance	d = 120 + x	$\frac{1}{2} * k * x^2 = m * g * h$
Acceleration	g = -9.8 m/s ²	$x^2 = 2 * m * g * h / k$
Kinetic energy	Ek = $\frac{1}{2} * m * v^2$	$x = \sqrt{(2 * m * g * h / k)}$
Potential energy	Ep = m * g * h	$x = \sqrt{(2 * 100 * 9.82 * 120 / 100)}$
Spring energy	If = $\frac{1}{2} * k * x^2$	x = 48.5
		d = 120 + 48.5 = 168.5 meters

M4. A person swings in a swing (air resistance is disregarded). The swing set is 4 m high and cord length is 3 m. What is the oscillation time? In the extreme position, the fluctuation is 50 degrees. What is the maximum speed? How far is the jump if the takeoff is in the bottom position?

Swing time	T = ? seconds	$T = 2 * \pi * \sqrt{l / g}$
Cord length	l = 3 m	$T = 2 * \pi * \sqrt{(3 / 9.82)}$
Acceleration	g = -9.8 m/s ²	T = 3.47 seconds

Rise height at 50 degree oscillation

Maximum speed at 0 degree oscillation:

s = ? meters	$s = l - l * \cos \alpha$	v = ? m/sec.	Ek = Ep
l = 3 meters	$s = 3 - 3 * \cos 50$	h = 1.07 m	$\frac{1}{2} * m * v^2 = m * g * h$
v = 50 degrees	s = 1.07 meters	g = 9.8 m/s ²	$v^2 = 2 * g * h$
			$v = \sqrt{(2 * g * h)}$
			$v = \sqrt{(2 * 9.82 * 1.07)}$
			v = 4.58 meters/seconds

Drop time at 0 degree oscillation

Jump length at 0 degree oscillation:

t = ? seconds	$s = \frac{1}{2} * g * t^2$	s = ? meter	s = v * t
s = 4 - 3 = 1 meters	$2 * s / g = t^2$	v = 4.58 m/s	s = 4.58 * 0.45
g = 9.8 m/s ²	$\sqrt{(2 * s / g)} = t$	t = 0.45 s	s = 2.06 meters
	$\sqrt{(2 * 1 / 9.82)} = t$		
	0.45 seconds = t		

The Economic Flow Diagram

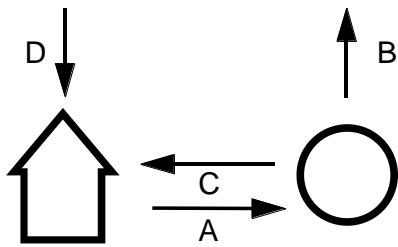
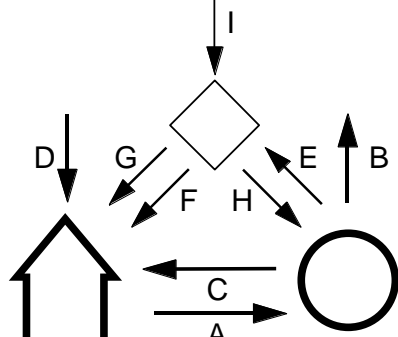
The basic economic flow consists of two sectors, production and private households. We have a number of needs that we meet by producing goods and services for others. In return, we receive an income that we can use to cover our own needs. This creates the basic economic flow consisting of the two money flows: Production creates an income A (wage), which is used for consumption C (food, clothing, etc.), which in turn leads to a new production, which in turn creates new consumption, etc. If income and consumption are in balance, the economic flow is stable. However, there is a drain and a source in the flow: Savings B and investments D. Savings are money that is not spent on consumption. Investment is money used to buy goods that cannot be consumed, e.g. buildings and machinery, etc.

The flow is stable if savings and investments are in balance. If savings are greater than investment, the flow will shrink, resulting in mass unemployment. This was the case after the First World War, when Germany was forced to send money to France as war reparations without France being obliged to buy German goods for the money. This caused the English economist J. M. Keynes to withdraw from the peace negotiations.

And this was the case in the United States during the Great Depression of the 1930s, where investment in stocks fell dramatically after the Great Wall Street crash of 1929, and where savings increased in order to repay the large loans taken out to participate in speculation on the stock market.

An economic flow with a state sector.

Keynes showed how a third public sector can balance a two-sector circuit. The public sector pulls taxes E out of the loop and uses this money to pump money back into the flow through transfer income H to the unemployed, public consumption F (more public employees, etc.) and public investment G (more roads, etc.). The public authorities may borrow loans, which will be repaid when the flow is back in balance.

 <p style="text-align: center;">production households</p>	 <p style="text-align: center;">production state households</p>																																																
<p>A model of the basic economic flow could contain 4 equations:</p> <p>First trip:</p> <table style="width: 100%; border: none;"> <tr> <td style="width: 50%;">1 Initial consumption</td> <td style="width: 50%; text-align: right;">Co = 100</td> </tr> <tr> <td>2 Initial savings</td> <td style="text-align: right;">Bo = 20</td> </tr> <tr> <td>3 The investment is assumed to be a constant percentage of consumption</td> <td style="text-align: right;">Do = d*Co</td> </tr> <tr> <td>4 Income is consumption plus investment</td> <td style="text-align: right;">Ao = Co+Do</td> </tr> </table> <p>Next trip:</p> <table style="width: 100%; border: none;"> <tr> <td style="width: 50%;">1 Consumption is assumed to be a constant percentage of income</td> <td style="width: 50%; text-align: right;">C1 = c*Ao</td> </tr> <tr> <td>2 Savings are the income that is not consumed</td> <td style="text-align: right;">B1 = Ao-C1</td> </tr> </table> <p>etc.</p> <p>2 and 4 are fact equations, 1 and 3 are fiction equations. That is, the model as such is a fiction that should be supplemented with alternative models and scenarios.</p> <p>For example. Could proportionality equations 1 and 3 be replaced by linearity equations:</p> <table style="width: 100%; border: none;"> <tr> <td style="width: 50%;">1 C1 = c*Ao -></td> <td style="width: 50%; text-align: right;">C1 = c*Ao + K</td> </tr> <tr> <td>4 D = d*C -></td> <td style="text-align: right;">D = d*C + L</td> </tr> </table> <p>Finally, an intervention can be made that changes the investment rate d from d to d+i</p> <table style="width: 100%; border: none;"> <tr> <td style="width: 50%;">D = d*C -></td> <td style="width: 50%; text-align: right;">D = (d+i)*C</td> </tr> </table> <p>This justifies the introduction of a state sector in the flow.</p> <p>In both cases, the systems of equations can be solved on an Excel spreadsheet:</p>	1 Initial consumption	Co = 100	2 Initial savings	Bo = 20	3 The investment is assumed to be a constant percentage of consumption	Do = d*Co	4 Income is consumption plus investment	Ao = Co+Do	1 Consumption is assumed to be a constant percentage of income	C1 = c*Ao	2 Savings are the income that is not consumed	B1 = Ao-C1	1 C1 = c*Ao ->	C1 = c*Ao + K	4 D = d*C ->	D = d*C + L	D = d*C ->	D = (d+i)*C	<p>A model for this 3-sector economic flow with 9 equations:</p> <p>First trip:</p> <table style="width: 100%; border: none;"> <tr> <td style="width: 50%;">1 Initial consumption</td> <td style="width: 50%; text-align: right;">Co = 100</td> </tr> <tr> <td>2 Initial savings</td> <td style="text-align: right;">Bo = 20</td> </tr> <tr> <td>3 Initial investment</td> <td style="text-align: right;">Do = 20</td> </tr> <tr> <td>4 Income is consumption plus investment</td> <td style="text-align: right;">Ao = Co+Do</td> </tr> <tr> <td>5 Initial transfers</td> <td style="text-align: right;">Ho = 4</td> </tr> </table> <p>Next trip:</p> <table style="width: 100%; border: none;"> <tr> <td style="width: 50%;">1 Taxes are assumed to be a constant percentage of income and transfers</td> <td style="width: 50%; text-align: right;">E1 = e*(Ao+Ho)</td> </tr> <tr> <td>2 Private consumption is assumed to be a constant percentage of availability C1</td> <td style="text-align: right;">= c*(Ao+Ho-E1)</td> </tr> <tr> <td>3 Savings are the disposable amount that is not consumed:</td> <td style="text-align: right;">B1 = Ao+Ho-E1-C1</td> </tr> <tr> <td>4 Government consump. assumed to be constant</td> <td style="text-align: right;">F1 = constant</td> </tr> <tr> <td>5 Public investment assumed to be a constant percentage of the investment gap</td> <td style="text-align: right;">G1 = g*(B1-Do)</td> </tr> <tr> <td>6 Private investment is assumed to be a constant percentage of consumption</td> <td style="text-align: right;">D1 = d*(C1+F1)</td> </tr> <tr> <td>7 The next income is that produced for consumption and investment</td> <td style="text-align: right;">A1 = Co+Do+Fo+Go</td> </tr> <tr> <td>8 Transfers are assumed to be a constant percentage of the employment gap</td> <td style="text-align: right;">H1 = h*(Ao-A1)</td> </tr> <tr> <td>9 Borrowing is the difference between taxes and government expenditure</td> <td style="text-align: right;">I1 = E1-F1-G1-H1</td> </tr> <tr> <td>10 Debt is the summed up borrowing</td> <td></td> </tr> </table> <p>3, 7 and 9 are fact equations, the rest are fiction equations. That is, the model as such is a fiction that should be supplemented by alternative models and scenarios.</p>	1 Initial consumption	Co = 100	2 Initial savings	Bo = 20	3 Initial investment	Do = 20	4 Income is consumption plus investment	Ao = Co+Do	5 Initial transfers	Ho = 4	1 Taxes are assumed to be a constant percentage of income and transfers	E1 = e*(Ao+Ho)	2 Private consumption is assumed to be a constant percentage of availability C1	= c*(Ao+Ho-E1)	3 Savings are the disposable amount that is not consumed:	B1 = Ao+Ho-E1-C1	4 Government consump. assumed to be constant	F1 = constant	5 Public investment assumed to be a constant percentage of the investment gap	G1 = g*(B1-Do)	6 Private investment is assumed to be a constant percentage of consumption	D1 = d*(C1+F1)	7 The next income is that produced for consumption and investment	A1 = Co+Do+Fo+Go	8 Transfers are assumed to be a constant percentage of the employment gap	H1 = h*(Ao-A1)	9 Borrowing is the difference between taxes and government expenditure	I1 = E1-F1-G1-H1	10 Debt is the summed up borrowing	
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Simulation of the economic cycle with and without a public sector

Change the rates in cells C5-C10

Don't edit the equations

Rates

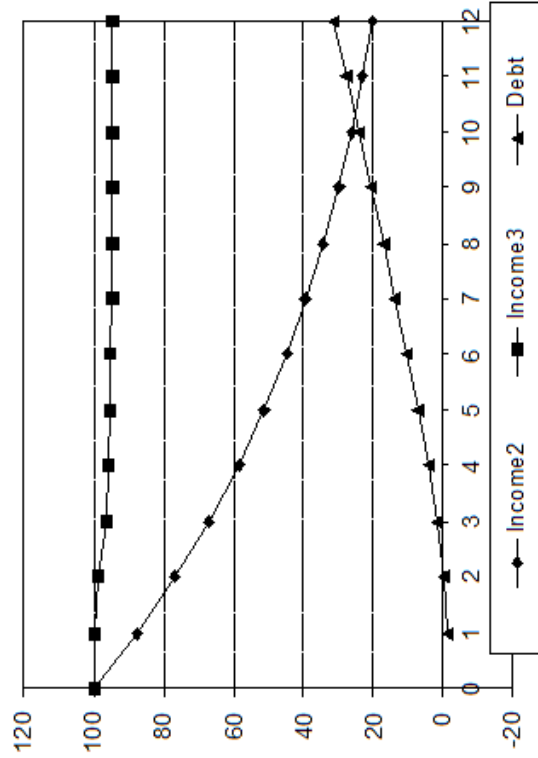
Time	n	0	1	2	3	4	5	6	7	8	9	10	11	12
Private consumption	c	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%	70%
Private Investment	d	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%	25%
Tax	e	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%	30%
Public investment	g	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%
Income transfer	h	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%	50%
Public consumption	F	20	20	20	20	20	20	20	20	20	20	20	20	20

2sector model

Time	n	0	1	2	3	4	5	6	7
Consumption	C	80	70	61,25	53,59	46,89	41,03	35,9	31,42
Savings	B	20	30	26,25	22,97	20,1	17,59	15,39	13,46
Investment	D	20	17,5	15,31	13,4	11,72	10,26	8,976	7,854
Income2	A	100	87,5	76,56	66,99	58,62	51,29	44,88	39,27

3sector model

Time	n	0	1	2	3	4	5	6	7
Income3	A	100	100	98,7	96,25	95,85	95,1	94,98	94,75
Transfers	H	4	0	0,65	1,875	2,074	2,449	2,51	2,625
Tax	E		31,2	30	29,81	29,44	29,38	29,27	29,25
Private consumption	C	80	50,96	49	48,68	48,08	47,98	47,8	47,77
Public consumption	F		20	20	20	20	20	20	20
Savings	B	20	21,84	21	20,86	20,61	20,56	20,49	20,47
Private Investment	D	20	17,74	17,25	17,17	17,02	17	16,95	16,94
Public Investment	G		10	10	10	10	10	10	10
Borrowing	I		1,2	-0,65	-2,07	-2,637	-3,071	-3,245	-3,378
Debt			-1,2	-0,55	1,52	4,157	7,228	10,47	13,85



Meeting many in a STEM context

OECD (2015b) says: “In developed economies, investment in STEM disciplines (science, technology, engineering and mathematics) is increasingly seen as a means to boost innovation and economic growth.” STEM thus combines knowledge about how humans interact with nature to survive and prosper: Mathematical formulas predict nature’s behavior, and this knowledge, logos, allows humans to invent procedures, techne, and to engineer artificial hands and muscles and brains, i.e., tools, motors and computers, that combined to robots help transforming nature into human necessities.

Nature as Things in Motion

To meet, we must specify space and time in a nature consisting of things at rest or in motion. So, in general, we see that what exists in nature is matter in space and time.

A falling ball introduces nature’s three main ingredients, matter and force and motion, similar to the three social ingredients, humans and will and obedience. As to matter, we observe three balls: the earth, the ball, and molecules in the air. Matter houses two forces, an electro-magnetic force keeping matter together when collisions transfer motion, and gravity pumping motion in and out of matter when it moves in the same or in the opposite direction of the force. In the end, the ball is at rest on the ground having transferred its motion through collisions to molecules in the air; the motion has now lost its order and can no longer be put to work. In technical terms: as to motion, its energy stays constant, but its disorder (entropy) increases. But, if the disorder increases, how is ordered life possible? Because, in the daytime the sun pumps in high-quality, low-disorder light-energy; and in the nighttime the space sucks out low-quality, high-disorder heat-energy; if not, global warming would be the consequence.

So, a core STEM curriculum could be about cycling water. Heating transforms water from solid to liquid to gas, i.e., from ice to water to steam; and cooling does the opposite. Heating an imaginary box of steam makes some molecules leave making gravity push up the lighter box until it becomes heavy water by cooling, now pulled down by gravity as rain in mountains, and through rivers to the sea. On its way down, a dam and magnets can transform moving water into moving electrons, electricity.

In the sea, water contains salt. Meeting ice at the poles, water freezes but the salt stays in the water making it so heavy it is pulled down by gravity, elsewhere pushing warm water up thus creating cycles in the ocean pumping warm water to cold regions.

The two water-cycles fueled by the sun and run by gravity leads on to other STEM areas: to the trajectory of a ball pulled down by gravity; to an electrical circuit where electrons transport energy from a source to a consumer; to dissolving matter in water; and to building roads on hillsides.

In nature, we count matter in kilograms, space in meters and time in seconds. Things in motion have a momentum = mass * velocity, a multiplication formula as most STEM formulas expressing recounting by per-numbers:

- kilogram = (kilogram/cubic-meter) * cubic-meter = density * cubic-meter
- meter = (meter/second) * second = velocity * second
- force = (force/square-meter) * square-meter = pressure * square-meter
- gram = (gram/mole) * mole = molar mass * mole
- mole = (mole/liter) * liter = molarity * liter
- energy = (energy/kg/degree) * kg * degree = heat * kg * degree
- Δ momentum = (Δ momentum/second) * second = force * seconds
- Δ energy = (Δ energy/meter) * meter = force * meter = work
- energy/sec = (energy/charge) * (charge/sec) or Watt = Volt * Amp.

Thus, STEM-subjects swarm with per-numbers: kg/m³ (density), meter/second (velocity), Joule/second (power), Joule/kg (melting), Newton/m² (pressure), etc.

Warming and Boiling Water

In a water kettle, a double-counting can take place between the time elapsed and the energy used to warm the water to boiling, and to transform the water to steam.

If pumping in 410 kiloJoule will heat 1.4 kg water 70 degrees we get a double per-number $410/70/1.4$ Joule/degree/kg or 4.18 kJ/degree/kg, called the specific heat capacity of water. If pumping in 316 kJ will transform 0.14 kg water at 100 degrees to steam at 100 degrees, the per-number is $316/0.14$ kJ/kg or 2260 kJ/kg, called the heat of evaporation for water.

Dissolving Material in Water

In the sea, salt is dissolved in water, described as the per liter number of moles, each containing a million billion billion molecules. A mole of salt weighs 59 gram, so recounting 100 gram salt in moles we get $100 \text{ gram} = (100/59) * 59 \text{ gram} = (100/59) * 1 \text{ mole} = 1.69 \text{ mole}$, that dissolved in 2.5 liter has a strength as 1.69 moles per 2.5 liters or 1.69/2.5 mole/liter, or 0.676 mole/liter.

Building Batteries with Water

At our planet life exists in three forms: black, green and grey cells. Green cells absorb the sun's energy directly; and by using it to replace oxygen with water, they transform burned carbon dioxide to unburned carbohydrate storing the energy for grey cells, releasing the energy by replacing water with oxygen; or for black cells that by removing the oxygen transform carbohydrate into hydrocarbon storing the energy as fossil energy. Atoms combine by sharing electrons. At the oxygen atom the binding force is extra strong releasing energy when burning hydrogen and carbon to produce harmless water H₂O, and carbon dioxide CO₂, producing global warming if not bound in carbohydrate batteries. In the hydrocarbon molecule methane, CH₄, the energy comes from using 4 oxygen atoms to burn it.

Technology & Engineering: Steam and Electrons Produce and Distribute Energy

A water molecule contains two hydrogen and one oxygen atom weighing $2 * 1 + 16$ units making a mole of water weigh 18 gram. Since the density of water is roughly 1 kilogram/liter, the volume of 1000 moles is 18 liters. With about 22.4 liter per mole, its volume increases to about $22.4 * 1000$ liters if transformed into steam, which is an increase factor of 22,400 liters per 18 liters = 1,244 times. But, if kept constant, instead the inside pressure will increase as predicted by the ideal gas law, $p * V = n * R * T$, combining the pressure p, and the volume V, with the number of moles n, and the absolute temperature T, which adds 273 degrees to the Celsius temperature. R is a constant depending on the units used.

The formula expresses different proportionalities: The pressure is direct proportional with the number of moles and the absolute temperature so that doubling one means doubling the other also; and inverse proportional with the volume, so that doubling one means halving the other.

Thus, with a piston at the top of a cylinder with water, evaporation will make the piston move up, and vice versa down if steam is condensed back into water. This is used in steam engines. In the first generation, water in a cylinder was heated and cooled by turn. In the next generation, a closed cylinder had two holes on each side of an interior moving piston thus increasing and decreasing the pressure by letting steam in and out of the two holes. The leaving steam is visible on e.g., steam locomotives.

Power plants use a third generation of steam engines. Here a hot and a cold cylinder are connected with two tubes allowing water to circulate inside the cylinders. In the hot cylinder, heating increases the pressure by increasing both the temperature and the number of steam moles; and vice versa in the cold cylinder where cooling decreases the pressure by decreasing both the temperature and the number of steam moles condensed to water, pumped back into the hot cylinder in one of the tubes.

In the other tube, the pressure difference makes blowing steam rotate a mill that rotates a magnet over a wire, which makes electrons move and carry electrical energy to consumers.

An Electrical Circuit

Energy consumption is given in Watt, a per-number double-counting the number of Joules per second. Thus, a 2000 Watt water kettle needs 2000 Joules per second. The socket delivers 220 Volts, a per-number double-counting the number of Joules per 'carrier' (charge-unit). Recounting 2000 in 220 gives $(2000/220)*220 = 9.1*220$, so we need 9.1 carriers per second, which is called the electrical current counted in Ampere, a per-number double-counting the number of carriers per second. To create this current, the kettle must have a resistance R according to a circuit law 'Volt = Resistance*Ampere', i.e., $220 = \text{Resistance}*9.1$, or Resistance = 24.2 Volt/Ampere called Ohm. Since Watt = Joule per second = (Joule per carrier)*(carrier per second) we also have a second formula, Watt = Volt*Ampere. Thus, with a 60 Watt and a 120-Watt bulb, the latter needs twice the energy and current, and consequently has half the resistance of the former, making the latter receive half the energy if connected in series.

How High Up and How Far Out

A spring sends a ping-pong ball upwards. This allows a recounting between the distance and the time to the top, e.g. 5 meters and 1 second. The gravity decreases the vertical speed when going up and increases it when going down, called the acceleration, a per-number counting the change in speed per second. To find its initial speed we turn the spring 45 degrees and count the number of vertical and horizontal meters to the top as well as the number of seconds it takes, e.g., 2.5 meters, 5 meters and 0,71 seconds. From a folding ruler we see that now the total speed is split into a vertical and a horizontal part, both reducing the total speed with the same factor $\sin 45 = \cos 45 = 0,707$. The vertical speed decreases to zero, but the horizontal speed stays constant. So we can find the initial speed u by the formula: Horizontal distance to the top position = horizontal speed * time, or with numbers: $5 = (u*0,707)*0,71$, solved as $u = 9.92$ meter/seconds by moving to the opposite side with opposite calculation sign, or by a solver-app. Compared with the horizontal distance, the vertical distance is halved, but the speed changes from 9.92 to $9.92*0.707 = 7.01$. However, the speed squared is halved from $9.92*9.92 = 98.4$ to $7.01*7.01 = 49.2$. So horizontally, the distance and the speed are proportional. Whereas vertically, there is a proportionality between the distance and the speed squared, so that doubling the vertical speed will increase the vertical distance four times.

How to construct a road up a steep hill side

On a 30-degree hillside, a 10-degree road is constructed. How many turns will there be on a 1 km by 1 km hillside?

We let A and B label the ground corners of the hillside. C labels the point where a road from A meets the edge for the first time, and D is vertically below C on ground level. We want to find the distance $BC = u$.

In the triangle BCD, the angle B is 30 degrees, and $BD = u*\cos(30)$. With Pythagoras we get $u^2 = CD^2 + BD^2 = CD^2 + u^2*\cos(30)^2$, or $CD^2 = u^2(1-\cos(30)^2) = u^2*\sin(30)^2$. In the triangle ACD, the angle A is 10 degrees, and $AD = AC*\cos(10)$. With Pythagoras we get $AC^2 = CD^2 + AD^2 = CD^2 + AC^2*\cos(10)^2$, or $CD^2 = AC^2(1-\cos(10)^2) = AC^2*\sin(10)^2$. In the triangle ACB, $AB = 1$ and $BC = u$, so with Pythagoras we get $AC^2 = 1^2 + u^2$, or $AC = \sqrt{1+u^2}$.

Consequently, $u^2*\sin(30)^2 = AC^2*\sin(10)^2$, or $u = AC*\sin(10)/\sin(30) = AC*r$, or $u = \sqrt{1+u^2}*r$, or $u^2 = (1+u^2)*r^2$, or $u^2*(1-r^2) = r^2$, or $u^2 = r^2/(1-r^2) = 0.137$, giving the distance $BC = u = \sqrt{0.137} = 0.37$.

Thus, there will be 2 turns: 370 meter and 740 meter up the hillside.

Jumping from a swing

When I jump from a swing I have the maximum speed at the bottom point. But here the angle with horizontal begins increasing from zero. But now the speed decreases. So a what point should I jump to obtain a maximum length?